On Nearly Euclidean Thurston Maps and the Halfspace Theorem

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Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

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November 3, 2016
Blacksburg, Virginia

Keywords: Halfspace theorem, nearly euclidean Thurston maps, fixed points

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(ABSTRACT)

A Thurston map whose postcritical set consists of exactly four points and for which the local degree at each of its critical points is 2 is called *nearly Euclidean*. These maps were specified to parse Thurston's combinatorial characterization of rational functions. We determine an extension of the half-space theorem which provides an open hyperbolic half-space such that the negative reciprocal of any fixed slope value is excluded from the boundary of the half-space.

On Nearly Euclidean Thurston Maps and the Halfspace Theorem

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(GENERAL AUDIENCE ABSTRACT)

Thurston proved necessary and sufficient conditions under which a certain class of mappings defined topologically are equivalent, in a precise sense which can be considered less strict than topological conjugacy, to a rational map. The conditions presented in the proof of this theorem are not ones for which computational algorithms are easily admitted in all settings. Nearly Euclidean Thurston maps are a sub-class of the maps to which this theorem is applicable and for which an abundance of information is algorithmically attainable. We extend a theorem in this setting. One main example which speaks to the utility of this extension is in determining when certain rational maps arise as matings of polynomials.

Acknowledgments

I am deeply grateful to have had the opportunity to learn from my advisor William J. Floyd. I thank him for both giving me the idea for the main result of this thesis and for his seemingly endless patience in guiding me through the details.

I would also like to thank Edgar Saenz Maldonado for inviting me to the world of mathematical research and for his endless help and guidance.

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0.1 Introduction

Nearly Euclidean Thurston maps are some of the simplest Thurston maps whose associated orbifolds have nontrivial Teichmüller space. The Teichmüller space of the orbifold of a nearly Euclidean Thurston map is in bijective correspondence with the upper halfplane $\mathbb{H}$. In [2] the authors prove a theorem which provides an open hyperbolic halfspace in $\mathbb{H}$ whose infinite boundary excludes Thurston obstructions. We extend this theorem to fixed slope values.

Chapter 1

Preliminaries

Below we discuss both preliminary definitions in the areas of hyperbolic geometry and Teichmüller theory. This is by no means an exhaustive or even detailed account of either field. We will simply provide some necessary background for the discussions to come.

1.1 Hyperbolic Geometry

In later sections, we will discuss the Teichmüller space of the orbifold of the four-punctured sphere. To work with this space, we will identify it with the complex upper half-plane. There are several common analytic models for hyperbolic space, though, for our purposes, becoming familiar with one planar model will suffice and so we choose the model that we will deal with directly, the upper half-plane model. We will first define the model, determine what hyperbolic lines correspond to in this model, discuss the transformations of our model which realize geometric motions, and finish by defining some special curves which we will use later.

1.1.1 The Upper Half-Plane Model

We choose as our model of hyperbolic space the following Riemannian manifold. As a set, it is given by

$$\mathbb{H} = \{(1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}.$$

As far as the topology on this set, we consider $\mathbb{H}$ as a subspace of the space $\mathbb{R}^3$ and with this, we may regard $\mathbb{H}$ as a topological manifold. The smooth structure on $\mathbb{H}$ is determined by the smooth atlas $\mathcal{A} = \{(\mathbb{H}, \varphi)\}$ where $\varphi(1, x_2, x_3) = (x_2, x_3) \in \mathbb{R}^2$.

To complete the construction of this model, we equip $\mathbb{H}$ with the following Riemannian metric $g : \mathbb{H} \to T^2T^*\mathbb{H}$, where $T^2T^*\mathbb{H}$ is the bundle of covariant 2-tensors on $\mathbb{H}$, and is given in coordinates by

$$g_z = \frac{1}{x_3^2}(dx_2^2 + dx_3^2),$$

where we see that this is just a scaled version of the standard Euclidean metric with no dx_1^2 component. So at each point, the Riemannian metric on $\mathbb{H}$, which we'll refer to henceforth as the ***hyperbolic metric***, is just the Euclidean metric scaled by $1/x_3^2$, that is, the scaling just depends on how far away you are from the axis $x_3 = 0$. We observe that the above metric defines a smooth symmetric covariant 2-tensor field on $\mathbb{H}$ that is positive definite at each point so is, by definition, a Riemannian metric.

Later on, we will find it more convenient to work with the connected analytic manifold of one complex dimension

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

where we've denoted this space by $\mathbb{H}$ since it is easily identified with the model we defined above. We will make this change in the section on the orientation-preserving isometries of $\mathbb{H}$. There we will identify the group of conformal automorphisms of the upper half-plane with a particular subgroup of the group of all fractional linear transformations which makes the complex setting most appropriate.

1.1.2 Geodesics in $\mathbb{H}$

We'd like to determine the analogs of straight lines in the above model. The role of lines in the hyperbolic plane are played by geodesics in the Riemannian manifold one uses to model the hyperbolic plane. In our case, there exist fundamental geodesics in the sense that all geodesics may be obtained from the fundamental ones through some isometry of the plane. In this section, we establish the existence of these fundamental geodesics and state a theorem classifying all geodesics in $\mathbb{H}$.

We shall use the following theorem to assert the existence of a fundamental geodesic,

Theorem 1.1 (The Retraction Principle). *Suppose that X is a Riemannian manifold, that $C : (a, b) \to X$ is an embedding of an interval (a, b) in X, and that there is a retraction $r : X \to \mathrm{image}(C)$ that is distance-reducing in the sense that, if one restricts the metric of X to $\mathrm{image}(C)$ and pulls this metric back via r to obtain a new metric on all of X, then at each point the pullback metric is less than or equal to the original metric on X. Then the image of C contains a shortest path (geodesic) between each pair of its points.*

Proof. Let X be a Riemannian manifold with metric g. Let $C : (a, b) \to X$ be an embedding of (a, b) in X and suppose $r : X \to \mathrm{image}(C)$ is a distance-reducing retraction in the sense defined above. Let $\gamma : [c, d] \to X$ be any arbitrary path between two points $p_1, p_2 \in \mathrm{image}(C)$, so $\gamma(c) = p_1$ and $\gamma(d) = p_2$. Since C is an embedding, there must exist $s_1, s_2 \in (a, b)$ such that $C(s_1) = p_1$ and $C(s_2) = p_2$. We start by computing the length of γ

in X,

$$L_g(\gamma) = \int_c^d |\gamma'(t)|_g dt$$

where $|\gamma'(t)|_g$ is the length of the tangent vector $\gamma'(t)$ in the tangent space $T_{\gamma(t)}X$ at $\gamma(t) \in X$. Now, by assumption, pulling back the restriction of g, which, by an abuse of notation we also denote g, via r yields a metric which is less than or equal to the original metric on X. Then the following implication must hold, for any $p \in X$ and any vectors v, w in the tangent space T_pX,

$$r^*g_p(v,w) \leqslant g_p(v,w) \Longrightarrow |\gamma'(t)|_{r^*g} \leqslant |\gamma'(t)|_g.$$

Note that, letting $p = \gamma(t)$ for notational convenience, we have

$$\begin{aligned} |\gamma'(t)|_{r^*g} &= \Big(r^*g_p(\gamma'(t), \gamma'(t))\Big)^{1/2} \\ &= \Big(g_{r(p)}(dr_p(\gamma'(t)), dr_p(\gamma'(t)))\Big)^{1/2} \\ &= \Big(g_{r(p)}(C'(s), C'(s))\Big)^{1/2} = |C'(s)|_g \end{aligned}$$

for some $s \in (a,b)$. That is, computing the length of γ under the metric obtained by pulling back g via r is the same as pushing the tangent vectors $\gamma'(t)$ forward to the tangent vectors on image(C), $C'(s)$, and integrating with respect to the restricted metric g on the embedded submanifold image(C). Therefore, continuing with the length computation,

$$\begin{aligned} L_g(\gamma) &= \int_c^d |\gamma'(t)|_g dt \\ &\geqslant \int_c^d |\gamma'(t)|_{r^*g} dt \\ &= \int_{s_1}^{s_2} |C'(s)|_g ds \\ &= L_g(C([s_1, s_2])) \\ &= L_g(r(\gamma)) \end{aligned}$$

which shows that the retraction of γ is at least as short as the original path and hence image(C) contains a shortest path (geodesic) between each pair of its points since the points we started with were arbitrary. $\blacksquare$.

The idea is illustrated in the following image.

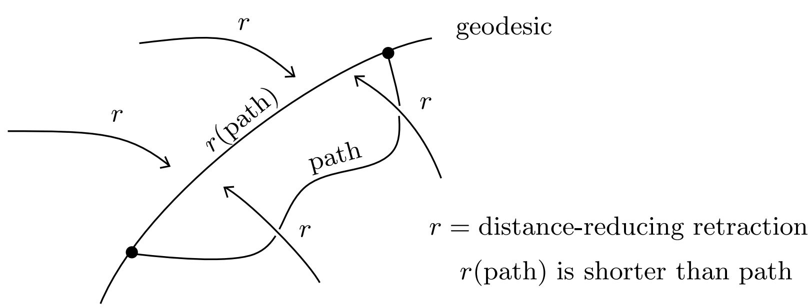


Figure 1.1: A distance reducing retraction showing that the length of the image of the path under the retraction is less than the length of the path.

Now we establish the existence of fundamental geodesics.

