On a Class of Parametrized Domain Optimization Problems with Mixed Boundary Condition Types

Cristina Felicitas Letona Bolivar

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Jeffrey T. Borggaard, Chair
Traian Iliescu
Tao Lin
Lizette Zietsman

September 22, 2016
Blacksburg, Virginia

Keywords: Domain Optimization, Shape Derivatives, PDE Constraints,
Mixed Boundary Conditions.

On a Class of Parametrized Domain Optimization Problems with Mixed Boundary Condition Types

Cristina Felicitas Letona Bolivar

(ABSTRACT)

Shape optimization problems have a long history of mathematical study and a wide range of applications. In recent decades there has been an interest in solving these problems with partial differential equation (PDE) constraints. We consider a special class of PDE-constrained shape optimization problems where different boundary condition types (Dirichlet and Neumann) are imposed on the same boundary segment. We also consider the case where the interface between these different boundary condition types may also be parameter dependent. This study also includes special cases where the shape of the region where the PDE is imposed does not change, but the domain of the partial differential operator is parameter dependent, due to the change in boundary condition type. Our treatment centers on the infinite dimensional formulation of the optimization problem. We consider existence of solutions as well as the calculation of derivatives of the associated shape functionals via adjoint solutions. These derivative formulations serve as a starting point for practical numerical approximations.

On a Class of Parametrized Domain Optimization Problems with Mixed Boundary Condition Types

Cristina Felicitas Letona Bolivar

(GENERAL AUDIENCE ABSTRACT)

Optimization problems arise in a number of areas and are usually posed as finding values of design parameters that minimize a given cost function. Examples include finding the shape of a car or airplane wing to reduce drag and improve fuel economy which maintaining a desired level of performance. This is an example of a constrained optimization problem where the constraint is described by a physical model known as a partial differential equation (PDE). For shape optimization problems, we want to find the best shape to minimizes a certain cost function, and the cost depends on the shape through the solution to the PDE. The strategy for solving a shape optimization problem depends on the particular problem at hand. In many cases, one assumes that the solution of an optimization problem exists, so the development of methods to find or approximate possible solutions is the first step.

In this dissertation, we study some theoretical aspects of the problem that can be used to guarantee the existence of an optimal (or locally optimal) solution to the problem. We focus our attention on a special class of PDE constraints where the cost function is calculated over a domain with an unknown portion that needs to be determined. We further consider a special case of boundary conditions for the PDE constraints known as mixed boundary conditions. In this work, we study the theoretical aspects to guarantee the existence of a solution, and then we provide formulations of the derivatives that permit algorithms to search for the shape of the domain that minimizes a given cost function. These formulations are important to develop efficient numerical approximations.

Dedication

To my lovely son Christopher.

Acknowledgments

I would like to thank my advisor Dr. Jeff Borggaard for his support and guidance during these years. His experience, patience, and encouragement has helped me to complete my thesis. I would also like to acknowledge the Institute for Mathematics and its Applications (IMA) for giving me the opportunity to participate in the workshop *Frontiers in PDE-Constrained Optimization*, this learning experience was unique.

To the members of my commitee, Dr. Lizette Zietsman for her comments and motivation in writing my thesis, as well as Dr. Tao Lin and Dr. Traian Iliescu for their time and important remarks in this work.

Thanks to my professors Dr. Leslie Kay, Dr. George Hagedorn, Dr. Michael Renardy, Dr. Yuriko Renardy and Dr. John Burns for all the feedback they provided me through their course assignments.

In the Math Department, I would particularly like to acknowledge Nicole Sutphin and Leigh Ann Teel for their assistance cutting through red tape as well as Eileen Shugart and Abigail Kohler for their experience and advice with my teaching duties.

To Edgar Saenz and our little Christopher for their time, support, and all the times they helped me stand up. My gratitude to my parents Ernesto and Feliciana, the strongest people I know. To my siblings for their love and emotional support: Gladys, Justina, Ernesto and Luisa.

To my roommates and friends for their help and motivation these years: Hans-Werner van Wyk, Moises Guerra, Bartleby Ordonez, Carlos Rautenberg, Boris Aguilar, Rosana Azpiroz, and Felita and David Murrugarra.

Contents

Notation

Ω	- An open subset of $\mathbb{R}^2$ with compact closure. It represents a domain in $\mathbb{R}^2$ whose shape is the goal of the optimization problem.
$V(\Omega)$	- Space of real functions defined in Ω.
$\mathbf{x}$	- The pair (x, y) in Ω.
n	- The unit normal vector on the boundary of Ω, outward to Ω.
J	- Shape functional defined on Ω (also known as domain functional or cost functional).
$\dot{J}$	- Derivative of the functional J.
Ω^*	- Domain that minimizes the shape functional J.
Γ	- Boundary of the domain Ω.
α	- Function defined on an interval contained in $[0, 1]$ whose graph describes a portion of the boundary that needs to be determined.
Ω_α	- Domain in $\mathbb{R}^2$ whose portion of the boundary is described by the graph of the function α.
$\mathscr{A}$	- Set that contains the functions α whose graph describes a portion of the boundary of Ω.
$\mathcal{O}$	- Set that contains all admissible domains ($\cup_{\alpha \in \mathscr{A}} \Omega_\alpha \subseteq \mathcal{O}$).
D	- A larger set that contains $\mathcal{O}$.
$C^k(\overline{D})$	- Set of continuous functions f such that $f^{(i)}(\cdot)$ is continuous for all $i = 0, \ldots, k$.
$C^\infty(\overline{D})$	- Set of continuous functions that are in the above set for every k.
$C^k_{per}([0, 2\pi])$	- Set of functions f in $C^k([0, 2\pi])$ such that $f^{(i)}(0) = f^{(i)}(2\pi)$ for all $i = 0, \ldots, k$.
$C^{0,1}[a, b]$	- Set of Lipchitz continuous functions in $[a, b]$.
Φ_t	- Mapping function from Ω to Ω_t.
$H^k(\Omega)$	- Set of functions whose derivatives up to order k (in the sense of distributions) are square integrable in Ω.
$H^k_0(\Omega)$	- Set of functions from $H^k(\Omega)$, whose derivatives up to order $(k-1)$ in the sense of traces are equal to zero on Γ.
$H^1_{loc}(\mathbb{R}^2)$	- Set of functions whose first derivatives are locally square integrable in Ω.
$L^2(\Omega)$	- Set of Lebesgue integrable function ($H^0(\Omega) = L^2(\Omega)$).

$L^2_{loc}(D)$	-	The set of locally square integrable functions.
$\vec{s}$	-	A smooth perturbation field defined on Ω.
$\mathcal{G}$	-	Set of pairs $(\Omega_\alpha, u_\alpha)$ where α is in $\mathscr{A}$ and u_α is the solution to a state problem in Ω_α.
$\dot{u}$	-	Material derivative of u.
u'	-	Shape derivative of u.
$\rightharpoonup$	-	Weakly convergences to.

Chapter 1

Introduction

1.1 Overview and Problem Description

We develop some theoretical results for shape optimization problems, the problem of determining the shape of the domain that minimizes an objective functional over a set of admissible domains. Shape optimization problems can be constrained or unconstrained. Constrained shape optimization problems, with constraints that take the form of partial differential equations and/or integral equations arise in many applications. These applications include drag reduction, shielding electromagnetic fields, and estimating the shape of cavities such as reservoirs or tumors. Variants of these applications can be posed as shape optimization problems with elliptic PDE constraints. As for any other optimization problem, solving a constrained problem where the dependence of the parameters on the functional is non-linear presents an interesting case for analysis. We also consider the case where we have either an infinite-dimensional parameter space or a complicated dependence on a single parameter.

In the literature we can find the following classification of domain optimization problems: sizing optimization, topology optimization, and shape optimization. In sizing optimization problems, the shape of the domain that minimizes a cost functional varies in size only (length, width, radius, etc.), so shape of the domain is not altered. For the case of topology optimization problems, the topology of the domain can be changed to find the shape of the domain that minimizes a cost function. Topology optimization is very active among practitioners, but usually relies on a specific underlying parametrization of the domain. In this dissertation, when we refer to shape optimization problems we imply that we can alter the shape of the domain to obtain the minimizing domain. In particular, we study the case where a portion of the boundary of the domain is fixed, and we need to determine another portion of the boundary that is described by the graph of a function.

To solve any optimization problem, we must identify the set that contains all possible solutions, then analyze if there is an absolute or local minimum so that we can start our search. Similarly, when working with domain optimization problems we define a set that contains all possible "domain" solutions, which is called the set of *admissible domains*. We prescribe additional assumptions to guarantee the existence of the minimizer and develop the theory of how this solution can be reached (approximated).

1.2 Summary of Prior Work

A number of theoretical results concerning the existence of solutions to domain optimization problems have been presented. These include pioneering results by [10], [23], [25], [33], and [35]. These studies vary in the choice of functionals or the form of the PDE constraints. The typical study of the existence of solutions begins with an analysis of a shape optimization problem subject to either homogeneous Dirichlet boundary conditions or homogeneous Neumann boundary conditions. Few results on finding the minimizing domain subject to PDE constraints with mixed boundary condition types are presented in the literature.

Since these problems deal with variations in the shape of the domains, either in the unconstrained or constrained case, a theory of *shape calculus* and the extended notion of first order necessary conditions have been developed. See for example, [13], [19], [26], [35], [37] and [39].

Obtaining second order necessary conditions and sufficient conditions depends on the problems of interest. For example [16] and [37] present a theory for unconstrained problems with star-shaped domains, such as

$$\text{find } \Omega \text{ that minimizes } J(\Omega) = \int_\Omega F(\mathbf{x})d\mathbf{x},$$

where Ω belongs to a set of admissible domains. The assumption of star-shaped domain admits a natural parameterization of the boundary. For example, the approximated domain solution can obtained using cubic splines.

Most approximation theory for shape optimization problems is developed for the discretized version of the PDE-constrained optimization problem. Optimizing the problem, then performing discretization is not well studied. However, there have been important developments using this approach, see for example [22] and [39]. Most of the time the shape optimization problem is discretized first, since computing the derivative of the shape functional seems to be a difficult task or impossible in some cases. However, a variety of different methods using the infinite dimensional version of the optimization problem have been developed, see for example [5], [20], and [23].

The optimization problems we usually have seen involves changing the shape of the whole domain (cf. [38]). However, we are going to focus our study in the theoretical aspects when a portion of the boundary of the domain is fixed.

1.3 Outline of the Dissertation

In this work we are going to consider shape optimization problems that contain specific PDE constraints. Thus, the definition of the admissible sets needs to be chosen carefully since we not only need to guarantee the uniqueness of solutions to the state problem, we also need to guarantee that the optimization problem of interest has at least a local solution. Hence, we cannot work with any arbitrary domain. As for the analysis of any other optimization problem, we desire compactness of the set over which we optimize, as well as the continuity of the shape functional over this set to conclude the existence of an optimum. However, since we are looking for a minimizer, an assumption of lower semi-continuity will be enough.

Chapter 2 presents a general approach of the abstract setting to the domain optimization problem subject to PDE constraints. In that chapter, we begin with the consideration of the unconstrained domain optimization problem. We present the unconstrained problem to develop the analytical framework as well as to contrast how unconstrained domain optimization problems differ from constrained cases. The set of admissible domains we considered for the unconstrained domain optimization problems is the set of star-shaped domains with center at the origin. We say that Ω is a star-shaped domain with respect to the origin (or center at the origin) if for all $\mathbf{x}$ in Ω the line segment that connects the origin with $\mathbf{x}$ is in Ω.

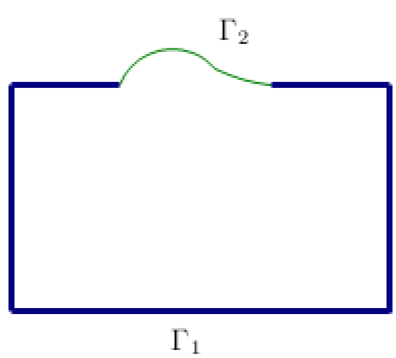
This leads us to the main emphasis of this dissertation, to find a domain Ω that minimizes the following functional:

$$J(\Omega) = \int_\Omega F(u)d\mathbf{x},$$

where u solves

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \Gamma_1 = \overline{\Omega} \backslash \Gamma_2 \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_2. \end{aligned}$$

We assume Γ_1 is a fixed portion of the boundary of the domain Ω, and Γ_2 is the unknown portion of the boundary of the domain that needs to be determined, see the picture below.



We assume lower semi-continuity of the shape functional J. For example, we will show that the case $J(\alpha, u) = ||u||^2_{0,\Omega_\alpha}$ satisfies our lower semi-continuity assumption for the optimization problem.

In Chapter 3, we cover the assumptions and theory about the existence of the minimizer for our shape (domain) optimization problem subject to PDE constraints. We define the set $\mathcal{G}$ that contains the pair $(\Omega_\alpha, u_\alpha)$ where α is a function whose graph describes Γ_2 and so Ω_α, and u_α is the solution to the state problem on Ω_α. We start describing the topology of $\mathcal{G}$ and verify the compactness property on $\mathcal{G}$, then we can guarantee the existence of solutions using the lower semicontinuity property. This framework allows us to formulate, to the best of our knowledge, the first results that hold for PDE constraints with mixed boundary condition types.

The goal in Chapter 4 is to compute the derivative of the shape functional J. Since the main ideas of solving one type of domain optimization problem can help us to solve a new one, we also compute the derivative of the shape functional J for important variations of our main problem.

Chapter 5 presents a different approach for computing the derivative of the shape functional J based only on perturbations of the boundary. The derivative of the shape functional J is obtained using our calculation from Chapter 4 and results given by Zochowski [38]. Thus we provide two different approaches for computing the derivative of J. The efficient computation of shape derivatives of J is often the main bottleneck to solving shape optimization problems. Thus, these approaches allow for flexibility in choosing an effective algorithm to compute the derivative.

Chapter 2

Domain Optimization Problems

The main goal of this chapter is to provide the necessary background material to develop our results on unconstrained and constrained domain optimization problems. We discuss the necessary conditions that guarantee the existence of locally optimal solutions for unconstrained problems. We also describe the abstract setting for our optimal shape design problems with PDE constraints.

2.1 Unconstrained Domain Optimization Problem

The goal is to find the domain that minimizes a shape functional

$$J(\Omega) = \int_{\Omega} F(u, \nabla u, \mathbf{x}) d\mathbf{x}, \tag{2.1}$$

where u is a given fixed function of $\mathbf{x}$. To find the minimizing domain, we need a perturbation of an initial domain, which is defined as:

$$\Omega_t[\vec{s}] = \{(I + t\vec{s})(\mathbf{x}) : \mathbf{x} \in \Omega\}, \tag{2.2}$$

where $\vec{s} : \Omega \to \mathbb{R}^d$ $(d = 2, 3)$ is a smooth perturbation field.

