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Comment on the Dispersion Relations used to Calculate $\Delta\rho$.

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ABSTRACT

We use the operator product expansion (OPE) to show that non-perturbative QCD corrections to $\Delta\rho$ can be calculated using unsubtracted dispersion relations for either the transverse or the longitudinal vacuum polarization functions. Recent calculations of the non-perturbative contribution to $\Delta\rho$ based on a non-relativistic calculation of corrections to the $t\bar{t}$ threshold are inconsistent with this result.

I. INTRODUCTION

An accurate calculation of the contribution of the t–b doublet to the ρ parameter is important in two respects. The first is that it will let us translate the tight constraint placed on the ρ parameter by LEP measurements to limits on the t quark mass, and the second is that in the event that the t quark mass is measured directly at the TEVATRON, it will help us constrain the contribution of new physics to $\Delta\rho$.

The contribution of the t–b doublet to the ρ parameter has been calculated to $O(\alpha\alpha_s)$, and the result is given by

$$\Delta\rho = \frac{3\alpha m_t^2}{16\pi s^2 c^2 m_Z^2}\left[1 - \frac{\alpha_s}{\pi}\left(\frac{2\pi^2+6}{9}\right)\right] \tag{1}$$

in the limit $m_b \to 0$ [1, 2].

Recently, Kniehl and Sirlin estimated the size of the higher order QCD corrections to Eq. (1) using a dispersion relation for $\Delta\rho$ [3]. Their approach has been to assume that the effect of non–perturbative QCD corrections is dominated by the change in the shape of the $t\bar{t}$ threshold. This change is then calculated using the leading non–relativistic approximation, and substituted into the dispersion relation for $\Delta\rho$. This work has lead to a certain amount of controversy since different authors found the effect to be different in magnitude, ranging from 10% to 80% of the $O(\alpha\alpha_s)$ correction, and sometimes even different in sign [4, 5].

The difference in sign comes from the fact the there are two possible dispersion relations for $\Delta\rho$. In order to explain what they are, we must introduce some notation. Following Ref. [3], we define

$$\begin{aligned}\Pi_{\mu\nu}^{V,A}(q,m_1,m_2) &= -i\int d^4x e^{iq\cdot x}\langle 0|T^*\left[J_\mu^{V,A}(x)J_\nu^{V,A\dagger}(0)\right]|0\rangle \\ &= g_{\mu\nu}\Pi^{V,A}(s,m_1,m_2) + q_\mu q_\nu \lambda^{V,A}(s,m_1,m_2) \\ &= \left(g_{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Pi^{V,A}(s,m_1,m_2) + \left(\frac{q^\mu q^\nu}{q^2}\right)\Delta^{V,A}(s,m_1,m_2)\end{aligned} \tag{2}$$

where $s = q^2$, and $J_\mu^{V,A}(x)$ represents the vector and axial vector currents constructed from the fermion fields, respectively. Note that

$$\Delta^{V,A}(s) = \Pi^{V,A}(s) + s\lambda^{V,A}(s) \tag{3}$$

so that

$$\Delta^{V,A}(0) = \Pi^{V,A}(0), \tag{4}$$

unless $\lambda^{V,A}(s)$ has a pole at $s=0$. We further introduce the notation

$$\begin{aligned}
\Pi_{\pm}^{V,A}(s) &= \Pi^{V,A}(s,m_1,m_2),\\
\lambda_{\pm}^{V,A}(s) &= \lambda^{V,A}(s,m_1,m_2),\\
\Delta_{\pm}^{V,A}(s) &= \Delta^{V,A}(s,m_1,m_2),\\
\Pi_{0}^{V,A}(s) &= \frac{1}{2}\left[\Pi^{V,A}(s,m_1,m_1)+\Pi^{V,A}(s,m_2,m_2)\right],\\
\lambda_{0}^{V,A}(s) &= \frac{1}{2}\left[\lambda^{V,A}(s,m_1,m_1)+\lambda^{V,A}(s,m_2,m_2)\right],\\
\Delta_{0}^{V,A}(s) &= \frac{1}{2}\left[\Delta^{V,A}(s,m_1,m_1)+\Delta^{V,A}(s,m_2,m_2)\right]. \qquad (5)
\end{aligned}$$

