SOME RELATIONSHIPS BETWEEN ELASTIC AND PLASTIC METHODS OF STRUCTURAL STEEL DESIGN

by

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INTRODUCTION

The solution of the beam equation $M = -EI \frac{d^2y}{dx^2}$ depends upon the loading condition, the length, and the boundary conditions for the member in question. In the design of a continuous frame composed of prismatic members, the loads and lengths are generally prescribed by considerations of plan, clearances, and finish materials. The variables at the disposal of the structural designer are, then, the boundary conditions.

For elastic design, the ends of the beams and beam-columns are usually neither completely fixed nor simply pinned, but their in-between condition can be represented by a spring of relative stiffness K; relative, that is, to the remainder of the joint. It is the designer's task to so apportion the stiffnesses of all the members of the frame that the bending moment will be distributed in the best manner, thus achieving a minimum-weight design. Due to the interrelationship of all the members of a given frame, this is necessarily a process of successive approximations.

In plastic design the inherent difficulty of variable boundary conditions is obviated by essentially assuming hinges at the ends of the members, thus reducing the problem to a statically determinate frame. The validity for such an assumption for a structural steel frame is given by the simple plastic theory, and the various theorems developed within its confines.

The purpose of this study is to determine the similarities or disparities of the final results of design by both methods. To accomplish this purpose the intrinsic nature of the methods is examined as related to a specific problem involving a continuous multi-story frame, so chosen as to include the effects of both horizontal and gravity loads. The frame is assumed to be laterally restrained in order to focus attention on the main problem.

By comparisons of the methods of design, it is concluded that for any given indeterminate frame to be built of prismatic steel members, solutions by elastic and by plastic design must yield quite similar results.

DESCRIPTION OF PROBLEM

A steel frame for a three-story building will be designed by both elastic and plastic methods. The dimensions to assumed centerlines of members are as shown in Figure 1, which also shows wind loads.

In order to avoid dealing with secondary effects such as the reduction in allowable bending stress due to lateral instability, both beams and columns will be considered to be laterally braced, perhaps by the floor deck or by intermediate struts, as the case may be. The frame will be considered to be fully continuous with pinned column bases and will be treated as an isolated plane frame, neglecting the strengthening effect of members and slabs perpendicular to it. The assumed loads are:

Dead load, roof	0.6 kips/ft,
Dead Load, floor	1.3 kips/ft,
Live load, roof	0.6 kips/ft,
Live load, center beam	3.0 kips/ft,
Live load, side beams	2.0 kips/ft.

The indicated loads are taken of the order of magnitude to be expected in an actual building, although they are not derived from an actual building circumstance. There are no architectural restrictions on sizes of members.

The ultimate goal is to achieve the greatest overall economy, which is not necessarily synonymous with the least weight. For this reason the columns of the second story will be carried to the roof, the additional weight costing less than splices.

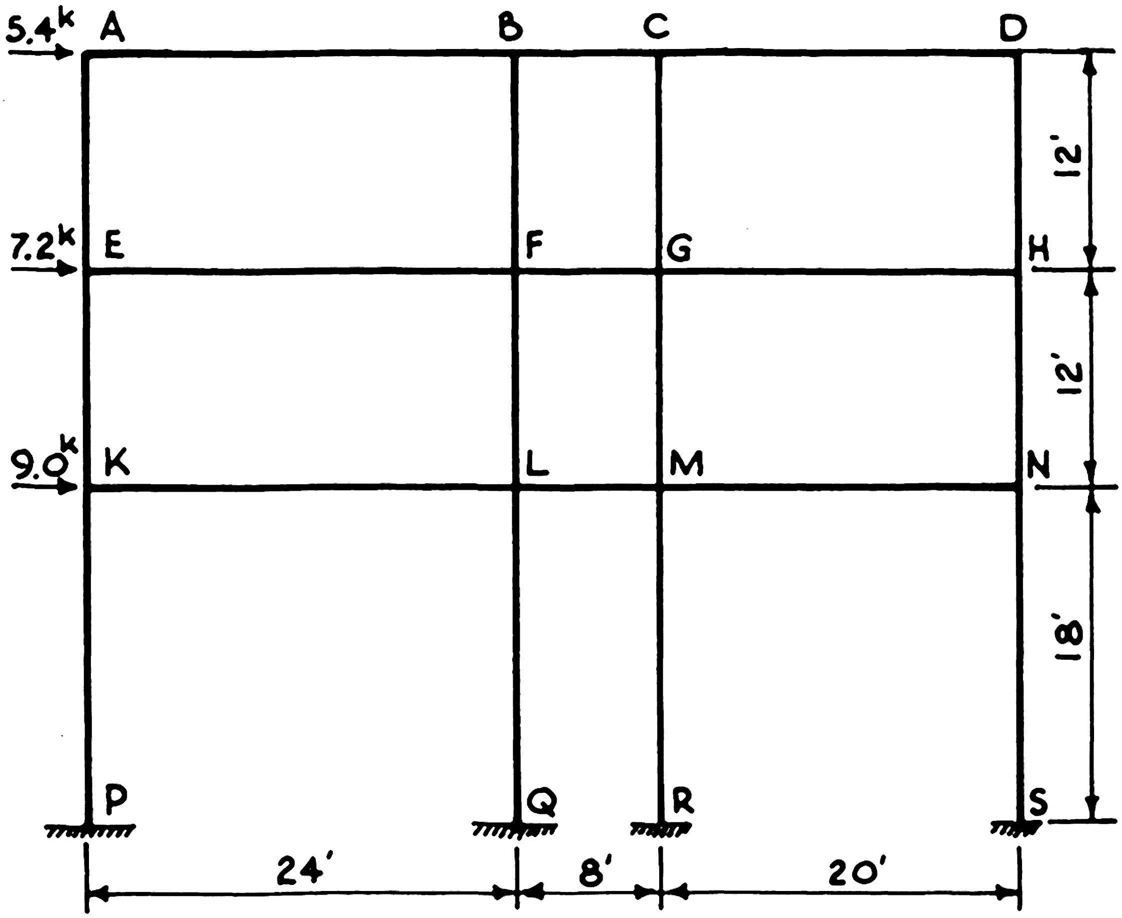


Figure 1 - Skeleton Frame

ELASTIC DESIGN

Method

The dimensions and loading being given, the frame must be well proportioned so as to provide an efficient and economic solution. The most economical structural design is undoubtedly the one which is the most efficient, that is, it produces maximum stress throughout each member. To determine these proper proportions, a simple rigid frame will first be studied, and this analysis then extended to a multi-story multi-bay bent.

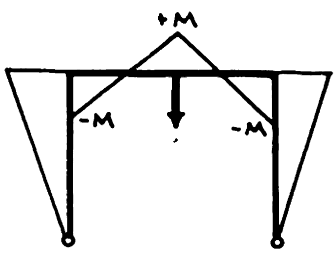


Figure 2
Rigid Frame

Referring to Figure 2, if the beam alone governed the cost of the structure, $-M/24$ should equal $+M/20$, based on AISC (1)* allowable stresses, since the beam would then be stressed to the same level at both ends and the center, since the allowable stress for positive bending is 20,000 psi, whereas this stress may be increased 20 percent for negative bending. Assuming constant cross-section, the required section modulus is $\frac{\text{bending moment}}{\text{allowable stress}}$, with the above results. These allowable stresses would have to be modified to some lesser values if the frame were not laterally braced, negating the ratio 24/20. (Throughout this study all members will be assumed to have adequate lateral support). The ratios of moment $-M = (24/20)(+M)$ occur if the stiffness of each of the columns is about twice that for the beam for a pin-ended frame, or if the stiffness of

*Parenthetic numbers refer to Bibliography.

each of the columns is about $1\frac{1}{2}$ that of the beam for a frame with fixed feet. If these ratios were used they would produce a large moment in the columns and result in an excessively heavy member. The compromise called for is to reduce the relative stiffness of the column, thus reducing its end moment while providing a somewhat inefficient design for the beam due to its relatively large positive bending moment. The final result is to have, for a fixed-ended frame, for a beam stiffness somewhat greater than unity a column stiffness reduced below $1\frac{1}{2}$.

Considering the multi-story frame as a set of superimposed bents, the bottoms of the columns in the individual bents become subject to rotations opposite to those which occur in one-story pin-ended frames. It is now necessary to further increase the relative stiffness of the beam while reducing again that of the column.

Since most frames are not of a single aisle in width, when combining frames laterally the problem of variable column spacing must be considered. The effect of unequal spans is dependent upon the relative lengths between supports and the loading pattern. For the usual loads of buildings, a loaded long span will tend to rotate a joint, whereas, even if it is loaded, the end of an adjacent short-span beam will tend to be rotated. The former might be considered as moment-producing, whereas the latter will be moment-receiving, yet contributing to the total stiffness of the joint. In illustration of this many reasonable designs could be shown which exhibit negative moment throughout the length of the short spans, even when they are fully loaded. If it is to do its full share of work, a moment-receiving beam should therefore have its stiffness increased above the preliminary estimates based solely on length and load. This increase will reduce the otherwise required weight of adjacent moment-producing beams by increasing joint stiffness, thus producing more of a fixed, rather than pinned end condition. That is to say, the moment-receiving beam will resist being rotated, and will tend to approach the behavior of a moment-producing beam. The increase in weight over the short span which undoubtedly accompanies the increase in moment of inertia is more than offset by the above-mentioned reduction in weight of the longer beams.

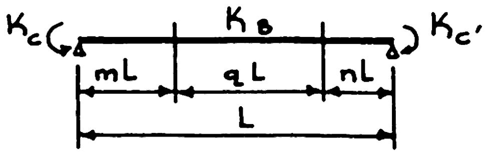
If there are no architectural restrictions as to its depth, a rough preliminary estimate of beam size is obtained by taking $M = wL^2/16$, for uniform load w on span L, M being the bending moment. Then the required section modulus is given by $S = M/20$, where S is the section modulus. Reference is now made to "Steel Construction" (1). Using the lightest weight member for the given section modulus, the I is tabulated, and an approximation for I/L computed.

As previously indicated, the stiffness of an exterior column supporting the roof should be about 1-1/3 that of the adjacent beam. At exterior floor joints the total stiffness of the column above and below the joint should be about one and one-half times that of the adjacent beam. Because interior beams provide additional stiffness to their joints, the interior columns can be less stiff than the exterior. It follows that, once the beam sizes have been estimated, the approximate required I of the columns can be calculated, and a corresponding depth selected.

The columns are now compared one to the other, the objectives being the same as for single bents, that is, to create a balanced design throughout the entire frame. Some column stiffnesses may be increased to prevent undue size in moment-receiving members. If a column at one end of a beam is unduly stiff it will prevent the column at the far end from receiving a good share of bending moment, and adjustment is called for. Adequate moment capacity for wind effects must also be provided. Finally, it is the primary duty of a building frame to provide an adequate path to the ground for gravity load forces, and the columns, as the essentials of the most direct route, must obviously be strong enough to sustain the expected axial loads, in addition to imposed bending moments.

If the frame is pinned at its footings, the first floor column size should be increased throughout, due to the abnormal effect of wind on pin-ended columns, and to a desire to maintain a total effective column stiffness at each second floor joint consistent with the floors above.

The tentative design is now checked by estimating points of contraflexure and determining bending moments. The essence of the analysis is to compare the tentative stiffness values with the following relationships between bending moments and certain simple stiffness ratios, assuming uniform loading:



Where K_c, $K_{c'}$ = column stiffness

K_B = beam stiffness

mL, nL = region of negative moment

qL = region of positive moment

Figure 3 - Moment vs. Stiffness

If K_B is set equal to unity, the following prevail:

K_c	$K_{c'}$	m	n	q
0	0	0	0	1.00
∞	∞	0.21	0.21	0.58
0	∞	0	0.25	0.75
2	2	0.15	0.15	0.70

Obviously straight line interpolation cannot be applied. True mathematical interpolation is also not possible within the four given sets of values, nor is it desirable, since this is a method of numerical convergence applied to standard rolled steel sections which form only a relatively small finite set of values of I. Approximations are therefore employed.

Some judgement must be exercised when computing the opposing stiffness at any given joint. If the continuation (on the far side of the column) of the beam in question is moment-receiving, it will get stiffer when it is loaded, so that for these purposes its apparent stiffness may be increased in lieu of estimating the effect of its load.

After the points of contraflexure of each end of the beam have been decided upon, they should be adjusted due to the continuity of the beam which provides that the ends do not act independently. That is to say, the affect on one end is carried over to the other so that the beam acts as an entity.

