QUANTUM $\mathbb{F}_{\mathbf{un}}$:
THE $q = 1$ LIMIT OF GALOIS FIELD QUANTUM MECHANICS, PROJECTIVE GEOMETRY & THE FIELD WITH ONE ELEMENT

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We argue that the $q = 1$ limit of Galois Field Quantum Mechanics, which was constructed on a vector space over the Galois Field $\mathbb{F}_q = GF(q)$, corresponds to its 'classical limit,' where superposition of states is disallowed. The limit preserves the projective geometry nature of the state space, and can be understood as being constructed on an appropriately defined analogue of a 'vector' space over the 'field with one element' $\mathbb{F}_1$.

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1. Introduction - the Classical Limit of Quantum Mechanics

How deterministic classical mechanics (CM) emerges from probabilistic quantum mechanics (QM) is a conundrum which is yet to be resolved. As first noticed by Dirac[1], the Poisson brackets of CM can be considered the $\hbar \to 0$ limit of commutators in QM, and the Heisenberg equation goes over to Hamilton's equation in that limit:

$$\frac{d\hat{A}}{dt} = \frac{1}{i\hbar}[\hat{A}, \hat{H}] \qquad \xrightarrow{\hbar \to 0} \qquad \frac{dA}{dt} = \{A, H\}. \tag{1}$$

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Despite this formal correspondence, however, it is far from clear how the full classical theory would be recovered from QM in such a limit. In particular, what happens to the Hilbert space of QM in the $\hbar \to 0$ limit is a difficult question.[a] Also, $\hbar$ being a dimensionful quantity, the limit $\hbar \to 0$ itself is not particularly well-defined.[b]

In order to see the CM$\leftrightarrow$QM correspondence more clearly, several attempts have been made to make CM look more like QM via the introduction of wave-functions, operators, and probability distributions. The WKB approximation[3,4,5] to the Schrödinger equation has been reinterpreted as the wave-function of CM by Van Vleck[6] and subsequently by Schiller,[7,8,9] with observables represented by commuting hermitian operators with continuous eigenvalues. However, utilizing the rewriting of the Schrödinger equation by de Broglie in his pilot-wave theory,[10] which was later elaborated on by Bohm to discuss a hidden variable interpretation of QM,[11,12] Rosen has shown that taking the $\hbar \to 0$ limit of the Schrödinger equation does not necessarily recover the Hamilton-Jacobi equation for superpositions of states.[13,14,15]

Koopman[16] and von Neumann[17,18], and later Sudarshan[19], developed a complete formulation of CM on a Hilbert space, which was again characterized by commuting hermitian operators as observables. Recent work on extending the Koopman-von Neumann-Sudarshan formalism with the introduction of path integrals, etc. include Refs. 20, 21, 22, 23, 24. It should be emphasized that the Koopman-von Neumann-Sudarshan theory is not the $\hbar \to 0$ limit of QM. Indeed, the operators that correspond to position and momentum in their formalism commute with each other without the taking of any limit, and respectively have canonically conjugate operators with which they do not commute but are deemed unobservable. The superposition of states also correspond to ensembles of classical states, and not macroscopic Schrödinger-cat like states. Thus, though the Koopman-von Neumann-Sudarshan theory is a formulation of CM on a Hilbert space, it does not (yet) provide much insight on how CM can emerge from QM, or vice versa.

Given the apparent absence of macroscopic Schrödinger-cat like states, they must somehow vanish in the classical limit. Several approaches have been used to address this problem, the most prominent being that of 'environmental decoherence' reviewed in Refs. 25, 26 by Zurek. There, the system under observation and its environment (rest of the universe) are both treated quantum mechanically, and it has been shown that the interaction between the two suppresses the off-diagonal terms in the density matrix in a preferred basis. The statistical nature of the theory remains, however, and leads to the many-worlds interpretation of QM developed from the pioneering work of Everett[27]. Other approaches to the system-environment interaction problem treat the environment classically, and the

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[a]The singularity of the $\hbar \to 0$ limit has been emphasized in an illuminating review by Berry[2].
[b]Given the dimensionfulness of $\hbar$, the $\hbar \to 0$ limit should be understood as $\hbar \ll S$ where S is the classical action that is used to describe the quantum formulation as, say, in the path integral approach.

interaction with the quantum system leads to classical-quantum 'hybrid' theories, the properties of which are still under investigation.[c]

Thus, despite impressive developments in our understanding of QM during the past century, the CM/QM divide remains and bridging that gap is still an intense area of investigation. What seems probable is that recovering CM from QM would require more than taking the $\hbar \to 0$ limit of QM in some yet to be discovered way, or the rewriting of CM or QM to look more like the other in some fashion. A more generic theory which encompasses both CM and QM may be necessary to understand the CM↔QM correspondence.

