

On Refinements of Van der Waerden's Theorem

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Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematics

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Keywords: Ramsey Theory, Van der Waerden's Theorem, Partition
Regularity, Topological Recurrence, Canonical Ramsey Theory,
Compactness Principle, Arithmetic Progression

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ABSTRACT

The purpose of this thesis is to examine the classical Theorem of Van der Waerden about the partition regularity of Arithmetic Progressions, and the different directions in which it can be generalized. We will provide a series of conjectures that indicate how to generalize the theory of the Large sets of Brown, Graham, and Landman in [1], to other variants of Van der Waerden's Theorem. In chapter 2, we will define Large sets rigorously, and provide a summary of the known results and conjectures about them. We will also show how to generalize the notion of a Large set in order to strengthen the Multidimensional Van der Waerden Theorem and the Canonical Van der Waerden Theorem in chapters 3 and 7 respectively. In chapter 4, we introduce yet another direction in which to yield a multidimensional generalization of the Van der Waerden Theorem, and the corresponding notion of a Large set. In chapter 5, we will demonstrate how to study Large sets (in all settings) from the perspective of finitistic combinatorics, and how this new perspective allows us to construct a broad class of Large sets. In chapter 6, we will discuss a closure operation motivated by the desire to impose algebraic structure on an arbitrary Large set that may a priori only have combinatorial structure. Lastly, we will introduce some of the known connections between Ramsey Theory and Large sets to Topological Dynamics and Recurrence in chapter 8, and propose a few more conjectures to further our understanding of this connection.

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GENERAL AUDIENCE ABSTRACT

Ramsey Theory is a subfield of mathematics in which randomness is studied from the perspective of partition regularity. We say that a structure \mathcal{A} is partition regular within some space, if for any partition of the space into to some finite number of pieces, one of the pieces contains a copy of \mathcal{A} . The simplest example of this, is letting \mathcal{A} be the collection of 2 point sets, then no matter how you partition the integers into a finite number of pieces, at least one of the pieces must contain some 2 point set. If we replace 2 in the previous example with some fixed number n , then we obtain what is commonly referred to as the pigeon hole principle, which is one of the earliest results of combinatorics. To be more precise, the pigeon hole principle tells us that given any number n , and any finite partition of the positive integers, at least one of the pieces contains some n point set. However, the pigeon hole principle does not tell us anything about the n point set other than its size. Ramsey Theory seeks to generalize the pigeon hole principle by imposing further restrictions, by asking questions such as if we can always find an n point set consisting of consecutive integers, even integers, perfect squares, and so on. One of the resulting generalizations is known as Van der Waerden's Theorem, which deals with structures known as arithmetic progressions. An arithmetic progression is a set of integers in which the difference between consecutive elements is the same, such as $\{3, 7, 11, 15, 19, 23, 27, 31\}$, or $\{a + jd\}_{j=0}^k$ is its most general form. Van der Waerden's Theorem states that we can generalize the pigeon hole principle by assuming that the n point sets we are finding are also arithmetic progressions. To be more precise, Van der Waerden's Theorem states that for any partition of the positive integers into a finite number of pieces, and any positive integer n , at least one of the pieces of the partition contains an arithmetic progression of n numbers. In this thesis, we will be examining how to further refine Van der Waerden's Theorem and its generalizations.

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Chapter 1

Introduction

Ramsey Theory concerns itself with questions about what substructures can be found in a finite partition of a large initial structure. To see what this means, let us consider some of the initial theorems that spurred the creation of the field.

Ramsey's Theorem: For any $k, r \in \mathbb{Z}^+$, there exists an integer $R(k, r)$, such that for any partition of a complete graph on $R(k, r)$ vertices into r sets, at least 1 of the partition classes contains a complete graph on k vertices.

Van der Waerden's Theorem: For any $k, r \in \mathbb{Z}^+$, there exists an integer $w(k, r)$, such that for any partition of $[1, w(k, r)]$ into r sets, at least 1 of the partition classes contains a k -term arithmetic progression.

Schur's Theorem: For any $r \in \mathbb{Z}^+$, there exists an integer $S(r)$, such that for any partition of $[1, S(r)]$ into r sets, at least 1 of the partition classes contains a solution to the equation $x + y = z$.

We see that in Ramsey's Theorem, the large initial structure is a complete graph, and the substructure that can always be found is also a complete graph. For Van der Waerden's Theorem, the large initial structure is an interval, and the substructure that can always be found is an arithmetic progression. For Schur's Theorem, the large initial structure is again an interval, and the substructure that can always be found is a solution to the equation $x + y = z$. These theorems are interesting, because they show that randomness is difficult to achieve, since structure can always be found regardless of how you partition the initial set. However, it is known that $w(3, 2) = 9$, and that $w(3, 3) = 27$, so when it is claimed that structure can be found in any partition of $[1, 9]$ into 2 sets, we see that we can avoid this structure if we instead partition $[1, 9]$ into 3 sets, and this seems to contradict the previous statement. To circumvent this problem, and more clearly realize the abundance of structure that can always be found, consider the following restatements of the above theorems as

theorems on infinite sets.

Ramsey's Theorem: For any $k, r \in \mathbb{Z}^+$, and any partition of a complete graph whose vertices are indexed by \mathbb{Z}^+ into r sets, at least 1 of the partition classes contains a complete graph on k vertices.

Van der Waerden's Theorem: For any $k, r \in \mathbb{Z}^+$, and any partition of \mathbb{Z}^+ into r sets, at least 1 of the partition classes contains a k -term arithmetic progression. In particular, at least 1 of the partition classes contains arbitrarily long arithmetic progressions.

Schur's Theorem: For any $r \in \mathbb{Z}^+$, and any partition of \mathbb{Z}^+ into r sets, at least 1 of the partition classes contains a solution to the equation $x + y = z$.

We can see that these infinite versions of the theorems are true as a result of the finite versions, simply because we can always restrict our attention to a large enough initial finite portion of the infinite space. We will later examine the far less obvious task of how to obtain the finite versions of Ramsey type theorems from the infinite versions. The main benefit of restating these theorems on infinite spaces, is that if we increase the values of k and r , we do not need to increase the size of the ambient space in order to still guarantee that our structures can be found. This allows us to introduce the following definitions that truly capture the essence of the structures examined in Ramsey Theory.

Definition 1.1: For a given set S , and some family $F \subseteq \mathcal{P}(S)$, we say that F is *weakly partition regular*, if for any finite partition of S as $S = \sqcup_{j=1}^n S_j$, there exist some $f \in F$ and some $1 \leq j_0 \leq n$, such that $f \subseteq S_{j_0}$.

Definition 1.2: For a given set S , and some family $F \subseteq \mathcal{P}(S)$, we say that F is *strongly partition regular*, if for any $f \in F$, and any finite partition of f as $f = \sqcup_{j=1}^n f_j$, there exist $f' \in F$, and $1 \leq j_0 \leq n$, such that $f' \subseteq f_{j_0}$.

Now let us see how these definitions apply to the previous theorems. For Ramsey's Theorem, let S be a complete graph on a set of vertices that are indexed by \mathbb{Z}^+ , and let F be the family consisting of all subgraphs of S that contain arbitrarily large complete subgraphs. The finite version of Ramsey's Theory tells us that F is strongly partition regular. For Van der Waerden's Theorem, we may take $S = \mathbb{Z}^+$, and let F be the family consisting of all subsets of S that contain arbitrarily long arithmetic progressions. The infinite version of Van der Waerden's Theorem tells us that F is weakly partition regular, but we can actually say more. We note that for a given n - term arithmetic progression given by $\{a + jd\}_{j=1}^n$, we may biject this arithmetic progression to the interval $[1, n]$ using the map $f(x) = \frac{x-a}{d}$. Given some fixed number of partition classes r , and some designated length k for an arithmetic progression, we see that any partition of an arithmetic progression $\{a + jd\}_{j=1}^{w(k,r)}$ into r sets, will induce a corresponding partition of $[1, w(k, r)]$ into r sets given by $\{w_j\}_{j=1}^r$, through the bijection f . By Van der Waerden's Theorem, some partition class

w_{j_0} contains a k -term arithmetic progression, so $f^{-1}(w_{j_0})$ also contains a k -term arithmetic progression, which shows us that the family F is actually strongly partition regular. Lastly, for Schur's Theorem, we may again take $S = \mathbb{Z}^+$, and let F be the family consisting of all subsets of S that contain a solution to the equation $x + y = z$. Schur's Theorem tells us that F is weakly partition regular, but we see that F is not strongly partition regular, as we may simply consider $\{1, 2\} \in F$.

The main question that we will be using to motivate our investigations is "What restrictions can be placed on F so that it is still weakly (strongly) partition regular?", or equivalently, "What subsets of F are still weakly (strongly) partition regular?". We will focus our attention on trying to generalize Van der Waerden's Theorems.

When talking about a partition of a set S as $S = \sqcup_{j=1}^n S_j$, it is often times useful to use its corresponding partition function $f : S \rightarrow [1, n]$ given by $f(s) = \{j \mid s \in S_j\} \forall s \in S$. Instead of saying "Consider the partition of S given by $S = \{S_j\}_{j=1}^n$ ", we may also say "Consider the partition of S given by f ", or "Let f be a partition of S ", where f is understood to be the partition function of the partition we are discussing. Instead of saying that a structure is contained in a single partition class of f when viewing f as a partition, we may view f as a function and say that a structure is constant under f . In the literature, many sources talk about colorings instead of partitions, where each partition class is assigned a color, and instead of seeking structure contained in a single partition class, we seek monochromatic structure. In this language we may restate Van der Waerden's Theorem as follows.

Van der Waerden's Theorem: For any finite coloring of \mathbb{Z}^+ , there must exist arbitrarily long monochromatic arithmetic progressions.

We will use the notation of partition functions instead of colorings.

Chapter 2

Van der Waerden's Theorem

Van der Waerden's Theorem on arithmetic progressions states that for any finite partition of \mathbb{Z}^+ , at least one of the partition classes contains arbitrarily long arithmetic progressions. The significance of arithmetic progressions other than their aesthetic appeal, is that an arithmetic progression is simply a scaling and a shift of the most basic structure in \mathbb{Z}^+ , the interval $[1, n]$. In particular, we see that any finite set of points in \mathbb{Z}^+ is contained in some interval $[1, n]$, so we may give an equivalent restatement of Van der Waerden's Theorem as follows. Let F be a finite set of points in \mathbb{Z}^+ , then for any partition of \mathbb{Z}^+ , at least one of the partition classes contains a scaled and shifted copy of F . This restatement of Van der Waerden's Theorem more clearly reveals its significance, because it shows that no matter how randomly you may think that you have partitioned the integers, you can still find scaled and shifted copies of any finite structure you choose in one of the partition classes, which is certainly not a random phenomenon. One way to generalize Van der Waerden's Theorem that we will be investigating, is what restrictions can be placed on the scale factors so that the Theorem still holds.

Definition 2.1: For any $S \subseteq \mathbb{Z}^+$, a k -term S -A.P. is a k -term arithmetic progression $\{a + id\}_{i=0}^{k-1}$, such that $d \in S$.

Definition 2.2: For $S \subseteq \mathbb{Z}^+$, we say that S is *Large* if for any $r, k \in \mathbb{Z}^+$, and any partition of \mathbb{Z}^+ into r sets, at least 1 partition class contains a k -term S -A.P..

With this new definition, we can again restate Van der Waerden's Theorem as follows. \mathbb{Z}^+ is a Large set. We can see that for any $n \in \mathbb{Z}^+$, the set of positive integers that are divisible by n , $n\mathbb{Z}^+$, is a Large set. To see this, we note that for any $k, r \in \mathbb{Z}^+$, and any partition of \mathbb{Z}^+ into r -sets, we can find a $(nk - n + 1)$ -term arithmetic progressions $\{a + jd\}_{j=0}^{nk-n}$ in some partition class as a result of Van der Waerden's Theorem, so the k -term arithmetic progression given by $\{a + jnd\}_{j=0}^{k-1}$ is a k -term $n\mathbb{Z}^+$ -A.P. that is contained in a single partition class. To obtain some more interesting examples of Large sets, we first require a few more

definitions.

Definition 2.3: For any $S \subseteq \mathbb{Z}^+$, and any sequence $X = \{x_s\}_{s \in S}$, let $FS(X)$ denote the set of all finite sums of distinctly indexed elements of X . To be more precise, define $FS(X) = \{\sum_{a \in A} x_a\}_{A \in \mathcal{P}_f(S)}$. If $S = \mathbb{Z}^+$, and X is given by $X = \{x_j\}_{j=1}^\infty$, we call $FS(X)$ the IP set generated by X , or just an IP set to be more concise. FS is an abbreviation for “Finite Sums”.

Theorem 2.4: Let $\{A_n\}_{n=1}^\infty$ be a sequence of finite subsets of \mathbb{Z}^+ , such that $\sup_{n \in \mathbb{Z}^+} |A_n| = \infty$. Then the set $A := \cup_{n=1}^\infty FS(A_n)$ is a Large set.

Proof: See [1].

Corollary 2.5: If S contains an IP set, then S is Large.

Proof: Let the IP set contained in S be generated by $\{x_j\}_{j=1}^\infty$. Apply Theorem 2.4 to the family $\{A_n\}_{n=1}^\infty$ given by $A_n = \{x_j\}_{j=1}^n$. Alternative proofs are provided in [2] and [3].

In [1], the 2 families of sets below are shown to be Large, by showing that they contain IP sets.

Example 1: For any $\alpha \in \mathbb{R}^+$, the set $\{\lfloor n\alpha \rfloor \mid n \in \mathbb{Z}^+\}$ is Large.

Example 2: For any $\alpha \in \mathbb{R}^+$, and any $\epsilon > 0$, the set $\{n \in \mathbb{Z}^+ \mid \{n\alpha\} \in [0, \epsilon)\}$ is Large.

To get yet another interesting family of Large sets, we must first consider another generalization of Van der Waerden’s Theorem that was first proven by Bergelson and Leibman in [4] as a corollary of a stronger result that was obtained through the use of ergodic theory. An alternative elementary proof is provided by Walters in [5].

Definition 2.6: We say that $p(x) \in \mathbb{Q}[x]_{j=1}^n$ is an *integral polynomial* if $p(\mathbb{Z}^n) \subseteq \mathbb{Z}$.

(Old) Polynomial Van der Waerden Theorem: Let $\{p_j(x)\}_{j=1}^n$ be a family of integral polynomials in $\mathbb{Q}[x]$ satisfying $p_j(0) = 0$ for all $1 \leq j \leq n$. For any finite partition of \mathbb{Z}^+ , there exist $a, d \in \mathbb{Z}^+$ such that the set $\{a + p_j(d)\}_{j=1}^n$ is contained in a single partition class.

We see that we can recover Van der Waerden’s Theorem from the (Old) Polynomial Van der Waerden Theorem by picking some length k for our arithmetic progression, and choosing the family of polynomials $\{p_j(x)\}_{j=1}^k$ to be given by $p_j(x) = jx$ for $1 \leq j \leq k$. With the aid of Lesigne, Bergelson and Leibman have further refined the (Old) Polynomial Van der Waerden Theorem in [6] to the (New) Polynomial Van der Waerden Theorem, once again as a corollary of a stronger theorem proven with advanced techniques, however, no elementary

proofs are currently known.

Definition 2.7: Let $P = \{p_j(x)\}_{j=1}^n$ be a family of polynomials in $\mathbb{Q}[x_i]_{i=1}^m$. We say that P is *jointly intersective* if for any $k \in \mathbb{Z}^+$, there exists $k' \in \mathbb{Z}^m$, such that $k \mid p_j(k')$ for all $1 \leq j \leq n$.

(New) Polynomial Van der Waerden Theorem: Let $P = \{p_j(x)\}_{j=1}^n$ be a family of jointly intersective integral polynomials in $\mathbb{Q}[x_i]_{i=1}^m$. For any finite partition of \mathbb{Z} , there exist $d \in \mathbb{Z}^m$ and $a \in \mathbb{Z}^+$ such that the set $\{a + p_j(d)\}_{j=1}^n$ is contained in a single partition class.

Corollary 2.8: If $p(x) \in \mathbb{Q}[x]$ is an integral polynomial, and for every $k \in \mathbb{Z}^+$, there exists $k' \in \mathbb{Z}$ such that $k \mid p(k')$, then $p(\mathbb{Z})$ is a Large set.

Proof: Given a finite partition of \mathbb{Z}^+ , apply the (New) Polynomial Van der Waerden Theorem with $m = 1$ to the family of polynomials $\{jp(x)\}_{j=1}^n$ to obtain an n -term $p(\mathbb{Z})$ -A.P..