The conservation of the neutral vector currents implies the Ward Identities:

$$\Pi_0^V(s) = -s\lambda_0(s), \qquad \Delta_0^V(s)\equiv 0. \qquad (6)$$

These definitions let us write the contribution of an $SU(2)$ fermion doublet, with masses m_1 and m_2, to the vacuum polarizations of the W and the Z at zero momentum transfer $s=0$ as

$$\begin{aligned}
\Pi_{WW}(0) &= \frac{g^2}{8}\left[\Pi_{\pm}^V(0)+\Pi_{\pm}^A(0)\right] = \frac{g^2}{8}\left[\Delta_{\pm}^V(0)+\Delta_{\pm}^A(0)\right],\\
\Pi_{ZZ}(0) &= \frac{g^2+g'^2}{8}\left[\Pi_0^V(0)+\Pi_0^A(0)\right] = \frac{g^2+g'^2}{8}\Delta_0^A(0). \qquad (7)
\end{aligned}$$

Note that $\Pi_0^V(0)$ is actually zero from current conservation (*cf.* Eq. (6)) but we will keep it in our expressions for later convenience. Inserting Eq. (7) into the definition of the ρ parameter, we find

$$\begin{aligned}
\Delta\rho &= \frac{\Pi_{WW}(0)}{M_W^2} - \frac{\Pi_{ZZ}(0)}{M_Z^2}\\
&= \frac{G_F}{\sqrt{2}}\left\{\left[\Pi_{\pm}^V(0)+\Pi_{\pm}^A(0)\right]-\left[\Pi_0^V(0)+\Pi_0^A(0)\right]\right\}\\
&= \frac{G_F}{\sqrt{2}}\left\{\left[\Delta_{\pm}^V(0)+\Delta_{\pm}^A(0)\right]-\Delta_0^A(0)\right\} \qquad (8)
\end{aligned}$$

Applying the 'unsubtracted' dispersion relation:

$$f(s) = \frac{1}{\pi}\int^{\infty} ds' \frac{\mathrm{Im} f(s')}{s'-s+i\epsilon} \qquad (9)$$

to the expression for $\Delta\rho$ using the $\Pi(s)$'s gives us

$$\Delta\rho = \frac{G_F}{\sqrt{2}}\frac{1}{\pi}\int^{\infty}\frac{ds}{s}\left[\left\{\mathrm{Im}\Pi^V_{\pm}(s) + \mathrm{Im}\Pi^A_{\pm}(s)\right\} - \left\{\mathrm{Im}\Pi^V_0(s) + \mathrm{Im}\Pi^A_0(s)\right\}\right]. \quad (10)$$

This dispersion relation is equivalent to that introduced in Ref. [6], and used in Ref. [2] to calculate the $O(\alpha\alpha_s)$ correction.

If we write down an unsubtracted dispersion relation for the expression of $\Delta\rho$ using the $\Delta(s)$'s, we get

$$\Delta\rho = \frac{G_F}{\sqrt{2}}\frac{1}{\pi}\int^{\infty}\frac{ds}{s}\left[\left\{\mathrm{Im}\Delta^V_{\pm}(s) + \mathrm{Im}\Delta^A_{\pm}(s)\right\} - \mathrm{Im}\Delta^A_0(s)\right]. \quad (11)$$

This is the relation that was introduced in Ref. [3]. Eqs. (10) and (11) are the two dispersion relations for $\Delta\rho$ that have appeared in the literature. Our point here is to illustrate that, although Eqs. (10, 11) should yield the same estimate for $\Delta\rho$, under certain commonly used approximations they do not.

In Sec. II we clarify the conditions that the $\Pi(s)$'s and $\Delta(s)$'s must satisfy for Eqs. (10) and (11) to be true. These conditions are indeed satisfied to order $\alpha\alpha_s$ in perturbation theory. In Sec. III, we argue that the operator product expansion (OPE) of the current–current correlators suggests that these conditions are satisfied for non–perturbative QCD corrections so that both Eqs. (10) and (11) are valid. In Sec. IV we show that the two dispersion relations Eqs. (10) and (11) give different answers when used in a threshold approximation to calculate non-perturbative effects. We argue that the disagreement between the two dispersion relations can be understood as a result of neglecting the non–threshold contribution of the $\mathrm{Im}\Pi(s)$'s and $\mathrm{Im}\Delta(s)$'s to $\Delta\rho$. Furthermore, the magnitude of the discrepancy between Eqs. (10, 11) can be used as a measure of the accuracy of the calculation. We conclude in Sec. V.