As with any difficult problem, finding a toy analog which simplifies the situation while maintaining the essence of the quandary is often instructive. In Refs. 40 and 41 we constructed a toy analog of N-level QM on the vector space $\mathbb{F}_q^N$, where $\mathbb{F}_q = GF(q)$ is the Galois field of order $q = p^n$, with p prime and $n \in \mathbb{N}$, which we dubbed "Galois Field Quantum Mechanics" (GQM).[d] GQM was necessarily different from canonical QM in many ways, due to the vector space $\mathbb{F}_q^N$ not possessing an inner product. Consequently, observables were not represented by operators, hermiticity being difficult to define without an inner product[e], and without operators there were no commutators, or $\hbar$ for that matter. However, in it, physical states were still represented by elements of a projective geometry, and the theory still predicted probabilities of the outcomes of a measurement which could not be mimicked by any hidden variable theory. Thus GQM, despite being constructed on a discrete and finite vector space $\mathbb{F}_q^N$ without an inner product, nevertheless captured some of the quantum-ness of canonical QM. A natural question to ask then is: does GQM have a 'classical' limit in which this quantum-ness is lost and replaced by classical-ness? An answer to this could help us understand how canonical QM becomes CM also.

In this paper, we present the observation that even though GQM does not have an $\hbar$ in its formulation, its 'classical' limit can still be defined by taking the limit $q \to 1$. In that limit, the Galois Field $\mathbb{F}_q$ can be expected to become $\mathbb{F}_1$, aka $\mathbb{F}_{un}$,[f] the 'field with one element,' an exotic and somewhat nebulous mathematical concept first suggested by Tits in 1957.[58] $\mathbb{F}_q$ in turn can be considered the 'quantum'-deformation of $\mathbb{F}_1$.[59] Though dormant for many decades, the study of $\mathbb{F}_1$, and efforts to actually define what it is, has intensified since the 1990's[g] under the expectation

[c]See Refs. 19, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 39 and references therein.

[d]See also Refs. 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55.

[e]It is possible to define analogs of hermitian operators using biorthogonal systems. See Refs. 51, 56, 57.

[f]Denoting $\mathbb{F}_1$ as $\mathbb{F}_{un}$ is a French-English bilingual pun. See, e.g. Refs. 75, 76.

[g]See, for instance, Refs. 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84.

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that it would lead to a proof of the Riemann hypothesis.[h][i]

This paper is organized as follows. In section 2 we review how GQM is constructed on $\mathbb{F}_q^N$. We will look at what happens to the theory if we let $q = 1$ for the case $N = 2$, and discuss its 'classical' properties. In section 3 we review the notion of $\mathbb{F}_1$, the 'field with one element,' following the treatment of Kurokawa and Koyama,[85] and show that the $q = 1$ limit of GQM can be constructed directly on the 'vector' space $\mathbb{F}_1^N$. The resulting state space is a projective geometry, just as with GQM and canonical QM. The theory on $\mathbb{F}_1^N$ also prohibits the superposition of states, precisely the property one expects in a 'classical' theory. Discussion on what this teaches us is given in section 4.

2. Galois Field Quantum Mechanics

2.1. *The Model*

Since the details of GQM can be found in Refs. 40 and 41, here we only present the basic outline.

Consider the vector space $\mathbb{F}_q^N$. Vectors in $\mathbb{F}_q^N$ represent states of the model system, while dual vectors in the dual vector space $\mathbb{F}_q^{N*}$ represent possible outcomes of measurements. The probability of obtaining the outcome represented by the dual-vector $\langle x| \in \mathbb{F}_q^{N*}$ when a measurement is performed on the state represented by the vector $|\psi\rangle \in \mathbb{F}_q^N$ is given by

$$P(x|\psi) \;=\; \frac{\left|\langle x|\psi\rangle\right|^2}{\sum_y \left|\langle y|\psi\rangle\right|^2} \;, \tag{2}$$

where the sum in the denominator runs over all the dual vectors $\langle y|$ in a basis of $\mathbb{F}_q^{N*}$ which includes $\langle x|$. The choice of basis of $\mathbb{F}_q^{N*}$ corresponds to an observable, each dual vector in the basis representing a different outcome.

The absolute value function in the above expression converts elements of $\mathbb{F}_q$ into either zero or one in $\mathbb{R}$:

$$|\underline{k}| \;=\; \begin{cases} 0 & \text{if } \underline{k} = \underline{0} \;, \\ 1 & \text{if } \underline{k} \neq \underline{0} \;. \end{cases} \tag{3}$$

Here, underlined numbers and symbols represent elements of $\mathbb{F}_q$, to distinguish them from elements of $\mathbb{R}$ or $\mathbb{C}$. Since $\mathbb{F}_q\backslash\{\underline{0}\}$ is a cyclic multiplicative group, this assignment of 'absolute values' is the only one consistent with the requirement that

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[h] According to Kurokawa and Koyama in Ref. 85, the proof for the analogue of the Riemann hypothesis on projective algebraic varieties over $\mathbb{F}_q$ is known (one of the Weil conjectures proved by Deligne in 1974). The hope is that reinterpreting the integers $\mathbb{Z}$ as a projective algebraic variety over $\mathbb{F}_1$ will lead to a proof of the Riemann hypothesis utilizing similar techniques. In a separate popular book[86], Kurokawa and Koyama state that Mochizuki's recent work on the ABC conjecture[87] also uses ideas based on $\mathbb{F}_1$.