Theorem 1.2 (Existence of a Fundamental Geodesic in $\mathbb{H}$). *In the half-space model of hyperbolic space, all vertical lines are geodesic. Such a line is the unique shortest path between any pair of points on it.* [3]

Proof. Let $C : (0, \infty) \to \mathbb{H}$, where $C(t) = (1, x_2, t) \in \mathbb{H}$ and where x_2 is a fixed constant, i.e., C is an arbitrary vertical line in $\mathbb{H}$. We define a retraction $r : \mathbb{H} \to \text{image}(C)$ given by

$$r(1, x_2', t) = (1, x_2, t)$$

for any $(1, x_2', t) \in \mathbb{H}$. See figure 1.2 for a cross-sectional view of the retraction. The original

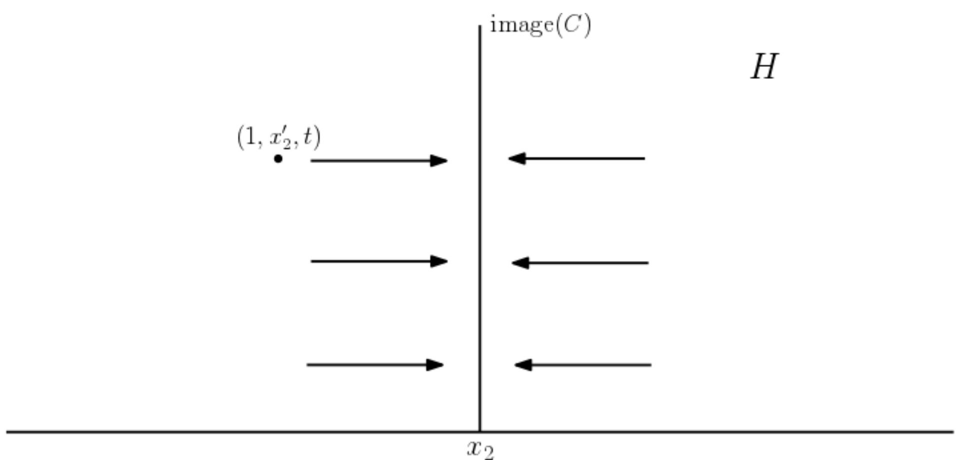


Figure 1.2: Cross-sectional view of retraction

hyperbolic metric was $g_z = \frac{1}{x_3^2}(dx_2^2 + dx_3^2)$. If we restrict to image(C) and pull-back we get,

$$r^* g_z = \frac{dx_3^2}{x_3^2}$$

which shows us that the retraction is distance-reducing. Therefore, by the retraction principle, the image of C contains a shortest path between each pair of its points. It remains to demonstrate that there is only one shortest path between any pair of points on image(C). Suppose one starts with an arbitrary path in $\mathbb{H}$ that does not remain in image(C), then, at some point, the path is not vertical. The pullback metric induced by the retraction would have to be smaller than the original hyperbolic metric at this point since g_z has a dx_2^2 component. Therefore, the retraction must be strictly shorter than the original path. This establishes that the shortest path between pairs of points on image(C) are vertical. The uniqueness of these shortest paths between arbitrary pairs of points on image(C) is clear since image(C) is itself a vertical line. ■

Now we state a classification theorem for all the geodesics in $\mathbb{H}$ and refer the reader to [3] for a proof.

Theorem 1.3 (Classification of Geodesics in $\mathbb{H}$). *The geodesics in the half-space model $\mathbb{H}$ of hyperbolic space are precisely the vertical lines in $\mathbb{H}$ and the Euclidean metric semicircles whose endpoints lie in and intersect the boundary $\{(1, x_2, 0)\} \subset \mathbb{R}^2$ of hyperbolic space $\mathbb{H}$ orthogonally.*

We finish the subsection by defining the hyperbolic distance. The ***hyperbolic distance from*** p ***to*** q, denoted by $d_g(p, q)$, is defined to be the infimum of $L_g(\gamma)$ where $L_g(\gamma)$ is the length of γ and the infimum is taken over all piecewise smooth curve segments γ from p to q.

1.1.3 Riemann Surfaces

In this subsection, we recall some basic definitions from the theory of Riemann surfaces. Informally speaking, Riemann surfaces are spaces which, locally, look like an open subset of the complex plane. The aesthetic features tacit in the term "looks like" include all the mathematical structure— algebraic, analytic, etc.— associated with such a set and so these surfaces are distinguished from mere two-dimensional topological manifolds. Let X be a topological space.

A ***complex chart***, or simply ***chart***, on X is a homeomorphism $\varphi : U \to V$, where $U \subseteq X$ is open in X, and $V \subseteq \mathbb{C}$ is open in $\mathbb{C}$. Two complex charts, $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ are said to be ***compatible*** if either $U_1 \cap U_2 = \varnothing$, or

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is analytic. The function $T = \varphi_2 \circ \varphi_1^{-1}$ is called the ***transition function*** between two charts. Note, that the transition function between compatible charts are always bijections and so, since any holomorphic bijection has a holomorphic inverse, the transition maps must be ***biholomorphic***.

A ***complex atlas*** (or simply ***atlas***) $\mathcal{A}$ on X is a collection $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha\}$ of pairwise compatible complex charts whose domains cover X, that is, $X = \cup_\alpha U_\alpha$ where α is taken to be an element of some index set whose name we omitted here for no special reason. Two complex atlases $\mathcal{A}$ and $\mathcal{B}$ are ***equivalent*** if every chart of one is compatible with every chart of the other.

Any atlas on X will determine a unique maximal atlas. That is, if $\mathcal{A}$ is an atlas on X, there exists an atlas $\mathcal{U}$ such that $\mathcal{A} \subseteq \mathcal{U}$ and if $\mathcal{U}'$ is another atlas with $\mathcal{A} \subseteq \mathcal{U}'$, then $\mathcal{U}' \subseteq \mathcal{U}$. This unique maximal atlas consists of all possible charts compatible with every chart of $\mathcal{A}$. A ***complex structure*** on X is a maximal complex atlas on X, or, equivalently, an equivalence class of complex atlases on X.

A ***Riemann surface*** is a second countable connected Hausdorff topological space together with a complex structure. We may refer to Riemann surfaces as 'surfaces' specifying the structure when necessary. Since connectedness and path-connectedness coincide for manifolds, all Riemann surfaces are path connected. It can be shown that all Riemann surfaces are orientable so we assume from here on that our surfaces are oriented.

1.1.4 Equivalence Among Riemann Surfaces

Here we discuss a notion of equivalence among compact Riemann surfaces.

See [12] for more on the following paragraph. Let X be a compact Hausdorff space. A ***curved triangle*** in X is a subspace A of X and a homeomorphism $h : T \to A$, where T is a closed triangular region in the plane $\mathbb{R}^2$. If e is an edge of T, then $h(e)$ is said to be an ***edge*** of A and if v is a vertex of T, then $h(v)$ is said to be a ***vertex*** of A. A ***triangulation*** of X is a colllection of curved triangles $A_1, \ldots, A_n$ in X whose union is X such that for $i \neq j$, the intersection $A_i \cap A_j$ is either empty, or a vertex of both A_i and A_j, or an edge of both.

It is well known [6] that all 2-dimensional topological manifolds with or without boundary, or *surfaces* for short, are homeomorphic to triangulated surfaces. Then, if X is a compact connected triangulable surface, then X is homeomorphic to a space obtained from a polygonal region in the plane by pasting the edges together in pairs. The following theorem is well-known.

Theorem 1.3 (The Classification Theorem). *Let X be the quotient space obtained from a polygonal region in the plane by pasting its edges together in pairs. Then X is homeomorphic either to S^2, to the n-fold torus T_n for some positive integer n, or to the m-fold projective plane P_m for some positive integer m.*

In light of this theorem, the classification of compact Riemann surfaces up to homeomorphism amounts to specifying the genus of the surface. This is too coarse a notion of equivalence for our needs because, for example, any two topological 1-holed tori are considered equivalent in the above sense but when one obtains these tori as quotients of $\mathbb{C}$ by distinct lattices, thereby endowing them with some complex structure, they represent geometrically distinct objects.

The idea then is to supplement mere topological equivalence with an additional condition which speaks to the preservation of the geometry provided by the complex structure. There are several ways to accomplish this, the simplest to state being that the homeomorphism also be conformal when restricted to charts.

Let S and S' be Riemann surfaces. A map $F : S \to S'$ is a ***conformal isomorphism*** if F is a homeomorphism and

$$\varphi_2 \circ F \circ \varphi_1^{-1} : \varphi(U) \to \psi(V)$$

is holomorphic where $\varphi : U \to \varphi(U)$ and $\psi : V \to \psi(V)$ are arbitrary charts on S and S', respectively, with $U \cap F^{-1}(V) \neq \varnothing$. In the particular case when $S = S'$, a conformal isomorphism between S and S' is called a ***conformal automorphism***. It is clear that the set of all conformal automorphisms of a given Riemann surface S forms a group under composition. We denote this group by $\mathrm{Aut}(S)$ and we shall denote the identity element of this group by I.

Now we consider two examples of Riemann surfaces that will be particularly useful.

Example 1.4 (The Riemann Sphere) Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where ∞ is a symbol not contained in $\mathbb{C}$. As a topological space, in addition to the standard open sets of $\mathbb{C}$, we let sets of the form $V \cup \{\infty\}$ where $V = \mathbb{C} \backslash K$ for some compact $K \subset \mathbb{C}$ be open. As a topological manifold, this space is homeomorphic to the 2-sphere, which we will denote by S^2. We define the following domains for the complex charts on $\hat{\mathbb{C}}$,

$$\begin{aligned} U_N &= \hat{\mathbb{C}} \backslash \{\infty\} = \mathbb{C} \\ U_S &= \hat{\mathbb{C}} \backslash \{0\} = \mathbb{C}^* \cup \{\infty\} \end{aligned}$$

where $\mathbb{C}^*$ denotes the set of all nonzero complex numbers. The subscripts N and S indicate that we're identifying these charts with the standard charts given as the domains for North and South pole stereographic projections when endowing S^2 with a smooth structure. The charts $\varphi_N : U_N \to \mathbb{C}$ and $\varphi_S : U_S \to \mathbb{C}$ are given by

$$\begin{aligned} \varphi_N(z) &= z; \\ \varphi_S &= \begin{cases} 1/z & \text{for } z \in \mathbb{C}^* \\ 0 & \text{for } z = \infty \end{cases}. \end{aligned}$$

We now check that these charts are compatible. Let $z \in U_N \cap U_S = \mathbb{C}^*$, we compute

$$\varphi_S \circ \varphi_N^{-1}(z) = \varphi_S(\varphi_N^{-1}(z)) = \varphi_S(z) = \frac{1}{z},$$

which is a holomorphic map. The atlas $\mathcal{A} = \{U_N, U_S\}$ gives rise to a complex structure on $\hat{\mathbb{C}}$. The connectedness of $\hat{\mathbb{C}}$ is clear so we now conclude that $\hat{\mathbb{C}}$, equipped with this structure, is a Riemann surface which we call the ***Riemann sphere***. //

Example 1.5 (The Complex Torus) We first define a ***lattice*** Λ as a subgroup of $\mathbb{R}^2$ isomorphic to $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent when regarding $\mathbb{C}$ as a vector space over $\mathbb{R}$. Under the natural identification of $\mathbb{C}$ with $\mathbb{R}^2$, ω_1 and ω_2 generate a lattice Λ given by

$$\Lambda = \langle \omega_1, \omega_2 \rangle = \{n_1\omega_1 + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}.$$

Now we define an equivalence relation $\sim$ on $\mathbb{C}$ given by

$$z_1 \sim z_2 \text{ if and only if } z_1 - z_2 \in \Lambda \text{ for } z_1, z_2 \in \mathbb{C}$$

and denote the set of all equivalence classes by $\mathbb{C}/\Lambda$. Let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ be the projection map which takes each point in $\mathbb{C}$ to its equivalence class in $\mathbb{C}/\Lambda$. Let $\mathbb{C}/\Lambda$ be endowed with the quotient topology induced by the surjective map π. The map π is an open map for if $V \subseteq \mathbb{C}$ is open, $\pi^{-1}(\pi(V)) = \bigcup_{\lambda \in \Lambda} (\lambda + V)$, which is open and π is clearly continuous by definition. We now develop the complex structure on $\mathbb{C}/\Lambda$. Let $V \subset \mathbb{C}$ be open such that for all $v_1, v_2 \in V$, $v_1 \nsim v_2$. Since π is open, $U = \pi(V)$ is open in $\mathbb{C}/\Lambda$. Since no two points in V were allowed to be equivalent, this makes $\pi\big|_V$ a homeomorphism onto its image. Let $\varphi : U \to V$ denote the inverse of π and note that φ defines a complex chart on $\mathbb{C}/\Lambda$.