II. THE DISPERSION RELATIONS FOR $\Delta\rho$

Let us begin this section by recalling that the analyticity of a vacuum polarization function $f(s)$ and Cauchy's theorem tell us that

$$f(s) = \frac{1}{\pi}\int^{\Lambda^2} ds' \frac{\mathrm{Im}f(s')}{s' - s + i\epsilon} + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds' \frac{f(s')}{s' - s}. \quad (12)$$

Hence for Eq. (9) to be true, $f(s)$ must satisfy

$$\lim_{\Lambda^2\to\infty}\oint_{|s|=\Lambda^2}\frac{f(s')}{s'-s} = 0. \quad (13)$$

Applying Eq. (12) to the $\Pi(0)$'s in Eq. (8), we find

$$\begin{aligned}
\Pi^V_\pm(0) &= \frac{1}{\pi}\int^{\Lambda^2}\frac{ds}{s}\mathrm{Im}\Pi^V_\pm(s) + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\frac{ds}{s}\Pi^V_\pm(s),\\
\Pi^A_\pm(0) &= \frac{1}{\pi}\int^{\Lambda^2}\frac{ds}{s}\mathrm{Im}\Pi^A_\pm(s) + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\frac{ds}{s}\Pi^A_\pm(s),\\
\Pi^V_0(0) &= \frac{1}{\pi}\int^{\Lambda^2}\frac{ds}{s}\mathrm{Im}\Pi^V_0(s) + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\frac{ds}{s}\Pi^V_0(s),\\
\Pi^A_0(0) &= \frac{1}{\pi}\int^{\Lambda^2}\frac{ds}{s}\mathrm{Im}\Pi^A_0(s). + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\frac{ds}{s}\Pi^A_0(s).
\end{aligned} \tag{14}$$

We regard the Π's in these relations to be *regularized* [1] and thus finite quantities so that both the left and right hand sides of these equations are well defined. Note that the Ward Identity, Eq. (6), ensures that

$$\begin{aligned}
\Pi^V_0(0) &= \frac{1}{\pi}\int^{\Lambda^2}\frac{ds}{s}\mathrm{Im}\Pi^V_0(s) + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\frac{ds}{s}\Pi^V_0(s),\\
&= -\left[\frac{1}{\pi}\int^{\Lambda^2} ds\mathrm{Im}\lambda^V_0(s) + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2} ds\lambda^V_0(s),\right]\\
&= -\frac{1}{2\pi i}\oint ds\lambda^V_0(s)\\
&= 0,
\end{aligned} \tag{15}$$

as required. This result follows simply from Cauchy's theorem and the (assumed) analyticity of $\lambda(s)$. If we define

$$\begin{aligned}
\Delta\rho_T(\Lambda^2) &\equiv \frac{G_F}{\sqrt{2}}\frac{1}{\pi}\int^{\Lambda^2}\frac{ds}{s}\left[\left\{\mathrm{Im}\Pi^V_\pm(s)+\mathrm{Im}\Pi^A_\pm(s)\right\} - \left\{\mathrm{Im}\Pi^V_0(s)+\mathrm{Im}\Pi^A_0(s)\right\}\right],\\
\Delta R_T(\Lambda^2) &\equiv \frac{G_F}{\sqrt{2}}\frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\frac{ds}{s}\left[\left\{\Pi^V_\pm(s)+\Pi^A_\pm(s)\right\} - \left\{\Pi^V_0(s)+\Pi^A_0(s)\right\}\right],
\end{aligned} \tag{16}$$

the substitution of Eq. (14) into Eq. (8) gives

$$\Delta\rho = \Delta\rho_T(\Lambda^2) + \Delta R_T(\Lambda^2). \tag{17}$$

(The subscript "T" stands for "transverse".) Therefore, for the dispersion relation Eq. (10) to be valid, we must have

$$\lim_{\Lambda^2\to\infty}\Delta R_T(\Lambda^2) = 0. \tag{18}$$

[1] With $\overline{\mathrm{MS}}$ or some other regularization scheme which respects the symmetry of the theory so that the Ward Identities are satisfied.

This requires the linear combination of the $\Pi(s)$'s in the integrand of $\Delta R_T(\Lambda^2)$ to vanish as $|s| \to \infty$.