[i] See also Ref. 88 for a quantum mechanical approach to the Riemann hypothesis.

the map from $\mathbb{F}_q$ to non-negative $\mathbb{R}$ be product preserving, that is: $|\underline{kl}| = |\underline{k}||\underline{l}|$.[j]
Since the same absolute value is assigned to all non-zero brackets, all outcomes $\langle x|$
for which the bracket with the state $|\psi\rangle$ is non-zero are given equal probabilistic
weight.

Note also that the multiplication of $|\psi\rangle$ with a non-zero element of $\mathbb{F}_q$ will not
affect the probability. Thus, vectors that differ by non-zero multiplicative constants
are identified as representing the same physical state, and the state space is endowed
with the finite projective geometry [89,90,91,92]

$$PG(N - 1, q) \; = \; \left(\mathbb{F}_q^N \backslash \{\underline{\mathbf{0}}\} \right) / \left(\mathbb{F}_q \backslash \{\underline{0}\} \right) , \qquad (4)$$

where each 'line' going through the origin of $\mathbb{F}_q^N$ is identified as a 'point,' in close
analogy to the complex projective geometry $\mathbb{C}P^{N-1}$ of canonical N-level QM de-
fined on $\mathbb{C}^N$.

2.2. An Example

To give a concrete example of our proposal, let us construct a 2-level system, anal-
ogous to spin, on $\mathbb{F}_q^2$ for which the state space is $PG(1, q)$. This geometry consists
of $q + 1$ 'points,' which can be represented by the vectors

$$|0\rangle = \begin{bmatrix} \underline{1} \\ \underline{0} \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} \underline{0} \\ \underline{1} \end{bmatrix}, \quad |r\rangle = \begin{bmatrix} \underline{a}^{r-1} \\ \underline{1} \end{bmatrix}, \qquad (5)$$

$r = 2, 3, \cdots, q$, where $\underline{a}$ is the generator of the multiplicative group $\mathbb{F}_q \backslash \{\underline{0}\}$ with
$\underline{a}^{q-1} = 1$. The number $q + 1$ results from the fact that of the $q^2 - 1$ non-zero
vectors in $\mathbb{F}_q^2$, every $q - 1$ are equivalent, thus the number of inequivalent vectors
is $(q^2 - 1)/(q - 1) = (q + 1)$. Similarly, the $q + 1$ inequivalent dual-vectors can be
represented as:

$$\begin{aligned} \langle \overline{0}| &= \begin{bmatrix} \underline{0} & -\underline{1} \end{bmatrix} , \\ \langle \overline{1}| &= \begin{bmatrix} \underline{1} & \underline{0} \end{bmatrix} , \\ \langle \overline{r}| &= \begin{bmatrix} \underline{1} & -\underline{a}^{r-1} \end{bmatrix} , \qquad r = 2, 3, \cdots, q , \end{aligned} \qquad (6)$$

where the minus signs are dropped when the characteristic of $\mathbb{F}_q$ is two.[k] From
these definitions, we find:

$$\begin{aligned} \langle \overline{r}|s\rangle &= \underline{0} \quad \text{if } r = s , \\ &\neq \underline{0} \quad \text{if } r \neq s , \end{aligned} \qquad (7)$$

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[j]The product preserving nature of the absolute value function guarantees that the probabilities of
product observables on product states factorize in multi-particle systems. This property is crucial
if we want to have isolated particle states.
[k]The 'characteristic' of a field is the smallest non-negative integer m such that $\underbrace{\underline{1} + \underline{1} + \cdots + \underline{1}}_{m \text{ times}} = \underline{0}$,
where $\underline{1}$ is the multiplicative unit and $\underline{0}$ is the additive unit. For example, the characteristic of $\mathbb{F}_q$
with $q = p^n$ is the prime p. The characteristics of $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ are defined to be zero.

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and

$$\left| \langle \bar{r} | s \rangle \right| \;=\; \begin{cases} 0 & \text{if } r = s \,, \\ 1 & \text{if } r \neq s \,. \end{cases} \tag{8}$$

Observables are associated with a choice of basis of $\mathbb{F}_q^{2*}$:

$$A_{rs} \;\equiv\; \{\, \langle \bar{r} |, \; \langle \bar{s} | \,\} \,, \qquad r \neq s \,. \tag{9}$$

We assign the outcome $+1$ to the first dual-vector of the pair, and the outcome -1 to the second to make these observables spin-like. This assignment implies $A_{sr} = -A_{rs}$. The indices rs can be considered as indicating the direction of the 'spin,' and the interchange of the indices as indicating a reversal of this direction.