Since the analysis so far is for gravity loads only, the columns will absorb moments from the beams. Assuming the moment to be applied to the midheight of the column, the following relationships are obtained:

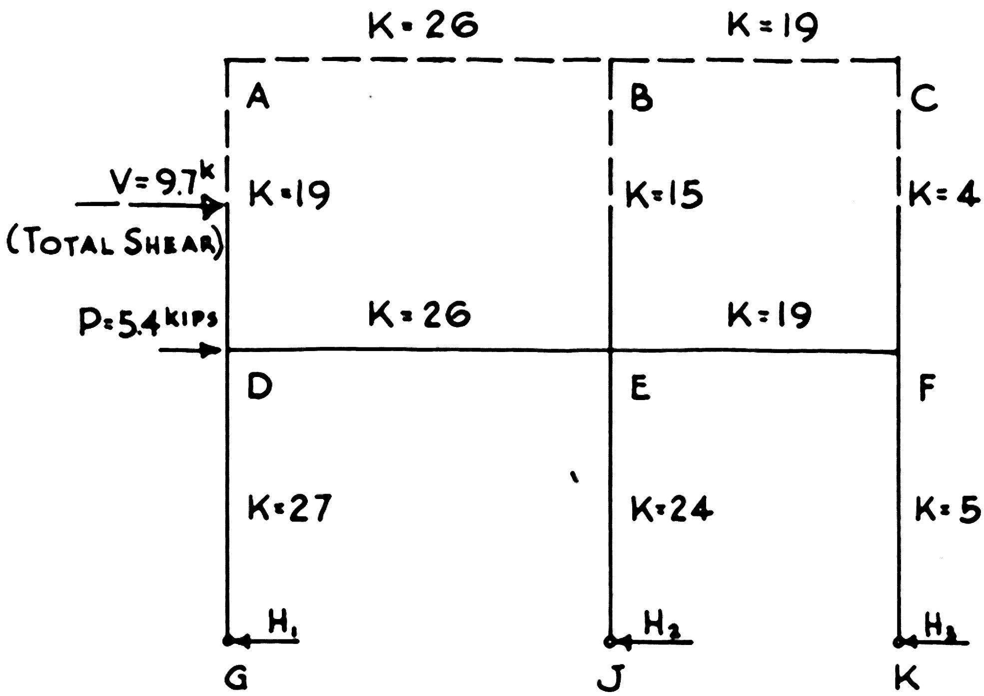


Figure 4 - Example Frame

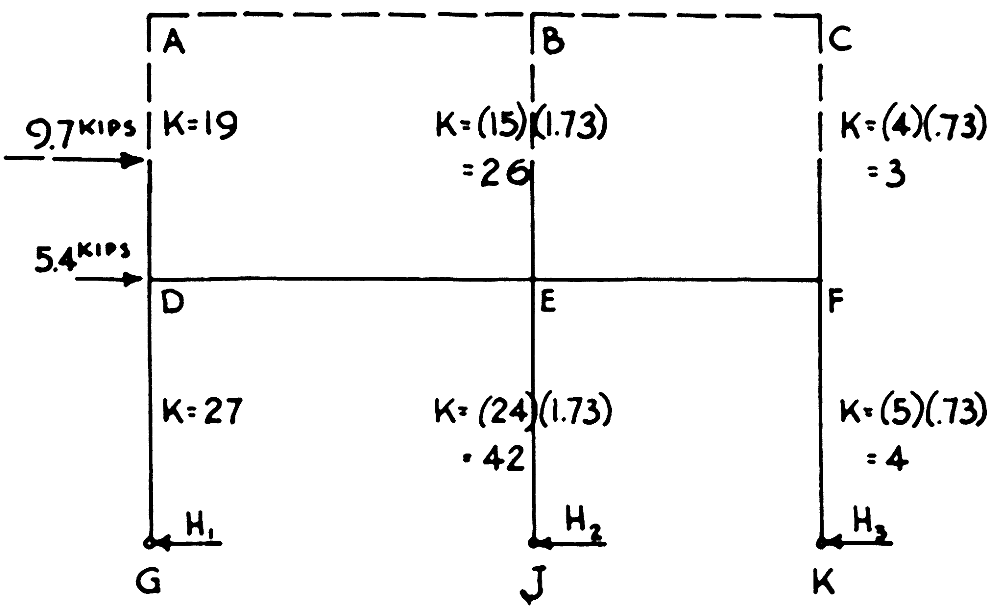
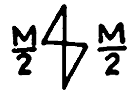
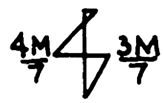


Figure 5 - Example Frame

or


yields


and


yields


where M accrues from a beam intersecting the column. If the stiffnesses of the columns above and below the joint differ, the moments must be modified accordingly.

The moments on the beams are now computed by assuming hinges at the points of contraflexure. Next the moments in the columns are determined. With a knowledge of probable moments and loads, all the members can be selected. In fact, for the beams several choices will usually be possible. Thus a second approximation is obtained.

At this point the entire design is reviewed, and the frame is considered as an entity. Certain seeming overstresses (due to apparently underdesigned members) will disappear, whereas some members will become overstressed when the probable effects of stiffnesses and loadings of members above and below the one in question are considered. This analysis will help decide which of the several possible choices to select.

The design being fairly secure, the wind stresses are determined. The relative column shear stiffnesses are determined by employing the restraint offered by the beams. A hypothetical numerical example follows, as shown in Figure 4. Assume the basis of stiffness to be column ADG. Then column BE is increased in stiffness due to the beam DEF in the ratio $\frac{(26 + 19)}{26} = 1.73$.

This value holds for column EJ. Column CF will be reduced in stiffness by 19/26 = 0.73; and the same holds for column FK. The relative stiffnesses are now as shown in Figure 5. The distribution of shear is assumed proportional to these new stiffnesses. Thus:

$$V_{AD} = 9.7\left[\frac{19}{19 + 26 + 3}\right] = 3.8 \text{ kips and}$$

$$H_1 = V_{DG} = (9.7 + 5.4)\left[\frac{27}{27 + 42 + 4}\right] = 5.6 \text{ kips.}$$

If the height of AD is ten feet, and height of DG is 11 feet, the column moments are, for example,

$$M_{AD} = (10/2)(3.8) = 19 \text{ foot kips, and}$$

$$M_{DG} = (11)(5.6) = 61.6 \text{ foot kips.}$$

The sum of the column moments is applied to the beams, and the entire design is reviewed on the basis of allowable stresses for wind action.

Redesign may be necessary if wind is a significant factor. When this is completed the frame is in a better state of approximation and is ready for a more precise analysis.

Computations

Floor beams, left side: $\frac{wL^2}{16} \doteq 120 \Rightarrow S \doteq 72 \Rightarrow 16$ WF 45

right side: $\frac{wL^2}{16} \doteq 85 \Rightarrow S \doteq 51 \Rightarrow 16$ WF 36

center span: Moment-receiving, try 12 WF for adequate stiffness as per previous discussion on moment-receiving beams.

Roof beam, left side: $\frac{wL^2}{16} \doteq 45 \Rightarrow S \doteq 27 \Rightarrow 12$ WF 27

Use same depth on right side, smaller in center

Column AP: At lower stories we have T rather than L, i.e. two columns per beam rather than one. For K (beam) = 2, total K (column) should be about 3, and therefore each K (column) $\doteq$ 1.50. Using a 16 WF 45 beam yields $K = I/L = 583/24 \doteq 24$. Then K (column) $\doteq$ $(24)(3/4) = 18$. For center story, I (column) $\doteq (18)(12) = 216$. For bottom story, I (column) $\doteq (18)(18) = 324$. K (roof beam), for 12 WF 27 $= 204/24 \doteq 9$. K (column) should be about $(1\text{-}1/3)$(K beam) $\doteq 12$. For top story, I (column) $\doteq (12)(12) = 144$. The most economical section for the bottom story column seems to be a 14 WF. Both a 10 WF and a 12 WF will produce the I required in the second story, but it is perhaps questionable to change the depth of the columns in so low a building. Anything shallower than a 14 WF used for the first story would greatly increase the total weight of the structure, so we will here assume that the difficulty in forming the splice is not as costly as the additional weight. To reduce splicing difficulties as much as possible, select a 12 WF for the upper portion.

Column DS: By a similar analysis, use the same set of values, 14 WF for first story, 12 WF above that.

Interior columns: Because the interior beams provide additional stiffness to the joints, the interior columns can be less stiff than the exterior. But column DS requires the lightest of the 12 WF sections, so for column CR choose a 10 WF for the upper two stories, and a 12 WF below them. Column BQ will be of the same series in each story as column AP, but not quite as stiff.

The total selections are shown in Figure 6. In accord with the previous discussion on pin ends the bottom story columns should be stiffer, and

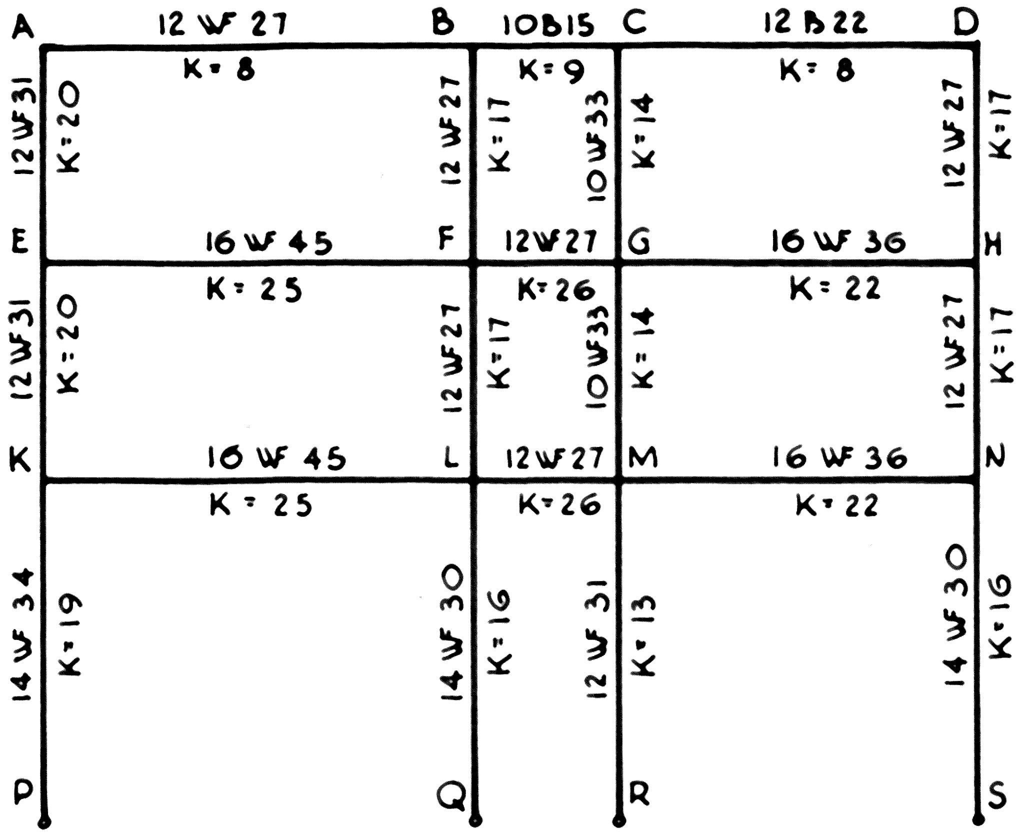


Figure 6 - First Trial of Sizes

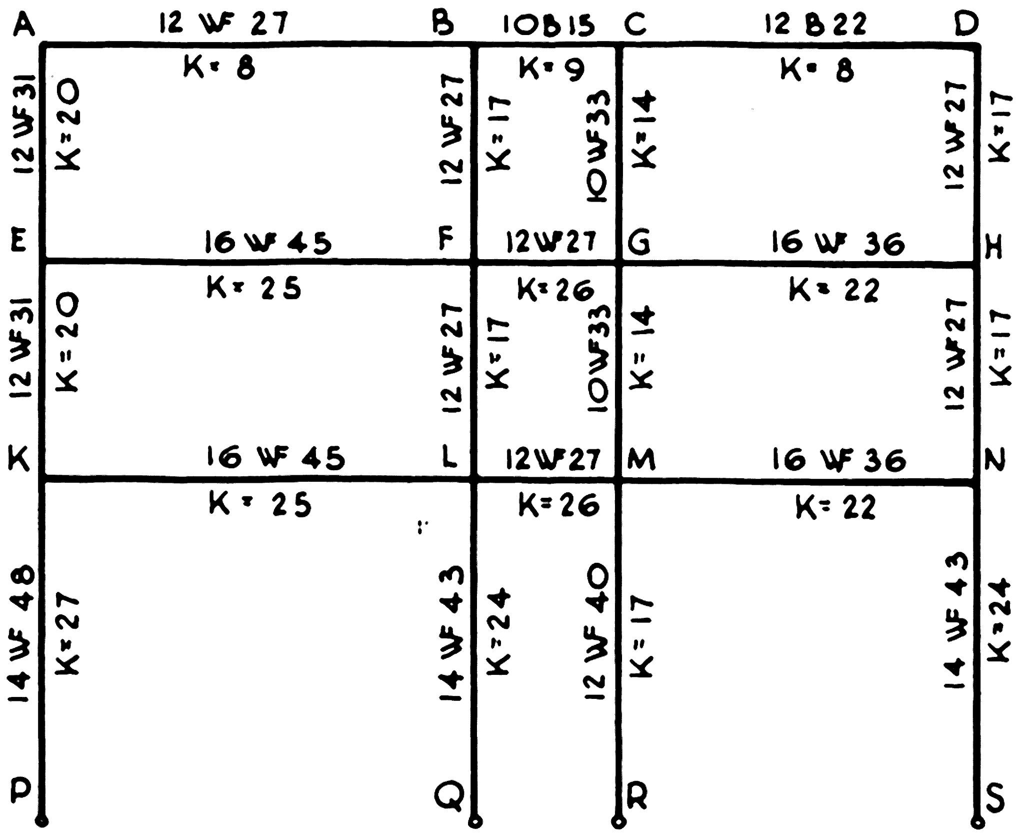


Figure 7 - Revised Trial of Sizes

are now increased as shown in Figure 7. The selections for KP, LQ, and NS were, in fact, controlled by limiting $\frac{L}{r}$ to 120.