Let $\mathcal{A}$ be the set of all charts defined as above. We'd like to verify that all charts in $\mathcal{A}$ are pairwise compatible thereby establishing $\mathcal{A}$ as an atlas which will then induce a complex structure. Let $\varphi_i : U_i \to V_i$, $i = 1, 2$, be two charts for which $U_1 \cap U_2 \neq \varnothing$. Let $T = \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ be the transition map between these charts. Let $z \in \varphi_1(U_1 \cap U_2)$, then one may computationally verify that $T(z) - z$ is constant on every connected component of $\varphi_1(U_1 \cap U_2)$. Thus the transition map is holomorphic and the charts are compatible. The other necessary topological properties are also clearly seen to hold. Thus, the space $\mathbb{C}/\Lambda$ with the complex structure defined above defines a Riemann surface called a ***complex torus***, or simply ***torus***. //

Let us compare the above method of obtaining complex tori as the quotient of $\mathbb{C}$ by a lattice Λ with the method of pasting together pairs of opposite edges of a parallelogram in the plane $\mathbb{R}^2$. As topological spaces, they are homeomorphic. In fact, due to the above characterization theorem, all compact genus—1 Riemann surfaces— which are all topologically tori- can be obtained as quotients of the plane by some lattices. It is the *complex* structure that distinguishes genus-1 tori as geometric objects since different lattices give rise to different complex tori.

1.1.5 Isometries of $\mathbb{H}$

Here we state some useful identifications.

A ***fractional linear transformation*** is a function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$w = f(z) = \frac{az+b}{cz+d}$$

where a, b, c, d are complex constants satisfying $ad - bc \neq 0$. These maps are also called ***Möbius transformations***. If c is zero giving us a fractional linear transformation of the form $f(z) = az + b$, we define $f(\infty) = \infty$. If not, we define

$$f(-\frac{d}{c}) = \infty \quad \text{and} \quad f(\infty) = \lim_{z\to\infty} f(z) = \frac{a}{c}.$$

The set of all fractional linear transformations turns out to form a group which we denote for the time being by $\mathcal{M}$. All fractional linear transformations may be expressed as a composition of three special types of Möbius transformations.

Proposition 1.6. *Every fractional linear transformation is a composition of maps given by $z \mapsto az$ called dilations, $z \mapsto z + b$ called translations, and $z \mapsto 1/z$.*

Proof. Let $f(z) \in \mathcal{M}$. Suppose f fixes ∞, that is $f(\infty) = \infty$. Then $f(z) = az + b$ for some nonzero $a \in \mathbb{C}$. Then $f(z)$ is given by

$$z \mapsto z + \frac{b}{a} \mapsto a(z + \frac{b}{a}) = az + b.$$

If $f(\infty) = w$ for some $w \neq \infty$, then $f(z) = (az + b)/(cz + d)$ where $c \neq 0$. Then $f(z)$ is given by

$$z \mapsto z + \frac{d}{c} \mapsto \frac{1}{z + d/c} \mapsto -\frac{(ad-bc)}{c^2}\left(\frac{1}{z+d/c}\right) \mapsto -\frac{(ad-bc)}{c^2}\left(\frac{1}{z+d/c}\right) + \frac{a}{c} = \frac{az+b}{cz+d}.$$

■

Note that dilations, translations, and inversion are all analytic maps and therefore continuous. We now state without proof another useful property of fractional linear transformations.

Proposition 1.7. *Given any pair of three distinct points z_0, z_1, z_2 and w_0, w_1, w_2 in $\hat{\mathbb{C}}$, there is a unique fractional linear transformation $w = w(z)$ such that $w(z_0) = w_0$, $w(z_1) = w_1$, and $w(z_2) = w_2$.*

The group $\mathcal{M}$ actually comprises the group of all conformal automorphisms of $\hat{\mathbb{C}}$.

Proposition 1.7 (Conformal Automorphisms of $\hat{\mathbb{C}}$). *The group $Aut(\hat{\mathbb{C}})$ of all conformal automorphisms of the Riemann sphere is equal to the group of all Möbius transformations $\mathcal{M}$.*

Proof. We will first show that $\mathcal{M}$ is a subgroup of Aut$\hat{\mathbb{C}}$. Let $g(z) \in \mathcal{M}$. The inverse of $g(z)$ is given by

$$g^{-1}(w) = \frac{-dw + b}{cw - a}$$

which is again a Möbius transformation and which satisfies $g(g^{-1}(z)) = g^{-1}(g(z)) = z$ thereby showing that $g(z)$ is bijective. By Proposition 1.6, $g(z)$ is a composition of analytic maps and therefore analytic and continuous. This also applies to the inverse g^{-1} which shows that g is a homeomorphism. We now want to show that this homeomorphism g is holomorphic in terms of the complex charts of $\hat{\mathbb{C}}$. Let φ_N and φ_S be as in example 1.4, then we compute for any $z \in \mathbb{C}$,

$$\varphi_S \circ g \circ \varphi_N^{-1}(z) = \frac{1}{g(z)} \qquad \varphi_S \circ g \circ \varphi_S^{-1}(z) = \frac{1}{g(1/z)}$$
$$\varphi_N \circ g \circ \varphi_N^{-1}(z) = g(z) \qquad \varphi_N \circ g \circ \varphi_S^{-1}(z) = g(1/z).$$

Note that each of these maps is a Möbius transformation. Thus, $g(z)$ is a conformal automorphism and since g was arbitrary we have $\mathcal{M}$ is a subgroup of Aut$(\hat{\mathbb{C}})$. For the converse, see [11].

1.1.6 Special Curves: Horocycles and Horoballs

What follows is a qualitative discussion on horoballs and horocycles. See, for instance,[13].

In the hyperbolic plane, one may define a hyperbolic circle as the set of points equidistant from some point in $\mathbb{H}$ using hyperbolic distance. Horocycles are curves that are unique to hyperbolic space in that they do not have an analog in Euclidean space. We will only provide a general pictorial description of these curves. Let p be a point fixed on a hyperbolic line (recall that these are the geodesics in $\mathbb{H}$ so we're thinking of a vertical line). Let a be the center of a hyperbolic circle C on the same line as p which passes through p. As we take a to a point infinitely far away on the horizon $\partial\mathbb{H}$ at A, we obtain the curve in figure [a].

This curve is neither a hyperbolic circle or a hyperbolic line. It is a curve which is called a ***horocycle***. The curve in figure [b] is obtained similarly except that we allow a to tend toward infinity in the opposite direction. We will make use of these curves by relating them to moduli of families of curves. When that time comes, we will construct formulae which describe these curves explicitly.

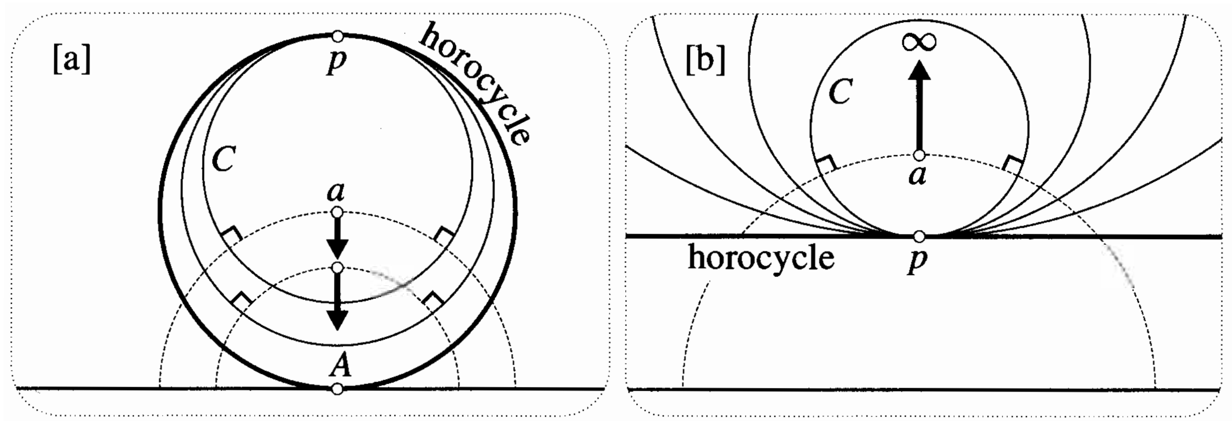


Figure 1.3: Qualitative depiction of horocycles

1.2 Teichmüller Space of a Punctured Sphere

Teichmüller theory has applications to many fields of mathematics. The general theory dates back to Riemann's work on Abelian functions [1]. The author's window into the subject was framed by Thurston's work which elucidated relations between Teichmüller space and the dynamics of rational maps of the Riemann sphere.

Since we will not have much occasion to utilize general results from Teichmmüller theory, we shall only provide the necessary information relevant for our purposes. More specifically, we will only need to know the Teichmüller space of the Riemann sphere $\hat{\mathbb{C}}$ with four points removed. The four points will be the four points of the postcritical set of a given nearly Euclidean Thurston map; for definitions, see chapters 2 and 3.

1.2.1 Identifying a Teichmüller Space with $\mathbb{H}$

We specialize to the Teichmüller space of a topological 2-sphere S^2 with a distinguished finite subset $P \subset S^2$. The Teichmüller space is defined by

$$\mathcal{T}(S^2, P) = \{\text{orientation-preserving homeomorphisms } \varphi : S^2 \to \hat{\mathbb{C}}\}/\sim$$

where $\varphi_0 \sim \varphi_1$ if there is a Möbius transformation $\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\mu \circ \varphi_1\Big|_P = \varphi_0\Big|_P$ and $\mu \circ \varphi_1$ is isotopic to φ_0 rel. P. There are several ways to think of elements in general Teichmüller spaces of surfaces of finite type, though, in our case, we think of elements as equivalence classes of conformal structures on (S^2, P). To reconcile the definition with this perspective on the points of $\mathcal{T}(S^2, P)$ we explain how an orientation-preserving homeomorphism $\varphi : S^2 \to \hat{\mathbb{C}}$ is representative of an equivalence class of conformal structures on S^2.

There exists a unique conformal structure on S^2 such that the homeomorphism $\phi : S^2 \to \hat{\mathbb{C}}$ is made analytic, see [5]. Having equipped S^2 with a complex structure we have that it is isomorphic to $\hat{\mathbb{C}}$ by uniformization and thus the equivalence relation defined above tells us that elements of $\mathcal{T}(S^2, P)$ are representing isomorphism classes of complex structures on (S^2, P).