Applying Eq. (2) to this system, it is straightforward to show that

$$\begin{aligned} P(A_{rs} = +1 \,|\, r) &= 0 \,, & P(A_{rs} = -1 \,|\, r) &= 1 \,, \\ P(A_{rs} = +1 \,|\, s) &= 1 \,, & P(A_{rs} = -1 \,|\, s) &= 0 \,, \\ P(A_{rs} = \pm 1 \,|\, t) &= \frac{1}{2} \,, & \text{for } t \neq r, s \,, \end{aligned} \tag{10}$$

and thus,

$$\begin{aligned} \langle A_{rs} \rangle_r &= -1 \,, \\ \langle A_{rs} \rangle_s &= +1 \,, \\ \langle A_{rs} \rangle_t &= 0 \,, \quad \text{for } t \neq r, s. \end{aligned} \tag{11}$$

So for each 'spin' A_{rs} there exist two 'eigenstates': $|s\rangle$ for $+1$ ('spin' up) and $|r\rangle$ for -1 ('spin' down). For all other states the two outcomes ± 1 are equally probable.

2.3. *Spin Correlations*

A two-'spin' system can be constructed on the tensor product space $\mathbb{F}_q^2 \otimes \mathbb{F}_q^2 = \mathbb{F}_q^4$. The number of non-zero vectors in this space is $q^4 - 1$, of which every $q - 1$ are equivalent, so the number of inequivalent states is $(q^4 - 1)/(q - 1) = q^3 + q^2 + q + 1$. Of these, $(q+1)^2$ are product states, leaving $(q^3 + q^2 + q + 1) - (q+1)^2 = q(q^2 - 1)$ that are entangled. Of these, there exists an analog of the spin-singlet state given by

$$|S\rangle \;=\; |r\rangle \otimes |s\rangle - |s\rangle \otimes |r\rangle \,, \qquad r \neq s \,, \tag{12}$$

for any two states $|r\rangle$ and $|s\rangle$ up to a multiplicative constant. If the characteristic of $\mathbb{F}_q$ is two, the minus sign is replaced by a plus sign.

Products of the 'spin' observables are defined as

$$A_{rs} A_{tu} \;=\; \{\, \langle \bar{r} | \otimes \langle \bar{t} | \,, \; \langle \bar{r} | \otimes \langle \bar{u} | \,, \; \langle \bar{s} | \otimes \langle \bar{t} | \,, \; \langle \bar{s} | \otimes \langle \bar{u} | \,\} \,, \tag{13}$$

the four tensor products representing the outcomes $++$, $+-$, $-+$, and $--$, and the expectation value giving the correlation between the two 'spins.' The probabilities of the four outcomes are particularly easy to calculate for the state $|S\rangle$ since [42]

$$\left(\langle \bar{r} | \otimes \langle \bar{s} | \,\right) |S\rangle = \underline{0} \quad \text{if } r = s \,,$$

Observable	$++$	$+-$	$-+$	$--$	E.V.
$A_{rs}A_{rs}$	0	$\dfrac{1}{2}$	$\dfrac{1}{2}$	0	-1
$A_{rs}A_{rt}$	0	$\dfrac{1}{3}$	$\dfrac{1}{3}$	$\dfrac{1}{3}$	$-\dfrac{1}{3}$
$A_{rs}A_{st}$	$\dfrac{1}{3}$	$\dfrac{1}{3}$	0	$\dfrac{1}{3}$	$+\dfrac{1}{3}$
$A_{rs}A_{tu}$	$\dfrac{1}{4}$	$\dfrac{1}{4}$	$\dfrac{1}{4}$	$\dfrac{1}{4}$	0

Table 1. Probabilities and expectation values of product observables in the state $|S\rangle$. The indices r, s, t, and u are distinct. Cases that can be obtained by flipping signs using $A_{rs} = -A_{sr}$ are not shown.

$$\neq \underline{0} \quad \text{if } r \neq s\,, \tag{14}$$

thus

$$\left| \left(\langle \bar{r}| \otimes \langle \bar{s}| \right)|S\rangle \right| \;=\; 1 - \delta_{rs}\,, \tag{15}$$

and we obtain the probabilities and correlations listed in Table. 1. It is straightforward to show that these probabilities cannot be reproduced by any classical hidden variable theory.[40,41] Thus, GQM is 'quantum' in this sense.

