

Universality classes in anisotropic non-equilibrium growth models

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We study the effect of generic spatial anisotropies on the scaling behavior in the Kardar-Parisi-Zhang equation. In contrast to its “conserved” variants, anisotropic perturbations are found to be relevant in $d > 2$ dimensions, leading to rich phenomena that include novel universality classes and the possibility of first-order phase transitions and multicritical behavior. These results question the presumed scaling universality in the strong-coupling rough phase, and shed further light on the connection with generalized driven diffusive systems.

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In non-equilibrium systems with *conserved* order parameter, spatial anisotropies emerge as highly relevant perturbations, leading to drastic changes in the universal scaling laws. For example, in driven diffusive systems (DDS) such as the driven Ising lattice gas [1], an external drive generates a steady-state current that explicitly singles out one spatial direction. This leads to characteristic singularities in the structure factor already in the disordered phase, implies anisotropic scaling at the phase transition, and generates rich ordered structures in the low-temperature phase [2]. Related anisotropic scaling behavior even ensues near the critical point of an Ising or Heisenberg system with conserved order parameter dynamics that is driven out of equilibrium simply by imposing different Langevin noise strengths in different spatial directions, thus violating detailed balance [3,4].

Yet, one might expect that in models with a *non-conserved* order parameter, the picture could rather look similar to the situation in equilibrium systems, where anisotropies have much less dramatic effects. In order to further explore this issue, we study the Kardar–Parisi–Zhang (KPZ) model for kinetic roughening [5], which has become another prototypical model for generic scale invariance far from equilibrium [6,7]. Its anisotropic generalization reads

$$\partial_t h = \nu_{\parallel} \partial_{\parallel}^2 h + \nu_{\perp} \partial_{\perp}^2 h + \frac{\lambda_{\parallel}}{2} (\partial_{\parallel} h)^2 + \frac{\lambda_{\perp}}{2} (\partial_{\perp} h)^2 + \eta, \quad (1)$$

with relaxation constants $\nu_{\parallel/\perp} > 0$, but no restrictions on the signs of the non-linearities $\lambda_{\parallel/\perp}$ that describe curvature-driven growth. We denote the dimensionalities of the longitudinal and transverse sectors with d_{\parallel} and d_{\perp} , respectively, with $d_{\parallel} + d_{\perp} = d$. η is a stochastic driving force with Gaussian correlations determined by the second moment $\langle \eta(\mathbf{x}', t') \eta(\mathbf{x}, t) \rangle = 2D \delta(\mathbf{x} - \mathbf{x}') \delta(t - t')$. After a simple length rescaling one may of course either choose $\nu_{\parallel} = \nu_{\perp}$ or $|\lambda_{\parallel}| = |\lambda_{\perp}|$.

A previous study of this model restricted to two spatial dimensions ($d_{\parallel} = d_{\perp} = 1$) found the anisotropy to be irrelevant for λ_{\parallel} and λ_{\perp} both positive [8]. Interestingly, if the non-linear terms come with opposite sign, the

anisotropy is still irrelevant, but in addition also the non-linearity itself scales to zero. In a related model, where one of the non-linear couplings vanishes, the relevance of the remaining non-linearity was shown to depend on the values of d_{\parallel} and d_{\perp} [9]. It would therefore appear that anisotropies play only a minor role in kinetic roughening phenomena: Only if we render the signs of the non-linear couplings different, or set one of them to zero, do we seem to obtain any change in the scaling behavior. We shall, however, see that in any physical dimension larger than 2, the above anisotropies destroy isotropic scaling at long wavelengths and in the long-time limit, and generate remarkably diverse behavior. This supports earlier conjectures that perhaps the strong-coupling scaling regime in the KPZ problem is *not* governed by universal scaling exponents [10], but rather subtly depends on microscopic details. These are of course captured only rudimentarily in our simple anisotropies. For the original KPZ non-equilibrium growth or driven interface problem, this issue would appear to be mostly of academic interest; yet in light of a recent suggestion that the asymptotic scaling properties of interfaces pulled into unstable regions should be described by the $(d + 1)$ -dimensional rather than the d -dimensional KPZ equation [11], our findings may well become directly accessible to real experiments.