There is a natural bijective correspondence between Teichmüller space of the simple torus T and the upper halfplane model, $\mathbb{H}$, of the hyperbolic plane, see, for instance chapter 10 of [4]. There is also a bijective correspondence between the conformal structures on the torus, and the conformal structures on (S^2, P). Hence, there is a bijective correspondence between $\mathcal{T}(S^2, P)$ and $\mathbb{H}$.

We will make use of this correspondence in chapter 4 when we discuss the halfspace theorem.

Chapter 2

Thurston's Characterization Theorem

In this chapter we define and elaborate on the setting of Thurston's characterization theorem. We state the theorem in the first section and note several aspects which make this theorem difficult to apply in practice. In the next chapter, we discuss some of the work done to parse this characterization in depth.

2.1 Characterization Theorem

We discuss here results contained in [**?**]. Let X and Y be two compact, connected, and oriented two-dimensional topological manifolds. A map $f : X \to Y$ is said to be a ***branched covering map*** if for each $x \in X$ there exist neighborhoods $U \subset X$ of x, $V \subset Y$ of $y = f(x)$, both homeomorphic to the unit disk D^2, $d \in \mathbb{N}$, and orientation-preserving homeomorphisms $\varphi : U \to D^2$ and $\psi : V \to D^2$ with $\varphi(x) = 0$ and $\psi(y) = 0$ such that $(\psi \circ f \circ \varphi^{-1})(z) = z^d$ for all $z \in D^2$. This definition is modelled on the local behavior of nonconstant holomorphic functions between Riemann surfaces, see [5].

For any branched covering map $f : X \to Y$, we call the integer $d \geqslant 1$ associated to any point $x \in S^2$ the ***local degree*** of f at x and denote it by $\deg_f(x)$. Let C_f be the ***critical set*** of f which consists of points x at which $\deg_f(x) \geqslant 1$. These are the points at which f fails to be locally injective and are called the ***critical points*** of f. If z is a critical point, we call $f(z)$ a ***critical value*** and note that critical values are isolated, that C_f is a discrete subset of X, and thus C_f must be finite since X is compact.

Every branched cover has a well-defined topological degree, $\deg(f)$. Let f denote a branched covering of degree $d \geqslant 2$. We may determine the order of C_f for any branched covering map $f : X \to Y$ counting multiplicity by the Riemann-Hurwitz formula which is given by

$$\chi(X) + \sum_{x \in C_f} (\deg_f(x) - 1) = \deg(f)\chi(Y)$$

where $\chi(X)$ and $\chi(Y)$ are the Euler characteristics of X and Y, respectively. The ***postcritical set*** P_f of f is given by

$$P_f := \bigcup_{n>0} f^{\circ n}(C_f)$$

and an orientation-preserving branched covering map $f : S^2 \to S^2$, with $\deg(f) \geqslant 2$, and whose postcritical set is finite is called a ***Thurston map***.

We will need a notion of equivalence that is weaker than topological conjugacy. Two Thurston maps $f : (S^2, P_f) \to (S^2, P_f)$ and $g : (S^2, P_g) \to (S^2, P_g)$ are ***combinatorially equivalent*** or ***Thurston equivalent*** if there are orientation-preserving homeomorphisms φ_o, $\varphi_1 : (S^2, P_f) \to (S^2, P_g)$ such that

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\varphi_1} & (S^2, P_g) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ (S^2, P_f) & \xrightarrow{\varphi_o} & (S^2, P_g) \end{array} \tag{2.1}$$

that is, $\varphi_o \circ f = g \circ \varphi_1$, with φ_o isotopic to φ_1 relative to P_f.

Let γ be a simple closed curve on $S^2 \backslash P_f$. Note that $f^{-1}(\gamma)$, the preimage of γ under a branched mapping f, is a union of disjoint simple closed curves. We say γ is ***non-peripheral*** if both components of $S^2 \backslash \gamma$ contains at least two points of P_f otherwise, γ is ***peripheral***.

A ***multicurve*** is a finite collection of simple, closed, pairwise disjoint, pairwise non-homotopic rel. P_f, non-peripheral curves $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$.

A multicurve is ***f-stable***, ***f-invariant***, or simply ***invariant*** if, for any $\gamma_i \in \Gamma$, every essential non-peripheral component of $f^{-1}(\gamma_i)$ is homotopic rel. P_f to some $\gamma_j \in \Gamma$.

Given any f-stable multicurve Γ we may construct an associated matrix called the ***Thurston matrix*** $M_\Gamma = (m_{ij})$, a linear transformation from $\mathbb{R}^\Gamma$, the set of functions from Γ to $\mathbb{R}$, to itself whose entries are given by

$$m_{ij} = \sum_{\gamma_{i,j,\alpha}} \frac{1}{(\deg f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j)} \tag{2.2}$$

where $\gamma_{i,j,\alpha}$ are the components of $f^{-1}(\gamma_j)$ homotopic to γ_i (indexed by α) in $S^2 \backslash P_f$. That is, to compute the ij-th entry, we take γ_i and γ_j, look at the preimages $f^{-1}(\gamma_j)$ of γ_j, find the ones homotopic to γ_i and take the sum of the reciprocals of the degrees with which f maps these homotopic preimages onto γ_j.

The entries of M_γ are non-negative and therefore, there must exist a largest positive real eigenvalue λ_Γ. We say that the multicurve Γ is a ***Thurston obstruction*** if $\lambda_\Gamma \geqslant 1$. We now state Thurston's characterization theorem:

THEOREM 2.1 (THURSTON'S THEOREM). *A postcritically finite branched map $f : S^2 \to S^2$ with hyperbolic orbifold is equivalent to a rational function if and only if for any f-stable multicurve Γ we have $\lambda_\Gamma < 1$.*

In other words, Thurston's theorem says that a postcritically finite branched map f with hyperbolic orbifold is combinatorially equivalent to a rational map if any only if there are no Thurston obstructions for f.

2.2 Problem Translation and Practical Difficulties

In this section we demonstrate how a Thurston map induces a map $\Sigma_f : \mathcal{T}_f \to \mathcal{T}_f$ called the ***Thurston pullback map***. We then recast the problem of determining the equivalence of a Thurston map with hyperbolic orbifold to a rational map as a question of when Σ_f has a fixed point. We shall merely state the theorem then move on to comment on the difficulties which arise out of applying this characterization in practice.

We obtain $\Sigma_f : \mathcal{T}(S^2) \to \mathcal{T}(S^2)$ as follows.

Let $f : S^2 \to S^2$ be a Thurston map and let $\mathcal{T}_f$ denote the associated Teichmüller space discussed in section 1.4.3. Let $\mathcal{A} \in \mathcal{T}_f$, so $\mathcal{A}$ is a conformal structure on (S^2, P_f). Consider a chart (U, φ) not containing any member of P_f. Then, we may define new charts on (S^2, P_f) by $(f^{-1}(U), \varphi \circ f)$. Now we must define new charts about points $x \in P_f$. We have a neighborhood U which appears locally as a punctured disk. Let $\varphi : U \to D_{r'}(z)$ denote the chart of (S^2, ν_f) equipped with the complex structure defined in section 1.2.1. We can extend to the punctured point by taking a branch of $(\varphi \circ f)^{1/\deg(f,x)}$. We thereby obtain a new complex structure $\mathcal{A}'$ in this way and thus set $\sigma_f([\mathcal{A}]) = [\mathcal{A}']$.

The problem becomes translated into demonstrating the following

THEOREM 2.2. *A Thurston map f is combinatorially equivalent to a rational function if and only if the Thurston pullback map σ_f has a fixed point*

For a proof, see [7].

To determine whether a Thurston map is combinatorially equivalent to a rational function, one must check to see if any Thurston obstructions exist. This is especially difficult since, in general, there may be infinitely-many conditions to check. It also isn't always clear how to parametrize the problem. In the next section we consider a class of maps for which the conditions necessary to check for obstructions become tractable.

Chapter 3

Nearly Euclidean Thurston Maps

3.1 Definitions and Examples

A ***nearly Euclidean Thurston map*** or ***NET map*** is a Thurston map f satisfying the following properties

(i) $\deg_f(x) = 2$ for all $x \in C_f$.

(ii) $\#P_f = 4$.

The first condition says the local degree of f at any critical point is 2 and the second condition says that the postcritical set is comprised of exactly 4 points. A Thurston map is called ***Euclidean*** if no postcritical point is critical, that is, if $P_f \cap \mathcal{C}_f = \varnothing$. We start with a few examples of NET maps.

1. Consider the function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by

$$f(z) = \frac{3z^2}{2z^3 + 1} \tag{3.1}$$

Since f is a nonconstant holomorphic map from S^2 to itself, it is an orientation-preserving branched covering map whose branch points correspond to C_f. We note $\deg(f) = 3$ and that $C_f = \{0, 1, w, \overline{w}\}$ where $w = \frac{-1}{2} + \frac{\sqrt{3}}{2}$. The determination of P_f is represented diagramatically by depicting the forward orbits of each point in C_f along with the degrees with which they map to their images and is shown below.

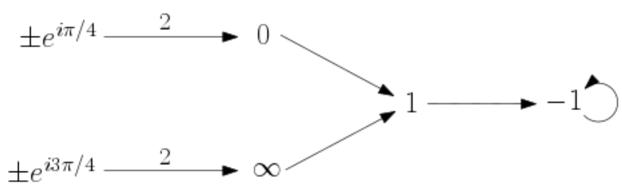
$$\infty \xrightarrow{1} 0 \circlearrowleft^2, \quad 1 \circlearrowleft^2, \quad w \underset{2}{\overset{2}{\rightleftarrows}} \overline{w} \tag{3.2}$$

We call the above scheme corresponding to the forward orbits of critical points the ***ramification portrait*** of f. From the ramification portrat, one may read off $P_f = \{0, 1, w, \overline{w}\} = C_f$. This map is, therefore, not a Euclidean Thurston map since every postcritical point is critical.

2. Consider the function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by

$$f(z) = \left(\frac{z^2 - i}{z^2 + i}\right)^2 \tag{3.3}$$

Again we have a Thurston map with $\deg(f) = 4$ with $C_f = \{\pm e^{i\pi/4}, \pm e^{i3\pi/4}\}$. The ramification portrait for f is shown below which shows $P_f = \{0, \pm 1, \infty\}$. Since we have



$P_f \cap C_f = \varnothing$ this map is a Euclidean Thurston map. This in fact makes it a NET map because, as we will show below, all Euclidean Thurston maps are also nearly Euclidean Thurston maps.