Similarly, applying Eq. (12) to the $\Delta(s)$'s gives us

$$\begin{aligned}
\Delta^V_\pm(0) &= \frac{1}{\pi}\int^{\Lambda^2}\frac{ds}{s}\mathrm{Im}\Delta^V_\pm(s) + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\frac{ds}{s}\Delta^V_\pm(s), \\
\Delta^A_\pm(0) &= \frac{1}{\pi}\int^{\Lambda^2}\frac{ds}{s}\mathrm{Im}\Delta^A_\pm(s) + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\frac{ds}{s}\Delta^A_\pm(s), \\
\Delta^A_0(0) &= \frac{1}{\pi}\int^{\Lambda^2}\frac{ds}{s}\mathrm{Im}\Delta^A_0(s). + \frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\frac{ds}{s}\Delta^A_0(s). \qquad (19)
\end{aligned}$$

Again, we consider the $\Delta(s)$'s to be regularized and finite quantities. If we define

$$\begin{aligned}
\Delta\rho_L(\Lambda^2) &\equiv \frac{G_F}{\sqrt{2}}\frac{1}{\pi}\int^{\Lambda^2}\frac{ds}{s}\left[\left\{\mathrm{Im}\Delta^V_\pm(s) + \mathrm{Im}\Delta^A_\pm(s)\right\} - \mathrm{Im}\Delta^A_0(s)\right], \\
\Delta R_L(\Lambda^2) &\equiv \frac{G_F}{\sqrt{2}}\frac{1}{2\pi i}\oint_{|s|=\Lambda^2}\frac{ds}{s}\left[\left\{\Delta^V_\pm(s) + \Delta^A_\pm(s)\right\} - \Delta^A_0(s)\right], \qquad (20)
\end{aligned}$$

the substitution of Eq. (19) into Eq. (8) gives us

$$\Delta\rho = \Delta\rho_L(\Lambda^2) + \Delta R_L(\Lambda^2). \qquad (21)$$

(The subscript "L" stands for "longitudinal".) Therefore, for the dispersion relation Eq. (11) to be valid, we must have

$$\lim_{\Lambda^2\to\infty}\Delta R_L(\Lambda^2) = 0, \qquad (22)$$

which requires the linear combination of the $\Delta(s)$'s in the integrand of $\Delta R_L(\Lambda^2)$ to vanish as $|s| \to \infty$.

The actual asymptotic forms of the $\Pi(s)$'s and $\Delta(s)$'s up to order $\alpha\alpha_s$ can by found in Ref. [7] and one can explicitly check that Eqs. (18) and (22) hold. Therefore, to order $\alpha\alpha_s$ in perturbation theory,

$$\Delta\rho = \Delta\rho_T(\infty) = \Delta\rho_L(\infty), \qquad (23)$$

III. THE OPERATOR PRODUCT EXPANSION

In the previous section, we have seen that

$$\Delta R_T(\infty) = \Delta R_L(\infty) = 0 \qquad (24)$$

implies

$$\Delta\rho = \Delta\rho_T(\infty) = \Delta\rho_L(\infty), \tag{25}$$

and that Eq. (24) holds to order $O(\alpha\alpha_s)$ in perturbation theory. Whether these equations continue to be valid to higher orders in perturbation theory, and non-perturbatively is a difficult problem in general. However, since we are only interested in QCD corrections, the non–perturbative asymptotic behavior of the vacuum polarization functions $\Pi(s)$ and $\Delta(s)$ as $|s| \to \infty$ can still be extracted from their operator product expansions.

The OPE's for $\Pi_{\pm,0}(s)$ and $\Delta_{\pm,0}(s)$ can be found in the appendix of Ref. [8]. They are:

$$\begin{aligned}
\Delta^V_\pm(-Q^2) &= \hat{C}_{\Delta 1}(Q)\left[\hat{m}_1(Q) - \hat{m}_2(Q)\right]^2 + \hat{C}_{\Delta 2}(\mu)\left[\hat{m}_1(\mu) - \hat{m}_2(\mu)\right]^2 + O\left(\frac{1}{Q^2}\right), \\
\Delta^A_\pm(-Q^2) &= \hat{C}_{\Delta 1}(Q)\left[\hat{m}_1(Q) + \hat{m}_2(Q)\right]^2 + \hat{C}_{\Delta 2}(\mu)\left[\hat{m}_1(\mu) + \hat{m}_2(\mu)\right]^2 + O\left(\frac{1}{Q^2}\right), \\
\Delta^A_0(-Q^2) &= \hat{C}_{\Delta 1}(Q)\left[2\hat{m}_1(Q)^2 + 2\hat{m}_2(Q)^2\right] \\
&\quad + \hat{C}_{\Delta 2}(\mu)\left[2\hat{m}_1(\mu)^2 + 2\hat{m}_2(\mu)^2\right] + O\left(\frac{1}{Q^2}\right),
\end{aligned} \tag{26}$$