We have analyzed the non-linear Langevin equation (1) by means of the dynamic renormalization group to one-loop order in the perturbation expansion with respect to the non-linear couplings $\lambda_{\parallel/\perp}$. The calculation is a straightforward generalization of the one-loop treatment for the isotropic KPZ equation or the equivalent noisy Burgers equation [12,5]. We map the Langevin equation to a dynamic functional, and proceed therefrom using standard field-theoretic tools [13]. The renormalization constants that track the ultraviolet singularities in $d \geq 2$ dimensions are determined at a finite external wave vector $q = \mu$ (or equivalently, at non-zero external frequency) [14]. The scaling behavior of the theory with respect to this normalization scale is then encoded in the associated RG equations, which are solved

by the method of characteristics ($\mu \rightarrow \mu l$). The ensuing first-order differential flow equations define the running couplings as functions of the flow parameter l (or momentum scale μ). We have to distinguish between the cases where both non-linearities have the same or opposite signs, $\lambda = \lambda_{\parallel} = \pm \lambda_{\perp}$. The RG recursion relations $l\partial\alpha/\partial l = \beta_{\alpha}(g, \gamma)$ for the *anisotropy ratio* $\gamma = \nu_{\parallel}/\nu_{\perp}$ and the *effective coupling* $g = A_d \mu^{d-2} \lambda^2 D \gamma^{d_{\perp}/2} / \nu_{\parallel}^3$, where $A_d = \Gamma(2 - \frac{d}{2}) / [2^{d+1} \pi^{d/2} d(d+2)]$ denotes a geometric factor, are determined by the RG beta functions

$$\beta_g = g \left(d - 2 + \zeta_D - 3\zeta_{\nu_{\parallel}} + \frac{d_{\perp}}{2} \zeta_{\gamma} \right), \quad (2)$$

$$\beta_{\gamma} = \gamma \zeta_{\gamma} = \gamma (\zeta_{\nu_{\parallel}} - \zeta_{\nu_{\perp}}), \quad (3)$$

with the explicit one-loop flow functions

$$\zeta_D = -g [d_{\parallel}(d_{\parallel} + 2) + d_{\perp}(d_{\perp} + 2)\gamma^2 \pm 2d_{\parallel}d_{\perp}\gamma], \quad (4)$$

$$\zeta_{\nu_{\parallel}} = g [d(d_{\parallel} \pm d_{\perp}\gamma) - 4], \quad (5)$$

$$\zeta_{\nu_{\perp}} = g\gamma [d(\pm d_{\parallel} + d_{\perp}\gamma) - 4\gamma]. \quad (6)$$

The recursion relations for the special case $\lambda_{\perp} = 0$ are readily obtained by setting $\gamma = 0$ in the expressions for these zeta functions. The zeros of the beta functions (2), (3) yield the RG fixed points, describing scale-invariant behavior. The universal infrared scaling exponents are then given by the corresponding fixed-point values of the zeta functions (4)–(6). Below the critical dimension $d_c = 2$, the theory is ultraviolet-finite, and the emerging RG fixed point is infrared-stable. For $d \geq d_c$, the field theory remains renormalizable as a consequence of the emergence of an infrared-*unstable* fixed point and the underlying infinitesimal tilt invariance of the interface (Galilean invariance for the Burgers equation) [15]. Technically, renormalizability applies only in a systematic ϵ expansion about the lower critical dimension for the roughening transition ($\epsilon = d - 2$). We shall nevertheless consider the theory also at fixed $d = d_{\parallel} + d_{\perp}$, with either sector dimension or $\Delta \equiv d_{\parallel} - d_{\perp}$ as parameters.

The first question to be addressed is whether the anisotropy is a relevant perturbation at the isotropic KPZ fixed point. A previous study established that the isotropic fixed point is stable in 2 + 1 dimensions ($d_{\parallel} = d_{\perp} = 1$) [8]. However, our one-loop analysis reveals this to be far from true in general. In fact, the RG flow equations, specifically Eq. (3) with (5) and (6), allow for *four* different scenarios, depending on the values of d and Δ . Without loss of generality we can restrict ourselves to $\Delta > 0$; for $\Delta < 0$ merely the roles of perpendicular and parallel components are switched. In an analysis at *fixed* d and Δ we find the following regimes (depicted in Fig. 1):

In *regime A*, $d < -\Delta/2 + \sqrt{(\Delta/2)^2 + 8}$, both the isotropic fixed point $\gamma_1 = 1$ and the anisotropic fixed point $\gamma_A = (4 - dd_{\parallel})/(dd_{\perp} - 4) < 0$ are stable with an intermediate unstable uniaxial fixed point $\gamma_0 = 0$. Note