Then the directional derivative of the shape functional is given by

$$\lim_{t \to 0} \frac{J(\Omega_t[\vec{s}]) - J(\Omega)}{t}. \tag{2.3}$$

If we consider the following shape functional

$$J(\Omega) = \int_{\Omega} h(\mathbf{x}) d\mathbf{x}, \tag{2.4}$$

one approach to solve this type of minimization problem is by parametrizing the boundary of the domain using polar coordinates:

$$\Gamma = \{\gamma(\phi) = (r(\phi)\cos(\phi), r(\phi)\sin(\phi)) : \phi \in [0, 2\pi]\}, \tag{2.5}$$

where $r \in C^1_{per}([0, 2\pi])$ is a positive function. Then consider the set of admissible domains $\mathcal{O}$ to be the set of star-shaped domains with respect to the origin. Also, note that the boundaries Γ_t are defined by $r_t = r(\phi) + t dr(\phi)$, with $dr \in C^1_{per}([0, 2\pi])$.

Lemma 2.1.1. *The shape functional (2.4) is twice Frechét differentiable with respect to $C^1_{per}([0, 2\pi])$, where the shape gradient and Hessian are*

$$\nabla J(\Omega)[dr] = \int_0^{2\pi} r(\phi) dr(\phi) h(r(\phi), \phi) d\phi,$$

$$\nabla^2 J(\Omega)[dr_1, dr_2] = \int_0^{2\pi} dr_1(\phi) dr_2(\phi) \left\{ h(r(\phi), \phi) + r(\phi) \frac{\partial h}{\partial n}(r(\phi), \phi) \right\} d\phi.$$

The proof can be found in [15], [19].

We say that Ω_0 is a **stationary domain** if $\nabla J(\Omega_0)[\delta r] = 0$ for all $\delta r \in C^1_{per}([0, 2\pi])$.

Furthermore, for unconstrained optimization problems with shape functional of the form (2.4), we have the following optimality condition. The proof can be found in [16].

Theorem 2.1.2. *For $\Omega_0 \in C^1$ and $h \in C^2$ the conditions $h|_{\Gamma_0} \equiv 0$ and $\left.\frac{\partial h}{\partial r}\right|_{r=r_0} > 0$ are sufficient for optimality.*

2.1.1 Example: A minimization problem with star-shaped admissible domains

The following example is stated in [19]. Note that we can solve this problem geometrically, but the main goal of stating this unconstrained optimization problem is to get familiar with the above results.

$$\text{Minimize } J(\Omega) = \int_\Omega \left(\frac{x^2}{8} + \frac{y^2}{4} - 1 \right) dxdy.$$

where Ω is a star-shaped domain with center at the origin. Note that in this problem

$$h(x, y) = \frac{x^2}{8} + \frac{y^2}{4} - 1.$$

Using Theorem (2.1.2), we can guarantee that the region enclosed by the ellipse $\frac{x^2}{8} + \frac{y^2}{4} - 1$ is an strict local minimizer.

Claim 2.1.3. *Under above assumptions,* $\frac{\partial h}{\partial r}(r^*(\theta),\theta) > 0$ *where* $r^*(\theta) = \frac{2\sqrt{2}}{\sqrt{1+\sin^2\theta}}$ *is the function determined by the ellipse* $\frac{x^2}{8} + \frac{y^2}{4} = 1$.

Proof. Consider the function

$$h(r,\theta) = h(r\cos\theta, r\sin\theta) = \frac{(r\cos\theta)^2}{8} + \frac{(r\sin\theta)^2}{4} - 1. \tag{2.6}$$

Then the derivative of h with respect to r is

$$\frac{\partial h}{\partial r} = \frac{2r\cos^2\theta}{8} + \frac{2r\sin^2\theta}{4} = \frac{r}{4}(\cos^2\theta + 2\sin^2\theta) = \frac{r}{4}(1+\sin^2\theta).$$

Therefore,

$$\frac{\partial h}{\partial r}(r^*(\theta),\theta) = \frac{1}{4}\frac{2\sqrt{2}}{\sqrt{1+\sin^2\theta}}(1+\sin^2\theta) = \frac{\sqrt{2}}{2}(\sqrt{1+\sin^2\theta}) > 0 \text{ on } [0,2\pi].$$

□

Next we show that the region enclosed by the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$ is the global minimizer. The shape functional in polar coordinates can be written as

$$J(\Omega) = \int_0^{2\pi}\int_0^{r(\theta)} h(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{r(\theta)} \Big[\frac{r^2\cos^2\theta}{8} + \frac{r^2\sin^2\theta}{4} - 1\Big] r\,dr\,d\theta$$

$$= \int_0^{2\pi} \Big[\frac{r^4\cos^2\theta}{32} + \frac{r^4\sin^2\theta}{16} - \frac{r^2}{2}\Big]\Big|_0^{r(\theta)} d\theta.$$

Thus

$$J(r(\cdot)) = \int_0^{2\pi} \Big[\frac{r(\theta)^4\cos^2\theta}{32} + \frac{r(\theta)^4\sin^2\theta}{16} - \frac{r(\theta)^2}{2}\Big] d\theta.$$

So the original problem is equivalent to

$$J(r(\cdot)) = \int_0^{2\pi} \Big[\frac{r(\theta)^4}{32}(\cos^2\theta + 2\sin^2\theta)\Big] - \frac{r(\theta)^2}{2} d\theta$$

$$= \int_0^{2\pi} \Big[\frac{r(\theta)^4}{32}(1+\sin^2\theta)\Big] - \frac{r(\theta)^2}{2} d\theta.$$

We know that the region Ω^*: $\dfrac{x^2}{8}+\dfrac{y^2}{4} \leq 1$ defines an extreme point for $J(\Omega) = \displaystyle\int\int_\Omega h(x,y)dydx$.
Since the boundary of Ω^* is determined by

$$\frac{[r^*(\theta)\cos\theta]^2}{8} + \frac{[r^*(\theta)\sin\theta]^2}{4} = 1$$

then $r^*(\theta)$ satisfies

$$[r^*(\theta)]^2 = \frac{8}{1+\sin^2\theta}.$$

Hence the shape functional at r^* is given by

$$J(r^*) = \int_0^{2\pi} \frac{(r^*)^4}{32}(1+\sin^2\theta) - \frac{(r^*)^2}{2}d\theta = \int_0^{2\pi} \frac{2}{1+\sin^2\theta} - \frac{4}{1+\sin^2\theta}d\theta.$$

That is,

$$J(r^*) = \int_0^{2\pi} \frac{-2}{1+\sin^2\theta}d\theta. \tag{2.7}$$

To conclude that the ellipse describes the global minimizer, we have the following claim.

Claim 2.1.4. *Under the above assumptions, $J(r(\cdot)) \geq J(r^*(\cdot))$ for all $r \in C^1_{per}[0,2\pi]$.*

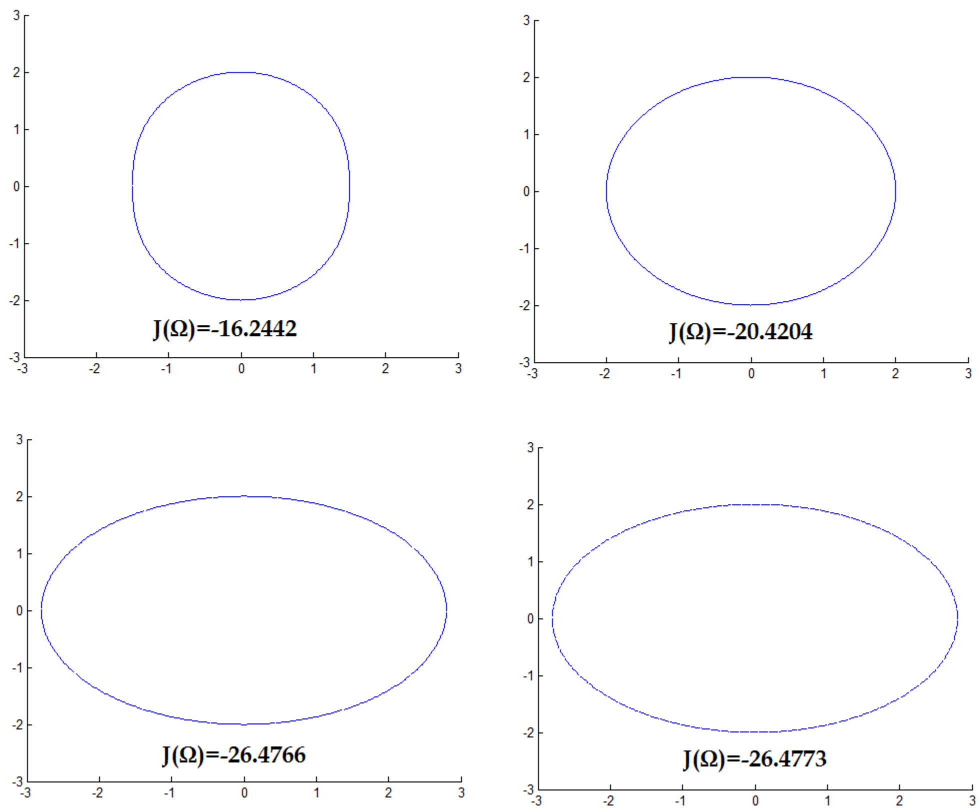
Proof. We just need to prove that $J(r(\cdot)) - J(r^*(\cdot)) \geq 0$.

$$\begin{aligned}
J(r(\cdot)) - J(r^*(\cdot)) &= \int_0^{2\pi} \left[\frac{r^4}{32}(1+\sin^2\theta) - \frac{r^2}{2} + \frac{2}{1+\sin^2\theta}\right]d\theta \\
&= \int_0^{2\pi} \frac{1}{2(1+\sin^2\theta)}\left[\frac{r^4(1+\sin^2\theta)^2}{4^2} - r^2(1+\sin^2\theta) + 4\right]d\theta \\
&= \int_0^{2\pi} \frac{1}{2(1+\sin^2\theta)}\left[\left[\frac{r^2(1+\sin^2\theta)}{4}\right]^2 - r^2(1+\sin^2\theta) + 4\right]d\theta \\
&= \int_0^{2\pi} \frac{1}{2(1+\sin^2\theta)}\left[\frac{r^2(1+\sin^2\theta)}{4} - 2\right]^2 d\theta \\
&\geq 0.
\end{aligned}$$

□

Remark 2.1.5. *Note that $J(r(\cdot)) - J(r^*(\cdot))$ is zero if and only if $r^2(1+\sin^2\theta) = 8$, that is $r^2 = \dfrac{8}{1+\sin^2\theta}$, which is the ellipse that defines Ω^*.*

Below are some calculations of how cubic splines can be used to approximate solutions to unconstrained domain optimization problem where the set of admissible domains is the set of star-shaped domains with center at the origin. We compare the values of $J(\Omega)$ using N nodes with the value of $J(\Omega^*)$.



N	$J(\Omega)$	$J(\Omega) - J(\Omega^*)$
5	-27.42600053	-0.76870290154680
15	-26.65658771	0.00070992086390
25	-26.65721841	0.00007921637130
35	-26.65727913	0.00001849931510
45	-26.65729119	0.00000643819480
55	-26.65729482	0.00000280420740
65	-26.65729622	0.00000141163590
75	-26.65729684	0.00000078647900
85	-26.65729716	0.00000047237090
95	-26.65729733	0.00000030063690

2.2 PDE-Constrained Domain Optimization

A general domain optimization problem with PDE constraints is described below.

Minimize the shape functional

$$J(\Omega) = \int_{\Omega} F(u, \nabla u, \mathbf{x}) d\mathbf{x}, \quad \Omega \in \mathcal{O},$$

where $\mathcal{O}$ is the set of admissible domains $\Omega \in \mathbb{R}^2$, and u satisfies

$$\mathcal{A}u = f \text{ in } \Omega$$
$$\mathcal{B}u = g \text{ on } \Gamma \subset \partial\Omega$$

where $\mathcal{A}$ is a well-posed partial differential operator in the domain Ω, and $\mathcal{B}$ operates on the functions supported at the free boundary Γ.

There are different approaches to solve this type of optimization problem, one method is by discretizing the problem first, and then optimizing. Another approach is optimizing first, then discretize. In this work, we begin by analyzing the existence of the solution to the optimization problem. Therefore, we first need to study the existence and uniqueness of solutions to the PDE problem. Then we compute the derivative of the shape functional for the abstract problem so that it can be used to approximate the solution numerically. Since we are dealing with non-linear problems, each optimization problem is treated individually, however the process of working with one problem can help us to solve a similar problem.

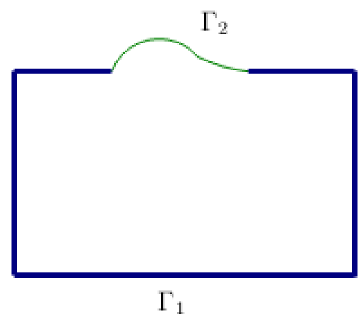
Our main goal is to study an optimization problem that considers mixed boundary conditions. That is, find Ω that solves

$$\min \ J(\Omega, u) = \int_{\Omega} F(u) d\mathbf{x}$$

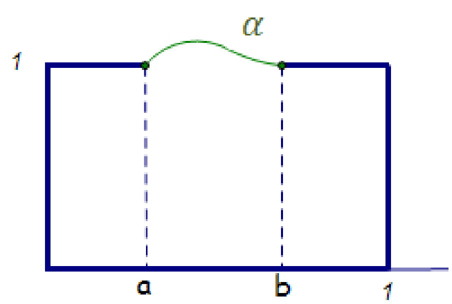
where Ω is in the set of admissible domains $\mathcal{O}$ and u solves the system:

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \Gamma_1 = \overline{\Omega} \backslash \Gamma_2 \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_2. \end{aligned} \tag{2.8}$$

Here Γ_1 is the fixed part of the boundary of the domain, and Γ_2 is the unknown part that needs to be determined.



To analyze this problem, we need to define our set of admissible domains $\mathcal{O}$. We begin by representing Γ_2 by the graph of a function α, where the function α is defined on a set $[a, b]$ contained in the interval [0,1].



Since the shape of the domain is the goal of the optimization problem, we need to define a set $\mathscr{A}$ that contains the functions α that describes Γ_2.

Let $c > 0$, we say that $\alpha \in \mathscr{A}$ if and only if $\alpha : [a, b] \to R$ is Lipschitz, the Lipschitz constant is $L(\alpha) \leq c_0$, and $\alpha_{\min} < \alpha(x) < \alpha_{\max}$, where $\alpha_{\min}$ and $\alpha_{\max}$ are two positive constants.

Then we define $\mathcal{O} = \{\Omega(\alpha) | \alpha \in \mathscr{A}\}$. It is the set of domains whose curved portion Γ_2 is represented by the graph of the function $\alpha \in \mathscr{A}$.