3. We may also define a NET map topologically using the following data taken from [8].

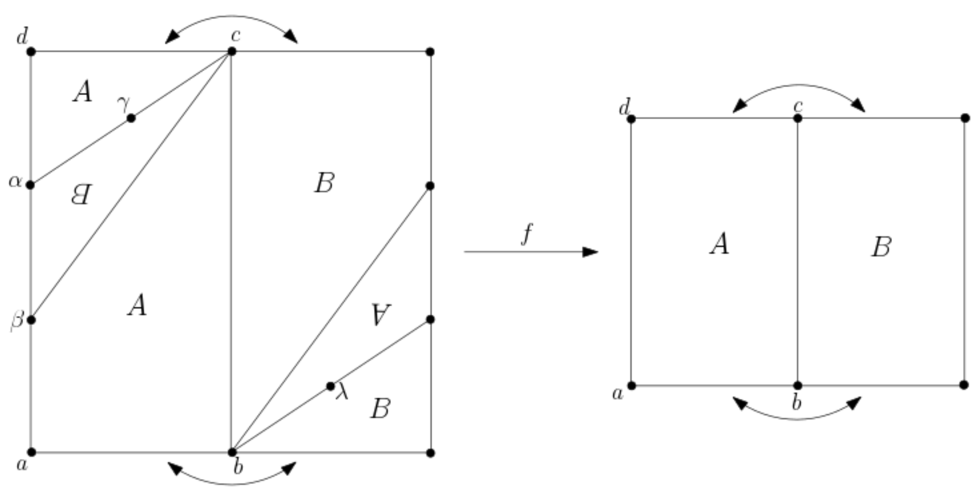


Figure 3.1: Subdivision rule for NET map f.

This map has the following ramification portrait:

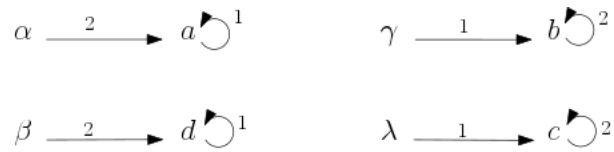


Figure 3.2: Ramification portrait of example 3.

4. And similar to the preceding example, a NET map may be defined topologically using the data given by

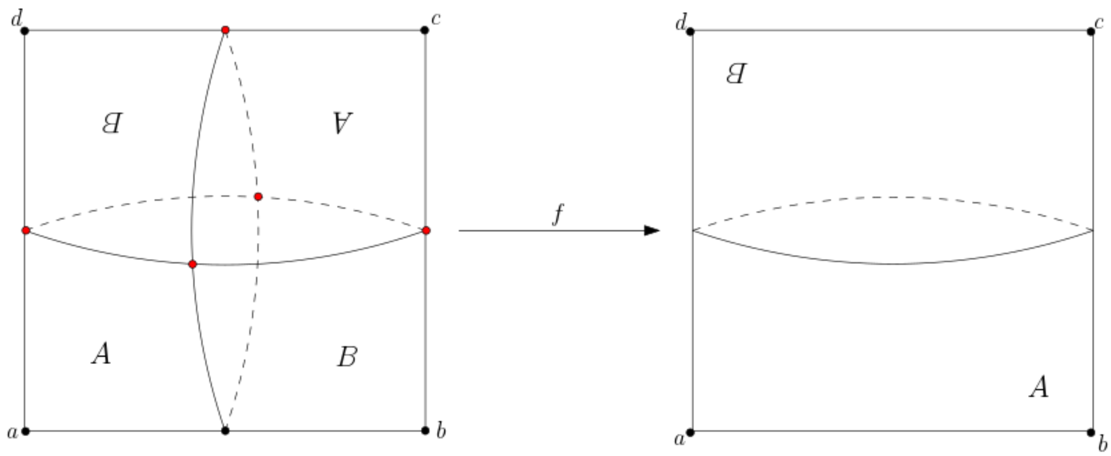


Figure 3.3: Cell structure data on topological sphere for NET map f.

and the ramification portrait

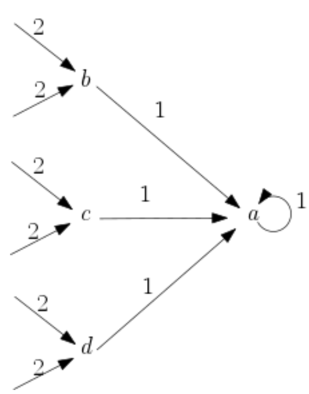


Figure 3.4: Ramification portrait of example 4.

Details as to why the data provided by the third and fourth examples suffice to define a NET map will be given in the following section through another example in which similar data is provided. The details will rely on the following properties of NET maps. We start with the following lemma taken from [2]

LEMMA 3.1 (NON-CRITICAL PREIMAGES). *Let $f : S^2 \to S^2$ be a NET map with postcritical set P_f. Then $f^{-1}(P_f)$ contains exactly 4 points which are not critical points. Furthermore, f is Euclidean if and only if these four points are the points of P_f.*

This lemma plays a crucial role in proving the following theorem which characterizes NET maps by the existence of lifts to maps of tori. The proof of this theorem can be found in [2].

THEOREM 3.2. *Let $f : S^2 \to S^2$ be a Thurston map. Then f is nearly Euclidean if and only if there exist branched covering maps $p_1 : T_1 \to S^2$ and $p_2 : T_2 \to S^2$ with degree 2 from tori T_1 and T_2 to S^2 such that the set of branch points of p_2 is the postcritical set of f and there exists a continuous map $\widetilde{f} : T_1 \to T_2$ such that $p_2 \circ \widetilde{f} = f \circ p_1$. If f is nearly Euclidean, then f is Euclidean if and only if the set of branch points of p_1 is the postcritical set of f.*

In section 2 of [2] it is shown how one may construct a NET map f from any given NET map g by applying what is described as a "twist" of g. The result is given in the following

THEOREM 3.3. *Let $g : S^2 \to S^2$ be a NET map and $h : S^2 \to S^2$ be an orientation-preserving homeomorphism such that $h(P_g) \subseteq g^{-1}(P_g)$. Define $f := h \circ g$. Then the following must hold:*

(i) If $\#P_f \geqslant 4$, then f is a NET map

(ii) If $deg(g) = 3$ or $deg(g) \geqslant 5$, then f is a NET map.

There is a partial converse to this theorem which is given by

THEOREM 3.4. *Let $f : S^2 \to S^2$ be a NET map with $P_1 = P_1(f)$ the set of points in the preimage of P_f which are not critical, $P_2 = P_f$, and let $h : S^2 \to S^2$ be an orientation-preserving homeomorphism such that $h(P_1) = P_2$. Then $f = h \circ g$ where g is a Euclidean Thurston map with $P_g = P_1$ and $P_2 \subseteq g^{-1}(P_g)$ so that $h(P_g) \subseteq g^{-1}(P_g)$.*

These theorems allow us to construct large collections of examples of NET maps using subdivision maps g of finite subdivision rules $\mathcal{Q}$ and orientation-preserving homeomorphisms $h : S^2 \to S^2$ with $h(P_g) \subseteq g^{-1}(P_g)$ which take the 1-skeleton of S^2 into the 1-skeleton of the first subdivision $\mathcal{Q}(S^2)$, see for example section 3 of [2].

The above results are utilized in analyzing the preimages of multicurves. In general, determining the preimages of multicurves is a difficult task. In the setting of the above theorems, however, looking at preimages of multicurves will be made in some sense easier. We illustrate this sense in the following section with an example. We construct NET maps given a relatively small amount of data.

3.2 NET Map Presentations

The results of the previous section suggest how one may construct a NET map. First, we start by obtaining a Euclidean Thurston map. We will do this by letting $\Lambda_2 = \mathbb{Z} \times \mathbb{Z}$ and choosing a proper sublattice $\Lambda_1 = \langle (2,0),(0,2) \rangle$ and constructing an orientation-preserving affine isomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi(\Lambda_2) = \Lambda_1$.

Let Γ_i be the group of isometries of $\mathbb{R}^2$ of the form $x \mapsto 2\lambda \pm x$ where $\lambda \in \Lambda_i$ and $i = 1, 2$. This is the group generated by 180 degree rotations about elements of Λ_i. A fundamental domain for the action of Γ_2 on $\mathbb{R}^2$ is the closed rectangle $[0,2] \times [0,1]$ and we denote this domain by F_2. We equip F_2 with a cell structure by specifying its vertices as the points $(0,0)$, $(1,0)$, $(2,0)$, $(2,1)$, $(1,1)$ and $(0,1)$ with its 1-skeleton being the boundary of F_2. The images of F_2 under elements of Γ_2 form a tilling of the plane which we denote by S_2.

Similarly, the fundamental domain for the action of Γ_1 on $\mathbb{R}^2$, which we denote F_1, is the closed rectangle $[0,4] \times [0,2]$. We equip this domain with a cell structure by making its 1-skeleton the boundary of F_1 and letting the vertices be the points $(0,0)$, $(2,0)$, $(4,0)$, $(4,2)$, $(2,2)$, and $(0,2)$. We let S_1 denote the tiling of the plane given by the images of F_1 under the action of Γ_1.

Let $j \in \{1,2\}$. Let $T_j = \mathbb{R}/2\Lambda_j$. Let $q_j : \mathbb{R}^2 \to T_j$ be the canonical quotient map from $\mathbb{R}^2$ to T_j and let $p_j : T_j \to \mathbb{R}^2/\Gamma_j$ be the canonical quotient map from T_j to $\mathbb{R}^2/\Gamma_j$. For $j \in \{1,2\}$, $\mathbb{R}^2/\Gamma_j$ has an induced tiling coming from S_j.

We obtain an identification map $\phi : \mathbb{R}^2/\Gamma_2 \to \mathbb{R}^2/\Gamma_1$ induced by the affine isomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi(\Lambda_2) = \Lambda_1$ which is clearly given by

$$\Phi(x) = 2x$$

for any $x \in \mathbb{R}^2$ so we have $\Phi(x) = Ax$ where $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. This map induces a homeomorphism $\phi : \mathbb{R}^2/\Gamma_2 \to \mathbb{R}^2/\Gamma_1$ and we identify the resulting spaces with S^2 consisting of a single tile. The identification is a cellular homeomorphism taking the induced 1-skeleton of $\mathbb{R}^2/\Gamma_2$ to the induced 1-skeleton of $\mathbb{R}^2/\Gamma_1$.

We define a Euclidean Thurston map $g : S^2 \to S^2$ by the composition $g = \overline{\Phi} \circ \overline{\mathrm{id}}$ where $\overline{\Phi}$ (resp. $\overline{\mathrm{id}}$) is the quotient of the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ (resp. $\mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2$). The postcritical set P_g of g is the set of branch points of p_1 and p_2. The map g is a subdivision map of a finite subdivision rule $\mathcal{Q}$. The subdivision complex of $\mathcal{Q}$ is S^2 with the cell structure consisting of the pushforward of S_1 under $p_1 \circ q_1$ which is equivalent to the pushforward of S_2 under $p_2 \circ q_2$. We take a push-map homeomorphism $h : \mathbb{R}^2/\Gamma_1 \to \mathbb{R}^2/\Gamma_1$ which fixes the image of $(0,2)$, sends the image of $(0,0)$ to the image of $(1,1)$, the image of $(2,0)$ to the image of $(3,2)$, and the image of $(2,2)$ to the image of $(4,2)$.