and

$$\Pi^{V,A}_{\pm,0}(-Q^2) = Q^2\lambda^{V,A}_{\pm,0}(-Q^2) + \Delta^{V,A}_{\pm,0}(-Q^2) \tag{27}$$

where

$$\begin{aligned}
\lambda^V_\pm(-Q^2) &= \hat{C}_{\lambda 1}(Q) + \hat{C}_{\lambda 2}(Q)\frac{[\hat{m}_1(Q) + \hat{m}_2(Q)]^2}{Q^2} \\
&\quad + \hat{C}_{\lambda 3}(Q)\frac{[\hat{m}_1(Q) - \hat{m}_2(Q)]^2}{Q^2} + O\left(\frac{1}{Q^4}\right), \\
\lambda^A_\pm(-Q^2) &= \hat{C}_{\lambda 1}(Q) + \hat{C}_{\lambda 2}(Q)\frac{[\hat{m}_1(Q) - \hat{m}_2(Q)]^2}{Q^2} \\
&\quad + \hat{C}_{\lambda 3}(Q)\frac{[\hat{m}_1(Q) + \hat{m}_2(Q)]^2}{Q^2} + O\left(\frac{1}{Q^4}\right), \\
\lambda^V_0(-Q^2) &= \hat{C}_{\lambda 1}(Q) + \hat{C}_{\lambda 2}(Q)\frac{[2\hat{m}_1(Q)^2 + 2\hat{m}_2(Q)^2]}{Q^2} + O\left(\frac{1}{Q^4}\right), \\
\lambda^A_0(-Q^2) &= \hat{C}_{\lambda 1}(Q) + \hat{C}_{\lambda 3}(Q)\frac{[2\hat{m}_1(Q)^2 + 2\hat{m}_2(Q)^2]}{Q^2} + O\left(\frac{1}{Q^4}\right).
\end{aligned} \tag{28}$$

The short distance physics is embodied in the Wilson Coefficients $\hat{C}_*(Q)$ which can be calculated perturbatively, and their explicit forms for the first few orders in $\alpha_s(Q)$ can be found in Ref. [8]. The long distance non–perturbative physics is embodied in the running masses $\hat{m}_{1,2}(Q)$ and the vacuum expectation values of the higher dimensional operators that appear at higher order in the OPE. The Wilson coefficients are independent of the masses except through the running of $\alpha_s(Q)$.

That only these combinations of running masses appear at dimension 2 in the OPE can be understood as follows: Consider the charged channel functions $\Delta_{\pm}^{V,A}(s)$ and $\lambda_{\pm}^{V,A}(s)$. Since they must be symmetric under the interchange $\hat{m}_1 \leftrightarrow \hat{m}_2$, they can only depend on $(\hat{m}_1 \pm \hat{m}_2)^2$. Changing the sign of one of the masses will interchange the vector and axial–vector cases so the coefficient of $(\hat{m}_1 \pm \hat{m}_2)^2$ in the vector channel is equal to the coefficient of $(\hat{m}_1 \mp \hat{m}_2)^2$ in the axial–vector channel. Since $\Delta_{\pm}^{V}(s)$ must vanish when $\hat{m}_1 = \hat{m}_2$, it can only depend on $(\hat{m}_1 - \hat{m}_2)^2$, which in turn means that $\Delta_{\pm}^{A}(s)$ can depend only on $(\hat{m}_1 + \hat{m}_2)^2$. The dependence of the neutral channel functions $\Delta_0^{V,A}(s)$ and $\lambda_0^{V,A}(s)$ on the running masses follows trivially from that of the charged channel functions.

Note that the Wilson Coefficients $\hat{C}_*(Q)$ and the running masses $\hat{m}_{1,2}(Q)$ depend only logarithmically on Q. Therefore, though these OPE's are derived in the deep Euclidean region $-s = Q^2 \gg 0$, we can expect the dependence of the $\Delta(s)$'s and $\lambda(s)$'s on the powers of s to be the same all around the circle $|s| = \Lambda^2$. We can see immediately that this implies $\Delta R_L(\infty) = \Delta R_T(\infty) = 0$. We can therefore conclude that Eq. (25) is correct even when higher order and non–perturbative QCD corrections are taken into account.