In [7], Bergelson, Furstenburg, and McCutcheon combined the facts that a polynomial image of the integers is Large, and IP sets are Large, to show that the polynomial image of an IP set is Large.

IP-Polynomial Van der Waerden Theorem: Let $\{p_j(x)\}_{j=1}^n$ be a family of integral polynomials in $\mathbb{Q}[x]$ satisfying $p_j(0) = 0$ for all $1 \leq j \leq n$. If S is any IP set, then for any finite partition of \mathbb{Z}^+ , there exist $a \in \mathbb{Z}^+$, and $d \in S$, such that the set $\{a + p_j(d)\}_{j=1}^n$ is contained in a single partition class.

Corollary 2.9: If S is any IP set, and $p(x) \in \mathbb{Q}(x)$ satisfies $p(\mathbb{Z}) \subseteq \mathbb{Z}$ and $p(0) = 0$, then $p(S)$ is a Large set.

Proof: Given a finite partition of \mathbb{Z}^+ , apply the IP-Polynomial Van der Waerden Theorem to the set of polynomials $\{jp(x)\}_{j=1}^n$ in order to find a monochromatic n -term $p(S)$ -A.P..

Among all of the aforementioned examples of Large sets, the Polynomial IP Van der Waerden Theorem arguably yields the most general class, so there seems to be hope to try and classify precisely which sets are Large, however, in chapter 5, we shall give strong evidence to suggest that a reasonable classification is not possible. The first step in this direction has already been taken in [1], where it is shown that the family of Large sets is strongly partition regular. Intuitively, it may seem that this result alone implies that it is difficult to classify which sets are Large, as one may think that any Large set with pleasant structure can be meticulously partitioned in such a way that none of the partition classes contain pleasant structure other than being Large, but this is not obvious from the above examples. To see this, we note that in [3], Furstenburg shows that in any finite partition of an IP set, at least 1 of the partition classes contains an IP set. Now let us consider $p(x) \in \mathbb{Z}[x]$ with $p(0) = 0$, and $X = \{x_j\}_{j=1}^\infty$, so $S := p(FS(X))$ is a Large set by the IP-Polynomial Van

der Waerden Theorem. We see that any finite partition of S as $S = \sqcup_{i=1}^n S_i$, induces a unique partition of $p(FS(X))$ as $p(FS(X)) = \sqcup_{i=1}^n P_i$ such that $S_i = p(P_i)$ for $1 \leq i \leq n$. Since a finite partition of an IP set results in at least 1 partition class containing an IP set, let P_{i_0} be a partition class that contains an IP set. It follows that S_{i_0} is not only a Large set, but a Large set that contains the polynomial image of some IP set, which is highly structured.

Another way in which we can continue our investigations of Large sets, is by generalizing to the notion of r -Large sets as defined in [1].

Definition 2.10: For any $r \in \mathbb{Z}^+$ and $S \subseteq \mathbb{Z}^+$, we say that S is r -Large, if for any $k \in \mathbb{Z}^+$, and any partition of \mathbb{Z}^+ into r -sets, at least 1 of the partition classes contains a k -term S -A.P..

We see that if S is an r -Large set, then for any $r' \leq r$, S is also an r' -Large set. Intuitively, we would not expect the converse to be true, we expect that for any $r \in \mathbb{Z}^+$, there exists a set S_r that is r -Large, but not $(r+1)$ -Large, but an example of S_r has not been found for any $r \geq 2$. This leads to the following conjecture posed in [1].

Two-Large Is Large Conjecture: If S is a 2-Large set, then S is a Large set.

In [8], Host, Kra, and Maass attempted to answer this conjecture, and yielded a new formulation of the conjecture, as well as some progress. Before discussing this, we require some more definitions.

Definition 2.11: Let $S \subseteq \mathbb{Z}^+$ be an infinite set, and let $\{s_j\}_{j=1}^\infty$ be an enumeration of the elements of S in increasing order. If $s_{j+1} - s_j \leq r$ for all $j \in \mathbb{Z}^+$, then we say that S is r -syndetic. If S is r -syndetic for some r , then we say that S is a *syndetic set*, and the minimal value of r for which S is r -syndetic is the *syndeticity constant* of S .

Definition 2.12: For any $r \in \mathbb{Z}^+$ and $S \subseteq \mathbb{Z}^+$, we say that S is r -syndetic Large if any r -syndetic set contains arbitrarily long S -A.P.s.

Theorem 2.13: If S is an r -Large set, then S is r -syndetic Large.

Proof: For $S \subseteq \mathbb{Z}^+$, and $j \in \mathbb{Z}^+$, let $S-j$ denote the set $\{s-j \mid s \in S \text{ \& } s > j\}$. Let T be an r -syndetic set, and consider $T_1 = T$, and for $2 \leq j \leq r$, define $T_j = (T_1 - (j-1)) - (\cup_{i=1}^{j-1} T_i)$. We see that $\{T_j\}_{j=1}^r$ is a partition of \mathbb{Z}^+ into r sets, so for some $1 \leq j \leq r$, T_j contains arbitrarily long S -A.P.s. Since $\{t+j-1 \mid t \in T_j\} \subseteq T$, we see that T must also contain arbitrarily long S -A.P.s.

Theorem 2.14: If S is a $(2r-1)$ -syndetic Large set, then S is r -Large.

Proof: See [8].

We now see that a set S is Large if and only if it is r -syndetic Large for all $r \in \mathbb{Z}^+$. This leads to the following restatement of the 2-Large is Large conjecture.

Conjecture 2.15: If S is a 2-syndetic Large set, then S is r -syndetic Large for all $r \in \mathbb{Z}^+$.

The following theorem shows us that multiplicative structure can strongly influence the Largeness of a set, and lets us see that the family of Large sets is strongly partition regular.

Theorem 2.16: If $A, B \subseteq \mathbb{Z}^+$ are such that A is not r -Large, and B is not s -Large, then $A \cup B$ is not rs -large.

A first step towards proving or disproving the 2-Large is Large conjecture, would be to determine whether or not the family of 2-Large sets are strongly partition regular. Is the family of r -Large sets strongly partition regular for any r ? What can be said about the partition regularity of r -syndetic Large sets? The following seemingly simple conjecture, which is a special case of the partition regularity of r -Large sets, remains unsolved.

Conjecture 2.17: If $S \subseteq \mathbb{Z}^+$ is an r -Large set, then $S_k = \{s \in S \text{ s.t. } k|s\}$ is also r -Large.

The method used in [1] to show that Large sets are strongly partition regular, cannot be used to show that r -Large sets are strongly partition regular for any r , as the method explicitly takes advantage of the fact that we are free to increase the number of colors when working with a Large set, as is often the case in Ramsey Theory. We propose the following technique to merge 2 partitions of the integers in an attempt to prove the partition regularity of r -Large sets. For any three 2-partitions χ_A, χ_B , and χ_C of \mathbb{Z}^+ , we may define

$$\chi_D := \chi_A \times_{\chi_C} \chi_B := \begin{cases} \chi_A(i) & \text{if } \chi_C(i) = 0 \\ \chi_B(i) & \text{if } \chi_C(i) = 1 \end{cases} .$$

Next, let $A, B \subseteq \mathbb{Z}^+$ be such that neither of them are 2-Large. Let χ_A and χ_B be partitions of \mathbb{Z}^+ that are not constant on arbitrarily long A -A.P.s and B -A.P.s respectively. Let L_A be the length of the longest A -A.P. on which χ_A is constant, and define L_B similarly. In order to show that $D := A \cup B$ is not a 2-Large set, we will try to construct a 2-partition χ_C of \mathbb{Z}^+ , such that the 2-partition χ_D given by $\chi_D = \chi_A \times_{\chi_C} \chi_B$ does not contain arbitrarily long D -A.P.s. A property of χ_C that would be sufficient to achieve this is the following. There exists some $k \in \mathbb{Z}^+$, such that any length k A -A.P. in χ_C , contains at least $L_A + 1$ consecutive elements in partition class 0, and any length k B -A.P. in χ_C , contains at least $L_B + 1$ consecutive elements in partition class 1. We see that if χ_C satisfies this condition, then in any k -term D -A.P. $\{a + jd\}_{j=0}^{k-1}$ in χ_D , we may assume without loss of generality that $d \in A$, so that some $L_A + 1$ consecutive terms of this A.P. given by $\{a + jd\}_{j=j_0}^{j_0+L_A}$, are colored 0 under χ_C , so $\chi_D(a + jd) = \chi_A(a + jd)$ for $j_0 \leq j \leq j_0 + L_A$. It follows that χ_D is not constant on $\{a + jd\}_{j=j_0}^{j_0+L_A}$, and is consequently not constant on $\{a + jd\}_0^{k-1}$, so χ_D is not constant on any length k D -A.P., and D is not 2-Large.

Chapter 3

Multidimensional Van der Waerden Theorem

It is natural to ask how to generalize Van der Waerden's Theorem to a multidimensional theorem on \mathbb{Z}^n . The natural definition for a k -term arithmetic progression in \mathbb{Z}^n , is to pick some $a, d \in \mathbb{Z}^n$, and consider the set $\{a + jd\}_{j=0}^{k-1}$. We may then ask if for any partition of \mathbb{Z}^n into a finite number of sets, does there exist at least 1 partition class containing arbitrarily long arithmetic progressions? We can see that the answer to this question is yes as follows. Given some partition of \mathbb{Z}^n , we may simply consider the induced partition along some coordinate axis, then apply Van der Waerden's Theorem on this induced partition in order to obtain our long arithmetic progressions. We can see that we have not formulated the question correctly, because the solution did not require the ambient space to be multidimensional. To obtain the correct formulation, we only have to recall our statement of Van der Waerden's Theorem as a theorem on finite sets rather than a theorem on arithmetic progressions. Our desired statement for the Multidimensional Van der Waerden Theorem, is that for any $n \in \mathbb{Z}^+$, and any finite partition of \mathbb{Z}^n , at least 1 partition class contains a scaled and shifted copy of any finite set of points in \mathbb{Z}^n . We see that if F is some finite set of points in \mathbb{Z}^n , then there exists some $m \in \mathbb{Z}^+$, such that $F \subseteq \prod_{j=1}^n [-m, m]$, so we only need to have scaled and shifted copies of arbitrarily large n -dimensional cubes in order to have scaled and shifted copies of any finite set of points.

Before we officially state the Multidimensional Van der Waerden Theorem, we will require some more notation. Let $\{e_j\}_{j=1}^n$ denote the standard basis in \mathbb{Z}^n . A length k n -cube is a set of the form $C = \{a + c_1de_1 + c_2de_2 + \cdots + c_nde_n \mid 0 \leq c_1, c_2, \dots, c_n < k\}$, where $a \in \mathbb{Z}^n$ is the base point, and $d \in \mathbb{Z}^+$ is the common difference. We say that C is generated by the base point a and the common difference d . To be more concise, we may sometimes say "Consider the n -cube C generated by (a, d) ". A length k S -gap n -cube is a length k

n -cube in which the common difference is an element of S .

Multidimensional Van der Waerden Theorem: For any $n \in \mathbb{Z}^+$, and any finite partition of \mathbb{Z}^n , at least 1 of the partition classes contains arbitrarily large n -cubes.

As with the theorems from the introduction, the Multidimensional Van der Waerden Theorem also has a finitistic version which we will now state.

Multidimensional Van der Waerden Theorem: For any $n, k, r \in \mathbb{Z}^+$, there exists an integer $w(n, k, r)$, such that for any partition of $\prod_{j=1}^n [1, w(n, k, r)]$ into r sets, at least 1 of the partition classes contains a length k n -cube.

We can investigate the Multidimensional Van der Waerden Theorem using the same line of questioning that we used to investigate Van der Waerden's Theorem. In particular, what restrictions can we place on the common difference of an n -cube so that we still have a weakly partition regular family?

Definition 3.1: Let us call $S \subseteq \mathbb{Z}^+$ *Multidimensionally Large*, if for any $n, r \in \mathbb{Z}^+$, and any partition of \mathbb{Z}^n into r sets, at least 1 of the partition classes contains arbitrarily large S -gap n -cubes.

Definition 3.2: Let us call $S \subseteq \mathbb{Z}^+$ *Multidimensionally r -Large*, if for any $n \in \mathbb{Z}^+$, and any partition of \mathbb{Z}^n into r sets, at least 1 of the partition classes contains arbitrarily large S -gap n -cubes.

Definition 3.3: Let us call $S \subseteq \mathbb{Z}^+$ *n -dimensionally r -Large*, if for any partition of \mathbb{Z}^n into r sets, at least 1 of the partition classes contains arbitrarily large S -gap n -cubes.

We can show as we did with Large sets, that for any $n \in \mathbb{Z}^+$, the set $n\mathbb{Z}^+$ is a Multidimensionally Large set. In [3], it is shown that IP sets are Multidimensionally Large. In [4], Bergelson and Leibman prove the Multidimensional Polynomial Van der Waerden Theorem, which we state below.

Multidimensional Polynomial Van der Waerden Theorem: Let $\{p_j(x)\}_{j=1}^n$ be a family of integral polynomials in $\mathbb{Q}^l[x]$ satisfying $p_j(0) = 0$ for all $1 \leq j \leq n$. Then for any finite partition of \mathbb{Z}^l , there exist $a \in \mathbb{Z}^l$ and $d \in \mathbb{Z}^+$ such that the set $\{a + p_j(d)\}_{j=1}^n$ is contained in a single partition class.

Corollary 3.4: If $p(x) \in \mathbb{Q}[x]$ is an integral polynomial satisfying $p(0) = 0$, then $p(\mathbb{Z}^+)$ is a Large set.

Proof: Let $p(x) = \sum_{j=0}^n a_j x^j$. For $q \in \mathbb{Q}$, let \hat{q}_k denote the element in \mathbb{Q}^l , all of whose components are 0, except for the k th component, which is q . We see that the polynomial $p_k(x) = \sum_{j=0}^n (\hat{a}_j)_k x^j$ is an element of \mathbb{Q}^l , and $p_k(\hat{0}) = 0$ for all $1 \leq k \leq l$. Now, we only need to apply the Multidimensional Polynomial Van der Waerden Theorem to the family of polynomials given by $\cup_{k=1}^l \{j p_k(x)\}_{j=0}^{n-1}$ in order to get a length n $p(\mathbb{Z}^+)$ -gap l -cube.

As before, we can combine the fact that IP sets are Multidimensionally Large and the fact that polynomial images of the integers are Multidimensionally Large into a single theorem which we give below.

Multidimensional IP-Polynomial Van der Waerden Theorem: Let $\{p_j(x)\}_{j=1}^n$ be a family of integral polynomials in $\mathbb{Q}^l[x]$ satisfying $p_j(0) = 0$ for all $1 \leq j \leq n$. Let S be any IP set. Then for any finite partition of \mathbb{Z}^l , there exist $a \in \mathbb{Z}^l$ and $d \in S$ such that the set $\{a + p_j(d)\}_{j=1}^n$ is contained in a single partition class.

Corollary 3.5: If $p(x) \in \mathbb{Q}[x]$ is an integral polynomial satisfying $p(0) = 0$, and S is an IP set, then $p(S)$ is a Large set.

We can see that almost any example of a set that we know is Large, is also Multidimensionally Large. The only possibility for an exception so far, is to determine whether or not there exists a Multidimensional Polynomial Van der Waerden Theorem for families of jointly intersective integral polynomials that do not necessarily map 0 to 0. This leads us to the following conjecture.

Conjecture 3.6: If $S \subseteq \mathbb{Z}^+$ is a Large set, then S is Multidimensionally Large.

We can also formulate the analog of the 2-Large is Large conjecture.

Conjecture 3.7: If $S \subseteq \mathbb{Z}^+$ is Multidimensionally 2-Large, then S is Multidimensionally Large.

Let $<_n$ denote the partial ordering on \mathbb{Z}^n , in which for any $a, b \in \mathbb{Z}^n$, we have $a <_n b$, if each component of a is strictly smaller than the corresponding component of b under the standard ordering of \mathbb{Z} . Let \leq_n denote the extension of the partial order $<_n$ such that for any $a, b \in \mathbb{Z}^n$, we have $a \leq_n b$ if $a <_n b$, or $a = b$. In general, we would like to know if a set S is n -dimensionally r -Large for some $n, r \in \mathbb{Z}^+$, then can we show that S is n' -dimensionally r' -Large for some $n', r' \in \mathbb{Z}^+$ satisfying $(n, r) <_2 (n', r')$? Conversely, for each $n, r \in \mathbb{Z}^+$, does there exist a set $S_{n,r}$ that is n -dimensionally r -Large, but is not n' -dimensionally r' -Large for any $n', r' \in \mathbb{Z}^+$ satisfying $(n, r) <_2 (n', r')$?