(see remark after flow equations) that a negative fixed point in the anisotropy ratio does not mean that one of the relaxation constants ν_{α} becomes negative, but that we are at a fixed point where the non-linearities have opposite sign. For the special case $d_{\parallel} = d_{\perp}$ one obtains $\gamma_A = -1$, and thus Wolf's earlier results [8] are recovered. In *regime B*, $-\Delta/2 + \sqrt{(\Delta/2)^2 + 8} < d < \sqrt{8}$, the anisotropic fixed point becomes positive, $0 < \gamma_A < 1$, and switches roles with the uniaxial fixed point γ_0 . In *regime C*, $\sqrt{8} < d < \Delta/2 + \sqrt{(\Delta/2)^2 + 8}$, the anisotropic fixed point becomes $\gamma_A > 1$ and again stable, whereas the isotropic fixed point is now unstable. Finally, in *regime D*, $d > \Delta/2 + \sqrt{(\Delta/2)^2 + 8}$, the anisotropic fixed point assumes again a negative value. Yet here the stability features of the fixed points are the converse of regime A.

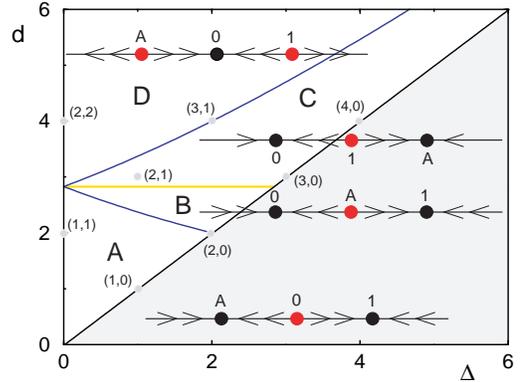


FIG. 1. Stability diagrams for the fixed points of the anisotropy ratio γ as functions of d and Δ ; also indicated are $(d_{\parallel}, d_{\perp})$ at some particularly interesting dimensions.

Note that the stability boundaries for the various fixed points of the anisotropy ratio γ , as determined from Eq. (3), merely require the *existence* of a finite fixed point $0 < g^* < \infty$, and are independent of its actual value. Hence one may speculate that a topologically similar if not identical stability diagram applies for the critical fixed point g_c even beyond our one-loop approximation, and might perhaps even extend to the “strong-coupling” scaling behavior in the rough phase.

The RG flows, projected onto the γ - g plane, are illustrated in Fig. 2 for scenario C. There are four critical fixed points, each one representing a different universality class (as discussed below). In the neighborhood of the uniaxial and the anisotropic fixed points, the flows along the critical surface are towards each of them, since they each possess only one relevant scaling variable, the effective coupling constant g . The fixed point at $\gamma_1 = 1$ (describing the *isotropic* kinetic roughening transition), located at the junction of two lines of critical points, is an example of a *non-equilibrium bicritical point*. The RG flows are quite reminiscent of a bicritical point in uniaxial magnetic systems, with the high- and low-temperature phases of the magnetic system corresponding to the smooth and rough phases of the kinetic roughening problem. By anal-

ogy this suggests that there might actually exist a *first-order* phase transition along the line $\gamma_1 = 1$ for $g > g_c$, separating two distinct non-equilibrium strong-coupling phases of different symmetry. Whereas the latter statement is somewhat speculative, one can systematically study the critical behavior of the new universality classes at the roughening transition.

If $\gamma \rightarrow 0$, i.e. $\nu_\perp \rightarrow \infty$, the model reduces to the uniaxial KPZ equation with $\lambda_\perp = 0$. This can be seen by direct inspection of the RG flow equation, but it is also suggested by intuition since an infinitely large surface tension ν_\perp will always dominate over any finite non-linearity. With $\mathbf{u} = \partial_\parallel h$ one realizes immediately that this *uniaxial KPZ equation* maps onto the *generalized driven diffusion equation* (GDDS)

$$\partial_t \mathbf{u} = (\nu_\parallel \partial_\parallel^2 + \nu_\perp \partial_\perp^2) \mathbf{u} + \frac{\lambda_\parallel}{2} \partial_\parallel \mathbf{u}^2 + \zeta, \quad (7)$$

where \mathbf{u} is a vector field with d_\parallel components and ζ is a conserved noise with correlator $\langle \zeta_i(\mathbf{x}, t) \zeta_j(\mathbf{x}', t') \rangle = -2D \delta_{ij} \nabla^2 \delta(\mathbf{x} - \mathbf{x}') \delta(t - t')$. Eq. (7) was originally introduced and analyzed to describe driven line liquids [9].