Now we can formulate our problem as follows: find the pair $(\Omega_\alpha, u_\alpha)$ that minimizes

$$J(\Omega_\alpha, u_\alpha) = \int_{\Omega_\alpha} F(u_\alpha) dx \tag{2.9}$$

where Ω_α is in the set of admissible domains $\mathcal{O}$ and u_α solves the system:

$$\begin{aligned} -\Delta u_\alpha &= f \text{ in } \Omega_\alpha \\ u_\alpha &= g \text{ on } \Gamma_1 = \overline{\Omega}_\alpha \backslash \Gamma_2 \\ \frac{\partial u_\alpha}{\partial n} &= 0 \text{ on } \Gamma_2. \end{aligned} \tag{2.10}$$

Next section we study the existence of solutions to our optimization problem. In order to guarantee the existence of a minimum, we want lower-semicontinuity of J and compactness of $\mathcal{G}$ where

$$\mathcal{G} = \{(\alpha, u_\alpha) : \alpha \in \mathscr{A} \text{ and } u_\alpha \text{ is the solution to (2.10) in } \Omega_\alpha\}.$$

It is important to remark that for the compactness of $\mathcal{G}$, we need to equip $\mathcal{G}$ with a topology. Also, calculating the derivative of the shape functional is not as easy as the unconstrained problem, so finding stationary domains may not be possible. However, finding this derivative can help us to find a descent direction. The calculation of the derivative of the shape functional depends on the problem but as we said before working with one problem can help us to solve a similar one.

2.2.1 Abstract setting of optimal shape design problems subject to PDE constraints

Since the goal of the optimization problem is the shape of the domain, we need to define convergence of sets through convergence of functions that describe the boundary of the domains.

Let $\tilde{\mathcal{O}}$ be a larger system containing $\mathcal{O}$, where $\mathcal{O}$ is the set of admissible domains. That is, $\mathcal{O} \subset \tilde{\mathcal{O}}$. Let $\{\Omega_n\}$ be a sequence in $\tilde{\mathcal{O}}$, and $\Omega \in \tilde{\mathcal{O}}$. The notation $\Omega_n \xrightarrow{\tilde{\mathcal{O}}} \Omega$, as $n \to \infty$ means $\{\Omega_n\}$ tends to Ω where the convergence satisfies the assumption: $\Omega_{n_k} \xrightarrow{\tilde{\mathcal{O}}} \Omega$, as $k \to \infty$ for any subsequence Ω_{n_k} of Ω_n.

For each $\Omega \in \tilde{\mathcal{O}}$ we associate a function space $V(\Omega)$ of real functions defined in Ω. Then we introduce convergence of functions that are in $V(\Omega)$ for different $\Omega \in \mathcal{O}$. So, if y_n is a sequence in $V(\Omega)$, $\Omega_n \in \tilde{\mathcal{O}}$, $y \in V(\Omega)$, $\Omega \in \tilde{\mathcal{O}}$, the convergence of y_n to y is denoted by $y_n \rightsquigarrow y$ and again we assume that for any subsequence $\{y_{n_k}\}$ of $\{y_n\}$: $y_{n_k} \rightsquigarrow y$ as $k \to \infty$.

In any $\Omega \in \tilde{\mathcal{O}}$ we solve a state problem, so we define $u : \Omega \mapsto u(\Omega)$ i.e. for any $\Omega \in \tilde{\mathcal{O}}$ we associate an element $u(\Omega) \in V(\Omega)$.

$$u : \Omega \mapsto u(\Omega) \in V(\Omega), \Omega \in \tilde{\mathcal{O}}. \tag{2.11}$$

Assume that 2.11 has a unique solution for $\Omega \in \tilde{\mathcal{O}}$.

Finally, let $J : (\Omega, y) \to J(\Omega, y) \in \mathbb{R}$, $\Omega \in \tilde{\mathcal{O}}$, $y \in V(\Omega)$ be a cost functional and define

$$\tilde{\mathcal{G}} = \{(\Omega, u(\Omega)) : \Omega \in \mathcal{O}\}.$$

The abstract optimal shape design problem is given by

$$\text{Find } \Omega^* \in \mathcal{O} \text{ such that } J(\Omega^*, u(\Omega^*)) \leq J(\Omega, u(\Omega)) \text{ for all } \Omega \in \mathcal{O}$$
$$\text{where } u(\Omega) \in V(\Omega) \text{ solves } (P(\Omega)).$$

To guarantee the existence of solutions to the abstract problem above we need compactness property of $\tilde{\mathcal{G}}$ and lower semicontinuity of J.

(I) Compactness property of $\tilde{\mathcal{G}}$: For any sequence $(\Omega_n, u(\Omega_n)) \in \mathcal{G}$, there is a subsequence $(\Omega_{n_k}, u(\Omega_{n_k}))$ and an element $(\Omega, u(\Omega)) \in \mathcal{G}$ such that $\Omega_{n_k} \to \Omega$ as elements of $\tilde{\mathcal{O}}$, and $u(\Omega_{n_k}) \rightsquigarrow u(\Omega)$, as $k \to \infty$.

(II) Lower semicontinuity of J: If $\Omega_n \xrightarrow{\tilde{\mathcal{O}}} \Omega$, $\Omega_n, \Omega \in \tilde{\mathcal{O}}$ and $y_n \rightsquigarrow y, y_n \in V(\Omega_n), y \in V(\Omega)$, then $\liminf_{n\to\infty} J(\Omega_n, y_n) \geq J(\Omega, y)$.

Theorem 2.2.1. *If the compactness property of $\mathcal{G}$ and the lower semicontinuity of J are satisfied, then the abstract problem has at least one solution.*

The proof can be found in [23].

Chapter 3

Existence of the Optimal Solution

3.1 Overview and Background

In this chapter we investigate the existence of the optimal solution of the domain functional (2.9) introduced in Chapter 2. Since the shape of the domain is the goal of the optimization problem, we need to define a set $\mathscr{A}$ that contains the functions α that describes $\Gamma_2(\alpha)$, and a set that contains all admissible domains.

H1. *Let* α_{min}, α_{max} *and* c_0 *be constants satisfying* $0 < \alpha_{min} < 1 < \alpha_{max}$ *and* $c_0 > 0$, $[a, b]$ *be a nonempty interval in* $]0, 1[$,

$$\mathscr{A} = \{\alpha \in C^{0,1}[a, b] : 0 < \alpha_{min} \leq \alpha \leq \alpha_{max} \text{ in } [a, b],\ |\alpha'| \leq c_0 \text{ a.e. in } [a, b], \alpha(a) = \alpha(b) = 1\}.$$

Here $C^{0,1}[a, b]$ is the set of all Lipschitz continuous functions in $[a, b]$ and the parameters α_{min}, α_{max}, c_0 are such that $\mathscr{A} \neq \emptyset$. The set of admissible domains is then defined by

$$\mathcal{O} = \{\Omega(\alpha) : \alpha \in \mathscr{A}\},$$

where $\Omega(\alpha) = \{(x, y) \in \mathbb{R}^2 : 0 < y < \alpha(x), x \in [a, b]\} \cup \{(x, y) \in \mathbb{R}^2 : 0 < y < 1, x \in (0, 1) \setminus [a, b]\}$ (see Figure 3.1). To state some intermediate results we shall also use the larger set $D =]0, 1[\times]0, Y[$ where Y is a constant such that $\alpha_{max} < Y$.

For each $\Omega(\alpha) \in \mathcal{O}$ let $\Gamma(\alpha)$ be its boundary. Set $\Gamma_2(\alpha) = \{(x, \alpha(x)) : x \in [a, b]\}$ and let Γ_1 be the complement of $\Gamma_2(\alpha)$ in Γ. Note that Γ_1 is the same for every $\Omega(\alpha) \in \mathcal{O}$. Now let $f \in L^2_{loc}(D)$, $g \in H^1_{loc}(\mathbb{R}^2)$, and let u_α be the weak solution to the state equation

$$-\Delta u = f \text{ in } \Omega(\alpha), \qquad u = g \text{ on } \Gamma_1, \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_2(\alpha). \tag{3.1}$$

We want to find α^* such that $\Omega(\alpha^*) \in \mathcal{O}$ and

$$J(\alpha^*, u_{\alpha^*}) \leq J(\alpha, u_\alpha) \tag{3.2}$$

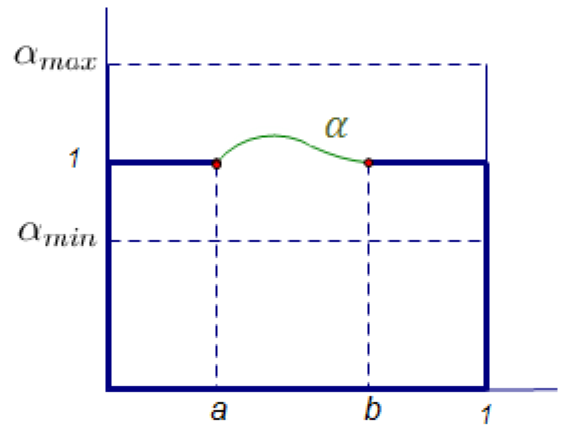


Figure 3.1: Admissible domain determined by $\alpha \in \mathscr{A}$

for all $\alpha \in \mathscr{A}$, where u_α is the weak solution to (3.1). As a concrete example consider $J(\alpha, u) = \int_{\Omega(\alpha)} u^2 dx$.

Therefore, we have the following optimization problems:

- Find Ω that minimizes $J(\Omega) = \int_\Omega F(u)\, d\mathbf{x}$ where u solves (3.1).
- Find Ω that minimizes $J(\Omega) = \int_\Omega u^2\, d\mathbf{x}$ where u solves (3.1).

We continue with a description of the topology space. Then, we will rely on Rellich's theorem (cf. [23], [28]), to prove the lower semicontinuity of the functional J.

3.2 Topology on $\mathcal{G}$

We first give some notation and definitions that will be used in the sequel. For each $\alpha \in \mathscr{A}$, the space V_α is defined by

$$V_\alpha = \{v \in H^1(\Omega(\alpha)) : v|_{\Gamma_1} = 0\},$$

where $H^1(\Omega(\alpha))$ is the usual Sobolev space equipped with the norms $||\cdot||_{1,\Omega(\alpha)}$, $||\cdot||_{0,\Omega(\alpha)}$, and seminorm $|v|_{1,\Omega(\alpha)}$ defined by

$$||v||_{1,\Omega(\alpha)} = \left(||v||^2_{0,\Omega(\alpha)} + ||\nabla v||^2_{0,\Omega(\alpha)}\right)^{1/2},$$

$$||v||_{0,\Omega(\alpha)} = \left(\int_{\Omega(\alpha)} |v|^2 dx \right)^{1/2},$$

$$|v|_{1,\Omega(\alpha)} = \left(\int_{\Omega(\alpha)} |\nabla v|^2 dx \right)^{1/2}.$$

Under the above settings, $u_\alpha \in H^1(\Omega(\alpha))$ is the weak solution to (3.1) if and only if

$$\begin{cases} u_\alpha - g \in V_\alpha \\ \displaystyle\int_{\Omega(\alpha)} \nabla u_\alpha \cdot \nabla v \; dx = \int_{\Omega(\alpha)} f v dx \quad \text{for all} \;\; v \in V_\alpha \; . \end{cases} \tag{3.3}$$

Since the admissible domains are Lipschitz and since the given functions f and g are smooth enough, the existence and the uniqueness of the solution to the state equation are ensured by the Lax-Milgram Theorem (cf. [28], [33]).

Thus, we can define the mapping $\alpha \to u_\alpha$ and denote its graph by

$$\mathcal{G} = \{(\alpha, u_\alpha) : \alpha \in \mathscr{A} \text{ and } u_\alpha \text{ is the solution to (3.3) on } \Omega(\alpha)\}.$$

The minimization problem is usually solved by endowing the set $\mathcal{G}$ with a topology for which $\mathcal{G}$ is compact and J is lower-semicontinuous. We first define the topology we will work with. First, we define the convergence of a sequence $\{\alpha_n\} \subset \mathscr{A}$ by

$$\alpha_n \to \alpha \quad \Leftrightarrow \quad \alpha_n \to \alpha \text{ uniformly on } [a, b]. \tag{3.4}$$

Then, the convergence of a sequence of domains $\Omega_n := \Omega(\alpha_n) \in \mathcal{O}$ to $\Omega := \Omega(\alpha) \in \mathcal{O}$ is simply defined by

$$\Omega_n \to \Omega \quad \Leftrightarrow \quad \alpha_n \to \alpha. \tag{3.5}$$

The domains belonging to $\mathcal{O}$ have some useful features. It follows from the assumptions on $\mathscr{A}$ that if $\alpha \in \mathscr{A}$ then $|\alpha(x_1) - \alpha(x_2)| \leq c_0 |x_1 - x_2|$ for all $x_1, x_2 \in [a, b]$. This uniform Lipschitz constraint prevents impractical oscillations of $\Gamma_2(\alpha)$. Furthermore, all $\Gamma_2(\alpha)$ remain in the open strip bounded by α_{min} and α_{max}. Hence the elements of $\mathcal{O}$ are uniformly Lipschitz open sets contained in D. So the family $\mathcal{O}$ is a special case of domains satisfying the uniform cone property. This class of domains possesses a very important uniform extension property, see [10].

Theorem 3.2.1. *Let $k \geq 0$, $p > 1$, and $\Omega \in \mathcal{O}$. Then, there exists an extension operator P_Ω,*

$$P_\Omega : W_p^k(\Omega) \to W_p^k(D),$$

such that

$$||P_\Omega u||_{W_p^k(D)} \leq M||u||_{W_p^k(\Omega)}$$

and M is independent of Ω.

Let $\Omega \in \mathcal{O}$. If $w \in H^1(\Omega)$, we denote a uniform extension of w from Ω to D by $\bar{w}$. The existence of such a uniform extension is ensured by Theorem (3.2.1). We now define the convergence of a sequence (u_n) of solutions of (3.1) on $\Omega_n := \Omega(\alpha_n)$ to the solution u of (3.1) on $\Omega(\alpha)$ as follows

$$u_n \to u \quad \Leftrightarrow \quad \bar{u}_n \rightharpoonup \bar{u} \text{ weakly in } H^1(D). \tag{3.6}$$

More precisely,

$$u_n \to u \quad \Leftrightarrow \quad P_{\Omega_n} u_n \rightharpoonup P_\Omega u \text{ weakly in } H^1(D). \tag{3.7}$$

The topology on $\mathcal{G}$ is now defined as the one induced by the convergence defined by

$$(\alpha_n, u_n) \to (\alpha, u) \Leftrightarrow \begin{cases} \alpha_n \to \alpha \\ u_n \to u \end{cases}. \tag{3.8}$$

3.3 Compactness of $\mathcal{G}$

Under the settings of Section 3.1 and based on the ideas given in [23], we begin with the following technical lemma.