The columns are now checked to determine whether they are large enough to sustain full axial load in addition to the estimated bending moment which exists at this stage. LQ and MR are found to have insufficient strength, and so must be increased. By the previous discussion on columns, such an increase necessitates making KP and NS stiffer. The stiffness of the columns has been increased at both ends of any one beam, thus increasing the negative moments in the respective beams and negating the original objective of balanced end and center moments. The design as it stands is made indefensible, and requires an increase in stiffness of beams KL and MN. At joint N, for example, the stiffness of beam MN should be about (2/3)(17+27)=29, which implies a 16 WF 45. The results are shown in Figure 8.

Points of contraflexure are now estimated, increasing the apparent stiffness of moment-receiving beams, since full uniform load is assumed throughout. For beam GH, for example, we have the following:

End	Stiffness Ratio	m'	n'	m	n	q
				(Modified) adjust interpolation for unity		
GH	$\frac{22}{22 + 2(26) + 14 + 14} \doteq 1/5$	.19	.16	.20	.15	.65
HG	$\frac{22}{22 + 17 + 17} \doteq 2/5$					

Now hinges are assumed at points of contraflexure, and the end and center moments are obtained. M_L is obtained by assuming mL as a cantilever with uniform load plus a concentrated load P (the reaction

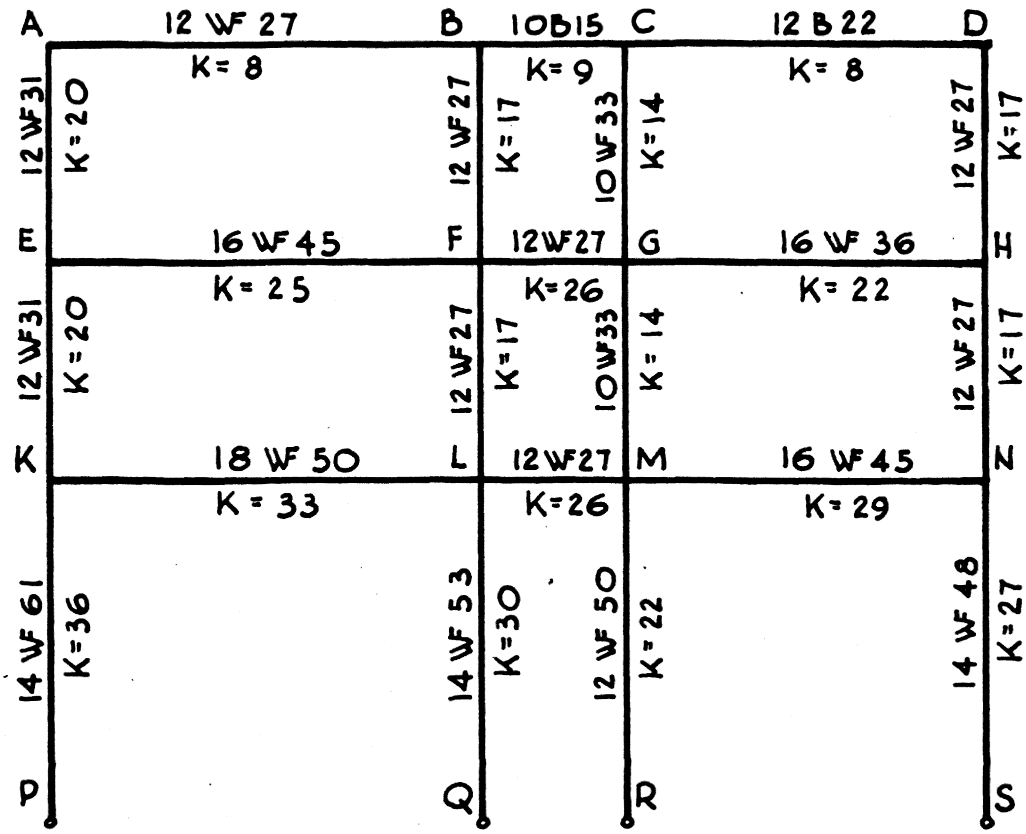
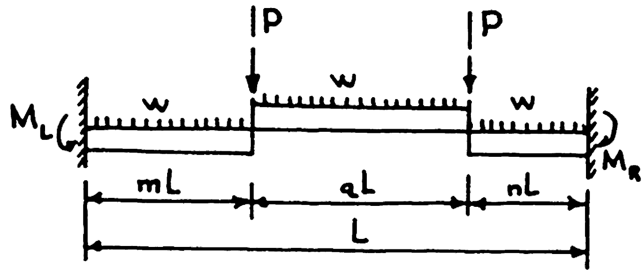


Figure 8 - Further Revisions

of qL) at its end. Thus, for GH,



$L = 20$ feet, $w = 3.30$ kips/foot;

$mL = (20)(.20) = 4.0$ ft,

$nL = (20)(.15) = 3.0$ ft,

$qL = (20)(.65) = 13.0$ ft.

Then $P = (3.30)(13/2) = 21.5$ kips.

The maximum positive moment is $\frac{2(qL)^2}{8} = 69.8$ kip feet.

$M_L = P(mL) + \frac{w(mL)^2}{2} = 112.4$ kip feet.

$M_R = P(nL) + \frac{w(nL)^2}{2} = 79.4$ kip feet.

The required section modulus is $\frac{(112.4)(12)}{24} = 56.2\ \text{in}^3$, and the section modulus provided by a 16 WF 36 is $56.3\ \text{in}^3$.

Proceeding in this fashion, it was found that all center beams were apparently too strong, but they will be retained for their stiffness values since they are so short. Beam CD can be reduced to a 14B 17.2 with a K value of 7.4, which is not significantly different from the assumed K of 8. All other beams show a good fit, so that the points of contraflexure need not be revised, and we can proceed with the columns.

The analysis of the columns will be exemplified by considering column DS. At D, $M_{DH} = M_{DC}$ by statics. As to H, there appears to be more resistance to rotation at N than at D, so more moment will go to the lower section than to the upper, since this effect increases the apparent stiffness of the lower part. Assume distribution in the ratio 55:45. As to N, the column is pinned at its base, but "more than fixed" at H, so the stiffnesses are adjusted accordingly. Take the stiffness of the pinned end column as a basis. Then the column with over-fixed end is about twice as stiff: $17 \times 2 = 34$. The moments are distributed according to these apparent stiffnesses. Finally, we have:

$$M_{DH} = M_{DC} = 30.8 \text{ ft. kips},$$

$$M_{HD} = (0.45)(79.4) = 35.7 \text{ ft. kips},$$

$$M_{HN} = (0.55)(79.4) = 43.7 \text{ ft. kips},$$

$$M_{NH} = (34/61)(84.5) = 47.0 \text{ ft. kips, and}$$

$$M_{NS} = (27/61)(84.5) = 37.5 \text{ ft. kips.}$$

Proceeding in this manner, and solving for the combined effect of axial loading plus bending moment, the anticipated strength of most of the columns seems to be insufficient. GM, for example, is apparently overstressed by 21 percent, so it will be increased to a 10 WF 45. But this will cause an increase in the negative moment at the left end of beam GH which is already stressed very close to capacity at that point. Therefore this beam will be increased to a 16 WF 45. With this increase in stiffness, which will tend to reduce the end moment, column GM need not be increased quite so greatly, and the compromise is to use a 10 WF 39. Now HN seems to be overstressed by 11 percent, so it will be increased to a 12 WF 31, since the increase in beam GH will not remove this overstress completely.

MR will not be changed in size, even though it shows an overstress of nine percent; the increased stiffness of both GM and HN should alleviate this condition.

Column NS, which is evidently stressed to only 83 percent capacity, can be reduced in size since HN is to be stiffer. Select a 14 WF 43. The left side of the frame is treated in a similar fashion, and the total results are displayed in Figure 9.

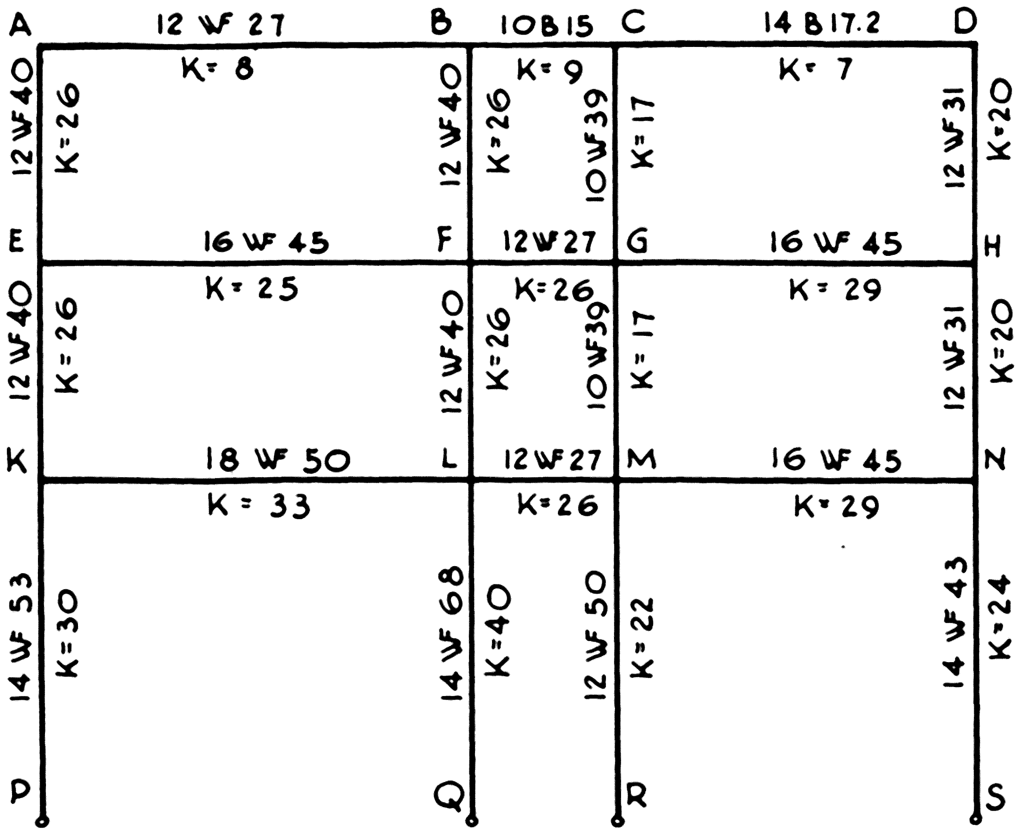


Figure 9 - Second Trial of Sizes

The frame was again analysed by estimating points of contraflexure. It was found that columns KP, LQ, and NS are seemingly overdesigned, but if they are reduced in stiffness their second-story continuations will be overstressed so they will be retained. Since columns HN and GM have some reserve strength, beams GH and MN, which are apparently overdesigned, will each be reduced to a 16 WF 40. With these two slight modifications the frame is subjected to wind analysis.

The results of this analysis indicated that columns LQ and MR were overstressed, as were beams KL and LM. All these members are, therefore, increased to the next weight of the same depth, so as to maintain about the same stiffnesses.

The increases in stiffness throughout the center and left bays of the first story are fairly proportional, and thus are acceptable for gravity load conditions, requiring no further modifications on that account. Subjected to wind load analysis the frame of Figure 10 proved adequate in all respects.