Let $f = h \circ g$. We see that $h(P_g) \subseteq g^{-1}(P_g)$ and so it follows that f is a NET map which is the subdivision map of a finite subdivision rule.

For any given NET map f, we can always model f using a presentation like the one outlined above. An essential nonperipheral multicurve must contain 2 postcritical points. This enables one to associate a core arc connecting the two points contained by a an essential nonperipheral multicurve. Understanding preimages of essential multicurves then amounts to understanding the preimages of these associated core arcs and lifting these core arcs to the plane enables one to associate a slope to each essential nonperipheral multicurve. See [2] for more information.

In the next section we discuss the halfspace theorem proved in [2] and make a modest extension.

Chapter 4

The Halfspace Theorem

We discuss in this section a theorem which provides halfspaces in the upper halfplane that do not contain any Thurston obstructions.

4.1 Horocycles in Teichmüller Space

In the preliminaries chapter, we discussed horocycles in the upper halfplane. These were the curves one obtains by considering a circle in the hyperbolic plane and allowing its radius to tend to infinity.

If one lets the center of the circle head in the direction opposite to the horizon, the horocycle is a horizontal line, given by $\text{Im}(z) = m$ for positive real numbers m. Let p and q be relatively prime integers with $q \neq 0$. Letting the center of a hyperbolic circle travel along a vertical geodesic, which passes through a fixed point on the same geodesic, tend to $\frac{p}{q}$ on the horizon, one obtains the horocycle in $\mathbb{H}$ centered at $\frac{p}{q}$ as shown in the figure below. As in the figure,

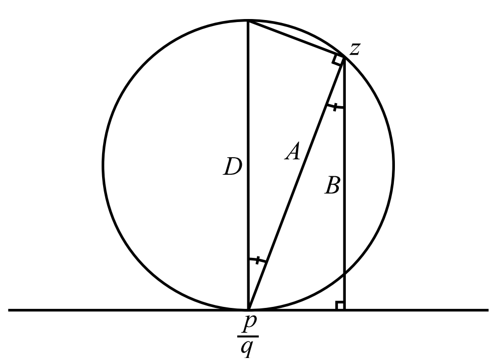


Figure 4.1: Euclidean geometry of horocycle centered at p/q

we denote the Euclidean diameter of the horocycle at $\frac{p}{q}$ by D. The two right triangles of the figure are similar which yields the following:

$$\frac{B}{A} = \frac{A}{D} \Longleftrightarrow \frac{B}{A^2} = \frac{1}{D} \Longleftrightarrow \frac{\text{Im}(z)}{|z - p/q|^2} = \frac{1}{D} \Longleftrightarrow \frac{\text{Im}(z)}{|qz - p|^2} = m,$$

where $m = \frac{1}{q^2 D}$ and $D = \frac{1}{q^2 m}$. We see that horoballs, of varying Euclidean diameter, in $\mathbb{H}$ at $\frac{p}{q} \in \hat{\mathbb{Q}}$ are subsets given by $\{z \in \mathbb{H} : \frac{\text{Im}(z)}{|qz-p|^2} > m\}$ for positive real numbers m. Our aim in the next section is to connect horoballs in $\mathbb{H}$ to moduli of curve families. A natural thing to consider then are the types of isometries of $\mathbb{H}$ which take horocycles to horocycles.

As we noted in the hyperbolic geometry section of the preliminaries chapter, the orientation-preserving isometries of $\mathbb{H}$ are in bijective correspondence with $\text{PSL}(2, \mathbb{R})$. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2, \mathbb{Z})$ with determinant equal to -1, then $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ is an orientation-reversing isometry of $\mathbb{H}$ so we view $\text{PGL}(2, \mathbb{Z})$ as a subgroup of the group of isometries of $\mathbb{H}$. The next lemma tells us how $\text{PGL}(2, \mathbb{Z})$ acts on horoballs in $\mathbb{H}$.

Lemma 4.1. *Let $\varphi \in PGL(2, \mathbb{Z})$, let $\frac{p}{q} \in \hat{\mathbb{Q}}$ and suppose that $\varphi(\frac{p}{q}) = \frac{p'}{q'} \in \hat{\mathbb{Q}}$. Then,*

$$\frac{\text{Im}(\varphi(z))}{|q'\varphi(z) - p'|^2} = \frac{\text{Im}(z)}{|qz - p|^2}$$

Proof. It suffices to prove this lemma for $z \mapsto -\bar{z}$, $z \mapsto z + 1$, and $z \mapsto -\frac{1}{z}$ as these three elements of $\text{PGL}(2, \mathbb{Z})$ generate $\text{PGL}(2, \mathbb{Z})$ and if the lemma is true for $\varphi_1, \varphi_2 \in \text{PGL}(2, \mathbb{Z})$ then it is true for $\varphi_1 \circ \varphi_2$.

1. Let $\varphi(z) = -\bar{z}$. Then we have that $\varphi(\frac{p}{q}) = -\frac{p}{q} = \frac{p'}{q'}$ so we may assume that $p' = p$ and $q' = -q$. Since $\text{Im}(\varphi(z)) = \text{Im}(-\bar{z}) = \text{Im}(z)$, we have

$$\frac{\text{Im}(\varphi(z))}{|q'\varphi(z) - p'|^2} = \frac{\text{Im}(z)}{|(-q)(-\bar{z}) - p|^2} = \frac{\text{Im}(z)}{|qz - p|^2}$$

   and the lemma holds for this case.

2. Let $\varphi(z) = z + 1$. Then, $\varphi(\frac{p}{q}) = \frac{p'}{q'} = \frac{p}{q} + 1$ so we may assume $p' = p + q$ and $q' = q$. Since $\text{Im}(\varphi(z)) = \text{Im}(z + 1) = \text{Im}(z)$, we compute

$$\frac{\text{Im}(\varphi(z))}{|q'\varphi(z) - p'|^2} = \frac{\text{Im}(z)}{|q(z + 1) - (p + q)|^2} = \frac{\text{Im}(z)}{|qz - p|^2}$$

   so the lemma holds for this case as well.

3. Lastly, let $\varphi(z) = -\frac{1}{z}$. Then $\varphi(\frac{p}{q}) = \frac{p'}{q'} = -\frac{q}{p}$ and so we may assume $p' = -q$ and $q' = p$. Since $\mathrm{Im}(\varphi(z)) = \mathrm{Im}(-\frac{1}{z}) = \frac{\mathrm{Im}(z)}{|z|^2}$, we compute

$$\frac{\mathrm{Im}(\varphi(z))}{|q'\varphi(z) - p'|^2} = \frac{(\mathrm{Im}(z)/|z|^2)}{|(p)(-1/z) - (-q)|^2} = \frac{\mathrm{Im}(z) \cdot \frac{1}{|z|^2}}{(\frac{1}{|z|^2})|qz - p|^2} = \frac{\mathrm{Im}(z)}{|qz - p|^2}$$

so the lemma holds in this case thus completing the proof. ■

This lemma tells us that the action of these orientation-reversing isometries also sends horoballs to horoballs. We state this in the following

Corollary 4.2. *Let $\varphi \in \mathrm{PGL}(2, \mathbb{Z})$, let $\frac{p}{q} \in \hat{\mathbb{Q}}$ and suppose that $\varphi(\frac{p}{q}) = \frac{p'}{q'} \in \hat{\mathbb{Q}}$. Then φ maps the horoball $\{z \in \mathbb{H} : \frac{Im(z)}{|qz-p|^2} > m\}$ bijectively onto the horoball $\{z \in \mathbb{H} : \frac{Im(z)}{|q'z-p'|^2} > m\}$ for every positive real number m.*

In the next section, we relate horocycles and horoballs to moduli of curve families.

4.2 Moduli of Curve Families and Horoballs

We shall not provide any justification for the lemma we state in this section but instead refer the reader to [2] for a proof. The modulus method is a tool often utilized in geometric function theory. One considers a family of curves in some domain space and certain metrics considered admissible with respect to this family. The admissibility criterion for a metric ρ is usually that the length of γ with respect to this metric, $l_\rho(\gamma)$, be at least 1 for all γ in the family.

Then one may define a quantity which depends on the length and area of the space in which the curves reside. Taking the infimum over all admissible metrics yields what's called the modulus of the curve family. What makes this quantity useful is its conformal invariance. That is, if one maps the space in which the curves of the family reside to another space in such a way as to preserve the angles between curves, the modulus for the images of the family of curves under the conformal map will be the same as the original modulus. Let us make these notions precise in our present context.

Let $\tau \in \mathbb{H}$, let $\Lambda_\tau = \langle 1, \tau \rangle$, and let $T_\tau = \mathbb{C}/\Lambda_\tau$. These constructions were discussed in section 1.2.2 and 1.2.3 of the preliminaries chapter. There we established the view of T_τ as a torus with complex structure. Recall that we can associate a slope to a simple closed curve γ in T_τ through choosing the ordered basis $(1, \tau)$ of Λ_τ and lifting γ to the universal covering space $\mathbb{R}^2$ of T_τ. Our families of curves will be defined as follows. We denote by $\Gamma_{\frac{p}{q},\tau}$ the set of simple closed curves in T_τ with slope $\frac{p}{q}$ which we may, at times, abbreviate to $\Gamma_{\frac{p}{q}}$.

Abusing notation, let dz denote the differential 1-form on T_τ induced by the standard 1-form on $\mathbb{C}$. Let ρ be any nonnegative Borel measurable function on T_τ. This condition on ρ

suffices to establish the existence of

$$L_\rho(\Gamma_{\frac{p}{q}}) = \inf_{\gamma \in \Gamma_{\frac{p}{q}}} \int_\gamma \rho |dz| \quad \text{and} \quad A_\rho = \iint_{T_\tau} \rho^2 |dz|^2.$$

The ***extremal length*** of the curve family $\Gamma_{\frac{p}{q}}$ on T_τ is

$$\sup_\rho \frac{L_\rho(\Gamma_{\frac{p}{q}})}{A_\rho}.$$

The modulus of the curve family $\Gamma_{\frac{p}{q}}$ on T_τ is defined as the reciprocal of the extremal length, given as

$$\text{mod}_\tau\left(\frac{p}{q}\right) = \inf_\rho \frac{A_\rho}{L_\rho(\Gamma_{\frac{p}{q}})}.$$

The conformal invariance of mod_τ is then stated more precisely by saying that if $\phi : T_\tau \to T_{\tau'}$ is a conformal isomorphism sending $\Gamma_{\frac{p}{q},\tau}$ to $\phi(\Gamma_{\frac{p}{q},\tau}) = \Gamma_{\frac{p'}{q'},\tau'}$, then

$$\text{mod}_\tau\left(\frac{p}{q}\right) = \text{mod}_{\tau'}\left(\frac{p'}{q'}\right).$$

Now we state the lemma we will not justify which relates the moduli of the curve families we defined above to horocycles in $\mathbb{H}$.