Lemma 3.3.1. *There exists a constant $\beta > 0$ such that $|v|_{1,\Omega(\alpha)} \geq \beta ||v||_{1,\Omega(\alpha)}$ for all $\alpha \in \mathscr{A}$, for all $v \in V_\alpha$.*

Proof. We proceed by contradiction. For each $k \in \mathbb{N}$, suppose there exist $\alpha_k \in \mathscr{A}$, $v_k \in V_{\alpha_k}$ such that

$$|v_k|_{1,\Omega(\alpha_k)} < \frac{1}{k} ||v_k||_{1,\Omega(\alpha_k)}.$$

Without loss of generality we may assume that $||v_k||_{1,\Omega(\alpha_k)} = 1$ for all $k \in \mathbb{N}$ and, in addition, considering a subsequence of $\{\alpha_k\}$ if necessary, we may assume that there is $\alpha \in \mathscr{A}$ such that

$$\alpha_k \to \alpha \quad \text{in} \quad [a, b], \quad k \to \infty.$$

Let $\bar{v}_k$ be the uniform extension of v_k from $\Omega(\alpha_k)$ to D. By Theorem (3.2.1) we would have

$$||\bar{v}_k||_{1,D} \leq M||v_k||_{1,\Omega(\alpha_k)} = M$$

for all $k \in \mathbb{N}$. Then the sequence $\{\bar{v}_k\}$ is bounded in $H^1(D)$. Since $H^1(D)$ is a Hilbert space, considering a subsequence of $\bar{v}_k$ if necessary, we may assume that there is $\bar{v} \in H^1(D)$ such that

$$\bar{v}_k \rightharpoonup \bar{v} \quad \text{in} \quad H^1(D), \quad k \to \infty.$$

For each $m \in \mathbb{N}$ we define the set $B_m(\alpha) = \{x \in \Omega(\alpha) : \mathrm{dist}(x, \Gamma(\alpha)) > 1/m\}$. Since $B_m(\alpha)$ is an open subset contained in D, $\bar{v}_k \rightharpoonup \bar{v}$ in $H^1(B_m(\alpha))$ as $k \to \infty$. Furthermore, for each $m \in \mathbb{N}$, $\overline{B_m(\alpha)}$ is a compact set contained in $\overline{\Omega(\alpha)}$. So there exists $k_0 := k_0(m)$ such that $\overline{B_m(\alpha)} \subset \overline{\Omega(\alpha_k)}$ for all $k \geq k_0$. Combining this fact with the weak lower semicontinuity of the seminorm $|\cdot|_{1,B_m(\alpha)}$ we have

$$|\bar{v}|_{1,B_m(\alpha)} \leq \liminf_{k\to\infty} |v_k|_{1,B_m(\alpha)} \leq \liminf_{k\to\infty} |v_k|_{1,\Omega(\alpha_k)}$$

for each $m \in \mathbb{N}$. Taking the limit as $m \to \infty$, we obtain

$$|\bar{v}|_{1,\Omega(\alpha)} \leq \liminf_{k\to\infty} |v_k|_{1,\Omega(\alpha_k)}.$$

Since $|v_k|_{1,\Omega(\alpha_k)} < 1/k$, it follows that the seminorm $|\bar{v}|_{1,\Omega(\alpha)} = 0$. Thus, $\bar{v}$ is constant in $\Omega(\alpha)$ and, since $\bar{v}|_{\Gamma_1} = 0$ in the sense of traces, we have $\bar{v} = 0$ in $\Omega(\alpha)$. On the other hand,

$$||v_k||^2_{1,\Omega(\alpha_k)} = ||v_k||^2_{0,\Omega(\alpha_k)} + |v_k|^2_{1,\Omega(\alpha_k)} = 1, \text{ for all } \ k \in \mathbb{N}.$$

Then $||v_k||^2_{0,\Omega(\alpha_k)} \geq 1/2$ for sufficiently large k. However, by Rellich's Theorem (see [28]), $\bar{v}_k \to \bar{v}$ in $L^2(D)$. Then $\{||\bar{v}||_{0,\Omega(\alpha_k)} - ||\bar{v}||_{0,\Omega(\alpha_k)}\} \to 0$. Combining this fact and the monotone convergence theorem, we have

$$||\bar{v}||^2_{0,\Omega(\alpha)} = \lim_{k\to\infty} ||v_k||^2_{0,\Omega(\alpha_k)} \geq 1/2.$$

This contradicts the result that $\bar{v} = 0$ in $\Omega(\alpha)$. □

Recall from (2.10) that u_α and u_{α^*} satisfy the Dirichlet boundary condition g on the boundary Γ_1. The result below states that all solutions remain in a closed ball about g.

Corollary 3.3.2. *There exists a constant $C > 0$ such that $||u_\alpha - g||_{1,\Omega_\alpha} \leq C$ for all $\alpha \in \mathscr{A}$.*

Proof. Since $v = u_\alpha - g \in V_\alpha$ for all $\alpha \in \mathscr{A}$, we replace v in the weak formulation (3.1):

$$\int_{\Omega(\alpha)} \nabla u_\alpha \cdot \nabla(u_\alpha - g)dx = \int_{\Omega(\alpha)} f(u_\alpha - g)dx.$$

Then

$$\int_{\Omega(\alpha)} [\nabla(u_\alpha - g)]^2 dx = \int_{\Omega(\alpha)} f(u_\alpha - g)dx + \int_{\Omega(\alpha)} \nabla g \cdot \nabla(u_\alpha - g)dx.$$

This yields

$$|u_\alpha - g|^2_{1,\Omega(\alpha)} \leq ||f||_{0,\Omega(\alpha)}||u_\alpha - g||_{0,\Omega(\alpha)} + ||\nabla g||_{0,\Omega(\alpha)}||\nabla(u_\alpha - g)||_{0,\Omega(\alpha)}.$$

By Lemma (3.3.1), $|u_\alpha - g|^2_{1,\Omega(\alpha)} \geq \beta^2||u_\alpha - g||^2_{1,\Omega(\alpha)}$, whence

$$\beta^2||u_\alpha - g||^2_{1,\Omega(\alpha)} \leq ||f||_{0,\Omega(\alpha)}||u_\alpha - g||_{1,\Omega(\alpha)} + ||g||_{1,\Omega(\alpha)}||u_\alpha - g||_{1,\Omega(\alpha)},$$

and so

$$||u_\alpha - g||_{1,\Omega(\alpha)} \leq \frac{1}{\beta^2}(||f||_{0,\Omega(\alpha)} + ||g||_{1,\Omega(\alpha)}).$$

Therefore,

$$||u_\alpha - g||_{1,\Omega(\alpha)} \leq \frac{1}{\beta^2}(||f||_{0,D} + ||g||_{1,D}).$$

□

Lemma 3.3.3. *Let (α_n, u_n) be a sequence in $\mathcal{G}$, where $u_n := u_{\alpha_n} \in V_{\alpha_n}$ is the solution of (3.1) on $\Omega_n := \Omega(\alpha_n)$. Then a subsequence (α_k, u_k) and element $\alpha \in \mathscr{A}$ and element $u \in H^1(D)$ exist such that*

$$\begin{cases} \alpha_k \to \alpha \\ \bar{u}_k \rightharpoonup u \quad in \quad H^1(D). \end{cases} \tag{3.9}$$

Furthermore, $u_\alpha := u|_{\Omega(\alpha)}$ is the solution to (3.1) on $\Omega := \Omega(\alpha)$.

Proof. By the Arzela-Ascoli Theorem there exists a subsequence $\{\alpha_k\}$ that converges uniformly in $C^{0,1}[a,b]$. Denote this limit by α. Then $\alpha_{min} \leq \alpha(x) \leq \alpha_{max}$ for all $x \in [a,b]$. Since $|\alpha_k'(x)| \leq c_0$ a.e. in $[a,b]$ and $\alpha_k \to \alpha$ uniformly in $[a,b]$, it follows that $|\alpha'(x)| \leq c_0$ a.e. in $[a,b]$. Thus, $\alpha \in \mathscr{A}$.

Now let $\bar{u}_k$ be the uniform extension of u_k from Ω_k to D. Combining Theorem (3.2.1) and Corollary (5.1.2) with the fact that $g \in H^1_{loc}(\mathbb{R}^2)$ leads to

$$||\bar{u}_k||_{1,D} \leq M||u_k||_{1,\Omega_k} \leq M||u_k - g||_{1,\Omega_k} + M||g||_{1,\Omega_k} \leq MC + M||g||_{1,D}.$$

So, $\{||\bar{u}_k||_{1,D}\}$ is bounded. Passing again to a new subsequence if necessary, there is an element $u \in H^1(D)$ such that

$$\bar{u}_k \rightharpoonup u \ \text{ in } H^1(D).$$

Since Γ_1 is a polygonal, the trace operator $|_{\Gamma_1} : H^1(D) \to L^2(\Gamma_1)$ is compact, cf. [23], so it takes weakly convergent sequences into strongly convergent sequences. Then, $\lim_{k\to\infty} \bar{u}_k|_{\Gamma_1} = u|_{\Gamma_1}$ in $L^2(\Gamma_1)$. Since $\bar{u}_k = u_k$ in Ω_k, it follows that

$$u|_{\Gamma_1} = \lim_{k\to\infty} \bar{u}_k|_{\Gamma_1} = \lim_{k\to\infty} u_k|_{\Gamma_1} = g.$$

Hence $u \in V_\alpha$. We now show that $u|_{\Omega(\alpha)}$ solves (3.1) on $\Omega(\alpha)$. Let $v \in V_\alpha$ and let $\bar{v}$ be its uniform extension to D. There is a sequence $\{V_m\} \subset C^\infty(\overline{D})$ such that supp $\phi_m \cap \Gamma_1 = \emptyset$,

$$\phi_m \to \bar{v} \ \text{ in } H^1(D).$$

Since $V_m|_{\Omega_k} \in V_{\alpha_k}$ for all m, it may be substituted in place of v in (3.3). This yields,

$$\int_D \chi_k \nabla \bar{u}_k \cdot \nabla V_m \, dx = \int_D \chi_k f \phi_m \, dx,$$

where χ_k is the characteristic function of Ω_k. Passing to the limit with m, we obtain

$$\int_D \chi_k \nabla \bar{u}_k \cdot \nabla \bar{v}\ dx = \int_D \chi_k f \bar{v}\ dx.$$

Since $\alpha_k \to \alpha$ uniformly, passing to the limit with k, we have

$$\int_D \chi \nabla u \cdot \nabla \bar{v}\ dx = \int_D \chi f \bar{v}\ dx,$$

where χ is the characteristic function of the domain $\Omega(\alpha)$. Since $\bar{v}|_{\Omega(\alpha)} = v$, we conclude that

$$\int_{\Omega(\alpha)} \nabla u \cdot \nabla v\ dx = \int_{\Omega(\alpha)} f v\ dx$$

for all $v \in V_\alpha$. Therefore, $u|_{\Omega(\alpha)}$ solves (3.3) on $\Omega(\alpha)$. □

3.4 Lower Semicontinuity

We assume lower semicontinuity of our cost functionals. This occurs in many practical cases. For example, $J(\alpha, u) = ||u||^2_{0,\Omega_\alpha}$.

Theorem 3.4.1. *The cost functional $J(\alpha, u) = ||u||^2_{0,\Omega_\alpha}$ is lower semicontinuous. That is, if the $(\alpha_n, u_n) \to (\alpha, u) \in \mathcal{G}$ in the sense of (3.8) then $\liminf_{n\to\infty} J(\alpha_n, u_n) \geq J(\alpha, u)$.*

Proof. Let $\bar{u}_n$ and $\bar{u}$ be the uniform extensions of u_n and u respectively. Since $\bar{u}_n \rightharpoonup \bar{u}$ in $H^1(D)$, by Rellich's theorem [28] it follows that $\lim_{n\to\infty} ||\bar{u}_n - \bar{u}||_{0,L^2(D)} = 0$. Since $\Omega(\alpha_n) \subset D$, we have $\lim_{n\to\infty} ||\bar{u}_n - \bar{u}||_{0,L^2(\Omega(\alpha_n))} = 0$. Hence,

$$\lim_{n\to\infty} \{||\bar{u}_n||_{0,L^2(\Omega(\alpha_n))} - ||\bar{u}||_{0,L^2(\Omega(\alpha_n))}\} = 0. \tag{3.10}$$

On the other hand, $\alpha_n \to \alpha$ uniformly, then the Lebesgue Dominated Convergence Theorem leads us to the relation

$$\lim_{n\to\infty} ||\bar{u}||_{0,L^2(\Omega(\alpha_n))} = ||\bar{u}||_{0,L^2(\Omega(\alpha))}. \tag{3.11}$$

Combining (3.10) and (3.11) we have $\lim_{n\to\infty} ||\bar{u}_n||_{0,L^2(\Omega(\alpha_n))} = ||\bar{u}||_{0,L^2(\Omega(\alpha))}$. This yields, $\lim_{n\to\infty} J(\alpha_n, u_n) = J(\alpha, u)$. So J is continuous. In particular, J is lower semicontinuous. □

3.5 Optimal Solution

We now show that problem (3.2) has at least one solution.

Theorem 3.5.1. *Problem (3.2) has at least one solution.*

Proof. Let us consider a minimizing sequence for J; i.e., (α_n, u_n) such that

$$\lim_{n\to\infty} J(\alpha_n, u_n) = \inf\{J(\alpha, u) : (\alpha, u) \in \mathcal{G}\}.$$

We apply Lemma 3.3.3 to obtain a subsequence (α_k, u_k) and an element $\alpha \in \mathscr{A}$ such that $\alpha_k \to \alpha$ uniformly in $[a, b]$, $\bar{u}_k \rightharpoonup u$ in $H^1(D)$, and $u|_{\Omega(\alpha)}$ solves (3.3) in $\Omega(\alpha)$. Combining this and the continuity of J we obtain

$$J(\alpha, u|_{\Omega(\alpha)}) = \lim_{k\to\infty} J(\alpha_k, u_k) = \inf\{J(\alpha, u) : (\alpha, u) \in \mathcal{G}\}.$$

□

Chapter 4

Derivative of the Shape Functionals

4.1 Overview

The main goal in this chapter is to compute the derivative of the shape functional J to obtain a descent direction to approximate the shape of the domain that provides the minimum value of J. In the first section we derive the calculation of the derivative of the shape functional for our main domain optimization problem subject to PDE constraints using adjoint solutions. Based on the result of Section 4.2 we modify the main problem by sliding the graph of a fixed function α, that is, we are sliding a portion of the boundary. The last section of this chapter is one approach about how the derivative of the shape functional is obtained by sliding first and then deforming a portion of the boundary.

4.2 Perturbation of the Parametrized Boundary Over a Fixed Interval

Let $\mathcal{O}$ be a family of admissible domains described by a smooth function. As a concrete example, consider the following assumptions.

H2. *Let Y, α_{min}, α_{max} be constants satisfying $0 < \alpha_{min} < 1 < \alpha_{max}$, $\alpha_{max} < Y$, $[a,b]$ be a nonempty interval in $(0,1)$, and*

$$\mathscr{A} = \{\alpha \in C^{1,1}[a,b] : 0 < \alpha_{min} < \alpha(x) < \alpha_{max} \text{ for } x \in [a,b],\ \alpha(a) = \alpha(b) = 1\},$$

such that elements α of $\mathscr{A}$ define admissible domains $\Omega(\alpha) \subset D =]0,1[\times]0,Y[$ belonging to $\mathcal{O}$, see Figure 4.1.