It should be noted, though, that GQM also has a common feature with CM when we look at the Clauser-Horne-Shimony-Holt (CHSH) version of Bell's inequality.[93,94] The CHSH bound[95] is the upper bound of the absolute value of the following combination of correlators:

$$\langle A, a\,; B, b\rangle \;\equiv\; \langle AB\rangle + \langle Ab\rangle + \langle aB\rangle - \langle ab\rangle\,, \tag{16}$$

where A and a are two observables of particle 1, and B and b are two observables of particle 2. All four observables are assumed to take on only the values ± 1 upon measurement. For classical hidden variable theory the bound on $|\langle A, a; B, b\rangle|$ is 2,[95] while for canonical QM it is $2\sqrt{2}$.[96,97] [1] Despite GQM not allowing a hidden variable minic, we can nevertheless show, again using Table 1, that its CHSH bound is the 'classical' value of 2. This bound is independent of the value of q chosen for our Galois field $\mathbb{F}_q$ and is thus also independent of the size of the vector space $\mathbb{F}_q^2 \otimes \mathbb{F}_q^2 = \mathbb{F}_q^4$.[40,41] So the limitation on the correlations is not due to the limited number of 'spin' direction available in the model. This GQM example shows that the absence of hidden variable mimics does not necessarily guarantee the violation of the classical CHSH bound.

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[1]The largest possible value of the CHSH bound is 4.[98] See Ref. 51 for a model which saturates this bound.

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2.4. *The $q = 1$ Limit*

The Galois fields $\mathbb{F}_q$ are only defined for $q = p^n$, that is, the order q must be a power of a prime p. Thus, setting $q = 1$ is illegitimate from the Galois field point of view. Indeed, $\mathbb{F}_q$ consists of q elements so taking the naive $q = 1$ limit, one expects $\mathbb{F}_1$ to consist of only one element (which could be denoted by either $\underline{0}$ or $\underline{1}$) with no distinction between addition and multiplication. Such an object is obviously not a "field." In other words, a "field with one element," in the usual sense of the term, does not and cannot exist. However, a different and more interesting picture emerges if instead of trying to define $\mathbb{F}_1$ first, we set $q = 1$ directly in the $N = 2$ 'spin' model we constructed above.

First, the number of states given in Eq. (5) will be reduced to $q + 1 = 2$,

$$| \uparrow \rangle \equiv |0\rangle = \begin{bmatrix} \underline{1} \\ \underline{0} \end{bmatrix} , \qquad | \downarrow \rangle \equiv |1\rangle = \begin{bmatrix} \underline{0} \\ \underline{1} \end{bmatrix} , \tag{17}$$

as are the possible outcomes listed in Eq. (6):

$$\langle \downarrow | \equiv \langle \overline{0}| = [\underline{0} \ \underline{1}] , \qquad \langle \uparrow | \equiv \langle \overline{1}| = [\underline{1} \ \underline{0}] . \tag{18}$$

The sole observable of the system, Eq. (9), will be

$$A \equiv A_{10} = -A_{01} = \{ \langle \overline{1}|, \langle \overline{0}| \} = \{ \langle \uparrow |, \langle \downarrow | \} , \tag{19}$$

for which $| \uparrow \rangle$ and $| \downarrow \rangle$ are 'eigenstates':

$$\langle A \rangle_\uparrow = 1 , \qquad \langle A \rangle_\downarrow = -1 . \tag{20}$$

Thus, a measurement of A on $| \uparrow \rangle$ will always yield $+1$, while that on $| \downarrow \rangle$ will always yield -1. No superpositions of these states exist, so the system reduces to a 'classical' one where each states has a definite outcome upon measurement.

The 'two'-spin system will reduce to $q^3 + q^2 + q + 1 = 4$ states, of which $q(q^2 - 1) = 0$ are entangled. These four are the product states

$$| \uparrow\uparrow \rangle \equiv | \uparrow \rangle \otimes | \uparrow \rangle = \begin{bmatrix} \underline{1} \\ \underline{0} \\ \underline{0} \\ \underline{0} \end{bmatrix} , \quad | \uparrow\downarrow \rangle \equiv | \uparrow \rangle \otimes | \downarrow \rangle = \begin{bmatrix} \underline{0} \\ \underline{1} \\ \underline{0} \\ \underline{0} \end{bmatrix} ,$$

$$| \downarrow\uparrow \rangle \equiv | \downarrow \rangle \otimes | \uparrow \rangle = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{1} \\ \underline{0} \end{bmatrix} , \quad | \downarrow\downarrow \rangle \equiv | \downarrow \rangle \otimes | \downarrow \rangle = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{0} \\ \underline{1} \end{bmatrix} . \tag{21}$$

The sole product observable, Eq. (13), is

$$AA = \{ \langle \uparrow\uparrow |, \langle \uparrow\downarrow |, \langle \downarrow\uparrow |, \langle \downarrow\downarrow | \} , \tag{22}$$

where

$$\langle \uparrow\uparrow | = \langle \uparrow | \otimes \langle \uparrow | = [\underline{1} \ \underline{0} \ \underline{0} \ \underline{0}] ,$$
$$\langle \uparrow\downarrow | = \langle \uparrow | \otimes \langle \downarrow | = [\underline{0} \ \underline{1} \ \underline{0} \ \underline{0}] ,$$

$$\langle\,\downarrow\uparrow\,| = \langle\,\downarrow\,|\otimes\langle\,\uparrow\,| \;=\; [\underline{0}\;\;\underline{0}\;\;\underline{1}\;\;\underline{0}]\,,$$
$$\langle\,\downarrow\downarrow\,| = \langle\,\downarrow\,|\otimes\langle\,\downarrow\,| \;=\; [\underline{0}\;\;\underline{0}\;\;\underline{0}\;\;\underline{1}]\,. \tag{23}$$

The four states are all 'eigenstates' of AA with definite outcomes when AA is measured. Thus, the system is 'classical' and its CHSH bound is trivially 2.