We take a first step towards answering these questions by providing a reformulation in terms of syndetic Large sets in a similar manner to that of Host, Kra, and Maass in [8]. As before, for $S \subseteq \mathbb{Z}^n$, and $j \in \mathbb{Z}^n$, let $S - j$ denote the set $\{s - j \mid s \in S\}$.

Definition 3.8: We say that $S \subseteq \mathbb{Z}^n$ is *r-syndetic*, if there exists a finite subset $\{a_j\}_{j=1}^r$ of \mathbb{Z}^n , such that $\mathbb{Z}^n \subseteq \cup_{j=1}^r (S - a_j)$. If S is *r-syndetic* for some r , then we also say that S is *syndetic*. If S is a syndetic set, then the minimal value of r for which S is *r-syndetic*, is the *syndeticity constant* of S .

Definition 3.9: We say that $S \subseteq \mathbb{Z}^+$ is *Multidimensionally r-syndetic Large*, if for any $n \in \mathbb{Z}^+$, and any *r-syndetic* subset T of \mathbb{Z}^n , T contains arbitrarily large S -gap n -cubes.

Definition 3.10: We say that $S \subseteq \mathbb{Z}^+$ is *n-dimensionally r-syndetic Large*, if for any *r-syndetic* subset T of \mathbb{Z}^n , T contains arbitrarily large S -gap n -cubes.

Theorem 3.11: If $S \subseteq \mathbb{Z}^+$ is *n-dimensionally r-Large*, then S is *n-dimensionally r-syndetic Large*.

Proof: Let $T \subseteq \mathbb{Z}^n$ be some *r-syndetic* set, and let $\{a_j\}_{j=1}^r$ be such that $\mathbb{Z}^n \subseteq \cup_{j=1}^r (T - a_j)$. We may construct a *r-partition* $\{T_j\}_{j=1}^r$ of \mathbb{Z}^n by setting $T_1 = T$, and inductively defining $T_j = (T - a_j) - (\cup_{k=1}^{j-1} T_k)$ for $2 \leq j \leq r$. Since S is *n-dimensionally r-Large*, there exists $1 \leq j_0 \leq r$ such that T_{j_0} contains arbitrarily large S -gap n -cubes, but $T_{j_0} + j_0 \subseteq T$, so T must also contain arbitrarily large S -gap n -cubes, so S is *n-dimensionally r-syndetic Large*.

Theorem 3.12: If $S \subseteq \mathbb{Z}^+$ is *n-dimensionally (2r - 1)ⁿ-syndetic Large*, then S is *n-dimensionally r-Large*.

Proof: Let f be a *r-partition* of \mathbb{Z}^n , and consider the set T given by $T = \{(rx_1 + f(x), rx_2 + f(x), \dots, rx_n + f(x)) \mid x \in \mathbb{Z}^n\}$. We can see that T is $(2r - 1)^n$ -syndetic since $r\mathbb{Z}^n \subseteq \cup_{k=1}^{2r-1} (T + \underbrace{(k, k, \dots, k)}_n)$. We see that rS is also a $(2r - 1)^n$ -syndetic Large set, so there exist

some base point $a_k \in \mathbb{Z}^n$ and some common difference $d_k \in rS$ that generate a length k n -cube in T , for any $k \in \mathbb{Z}^+$. For $x, y \in \mathbb{Z}^n$, we see that $(ry_1 + f(y), ry_2 + f(y), \dots, ry_n + f(y)) + d_k e_j = (rx_1 + f(x), rx_2 + f(x), \dots, rx_n + f(x))$ if and only if $(y_1, y_2, \dots, y_n) + \frac{d_k}{r} e_j = (x_1, x_2, \dots, x_n)$ and $f(x) = f(y)$, since $r \mid d_k$ and $f(\mathbb{Z}^n) = [1, r]$. We now see that the base point $(\lfloor \frac{(a_k)_1}{r} \rfloor, \lfloor \frac{(a_k)_2}{r} \rfloor, \dots, \lfloor \frac{(a_k)_n}{r} \rfloor)$ and the common difference $\frac{d_k}{r}$ generate a length k n -cube contained in a single partition class of f .

We now see that a set S is *n-dimensionally Large* if and only if it is *n-dimensionally syndetic Large*, which let us restate conjecture 3.7 as follows.

Conjecture 3.13: If $S \subseteq \mathbb{Z}^+$ is *Multidimensionally 2-syndetic Large*, then S is *Multidimensionally r-syndetic Large* for all $r \in \mathbb{Z}^+$.

Furthermore, using the notion of syndeticity, we can introduce a natural intermediate step to conjecture 3.13.

Definition 3.14: Let us call $S \subseteq \mathbb{Z}^n$ *k-dimensionally r-syndetic*, if S is an r -syndetic set, and there exists a set $\{a_j\}_{j=1}^r$ contained in a k -dimensional linear subspace of \mathbb{Z}^n , such that $\mathbb{Z}^n \subseteq \cup_{j=1}^r (S - a_j)$.

Definition 3.15: Let us call $S \subseteq \mathbb{Z}^+$ *(n, k)-dimensionally r-syndetic Large*, if any k -dimensionally r -syndetic subset of \mathbb{Z}^n contains arbitrarily large S -gap n -cubes.

We can now prove conjecture 3.6 by first showing that all Large sets are $(n, 1)$ -dimensionally Large for any $n \in \mathbb{Z}^+$, then showing that any (n, k) -dimensionally Large set is also $(n, k + 1)$ -dimensionally Large. The following theorem gives good evidence to suggest that this approach is likely to work. First, recall that for $S \subseteq \mathbb{Z}^+$, we let S_r denote the set of elements of S that are divisible by r .

Theorem 3.16: If S is a $(n, 1)$ -dimensionally $(2r - 1)$ -syndetic Large set, such that S_r is also a $(n, 1)$ -dimensionally $(2r - 1)$ -syndetic Large set, then S is n -dimensionally r -Large.

Proof: Let f be a r -partition of \mathbb{Z}^n , and consider the set T given by $T = \{(rx_1 + f(x), x_2, \dots, x_n) \mid x \in \mathbb{Z}^n\}$. We can see that T is 1-dimensionally $(2r - 1)$ -syndetic since $r\mathbb{Z}^n \subseteq \cup_{k=1}^{2r-1} (T + ke_1)$, so there exist some base point $a_k \in \mathbb{Z}^n$ and some common difference $d_k \in S_r$ that generate a length rk n -cube in T , for any $k \in \mathbb{Z}^+$. For $x, y \in \mathbb{Z}^n$, we see that $(ry_1 + f(y), y_2, \dots, y_n) + d_k e_1 = (rx_1 + f(x), x_2, \dots, x_n)$ if and only if $(y_1, y_2, \dots, y_n) + \frac{d_k}{r} e_1 = (x_1, x_2, \dots, x_n)$ and $f(x) = f(y)$, and for $2 \leq j \leq n$ we have $(ry_1 + f(y), y_2, \dots, y_n) + d_k e_j = (rx_1 + f(x), x_2, \dots, x_n)$ if and only if $(y_1, y_2, \dots, y_n) + d_k e_j = (x_1, x_2, \dots, x_n)$ and $f(x) = f(y)$, since $r \mid d_k$ and $f(\mathbb{Z}^n) = [1, r]$. We now see that for $a' = (\lfloor \frac{(a_k)_1}{r} \rfloor, (a_k)_2, \dots, (a_k)_n)$, the set

$C = \{a' + c_1 \frac{d_k}{r} e_1 + \sum_{j=2}^n c_j d_k e_j \mid 0 \leq c_1, c_2, \dots, c_n \leq rk - 1\}$ is contained in a single partition

class of f . Taking c_1 from C to be an element of $\{rj\}_{j=0}^{k-1}$ shows us that the base point a' and the common difference d_k generate a length k n -cube contained in a single partition class of f .

We now see that if we can prove for any $S \subseteq \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$ that passing from S to S_k does not decrease the ‘‘Largeness’’ of S , then we will have shown that $(n, 1)$ -dimensionally syndetic Large sets are also n -dimensionally Large, so it will only remain to show that Large sets are $(n, 1)$ -dimensionally syndetic Large. Showing that S_k is just as ‘‘Large’’ as S for any k , is a special case of partition regularity, since we know that any 2-Large must contain a multiple of every positive integer. As far as partition regularity is concerned, we present 2 results, either of which can be used to show that the family of Multidimensionally Large sets is strongly partition regular.

Theorem 3.17: For any $n \in \mathbb{Z}^+$, the family of n -dimensionally Large sets is strongly partition regular. In particular, given $A, B \subseteq \mathbb{Z}^+$ such that A is not n -dimensionally r -Large, and B is not n -dimensionally s -Large, then $A \cup B$ is not n -dimensionally rs -Large.

Proof: Let f_A and f_B be a r -partition of \mathbb{Z}^n and a s -partition of \mathbb{Z}^n respectively, such that f_A does not contain arbitrarily large A -gap n -cubes in any partition class, and f_B does not contain arbitrarily large B -gap n -cubes in any partition class. Let M_A be the length of the largest n -cube on which f_A is constant, and define M_B similarly. Create the rs -partition f_C of \mathbb{Z}^n by setting $f_C(x) = (f_A(x), f_B(x))$. If $a \in \mathbb{Z}^n$ and $d \in A \cup B$ are such that the base point a and the common difference d generate a length k n -cube \mathcal{C} , that is constant under f_C , then \mathcal{C} will be constant under f_A and f_B as well. If $d \in A$, then $k \leq M_A$, and if $d \in B$, then $k \leq M_B$, so $k \leq \max(M_A, M_B)$, which yields the desired result.

Theorem 3.18: For any $r \in \mathbb{Z}^+$, the family of Multidimensionally r -Large sets is strongly partition regular. In particular, given $A, B \subseteq \mathbb{Z}^+$ such that A is not n -dimensionally r -Large, and B is not m -dimensionally r -Large, then $A \cup B$ is not $(n + m)$ -dimensionally r -Large.

Proof: Let f_A and f_B be r -partitions of \mathbb{Z}^n and \mathbb{Z}^m respectively, such that f_A does not contain arbitrarily large A -gap n -cubes in any partition class, and f_B does not contain arbitrarily large B -gap m -cubes in any partition class. Let M_A be the length of the largest n -cube on which f_A is constant, and define M_B similarly. Create the r -partition f_C of \mathbb{Z}^{n+m} by setting $f_C(x) = (f_A((x_j)_{j=1}^n), f_B((x_j)_{j=n+1}^{n+m}))$. If $a \in \mathbb{Z}^{n+m}$ and $d \in A \cup B$ are such that the base point a and the common difference d generate a length k $(n + m)$ -cube that is constant under f_C , then the base point $(a_j)_{j=1}^n$ and the common difference d generate a length k n -cube that is constant under f_A , and the base point $(a_j)_{j=n+1}^{n+m}$ and the common difference d generate a length k m -cube that is constant under f_B . If $d \in A$, then $k \leq M_A$, and if $d \in B$, then $k \leq M_B$, so $k \leq \max(M_A, M_B)$, which yields the desired result.

Chapter 4

New Multidimensional Structures

We have already seen that the significance of arbitrarily long arithmetic progressions in \mathbb{Z}^+ , is that they will contain an affine image of any finite set of points in \mathbb{Z}^+ , which is why we look for arbitrarily large k -cubes in the Multidimensional Van der Waerden Theorem on \mathbb{Z}^k . We also know that it is trivial to ask about the partition regularity of arbitrarily long arithmetic progressions in \mathbb{Z}^k , but, if we were to place restrictions on the allowable common differences d in the arithmetic progressions we are seeking, then the question becomes interesting once more.

Definition 4.1: A subset S of \mathbb{Z}^k is called *weakly k -dimensionally large*, if for any finite coloring of \mathbb{Z}^k , there exist arbitrarily long monochromatic arithmetic progressions whose common differences belong to S .

Definition 4.2: A subset S of \mathbb{Z}^k is called *weakly k -dimensionally r -large*, if for any r -coloring of \mathbb{Z}^k , there exist arbitrarily long monochromatic arithmetic progressions whose common differences belong to S .

Definition 4.3: For $S \subseteq \mathbb{Z}^+$, let (S, \mathbb{Z}^k) denote the set $\{s \in \mathbb{Z}^k \mid \gcd((s_i)_{i=1}^k) \in S\}$. We say by abuse of language that S is *weakly k -dimensionally large*, or *weakly k -dimensionally r -Large* if and only if the same property holds for (S, \mathbb{Z}^k) .

The motivation for the definition of (S, \mathbb{Z}^k) , is that when we restrict our attention to affine 1-dimensional subspace of \mathbb{Z}^k , it appears as if we are looking for S -A.P.s in \mathbb{Z} . We see that requiring a set to be weakly n -dimensionally large becomes less restrictive as n increases, because we can always restrict ourselves to subspaces of lower dimension. We are led to wonder about what is the interplay between r -large sets, and weakly k -dimensionally r -large sets.

Theorem 4.4: Let S be a weakly t -dimensionally 2-Large set. For any set of primes $\{p_i\}_{i=1}^t$

and any set of positive integers $\{k_i\}_{i=1}^t$, there exists $d \in S$ such that $p_i^{k_i} \mid d_i$ for $1 \leq i \leq t$.

Proof: We will proceed by induction on t , with the base case being $t = 1$, but our inductive hypothesis will be stronger than the statement of the Theorem. In order to motivate this new inductive hypothesis, let us offer an alternative proof for the case of $t = 1$, which we already know to be true.

For any prime p and any $n, k \in \mathbb{Z}^+$ with $k < n$, let $C[p, n, k]$ be the 2-partition of the cyclic group \mathbb{Z}_{p^n} , in which $C[p, n, k](j) = 0$ if $0 \leq j \leq p^k - 1$, and $C[p, n, k](j) = 1$ if $p^k \leq j \leq p^n - 1$. Let $A = \{a + dj\}_{j=0}^{M-1}$ be any arithmetic progression in \mathbb{Z}_{p^n} of length M , where p^n divides M , and let $p^m = \gcd(d, p^n)$. We will show that if $m \leq k$, then $C[p, n, k]$ will take the value 0 $p^k \binom{M}{p^n}$ times on A , and the value 1 will be taken $(p^n - p^k) \binom{M}{p^n}$ times on A , therefore $C[p, n, k]$ cannot be constant on A unless $p^{k+1} \mid d$. To see why this is true, we see that $\gcd(\frac{d}{p^m}, p^{n-m}) = 1$, so the arithmetic progression $A' = \{a + \frac{d}{p^m} j\}_{j=0}^{M-1}$ in $\mathbb{Z}_{p^{n-m}}$ takes on every value precisely $\binom{M}{p^{n-m}}$ times. In particular, since any individual value in the range $[1, p^{k-m}]$ is taken on precisely $\binom{M}{p^{n-m}}$ times in A' , we see that A' takes on values in $[1, p^{k-m}]$ a total of $p^{k-m} \binom{M}{p^{n-m}}$ times, therefore A takes on values in the range $[1, p^k]$ a total of $p^{k-m} \binom{M}{p^{n-m}} < M$ times, from which the desired result follows. In particular, we notice that there must be some interval I of length p^{m-k-1} such that $C[p, n, k](a + jd) = 1$ for $j \in I$. To see why this is true, we notice that there are p^{k+1} intervals of the form $[cp^{n-k-1}, (c+1)p^{n-k-1} - 1]$, and at most p^k of them can contain some point g such that $C[p, n, k](g) = 0$.