Lemma 4.3. *For every $\frac{p}{q} \in \hat{\mathbb{Q}}$ and $\tau \in \mathbb{H}$ we have that*

$$\text{mod}_\tau\left(\frac{p}{q}\right) = \frac{\text{Im}(\tau)}{|p\tau + q|^2}$$

This tells us that the modulus of the curve family $\Gamma_{\frac{p}{q},\tau}$ is equal to $\frac{1}{q^2 D}$ where D is the Euclidean diameter of the horocycle at $-\frac{q}{p}$ and that τ is on this horocycle. For every $\frac{p}{q} \in \hat{\mathbb{Q}}$ and every positive real number m, let

$$B_m\left(\frac{p}{q}\right) = \{\tau \in \mathbb{H} : \text{mod}_\tau\left(\frac{p}{q}\right) > m\}.$$

Then Lemma 4.3 implies the following

Corollary 4.4. *If $\frac{p}{q} \in \hat{\mathbb{Q}}$ and if m is a positive real number, then*

$$B_m\left(\frac{p}{q}\right) = \{\tau \in \mathbb{H} : \frac{\text{Im}(\tau)}{|p\tau + q|^2} > m\}$$

which is a horoball in $\mathbb{H}$ at $-\frac{q}{p}$.

The families of curves we defined the modulus for above were simple closed curves in the torus. We would like a way of making the family of curves we study be the system of multicurves we defined earlier. We must make a slight alteration to the modulus when this is the case.

Consider the map $z \mapsto -z$ and the quotient space S_τ of T_τ it determines. There is a corresponding degree 2 branched covering map $p_\tau : T_\tau \to S_\tau$ and we let P_τ be the set of four branch points of p_τ in S_τ. Let $\frac{p}{q} \in \hat{\mathbb{Q}}$. The set of essential (meaning not homotopic to a point, puncture, or boundary component) simple closed curves in $S_\tau \backslash P_\tau$ with slope $\frac{p}{q}$ lifts under p_τ to $\Gamma_{\frac{p}{q},\tau}$. Lengths of curves are unaltered under pullback via p_τ but area doubles so we can define moduli of systems of multicurves in $S_\tau \backslash P_\tau$ as we did above but with a factor of $\frac{1}{2}$. That is, the new moduli are equal to $\frac{1}{2}\text{mod}_\tau(\frac{p}{q})$.

In the next section we discuss the halfpsace theorem which will make use of the above constructions and give us a way of determining halfspaces in $\mathbb{H}$ which do not contain any Thurston obstructions.

4.3 The Half-space Theorem

Let f be a NET map. We define a function $\delta_f : \hat{\mathbb{Q}} \to \mathbb{Q}$ whose value will determine if the associated multicurve is an obstruction. Let $\frac{p}{q} \in \hat{\mathbb{Q}}$. Let γ be an essential simple closed curve in $S^2 \backslash P_2$ with slope $\frac{p}{q}$. Note, f will map each connected component of $f^1(\gamma)$ to γ with the same degree, which we denote by d. Some of these connected components may end up being inessential or peripheral (meaning some component of $S^2 \backslash \gamma$ contains less than 2 points of P_2) after being pulled back. We let c be the number of these connected components which are essential and non-peripheral. We then set $\delta_f(\frac{p}{q}) = \frac{c}{d}$.

Since the cardinality of P_2 is 4, a multicurve in $S^2 \backslash P_2$ can have at most 1 element, since, in general, a multicurve can have at most $|P_f| - 3$ elements. Recall that a multicurve is f-stable if every component of $f^{-1}(\gamma_i)$ is either homotopic to some $\gamma_j \in \Gamma$ or is inessential or peripheral. Then in the context of the NET map f, the multicurve, which only consists of a single γ is f-stable if and only if either $\sigma_f(\frac{p}{q}) = \frac{p}{q}$ or $\sigma_f(\frac{p}{q}) = o$. If Γ is f-stable, then the Thurston matrix A^Γ is 1×1 with entry $\delta_f(\frac{p}{q})$. Therefore, Γ is a Thurston obstruction if and only if $\frac{p}{q} \in \text{Fix}(\sigma_f)$ and $\delta_f(\frac{p}{q}) \geqslant 1$.

Continuing in this setting, recall the map f induces $\Sigma_f : \mathbb{H} \to \mathbb{H}$ on Teichmüller space, now identified with the upper halfplane as was done in the preliminaries chapter, given by pulling back complex structures. Let $\tau \in \mathbb{H}$, $\tau' = \Sigma_f(\tau)$, $\frac{p'}{q'} = \sigma_f(\frac{p}{q})$ and let $\delta = \delta_f(\frac{p}{q})$. Then, the subadditivity of moduli implies

$$\text{mod}_{\tau'}\left(\frac{p'}{q'}\right) \geqslant \delta \, \text{mod}_\tau\left(\frac{p}{q}\right).$$

This implies the following

$$\Sigma_f\left(B_m\left(\frac{p}{q}\right)\right) \subseteq B_{\delta m}\left(\frac{p'}{q'}\right) \tag{4.1}$$

telling us where the horoball will go under Thurston's pullback map. This relation will be vital in proving the Half-space Theorem. Let $d(\cdot,\cdot)$ denote the hyperbolic distance function, related but ultimately distinguished from the Riemannian metric which we named the hyperbolic metric in the preliminaries chapters. If H is a half-space in $\mathbb{H}$, let $\partial_\infty H$ denote the set of points in the boundary of $\mathbb{H}$ which are limits of sequences in H.

Theorem 4.5 (The Half-space Theorem).

1. *If $\frac{p}{q} \neq \frac{p'}{q'}$, then for every sufficiently large m, the closed horoballs $B = \overline{B_m(\frac{p}{q})}$ and $B' = \overline{B_{\delta m}(\frac{p'}{q'})}$ are disjoint. When they are disjoint, the set $H = \{\tau \in \mathbb{H} : d(\tau, B) < d(\tau, B')\}$ is an open hyperbolic half-space which is independent of m.*

2. *If $\frac{r}{s} \in Fix(\sigma_f)$ and $-\frac{s}{r} \in \partial_\infty H$, then $\delta_f(\frac{r}{s}) < 1$; that is, there is no Thurston obstruction with slope $\frac{r}{s}$.*

3. *If $\tau_0 \in Fix(\Sigma_f)$, then $\tau_0 \notin H$.*

We give the proof which appears in [2] not just for completeness but because the proof of the modest extension in the next section will be aided by a picture of the geometry associated with proving the Half-space Theorem.

Proof.

1. Since m is inversely proportional to the Euclidean diameter of the horoball at $\frac{p}{q}$, it is clear that there exists an m sufficiently large such that B and B' are disjoint. Assume m to be this large. In general, for all $m > 0$, $\frac{a}{b} \in \hat{\mathbb{Q}}$, and $t > 0$, the hyperbolic distance between the horocycles $\partial B_m(\frac{a}{b})$ and $\partial B_{tm}(\frac{a}{b})$ is equal to $|\ln(t)|$; in particular, this is independent of m. Let l be the hyperbolic geodesic joining $-\frac{q}{p}$ and $-\frac{q'}{p'}$, let $l_m \subseteq l$ be the closure of the geodesic segment lying outside $B \cup B'$ and let $l_m^\perp$ be its perpendicular bisector. Then $l_m^\perp$ is independent of m, and so H is an open hyperbolic half-space independent of m proving statement 1.

2. Let $\frac{r}{s} \in \text{Fix}(\sigma_f)$, $-\frac{s}{r} \in \partial_\infty H$ and, for the sake of contradiction, suppose $\delta_f(\frac{r}{s}) \geqslant 1$. Choose m such that B and B' intersect in a single point. Now let $m^* > 0$ be such that B and $\overline{B_{m^*}(\frac{r}{s})}$ also intersect in a single point η. The assumption $-\frac{s}{r} \in \partial_\infty H$ implies that $\overline{B_{m^*}(\frac{r}{s})} \cap B' = \varnothing$. Equation (4.1) implies that $\Sigma_f(B) \subseteq B'$ and $\Sigma_f(\overline{B_{m^*}(\frac{r}{s})}) \subseteq \overline{B_{m^*}(\frac{r}{s})}$ since $\delta_f(\frac{r}{s}) \geqslant 1$. This, however, would imply that $\Sigma_f(\eta) \in B' \cap \overline{B_{m^*}(\frac{r}{s})}$, a contradiction. This proves 2.

3. Lastly, suppose $\tau_0 \in \text{Fix}(\sigma_f)$. Let $\tau_1 \in B$ realize the distance between τ_0 and B. Equation 4.1 implies that $\Sigma_f(B) \subseteq B'$ and Σ_f is distance nonincreasing, so

$$d(\tau_0, B) = d(\tau_0, \tau_1) \geqslant d(\tau_0, \Sigma_f(\tau_1)) \geqslant d(\tau_0, B').$$

Therefore, $\tau_0 \notin H$, which proves 3.

This proves the Half-space theorem. ■. The figures below depict both the general circumstances of the proof and the case of part 2 where $\frac{p}{q} = 0$.

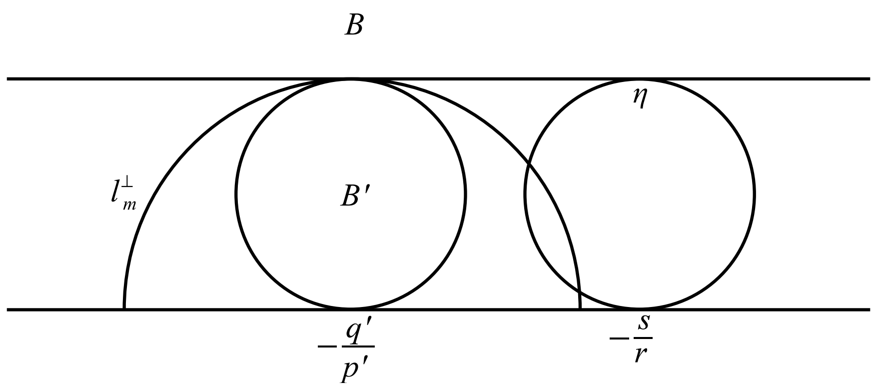


Figure 4.2: Setting for the proof of the second part of the half-space theorem.