This discussion can be easily generalized to generic N-level GQM, in which the $q = 1$ case reduces to a situation where all surviving states are simultaneous 'eigenstates' of all surviving observables.[m] What this exercise shows is that the $q = 1$ limit of GQM, though admittedly fairly trivial, does make sense as a 'classical' theory. Note, however, that in letting $q = 1$ we have retained the formalism of GQM as is from the 'quantum' $q \neq 1$ cases. The only thing that has changed is the value of q, which changes the number field over which the state space is constructed, formally, from $\mathbb{F}_q$ to $\mathbb{F}_1$. But what is $\mathbb{F}_1$?

3. The Field with One Element $\mathbb{F}_1$

3.1. *The Observation of Tits*

The concept of $\mathbb{F}_1$ first appeared in a 1957 paper by Tits[58] as the "corps de caractéristique 1." The main objective of the Tits paper was to use projective geometries to define the then recently discovered Chevalley groups geometrically.[n] To each Dynkin diagram and choice of finite field $\mathbb{F}_q$ a projective geometry was associated, and the Chevalley group identified with the group of projective transformations on the geometry. For instance, the projective geometry $PG(N-1,q)$ is associated with the Dynkin diagram for A_{N-1}, and the corresponding Chevalley group is the projective linear group $PGL(N,q)$. Towards the end of the paper, Tits argues that the $q = 1$ limit of $PG(N-1,q)$ actually makes sense as a projective geometry,[o] and that the corresponding group of projective transformations is S_N.

To see this, let us first list a few properties of $PG(N-1,q)$ for $q \neq 1$:

- The number of points in the geometry $PG(N-1,q)$ is

$$\begin{bmatrix} N \\ 1 \end{bmatrix}_q \;=\; [N]_q \;=\; \frac{q^N - 1}{q - 1} \;=\; 1 + q + q^2 + \cdots + q^{N-1}\,. \tag{24}$$

  (See chapter 9 of Ref. 101.) Here, the notation is:

$$\begin{bmatrix} N \\ M \end{bmatrix}_q \;=\; \frac{[N]_q!}{[M]_q![N-M]_q!}\,, \tag{25}$$

  with

$$[N]_q \equiv 1 + q + q^2 + \cdots + q^{N-1} \;=\; \frac{q^N - 1}{q - 1}\,,$$

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[m]Note that the vector space $\mathbb{F}_q^N$ has $q^N - 1$ non-zero vectors, of which every $q - 1$ of them are physically equivalent in GQM. Thus, the number of physically distinct states is $(q^N - 1)/(q - 1) = [N]_q$, which reduces to N in the $q = 1$ limit.

[n]Chevalley originally used purely algebraic methods in his definition.[99]

[o]With a minor modification to the axioms. See Ref. 100.

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$$[N]_q! \equiv [N]_q [N-1]_q \cdots [2]_q [1]_q \,. \tag{26}$$

Eq. (25) is known as the Gaussian binomial coefficient, $[N]_q$ the q-analogue of the natural number N, and $[N]_q!$ the q-factorial.

- The number of k-dimensional subspaces (points, lines, planes, etc.) in $PG(N-1, q)$ are

$$\begin{aligned} \begin{bmatrix} N \\ k+1 \end{bmatrix}_q &= \frac{[N]_q [N-1]_q \cdots [N-k]_q}{[k+1]_q [k]_q \cdots [1]_q} \\ &= \frac{(q^N-1)(q^{N-1}-1)\cdots(q^{N-k}-1)}{(q^{k+1}-1)(q^k-1)\cdots(q-1)} \,. \end{aligned} \tag{27}$$

Each subspace contains

$$[k+1]_q \;=\; \frac{q^{k+1}-1}{q-1} \;=\; 1 + q + q^2 + \cdots + q^k \tag{28}$$

points.

In the limit $q \to 1$, the q-analog $[N]_q$ reduces to N, the q-factorial to the usual factorial, and the Gaussian binomial coefficients to the usual binomial coefficients. So the 'projective' space $PG(N-1,1)$ should be a space consisting of N 'points,' and the number of k-dimensional subspaces should be $_NC_{k+1}$, each consisting of $k+1$ points. This would simply be a set consisting of N elements, and all the subsets consisting of $k+1$ elements are the k-dimensional subspaces. The group of all possible transformations, projective or not, for a set with N elements is obviously S_N.