We are now ready to introduce the inductive hypothesis. For any set $P = \{(p_i, n_i, k_i)\}_{i=1}^{t+1}$ where all p_i are prime, and n_i and k_i are positive integers for all i that satisfy $k_i < n_i$, let us define the 2-partition C_P of $D = \prod_{i=1}^t \mathbb{Z}_{p_i^{n_i}}$ given by $\prod_{i=1}^t C[p_i, n_i, k_i]$. To be more precise, we

have $C_P((x_i)_{i=1}^t) = (\sum_{i=1}^t C[p_i, n_i, k_i](x_i))$, where the summation takes place in \mathbb{Z}_2 , so that

the result is either a 0 or a 1. We will show that for any set $Q = \{(p_i, k_i)\}_{i=1}^t$ where the p_i are primes and the k_i are positive integers, there exist a set $\{n_i\}_{i=1}^t$ and some $M \in \mathbb{Z}^+$, such that for the set $P = \{(p_i, n_i, k_i)\}_{i=1}^t$, any M -term arithmetic progression $A = \{a + jd\}_{j=0}^{M-1}$ that is constant under C_P , must have $p_i^{k_i} \mid d_i$ for $1 \leq i \leq t$. We have already proven this to be true for the base case of $t = 1$, so let us assume that the hypothesis has been proven for $1 \leq s \leq t$, and that we are trying to prove it for $s = t + 1$. Consider $Q = \{(p_i, k_i)\}_{i=1}^{t+1}$, and $Q_t = \{(p_i, k_i + 1)\}_{i=1}^t$. By the inductive hypothesis, we can find $\{n_i\}_{i=1}^t$ satisfying the problem for Q_t , so we will now find n_{t+1} such that $\{n_i\}_{i=1}^{t+1}$ satisfies the problem for Q . Let $P_t = \{(p_i, n_i, k_i + 1)\}_{i=1}^t$, and let M_t be a positive integer such that if the arithmetic

progression $\{b + jf\}_{j=0}^{M_t-1}$ is constant under C_{P_t} , then $p_i^{k_i+1} \mid f_i$ for $1 \leq i \leq t$. Let $N = \prod_{i=1}^t p_i^{n_i}$ us take $n_{t+1} = M_t + k_{t+1} + 1$, and $M_{t+1} = p^{n_{t+1}} M_t N$.

Let $A = \{a + jd\}_{j=0}^{M_{t+1}}$ be an arithmetic progression, and let $q = v_{p_{t+1}}(\gcd(d_{t+1}, p^{n_{t+1}}))$. If $q \leq k_{t+1}$, then $C[P_{t+1}, n_{t+1}, k_{t+1}](a_{t+1} + jd_{t+1}) = 1$ for $j \in I$, for some interval I of length $p^{n_{t+1}-k_{t+1}-1} \geq M_t$, as seen by our analysis in the $t = 1$ case. If A was to be constant under C_{P_t} , then we must have $p_i^{k_i} \mid d_i$ for $1 \leq i \leq t$ according to the inductive hypothesis. We also see that $\{a_{t+1} + cd_{t+1} + jNd_{t+1}\}_{j=1}^{p^{n_{t+1}}M_t}$ is constant under $C[p_{t+1}, n_{t+1}, k_{t+1}]$, for any c , but this is a contradiction because c can be chosen such that $C[p_{t+1}, n_{t+1}, k_{t+1}](a_{t+1} + cd_{t+1}) = 0$ and $C[p_{t+1}, n_{t+1}, k_{t+1}](a_{t+1} + cd_{t+1} + Nd_{t+1}) = 1$. For our next case, we assume that $q = n_{t+1}$, so the $t + 1$ component of A is constant, and therefore the partition class of any point in A is determined by its first t coordinates. Since $M_{t+1} > M_t$, we see by the inductive hypothesis that we must have $p_i^{k_i} \mid d_i$ for all $1 \leq i \leq t$ as desired. For the final case we assume that $k_{t+1} < q < n_{t+1}$. If $\{a_{t+1} + jd_{t+1}\}_{j=0}^{M_t}$ is constant under $C[p_{t+1}, n_{t+1}, k_{t+1}]$, then we are done as we were in the previous situation. If this is not the case, then let c be such that $C[p_{t+1}, n_{t+1}, k_{t+1}](a_{t+1} + cd_{t+1}) = 0$. Since $q < n_{t+1}$, we see that $C[p_{t+1}, n_{t+1}, k_{t+1}](a_{t+1} + (c + j)d_{t+1}) = 1$ for some j , so let us take the minimal value of j that satisfies this and denote it by j_0 . We see that $C[p_{t+1}, n_{t+1}, k_{t+1}](a_{t+1} + (c + j_0)d_{t+1} + jp_{t+1}d_{t+1}) = 1$ for all j , so by the inductive hypothesis we have that $p_i^{k_i+1} \mid p_{t+1}d_i$ for all $1 \leq i \leq t$, so $p_i^{k_i} \mid d_i$ for all $1 \leq i \leq t + 1$ as desired.

Theorem 4.5: If S is a weakly t -Dimensionally 2-Large set, then for any $n \in \mathbb{Z}^+$, there exists $s \in S$ such that $n \mid s$.

Proof: Let the prime factorization for n be $n = \prod_{i=1}^m p_i^{k_i}$. For any 2-partition of \mathbb{Z}^{tm} , and any

$l \in \mathbb{Z}^+$, there must exist an arithmetic progression $A = \{a + jd\}_{j=0}^{l-1}$ where d is of the form $d = (\underbrace{d_1, \dots, d_1}_m, \underbrace{d_2, \dots, d_2}_m, \dots, \underbrace{d_t, \dots, d_t}_m)$, such that $(d_i)_{i=1}^t \in S$, since this is equivalent

to finding an S -A.P. in a t -dimensional subspace of \mathbb{Z}^{tm} . Consider $Q = \{(p_i, k_i)\}_{i=1}^m$ and $Q_t = \{(q_i, w_i)\}_{i=1}^{mt}$, where $(q_i, w_i) = (p_i \pmod{m}, k_i \pmod{m})$ for $1 \leq i \leq mt$. By Theorem 4.4, there exist a coloring C_P of \mathbb{Z}^{mt} , and $M \in \mathbb{Z}^+$, such that any monochromatic arithmetic progression of length M or more must satisfy $p_i^{k_i} \mid d_i$ for $1 \leq i \leq mt$. The structure of d shows us that $p_i^{k_i} \mid d_j$ for all $1 \leq i \leq m$, and any $1 \leq j \leq t$, from which it follows that S contains a multiple of n .

We can continue to search for the relationships between weakly n -dimensionally r -Large sets, and r -Large sets through the following series of conjectures.

Conjecture 4.6: If S is weakly n -dimensionally 2-Large, then S is a Bohr(1) set as defined

in chapter 8.

Conjecture 4.7: If a 2-Large set is partitioned into 2 sets, at least 1 of the sets will be weakly 2-dimensionally 2-Large.

Conjecture 4.8: If S is weakly n -dimensionally 2-Large, then S is weakly n -dimensionally Large.

Question 4.9: If S is weakly n -dimensionally 2-Large for some n , then is S weakly m -dimensionally Large for some $m < n$, and in particular, is S 2-Large?

While the ideas in this chapter may already seem to be multidimensional, we so far have only been examining arithmetic progressions, which are 1-dimensional subsets, but we can also look for structure in subsets of higher dimension.

Definition 4.10: Let $S \subseteq \mathbb{Z}^+$ be *m -weakly n -dimensionally Large*, if for any finite coloring of \mathbb{Z}^n and any positive integer k , there exist $a, d_1, \dots, d_m \in \mathbb{Z}^n$ with $\{d_i\}_{i=1}^m$ being a \mathbb{Q} -linearly independent set, and $s \in S$, such that the set $\{a + c_1sd_1 + \dots + c_msd_m \mid 0 \leq c_1, \dots, c_m \leq k\}$ is monochromatic.

We see from the definition that we must require $m \leq n$ in order for it to make sense. Furthermore, we once again, see from the Multidimensional Van der Waerden Theorem that $S = \mathbb{Z}^+$ is a m -weakly n -dimensionally Large set for any $m, n \in \mathbb{Z}^+$.

Question 4.11: If S is a m -weakly n -dimensionally Large set, is S also a m -weakly $(n+1)$ -dimensionally Large set?

Chapter 5

The Compactness Principle

Many of the ideas that we have been discussing, are set in the framework of the infinite. This is convenient from the perspective of dynamics and analysis, because it allows for the thought process of an analyst, in which you allow an ϵ of error, and show that you are within this error, if you “go far enough out”, and the infinite is needed in order to “go far enough out”. We can show that finitistic versions of these ideas exist as well, and they result from the compactness that will be assumed in the topological dynamics of chapter 8. We can use these finitistic formulations to show that a classification of “Large” sets (of any kind) purely from the perspective of the infinite, is an unreasonable expectation. Let us first recall the finitistic version of Van der Waerden’s Theorem, and see how we can obtain it as a consequence of the infinite version. The technique that we will use is discussed in its full generality in [9], and is referred to as “The Compactness Principle”. We will only use this technique in the scope of generalizations of Van der Waerden’s Theorem, so we will not discuss the most general form of this for the sake of simplicity.

Finitistic Van der Waerden Theorem: For any $k, r \in \mathbb{Z}^+$, there exists an integer $w(k, r)$, such that for any $n \geq w(k, r)$, and any r -coloring of $[1, n]$, there exists a k -term monochromatic arithmetic progression. The numbers $w(k, r)$ are known as the Van der Waerden numbers.

Proof: Assume for the sake of contradiction, that we may choose $k, r \in \mathbb{Z}^+$, such that $w(k, r)$ does not exist. For each $m \in \mathbb{Z}^+$, let C_m be an r -coloring of $[1, m]$ that does not contain a k -term monochromatic arithmetic progression. We will now construct an r -coloring C of \mathbb{Z}^+ that does not contain a k -term arithmetic progression, and this will contradict the infinite version of the Van der Waerden Theorem. Pick $s \in [0, r - 1]$, such that s appears infinitely many times in the set $\{C_m(1)\}_{m=1}^\infty$. Let $\{n_{1,k}\}_{k=1}^\infty$ be a subsequence of $\{m\}_{m=1}^\infty$, such that $C_{n_{1,k}}(1) = s$ for all k . Having defined C on $[1, t]$, and created the sequences $\{C_{n_{j,k}}\}_{k=1}^\infty\}_{j=1}^t$ satisfying $C_{n_{j,k}}(j) = C(j)$ for all j and k , we will proceed by induction to define $C(t+1)$ and create a subsequence $\{n_{t+1,k}\}_{k=1}^\infty$ of $\{n_{t,k}\}_{k=1}^\infty$ satisfying $C_{n_{t+1,k}}(t+1) = C(t+1)$ for all k . It

will follow from construction that $C_{n_{j,k}}(s) = C(s)$ for all $1 \leq s \leq j$, and any k . The base case of $t = 1$ has already been handled. Let $s \in [0, r - 1]$ be such that s appears infinitely many times in the set $\{C_{n_{t,k}}\}_{k=1}^{\infty}$, and let $\{n_{t+1,k}\}_{k=1}^{\infty}$ be a subsequence of $\{n_{t,k}\}_{k=1}^{\infty}$, such that $C_{n_{t+1,k}}(t+1) = s$ for all k . We have thus defined $C(t)$ for all $t \in \mathbb{Z}^+$. If $\{a + qd\}_{q=0}^{k-1}$ is a k -term arithmetic progression such that $C(a + qd) = C(a)$ for all $1 \leq q \leq k - 1$, then we note that $C_{n_{a+(k-1)d,1}}(s) = C(s)$ for any $1 \leq s \leq a + (q-1)d$, so $C_{n_{a+(k-1)d,1}}(a) = C_{n_{a+(k-1)d,1}}(a + qd)$ for any $1 \leq q \leq k - 1$, which contradicts the fact that $C_{a+(k-1)d,1}$ contains no monochromatic k -term arithmetic progression. ■

We note that we have not assumed any special properties about d , the common difference in the arithmetic progression. We have simply proven that if a monochromatic k -term arithmetic progression with common difference d , does not appear in any of the colorings $\{C_m\}_{m=1}^{\infty}$, then a monochromatic k -term arithmetic progression does not occur in the coloring C that was constructed in the above proof. This leads to the stronger theorem stated below.

Theorem 5.1: For a given $S \subseteq \mathbb{Z}^+$, let $k, r \in \mathbb{Z}^+$ be such that for any r -coloring of \mathbb{Z}^+ , there exists a monochromatic k -term S -A.P.. Then there exists an integer $w(S, k, r)$, such that for any $n \geq w(S, k, r)$, and any coloring of $[1, n]$, there exists a monochromatic k -term S -A.P..

Corollary 5.2: Let S be a Large set. Then for any $k, r \in \mathbb{Z}^+$, there exists an integer $w(S, k, r)$, such that for any $n \geq w(S, k, r)$, and any coloring of $[1, n]$, there exists a monochromatic k -term S -A.P..

We can easily adapt the proofs above to prove analogous statements for colorings of \mathbb{Z}^m , and Multidimensionally Large sets, but we shall do so from the viewpoint of topology instead to illustrate why this idea is called the Compactness Principle.

Theorem 5.3: For any integers $k, r, n \in \mathbb{Z}^+$, there exists an integer $w(k, r, n)$, such that for any $m \geq w(k, r, n)$, and any r -coloring of $\prod_{j=1}^n [1, m]$, there exists a monochromatic length k n -cube.

Proof: Let us assume for the sake of contradiction that the number $w(k, r, n)$ does not exist.

For each $m \in \mathbb{Z}^+$, let C'_m be an r -partition of $\prod_{j=1}^n [-m, m]$ that does not contain a length

k n -cube in any partition class, and let C_m be any extension of C'_m to an r -partition of all of \mathbb{Z}^n . Since Ω_r^n is compact, we see that the sequence $\{C_m\}_{m=1}^{\infty}$ must have at least 1 limit point C . Assume that (a, d) generates a length k n -cube \mathcal{C} , that is constant under C , and

let $N \in \mathbb{Z}^+$ be such that $\mathcal{C} \subseteq \prod_{j=1}^n [-N, N]$. Let M be such that $C_M \in B_{\frac{1}{N+1}}(C)$, so that \mathcal{C} is

constant under C_M as well, but this contradicts the fact that C_M does not contain a length k n -cube in any partition class. Since arithmetic progressions are shift invariant, we may take $w(k, r, n) = 2M$.

We once again notice that we did not assume any special properties about d , the gap of the n -cubes in the proof above. We have simply proven that if a monochromatic length k $\{d\}$ -gap n -cube does not appear in any of the colorings $\{C_m\}_{m=1}^\infty$, then any limit point C of the sequence also does not contain a monochromatic length k d -gap n -cube. This once again leads to a stronger theorem which is stated below.

Theorem 5.4: Let $S \subseteq \mathbb{Z}^+$ be m -dimensionally Large. Then there exists an integer $w(S, k, r, m)$, such that for any $n \geq w(S, k, r, m)$, and any r -coloring of $\prod_{j=1}^m [1, n]$, there exists a monochromatic length k S -gap m -cube.

Corollary 5.5: Let S be a Multidimensionally Large Set. For any $k, r, n \in \mathbb{Z}^+$, there exists an integer $w(S, k, r, n)$, such that for any $m \geq w(S, k, r, n)$, and any r -partition of $\prod_{j=1}^n [1, m]$, there exists a length k S -gap n -cube contained in a single partition class.

Before we can illustrate the usage of these Van der Waerden like numbers to construct Large sets, we require the following lemma.

Theorem 5.6: If $S \subseteq \mathbb{Z}^+$ and $k, r, n \in \mathbb{Z}^+$ are such that $w(S, k, r, n) < \infty$, then for any $c \in \mathbb{Z}^+$, we have $w(cS, k, r, n) = cw(S, k, r, n) - c + 1$

Proof: Let f be a r -partition of $\prod_{j=1}^n [1, w(S, k, r, n) - 1]$ that is not constant on any length k

S -gap n -cube. It follows that the r -partition f_c of $\prod_{j=1}^n [1, cw(S, k, r, n) - c]$ given by $f_c(x) = f(\lfloor \frac{x_j}{c} \rfloor)_{j=1}^n$ is not constant on any length k cS -gap n -cube, so $w(cS, k, r, n) \geq cw(S, k, r, n) - c + 1$. To obtain the reverse inequality, let f_c be any r -partition of $\prod_{j=1}^n [1, cw(S, k, r, n) - c + 1]$,

and define the r -partition f of $\prod_{j=1}^n [1, w(S, k, r, n)]$ by $f(x) = f_c((cx_j - c + 1)_{j=1}^n)$. Let $a \in \mathbb{Z}^n$

and $d \in S$ be such that (a, d) generates a length k S -gap n -cube that is constant under f , so $((ca_j - c + 1)_{j=1}^n, cd)$ generates a length k cS -gap n -cube that is constant under f_c , thus $w(cS, k, r, n) \leq cw(S, k, r, n) - c + 1$ as desired.

We are now ready to show that the union of an almost arbitrary family of finite sets to

create a Large set. The family cannot be totally arbitrary, because we may always take the family $\{S_j\}_{j=1}^\infty$ given by $S_j = \{2j + 1\}$ for all j , whose union is most certainly not a Large set. We will see that the family may be seen as arbitrary, in that the finite sets need not have any relationship with each other, but only need to satisfy the given conditions individually.