Fix any $\Omega = \Omega(\alpha) \in \mathcal{O}$ and let Γ be its boundary. Set $\Gamma_2 = \{(x, \alpha(x)) : x \in [a,b]\}$ and let Γ_1 be the complement of Γ_2 in Γ. Let $\vec{s} = (s_1, s_2)$ be a C^1-regular vector field defined on D

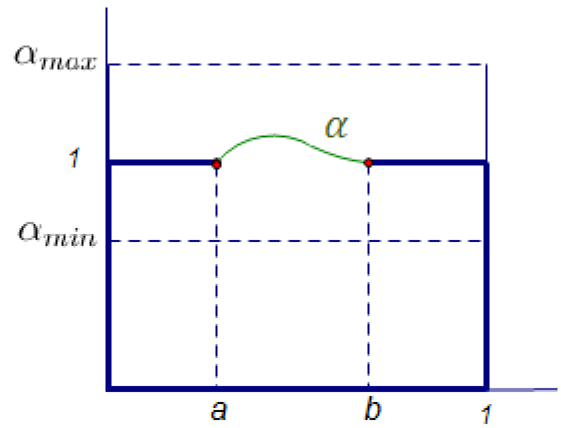


Figure 4.1: Admissible domain determined by $\alpha \in C^{1,1}[a,b]$

such that $(s_1, s_2) = (0,0)$ on Γ_1. Now define the domain mapping function

$$\Phi_t : \Omega \to \Omega_t \quad \text{by} \quad \Phi_t(\mathbf{x}) = \mathbf{x} + t\vec{s}(\mathbf{x}). \tag{4.1}$$

For sufficiently small ϵ, Φ_t is a diffeomorphism, Ω_t is a subset of D and $\Omega_t \in \mathcal{O}$ for all $-\epsilon \le t \le \epsilon$.[1]

Let $f \in L^2_{loc}(D)$, $g \in H^1_{loc}(\mathbb{R}^2)$, and $F : \mathbb{R} \to \mathbb{R}$ be a smooth function. We shall determine the derivative of the functional

$$J_t = \int_{\Omega_t} F(u_t)\ dx_t$$

with respect to t, where u_t is the solution of the following state equation:

$$-\Delta u_t = f \quad \text{in} \quad \Omega_t \tag{4.2}$$

with boundary conditions

$$u_t = g \quad \text{on} \quad \Gamma_1$$
$$\frac{\partial u_t}{\partial n} = 0 \quad \text{on} \quad \Gamma_2^t.$$

Here $\Gamma_2^t = \Phi_t(\Gamma_2)$. Note that Γ_1 is invariant under the mapping Φ_t. Our goal is to determine

$$\dot{J} = \frac{d}{dt} J_t \Big|_{t=0}.$$

To compute $\dot{J}$ we add some regularity assumptions to u_t. Let $Q = [-\epsilon, \epsilon] \times D$ and let Q_r be an open neighborhood of Q. We assume that there exists $u : Q_r \to \mathbb{R}$ smooth enough

[1]Note that we could weaken the assumption on $\vec{s}$ to belong to $H^1(D)$, satisfy $\vec{s} = \vec{0}$ on Γ_1 and admit a diffeomorphism in (4.1)

such that $u(t, \mathbf{x} + t\vec{s}(\mathbf{x})) = u_t(\mathbf{x} + t\vec{s}(\mathbf{x}))$. Note that $u(0, \cdot) = u_0$ is the solution to the state equation (4.28) on the fixed domain Ω. Furthermore, by chain rule one sees that

$$\dot{u}(0, \mathbf{x}) = \frac{d}{dt}\{u(t, \mathbf{x} + t\vec{s}(\mathbf{x}))\}\Big|_{t=0} = \frac{\partial u}{\partial t}(0, \mathbf{x}) + \nabla_{\mathbf{x}} u(0, \mathbf{x}) \cdot \vec{s}(\mathbf{x}).$$

In short notation $\dot{u} = u' + \nabla u \cdot \vec{s}$. We now proceed to compute $\dot{J}$. First of all, note that

$$J_t = \int_\Omega F(u(t, \Phi_t(\mathbf{x})))I_t \, d\mathbf{x}$$

where $I_t = |\det(D\Phi_t)|$. Without loss of generality we may assume that $I_t = \det(D\Phi_t)$. Then

$$\begin{aligned}\dot{J} &= \int_\Omega \frac{d}{dt}[F(u(t, \Phi_t))I_t]\Big|_{t=0} d\mathbf{x} \\ &= \int_\Omega [F_u(u(0, \mathbf{x}))\dot{u}(0, \mathbf{x}) + F(u(0, \mathbf{x})) \operatorname{div}(\vec{s})] d\mathbf{x}\end{aligned}$$

so

$$\dot{J} = \int_\Omega [F_u(u_0)\dot{u} + F(u_0) \operatorname{div}(\vec{s})] d\mathbf{x}. \tag{4.3}$$

This computation depends on $\dot{u}$ the material derivative of u_t. To eliminate $\dot{u}$ we introduce the adjoint equation below.

For each t, we define $H^1_{\Gamma_1}(\Omega_t) = \{v \in H^1(\Omega_t) : v = 0 \text{ on } \Gamma_1\}$. Here $v = 0$ on Γ_1 in the sense of traces. Then the weak formulation of the state equation on Ω_t is derived as follows

$$-\int_{\Omega_t} (\Delta u_t)\, v_t \, dx_t = \int_{\Omega_t} f v_t \, dx_t. \tag{4.4}$$

for all $v_t \in H^1_{\Gamma_1}(\Omega_t)$. The boundary of Ω_t is a $C^{1,1}$ curvilinear polygonal, so Green's identities work. Thus the left hand side of the weak formulation can be written as

$$\begin{aligned}-\int_{\Omega_t} (\Delta u_t) v_t \, dx_t &= \int_{\Omega_t} \nabla u_t \cdot \nabla v_t - \int_{\partial\Omega_t} \frac{\partial u_t}{\partial n} v_t \, ds \\ &= \int_{\Omega_t} \nabla u_t \cdot \nabla v_t - \int_{\Gamma_1} \frac{\partial u_t}{\partial n} v_t ds - \int_{\Gamma_2^t} \frac{\partial u_t}{\partial n} v_t \, ds.\end{aligned}$$

Since $v_t \in H^1_{\Gamma_1}(\Omega_t)$ and $\dfrac{\partial u_t}{\partial n} = 0$,

$$\begin{aligned}-\int_{\Omega_t} (\Delta u_t) v_t \, dx_t &= \int_{\Omega_t} \nabla u_t \cdot \nabla v_t \, dx_t \\ &= \int_\Omega (\nabla u_t \circ \Phi_t) \cdot (\nabla v_t \circ \Phi_t) I_t \, d\mathbf{x}.\end{aligned}$$

Hence

$$-\int_{\Omega_t}(\Delta u_t)v_t\,dx_t = \int_{\Omega}\langle (D\Phi_t)^{-1}(D\Phi_t)^{-T}\nabla(u_t\circ\Phi_t), \nabla(v_t\circ\Phi_t)\rangle_{\mathbb{R}^2}\, I_t\, d\mathbf{x}. \tag{4.5}$$

The right hand side of the weak formulation is:

$$\int_{\Omega_t} f v_t\, dx_t = \int_{\Omega} f(\Phi_t(\mathbf{x}))v_t(\Phi_t(\mathbf{x}))\, I_t\, d\mathbf{x}. \tag{4.6}$$

Combining (4.4), (4.5), and (4.6) we have

$$\int_{\Omega}\langle (D\Phi_t)^{-1}(D\Phi_t)^{-T}\nabla(u_t\circ\Phi_t), \nabla(v_t\circ\Phi_t)\rangle_{\mathbb{R}^2}\, I_t\, d\mathbf{x} = \int_{\Omega} f(\Phi_t(\mathbf{x}))v_t(\Phi_t(\mathbf{x}))\, I_t\, d\mathbf{x}. \tag{4.7}$$

Let $v \in H^1_{\Gamma_1}(\Omega)$ be arbitrary and define $v_t = v\circ\Phi_t^{-1}$. It is clear that $v_t \in H^1_{\Gamma_1}(\Omega_t)$ and $v_t = 0$ on Γ_1. Then $v_t(\Phi_t(\mathbf{x})) = v(\mathbf{x})$ on Ω and so

$$\dot{v} := \frac{d}{dt}v_t(\Phi_t(\mathbf{x}))\Big|_{t=0} = \frac{d}{dt}(v(\mathbf{x})) = 0. \tag{4.8}$$

We now take the material derivative of (4.7) when $v_t = v\circ\Phi_t^{-1}$ and $v \in H^1_{\Gamma_1}(\Omega)$. Using the formulas for the transport of differential forms, see Appendix A, and (4.8) we obtain

$$\int_{\Omega}\langle\nabla\dot{u}, \nabla v\rangle_{\mathbb{R}^2}\, d\mathbf{x} + \int_{\Omega}\langle k(\vec{s})\nabla u_0, \nabla v\rangle_{\mathbb{R}^2}\, d\mathbf{x} = \int_{\Omega}\mathrm{div}(f\vec{s})\, v\, d\mathbf{x},$$

where $k(\vec{s}) = \mathrm{div}(\vec{s})I - D\vec{s} - (D\vec{s})^T$. So for all $v \in H^1_{\Gamma_1}(\Omega)$ we have

$$\int_{\Omega}\langle\nabla\dot{u}, \nabla v\rangle_{\mathbb{R}^2}\, d\mathbf{x} = \int_{\Omega}\mathrm{div}(f\vec{s})v\, d\mathbf{x} - \int_{\Omega}\langle k(\vec{s})\nabla u_0, \nabla v\rangle_{\mathbb{R}^2}\, d\mathbf{x}. \tag{4.9}$$

Keeping in mind that $u_t(\Phi_t(\mathbf{x})) = g(\Phi_t(\mathbf{x}))$ on Γ_1, we have $u(t,\mathbf{x}) = g(\mathbf{x})$ for all $t \in [-\epsilon, \epsilon]$. This leads us to

$$\dot{u} = \nabla g\cdot\vec{s} = 0 \quad \text{on} \quad \Gamma_1. \tag{4.10}$$

Lemma 4.2.1. *Let $f \in L^2_{loc}(D)$ and $g \in H^1_{loc}(\mathbb{R}^2)$. Then the integral equation (4.9) above*

$$\int_{\Omega}\langle\nabla z, \nabla v\rangle_{\mathbb{R}^2}\, d\mathbf{x} = \int_{\Omega} div(f\vec{s})v\, d\mathbf{x} - \int_{\Omega}\langle k(\vec{s})\nabla u_0, \nabla v\rangle_{\mathbb{R}^2}\, d\mathbf{x} \quad \textit{for all } v \in H^1_{\Gamma_1}(\Omega)$$

has a unique solution in $H^1_{\Gamma_1}(\Omega)$.

Proof. Since $\Omega \in \mathcal{O}$, Γ is Lipschitz continuous. Furthermore, since Γ is a curvilinear polygon, the trace operator $\gamma : H^1(\Omega) \to L^2(\Gamma_1)$ is continuous (see [22]). Hence $\mathbb{V} := H^1_{\Gamma_1}(\Omega)$ together with $||\cdot||_1$ is a Hilbert space. Now let $a : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ and $G : \mathbb{V} \to \mathbb{R}$ be the mappings defined by

$$a(z, v) = \int_\Omega \langle \nabla z, \nabla v \rangle_{\mathbb{R}^2} \, d\mathbf{x}$$

and

$$G(v) = \int_\Omega \operatorname{div}(f\vec{s}) v \, d\mathbf{x} - \int_\Omega \langle k(\vec{s}) \nabla u_0, \nabla v \rangle_{\mathbb{R}^2} \, d\mathbf{x}.$$

It is obvious that $a(\cdot, \cdot)$ is bilinear and $|a(z, v)| \leq ||z||_1 ||v||_1$ for all $z, v \in \mathbb{V}$. By Friedrich's first inequality, there exists a positive constant c such that

$$a(v, v) \geq c \int_\Omega v^2 d\mathbf{x}.$$

It follows that $a(v, v) \geq \rho ||v||_1^2$, where $\rho = \{1/2, c/2\}$. Since $u_0 \in H^1(\Omega)$ and $\operatorname{div}(\vec{s}) \in C(D)$ we have

$$|G(v)| \leq c_1 ||v||_{L^2(\Omega)} + c_2 ||\nabla v||_{L^2(\Omega)} \leq C ||v||_1$$

for all $v \in \mathbb{V}$. On the other hand, it is clear that G is linear. Hence, by the Lax-Milgram Theorem, the above integral equation has a unique solution in $H^1_{\Gamma_1}(\Omega)$. □

Theorem 4.2.2. *Under the above settings, $\dot{u} \in H^1_{\Gamma_1}(\Omega)$.*

Proof. It follows from equations (4.9), (4.10), and Lemma 4.2.1. □

We finally introduce the adjoint equation

$$-\Delta w = F_u(u_0) \text{ in } \Omega \tag{4.11}$$

with boundary conditions

$$w = 0 \quad \text{on} \quad \Gamma_1$$
$$\frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \Gamma_2,$$

where u_0 is the solution to the state equation in Ω (the fixed domain). Let $\mathbf{w}$ be the solution to the adjoint equation. Since $\dot{u} \in H^1_{\Gamma_1}(\Omega)$, we multiply the adjoint equation by $\dot{u}$ and integrate, obtaining

$$\begin{aligned}\int_\Omega F_u(u_0) \dot{u} \, d\mathbf{x} &= -\int_\Omega (\Delta \mathbf{w}) \dot{u} \, d\mathbf{x} \\ &= \int_\Omega \nabla \dot{u} \cdot \nabla \mathbf{w} \, d\mathbf{x} - \int_{\partial\Omega} \frac{\partial \mathbf{w}}{\partial n} \dot{u} \, dS.\end{aligned}$$

By equation (4.10) and the boundary condition of $\mathbf{w}$ on Γ_2 it follows that

$$\int_\Omega F_u(u_0)\dot{u}\, d\mathbf{x} = \int_\Omega \langle \nabla \dot{u}, \nabla \mathbf{w} \rangle_{\mathbb{R}^2}\, d\mathbf{x}. \tag{4.12}$$

Since $\mathbf{w} \in H^1_{\Gamma_1}(\Omega)$, it may be substituted in place of v in (4.9). That is

$$\int_\Omega \langle \nabla \dot{u}, \nabla \mathbf{w} \rangle_{\mathbb{R}^2}\, d\mathbf{x} = \int_\Omega \operatorname{div}(f\vec{s})\mathbf{w}\, d\mathbf{x} - \int_\Omega \langle k(\vec{s})\nabla u_0, \nabla \mathbf{w} \rangle_{\mathbb{R}^2}\, d\mathbf{x}. \tag{4.13}$$

Combining equations (4.3), (4.12) and (4.13) we finally get

$$\dot{J} = \int_\Omega \operatorname{div}(f\vec{s})\, \mathbf{w}\, d\mathbf{x} - \int_\Omega (k(\vec{s})\nabla u_0) \cdot \nabla \mathbf{w}\, d\mathbf{x} + \int_\Omega F(u_0)\operatorname{div}(\vec{s})\, d\mathbf{x}, \tag{4.14}$$

where $\mathbf{w}$ is the solution to the adjoint equation and u_0 is the solution to the state equation on Ω (fixed domain). Note that the above expression is linear with respect to C^1-regular vector fields $\vec{s} = (s_1, s_2)$ defined on D with the property $\vec{s} = \vec{0}$ on Γ_1.