IV. CALCULATING $t\bar{t}$ THRESHOLD EFFECTS

Let us now turn to the problem of using dispersion relations to calculate the non–perturbative QCD corrections to $\Delta\rho$. In order to make use of the dispersion relations $\Delta\rho = \Delta\rho_T(\infty)$ and/or $\Delta\rho = \Delta\rho_L(\infty)$, we need to know the spectral functions $\mathrm{Im}\Pi(s)$ and/or $\mathrm{Im}\Delta(s)$ when non–perturbative corrections are taken into account. Since it is impossible to calculate them exactly, this means that we must make some assumptions and approximations about their non–perturbative behavior.

Let us denote the difference between the non–perturbative and perturbative vacuum polarization functions, by

$$\begin{aligned} \delta\mathrm{Im}\Pi(s) &\equiv \mathrm{Im}\Pi_{NP}(s) - \mathrm{Im}\Pi_P(s), \\ \delta\mathrm{Im}\Delta(s) &\equiv \mathrm{Im}\Delta_{NP}(s) - \mathrm{Im}\Delta_P(s), \end{aligned} \tag{29}$$

In Ref. [3], it was assumed that the most important effect of non–perturbative

QCD corrections on $\mathrm{Im}\Pi(s)$ and $\mathrm{Im}\Delta(s)$ is to modify the shape of the $t\bar{t}$ threshold. The threshold region is also the place where higher order corrections in α_s can be resummed in the leading non–relativistic approximation and calculated reliably in terms of a simple non-relativistic Schrödinger Green's function, as in Ref. [9]. Far from the threshold region, the $\mathrm{Im}\Pi(s)$'s are assumed to be well approximated by their $O(\alpha\alpha_s)$ perturbative results. Therefore, in this approximation, the functions $\delta\mathrm{Im}\Pi(s)$ and $\delta\mathrm{Im}\Delta(s)$ have their support only in the region near the threshold. Furthermore, in the leading non–relativistic approximation, only the s–wave states contribute so that

$$\delta_{t\bar{t}}\mathrm{Im}\Pi_0^V(s) = -s\delta_{t\bar{t}}\mathrm{Im}\lambda_0^V(s) = -s\delta_{t\bar{t}}\mathrm{Im}\lambda_0^A(s) = -\delta_{t\bar{t}}\mathrm{Im}\Delta_0^A(s), \tag{30}$$

while all the other $\delta_{t\bar{t}}\mathrm{Im}\Pi(s)$'s, $\delta_{t\bar{t}}\mathrm{Im}\Delta(s)$'s, and $\delta_{t\bar{t}}\mathrm{Im}\lambda(s)$'s are zero. (We have added a subscript to δ to indicate that since we don't expect large threshold corrections in the $b\bar{b}$ or $t\bar{b}$ channels, only $t\bar{t}$ threshold effects are included. This would not be the case if we were considering threshold effects for say a heavy fourth generation of quarks, but including the other channels does not alter our conclusions). Note that the first equality in Eq. (30) comes from the Ward Identity Eq. (6), the second equality comes from the spin independence of QCD interactions in the non–relativistic limit, and the third equality comes from Eq. (3), and the fact that $\delta_{t\bar{t}}\mathrm{Im}\Pi_0^A(s) = 0$ in this approximation. We will not specify what these non–zero terms look like in any detail since it is irrelevant to the following discussion.

Now, let us see what happens when we substitute Eq. (30) into the definitions of $\Delta\rho_T(\infty)$ and $\Delta\rho_L(\infty)$. We find:

$$\delta_{t\bar{t}}\left[\Delta\rho_T(\infty)\right] = -\frac{G_F}{\sqrt{2}}\left[\frac{1}{\pi}\int^\infty \frac{ds}{s}\delta_{t\bar{t}}\mathrm{Im}\Pi_0^V(s)\right] \equiv X, \tag{31}$$

and

$$\delta_{t\bar{t}}\left[\Delta\rho_L(\infty)\right] = -\frac{G_F}{\sqrt{2}}\left[\frac{1}{\pi}\int^\infty \frac{ds}{s}\delta_{t\bar{t}}\mathrm{Im}\Delta_0^A(s)\right] = \frac{G_F}{\sqrt{2}}\left[\frac{1}{\pi}\int^\infty \frac{ds}{s}\delta_{t\bar{t}}\mathrm{Im}\Pi_0^V(s)\right] = -X, \tag{32}$$

which are of the same magnitude but opposite in sign. This is the disagreement in sign that was mentioned in the introduction.