Theorem 5.7: Let $\{S_n\}_{n=1}^\infty$ be a collection of subsets of the positive integers. If for any $k, r \in \mathbb{Z}^+$, there exists n such that $w(S_n, k, r) < \infty$, then $S := \cup_{n=1}^\infty S_n$ is a large set.

Proof: We may assume without loss of generality that $w(S_n, n, n) < \infty$ by taking a sub-collection of $\{S_n\}_{n=1}^\infty$ if necessary. Let $k, r \in \mathbb{Z}^+$ be arbitrary, and consider $n = \max(k, r)$. $w(S, k, r) \leq w(S, n, n) \leq w(S_n, n, n) < \infty$, as desired. ■

Corollary 5.8: Let $\{S_n\}_{n=1}^\infty$ be a collection of subsets of the positive integers, such that for any $k \in \mathbb{Z}^+$, there exists n such that $w(S_n, k, k) < \infty$. Then for any sequence $\{c_n\}_{n=1}^\infty$ of positive integers, the set $S := \cup_{n=1}^\infty c_n S_n$ is Large.

Proof: Let us assume once again that $w(S_n, n, n) < \infty$. Since $w(cS, k, r) = cw(S, k, r) - c + 1$ for any $c \in \mathbb{Z}^+$, we see that $w(c_n S_n, n, n) < \infty$ for all n , so we may simply apply the previous Theorem. ■

To illustrate the pathology that can arise when studying Large sets, consider the following example. Consider $B_{m,n} := \{k^n\}_{k=m}^\infty$, which is a large set for any $m, n \in \mathbb{Z}^+$. Let C_1 be given by $C_1 := B_{1,2} \cap [1, w(B_{1,2}, 2, 2)]$, and let $c_1 := \max(C_1)$. Having defined $\{C_n\}_{n=1}^k$ and $\{c_n\}_{n=1}^k$, such that $w(C_n, n + 1, n + 1) < \infty$ and $c_n = \max(C_n)$ for all n , let us inductively define by C_{k+1} by $C_{k+1} = B_{c_k, k+2} \cap [1, w(B_{c_k, k+2}, k + 2, k + 2)]$, and c_{k+1} by $c_{k+1} = \max(C_{k+1})$. By Theorem 5.7, we see that $C := \cup_{k=1}^\infty C_k$ is a Large set, but there is no obvious global structure of this set, there only exists the local structure from which it was constructed. We may also apply corollary 5.8 to the sets $\{C_n\}_{n=1}^\infty$ for any sequence $\{c'_n\}_{n=1}^\infty$ of our choosing, to create a Large set whose structure is even further masked.

In the opposite direction of the preceding theorems, we see that the Van der Waerden like numbers can be used not only to construct Large sets, but to create meaningful decompositions of Large sets as well. We already know that there exist disjoint Large sets as a result of the Polynomial Van der Waerden Theorem. To see this, simply take the polynomials $p_1(x) = x^2$ and $p_2(x) = x^2 + x$, and note that the sets $S_1 = p_1(\mathbb{Z}^+)$ and $S_2 = p_2(\mathbb{Z}^+)$ are disjoint Large sets. It follows that we cannot find some “minimal” Large set that is contained in all other Large sets, but we can still search for a “minimal” Large subset of any given Large set. Even this task is not trivial to formulate, because we know that removing any finite set from a Large set will still leave us with a Large set, so what would happen if we were to remove some finite set from the “minimal” Large subset that we found? In fact, we already know that Large sets are strongly partition regular, so we could always finitely partition any “minimal” Large set, and still find a Large set within 1 of the partition classes. The following theorem further suggests that it is futile to search for a “minimal”

Large subset that contains the crucial structure of a given Large set.

Theorem 5.9: Any Large set S can be partitioned into an infinite number of disjoint Large sets.

Proof: First, consider an infinite partition of the integers $\{A_k\}_{k=1}^{\infty}$, such as that given by $A_k = \{m \mid m \equiv 2^{k-1} \pmod{2^k}\}$ for all k . Consider $S_1 := S \cap [1, w(S, 2, 2)]$, and note that $w(S_1, 2, 2) < \infty$. Since S_1 is a finite set, it is not Large, so $S'_1 := S - S_1$ is a Large set. Having defined $\{S_n\}_{n=1}^k$ and $\{S'_n\}_{n=1}^k$ such that $w(S_n, n+1, n+1) < \infty$ for all n , and S'_n is Large for all n , we inductively define S_{k+1} and S'_{k+1} by $S_{k+1} := S'_k \cap [1, w(S'_k, k+2, k+2)]$ and $S'_{k+1} = S'_k - S_{k+1}$. Consider the family of sets $\{B_n\}_{n=1}^{\infty}$ given by $B_n = \cup_{k \in A_n} S_k$ for all n . We see that $\{B_n\}_{n=1}^{\infty}$ is an infinite partition of S . Furthermore, we see that for any $k, n \in \mathbb{Z}^+$, we have that $w(B_n, k, k) \leq w(S_{2^{n-1}+k2^n}, k, k) \leq w(S_{2^{n-1}+k2^n}, 2^{n-1}+k2^n+1, 2^{n-1}+k2^n+1) < \infty$, so by Theorem , B_n is a Large set for any n . ■

Theorem 5.10: Any set S of topological recurrence, can be partitioned into an infinite number of disjoint sets of topological recurrence.

Theorem 5.11: Any Multidimensionally Large set S can be partitioned into an infinite number of disjoint Multidimensionally Large sets.

These Theorems imply that there cannot exist a “smallest” set of recurrence in any reasonable sense, as it can always be broken down into disjoint sets of recurrence.

Despite the pathology that we have found, these Van der Waerden like numbers give us an approach to solving the 2-Large is Large Conjecture. Let $S \subseteq \mathbb{Z}^+$ be arbitrary. We see that if for each $k \in \mathbb{Z}^+$, there exists $k' \in \mathbb{Z}^+$ independent of S , such that $w(S, k, k)$ exists when $w(S, k', 2)$ exists, then all 2-Large sets will be Large. Conversely, if there exist $k \in \mathbb{Z}^+$ and $S \subseteq \mathbb{Z}^+$, such that $w(S, k, k) = \infty$, but for any $k' \in \mathbb{Z}^+$ we have $w(S, k', 2) < \infty$, then S would be a set that is 2-Large, but not Large. These ideas are not novel restatements of the 2-Large is Large conjecture, but the benefit of the Van der Waerden like numbers is that they can guide us as to how to construct a 2-Large set that is not Large. Suppose that for each $k \in \mathbb{Z}^+$, we can find a set $S_k \subseteq \mathbb{Z}^+$, such that $w(S_k, k, 2) < \infty$, and $w(S_k, 3, 3) = \infty$. We see that $S = \cup_{k=2}^{\infty} S_k$ is a 2-Large set, and we would like to say that $w(S, 3, 3) = \infty$ as well so that S is not 3-Large, but there is no reason to expect this to be the case. However, with the aid of Theorem 5.8, we will show how to achieve this expectation in the following theorem, but first, we require some notation. Given r -partitions f and g of $[1, n]$ and $[1, m]$ respectively, let $f + g$ denote the r -Partition of $[1, n + m]$ given by $(f + g)(x) = f(x)$ if $1 \leq x \leq n$, and $(f + g)(x) = g(x - n)$ if $n < x \leq n + m$. Furthermore, let $f + 1$ denote the r -partition of $[1, n]$ given by $(f + j)(x) = (f(x) + j) \pmod{r}$.

Theorem 5.12: Let $k', r', r \in \mathbb{Z}^+$ be arbitrary. Suppose that for any $k \in \mathbb{Z}^+$, there exists a finite set $S_k \subseteq \mathbb{Z}^+$ such that $w(S_k, k, r) < \infty$ and $w(S_k, k', r') = \infty$. Then there

exists a sequence $\{c_k\}_{k=1}^{\infty}$ such that the set $S = \cup_{k=2}^{\infty} c_k S_k$ is a r -Large set that also satisfies $w(S, 2k' + 1, r') = \infty$, and is consequently not a r' -Large set.

Proof: Let $M_2 = \max(S_2)$, and let $l_2 = w(S_2, 2, r) + (k' + 1)M_2 + 1$. Let f_2 be any r' -partition of $[1, l_2]$ that does not contain a length $(k' + 1)$ S_2 -A.P.. Let g_2 be the r' -partition of $[1, l_2]$ that coincides precisely with f_2 , $c_2 = 1$, and $S'_2 = S_2$. We now proceed by induction to define M_n, l_n, c_n, S'_n, f_n , and g_n with g_n being constant on no length $k' + 1$ $c_n S_n$ -A.P., with the base case of $n = 2$ having already been handled. Let $c_{n+1} = l_n$, $M_{n+1} = \max(c_{n+1} S_{n+1})$, $l_{n+1} = w(S_{n+1}, n + 1, r) + (k' + 1)M_{n+1} + 1$, and let f_{n+1} be any r' -partition of $[1, l_{n+1}]$ that does not contain a length $(k' + 1)$ S_{n+1} -A.P.. Let $S'_{n+1} = S'_n \cup c_{n+1} S_{n+1}$. Let $g_{n+1} = \sum_{j=1}^{l_{n+1}} (g_n + f_{n+1}(j))$.

We will show that g_{n+1} does not contain a length $(2k' + 1)$ S'_{n+1} -A.P.. We see by construction that g_{n+1} does not contain a length $(k' + 1)$ $c_{n+1} S_{n+1}$ -A.P., and for $1 \leq i \leq n$, we see that $g_{n+1} = \sum_{j=1}^N (g_i + h_i(j))$, for some $N \in \mathbb{Z}^+$, and some set of functions $\{h_i\}_{i=1}^n$. It follows that any length $(2k' + 1)$ $c_i S_i$ -A.P. that is constant under g_{n+1} , must have at least $(k' + 1)$ terms that are contained in a block of the form $g_i + h_i(j)$, since the length of each such block is larger than $(k' + 1) \max(S_i)$, but this contradicts the induction hypothesis, so the claimed $(2k' + 1)$ term arithmetic progression does not exist. Now let $S = \cup_{j=1}^{\infty} S'_j$. We see that S is r -Large by construction. We also see that for any $n \in \mathbb{Z}^+$, there is a r' -partition of $[1, n]$ that does not contain a length $(2k' + 1)$ S -A.P., so $w(S, 2k' + 1, r') = \infty$, and S is not r' -Large.

It is also natural to wonder what the relationship is between the Van der Waerden numbers of different dimensions. Theorem 5.13 below shows that higher dimensional Van der Waerden numbers are at least proportional to the 1-dimensional Van der Waerden numbers, where we compare $w(\mathbb{Z}^+, k, r, n)$ with $w(\mathbb{Z}^+, k^n, r)$ instead of $w(\mathbb{Z}^+, k, r)$ because a length k n -cube contains k^n points.

Theorem 5.13: If S is a Large set, then for any $k, r \in \mathbb{Z}^+$, we have that $w(S, k, r, n) \geq \frac{k-1}{k^n-1} w(S, k^n, r)$.

Proof: Let C be a coloring of $[1, w(k^n, r, S)]$ that contains no monochromatic k^n -term S -A.P., and define the coloring C_n of $\prod_{i=1}^n [1, \lfloor \frac{k-1}{k^n-1} w(S, k^n, r) \rfloor]$ by $C_n(x_1, \dots, x_n) = C(\sum_{i=1}^n x_i k^{i-1})$. Suppose for the sake of contradiction that there exists a monochromatic length k S -gap n -cube in C_n generated by the base point $a \in \mathbb{Z}^n$ and the common difference $d \in S$. If $a = (a_1, \dots, a_n)$, then for any $(j_{1,1}, j_{1,2}, \dots, j_{1,n}), (j_{2,1}, j_{2,2}, \dots, j_{2,n}) \in [0, k-1]^n$, we have that $C((\sum_{i=1}^n a_i) + (\sum_{i=1}^n j_{1,i} d k^{i-1})) = C_n(a + \sum_{i=1}^n j_{1,i} d e_i) = C_n(a + \sum_{i=1}^n j_{2,i} d e_i) = C((\sum_{i=1}^n a_i) +$

$(\sum_{i=1}^n j_{2,i} dk^{i-1}))$, so $\{(\sum_{i=1}^n a_i) + dj\}_{j=0}^{k^n-1}$ would be a monochromatic k^n -term S -A.P. in C , which yields the desired contradiction.

We note that Theorem 5.13 tells us that it is much harder to find higher dimensional structure in partitions of higher dimensional lattices. To find a structure of size k^n in an r -partition of \mathbb{Z}^+ , it suffices to examine the interval $[1, w(k^n, r)]$, but to find a n -dimensional structure of k^n points in \mathbb{Z}^n , we must examine a length $w(k, r, n)$ n -cube, which contains at least $w(k, r, n)^n \geq (\frac{k-1}{k^n-1} w(k^n, r))^n$ points. We close this chapter with the following question as to whether or not Van der Waerden like numbers are sufficient for distinguishing Large sets.

Question 5.14: If S_1 and S_2 are Large sets such that $w(S_1, k, r) = w(S_2, k, r)$ for all $k, r \in \mathbb{Z}^+$, then must we have $S_1 = S_2$? If not, what can be said of the relationship of S_1 and S_2 if we further suppose that $w(S_1, k, r, n) = w(S_2, k, r, n)$ for all $k, r, n \in \mathbb{Z}^+$?

Chapter 6

A Closure Operation

It is shown in [1], that if $A \subseteq \mathbb{Z}^+$ is not 2-Large, then $A \cup [1, n]$ and nA are not 2-Large sets for any $n \in \mathbb{Z}^+$. Furthermore, there exist large sets which are disjoint, and as we have seen in the previous chapter, any Large set can be partitioned into an infinite number of disjoint Large sets. For these reasons, it is difficult to understand what is the inherent structure that makes a set Large or not. In an attempt to remedy this problem, we develop a closure operation on the space Ω_2 , and a closure operation on $\mathcal{P}(\mathbb{N})$.

Let $\mathbb{S} \subset \mathcal{P}(\mathbb{N})$ be a family of sets that are not 2-large. We define $f(\mathbb{S})$ to be the maximal (with respect to inclusion) subset of Ω_2 , such that for any $S \in \mathbb{S}$, we have that any $\chi \in f(\mathbb{S})$ is a 2-validating coloring for S . Similarly, if $\hat{\chi} \subset \Omega_2$ is a set of finite partitions of \mathbb{Z}^+ , we can define $g(\hat{\chi})$ to be the maximal family of subsets of \mathbb{N} , such that for any $\chi \in \hat{\chi}$, we have that χ is a 2-validating coloring for every $S \in g(\hat{\chi})$. Let us define $h : \mathbb{P}(\mathbb{N}) \rightarrow \mathbb{P}(\mathbb{N})$ by $h := g \circ f$, and $k : \Omega_2 \rightarrow \Omega_2$ by $k := f \circ g$.

Theorem 6.1: $g \circ k \equiv g$ and $f \circ h \equiv f$.

Proof: Let $\hat{\chi} \subset \Omega_2$ and $\mathbb{S} \subset \mathbb{P}(\mathbb{N})$ both be arbitrary. First, we note that f and g are both decreasing functions, in the sense that if $\hat{\chi} \subseteq \hat{\chi}'$ and $\mathbb{S} \subseteq \mathbb{S}'$, then $g(\hat{\chi}') \subset g(\hat{\chi})$ and $f(\mathbb{S}') \subseteq f(\mathbb{S})$. Next, we note that h and k are both increasing functions in the analogous sense. Furthermore, we see that $\mathbb{S} \subseteq h(\mathbb{S})$ and $\hat{\chi} \subseteq k(\hat{\chi})$. To see this, let $S \in g(\hat{\chi})$ be arbitrary, and we note that by the definition of g , any element of $\hat{\chi}$ is a 2-validating coloring for S , but S was arbitrary, so $\hat{\chi} \subset f(g(\hat{\chi})) = k(\hat{\chi})$. Similarly, let $\chi \in f(\mathbb{S})$ be arbitrary, and we note that by the definition of f , χ is a 2-validating coloring for any element of \mathbb{S} , so $\mathbb{S} \subseteq g(f(\mathbb{S})) = h(\mathbb{S})$. Now, let us prove that $g \circ k \equiv g$. Since $\hat{\chi} \subseteq k(\hat{\chi})$, we see that $g(\hat{\chi}) \supseteq g(k(\hat{\chi}))$, so we now only need to show that $g(\hat{\chi}) \subseteq g(k(\hat{\chi}))$. To see this, we note that $g(\hat{\chi}) \subseteq h(g(\hat{\chi})) = g(f(g(\hat{\chi}))) = g(k(\hat{\chi}))$. Next, we can prove that $f \circ h \equiv f$ with a similar method of proof. Since $\mathbb{S} \subseteq h(\mathbb{S})$, we see that $f(\mathbb{S}) \supseteq f(h(\mathbb{S}))$, and as before, we only need to show that $f(\mathbb{S}) \subseteq f(h(\mathbb{S}))$. To see this, we note that $f(\mathbb{S}) \subseteq k(f(\mathbb{S})) = f(g(f(\mathbb{S}))) = f(h(\mathbb{S}))$,

which yields the desired result. ■

Corollary 6.2: k and h are idempotent operations.