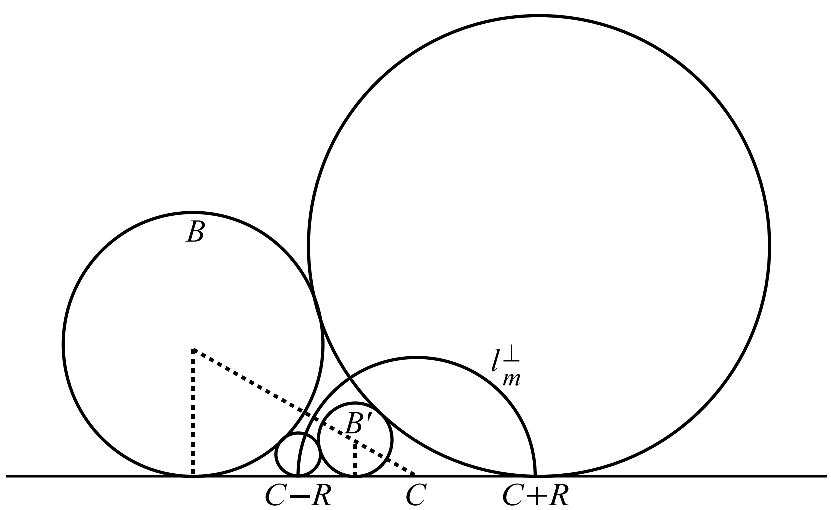


Figure 4.3: General setting of halfspace theorem. Halfspace centered at C with radius R

In the next section we shall prove a refinement of the Half-space theorem which gives us a statement explicitly about curves whose slopes are fixed by σ_f.

Chapter 5

New Halfspaces from Old

We discuss in this chapter a modest extension of the Half-space Theorem. We will start by deriving a bound on the multipliers of fixed points. We then use this to determine open hyperbolic half spaces whose infinite boundaries do not contain fixed points.

5.1 Multipliers of Fixed Points

We recall the setting of the Half-space Theorem of section 4.3. Let f be a NET map. Let $s \in \hat{\mathbb{Q}}$ be the slope of a simple closed curve in $S^2 \backslash P_f$ whose preimages under f contain an essential and nonperipheral connected component. Let $s' = \sigma_f(s)$ and suppose $s' \neq s$. The closed horoballs $B = \overline{B_m(s)}$ and $B' = \overline{B_{\delta m}(s')}$ are disjoint for sufficiently large m. We let $H = \{\tau \in \mathbb{H} : d(\tau, B) < d(\tau, B')\}$ be the halfspace determined by the halfspace theorem, independent of m.

We define a function $\delta_f : \hat{\mathbb{Q}} \to \mathbb{Q}$ as follows. Let γ be an essential simple closed curve in $S^2 \backslash P_f$ with slope $s \in \hat{\mathbb{Q}}$. Let c be the number of essential and nonperipheral connected components of $f^{-1}(\gamma)$ and let d be the degree with which f maps these connected component of $f^{-1}(\gamma)$ to γ. Then δ_f is given by $\delta_f(s) = \frac{c}{d}$. We now obtain a lower bound on the value of this function in the case when $s \in \text{Fix}(\sigma_f)$.

Lemma 5.1. Let f be a NET map and let $\beta = \deg(f)$ denote its topological degree. Suppose $t \in \hat{\mathbb{Q}}$ with $\sigma_f(t) = t$ and let $\delta_f(t) = \delta$ be the multiplier of t. Then,

$$\delta_f(t) \geqslant \frac{1}{\beta}.$$

Proof. First, let γ be an essential nonperipheral simple closed curve in $S^2 \backslash P_f$ with slope t. We obtain a lower bound on c, the number of essential and nonperipheral connected components of $f^{-1}(\gamma)$. Since $t \in \mathrm{Fix}(\sigma_f)$, there is at least one essential and nonperipheral connected component of $f^{-1}(\gamma)$, namely γ.

We obtain an upper bound on d, the degree with which f maps these connected component of $f^{-1}(\gamma)$ to γ. It is clear that the topological degree of the map is at least d and hence,

$$\delta_f(t) = \frac{c}{d} \geqslant \frac{1}{\beta}$$

as desired. ■

In the next section, we will use this bound to obtain our extension of the halfspace theorem. Recall that the second part of the halfspace theorem tells us that a slope r/s, fixed by the slope map σ_f, whose negative reciprocal resides in $\partial_\infty H$, does not represent an obstruction. Our aim is to determine a halfspace whose infinite boundary does not contain the negative reciprocal of any fixed slope.

5.2 Fixed Points and the Halfspace Theorem

We state and prove our extension of the Half-space theorem.

Theorem 5.2. Let f be a NET map. Let $s \in \hat{\mathbb{Q}}$ be the slope of an essential simple closed curve in $S^2 \backslash P_f$ and suppose $\sigma_f(s) = s'$ with $s \neq s'$. Then there exists an open hyperbolic halfspace $\widetilde{H}$ such that if $t \in \mathrm{Fix}(\sigma_f)$, then $-1/t \notin \partial_\infty \widetilde{H}$.

Proof. By corollary 6.2 of [NET maps], we may assume, without loss of generality that $s = 0$ so that $B = \overline{B_m(s)}$ is a horoball centered at infinity. Then ∂B is a horocycle centered at infinity and hence is given by an equation $\mathrm{Im}(z) = r$ for some positive real number r. Let $\widetilde{B} = \overline{B_n(t)}$, so $\widetilde{B}$ represents a horoball centered at $-1/t$. Let $\beta = \deg(f)$ be the topological degree of f and note $\beta \geqslant 2$.

Figure 5.1 depicts our current setting. Let Δ be the Euclidean radius of the halfspace H. We choose $m > 0$ such that B and B' are tangent and $m^* > 0$ such that B and $\widetilde{B}$ are tangent and let η be the single point at which B and $\widetilde{B}$ intersect. Note, the Euclidean diameters of B' and $\widetilde{B}$ are both equal to Δ. Let C be the horoball centered at $-1/t$ whose Euclidean diameter is $\widetilde{\Delta} = \beta\Delta$.

We construct $\widetilde{H}$ as follows. We define the complement $\widetilde{H}^c$ to be the closure of the intersection of the Euclidean disk centered at $-1/s'$ with radius $\Delta + \ln(\beta)$. Note, in general, one obtains $\widetilde{H}$ by shrinking the halfspace H by the hyperbolic distance $\ln(\beta)$.

Suppose, for the sake of contradiction, $-1/t \in \partial_\infty \widetilde{H}$. Note, $C \cap B' = \varnothing$ since for these two horoballs to intersect we would need $d_E(-1/s', -1/t) \leqslant \frac{1}{2}\Delta + \frac{1}{2}(\Delta + \ln(\beta)) = \Delta + \frac{1}{2}\ln(\beta)$, where d_E denotes the standard Euclidean metric, but we have $d_E(-1/s', -1/t) > \delta + \ln(\beta) > \Delta + \frac{1}{2}\ln(\beta)$ because we've assumed $-1/t \in \partial_\infty \widetilde{H}$.

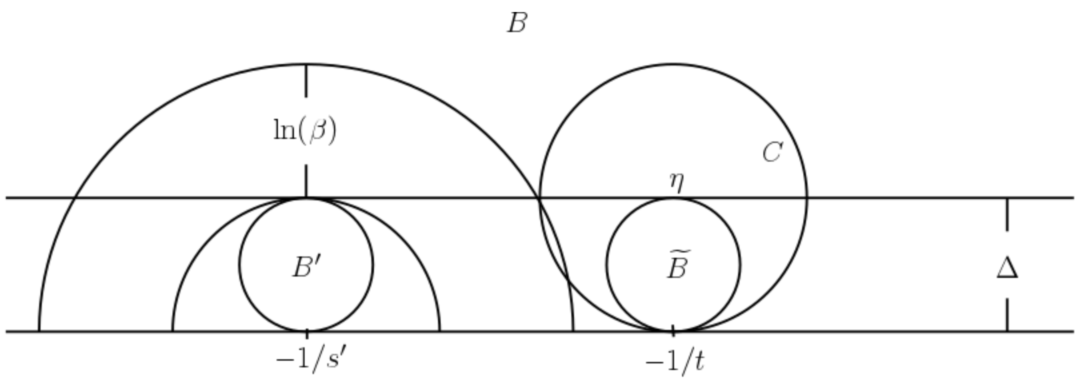


Figure 5.1: Setting for the proof of the extension of the half-space theorem.

(4.1) implies that $\Sigma_f(B) \subseteq B'$ and $\Sigma_f(\widetilde{B}) \subseteq \overline{B_{\widetilde{\delta}m^*}(t)} \subseteq C$. The last containment follows from lemma 5.1 where $\widetilde{\delta} \geqslant \frac{1}{\beta}$ implies $1/\widetilde{\delta} \leqslant \beta$. Letting D_1 be the Euclidean diameter of $\overline{B_{\widetilde{\delta}m^*}(t)}$, the equations of horocycles derived in the beginning of section 4.1 give us

$$D_1 = \frac{1}{\widetilde{\delta}m^*t^2} \leqslant \beta\frac{1}{m^*t^2} = \beta\Delta$$

where the last quantity is the Euclidean diameter of C. Therefore, $\Sigma_f(\eta) \in C \cap B'$ and so $C \cap B' \neq \varnothing$, a contradiction. Therefore, $-1/t \notin \partial_\infty \widetilde{H}$ as desired. ■

5.3 Matings of Polynomials

In this section, we describe the connection of our extension of the halfspace theorem to a result contained in [9]. See [10] for a resource on matings. Theorem 4.2 of [9] gives necessary and sufficient criteria which determine exactly when a hyperbolic postcritically finite rational map f arises as a mating. To state the condition, we require a definition.

Definition (Equator). Let $f : S^2 \to S^2$ be a Thurston map. A Jordan curve $\mathcal{E} \subset S^2 \backslash P_f$ is an *equator* for f if the following three conditions are satisfied.

(1) $\widetilde{\mathcal{E}} := f^{-1}(\mathcal{E})$ consists of a single component

(2) $\widetilde{\mathcal{E}}$ is isotopic to $\mathcal{E}$ rel. P_f

(3) $\widetilde{\mathcal{E}}$ is *orientation-preserving* isotopic to $\mathcal{E}$ rel. P_f

where *orientation-preserving* isotopic to $\mathcal{E}$ rel. P_f simply means that the two orientations on $\widetilde{\mathcal{E}}$ given above agree.

It is clear from conditions (1) and (2) of the definition of an equator that the slope of an equator of f must be fixed by the slope function μ_f described in section 5 of [2] (denoted there by σ_f). Thus, since theorem 5.2 provides one with an open interval in which the negative reciprocal of a slope value fixed by μ_f cannot reside, if one manages to obtain a covering of the extended real line $\hat{\mathbb{R}}$ by some collection of these intervals, one may further conclude that μ_f does not have any fixed points and hence cannot arise as a mating.

Chapter 6

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