The tentative frame is now ready for a more precise analysis. Moment distribution is employed with spotted live loads (see Figure 11), and full dead load (see Figure 12). The wind stresses may be computed in other ways, but they are here determined by the method of successive corrections (see Figures 13 and 14), in the manner suggested by Dr. Grinter (14), wherein the initial fixed end moments are increased by 50%. The final tabulation of stresses indicated that column CGM and beams BC, FG, and GH were too strong, whereas beam LM is overstressed by about 8%. As previously discussed, FG and BC will be retained for their stiffness values. If GH is reduced to a 16 WF 36, the moment in column DHN will

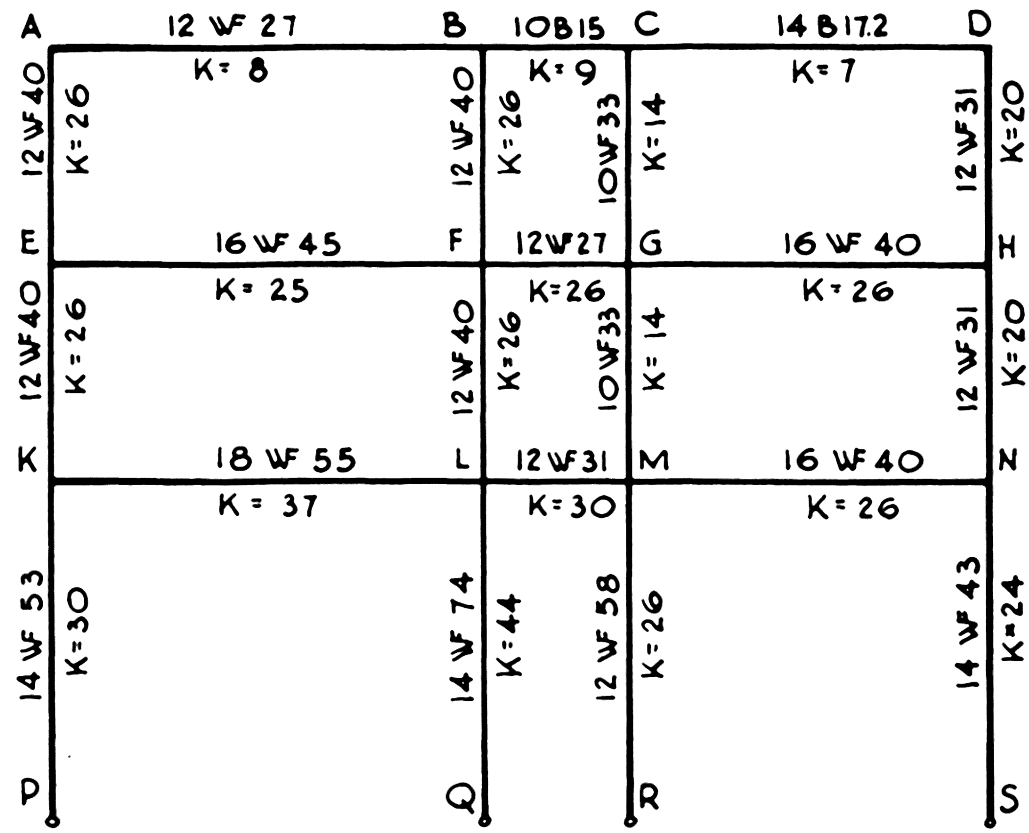
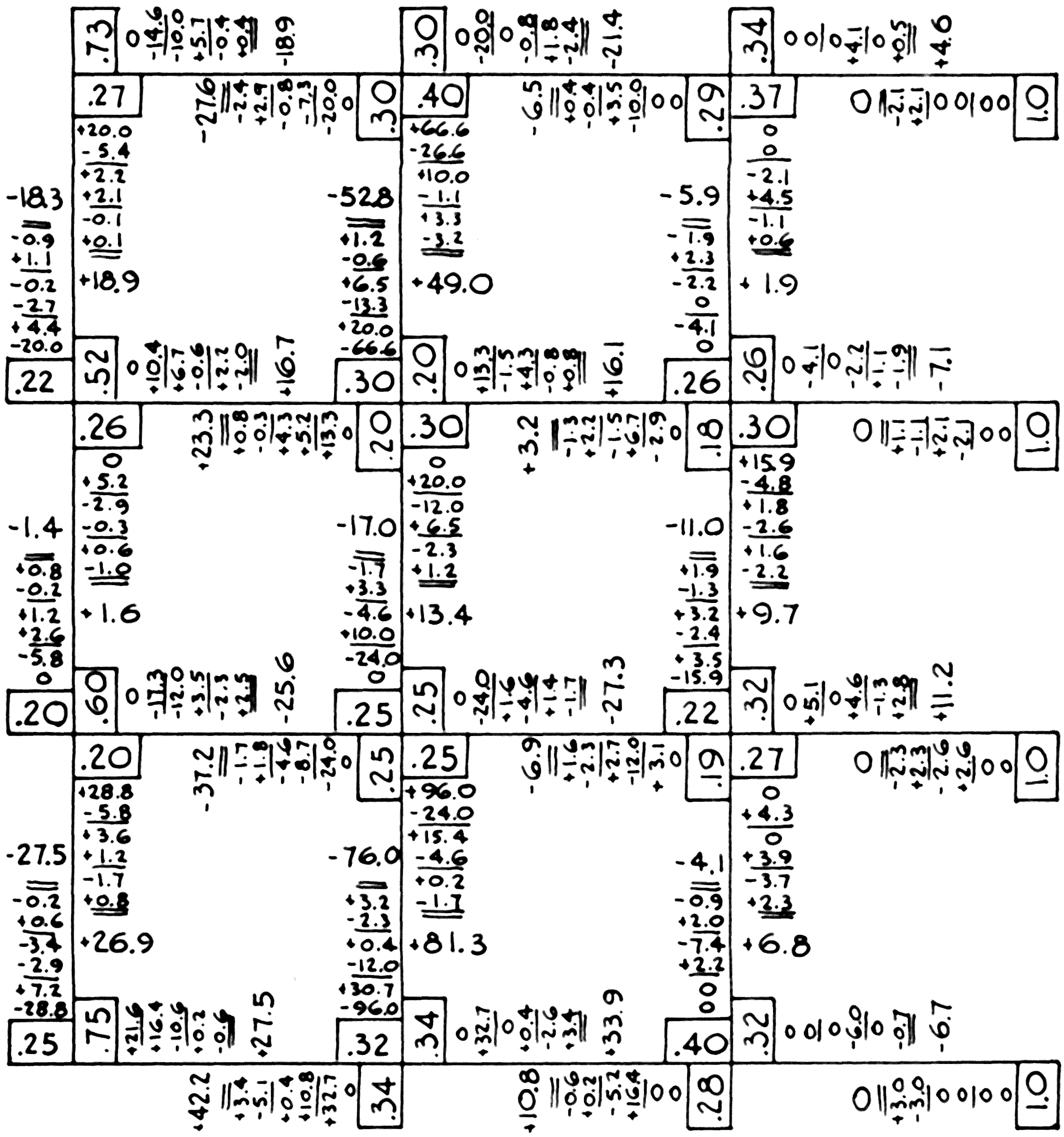


Figure 10 - Revised Second Trial



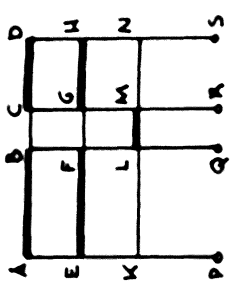


Figure 11 - Spotted Live Load Distribution

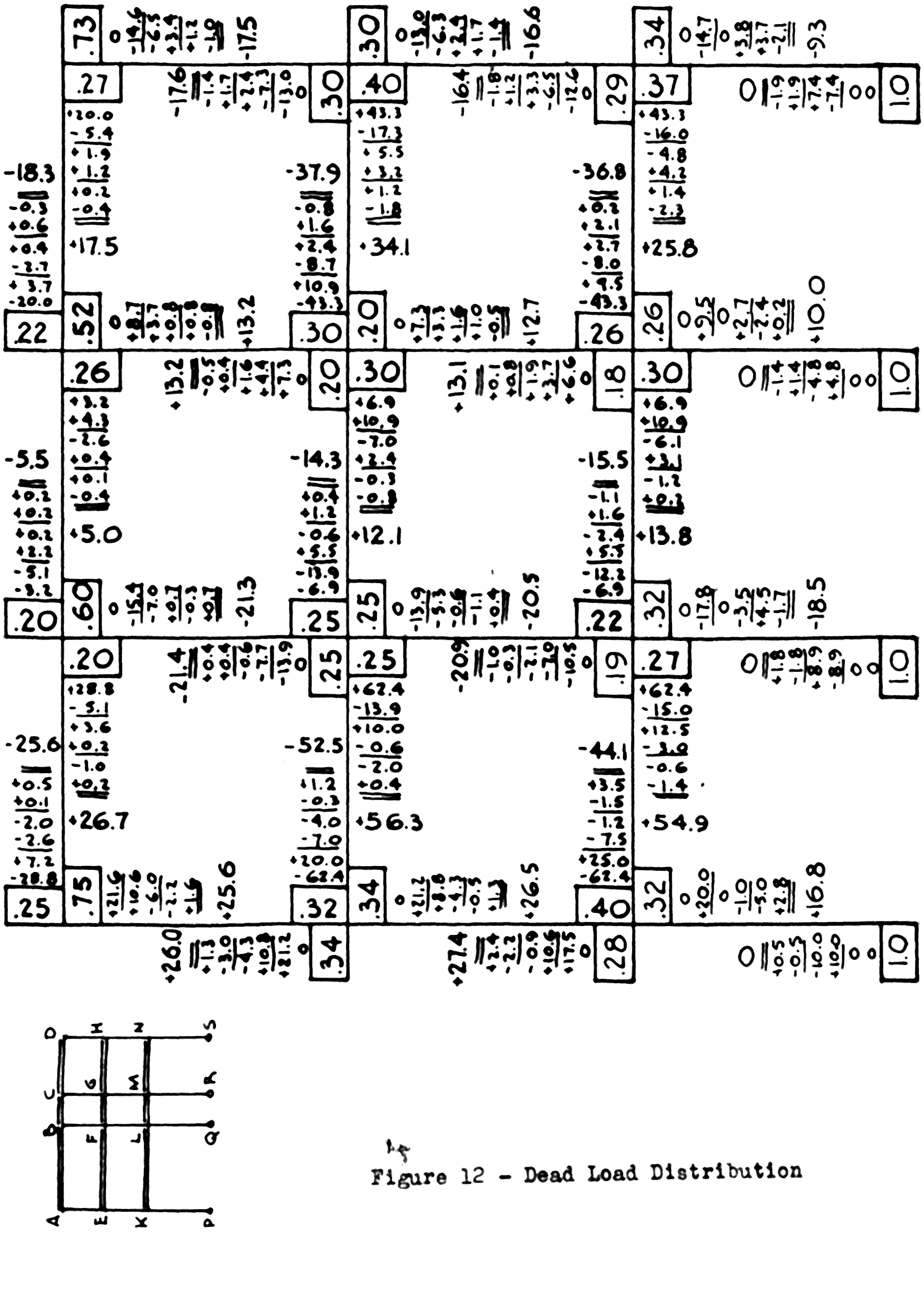
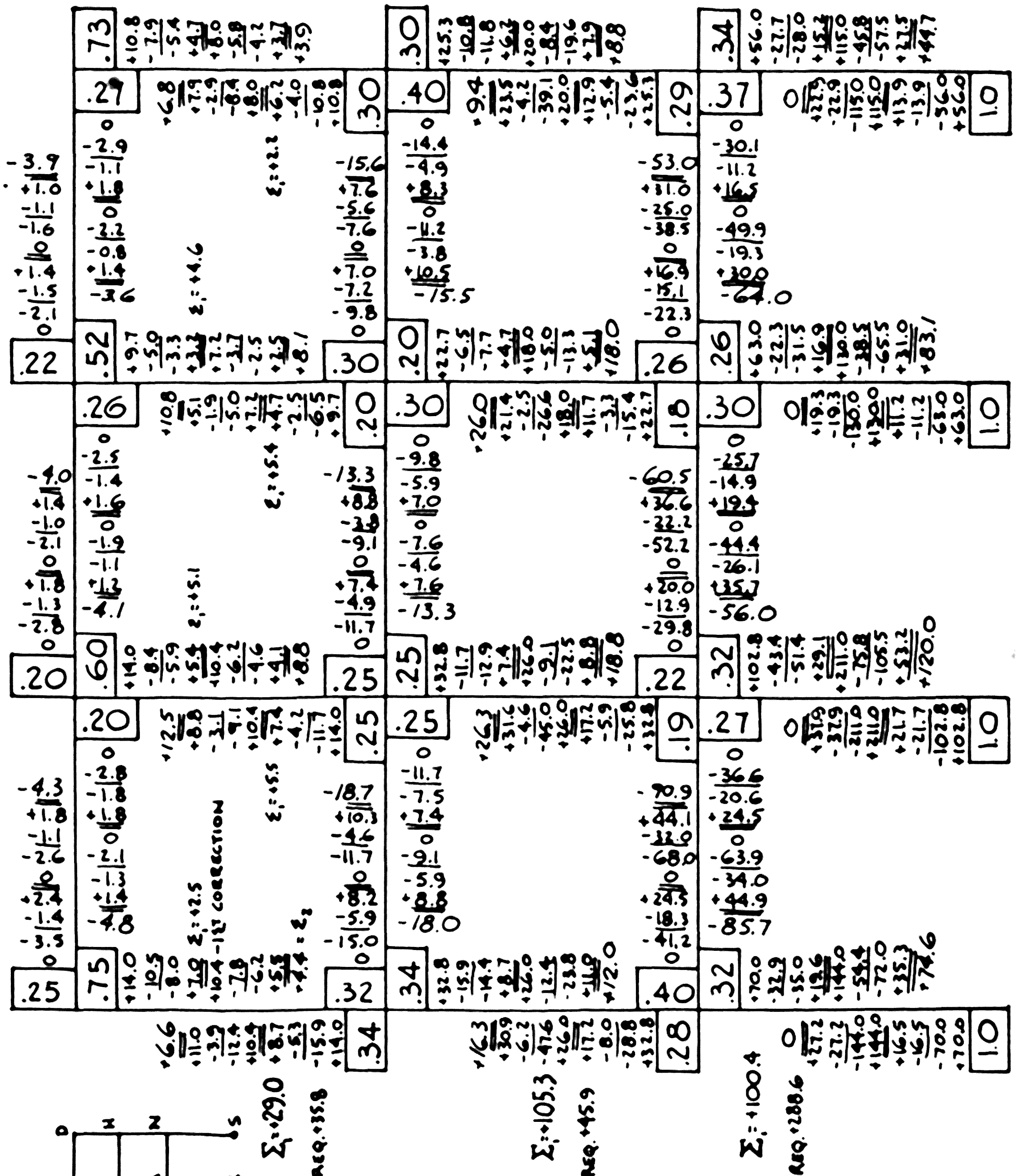