4.3 Sliding a Portion of the Boundary

In this section we compute the derivative of the shape functional

$$J(\Omega) = \int_\Omega F(u_\Omega)\; dx,$$

subject to a state equation of the type

$$-\Delta u_\Omega = f \quad \text{in} \quad \Omega, \tag{4.15}$$

with boundary conditions

$$u_\Omega = g \quad \text{on} \quad \Gamma_1(\Omega)$$
$$\frac{\partial u_\Omega}{\partial n} = 0 \quad \text{on} \quad \Gamma_2(\Omega).$$

Let Y and c be positive constants such that $c \in]0, 1[$. Let $\alpha : [0, c] \to \mathbb{R}^+$ be a C^1 function verifying $0 < \alpha < Y$ in $[0, c]$. For each a in $]0, 1 - c[$, denote the interval $[a, a + c]$ by I_a, define $\alpha_a : I_a \to \mathbb{R}$ by $\alpha_a(x) = \alpha(x - a)$ and define the sets R_a, G_a as follows.

$$R_a = (]0, 1[\backslash I_a) \times]0, 1[, \qquad G_a = \{(x, y) : x \in I_a \text{ and } 0 < y < \alpha_a(x)\}.$$

We consider the set of admissible domains as $\mathcal{O}_\alpha = \{\Omega_a : a \in]0, 1-c[\}$, where $\Omega_a = R_a \cup G_a$. Note that $\Omega_a \subset D =]0,1[\times]0,Y[$, see Figure 4.2.

For each $\Omega_a \in \mathcal{O}_\alpha$, let Γ_a be its boundary. Set $\Gamma_{a,2} = \{(x, \alpha_a(x)) : x \in I_a\}$ and let $\Gamma_{a,1}$ be the complement of $\Gamma_{a,2}$ in Γ_a. We would like to produce domains in $\mathcal{O}_\alpha$ whose $\Gamma_{\cdot,2}$ is essentially a translation of $\Gamma_{a,2}$. To do so we will define suitable perturbations of the identity map. Fix $\Omega = \Omega_a \in \mathcal{O}_\alpha$ and let $s_1 \in C^1([0,1])$ and take values in the interval $[0,1]$ such that $s_1(0) = s_1(1) = 0$, and $s_1(x) = 1$ on I_a. Now define the domain mapping function

$$\Phi_t : \Omega \to \Phi_t(\Omega) \quad \text{by} \quad \Phi_t(x,y) = (x,y) + t\vec{s}(x,y), \tag{4.16}$$

where $\vec{s}(x,y) = (s_1(x), 0)$. It is clear that $\Phi_t(\Omega) = \Omega_{a+t}$, which is essentially a translation of Ω_a in D. Also, note that Φ_t is a diffeomorphism for all $|t| < \epsilon < \min\{1-(a+c), 1/||s_1'||_\infty\}$.

As in Section 3.1 let $f \in L^2_{loc}(D)$, $g \in H^1_{loc}(\mathbb{R}^2)$, and $F : \mathbb{R} \to \mathbb{R}$ be a smooth function. We shall determine the derivative of the functional

$$J_t = \int_{\Omega_{a+t}} F(u_t)\, dx_t$$

with respect to t, where u_t is the solution of the following state equation:

$$-\Delta u_t = f \quad \text{in} \quad \Omega_{a+t} \tag{4.17}$$

with boundary conditions

$$\begin{aligned} u_t &= g \quad \text{on} \quad \Gamma_{a+t,1}, \\ \frac{\partial u_t}{\partial n} &= 0 \quad \text{on} \quad \Gamma_{a+t,2}. \end{aligned}$$

We can mimic the ideas as the previous section to obtain that $\dot{J}$. We assume the same regularity on u_t and for each t we define $H^1_{\Gamma_{a+t,1}} = \{v \in H^1(\Omega_{a+t}) : v = 0 \text{ on } \Gamma_{a+t,1}\}$. Here $v = 0$ in the sense of traces. Then we derive the weak formulation of the state equation on Ω_{a+t}

$$-\int_{\Omega_{a+t}} (\Delta u_t)\, v_t\, dx_t = \int_{\Omega_{a+t}} f v_t\, dx_t, \tag{4.18}$$

for all $v_t \in H^1_{\Gamma_{a+t,1}}$. The boundary of Ω_{a+t} is a C^1 curvilinear polygonal, so Green's identities hold. Then the left hand side of (4.18) can be written as

$$\begin{aligned} -\int_{\Omega_{a+t}} (\Delta u_t) v_t\, dx_t &= \int_{\Omega_{a+t}} \nabla u_t \cdot \nabla v_t - \int_{\partial\Omega_{a+t}} \frac{\partial u_t}{\partial n} v_t\, ds \\ &= \int_{\Omega_{a+t}} \nabla u_t \cdot \nabla v_t - \int_{\Gamma_{a+t,1}} \frac{\partial u_t}{\partial n} v_t ds - \int_{\Gamma_{a+t,2}} \frac{\partial u_t}{\partial n} v_t\, ds \end{aligned}$$

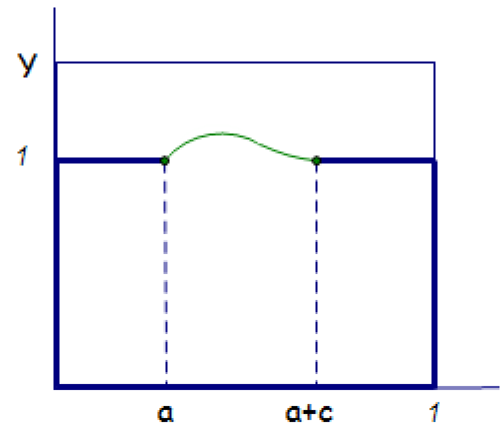


Figure 4.2: Admissible domain determined by $\alpha \in C^1[a, a+c]$

Since $v_t \in H^1_{\Gamma_{a+t,1}}$ and $\dfrac{\partial u_t}{\partial n} = 0$,

$$-\int_{\Omega_{a+t}} (\Delta u_t) v_t \, dx_t = \int_{\Omega_{a+t}} \nabla u_t \cdot \nabla v_t \, dx_t$$
$$= \int_{\Omega} (\nabla u_t \circ \Phi_t) \cdot (\nabla v_t \circ \Phi_t) \, dx_t.$$

Thus we obtain the following equality

$$\int_{\Omega} \langle (D\Phi_t)^{-1}(D\Phi_t)^{-T} \nabla(u_t \circ \Phi_t), \nabla(v_t \circ \Phi_t) \rangle_{\mathbb{R}^2} \, I_t \, d\mathbf{x} = \int_{\Omega} f(\Phi_t(\mathbf{x})) v_t(\Phi_t(\mathbf{x})) \, I_t \, d\mathbf{x}. \qquad (4.19)$$

This lead us to

$$\int_{\Omega} \langle \nabla \dot{u}, \nabla v \rangle_{\mathbb{R}^2} \, d\mathbf{x} + \int_{\Omega} \langle k(\vec{s}) \nabla u_0, \nabla v \rangle_{\mathbb{R}^2} \, d\mathbf{x} = \int_{\Omega} \operatorname{div}(f \vec{s}) \, v \, d\mathbf{x}$$

for all $v \in H^1_{\Gamma_{a,1}}$, where $k(\vec{s}) = \operatorname{div}(\vec{s}) I - D\vec{s} - (D\vec{s})^T$.

Rearranging terms, we find

$$\int_{\Omega} \langle \nabla \dot{u}, \nabla v \rangle_{\mathbb{R}^2} \, d\mathbf{x} = \int_{\Omega} \operatorname{div}(f \vec{s}) v \, d\mathbf{x} - \int_{\Omega} \langle k(\vec{s}) \nabla u_0, \nabla v \rangle_{\mathbb{R}^2} \, d\mathbf{x} \qquad (4.20)$$

for all $v \in H^1_{\Gamma_{a,1}}$.

Because $\Phi_t(\Gamma_{a,1}) = \Gamma_{a+t,1}$ and $u_t(\Phi_t(\mathbf{x})) = g(\Phi_t(\mathbf{x}))$ on $\Gamma_{a,1}$, we have $u(t, \mathbf{x}) = g(\mathbf{x})$ for all $t \in [-\epsilon, \epsilon]$. This yields

$$\dot{u} = \nabla g \cdot \vec{s} \quad \text{on} \quad \Gamma_{a,1}. \tag{4.21}$$

Lemma 4.3.1. *Let $f \in L^2_{loc}(D)$ and $g \in H^1_{loc}(\mathbb{R}^2)$. Then the integral equation*

$$\begin{cases} z - \nabla g \cdot \vec{s} = 0 \ \text{on}\ \Gamma_{a,1}. \\ \int_\Omega \langle \nabla z, \nabla v \rangle_{\mathbb{R}^2} \, d\mathbf{x} = \int_\Omega div(f\vec{s}) v \, d\mathbf{x} - \int_\Omega \langle k(\vec{s}) \nabla u_0, \nabla v \rangle_{\mathbb{R}^2} \, d\mathbf{x} \ \ \text{for all}\ v \in H^1_{\Gamma_{a,1}} \end{cases} \tag{4.22}$$

has a unique solution in $H^1(\Omega)$. Here $\Omega := \Omega_a$.

Proof. Follows from similar arguments to those given in Lemma 4.2.1. □

Theorem 4.3.2. *Under the above settings, $\dot{u} \in H^1(\Omega_{a,1})$ and $\dot{u} = \nabla g \cdot \vec{s}$ on $\Gamma_{a,1}$.*

Proof. It follows from equations (4.20), (4.21), and Lemma 4.3.1. □

We now define the adjoint equation as follows

$$-\Delta w = F_u(u_0) \ \text{ in } \ \Omega = \Omega_a \tag{4.23}$$

with boundary conditions

$$\begin{aligned} w &= 0 \quad \text{on} \ \ \Gamma_{a,1} \\ \frac{\partial w}{\partial n} &= 0 \quad \text{on} \ \ \Gamma_{a,2}, \end{aligned}$$

where u_0 is the solution to the state equation in $\Omega = \Omega_a$ (fixed domain). Let $\mathbf{w}$ be the solution to the adjoint equation. Since $\dot{u} \in H^1(\Omega_a)$, we multiply the adjoint equation by $\dot{u}$ and integrate, obtaining

$$\begin{aligned} \int_{\Omega_a} F_u(u_0)\dot{u} \, d\mathbf{x} &= -\int_{\Omega_a} (\Delta \mathbf{w})\dot{u} \, d\mathbf{x} \\ &= \int_{\Omega_a} \nabla \dot{u} \cdot \nabla \mathbf{w} \, d\mathbf{x} - \int_{\partial\Omega_{a,1}} \frac{\partial \mathbf{w}}{\partial n} \dot{u} \, dS. \end{aligned}$$

This relation and equation (4.20) lead us to

$$\int_{\Omega_a} F_u(u_0)\dot{u} \, d\mathbf{x} = \int_{\Omega_a} \nabla \dot{u} \cdot \nabla \mathbf{w} \, d\mathbf{x} - \int_{\Gamma_{a,1}} (\nabla g \cdot \vec{s}) \frac{\partial \mathbf{w}}{\partial n} \, dS \tag{4.24}$$

Since $\mathbf{w} \in H^1_{\Gamma_{a,1}}$, it may be substituted in place of v in (4.20). That is

$$\int_{\Omega_a} \langle \nabla \dot{u}, \nabla \mathbf{w} \rangle_{\mathbb{R}^2} \, d\mathbf{x} = \int_{\Omega_a} \operatorname{div}(f\vec{s})\mathbf{w} \, d\mathbf{x} - \int_{\Omega_a} \langle k(\vec{s})\nabla u_0, \nabla \mathbf{w} \rangle_{\mathbb{R}^2} \, d\mathbf{x}. \tag{4.25}$$

Combining equations (4.29), (4.24) and (4.25) we obtain

$$\dot{J} = \int_{\Omega_a} \operatorname{div}(f\vec{s}) \, \mathbf{w} \, d\mathbf{x} - \int_{\Omega_a} (k(\vec{s})\nabla u_0) \cdot \nabla \mathbf{w} \, d\mathbf{x} + \int_{\Omega_a} F(u_0) \operatorname{div}(\vec{s}) \, d\mathbf{x} - \int_{\Gamma_{a,1}} (\nabla g \cdot \vec{s}) \frac{\partial \mathbf{w}}{\partial n} \, dS, \tag{4.26}$$

where $\mathbf{w}$ is the solution to the adjoint equation (4.23) and u_0 is the solution to the state equation on $\Omega = \Omega_a$ (fixed domain).

We now consider a different type of transformation. For large n let $s_{1,n} \in C^1([0,1])$ and take values in $([0,1])$ with compact support such that $s_{1,n}(x) = 1$ on the interval $[a - 1/n, a + c + 1/n]$. The formula of $\dot{J}$ in the direction of $\vec{s}_n = (s_{1,n}, 0)$ is exactly the same obtained in (4.26), just replace $\vec{s}$ by $\vec{s}_n$. In particular, if $g = 0$ in $\overline{D}$, taking the limit as $n \to \infty$ we obtain

$$\dot{J} = \int_{\Omega_a} \operatorname{div}(f\vec{s}) \, \mathbf{w} \, d\mathbf{x},$$

where $\vec{s}$ is the constant vector field $\vec{s} = (1, 0)$. This lead us to the following corollary.

Corollary 4.3.3. *Under the above settings suppose $g = 0$ in $\overline{D}$, and f is constant in x. Then for each a in the interval $]0, 1 - c[$,*

$$\dot{J}(\Omega_a, \vec{s}) = 0$$

in the direction of $\vec{s} = (1, 0)$.

Proof. It follows from the identity

$$\dot{J} = \int_{\Omega_a} \operatorname{div}(f\vec{s}) \, \mathbf{w} \, d\mathbf{x} = \int_{\Omega_a} \frac{\partial f}{\partial x} \mathbf{w} \, d\mathbf{x} = \int_{\Omega_a} 0 \, \mathbf{w} \, d\mathbf{x} = 0.$$

□

4.4 Sliding and Deforming a Portion of the Boundary

In this section we combine the two cases discussed in the preceding sections. We first define the parameters needed and the admissible set of functions in order to compute the shape functional derivative of a functional of the type presented in (4.15).

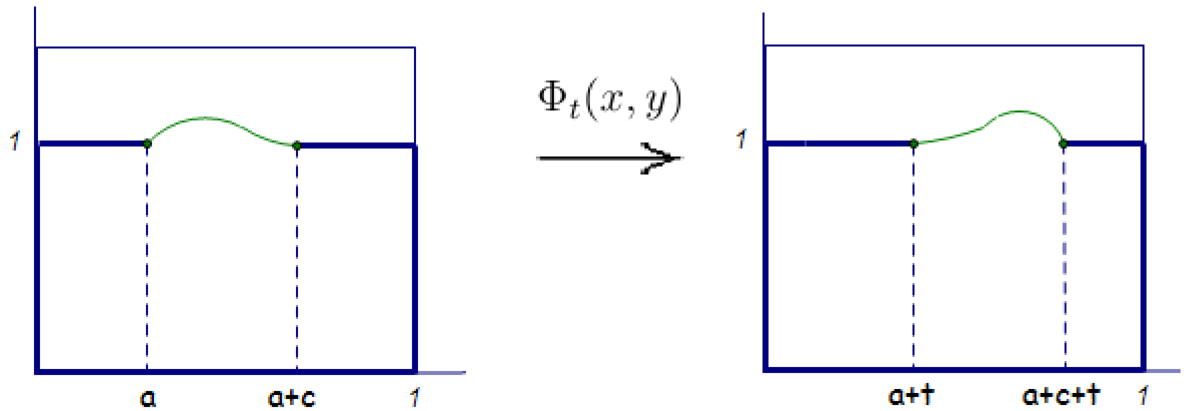


Figure 4.3: Admissible domain after applying the domain mapping function.