There are two ways to interpret this result. The first is that either one, or both of the dispersion relations Eq. (10) and (11) are wrong. This means that Eqs. (18) and (22) cannot be both correct. This is the approach adopted by the authors of Ref. [3] who, for a number of reasons, prefer Eq. (11) to Eq. (10). However, we do not think this possibility is very likely as we don't see where the OPE argument of the previous section could have failed.

A second and more plausible possibility is that using only the leading non–relativistic limit to calculate non–perturbative contributions to $\Delta\rho$ is simply not a good approximation and that the disagreement between Eq. (31) and (32) is a reflection of that fact.

Let us take a closer look at where this apparent inconsistency between the two approaches comes from. The reason we get the same magnitude but opposite signs in Eqs. (31) and (32) is because of Eq. (30), which is only true at leading order in the non–relativistic approximation in a small region near the threshold. If we could increase the range where we can calculate non-perturbative effects, Eq. (30) will not be true over the entire integration interval and we expect the difference between the two results to be reduced.

We illustrate this with an example: Consider the non-relativistic limit of the vacuum polarizations calculated to one loop in the $t\bar{t}$ channel,

$$
\begin{aligned}
\frac{\mathrm{Im}\Pi^V_{0,t\bar{t}}(s)}{s} &= -\frac{3\beta}{16\pi} + O(\beta^3), \\
\frac{\mathrm{Im}\Pi^A_{0,t\bar{t}}(s)}{s} &= O(\beta^3), \\
\frac{\mathrm{Im}\Delta^A_{0,t\bar{t}}(s)}{s} &= \frac{3\beta}{16\pi} + O(\beta^3),
\end{aligned}
\tag{33}
$$

where $\beta = \sqrt{1 - 4m_t^2/s}$. So, if we decided to calculate the perturbative contribution to $\Delta\rho$ using the leading non-relativistic approximation (and including all the other channels), we would again get answers of the same magnitude but opposite sign depending on whether we chose to calculate using the $\Pi(s)$'s or the $\Delta(s)$'s. However, from the full perturbative calculation, we know that both techniques should give the same answer. For similar reasons, (although there are important differences in the perturbative and non-perturbative cases), we believe pushing the cutoff further and further away from the threshold would cause the non perturbative results obtained using either the $\Pi(s)$'s or the $\Delta(s)$'s to converge towards each other, though perhaps very slowly. (The importance of the contribution from regions away from the threshold has already been noted in Ref. [5].) We are currently studying how much of this convergence can be achieved by taking the non–relativistic approximation to higher orders in β and thus expanding its region of applicability.

In any approximation, a good way to test its accuracy is to see how well it reproduces a known result. In the present case, a good way to test how well our approximation gives the correct value for $\Delta\rho_T(\infty)$ or $\Delta\rho_L(\infty)$ is to see how well it reproduces the known value for $\Delta\rho_T(\infty) - \Delta\rho_L(\infty)$, namely zero. When seen in this context, Eqs. (31, 32) are an indication that the $t\bar{t}$ threshold approximation fails, indeed we cannot even determine the sign of the additional contribution to $\Delta\rho$.

Of course, the bright side of it is that now we may understand the reason why the two dispersion relations Eq. (10) and Eq. (11) give seemingly contradicting results for the $t\bar{t}$ threshold effects. In fact, they are not contradicting at all. The difference between the two values is the error that should be associated with the approximation.

V. CONCLUSION

We have used the OPE to show that it is equally valid to calculate $\Delta\rho$ using either the dispersion relations of Ref. [6], or those of Ref. [3]. These correspond to using unsubtracted dispersion relations for either the transverse or the longitudinal parts respectively of the vacuum polarization functions that appear in the expression for $\Delta\rho$, and includes the consideration of non-perturbative effects.

When the two dispersion relations are used to calculate the effect of the $t\bar{t}$ threshold on $\Delta\rho$, they give results which are equal in magnitude but opposite in sign. This disagreement should not be interpreted as a sign that one of the dispersion relations is wrong, but as a sign that neglecting the non–threshold region when calculating non–perturbative effects to $\Delta\rho$ is a poor approximation.

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