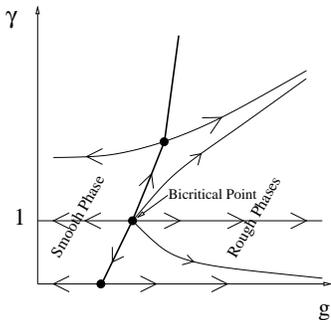


FIG. 2. Sketch of the projected renormalization group flows for the anisotropic KPZ equation for scenario C.

A crucial observation is that the nature of the fixed point of the effective coupling constant g changes drastically as a function of d and Δ . This can be seen by inspecting the RG beta function for g in Eq. (2). Note that the fixed-point value of the effective coupling constant diverges and changes sign at $\zeta_D = (3 - d_\perp/2)\zeta_{\nu_\parallel}$. In an ϵ expansion with respect to the critical dimension $d_c = 2$ one then finds two completely different regimes: For $\zeta_D < (3 - d_\perp/2)\zeta_{\nu_\parallel}$, i.e., to leading order in ϵ for $\Delta \geq 0.47 - 1.89\epsilon$, there exists a critical fixed point of order $\epsilon = d - 2$ above two dimensions marking a non-equilibrium phase transition from a smooth to a rough phase. In the converse case, the non-linearity is perturbatively irrelevant above $d_c = 2$, and consequently the roughening transition disappears. In addition, below two dimensions one finds a stable fixed of order $\epsilon' = 2 - d$ describing anomalous scaling behavior. In other words, there exists a critical value $\Delta_c \approx 0.47$ where $d_c = 2$ changes its role from a *lower* to an *upper* critical dimension,

and the elusive strong-coupling scenario is transformed into a weak-coupling one.

These observations have several implications. First, owing to the GDDS connection, it is possible to access the one-dimensional KPZ scaling ($d_\parallel = 1$, $d_\perp = 0$) *perturbatively* by expanding around the upper critical dimension $d_c = 2$ of the standard non-critical DDS model ($d_\parallel = 1$, $d_\perp = 1$). Second, a naive extrapolation of $\Delta = 0.47 - 1.89\epsilon$ to $\epsilon = 1$ ($d = 3$) would suggest that DDS with $d_\parallel = 1$ and $d_\perp = 2$ might display a non-equilibrium phase transition. However, this is most likely an artifact of the ϵ expansion, and higher-order terms in the perturbation series will probably result in a boundary line for the critical fixed point that asymptotically approaches $d_\parallel = 1$ as $d_\perp \rightarrow \infty$ such that purely mean-field behavior ensues for $d > 2$. Yet careful numerical simulations of the GDDS model that map out the (d, Δ) -plane and determine the exact location of the boundary line, beyond the regime close to $d = 2$ accessible in a perturbative RG approach, would certainly be very desirable.

There are now several ways to calculate the exponents in an ϵ expansion with respect to $d_c = 2$. Keeping, e.g., $\Delta = d_\parallel - d_\perp$ fixed, we find $z_\parallel = 2 + 2(\Delta - 2)\epsilon/(\Delta^2 + 8\Delta - 4)$ and $z_\perp = 2$, which describes, depending on the value of Δ , the critical fixed point of the roughening transition or the stable fixed point below d_c . For example, extrapolating to $(d_\parallel, d_\perp) = (2, 1)$ yields $z_\parallel \approx 1.6$. The actual value for z found here is actually not crucial — the one-loop analysis should not be expected to be very accurate — but the very fact that it differs from $z_c = 2$, the *exact* result for the roughening transition of the isotropic KPZ equation. Furthermore, the RG analysis at the uniaxial fixed point shows no sign of an upper critical dimension, while it is known that $d = 4$ is the upper critical dimension for the roughening transition of the isotropic KPZ model [14,16].

Besides the uniaxial fixed points $\gamma_0 = 0$ and $\gamma_\infty = \infty$ there exists also an anisotropic fixed point with a *finite* anisotropy ratio $\gamma_A = (4 - dd_\parallel)/(dd_\perp - 4)$. We find that the non-equilibrium dynamics described by the anisotropic and isotropic fixed point are markedly different. At first sight this may seem quite surprising since at both fixed points the parallel and perpendicular surface tension term $\nu_{\parallel,\perp}$ show *identical* scaling behavior under a RG transformation; the only difference resides in their amplitudes. The reason for this unusual behavior is that $\gamma = 1$ is a *non-generic* high-symmetry point in parameter space, allowing the isotropic KPZ problem to be mapped onto the statistical mechanics of a directed polymer in a random medium. The high symmetry is also reflected in the mean-field value for the dynamic exponent $z_c = 2$ at the *non-trivial* fixed point describing the roughening transition. In contrast, our RG analysis at the anisotropic fixed point γ_A yields a dynamic critical exponent *different* from 2, to leading order in a $2 + \epsilon$ expansion at fixed Δ given by $z = 2 + \epsilon(\Delta^2 - 4)/(10\Delta^2 - 8)$.