For instance, say $N = 3$: $PG(2,1)$ would consist of three 'points.' Let us call them a, b, and c. The 'lines' in this geometry would be $\{a,b\}$, $\{b,c\}$, and $\{c,a\}$, and the 'plane' would be the entire space $\{a,b,c\}$. There is a nice discussion in Ref. 100 on how this definition of 'geometry' satisfies all the required axioms. All possible ways to map the three points onto themselves form the group S_3.

Now, recalling that $PG(N-1,q)$ was obtained by identifying the lines through the origin of $\mathbb{F}_q^N$ as points, Tits argues that the 'projective geometry' $PG(N-1,1)$ should be interpretable as

$$PG(N-1,1) \;=\; \left(\mathbb{F}_1^N \backslash \{\underline{\mathbf{0}}\}\right) \Big/ \left(\mathbb{F}_1 \backslash \{\underline{0}\}\right) \,, \tag{29}$$

where $\mathbb{F}_1$ is the "corps de caractéristique 1," aka the field with one element.

3.2. $\mathbb{F}_1$ according to Kurokawa and Koyama

Taken literally, "corps de caractéristique 1" implies[p]

$$\underline{1} \;=\; \underline{0} \,, \tag{30}$$

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[p]See footnote k.

that is, the multiplicative unit and the additive unit are the same number, so it may seem that $\mathbb{F}_1 = \{\underline{1}\} = \{\underline{0}\}$, where

$$\underline{1} \times \underline{1} = \underline{1} , \qquad \underline{1} + \underline{1} = \underline{1} . \tag{31}$$

Such an object would not have any structure to speak of and it is difficult to envision how one could construct any geometry, projective or otherwise, on it. Thus, a more sophisticated definition of $\mathbb{F}_1$ is called for.

Various definitions of $\mathbb{F}_1$ and $\mathbb{F}_1$-geometries exist in the literature, which endeavor to make sense of these concepts, a survey of which can be found in Ref. 74 by Lorscheid and Lopez-Pena. Here, we adopt the definition of $\mathbb{F}_1$ by Kurokawa and Koyama in Ref. 85 who argue that $\mathbb{F}_1$ should be interpreted as the set $\mathbb{F}_1 = \{\underline{1}\} + \{\underline{0}\}$ on which only multiplication is defined. Note that $\underline{1}$ is the multiplicative unit, and $\underline{0}$ is the multiplicative zero, that is:

$$\underline{1}\,\underline{k} = \underline{k}\,\underline{1} = \underline{k} , \qquad \underline{0}\,\underline{k} = \underline{k}\,\underline{0} = \underline{0} , \qquad \underline{k} = \underline{0} \text{ or } \underline{1} . \tag{32}$$

This is different from $\mathbb{F}_2 = \{\underline{0}, \underline{1}\}$ in that addition is not allowed.[q] This is the reason for the unfamiliar notation $\{\underline{1}\} + \{\underline{0}\}$: $\{\underline{1}\}$ is the multiplicative group by itself, and $\{\underline{0}\}$ is the multiplicative zero (but not the additive unit since there is no addition) which is tacked on. Thus, $\mathbb{F}_1$ is not really a field but what is known as a 'monoid.'

Since addition is not allowed, 'vector' spaces over $\mathbb{F}_1^N$ are not truly 'vector' spaces in the usual sense of the term. However, lacking a better alternative, let us use vector space terminology anyway. Since $\mathbb{F}_1^N$ is an N-dimensional space, it will have N basis vectors, which we denote

$$|1\rangle, \ |2\rangle, \ \cdots, \ |N\rangle . \tag{33}$$

The linear combinations of these basis vectors are of the form

$$\sum_{i=1}^{N} c_i |i\rangle , \qquad c_i \in \mathbb{F}_1 , \tag{34}$$

but since we should have no addition, only one of the c_i's is allowed to be non-zero at a time. So the N basis vectors are the only (non-zero) vectors in this space. They will each be represented by column vectors with only one $\underline{1}$, the remaining elements being all $\underline{0}$. Note that these are precisely the states we obtained in taking the limit $q = 1$ in GQM.

Linear transformations on $\mathbb{F}_1^N$ must not lead to addition of the basis vectors. So the only allowed transformations are those for which the $N \times N$ matrix representation has at most one $\underline{1}$ in each row. The non-singular ones, i.e. the automorphisms, are the ones in which there is one $\underline{1}$ in each row and each column. These are simply the matrices that permute the N basis vectors. Since $\mathbb{F}_1 \backslash \{\underline{0}\} = \{\underline{1}\}$, the projective

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[q]It may be more precise to say that only the addition of $\underline{0}$ is allowed. $\underline{1} + \underline{1}$ is prohibited. Another definition referred to in the literature[62,82,83,84] defines $\underline{1} + \underline{1} = \underline{1}$ with all other additions and multiplications among $\underline{0}$ and $\underline{1}$ defined in the usual way, cf. Eq. (31).

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space $PG(N-1,1)$ will simply consist of the non-zero vectors in $\mathbb{F}_1^N$. There are N of these, as required for $PG(N-1,1)$. The projective linear group $PGL(N,1)$ would be S_N.