Figure 12 - Dead Load Distribution



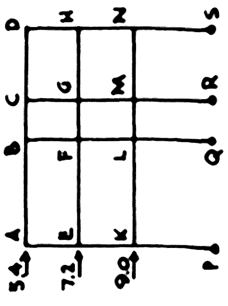


Figure 13 - Wind Load Analysis

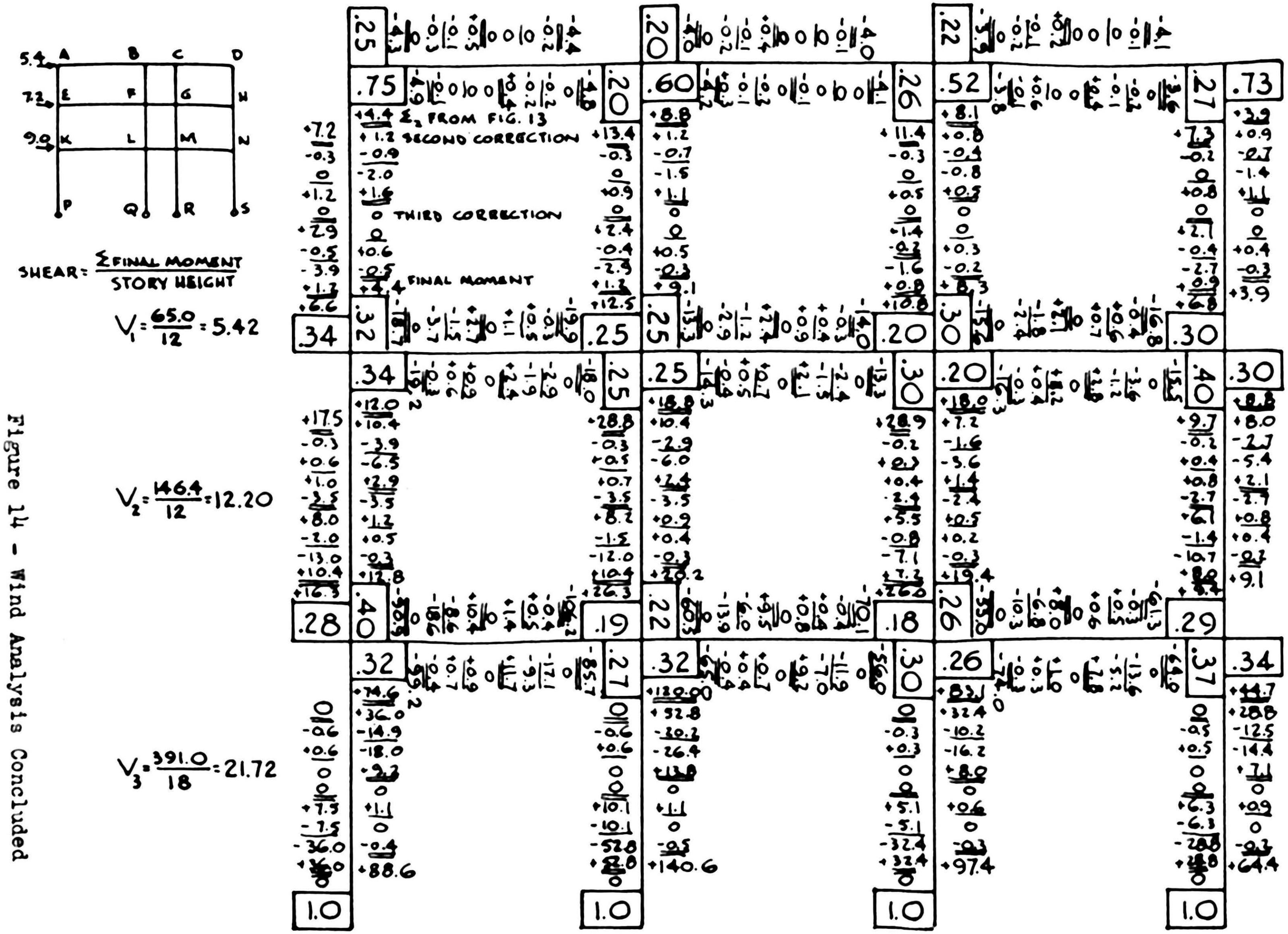


Figure 14 - Wind Analysis Concluded

increase due to the change of relative stiffnesses. However, there is sufficient reserve strength in the column to permit an increase in the bending moment at H, since the maximum moment at present is at N where the column is stressed to 98.5 percent. Beam GH will therefore be reduced. Column CGM appears to be stressed to only 84.5 percent, and, considering the change in beam GH, it will be reduced one weight listing, that is, to a 10 WF 33. Since the maximum moment in beam LM is at L, this change will not help that beam, and it will be increased to a 12 WF 36. All other members are either stressed close to 100 percent, or else are each the best available choice. The frame is now designed, and the final results are shown in Figure 15.

Sample Computations:

Beam MN:

Maximum gravity load moment 98.6 foot kips

Maximum wind load moment 61.3 foot kips

S required = $\frac{(98.6 + 61.3)(12)}{(24)(1.33)}$ = 60.0 in^3, S provided = 64.4 in^3

Column NS:

Maximum gravity load moment 28.1 foot kips

Maximum wind load moment 64.4 foot kips

Axial load 78.0 kips

For a 14 WF 43,

$$\frac{P}{Af_a} + \frac{M}{Sf_b} = \frac{78.0}{(12.65)(10.70)(1.33)} + \frac{(28.1 + 64.4)(12)}{(62.7)(24.0)(1.33)}$$

$$= .431 + .552$$

$$= .983, \text{ or } 98.3\% \text{ stress.}$$

Using the same member, with gravity loads only, we get:

$$\frac{P}{Af_a} + \frac{M}{Sf_b} = \frac{78.0}{(12.65)(10.70)} + \frac{(38.1)(12)}{(62.7)(24.0)}$$

$$= .575 + .223$$

$$= .798, \text{ or } 79.8\% \text{ stress.}$$

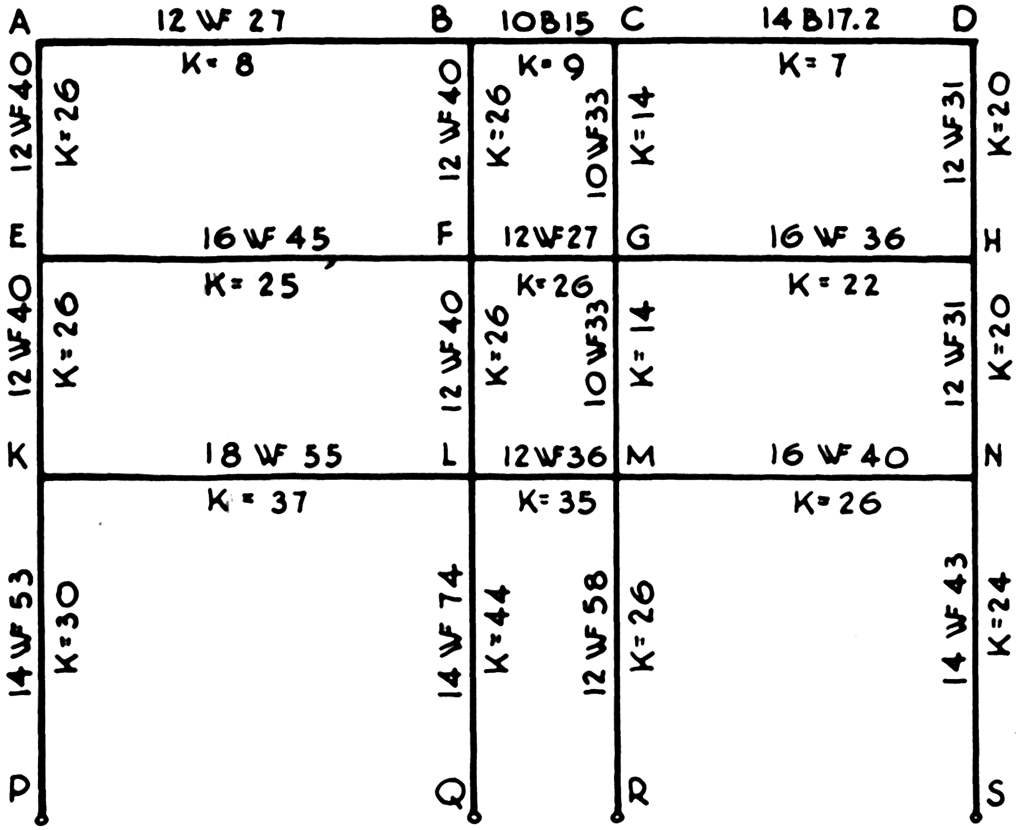


Figure 15 - Elastic Design

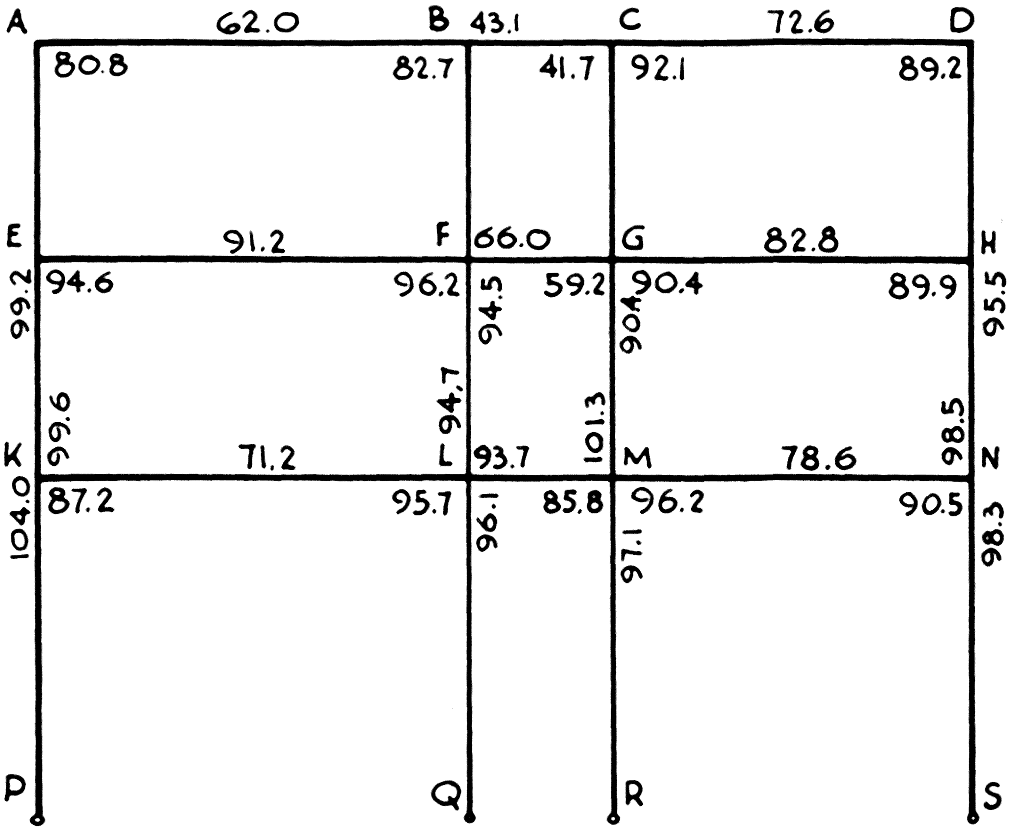


Figure 16 - Probable Maximum Percentages of Allowable Stresses, Elastic Design

PLASTIC DESIGN

Although there are several methods of analysis within the framework of the simple plastic theory, there have been only two methods devised for design, a work-energy procedure and a method of moment distribution. The former is based on the proposition (3) "In a structure at collapse, the rate at which work is done by the external loads is equal to the work absorbed in deformation at the plastic hinges", which, together with the use of several theorems, will lead to a minimum weight solution. However, the structure which contains the least weight of steel is not always the most economical since design and fabricating costs must be considered, at times at the expense of steel economy, as has previously been stated. A minimum weight design would nonetheless provide an excellent point of departure.