Since k and h are idempotent operations, they are good candidates for closure operations. To see an example of how these operations work, consider the set $S = (2\mathbb{Z}^+ - 1) - \{9^j\}_{j=1}^\infty$. If $\chi \in f(S)$ is constant on a length k arithmetic progression with common difference 9^j , then χ is constant on a length $\lfloor \frac{k}{3} \rfloor$ arithmetic progression of common difference $3 * 9^j$. Since χ is not constant on arbitrarily long arithmetic progressions with common difference of the form $3 * 9^j$, we see that χ is not constant on arbitrarily long arithmetic progressions with common difference of the form 9^j , so $(2\mathbb{Z}^+ - 1) \subseteq h(S)$. To obtain the reverse inequality, let χ denote the 2-partition of \mathbb{Z}^+ given by putting the even and the odd numbers in different partition classes. We certainly have that $\chi \in g(S)$, so no even number can be contained in $h(S)$, thus $h(S) = (2\mathbb{Z}^+ - 1)$. We see in this case that the operation h has filled in the unnatural holes given by the powers of 9 being removed from the odd numbers. Now to show that the union of 2 sets $A, B \subseteq \mathbb{Z}^+$ is not 2-Large when neither of A and B are 2-Large, it suffices to show that $g(A) \cap g(B) \neq \emptyset$. However, since $g \circ h \equiv g$, we only need to show that $g(h(A)) \cap g(h(B)) \neq \emptyset$, which will hopefully be an easier task to accomplish, since it would be easier to construct pleasant elements of $g(h(A))$ in comparison to $g(A)$, given that $h(A)$ should have more of its inner structure revealed.

We see from corollary 6.2 and the fact that $A \subseteq h(A)$ for all $A \in \mathbb{P}(\mathbb{N})$, that h behaves like a closure operation on $\mathbb{P}(\mathbb{N})$. Similarly, we see that k behaves like a closure operation on Ω_2 . The duality between h and k can be seen from the relations $g \circ k \circ f \equiv g \circ f \equiv h$ and $f \circ h \circ g \equiv f \circ g \equiv k$, both of which follow immediately from theorem 6.1.

We see that we can extend these ideas to study other classes of Large sets by defining f_r^t , g_r^t , h_r^t and k_r^t as we did f , g , h and k , by simply replacing Ω_2 with Ω_r^t , and 2-validating with $t^* - r$ -validating. We can see that $g_r^t \circ h_r^t \equiv g_r^t$ and $f_r^t \circ k_r^t \equiv f_r^t$, since the proof of $g \circ h \equiv g$ and $f \circ k \equiv f$ was independent of the number of partition classes, or the dimension of the integer lattice being partitioned. We list some possible directions for future research with these ideas stated with f , g , h , and k for simplicity. Can we create a σ -algebra of measurable sets on ω_2 such that for any $S \subset \mathbb{N}$, we can show that $f(S)$ is a measurable set? If this is possible, we can define how small a set is with the function $T : \mathbb{P}(\mathbb{N}) \rightarrow [0, 1]$ given by $T(S) = 1 - T(f(S))$. Even if $f(S)$ is not always measurable, for 2 subsets $S_1, S_2 \subset \mathbb{N}$, we can define the partial ordering $S_1 \leq S_2$ if and only if $f(S_2) \subset f(S_1)$. We can also define the equivalence relation $S_1 \sim S_2$ if and only if $f(S_1) = f(S_2)$, which is equivalent to requiring $h(S_1) = h(S_2)$. What do these relations tell us about the structure of sets that are not 2-Large? Can we show that $h_3(S) \subsetneq h(S)$ for any $S \subset \mathbb{N}$? This question is in some sense an analogue to the 2-Large is Large conjecture, since it is an attempt to measure if searching for arithmetic progressions in a 3-partition is truly more restrictive than searching for arithmetic progressions in a 2-partition.

Chapter 7

Canonical Ramsey Theory

In all of our discussions so far, we have assumed that the number of partition classes is finite, so it is natural to wonder what happens when we allow for an infinite number of partition classes. The first partition that comes to mind is allowing each integer to have its own partition class, so that we cannot even find a 2-term arithmetic progression in any partition class. However, in this particular scenario, another interesting event occurs, we find arbitrarily long Rainbow arithmetic progressions in which each term is in a different partition class from any other term in the progression. Graham and Erdos have proved that this is the only other scenario that needs to be considered in what they call the Canonical Van der Waerden Theorem.

Canonical Van der Waerden Theorem: For any (not necessarily finite) coloring C of the positive integers, and any $k \in \mathbb{Z}^+$, there either exists a monochromatic progression of length k in C , or a rainbow progression of length k in C .

A copy of the original proof of this Theorem is provided in [10], but a much simpler elementary proof is provided in [11]. We define a Canonically Large set, to be the analog of a Large set in this new setting. To be more precise, we give the following definition.

Definition: A subset S of the positive integers is Canonically Large, if for any coloring of \mathbb{Z}^+ , there are arbitrarily long monochromatic S -A.P.s, or arbitrarily long rainbow S -A.P.s.

It is natural to wonder if we can apply the compactness principle in this situation to get Canonical Van der Waerden numbers. In particular, we want to prove the following Theorem.

Theorem 7.1: For any $k \in \mathbb{Z}^+$, there exists an integer $N(k)$, such that for any $n \geq N(k)$, and any coloring of $[1, n]$, there either exists a monochromatic k -term arithmetic progression, or a rainbow k -term arithmetic progression.

To prove this, we cannot simply apply the compactness principle, because symbolic space on an infinite number of characters is not a compact topological space. To illustrate some of the issues that can occur, let $\{p_n\}_{n=1}^\infty$ be any enumeration of the primes, and let the coloring C_n of \mathbb{Z}^+ be given by $C_n(j) = p_n^j$, for all $n, j \in \mathbb{Z}^+$. It is intuitively obvious that the sequence of colorings $\{C_n\}_{n=1}^\infty$ is really a constant sequence of colorings that converges to the coloring in which no 2 elements share the same color, but this intuition has been shrouded by the names of the colors. In the proof below, we will demonstrate how to circumvent this problem.

Proof of Theorem 7.1: Assume for the sake of contradiction that $N(k)$ does not exist for some $k \in \mathbb{Z}^+$. For each $n \in \mathbb{Z}^+$, let C'_n be a coloring of $[1, n]$ that does not contain any k -term monochromatic arithmetic progression, or any k -term rainbow arithmetic progression. For each $n \in \mathbb{Z}^+$, we create a coloring C_n from C'_n as follows. Let the partition classes of C_n be precisely the same as those of C'_n , but relabel the colors of C_n , such that the label of a partition class is determined by its smallest element, i.e., the partition class containing 1 is labeled with the color 1, the partition class containing the smallest integer that is not contained in the color 1, is labeled with the color 2, and so forth. We now see that for any $m \geq n$, C_m is a n -coloring of $[1, n]$, so we are now ready to proceed as we did in the proof of the existence of the Van der Waerden numbers. We will construct a coloring C of \mathbb{Z}^+ from the sequence of colorings $\{C_n\}_{n=1}^\infty$. Since $C_n(1) = 1$ for all n , let $C(1) = 1$. Let $t \in \{1, 2\}$ be such that the set $S_2 := \{n \geq 1 \mid C_n(2) = t\}$ is infinite, and define $C(2) = t$. Having defined $C(j)$ for $1 \leq j \leq k$, and constructed a descending sequences of sets $\{S_j\}_{j=2}^k$ satisfying $C_n(j) = C(j)$ for all $n \in S_j$, we will proceed inductively to construct S_{k+1} and define $C(k+1)$. Let $t \in [1, k+1]$ be such that the set $S_{k+1} := \{n \in S_k \cap [k+1, \infty) \mid C_n(k+1) = t\}$ is infinite, and define $C(k+1) = t$ to obtain the desired result. By the Canonical Van der Waerden Theorem, let $\{a + qd\}_{q=0}^{k-1}$ be a k -term arithmetic progression that is either monochromatic or rainbow under C . Let $n \in S_{a+(k-1)d}$ be arbitrary. Since $C_n(a + qd) = C(a + qd)$ for $0 \leq q \leq k-1$, we see that $\{a + qd\}_{q=0}^{k-1}$ is also a monochromatic or rainbow arithmetic progression under C_n as well, which is a contradiction. ■

We notice as in the proof of the finitistic Van der Waerden Theorem, that we have not made any special assumptions about the common difference d of the arithmetic progressions we are examining. What we have actually proven, is the following theorem.

Theorem 7.2: If S is a Canonically Large set, then for any $k \in \mathbb{Z}^+$, there exists an integer $N(S, k)$, such that for any $n \geq N(S, k)$, and any coloring of \mathbb{Z}^+ , there either exist arbitrarily long monochromatic S -A.P.s, or arbitrarily long rainbow S -A.P.s.

Theorem 7.3: If S is a Canonically Large set, then S can be partitioned into infinitely many disjoint Canonically Large sets.

This leads us to wonder if Canonically Large sets are strongly partition regular as Large

sets and Multidimensionally Large sets are, but this is not yet known, which gives us the following conjecture.

Conjecture 7.4: The family of Canonically Large sets is strongly partition regular.

We also note that we can show that the set $n\mathbb{Z}^+$ is Canonically Large for any n , but other than this, there are no known examples of Canonically Large sets. Are all Large sets also Canonically Large? Can we show that IP sets are Canonically Large? What about the polynomial image of the positive integers for some nice polynomial? In particular, if $p(x) \in \mathbb{Q}[x]$ is an integral polynomial that satisfies $p(0) = 0$, then is $p(\mathbb{Z}^+)$ a Canonically Large set? Using Theorem 5.8, we can give a method analogous to that of Theorem 5.12, which instructs us on how to try and construct a Large set that is not Canonically Large. Once again, given partitions f and g of $[1, n]$ and $[1, m]$ respectively, we will define the partition $f + g$ of $[1, n + m]$ by $(f + g)(x) = f(x)$ if $1 \leq x \leq n$, and $(f + g)(x) = g(x)$ if $n < x \leq n + m$. We will also define for any $j \in \mathbb{Z}$ the partition $f + j$ by $(f + j)(x) = f(x) + j$, where we are assuming that the partition classes are labeled by integers.

Theorem 7.5: Let $k_0 \in \mathbb{Z}^+$ be such that for any $k \in \mathbb{Z}^+$, there exists a finite set $S_k \subseteq \mathbb{Z}^+$ satisfying $w(S_k, k, k) < \infty$ and $N(S_k, k_0) = \infty$. Then there exists some $\{c_k\}_{k=1}^\infty \subseteq \mathbb{Z}^+$ such that $S := \cup_{k=1}^\infty c_k S_k$ is a Large set satisfying $N(S, 2k_0 - 1) = \infty$, and is consequently not a Canonically Large set.

Proof: By Theorem 5.8, we see that S will be a Large set regardless of what values are chosen for the sequence $C = \{c_k\}_{k=1}^\infty$, so we only need to specify the values of C to ensure that S is not Canonically Large. Let $M_2 = \max(S_2)$, let $l_2 = w(S_2, 2, 2) + (k_0 + 1)M_2 + 1$, let f_2 be any partition of $[1, l_2]$ that does not contain a length k_0 canonical S -A.P., let g_2 be the partition of $[1, l_2]$ that coincides precisely with f_2 , let $c_2 = 1$, and $S'_2 = S_2$. We now proceed by induction to define M_n, l_n, c_n, S'_n, f_n , and g_n with g_n containing no canonical length k_0 $c_n S_n$ -A.P., with the base case of $n = 2$ having already been handled. Let $c_{n+1} = l_n$, let $M_{n+1} = \max(c_{n+1} S_{n+1})$, let $l_{n+1} = w(S_{n+1}, n + 1, n + 1) + (k' + 1)M_{n+1} + 1$, and let f_{n+1} be any partition of $[1, l_{n+1}]$ that does not contain a canonical length k_0 S_{n+1} -A.P.. Let

$S'_{n+1} = S'_n \cup c_{n+1} S_{n+1}$. Let $g_{n+1} = \sum_{j=1}^{l_{n+1}} (g_n + f_{n+1}(j))$. We will show that g_{n+1} does not contain a canonical length $2k_0 + 1$ S'_{n+1} -A.P.. We see by construction that g_{n+1} does not contain a

canonical length k_0 $c_{n+1} S_{n+1}$ -A.P., and for $1 \leq i \leq n$, we see that $g_{n+1} = \sum_{j=1}^{N_i} (g_i + h_i(j))$, for

some $N_i \in \mathbb{Z}^+$, and some set of functions $\{h_i\}_{i=1}^n$. It follows that any canonical length $2k_0 - 1$ $c_i S_i$ -A.P. that is contained in g_{n+1} , must have at least k_0 terms that are contained in a block of the form $g_i + h_i(j)$, since the length of each such block is larger than $(k' + 1) \max(S_i)$, but this contradicts the induction hypothesis, so the claimed $(2k_0 - 1)$ term arithmetic progression does not exist. Now let $S = \cup_{j=1}^\infty S'_j$. We see that $N(S, 2k_0 - 1) = \infty$, so S is

not Canonically Large.

In addition to examining Canonically Large sets, we may try to generalize the Canonical Van der Waerden Theorem by polynomializing it in a manner similar to that of the Polynomial Van der Waerden Theorem.

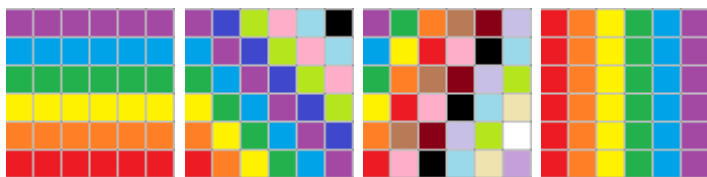
Canonical Polynomial Van der Waerden Conjecture: Let $\{p_i(x)\}_{i=1}^n$ be a family of polynomials with integer coefficients satisfying $p_i(0) = 0$ for all $1 \leq i \leq n$. Then for any coloring of \mathbb{Z}^+ , there exist $a, d \in \mathbb{Z}^+$, such that $\{a + p_i(d)\}_{i=1}^n$ is either monochromatic, or rainbow.

We would also like to state a Canonical Multidimensional Polynomial Van der Waerden Conjecture, but that means we must first state the Canonical Multidimensional Van der Waerden Theorem. The first proof of this theorem can be found in [10], but an alternative elementary proof can be found in [11].

Canonical Multidimensional Van der Waerden Theorem: For any $n \in \mathbb{Z}^+$, and any partition \mathcal{C} of \mathbb{Z}^n , at least 1 of the following 3 possibilities must occur.

- (1) There exist arbitrarily large n -cubes that are constant under \mathcal{C} .
- (2) There exist arbitrarily large n -cubes that are rainbow under \mathcal{C} .
- (3) There exists a linear subspace S of \mathbb{Z}^n satisfying the following. For each $k \in \mathbb{Z}^n$, there exists a length k n -cube denoted by T_k , such that for any points $p, q \in T_k$, we have $\mathcal{C}(p) = \mathcal{C}(q)$ if and only if $p - q \in S$.

Below are some pictures of the partitions described in situation (3) in \mathbb{Z}^2 .



What would be a good formulation for the Canonical Multidimensional Polynomial Van der Waerden Theorem? To see why this is much harder to formulate, we note that an equivalent formulation of the Canonical Van der Waerden Theorem, is that for any finite set of points $F \subset \mathbb{Z}^+$, and any coloring C of \mathbb{Z}^+ , there exist $a, d \in \mathbb{Z}^+$, such that $a + dF$ is either monochromatic or rainbow. This is analogous to the formulation of Van der Waerden's Theorem we used to see why squares are the appropriate structures to seek in the Multidimensional Van der Waerden Theorem. However, this statement fails to generalize appropriately to the Canonical Multidimensional Van der Waerden Theorem, since the

canonical coloring obtained from a given coloring of \mathbb{Z}^m , may be constant along translates of some subspace, so the structure of the points in our finite set F play a significant role.