Let δ, α_{min}, α_{max}, c be positive constants with $0 < c < 1$. For each a in $]0, 1-c[$, we define the set of admissible functions as

$$U_a = \{\alpha \in C^{1,1}[a, a+c] : \alpha_{min} < \alpha < \alpha_{max} \text{ for } x \in [a, a+c]\}$$

and the set of admissible domains

$$\mathcal{O} = \{\Omega(\alpha, a) : \alpha \in U_a, 0 < a < 1-c\}$$

where $\Omega(\alpha, a) = R(\alpha, a) \cup G(\alpha, a)$, and the sets $R_{\alpha,a}$, $G_{\alpha,a}$ are defined by $R(\alpha, a) = (]0,1[\backslash[a, a+c]) \times]0,1[$, $G(\alpha, a) = \{(x, y) : x \in [a, a+c] \text{ and } 0 < y < \alpha(x)\}$ respectively. Under the assumption $\alpha_{max} < \delta$, $\Omega(\alpha, a) \subset D =]0,1[\times]0,\delta[$, see Figure 4.1.

Fix $\Omega = \Omega(\alpha, a) \in \mathcal{O}$ let Γ be its boundary. Set $\Gamma_2 = \{(x, \alpha(x)) : x \in [a, a+c]\}$ and let Γ_1 be the complement of Γ_2 in Γ. Let $s_1 \in C^1([0,1],[0,1])$ such that $s_1(0) = s_1(1) = 0$, and $s_1(x) = 1$ on $[a, a+c]$. Now define the domain mapping function

$$\Phi_t : \Omega \to \Omega_t \quad \text{by} \quad \Phi_t(x, y) = (x, y) + t(s_1(x), s_2(x, y)) \tag{4.27}$$

where $s_2 \in C^1(\overline{D})$ and $s_2 = 0$ on Γ_1. For sufficiently small ϵ, Φ_t is a diffeomorphism, Ω_t is a subset of D and $\Omega_t \in \mathcal{O}$ for all $-\epsilon \leq t \leq \epsilon$.[2]

As in the preceding sections, let $f \in L^2_{loc}(D)$, $g \in H^1_{loc}(\mathbb{R}^2)$, and $F : \mathbb{R} \to \mathbb{R}$ be a smooth function. We shall determine the derivative of the functional

$$J_t = \int_{\Omega_t} F(u_t)\, dx_t$$

[2]Note that we could weaken the assumption on $\vec{s} = (s_1, s_2)$ to have derivatives in $H^1(D)$, satisfy $s_2 = 0$ on Γ_1 and admit a diffeomorphism in (4.27)

with respect to t, where u_t is the solution of the following state equation:

$$-\Delta u_t = f \quad \text{in} \quad \Omega_t \tag{4.28}$$

with boundary conditions

$$u_t = g \quad \text{on} \quad \Gamma_1^t$$
$$\frac{\partial u_t}{\partial n} = 0 \quad \text{on} \quad \Gamma_2^t.$$

Under regular assumptions on u_t, we obtain

$$\dot{J} = \int_\Omega [F_u(u_0)\dot{u} + F(u_0)\,\text{div}(\vec{s})] d\mathbf{x}. \tag{4.29}$$

We get rid off $\dot{u}$ by using an adjoint equation. Mimicking the arguments given in the preceding sections, it turns out that

$$\dot{J} = \int_\Omega \text{div}(f\vec{s})\,\mathbf{w}\,d\mathbf{x} - \int_\Omega (k(\vec{s})\nabla u_0)\cdot\nabla\mathbf{w}\,d\mathbf{x} + \int_\Omega F(u_0)\text{div}(\vec{s})\,d\mathbf{x} - \int_{\Gamma_1} (\nabla g\cdot\vec{s})\frac{\partial \mathbf{w}}{\partial n}\,dS \tag{4.30}$$

where

$$k(\vec{s}) = \text{div}(\vec{s})I - D\vec{s} - (D\vec{s})^T,$$

u_0 is the unique solution to the state equation in Ω (fixed domain); i.e., u_0 solves

$$-\Delta u = f \quad \text{in} \quad \Omega \tag{4.31}$$

with boundary conditions

$$u = g \quad \text{on} \quad \Gamma_1,$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_2,$$

and $\mathbf{w}$ is the solution to the adjoint equation

$$-\Delta w = F_u(u_0) \quad \text{in} \quad \Omega, \tag{4.32}$$

with boundary conditions

$$w = 0 \quad \text{on} \quad \Gamma_1,$$
$$\frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \Gamma_2.$$

Chapter 5

Derivative of the Shape Functional Using Variations of the Boundary

The goal of this chapter is the calculation of the derivative of the shape functional J in terms of perturbations of the boundary only. We use the definitions and theorems described by [4] and [22] about curvilinear polygons. Note that this calculation corresponds to the case where α is defined over a fixed interval $[a, b]$ contained in $[0, 1]$.

5.1 Calculation of $\dot{J}$ Over a Fixed Interval

Let $\mathcal{O}$ be a family of admissible domains described by a smooth function, under the **H2** assumptions (see Section 4.2).

Fix any $\Omega = \Omega(\alpha) \in \mathcal{O}$ and let Γ be its boundary. Note that Γ is a curvilinear polygon of class $C^{1,1}$ with $k \geq 1$. Each of the $C^{1,1}$ curves which constitute the boundary of Ω, is denoted by $\overline{\Gamma}_j$ (the closure of the open arc Γ_j) and some $j \in I := \{1, \cdots, 6\}$. The curve $\overline{\Gamma}_{j+1}$ follows $\overline{\Gamma}_j$ according to the positive (counterclokwise) orientation. See Figure 5.1.

Let $V_j := \overline{\Gamma_j} \cap \overline{\Gamma_{j+1}}$ be the jth vertex and let w_j be the measure of the angle at V_j towards the interior of Ω. Since Ω is an admissible domain the internal angles w_j between arcs must satisfy the inequalities $2\theta \leq w_j \leq 2\pi - 2\theta$, where θ is the angle associated to the cone property on $\mathcal{O}$. In our particular example, we can be more specific

$$w_1 = w_2 = w_5 = w_6 = \pi/2 \quad \text{and} \quad \pi/2 < w_3, w_4 < 3\pi/2 \tag{5.1}$$

Let $f \in L^2_{loc}(D)$, $g \in H^1_{loc}(\mathbb{R}^2)$, and $F : \mathbb{R} \to \mathbb{R}$ be a smooth function. Our state problem on Ω can be written as:

$$-\Delta u = f \quad \text{in} \quad \Omega \tag{5.2}$$

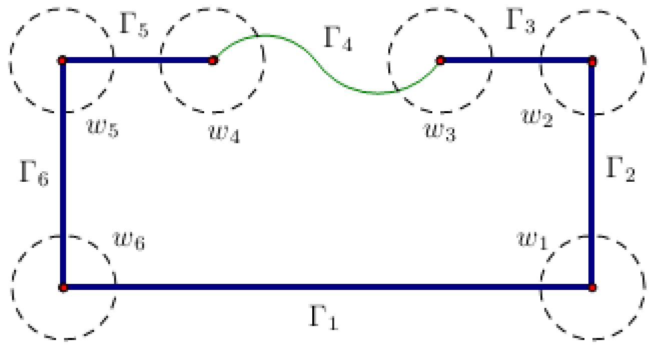


Figure 5.1: Admissible domain determined by $\alpha \in C^{1,1}[a,b]$

with boundary conditions

$$\begin{aligned} u = g &\quad \text{on} \quad \Gamma \setminus \Gamma_4, \\ \frac{\partial u}{\partial n} = 0 &\quad \text{on} \quad \Gamma_4. \end{aligned}$$

Following Grisvard [22] and Banasiak-Roach [4], we introduce the following notation:

$$\phi_j := \begin{cases} \pi/2, & \text{if } j \neq 4 \\ 0, & \text{if } j = 4 \end{cases},$$

$$\lambda_{j,m} := \frac{\phi_j - \phi_{j+1} + m\pi}{w_j}, \quad \text{for } j \in I, \ m \in \mathbb{Z},$$

and (for the same (j,m) pairs)

$$S_{\lambda_{j,m}}(x) = r_j^{\lambda_{j,m}} \cos(\lambda_{jm}\theta + \phi_{j+1}).$$

We identify $x = r_j \exp(i\theta)$. Here (r_j, θ) are polar coordinates with origin at V_j, i.e., r_j is the distance from the jth vertex and θ is the angle measured from the $(j+1)$th arc. Then there exist numbers $c_{j,m}$ such that

$$u - \sum_{\substack{1 \leq j \leq 6 \\ 0 < \lambda_{j,m} < 2/q \\ \lambda_{j,m} \neq 1}} c_{j,m} S_{j,m} \in W_p^2(\Omega), \tag{5.3}$$

where u is the solution to the state equation and $1/p + 1/q = 1$.

We are particularly interested in the case $p = 2$. If $p = 2$, then $q = 2$. Under our setting, $\lambda_{jm} = \dfrac{m\pi}{w_j} = 2m$ for $j \neq 3, 4$; $\lambda_{3,m} = \dfrac{\pi/2 + m\pi}{w_3}$ and $\lambda_{4,m} = \dfrac{-\pi/2 + m\pi}{w_4}$. Then (5.3) reduces to the condition

$$u - c_{3,0}S_{3,0} - c_{4,1}S_{4,1} \in W_p^2(\Omega).$$

Note that $\lambda_{3,0} = \pi/(2w_3)$ and $\lambda_{4,1} = \pi/(2w_4)$. Since $0 < w_3, w_4 < 2\pi$, in particular we have $\lambda_{3,0} - 1/4 > 0$ and $\lambda_{4,1} - 1/4 > 0$. In this case we say that u has singularities of the type $r^{1/4+\delta}$.

Remark 5.1.1. *For an equation of Dirichlet type, we set $\phi_j = 0$ for all j. This leads us to $\lambda_{jm} = m\pi/w_j$ for all j. In this case, $\lambda_{3,1} = \pi/w_3$ and $\lambda_{4,1} = \pi/w_4$. Since $\pi/2 < w_3, w_4 < 3\pi/2$, we have $\lambda_{3,1} - 1/2 > 0$ and $\lambda_{4,1} - 1/2 > 0$. In this case we say that the solution to the Dirichlet equation has singularities of the type $r^{1/2+\delta}$.*

Now, let J be the shape functional defined by

$$J = \int_\Omega F(u_\Omega)dx. \tag{5.4}$$

Let $\vec{s} = (s_1, s_2)$ defined on D with the property $\vec{s} = \vec{0}$ on $\Gamma \setminus \Gamma_4$. We know that

$$\dot{J}(\vec{s}) = \int_\Omega \mathrm{div}(f\vec{s})\,\mathbf{w}\,d\mathbf{x} - \int_\Omega (k(\vec{s})\nabla u)\cdot\nabla\mathbf{w}\,d\mathbf{x} + \int_\Omega F(u)\mathrm{div}(\vec{s})\,d\mathbf{x}, \tag{5.5}$$

where $k(\vec{s}) = \mathrm{div}(\vec{s})I - D\vec{s} - (D\vec{s})^T$ and $\mathbf{w}$ satisfies the adjoint equation

$$-\Delta\mathbf{w} = F_u(u) \quad \text{in} \quad \Omega, \tag{5.6}$$

with boundary conditions

$$\begin{aligned} \mathbf{w} &= 0 \quad \text{on} \quad \Gamma \setminus \Gamma_4 \\ \frac{\partial\mathbf{w}}{\partial n} &= 0 \quad \text{on} \quad \Gamma_4, \end{aligned}$$

where u is the solution to the state equation in Ω (fixed domain). Since $\mathbf{w}$ verifies the same type of equation as u, under suitable assumptions on $F_u(u)$ we can say that the adjoint solution $\mathbf{w}$ also has singularities of the form $r^{1/4+\delta}$.

Theorem 5.1.2. *Let $\vec{d} = (d_1, d_2)$ be a field defined on Γ, such that $\vec{d}$ is $C^{1,1}$ on Γ_4, $\vec{d} = (0, 0)$ on $\Gamma \setminus \Gamma_4$ and $\vec{d}$ continuous on Γ. Let $\vec{v} = (v_1, v_2)$ be the vector field that satisfies*

$$\Delta v_i = 0 \qquad \text{in } \Omega, \quad i = 1, 2, \tag{5.7}$$

$$v_i = d_i \qquad \text{on } \Gamma, \quad i = 1, 2, \tag{5.8}$$

Then, the derivative of J corresponding to the boundary variations $\vec{d}$ can be written as

$$\dot{J} = \int_\Omega div(f\vec{v})\,\mathbf{w}\,d\mathbf{x} - \int_\Omega (k(\vec{v})\nabla u)\cdot\nabla\mathbf{w}\,d\mathbf{x} + \int_\Omega F(u)\,div(\vec{v})\,d\mathbf{x} \tag{5.9}$$

Proof. If $f, g, F(u)$ are regular enough, the singularities depend on the geometry of Ω. These are precisely the vertices V_3, V_4. Under the preceding discussion, u and $\mathbf{w}$ have singularities at the vertices V_3, V_4 of the type $r^{1/4+\delta}$ and v_1, v_2 have singularities at V_3, V_4 of the type $r^{1/2+\delta}$. For each $i \in \{3,4\}$ take $\epsilon_{1,i}$ in the open interval $(0, \pi/(2w_i) - 1/4)$ verifying

$$\frac{\pi}{2w_3} - \epsilon_{1,3} = \frac{\pi}{2w_4} - \epsilon_{1,4}$$

Set $s_1 = \dfrac{\pi}{2w_4} + 1 - \epsilon_{1,4}$. For each $i \in \{3,4\}$ take $\epsilon_{2,i}$ in the open interval $(0, \pi/w_i - 1/2)$ verifying

$$\frac{\pi}{w_3} - \epsilon_{2,3} = \frac{\pi}{w_4} - \epsilon_{2,4}$$

Set $s_2 = \dfrac{\pi}{w_4} + 1 - \epsilon_{2,4}$.

By Theorem 1.4.5.3 of [22], we have $u, \mathbf{w} \in W_2^{s_1}; v_i \in W_2^{s_2}; \nabla u, \nabla \mathbf{w} \in W_2^{s_1}; \nabla \cdot \vec{v} \in W_2^{s_2-1}$. As in Zochowski [38] we may find sequences $s_{n,1}, s_{n,2}$ of functions in $W_2^4(\Omega)$ such that

$$||s_{n,i} - v_i||_{W_2^{s_2}(\Omega)} \to 0, \quad \text{and} \quad ||\nabla s_{n,i} - \nabla v_i||_{W_2^{s_2-1}(\Omega)} \to 0.$$

Since $(d_1, d_2) = (0,0)$ on $\Gamma \setminus \Gamma_4$ we can choose $s_{n,i}$ so that the $\text{dist}(\text{supp}\, s_{n,i}, \Gamma \setminus \Gamma_4) > 0$. Now we can mimic the argument given by Zochowski (see page 396 of [38]).