Although Boulton (5) has extended the method of multi-story frames, it will not be used here since moment-distribution is a simpler and more familiar concept.

The method used herein is that devised by Horne (3). Essentially it consists in distributing the moments arising from the load and span of each member of a frame in any statically admissible manner whatsoever, with only the lengths and loads being prescribed. The upper and lower bounds, i.e. the maximum and minimum moment that each member could be required to sustain, are determined separately, and then a simultaneous distribution is devised, satisfying at least the lower bound for the full plastic moment M_p. Theoretically, an infinite number of solutions is possible. However, in practice due regard is paid to the available sections, and wherever possible a member is given its full plastic moment. Some adjustment to this criterion is often necessary in order to minimize adjacent sections.

The starting moments are determined on the basis of local mechanisms, so that for a uniform load on a beam the end moments equal the center moment, $wL^2/16$. The wind moment in the columns is assumed to be equally distributed among all the columns. The load w used in the starting moments is determined by multiplying the true load by a load factor of 1.88 when wind is not operating. This factor is obtained by considering the ratio of yield stress to working stress, i.e. 33:20, and that of the plastic section modulus to the elastic section modulus, which ratio for wide flange sections is 1.14 on the average. The plastic section modulus is obtained by dividing the distance from centroid to outermost fiber into the static, or first moment of the section, $\int y dA$, since the stress in plasticity is assumed constant throughout the cross-section. So we have the load factor of safety as $\frac{33}{20}(1.14) = 1.88$. When wind is operating we get $(3/4)(1.88) = 1.41$ as the load factor.

The most economical design will obtain when every member has a plastic moment at every potential hinge, so that as many points as possible will be stressed to the maximum allowable. This implies a maximum loading condition for every member, and a large number of local mechanisms. Evidently we must have full live load throughout each member. Since for the theory of plastic design the principle of superposition is inadmissable, we conclude that we must apply the full live and dead loads simultaneously to every beam. In this manner the following initial moments are obtained:

MEMBER	M_p, ft-kips	WITH WIND	WITHOUT WIND
	$wL^2/16$	x1.41	x1.88
AB	$\frac{(1)(24)(24)}{16} = 36.0$	50.8	67.7
FG, LM	$\frac{(4.3)(8)(8)}{16} = 17.2$	24.3	32.3
GH, MN	$\frac{(3.3)(20)(20)}{16} = 82.5$	116.3	155.1

etc, the total gravity loads being as before, one kip per foot for roof beams, 3.30 kips per foot for floor beams in the side bays, and 4.30 kips per foot for floor beams in the center bay.

Adequate lateral support is still assumed.

Observing the previously mentioned assumption regarding wind moment in the columns, since we have four columns on each story, the column moments at the joints will be $\frac{PL(1.41)}{8}$. Then for the top story,

$M_p = \frac{(5.4)(12)(1.41)}{8} = 1.50$; for the middle story

$M_p = \frac{(12.6)(12)(1.41)}{8} = 26.8$; and for the bottom story

$M_p = \frac{(21.6)(18)(1.41)}{4} = 144.7$, since there is no moment at pinned ends.

The sign convention used is positive moments correspond to moments acting clockwise on the member at the joint, and positive moments correspond to tension in the bottom fiber at the centers of beams.

The method of distribution of moments on a beam will be that as stated by Horne (3). "Consider any beam ACB (Figure 17) subjected to a set of loads which would, if the beam were simply supported, produce a bending moment M at the

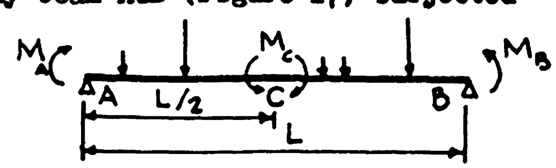


Figure 17 - Bending Moments

center C. Then if the bending moments at A, B and C are denoted by M_A, M_B, and M_C respectively, positive according to the accepted sign convention, then the condition of equilibrium is that $2M = -M_A + M_B + 2M_C$. If now M_A, M_B, and M_C are changed by the finite amount, ΔM_A, ΔM_B, and ΔM_C respectively, the applied loads remaining unchanged, $-\Delta M_A + \Delta M_B + 2\Delta M_C = 0$. In plastic moment distribution it is convenient to change only two bending moments in a beam at the same time. The permissible changes are therefore:

$$\Delta M_C = -\tfrac{1}{2}\Delta M_B \quad \text{WHEN} \quad \Delta M_A = 0,$$

$$\Delta M_C = +\tfrac{1}{2}\Delta M_A \quad \text{WHEN} \quad \Delta M_B = 0, \text{ and}$$

$$\Delta M_A = \Delta M_B \quad \text{WHEN} \quad \Delta M_C = 0."$$

The method of distribution of moments among the columns is that any arbitrary distribution whatsoever is permitted that does not violate the laws of statics concerning summation of bending moments at any joint, and summation of horizontal forces through any story.

To establish a minimum weight design, it is first necessary to determine the minimum bending moments that each member will be required to sustain. In the distribution of moments throughout the entire frame, each member will be subjected to at least this minimum. Thus the final objective will be to have all moments approach these minimal values from above, this being possible by the utilization of moment redistribution.

First, the minimum moments in the columns is established, allowing the beams to take on such moment values as satisfy the laws of statics, without regard to limit. Next, the minimum moments in the beams are determined by requiring the moments in the columns to establish equilibrium, again without regard to upper limit. There is now a minimum and a maximum for the bending moment on every member. Finally, a combined distribution is made, with the objective stated above.

The first computation was made to obtain minimum columns and maximum beams (Figure 18). Starting with full wind moments throughout, all joints were balanced into the side beams. The moment in these beams was then reduced to the maximum without wind, and the central joints now balanced into the center beam.

Next the minimum moments in the beams were established by permitting the columns to attain maximum moment (Figure 19). Starting with full beam loads without wind, the joints were balanced into the columns, due consideration being given to minimum column moments established previously. In this way, the moments in JP, KQ, MR, and NS were increased to at least 144.5 foot kips.

Finally, a simultaneous distribution was undertaken (Figure 20). The true maximum "sagging" moments do not occur at the center of the beam, but somewhere near it. They were determined in the manner suggested by Horne from charts prepared by him (3), and are shown in parenthesis in Figure 20. An ideal solution would have the maximum "sagging" moment exactly numerically equal to an end moment at each beam; the results shown are satisfactory for engineering purposes. The endeavor to satisfy the two preceding criteria on upper and lower values for bending moment failed in two places, beams KL and LM. If an attempt is made to reduce these moments, it can only be done by increasing those of some column sections, and possibly certain other beam sections, in a manner similar to the reduction of stiffness for a given beam in the elastic design.