Linear maps from $\mathbb{F}_1^N$ to $\mathbb{F}_1$, i.e. the dual vectors, must also satisfy the no-addition requirement, so they would be $N \times 1$ row-vectors, each with only one $\underline{1}$, the remaining elements all $\underline{0}$. These are precisely the dual vectors representing outcomes in the $q=1$ limit of GQM.

Thus, the Kurokawa-Koyama approach matches well with the GQM construction, and the $q=1$ 'classical' limit can be constructed directly on $\mathbb{F}_1^N$ defined in this fashion. The resulting state space has the projective geometry $PG(N-1,1)$. Note, particularly, that the prohibition of addition in Kurokawa-Koyama leads to the lack of superposition of states in the 'classical' limit of GQM. Therefore, the $q=1$ limit of GQM can be considered to be 'classical' due to this very special feature of $\mathbb{F}_1$.

4. Summary and Discussion

Illuminating the precise relation between CM and QM remains one of the outstanding problems in the foundations of physics. In this note we point out that a simple discrete toy model sheds new light on this important problem.

In order to compare the two seemingly disparate formalisms of physics, first, one needs a unifying mathematical language for such a comparison. This idea, that classical theory should be rewritten using the mathematics of QM goes back at least to Koopman[16] and von Neumann.[17,18] In our proposal this unifying mathematical language is provided by the projective spaces of GQM.[40,41]

We have demonstrated that GQM defined on $\mathbb{F}_q^N$ becomes 'classical' in the limit $q=1$. The field with one element $\mathbb{F}_1$, over which this limit can be constructed, have all the 'classical' properties of consisting only of zero and one, and addition being prohibited. In other words, the 'classical'-ness of $q=1$ GQM can be traced to the 'classical'-ness of $\mathbb{F}_1$. This way of achieving 'classicality' from 'quantum'-ness is quite different from the usual methodology of taking $\hbar \to 0$, in the limit of which operators become commuting. In GQM, there is no $\hbar$ and there are no operators.

Our result suggests that the pathway from a 'quantum' theory to a 'classical' one may not be unique. Looking at canonical QM, taking the $\hbar \to 0$ limit (in some yet unknown pathway) may not be the only way for the theory to become classical. Perhaps replacing $\mathbb{C}$ with some other mathematical structure, perhaps a continuous analog of $\mathbb{F}_1$, though there is no telling such an entity exists, could lead to a theory which is 'classical' in some new way.

The basic premise of this paper is that the relation between CM and QM lies in the differences in the structure and geometry of the underlying number fields upon which the descriptions are predicated. While we have not discussed the limit $q \to 1$ within the context of the Biorthogonal Quantum Mechanics we defined in Ref. 51, the premise is quite apparent there as well. The approach there was an alternative

method of constructing a quantum-like theory on $\mathbb{F}_q^N$. The CHSH bound on 'spin' systems defined on $\mathbb{F}_q^4 = \mathbb{F}_q^2 \times \mathbb{F}_q^2$ in this approach is q-dependent. We were able to demonstrate that this bound takes on the classical value of 2 for $q = 3$, and the super-quantum value of 4 for $q = 9$. Once again, it is the differences in the structure of the underlying number fields $\mathbb{F}_3$ and $\mathbb{F}_9 = \mathbb{F}_3[\underline{i}]$, the latter of which accommodates the analog of the imaginary unit i, and their associated geometries, that presage classical, quantum, and super-quantum correlations.[r]

QM is our current fundamental framework of physics. As such it has passed all available observational tests. And yet our understanding of QM remains limited. In particular, one would still like to understand if QM follows from few elementary axioms, in analogy what happens in the special theory of relativity. Also, one would like to understand in more detail the precise relation between QM and its classical limit, the problem addressed in this paper. Finally, could QM be part of a larger framework, the way CM appears as a formal limit of QM (or in another context, similar to the way the special theory of relativity can be naturally generalized to the general theory of relativity)? It is very difficult to answer all three questions within the framework of canonical QM of the real world as we understand it today, so a simpler model was called for. This was the main motivation of our effort to build a toy model of QM using Galois fields. Surprisingly, all three questions can be addressed in this context. In our previous publications on this subject[40,41] we have constructed the precise mathematical rules of the game in this toy world, and we have shed light on the nature of quantum correlations, on the nature of super-quantum correlations[51] (which generalize the usual quantum framework) as well as on the difference between classical and quantum correlations. In this paper, we complete this picture by addressing the subtle question of the 'classical' limit in the context of GQM. Understandably enough, our toy world is not the real world. However, for the first time, the two known theories of the real world (the classical and quantum theory) as well as the conjectured third theory (based on stronger than quantum, or superquantum, correlations) can be addressed in one coherent theoretical framework, that is simple yet illuminating. And this is where we find the most useful lessons of our work.

Acknowledgments

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[r]See also Refs. 48, 49, 50.

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