To see the issues that can arise, let F be a set of 4 points in general position in \mathbb{Z}^3 . We note that for any 3 points in F , there exists a plane passing through them, so there exist colorings of \mathbb{Z}^3 , for which the induced canonical coloring on F results in precisely 3 points having the same color. Similarly, for any 2 points in F , there is a line passing through them, so there exists colorings of \mathbb{Z}^3 , for which the induced canonical coloring on F results in precisely 1 set of 2 points having the same color. We may of course still find colorings of \mathbb{Z}^3 , for which the induced canonical coloring on F is monochromatic, or rainbow, and lastly, we may find colorings of \mathbb{Z}^3 , for which the induced canonical coloring on F results in 2 points $\{f_1, f_2\}$ of a given color, and 2 points $\{f_3, f_4\}$ of another color, if and only if the line passing through f_1 and f_2 , is parallel to the line passing through f_3 and f_4 . We see now that depending on the geometry of the points in F , we can have the induced canonical coloring of F be anything we would like, which makes the Canonical Multidimensional Van der Waerden Theorem difficult to state for an arbitrary finite set of points. Consequently, if we cannot state the Canonical Multidimensional Polynomial Van der Waerden Theorem for an arbitrary set of integer coefficient polynomials $\{p_i(x)\}$ satisfying $p_i(0) = 0$ for all i , then for what sets of polynomials do we get a nice statement of the Theorem?

Chapter 8

Topological Dynamics

A dynamical system is a pair (X, T) , where X is a compact topological space, and $T : X \rightarrow X$ is a homeomorphism. Some sources only require that T be a continuous map, but we shall see in the proofs below, that for our purposes it suffices to restrict ourselves to the case in which T is a homeomorphism. Nonetheless, to be consistent with the notation of other sources, many of our theorems will be stated using the action of T^{-1} instead of T , which is only necessary when T is assumed to be continuous, but not necessarily a homeomorphism. We shall soon see that there is a strong interplay between many of the “Large” sets that were discussed in the previous chapters, and topological dynamics.

In dynamics, we generally take some point $x \in X$, and examine the set $\{T^n(x)\}_{n=1}^{\infty}$, which is called the orbit of x under T . We want to see the effect of iterating T on X by examining the properties of the iterations on any given point x . It is also often times the case in topological dynamics that we study the orbit of some open set O being acted on by T , or T^{-1} . One of the first questions we can ask is, must the orbit of a given point x return arbitrarily close to x ? This is equivalent to requiring that for any open set O , there exists some $n \in \mathbb{Z}^+$ such that $O \cap T^{-n}(O) \neq \emptyset$. We see that this is not the case by considering the dynamical system (X, T) , in which X is the closed unit ball in \mathbb{R}^2 under the standard topology, and T is the map given by $T(x, y) = \frac{1}{2}(x, y)$. We see that any point that is not the origin, and any open set that does not contain the origin, fail to satisfy the desired properties. The problem can be attributed to the fact that for any open set O about the origin, there is some $n \in \mathbb{Z}^+$ such that $T^n(X) \subseteq O$, so most regions of our space X are completely ignored after a finite amount of time, and are arguably useless. To circumvent this problem, we need the idea of a minimal dynamical system. A dynamical system (X, T) is said to be minimal if for any open set O , there exists $n \in \mathbb{Z}^+$ such that $\cup_{j=1}^n T^j(O) \supseteq X$. This condition is equivalent to requiring the orbit of any point $x \in X$ to be a dense set. While these may seem like strong conditions to impose, it is known that for any dynamical system (X, T) , there exists $Y \subseteq X$ such that the dynamical system (Y, T) is well defined and minimal. A detailed discussion of these ideas and their proofs can be found in [3]. We

are now ready to state our first theorem.

Theorem 8.1: Let (X, T) be a minimal dynamical system. For any open set O , there exists $n \in \mathbb{Z}^+$ such that $T^{-n}(O) \cap O \neq \emptyset$.

Proof: Let $N \in \mathbb{Z}^+$ be such that $\cup_{j=1}^N T^{-j}(O) \supseteq X \supseteq T^{-(N+1)}(O)$. It follows that $T^{-(N+1)}(O) \cap T^{-j}(O) \neq \emptyset$ for some $1 \leq j \leq N$, so $T^{-(N+1-j)}(O) \cap O \neq \emptyset$.

An attempt to generalize this theorem in the same direction as that of Large sets for Van der Waerden's Theorem leads to the following definition.

Definition 8.2: $S \subseteq \mathbb{Z}^+$ is *topologically recurrent*, if for any minimal dynamical system (X, T) , and any open subset O , there exists some $s \in S$ such that $T^{-s}(O) \cap O \neq \emptyset$.

To see the connections of topologically recurrent sets and Ramsey Theory, we consider the following theorem.

Theorem 8.3: S is topologically recurrent, if and only if $w(S, 2, r) < \infty$ for every positive integer r . In particular, Large sets are topologically recurrent.

Proof: We will prove the contrapositive of each statement. For the first direction, let us assume that S is not topologically recurrent. Let (X, T) be a minimal invertible dynamical system, and O an open subset of X such that $T^s O \cap O = \emptyset$ for every $s \in S$. Let N be such that $\{T^i O\}_{i=0}^N$ is an open cover of X , and let $p \in X$ be arbitrary. Now consider the partition of X into $N + 1$ classes $\{O'_i\}_{i=0}^N$ as follows. $O'_0 = O$, and for $1 \leq i \leq N$, we have $O'_i = T^i O - (\cup_{j=0}^{i-1} O'_j)$. We can now construct the $(N + 1)$ -coloring C_p of \mathbb{Z} by setting $C_p(n) = m$ where $0 \leq m \leq N$ is such that $T^n(p) \in O'_m$. Suppose $C_p(n_1) = C_p(n_2)$ for some distinct positive integers n_1 and n_2 , then let $m = C_p(n_1)$, so we note that $T^{n_1}(p) \in O'_m \subseteq T^m O$ and $T^{n_2}(p) \in O'_m \subseteq T^m O$, thus $T^{n_1-m}(p) \in O$ and $T^{n_2-m}(p) \in O$, so $T^{n_2-n_1} O \cap O \neq \emptyset$. It follows by our choice of O that $n_2 - n_1 \notin S$, so we see that $w(S, 2, N + 1) = \infty$. For the converse, let us assume that $w(S, 2, r_0) = \infty$ for some positive integer r_0 . Let C be a minimal r_0 -coloring of \mathbb{Z} with no 2-term S -A.P., and let us view C as an element of Ω_{r_0} . Consider the minimal invertible dynamical subsystem (X, T) of (Ω_{r_0}, T) where $X = \overline{\{T^i C \mid i \in \mathbb{Z}\}}$. Consider $O = B_{\frac{1}{2}}(C) \cap X$, and let C' and s be arbitrary elements of O and S respectively. By construction of X , we see that for any positive integer M , there exists an integer j , such that $C'(i) = C(j + i)$ for $-M \leq i \leq M$, so let us pick some $M > s$ and the corresponding value of j , and we see that $C(0) = C'(0) = C(j) \neq C(j + s) = C'(s) = (T^s C')(0)$, so $T^s C' \notin B_{\frac{1}{2}}(C)$, but C' and s were arbitrary, so S is not topologically recurrent.

We now see that any Large set is a set of topological recurrence, so we have an extensive list of topologically recurrent sets other than \mathbb{Z}^+ . It is natural to wonder if any topologically recurrent set is also a Large set. To see that this is not true, we will first construct a family of topologically recurrent sets that we have not shown to be Large.

Theorem 8.4: Let S be an infinite subset of the positive integers, and let $H = S - S = \{s_1 - s_2 \mid s_1, s_2 \in S, s_2 < s_1\}$, then T is topologically recurrent.

Proof: Let (X, T) be a minimal dynamical system and O an open set. Let N be such that $\{T^i O\}_{i=0}^N$ is an open cover of X , and let $\{O'_i\}_{i=0}^N$ be as in the proof of theorem 5.1.1. Let $p \in X$ be arbitrary, and let the $(N + 1)$ - coloring C_p also be defined as in the proof of theorem 5.1.1. If $\{s_i\}_{i=1}^\infty$ is an enumeration of the elements of S , we may apply the pigeon hole principle to see that $C_p(s_i) = C_p(s_j)$ for some $1 \leq i < j \leq N + 2$, so it follows that $T^{s_j - s_i} O \cap O \neq \emptyset$, and H is topologically recurrent. ■

In [12], V. Jungic constructs a set of the form $H = S - S$, such that for some integer k , we have that $w(H, 3, k) = \infty$, so we see that topological recurrence is a strictly weaker notion than that of largeness. In order to obtain the topologically equivalent notion of a Large set, we must first introduce the topological Van der Waerden Theorem, whose proof can be found in [3].

Topological Van der Waerden Theorem: Let (X, T) be a minimal dynamical system. For any open subset O of X , and any $k \in \mathbb{Z}^+$, there exists $n \in \mathbb{Z}^+$ such that $\bigcap_{j=0}^k T^{-nj} O \neq \emptyset$.

Definition 8.5: $S \subseteq \mathbb{Z}^+$ is a set of k -topological recurrence, if for any minimal dynamical system (X, T) , any open set O , and any $k \in \mathbb{Z}^+$, there exists some $s \in S$ such that $\bigcap_{j=0}^{k-1} T^{-sj} O \neq \emptyset$. If S is a set of k -topological recurrence for all k , then S is a set of multiple topological recurrence.

We will show that the notions of Large sets and sets of multiple topological recurrence are equivalent. In order to do this, we will show an extension of theorem 8.3 in which we show that S is a set of k -topological recurrence if and only if $w(S, k, r) < \infty$ for all $r \in \mathbb{Z}^+$. We see that the notion of k -topological recurrence is dual to that of r -Large sets, because instead of fixing the number of colors and letting the length of the progression become unbounded, we are fixing the length of the progression, and letting the number of colors become unbounded. In this setting, the natural dual of the 2-Large is Large conjecture is whether or not there exists some $k_0 \in \mathbb{Z}^+$, such that any set of k_0 -topological recurrence is a set of multiple topological recurrence. We have already seen that we must have $k_0 \geq 3$ in order for this conjecture to have any hope of being true. However, we conjecture the opposite of this phenomenon below.

Conjecture 8.6: For any $k \geq 2$, there exists $S \subseteq \mathbb{Z}^+$ such that S is a set of k -topological recurrence, but not a set of $(k + 1)$ -topological recurrence.

The reason for this conjecture, is that in [13], this result was proven for the analogous statements of k -measure recurrence, which corresponds combinatorially to density statements rather than partition regular statements. While we will not be discussing density Ramsey Theory, the reader may find a dynamical introduction to the subject in [3] and a

combinatorial introduction to the subject in [9].

Theorem 8.7: $S \subseteq \mathbb{Z}^+$ is a set of k -topological recurrence if and only if $w(S, k, r) < \infty$ for all $r \in \mathbb{Z}^+$.

Proof: We will prove the contrapositive of each statement. For the first direction, let us assume that S is not k -topologically recurrent. Let (X, T) be a minimal invertible dynamical system, and O an open subset of X such that $\bigcap_{j=0}^{k-1} T^{-sj}O = \emptyset$ for every $s \in S$. Let N be such that $\{T^i O\}_{i=0}^N$ is an open cover of X , and let $p \in X$ be arbitrary. Now consider the partition of X into $N + 1$ classes $\{O'_i\}_{i=0}^N$ as follows. $O'_0 = O$, and for $1 \leq i \leq N$, we have $O'_i = T^i O - (\bigcup_{j=0}^{i-1} O'_j)$. We can now construct the $(N + 1)$ -partition C_p of \mathbb{Z} by setting $C_p(n) = m$ where $0 \leq m \leq N$ is such that $T^n(p) \in O'_m$. Suppose C_p is constant on the arithmetic progression $\{a + jd\}_{j=0}^{k-1}$ with $d \in S$, then let $m = C_p(a)$, so we note that $T^{a+jd}(p) \in O'_m \subseteq T^m O$ for all $0 \leq j < k$. It follows that $p \in \bigcap_{j=0}^{k-1} T^{-(a-m+jd)}$, which contradicts the assumption about O . It follows that C_p is not constant on any k -term S -A.P., so $w(S, k, N + 1) = \infty$. For the converse, let us assume that $w(S, k, r_0) = \infty$ for some positive integer r_0 . Let C be a minimal r_0 -coloring of \mathbb{Z} with no k -term S -A.P., and let us view C as an element of Ω_{r_0} . Consider the minimal invertible dynamical subsystem (X, T) of (Ω_{r_0}, T) where $X = \overline{\{T^i C \mid i \in \mathbb{Z}\}}$. Consider $O = B_{\frac{1}{2}}(C) \cap X$, and let $C' \in O$, $s \in S$, and $a \in \mathbb{Z}$ all be arbitrary. By construction of X , we see that for any positive integer M , there exists an integer j , such that $C'(i) = C(j + i)$ for $-M \leq i \leq M$, so let us pick some $M > |a| + ks$ and the corresponding value of j . We see that $C'(a + is) = C(j + a + is)$ for $0 \leq i < k$, but C is not constant on any k -term S -A.P., so C' is not constant on $\{a + is\}_{i=0}^{k-1}$. Let $0 \leq i_1 < i_2 < k$ be such that $C'(a + i_1 s) \neq C'(a + i_2 s)$, so $T^{-(a+i_1 s)}(B_{\frac{1}{2}}(C')) \cap T^{-(a+i_2 s)}(B_{\frac{1}{2}}(C')) = \emptyset \rightarrow T^{-(a+i_1 s)}(B_{\frac{1}{2}}(C)) \cap T^{-(a+i_2 s)}(B_{\frac{1}{2}}(C)) = \emptyset$, hence $\bigcap_{i=0}^{k-1} T^{-(a+is)}(O) = \emptyset$, and S is not k -topologically recurrent.

Next, we want to see what topological notions are equivalent to the Multidimensional Van der Waerden Theorem and Multidimensionally Large sets, which leads us to a generalization of topological recurrence called strong topological recurrence. Let X be a compact topological space, and G a commuting group of homeomorphisms of X . We say that the dynamical system (X, G) is minimal, if no closed subset of X is fixed under every element of G . This is equivalent to requiring the existence of a subset $\{S_i\}_{i=1}^N$ of G for any open set O , such that $\{S_i(O)\}_{i=1}^N$ is an open cover of X . Both of the previous conditions are also equivalent to requiring a dense orbit for any point $x \in X$, where the orbit in this case is defined as $O_x = \{g(x) \mid g \in G\}$. We see that in a general group, there is the possibility of having more than 1 generator, which is what happens with the group of translations in \mathbb{Z}^n , which is what also motivates the commutativity condition. In [17], nilpotent versions of these recurrence theorems have been proven, but we will not focus on these. Below we state the Topological Multidimensional Van der Waerden Theorem, a proof of which can be found in [3].

Topological Multidimensional Van der Waerden Theorem: Let (X, G) be a minimal dynamical system, and let O be any open set. For any $\{T_i\}_{i=1}^n \subseteq G$, and any $k \in \mathbb{Z}^+$, there exists some $j \in \mathbb{Z}^+$ such that $\cap_{i=1}^n T^{-ji}(O) \neq \emptyset$.

Definition 8.8: $S \subseteq \mathbb{Z}^+$ is a set of *strong topological recurrence* if for any minimal dynamical system (X, G) , any open set O , any $\{T_i\}_{i=1}^n \subseteq G$, and any $k \in \mathbb{Z}^+$, there exists some $s \in S$ such that $\cap_{i=1}^n T^{-si}(O) \neq \emptyset$.

Theorem 8.9: A set S is Multidimensionally Large if and only if it is strongly topologically recurrent.

Proof: For the first direction, let us assume that S is Multidimensionally Large. Let (X, G) be a minimal dynamical system, and let O be an arbitrary non-empty open set. Let $\{T_i\}_{i=1}^M$ be any finite subset of G . Let $\{S_i\}_{i=1}^N$ be a subset of G such that $\{S_i(O)\}_{i=1}^N$ is an open cover of X . We will use this open cover to construct a partition $\{O'_i\}_{i=1}^N$ of X as in Theorem 5.1.1. Let $O'_1 = S_1(O)$, and for $2 \leq i \leq N$, let $O'_i = S_i(O) - (\cup_{j=1}^{i-1} O'_j)$. Let $p \in X$ be arbitrary, and let C_p be the N -coloring of \mathbb{Z}^M given by $C_p(x_1, x_2, \dots, x_M) = n$, where $1 \leq n \leq N$ is such that $(T_1^{-x_1} \circ T_2^{-x_2} \circ \dots \circ T_M^{-x_M})(p) \in O'_n$. Since S is Multidimensionally Large, let $s \in S$ and $a \in \mathbb{Z}^M$ be such that $\{C_p(a + se_i)\}_{i=1}^M$ is a monochromatic set of color n . Letting $T' = T_1^{-a_1} \circ T_2^{-a_2} \circ \dots \circ T_M^{-a_M}$, we see that $T'(p) \in T_i^s(O'_n) \subseteq T_i^s(S_n(O))$ for all i , so we see that $(S_n^{-1} \circ T')(p) \in T_i^s(O)$ for all i after recalling that G is a commutative group, which yields the desired result.