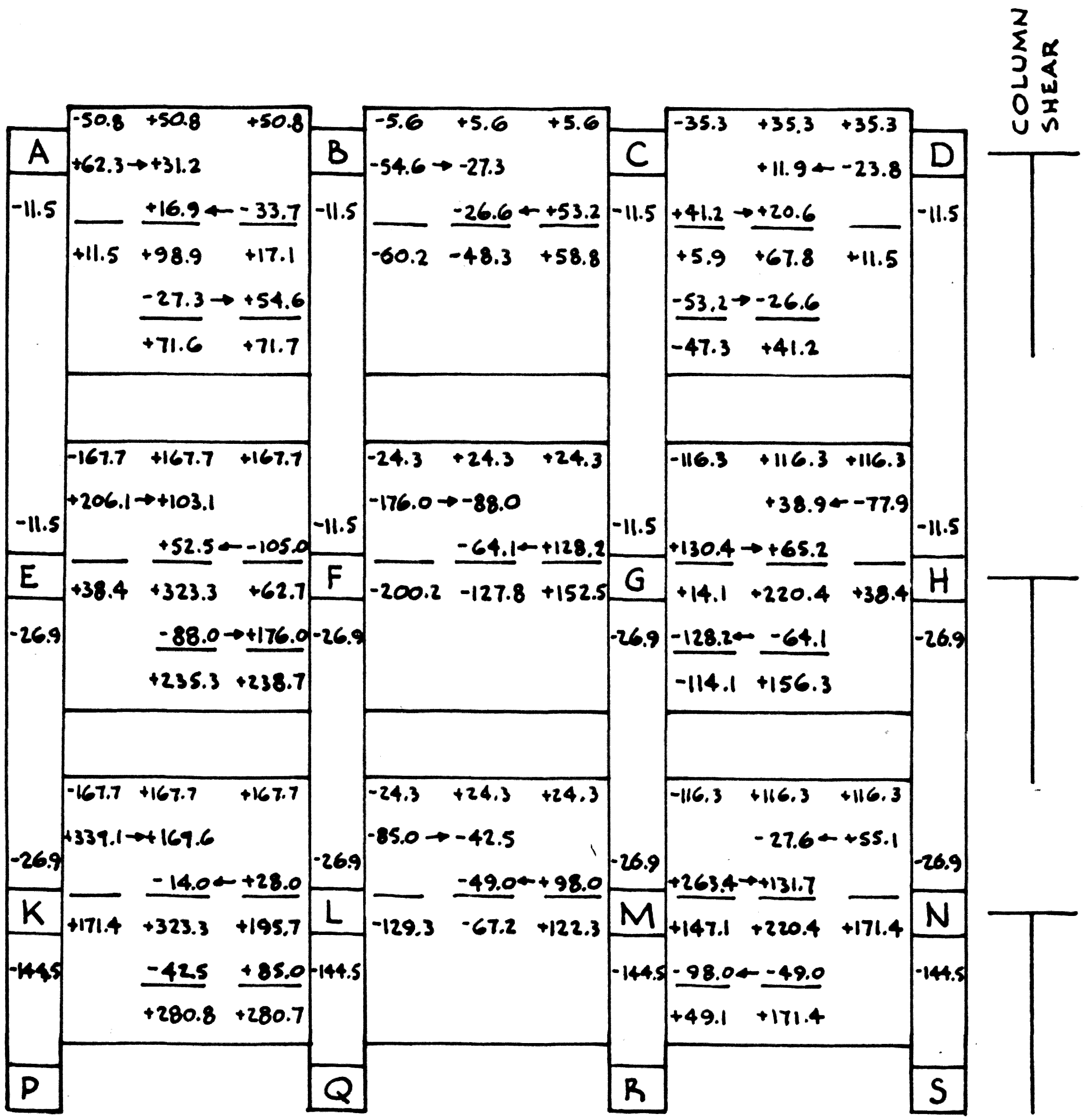


Figure 18 - First Plastic Distribution

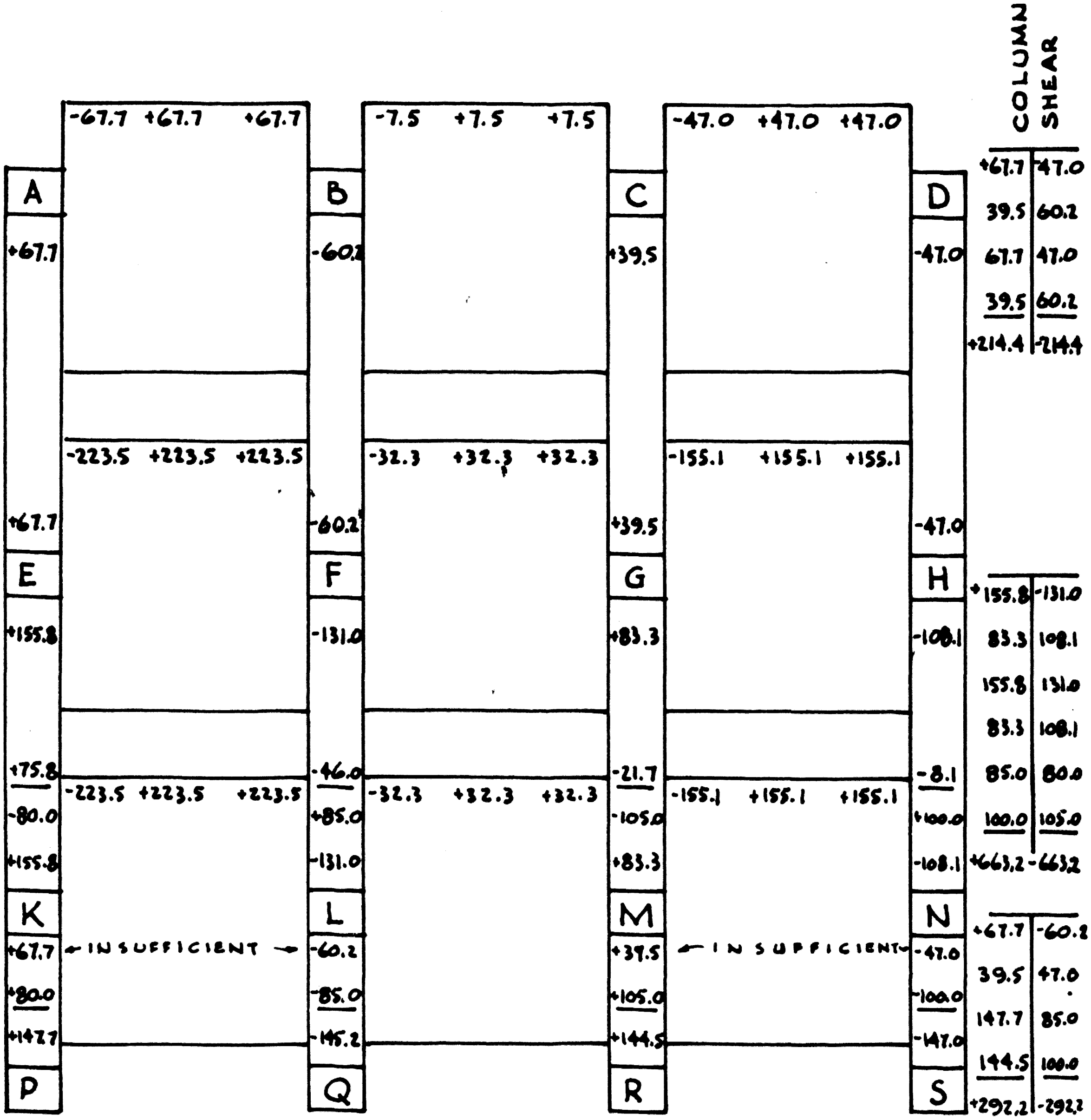


Figure 19 - Second Plastic Distribution

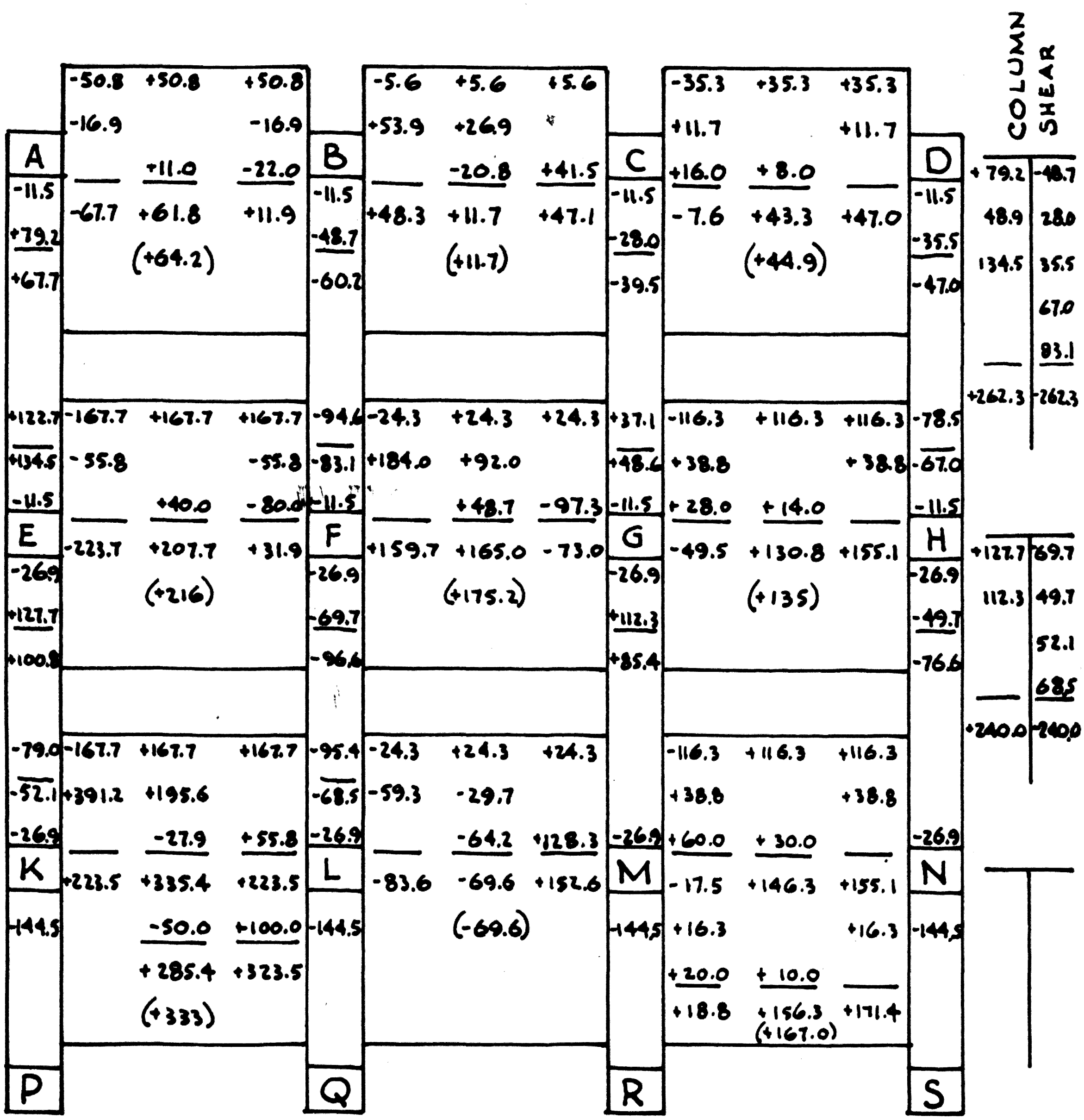


Figure 20 - Simultaneous Plastic Distribution

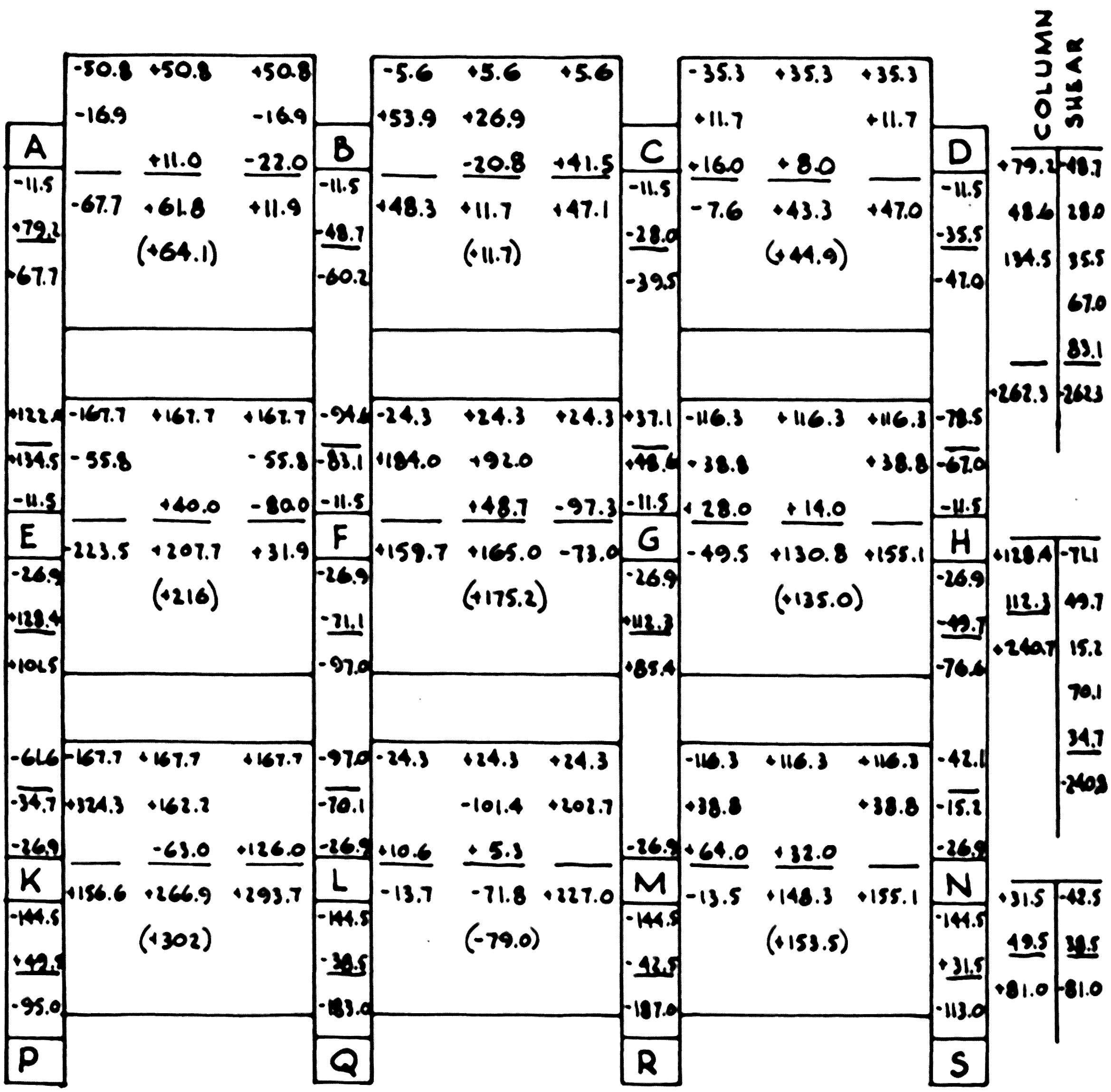


Figure 21 - Second Simultaneous Plastic Distribution

It was now noted that, for the 18 foot columns, the center two showed less than 80 percent utilization, in accordance with the design procedure described below. The moment in these columns was redistributed to obtain a more efficient solution, and a reduction in bending moment is observed in several other members as displayed in Figure 21.

Once the moment is prescribed, the required section modulus on an elastic basis is determined by the conversion (neglecting consideration of axial load and shear) $S = \frac{Mp(\text{inch pounds})}{Ff_{yp}}$ where M_p is the plastic moment, F is the shape factor, and f_{yp} is the yield stress. Thus $S = \frac{(M_p)(12)}{(1.14)(33)}$. For example, member FG has an M_p of 175.2 foot kips, requiring a section modulus of 56.0 in^3, so that a 16 WF 36 is selected with an S of 56.3 in^3, indicating that it is 99.5 percent fully utilized. The required section modulus of the columns is revised to include the effects of the axial load according to the recommendations of Beedle, Thurlimann, and Ketter (4). For example, column HN has a plastic moment of 76.6 foot kips and an axial load of 43.0 kips, indicating a required S of at least 24.5 in^3. Trying a 12 WF 27, $\frac{P}{P_y} = \frac{P}{AF_{yp}} = 0.163$, where P is the axial load, A is the area of the section, f_{yp} is the yield stress, and $\frac{L}{r} = 100$, where L is the length in inches and r is the least radius of gyration. With these numbers, use is made of Figure 9.12 of <u>Plastic Design in Structural Steel</u> (4), and $\frac{M_e}{M_p} = 0.72$. The true section modulus required is then $\frac{S(\text{bending})}{M_e/M_p} = \frac{24.5}{0.72} = 34.0 \text{ in}^3$. The section modulus provided is 34.1, so a 12 WF 27, with a utilization of 99.8 percent is accepted. All sections were selected from the fifth edition of Steel Construction (1).

In some instances the full plastic moment must again be reduced due to considerations of shear stress, in a manner similar to the procedure involving axial load. Actually there should be some slight reduction for all beams, but within accepted limits of engineering accuracy, shear effect need be investigated only for heavily loaded, short, deep beams where $\frac{\text{length from support to inflection point}}{\text{depth of beam}} < 3$.

Since this criterion does not apply to any beam of the frame, and since adequate lateral support has been assumed, no further checks are necessary.