Similar to the GDDS equation, there exists a critical line $\Delta_c(d)$ such that for $|\Delta| < \Delta_c(d)$ there is no roughening transition above $d = 2$. To leading order in ϵ we obtain $\Delta_c(d) = \pm(2/\sqrt{5} - 13\epsilon/5\sqrt{5})$. Since $|\Delta_c(d)| < 1$ for $d > 2$, a scenario with no roughening transition appears unlikely for any integer dimensions d_{\parallel} and d_{\perp} . For $\Delta > \Delta_c(d)$ there is, as for the uniaxial fixed point, one set of dimensions which is particularly interesting, namely $(d_{\parallel}, d_{\perp}) = (2, 1)$. Upon extrapolating the result obtained from our ϵ expansion we arrive at an unphysically low critical value $z_c = 1/2 \pm O(\epsilon^2)$. Unfortunately, this leading-order result is not accurate enough to wager a reliable quantitative prediction for the actual value of the dynamic critical exponent. But, as before, the important conclusion to be drawn from the RG analysis is again that the dynamic critical exponent differs from the isotropic value $z_c = 2$. In order to gain reliable quantitative estimates for the exponents of this novel universality class, careful numerical simulations would be invaluable.

From the above analysis of the various universality classes and our understanding of the isotropic KPZ equation we derive the following picture. With the exception of the particular case $d_{\parallel} = d_{\perp}$, the anisotropic fixed point always displays strong-coupling behavior, i.e. we find a roughening transition above the lower critical dimension $d_c = 2$ similar to the isotropic fixed point. However, the universality class of the roughening transition is markedly different from the isotropic KPZ model. In contrast to the anisotropic and the isotropic fixed points, the uniaxial fixed point $\gamma_0 = 0$ (GDDS) shows extended regions in the (d, Δ) plane with strong- and weak-coupling behavior, respectively. In particular, for $d_{\parallel} \leq d_{\perp}$ the roughening transition is entirely absent and we have anomalous scaling only below $d_c = 2$.

Some specific examples serve to illustrate the implications of the above results. For $(d_{\parallel}, d_{\perp}) = (3, 1)$ the isotropic fixed point $\gamma_1 = 1$ is unstable. If $\gamma < 1$, the RG flow tends towards the uniaxial fixed point. The corresponding GDDS universality class lacks the special symmetry of the isotropic KPZ equation implying that the dynamic critical exponent differs from $z_c = 2$. If $\gamma > 1$, the RG trajectories flow towards $\gamma = \infty$. This fixed point lies again in the GDDS universality class, but with interchanged roles of d_{\parallel} and d_{\perp} . Because of the symmetry of the model, the asymptotic behavior is equivalent to that at the fixed point $\gamma_0 = 0$ in $(d_{\parallel}, d_{\perp}) = (1, 3)$ space dimensions, where *no* roughening transition occurs above $d_c = 2$. This implies not only that the isotropic KPZ equation is *unstable* in $d = 4$ with respect to even the slightest amount of anisotropy, but more dramatically also that there exists *no* phase transition and hence *no* strong-coupling behavior at all, provided the relaxation amplitudes satisfy $\nu_{\parallel} < \nu_{\perp}$ for $d_{\parallel} = 1$. Similar critical end points are found throughout domain D of Fig.1.

Finally, we note that we have performed a similar one-loop RG analysis for the ‘‘conserved’’ KPZ variants, with

essentially an additional Laplacian on the right-hand-side of Eq. (1), both with conserved [17] and non-conserved white noise [18]. Neither of these models displays a roughening transition. Rather, there is a single universal scaling regime with non-trivial exponents below the upper critical dimensions $d_c = 4$ and 2, respectively. It then turns out that anisotropic perturbations do not matter crucially: The standard *isotropic* fixed point remains *stable* in both cases [19]. It would thus appear that the novel features reported above for the anisotropic KPZ equation are crucially connected with the very existence of a roughening transition, and thus with the scaling properties of a strong-coupling rough phase.

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