For the converse, let us assume that S is strongly topologically recurrent, let $M \in \mathbb{Z}^+$ be arbitrary, and let C be an arbitrary minimal r -coloring of \mathbb{Z}^M . Let (X, G) be the minimal dynamical subsystem of (Ω_r^M, G) , where G is the group of integer translations, and $X = \{g(C) \mid g \in G\}$. Consider the open set $O = B_{\frac{1}{2}}(C) \cap X$, and let $\{T_i\}_{i=1}^N$ be some finite subset of G . Since S is strongly topologically recurrent, let $s \in S$ be such that $O \cap (\cap_{i=1}^N T_i^s(O)) \neq \emptyset$. Let $C' \in O \cap (\cap_{i=1}^N T_i^s(O))$ be arbitrary. Since $T_i^s(C') \in B_{\frac{1}{2}}(O)$ for all i , we see that $C(0) = C'(0) = (T_i^s(C'))(0)$ for all i . Since C is a minimal coloring, and $C' \in \{g(C) \mid g \in G\}$, we have that $C \in \{g(C') \mid g \in G\}$, so for any $\epsilon > 0$, there exists $g_\epsilon \in G$ such that $g_\epsilon(C') \in B_\epsilon(C)$. For ϵ sufficiently small, we will have that $T_i^s(g_\epsilon(C'))(0) = T_i^s(C')(0) = C(0)$ for all i , which yields the desired result.

Most theorems in standard Ramsey Theory have formulations in terms of topological dynamics, but can we find a dynamical formulation of theorems in canonical Ramsey Theory? This would involve the loss of compactness in the most general case, so new methods would have to be developed.

We will now examine a concrete case of topological recurrence known as Bohr recurrence. Let us identify $[0, 1)$ with the unit torus \mathbb{T} . A subset S of the integers is called Bohr(1) if for any $\alpha \in \mathbb{T}$, and any $\epsilon > 0$, there exists $s \in S$ such that $s\alpha \in (-\epsilon, \epsilon)$. In general, S is called Bohr(n) if for any $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{T}^n$, and any $\epsilon > 0$, there exists $s \in S$ such

that $s\alpha_i \in (-\epsilon, \epsilon)$ for $1 \leq i \leq n$. If S is Bohr(n) for all n , then we simply say that S is Bohr recurrent. We see that Bohr(n) recurrence is just the special case of topological recurrence where the ambient space X is the n -dimensional torus \mathbb{T}^n , and the allowable homeomorphisms are the translations. We can construct a family of colorings out of the dynamical systems created by these tori as follows.

For any $\alpha \in [0, 1)$, let C_α denote the coloring of the integers given by

$$C_\alpha(i) = \begin{cases} 0 & \text{if } \{i\alpha\} \in [0, \frac{1}{2}) \\ 1 & \text{if } \{i\alpha\} \in [\frac{1}{2}, 1) \end{cases} .$$

It is shown in [3], that in Ω_2 , under the standard shift operator T , C_α is a minimal coloring for any α . If a set S is 2-Large, then there exist arbitrarily long monochromatic S -A.P.s in any C_α simply from the definition of 2-Large. Since the set of colorings $\hat{C} = \{C_\alpha\}_{\alpha \in [0, 1)}$ is a massive set of colorings, we are led to wonder if the converse is true?

Theorem 8.10: S is Bohr(1) if and only if any coloring in \hat{C} contains arbitrarily large S -A.P.s.

Proof: Let us first assume that S is Bohr(1), and let $C_\alpha \in \hat{C}$ be an arbitrary coloring. Since S is Bohr(1), for an arbitrary positive integer k , let $s \in S$ be such that $s\alpha \in (-\frac{1}{2k}, \frac{1}{2k})$. If $s\alpha \in [0, \frac{1}{2k})$, then $ns\alpha \in [0, \frac{1}{2})$ for $1 \leq n \leq k$, so $C_\alpha(ns) = 0$ for all such n , and C_α contains a k -term S -A.P.. If $s\alpha \in (-\frac{1}{2k}, 0)$, then $ns\alpha \in (-\frac{1}{2}, 0) = (\frac{1}{2}, 1)$ for $1 \leq n \leq k$, so $C_\alpha(ns) = 1$ for all such n , and C_α contains a k -term S -A.P., thus every coloring in \hat{C} contains arbitrarily long S -A.P.s. For the converse, let us assume that every coloring in \hat{C} contains arbitrarily long S -A.P.s. To show that S is Bohr(1), let $\alpha \in [0, 1)$ and $\epsilon > 0$ both be arbitrary. Let k be a positive integer such that $\frac{1}{2k} < \epsilon$, and let $s \in S$ be such that $\{a + si\}_{i=0}^k$ is a monochromatic $(k+1)$ -term S -A.P. in C_α . If $s\alpha \in [0, \frac{1}{2})$, then assume for the sake of contradiction that $s\alpha \in (\frac{1}{2k}, \frac{1}{2})$. We note that $a\alpha \in [0, \frac{1}{2})$, and that for some $0 \leq j < k$, we must have $(a + js)\alpha \in (\frac{1}{2} - \frac{1}{2k}, \frac{1}{2})$ since $s\alpha \in (\frac{1}{2k}, \frac{1}{2})$, so we must also have $(a + (j+1)s)\alpha \in (\frac{1}{2}, 1)$, thus $C_\alpha(a + (j+1)s) = 1$, which is the desired contradiction. If $s\alpha \in (\frac{1}{2}, 1)$, then $-s\alpha \in [0, \frac{1}{2})$, and the $(k+1)$ -term S -A.P. $\{a + ks + i(-s)\}_{i=0}^k$ is monochromatic in C_α , so we have reduced this to the previous case. ■

Corollary 8.11: Every 2-Large set is Bohr(1).

Corollary 8.12: Lacunary sets are not 2-Large.

Proof: According to [14], Lacunary sets are not Bohr(1), and are consequently not 2-Large. This solves a conjecture of Brown, Graham, and Landman posed in [1].

Is every Bohr(1) set a 2-Large set as well? In order to investigate this, we will investigate Bohr(1) recurrence from another perspective. Note that for any $\epsilon > 0$, the open set $I_n(\epsilon) =$

$\cup_{i=1}^n (\frac{i-\epsilon}{n}, \frac{i+\epsilon}{n})$ is the set of $\alpha \in [0, 1)$ such that $\{n\alpha\} \in I_1(\epsilon) = (-\epsilon, \epsilon)$. We can see an alternative formulation of a set S being Bohr(1) if and only if $\{I_s(\epsilon)\}_{s \in S}$ is an open cover of \mathbb{T} for every $\epsilon > 0$. This is equivalent to saying that $[0, 1) = \bar{I}(S) := \cap_{\epsilon > 0} (\cup_{s \in S} I_s(\epsilon))$. With this new definition, we can verify the known fact that $S = m\mathbb{Z}^+$ is Bohr(1) for any positive integer m as follows. It is clear that $\mathbb{Q} \cap [0, 1) \subset \bar{I}(m\mathbb{Z}^+)$, as for any $\frac{p}{q} \in \mathbb{Q}$, we see that $mq \in m\mathbb{Z}$, and $(\frac{p}{q})mq = pm \equiv 0$. In fact, we see from this proof that $\mathbb{Q} \cap [0, 1) \subseteq \bar{I}(S)$ if S contains a multiple of any positive integer. Next, consider any irrational $\alpha \in [0, 1)$ and any $\epsilon > 0$. Let $\frac{p}{q}$ be a continued fraction convergent of α with $q > \frac{m}{\epsilon}$, so $\frac{\epsilon}{mq} > \frac{1}{q^2} > |\alpha - \frac{p}{q}| = |\alpha - \frac{mp}{mq}|$, hence $\alpha \in I_{mq}(\epsilon)$, which gives the desired result.

While there exist more direct proofs of this result, this proof raises the question of how much the theory of continued fractions can tell us about recurrence? Furthermore, we can verify that any Bohr(1) set S must contain a multiple of every positive integer n . To see this, we will show that if S does not contain a multiple of some n_0 , then $\frac{1}{n_0} \notin I_n(\frac{1}{n_0})$ for any n with $n_0 \nmid n$, which will give the desired result. Let $n = mn_0 + q$ with $m \geq 0$ and $1 \leq q < n_0$. We see that $\frac{m}{mn_0+q} < \frac{1}{n_0} < \frac{m+1}{mn_0+q}$, so we now have that $\frac{1}{n_0} - \frac{m}{mn_0+q} = \frac{q}{n_0(mn_0+q)} \geq \frac{1}{nn_0}$ and $\frac{m+1}{mn_0+q} - \frac{1}{n_0} = \frac{n_0-q}{n_0(mn_0+q)} \geq \frac{1}{nn_0}$, so $\frac{1}{n_0} \notin I_n(\frac{1}{n_0})$ as desired. Now let us consider the sets $B_{n,m} = \{nx + m | x \in \mathbb{Z}^+\}$. We already know that $B_{n,m}$ is not Bohr(1) if $n \nmid m$, but we will be able to show that $[0, 1) - \mathbb{Q} \subseteq \bar{I}(B_{n,m})$ in this case. Let $\alpha \in [0, 1) - \mathbb{Q}$ be arbitrary. We note that $n\alpha$ is also irrational, so $\{xn\alpha\}_{x \in \mathbb{Z}^+}$ is dense in $[0, 1)$, so let x be chosen so that $xn\alpha \in (-m\alpha, -m\alpha + \epsilon)$, and it follows that $(nx + m)\alpha \in (0, \epsilon)$, which yields the desired result. The set $\hat{N} := \{n! | n \in \mathbb{Z}^+\}$ is known to not be Bohr(1) because it is a lacunary set, but $\mathbb{Q} \subseteq \bar{I}(\hat{N})$ since \hat{N} contains a multiple of every positive integer, so it follows that $S_{n,m} := B_{n,m} \cup \hat{N}$ is Bohr(1) for any choice of n and m , thus Bohr(1) sets are not partition regular.

While Bohr(1) sets are not partition regular, we can see that sets of Bohr recurrence are partition regular. To see this, let R and S be sets of positive integers that are not Bohr(r) and Bohr(s) respectively. Let $\hat{\alpha}_1 = (\alpha_{1,1}, \alpha_{1,2}, \dots, \alpha_{1,r})$ be an element of \mathbb{T}^r such that for some fixed $\epsilon > 0$, and any $x \in R$, there exists $1 \leq i \leq r$ such that $x\alpha_{1,i} \notin (-\epsilon, \epsilon)$, and let $\hat{\alpha}_2 = (\alpha_{2,1}, \alpha_{2,2}, \dots, \alpha_{2,s})$ be defined similarly for S . Consider $\hat{\alpha}_3 = (\alpha_{1,1}, \alpha_{1,2}, \dots, \alpha_{1,r}, \alpha_{2,1}, \alpha_{2,2}, \dots, \alpha_{2,s})$, which is an element of \mathbb{T}^{r+s} that demonstrates $R \cup S$ is not Bohr($r+s$). An intermediate step to determining the relationship between 2-Large sets and sets of topological recurrence, would be to determine the relationship between 2-Large sets and sets of Bohr recurrence, which leads to the following conjecture.

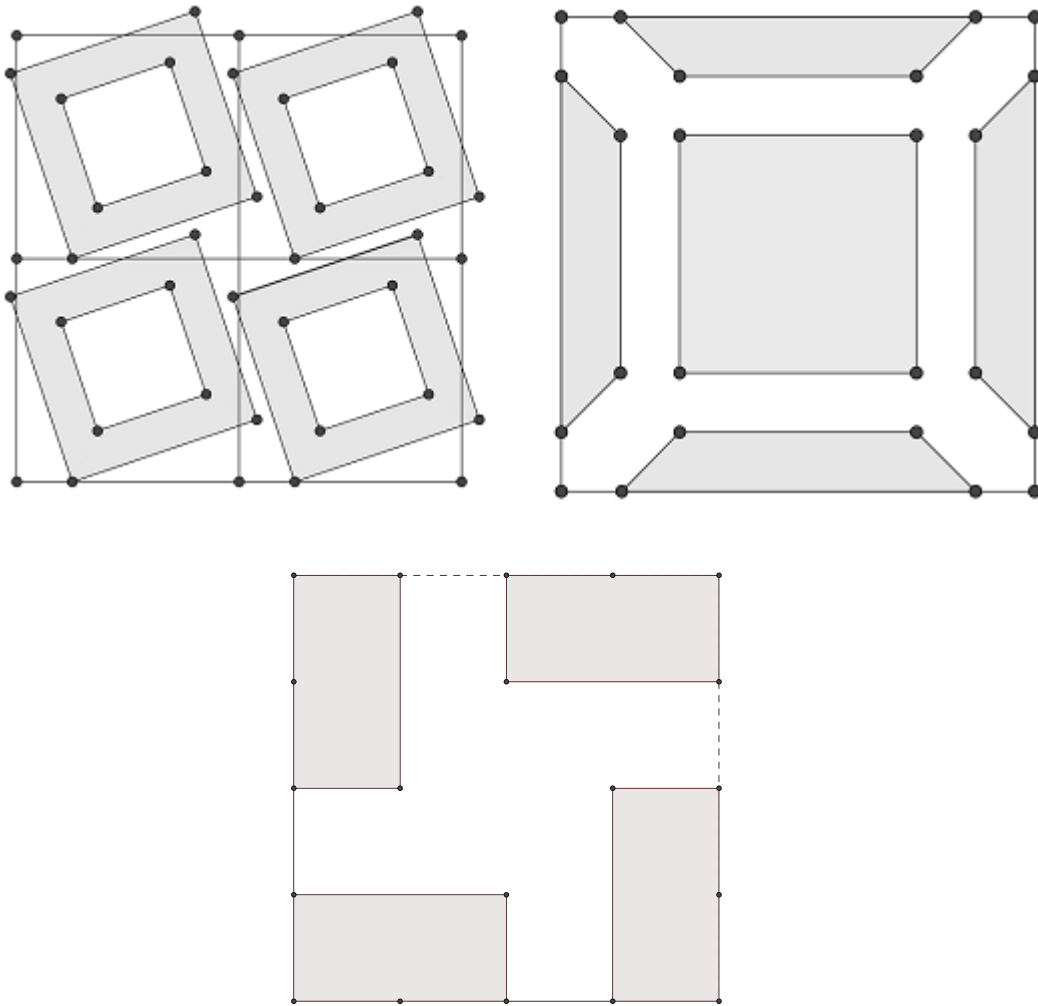
Conjecture 8.13: Every 2-Large set is also a set of Bohr recurrence.

Since we already know that all 2-Large sets are Bohr(1), the next step would be to determine if 2-Large sets are Bohr(2). From our discussion of the partition regularity of sets of Bohr recurrence, we see that the set $S_{2,1}$ from above is Bohr(1) but not Bohr(2), so it is natural to wonder if $S_{2,1}$ is a 2-Large set. In order to investigate this conjecture in full generality, we need to create a family of partitions of \mathbb{T}^2 similar to the partitions above of

T. Let $\mathbb{T}^2 = A \cup B$ be a partition of \mathbb{T}^2 , and let the coloring $C_{\alpha,\beta,A,B}$ be given by

$$C_{\alpha,\beta,A,B}(i) = \begin{cases} 0 & \text{if } (i\alpha, i\beta) \in A \\ 1 & \text{if } (i\alpha, i\beta) \in B \end{cases} .$$

In order to show that every 2-Large set is Bohr(2), it would suffice to find a partition A, B of \mathbb{T}^2 , such that a long monochromatic A.P. is found in the coloring $C_{\alpha,\beta,A,B}$, only if the common difference d of the arithmetic progression satisfies $(d\alpha, d\beta) < \epsilon$, where ϵ depends only on the length of the arithmetic progression. Below are 3 partitions of \mathbb{T}^2 , which I believe satisfy this criteria, but for which I am unable to prove whether or not it is true.



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