TABLE 1

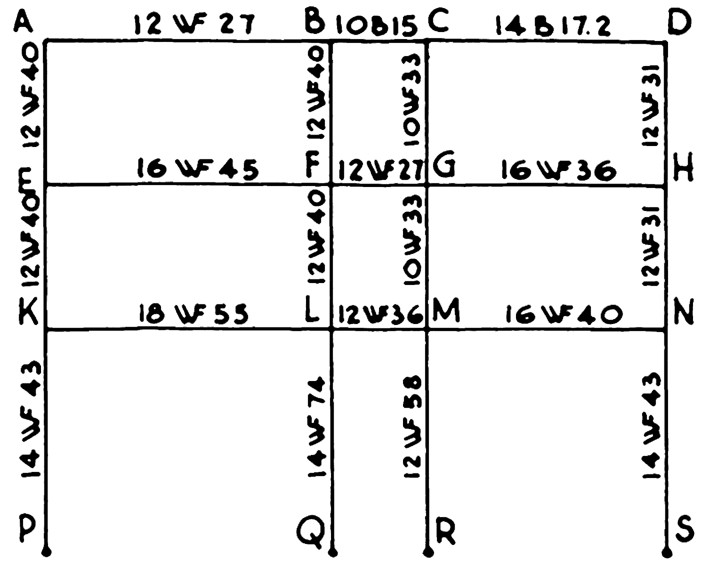
Plastic Design

MEMBER	M_p	S(REQD)	SECTION	S(PROVIDED)	% UTILIZED
AB	67.7	21.6	12 B 22	25.3	86.2
BC	48.3	15.4	12 B 16.5	17.5	88.0
CD	47.0	15.0	12 B 16.5	17.5	85.8
EF	223.5	71.3	16 WF 45	72.4	98.1
FG	175.2	56.0	16 WF 36	56.3	99.5
GH	155.1	49.5	16 WF 36	56.3	88.0
JK	293.7	93.6	18 WF 55	98.2	95.3
KM	227.0	72.4	16 WF 45	72.4	100.0
MN	155.1	49.5	16 WF 36	56.3	88.0

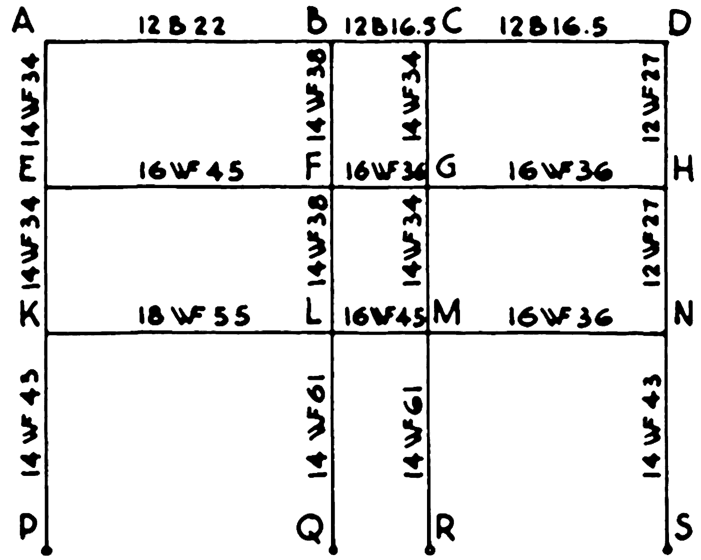
Table 1 - Plastic Design (Cont'd)

MEMBER	M_p	AXIAL LOAD	S(BENDING)	SECTION	S(P + M_p)	S(PROVIDED)	% UTILIZED
AE	122.0	12.0	39.0	14 WF 34			
EJ	101.5	51.6	33.4	14 WF 34	45.6	48.5	94.2
JP	95.0	91.2	30.3	14 WF 43	46.0	62.7	73.4*
BF	94.6	16.0	30.2				
FK	97.0	72.8	30.9	14 WF 38	49.1	54.6	90.0
KQ	183.0	129.6	58.4	14 WF 61	92.8	92.2	100.6
CG	39.5	14.0	12.6				
GM	85.4	64.2	27.2	14 WF 34	43.9	48.5	90.7
MR	187.0	114.4	59.7	14 WF 61	87.9	92.2	95.4
DH	78.5	10.0	25.0				
HN	76.6	43.0	24.5	12 WF 27	34.0	34.1	99.8
NS	113.0	76.0	36.1	14 WF 43	60.2	62.7	96.1

*Next lighter section has L/r > 120, which is inadmissible.



Elastic Design



Plastic Design

Figure 22 - Comparison of Final Results

PHYSICAL COMPARISON OF RESULTS

The final results of both methods of design are displayed in Figure 22. It is noted that fairly close agreement is shown in many of the beams, the major differences occuring in the short center beams. Had these same beams been given additional stiffness in the elastic design, the column sizes would have been remarkably similar in both cases. It is apparent that the limit design columns are almost all considerably stiffer than their counterparts, so that additional stiffness is provided throughout the frame.

An attempt was made to obtain a plastic solution identical to the elastic. While successful in establishing beam sizes, the design showed column sizes far in excess of the elastic, and so was discarded.

Obviously an almost infinite variety of solutions is possible by both methods. There is a region of minimum weight for each, and a good solution will obtain approximately a certain tonnage for any particular frame. Tachau (28) analyzed a two-story, single-bay frame of constant cross-section by both methods, and showed that a remarkable similarity of bending moment pattern obtained. In our case, the total weight of the frame designed by elastic methods is 6.45 tons, while the plastic design weighs 6.21 tons, a difference of less than 4 percent.

ANALYTIC COMPARISON OF RESULTS

The Mechanism of Design, Theory of Elasticity

According to Van den Broek (31), there are only two theories for the analysis of indeterminate structures, the elastic curve and the elastic energy theories. All others are derived from these, In any analysis on the basis of a limiting stress within the elastic range, dimensions are assumed for the members and the stresses are computed, being careful not to allow the stress in any member to exceed the safe working stress. Furthermore, the designer must also make certain that the computed stresses generally are not unreasonably low. If one or the other of these conditions cannot be satisfied, some member (or members) of the structure is changed and the analysis repeated.

The design preceding the analysis is empirical, involving a capricious choice of commercially available sections. The designer's goal is to obtain a balance of stiffnesses in the network under consideration such that the above criteria will prevail under some combination of loading. The stiffness of any one member is not independent of the remainder of the structure, since, in its usual definition, it is a measure of the relative moment which will be distributed into that member. The stiffness must be considered, not only on a geometric basis, but also in light of the load and the adjacent members (for example, the short center beams of the solved problem).

The minimal beam sizes are approximated first by evaluating gravity load requirements, and the columns are fitted to them. Probable interaction then results in adjustments. Lastly, the effects of lateral forces produce minimal column sizes, requiring further revisions. We have, then, a procedure of empirical successive approximations which has been guided by estimating points of contraflexure. Finally, a tentative frame is selected. This being done, the relative stiffnesses are established, and the frame can be subjected to a rigorous analysis. The resulting pattern of bending moments is unique, serves as a check on the design, and, in fact, may require further revisions. If these revisions are quite large, the entire design and analysis process should be repeated.

The Mechanism of Design, Simple Plastic Theory

The assumptions made for design under the simple plastic theory are quite similar to those of the elastic theory. Timoshenko (29) phrases them as "...during plasticity...we assume (1) that cross sections of the beam during bending remain plane and normal to the deflection curve, and (2) that the longitudinal fibers of the beam are in the condition of simple tension or compression and do not press on one another laterally." Based on these assumptions, the various theorems of plastic design were formed (see, for example, (3), (4) and (12)). These postulates resulted in the weakest-link concept discussed by Freudenthal (10) and Johnson (16). We conclude that the best design is the one which exhibits no one weakest link or local mechanism, but one which will collapse by failure of all its component parts simultaneously.

To this end it is necessary to establish a design wherein every member is just capable of withstanding a maximum load. This cannot be accomplished with complete disregard of the strength of adjacent members, since the laws of statics are inviolate.

Bending moments are arbitrarily assigned in the manner previously discussed, fixing upper and lower limits on both the beams and the columns. The two distributions are then made compatible, with a final design which contains, if possible, at least one hinge in every member. The validity of the procedure is demonstrated by Freudenthal (10), who proves that if in a redundant structure a set of redundants can be selected so as to define a condition of the structure that would represent the mobilization by plasticity of the possible maximum of self-help, the structure will actually tend to attain that condition.

The final distribution of moments is made compatible with member sizes, so selected as to have an M_p within the prescribed bounds.

Column Design, Both Cases

In both elastic and plastic analysis axial forces are omitted from consideration until after moments are distributed. Then the columns are checked for the effect of these forces, and the design is done again, if need be. Under the elastic theory, the stresses due to axial loads are measured by buckling standards, but still have the effect of reducing the moment-carrying capacity of the column. In other words, instead of considering the moment as reducing the critical buckling load, we could just as well state that an axial force reduces the moment capacity of a member (23). Similarily, design standards under the plastic theory provide that the plastic moment capacity shall be reduced under an axial load according to the formula given by Baker et al (3).

Allowable Stresses

In the United States the method of design in the plastic theory has been fixed up so as to produce an allowable stress of 20 ksi. Thus $\frac{\text{(form factor)(yield stress)}}{\text{load factor}} = \frac{(1.14)(33)}{1.68} = 20$. Since for a beam with uniform load fixed at both ends the plastic moment is $M_p = wL^2/16$, the required section modulus becomes $S_p = \frac{wL^2(12)}{(16)(20)}$.

For designs predicated on elastic behavior, while the allowable stress for positive bending remains at 20 ksi, for negative bending it is 24 ksi. Then, for an encastre beam with uniform load the negative end moment is $M_E = wL^2/12$, and the required section modulus is $S_E = \frac{wL^2(12)}{(12)(24)}$.

Comparison reveals that the ratio $S_p/S_E = 0.90$, indicating that the required section modulus for the former is ten percent less than that of the latter. Since there is a variation of approximately ten percent between any two weight listings in bold-face print in the Section Economy Table (1), there should be very little difference in the final designs by the two methods. In fact, if in the elastic design the stiffnesses have been proportioned to achieve the result $-M = +M$, the required section modulus will be $S_E = \frac{wL^2(12)}{(16)(20)}$, which is identical to S_p.

In all of the above pure bending is assumed, with adequate lateral support.

Comparison of Mechanisms of Design

A consideration of rotations and deflections would lead to, perhaps, a work-energy solution. Kachanow (18) proved that regardless of whether the material is perfectly elastic, perfectly plastic, or strain hardening, for deformations such as may occur in bodies of this kind, we may assume that the increase in internal energy equals the work of deformation. Such an analysis is thus valid in both cases.

In elasticity we would have some sort of summation of functions involving loads, lengths, and EI (13), whereas in plasticity the internal energy is dependent on the static moment due to the constant stress over the cross section. But the difference between the first and second moments integrated over the cross sectional area has been accounted for in the modus operandi for determining required sizes once the bending moments are prescribed. Thus again we should get identical results.

The method of ascertaining moments employed in the problem presented herein is basically similar in both cases. In the limit design the moment was distributed according to plastic behavior. A safety factor was then employed, reducing the plastic moments so as to ensure elastic behavior. An empirical distribution of moments is permissible, for, according to N. C. Kist (18), every assumption of statically indeterminate values is correct if the dimensions of the structure are designed accordingly. It is obvious that the above process is reversible, and we can go from a given moment of inertia to an elastic bending moment, to a plastic moment. The conclusion is that capricious choice of stiffnesses for a given frame is identical to arbitrary distribution of plastic moments, provided that, for the latter, the laws of statics are not violated.

By its nature and mathematical basis, the minimum-weight design procedure of the simple plastic theory must lead to an economical solution. That such a solution is also evolved from Horne's method is apparent when consideration is given to the basic theorems of limit design previously cited. The upper and lower bound theorems lead to the uniqueness theorem, thus defining the collapse load. Since the design load is given as a maximum condition initially, we are in this way defining the plastic moment by Horne's method, and therefore, by use of appropriate factors, the minimum sections required to sustain the given load. Should any member exceed the minimum section determined by the bounds, as for example, member JK, or if a member be purposely overdesigned for architectural or other reasons, there is no need for review because of the corollary to the upper bound theorem which states that extra strength cannot weaken a structure (see, for example, (3), (10), and (17)).

Freudenthal (8), Neal (20), and others have pointed out that this corollary is not applicable under Hooke's law, since increasing the stiffness of some one member of a redundant frame might increase the stress in another member which is already stressed to the proportional (or allowable) limit, thus theoretically precipitating failure and violating the rules of design under the governing criteria of elasticity. This is undoubtedly one of the difficulties encountered in elastic design, but the greatest stumbling block is the designer's lack of perspicacity, intelligence, experience, or, perhaps it might be termed, intuitive insight.

The rapidity of the convergence of any method of successive approximations depends upon intelligent guesswork, both for the initial values and for their refinements. The method used herein produces an excellent first trial, with sizes usually quite close to the final selections. Obviously, it is possible to produce exactly the ultimate design at the first attempt, but such a fortuitous circumstance is not commonplace, mainly being the result of chance. It is equally apparent that the elastic interaction of all of the members of a frame is quite complex, and so, perhaps, it is beyond the scope of the human mind to predict absolutely and exactly the resulting stresses for a given selection of members without resorting to laborious computations. Therefore the designer is not quite sure of his preliminary estimates, and as previously stated, must verify the true state of stress distribution, which process may result in revisions of the design.

CONCLUSION

For any given indeterminate frame to be built of prismatic steel members, solutions by elastic and by plastic design must yield quite similar results.

In elastic design we are dealing with the probability of causal relationships evolved from selections of values within a relatively large finite set of available sections, i.e., moments of inertia. The perfect balance of stiffnesses which results in the greatest efficiency and economy is well-nigh impossible to achieve on the first attempt, but it is the ultimate goal. In plastic design we note what bending moments would result if it were a fait accompli. These bending moments are susceptible to direct computation, as demonstrated in the problem, and lead immediately to an economic solution.

From test results (2) we can only conclude, however, that Hooke's law and the theory of limit design are both simplifications of a very intricate reality.

ACKNOWLEDGEMENT

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