□

Chapter 6

Conclusion

6.1 Summary of Main Results

The usual framework to approximate the domain that solves a shape optimization problem is to assume that solutions exist. In this work, we develop an existence theory for the optimization problem by defining a set of admissible domains that guarantee the existence of weak solutions to the problem. Of particular interest, we provide a new theoretical framework that can be used to show the existence of solutions for a parametrized domain optimization problem subject to PDE constraints with mixed boundary conditions.

The calculation of the derivative of the shape functional in the infinite dimensional case is a difficult task for general shape optimization problems, and the analysis depends on the precise problem setting. In most work in this field, the estimated solutions come from differentiating the discretized version of the problem. By building upon results for optimization problems that involve only Dirichlet conditions or Neumann conditions, we calculate the derivative of the shape functional for our problem of interest for the infinite dimensional case that involves mixed boundary conditions. We also explain the assumptions required for its calculation. Additionally, we calculate the derivative of the shape functional for an interesting collection of similar problems to the main problem of interest (for example when the parameter dependent portion of the boundary slides with parametric changes).

Given a parametrized domain optimization problem with mixed boundary constraints as the main problem presented in this work, we have the following steps to approximate the domain solution.

(1) Set $k = 0$. Start with an initial parametrization for Γ_2 that describes the initial domain Ω_k.

(2) Choose an initial regular vector field $\vec{s} = (s_1.s_2)$.

(3) Compute $\dot{J}(\Omega_0)$ using (4.14).

(4) If $\dot{J}(\Omega_0) > 0$, then define a new direction $\vec{s} = (s_1, s_2)$, go to step (3).

(5) If $\dot{J}(\Omega_0) < 0$, set $k = k + 1$. We have a new shape of the domain Ω_k. Test for convergence.

The results obtained in this research provide a method to calculate the derivative of the shape functional. This is an important step to create efficient algorithms to approximate the domain solution when mixed boundary conditions are presented.

6.2 Future Research Directions

The recent special workshop organized by Institute for Mathematics and its Applications, "Frontiers in PDE-constrained Optimization," featured a number of talks related to shape optimization and topology optimization problems subject to PDE-constraints. The workshop presented a number of approaches to solve various shape problems. It is clear that a number of theoretical aspects are still open due to the nature of the wide range of problems from different disciplines. Not much is done about the theoretical approach of parametrized domain optimization problems, so extending our main problem from the two dimensional case to the three dimensional case is one task to be attained, the set of admissible domains needs to be chosen so that we can guarantee the solution to the state problem. A natural purpose is to extend these results to problems that comes from other disciplines, analyzing the infinite dimensional case as a first approach. Also, using the infinite dimensional approach can serve as a guide to develop means in other areas where shape optimization is useful such as the design of electrical equipment, or shape reconstruction techniques.

Appendix A

Some Definitions and Theorems

Most of the definitions and theorems presented here and used in Chapter 3 can be found in [22],[23], [35], [39].

Theorem A.0.1. *(Trace Theorem) There exists a unique linear compact mapping T of $H^1(\Omega)$ into $L^2(\Gamma)$, where Ω is in the set of all domains in $\mathbb{R}^2$ with Lipschitz boundary, such that $Tv = v_{|\Gamma}$ for any v in $C^\infty(\overline{\Omega})$.*

Let $h > 0, \theta \in (0, \pi/2)$, and $\xi \in R^2, ||\xi|| = 1$, be given. The set $C(\xi, \theta, h) = \{x \in R^2 | (x, \xi) > ||x|| \cos(\theta), ||x|| < h\}$ is called the cone of angle θ, height h, and axis ξ.

Definition A.0.2. *(Cone property.) A domain $\Omega \subset \mathbb{R}^2$ is said to satisfy the cone property if and only if there exist numbers $\theta \in (0, \pi/2),\ h > 0,\ r \in (0, h/2)$ with the property that for all $x \in \Gamma$, there exist $C_x := C(\xi_x, \theta, h)$ such that for all $y \in B_r(x) \cap \Omega$ the set $y + C_x \subset \Omega$.*

Theorem A.0.3. *(Rellich's Theorem) The embedding of $H^k(\Omega)$ into $H^{k-1}(\Omega)$, with $k \in \mathbb{N}$, Ω in the set of all domains in $\mathbb{R}^2$ with Lipschitz boundary, is compact with the following convention of notation: $H^0(\Omega) := L^2(\Omega)$*

Corollary A.0.4. *If $v_n \rightharpoonup v$ (weakly convergence) in $H^k(\Omega)$ it follows that $v_n \to v$ in $H^{k-1}(\Omega)$, $k \in \mathbb{N}$, Ω in the set of domains with Lipschitz boundary.*

The following are well-known formulas in shape calculus that have been used in Chapter 4 to calculate the derivative of shape functional J, see [23], [35], or [39].

- If $z_t = D\Phi_t$, then $\frac{d}{dt} z_t \Big|_{t=0} = D\vec{s}$.
- If $I_t = \det(D\Phi_t)$, then $\frac{d}{dt} I_t \Big|_{t=0} = \text{div}\vec{s}$.

- If $z_t = \nabla(u \circ \Phi_t)$, then $\frac{d}{dt} z_t\Big|_{t=0} = -D\vec{s} \cdot \nabla u_0 + \nabla \dot{u}$.
- If $z_t = f \circ \Phi_t$, then $\frac{d}{dt} z_t\Big|_{t=0} = \nabla f \cdot \vec{s}$.
- If $z_t = n_t \circ \Phi_t$, then $\frac{d}{dt} z_t\Big|_{t=0} = n(n^T \cdot D\vec{s} \cdot n) - D\vec{s} \cdot n$.

Appendix B

On Curvilinear Polygons

In Chapter 5 we used the definition and theorem below to calculate the derivative of the shape functional J in terms of perturbations of the boundary (cf. [4] and [22]).

Definition B.0.1. *Let Ω be a bounded open subset of $\mathbb{R}^2$. We say that the boundary Γ is a curvilinear polygon of class C^m, where m is an integer, $m \geq 1$ (respectively $C^{k,\alpha}$, k is an integer, $k \geq 1,\ 0 < \alpha \leq 1$) if for every $x \in \Gamma$ there exists a neighborhood B of x in $\mathbb{R}^2$ and a mapping ψ from B in $\mathbb{R}^2$ such that*

- *ψ is injective,*
- *ψ and ψ^{-1} (defined on $\psi(B)$) belongs to the class C^m (respectively $C^{k,\alpha}$),*
- *$\Omega \cap B$ is either $\{y \in \Omega : \psi_2(y) < 0\}, \{y \in \Omega : \psi_1(y) < 0$ and $\psi_2(y) < 0\}$ or $\{y \in \Omega : \psi_1(y) < 0$ or $\psi_2(y) < 0\}$ where $\psi_j(y)$ denotes the jth component of ψ.*

The following theorem is used in the proof of Theorem 5.1.2.

Theorem B.0.2. *(1.4.5.3 [22]) Let Ω be a bounded open subset of $\mathbb{R}^2$, whose boundary Γ is a curvilinear polygon. Assume that $0 \in \Gamma$. Let B be a neighborhood of 0 such that*

$$B \cap \overline{\Omega} \subseteq \{(r\cos\theta, r\sin\theta) : r \geq 0, a \leq \theta \leq b\},$$

with $b - a < 2\pi$.
Finally let u be a function which is smooth in $\overline{\Omega} \setminus \{0\}$ and which coincides with $r^\alpha \varphi(\theta)$ in $B \cap \Omega$, where $\varphi \in C^\infty([a, b])$.

Then

$$u \in W_p^s(\Omega) \text{ for } Re\,\alpha > s - \frac{2}{p}$$

while

$$u \notin W_p^s(\Omega) \text{ for } Re\,\alpha \leq s - \frac{2}{p}, \text{ when } Re\,\alpha \text{ is not an integer.}$$

Bibliography

[1] W. Alt. *The Lagrange-Newton Method for Infinite-Dimensional Optimization Problems.* Functional Analysis and Optimization, vol. 11 (1990), pp. 201–224.

[2] E. Allgower, K. Böhmer, F. Potra, and W.C. Rheinboldt. *A Mesh-Independence Principle for Operator Equations and Their Discretizations.* SIAM Numerical Analysis, vol. 23, (1986), pp. 160–169.

[3] O. Axelson and V. A. Baker. *Finite Element Solution of Boundary Value Problems. Theory and Computation.* SIAM Classics in Applied Mathematics, Philadelphia (2001).

[4] J. Banasiak and G.F. Roach. *On Mixed Boundary Value Problems of Dirichlet Oblique-Derivative Type in Plane Domains with Piecewise Differentiable Boundary.* Journal of Differential Equations, vol. 79 (1989), pp. 111–131.

[5] A. Borzi and V. Schulz. *Computational Optimization of Systems Governed by Partial Differential Equations.* SIAM, Philadelphia (2012).

[6] J. Cagnol, M. Polis, and J.-P. Zolesio. *Shape Optimization and Optimal Design.* Marcel Dekker, Inc., New York (2001).

[7] E. Casas and J. Sokolowski. *Approximation of Boundary Control Problems on Curved Domains.* SIAM Journal on Control and Optimization, vol. 48 (2010), pp. 3746–3780.

[8] A. Chakib, A. Ellabib, A. Nachaoui, and M. Nachaoui. *A Shape Optimization Formulation of Weld Pool Determination.* Elsevier, Applied Mathematics Letters 25 (2012), pp. 374–379.

[9] J. Chaskalovic. *Mathematical and Numerical Methods for Partial Differential Equations.* Springer, (2014)

[10] D. Chenais. *On the Existence of a Solution in a Domain Identification Problem.* Journal of Mathematical Analysis and Applications, vol. 52 (1975), pp. 189–219.

[11] M.C. Delfour and G. Sabidussi, eds. *Shape Optimization and Free Boundaries.* Series C: Mathematical and Physical Sciences, vol. 380, Kluwer Academic Publisher, Dordrecht (1992).

[12] M.C. Delfour and J.P. Zolesio. *Structure of Shape Derivatives for Nonsmooth Domains*. Journal of Functional Analysis, vol. 104 (1992), pp. 1–33.

[13] M. C. Delfour and J.P. Zolesio. *Shapes and Geometries : Metrics, Analysis, Differential Calculus, and Optimization*. SIAM Advances in Design and Control, 2nd ed., Philadelphia (2011).

[14] G. Dogan, P. Morin, R.H. Nochetto, and M. Verani. *Finite Element Methods for Shape Optimization and Applications*. Preprint (2005).

[15] K. Eppler. *Optimal Shape Design for Elliptic Equations via BIE-Methods*. J. of Applied Mathematics and Computer Science, vol. 10, no. 3 (2000), pp. 487–516.

[16] K. Eppler. *Frechet-Differentiability and Sufficient Optimality Conditions for Shape Functionals*. Optimal Control of Partial Differential Equations, vol. 133 (1999), pp. 133–143.

[17] K. Eppler and H. Harbrecht. *Coupling of FEM and BEM in Shape Optimization*. Numerische Mathematik, vol. 104, no. 1 (2006), pp. 47–68.

[18] K. Eppler and H. Harbrecht. *Shape Optimization for Free Boundary Problems*. in Proceedings of the International Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering, K. Gürlebeck and C. Könke eds. (2009).

[19] K. Eppler, H. Harbrecht, and R. Schneider. *On Convergence in Elliptic Shape Optimization*. SIAM J. Control Optim., vol. 46 (2007), pp.61–83.

[20] K. Eppler, H. Harbrecht, and M. Mommer. *A New Fictitious Domain Method in Shape Optimization*. Springer Comput Optim Appl., vol. 40 (2008), pp.281–298.

[21] N. Fujii. *Second-Order Necessary Conditions in a Domain Optimization Problem*. Journal of Optimization Theory and Applications, vol. 65 (1990).

[22] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman Advanced Publishing Program, London, Great Britain (1985).

[23] J. Haslinger and R.A.E. Mäkinen. *Introduction to Shape Optimization. Theory, Aprroximation and Computation*. SIAM Advances in Design and Control, vol. 7, Philadelphia (2003).

[24] S. Hazra. *Multigrid One-Shot Method for PDE-Constrained Optimization Problems*. Applied Mathematics, vol. 3 (2012), pp. 1565–1571.

[25] Y. He and B. Z. Guo *The Existence of Optimal Solution for a Shape Optimization Problem on Starlike Domain*. Journal of Optimization Theory and Applications, vol. 152 (2012), pp. 21–30.

[26] K. Ito, K. Kunisch, and G. Peichl. *Variational Approach to Shape Derivatives for a Class of Bernoulli Problems*. Elsevier, Journal of Mathematics Analysis and Applications, vol. 314, no. 1 (2006), pp.126–149.

[27] D. Luenberger. *Optimization by Vector Space Methods*. John Wiley & Sons, Inc. (1969).

[28] J. Necas. *Direct Methods in the Theory of Elliptic Equations*. Springer-Verlag (2011).

[29] P. Neittaanmaki, J. Sprekels, and D. Tiba. *Optimization of Elliptic Systems. Theory and Applications*. Springer Monographs in Mathematics, New York (2006).

[30] A. Novruzi and J. Roche. *Newton's Method in Shape Optimization: A Three-Dimensional Case*. BIT Numerical Mathematics, vol. 40, (2000), pp. 102–120.

[31] K. Ohtsuka. *Shape optimization for partial differential equations/system with mixed boundary conditions*. 1791-20 (2012), pp. 172–181.

[32] P. Philip and D. Tiba. *Shape Optimization via Control of a Shape Function on a Fixed Domain: Theory and Numerical Results*. Numerical Methods for Differential Equations Optimization and Technological Problems, Springer Netherlands (2013), pp. 305–320.

[33] O. Pironneau. *Optimal Shape Design for Elliptic Systems*. Springe Series in Computational Physics, New York Inc. (1984).

[34] M. Reed and B. Simon. *I: Functional Analysis*. Academic Press (1981).

[35] J. Sokolowski and J.P. Zolesio. *Introduction to Shape Optimization*. Springer, Berlin (1992).

[36] F. Velazquez-Villegas and S. Santillan-Gutierrez. *Shape Optimization of a Hanging Body Based on Evolutive Rules and Parametric Modeling: The Shape of a Fruit*. Ingenieria Investigacion y Tecnologia, vol. XIV (2013), pp. 23–35.

[37] S. Walker. *The Shapes of Things: A Practical Guide to Differential Geometry and the Shape Derivative*. SIAM Advances in Design and Control, Philadelphia (2015).

[38] A. Zochowski. *Domain Sensitivity for Singular Elliptic Problems: Harmonic Transformation Approach*. Journal of Optimization Theory and Applications, vol. 73 (1992).

[39] J.P. Zolesio. *The Material Derivative (or Speed) Method for Shape Optimization*. Optimization of Distributed Parameter Structures, vol. II, Iowa City, Iowa (1980), pp. 1089–1151.