A NEW APPROACH TO KNESER'S THEOREM ON ASYMPTOTIC DENSITY

by

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INTRODUCTION

In this dissertation capital letters denote non-empty sets of non-negative integers. Sets of non-negative integers containing zero play a central role; therefore, unless specifically stated to the contrary, it is assumed that all such sets contain zero. $\underline{J}$ denotes the set of non-negative integers and $\underline{J^+}$ the set of positive integers.

If A is a set of integers and $n \in J$, A(n) denotes the number of positive elements of A which are less than or equal to n. When $g \in J^+$, $\underline{A^{(g)}}$ is the union of all residue classes, mod g, which have a representative in A.

$\underline{A + B} = \{a + b : a \in A, b \in B\}$ is called the <u>sum of A and B</u>. This sum is associative and commutative. $\underline{A - B} = \{a - b : a \in A, b \in B\}$ is the <u>difference of A and B</u>. When $A = \{a\}$, a singleton set, $A \pm B$ is abbreviated to $\underline{a \pm B}$ and a similar convention is observed when B is a singleton set.

[illegible] <u>asymptotically equal to B</u>, denoted $\underline{A \sim B}$, if there exists [illegible] that $A \cap [n, \infty) = B \cap [n, \infty)$. $\sim$ is an equivalence relation on [illegible] sets of non-negative integers.

[illegible] <u>Schnirelman density of A</u>, [illegible], is defined by $d(A) = \inf \{A(n)/n : n \in J^+\}$ and the <u>asymptotic density of A</u>, $\underline{\delta(A)}$, is given by $\delta(A) = \liminf_{n\to\infty} \frac{A(n)}{n}$. For two sets, A and B, the <u>two-fold asymptotic density of A and B</u>, $\underline{\delta(A, B)} = \liminf_{n\to\infty} \frac{A(n) + B(n)}{n}$.

An early problem in density theory was to determine a lower bound for $d(A + B)$ in terms of $d(A)$ and $d(B)$. The first result in this direction, $d(A + B) \geq d(A) + d(B) - d(A)d(B)$, was obtained by Schnirelman in 1930. Two years later, Landau and Schnirelman, on the basis of empirical evidence, advanced the conjecture that $d(A + B) \geq \min\{1, d(A) + d(B)\}$. During the next ten years, this conjecture attracted widespread attention throughout the mathematical community and achieved the status of a major unsolved problem.

In 1942, H. B. Mann published his celebrated theorem which affirmed the conjecture of Landau and Schnirelman. Proven through use of ingenious elementary methods, Mann's Theorem represents a major mathematical accomplishment.

This theorem and the results following from it aroused interest in the circumstances surrounding an asymptotic analog of it.

It is easy to see that an exact asymptotic analog of Mann's Theorem would be false. If $A = B = \{2j : j \in J\}$, then $A = B = A + B$ and $\delta(A) = \delta(B) = \delta(A + B) = 0.5$. Therefore, $\delta(A + B) < \min\{1, \delta(A) + \delta(B)\} = 1$. It is worth noting in this example that A and B equal the residue class, mod 2, represented by zero.

In 1953, Martin Kneser showed that the circumstances of the preceding example or ones very similar are the only situations in which an exact asymptotic analog of Mann's Theorem will fail. Kneser's Theorem asserts that $\delta(A + B) \geq \delta(A, B)$ or there exists $g \in J^+$ such that $A + B \sim (A + B)^{(g)}$ and $\delta(A + B) = \delta((A + B)^{(g)}) \geq \delta(A^{(g)}, B^{(g)}) - 1/g \geq \delta(A, B) - 1/g$.

The techniques of Kneser are innovative and clever. He introduced the concept of an e-transformation on a pair of sets and two set-functions, f and g, to accomplish his analysis. The set-functions are defined as follows: if C and $C \neq \{0\}$, $f(C) = \min\{|c - c'| : c, c' \in C, c \neq c'\}$, the minimum distance between distinct elements of C, and g(C) is the greatest common divisor of the elements of C. An e-transformation on a pair of sets A and B is given as follows: let $e \in A$, then the e-transformation, $A, B \xrightarrow{e} A', B'$ is defined by $A' = A \cup (B + e)$ and $B' = B \cap (A - e)$. A sequence of e-transformations is constructed according to the following diagram: $A, B \xrightarrow{e_0} A_0, B_0 \xrightarrow{e_1} A_1, B_1 \xrightarrow{e_2} \cdots$ Such a sequence may be finite or infinite.

Kneser first considered the sequence $\{f(B_j) : j \in J\}$, obtained from a sequence of e-transformations on A, B. If this sequence is unbounded, it is readily shown that $\delta(A + B) \geq \delta(A, B)$. Under the assumption that $\{f(B_j) : j \in J\}$ is bounded, Kneser next considered the special case where the sequence $\{g(B_j) : j \in J\}$ is constant with $g = g(B_j)$ for all $j \in J$ and A contains g consecutive integers. In this case, $\delta(A + B) \geq \delta(A, B)$ or $A + B \sim J$. Returning to the general situation, Kneser used an ingenious "shifting" operator to demonstrate $\delta(A + B) \geq \delta(A, B)$ or $A + B \sim (A + B)^{(h)}$ for some $h \in J^+$. A penetrating analysis of the structure of A + B when $\delta(A + B) < \delta(A, B)$ shows the existence of $g \in J^+$ such that $A + B \sim (A + B)^{(g)}$ and $\delta(A + B) = \delta((A + B)^{(g)}) \geq \delta(A^{(g)}, B^{(g)}) - 1/g \geq \delta(A, B) - 1/g$ which is Kneser's Theorem, the asymptotic analog of Mann's Theorem.

This dissertation will present a new and simplified approach to Kneser's Theorem through the introduction of three major innovations into the analysis of the asymptotic and [illegible] of Mann's Theorem.

The first innovation is replacing the original sets A, B with A^M, B^M, the maximal sets associated with A, B, which are the largest supersets of A and B, respectively, for which $A^M + B^M = A + B$. The transformation $A, B \longrightarrow A^M, B^M$ is called a maximal transformation and is denoted $A, B \xrightarrow[M]{} A^M, B^M$.

The second innovation, a new transformation, the maximal e-transformation, is constructed by taking the composition of an e-transformation followed by a maximal transformation according to the following diagram: $A, B \xrightarrow{e} A^*, B^* \xrightarrow[M]{} A', B'$. After replacing the original sets with maximal sets, a specific sequence of maximal e-transformations, the basic sequence of maximal e-transformations, is generated as shown in the next diagram: $A, B \xrightarrow[M]{} A^M, B^M \xrightarrow[M]{e_0} A_0, B_0 \xrightarrow[M]{e_1} A_1, B_1 \xrightarrow[M]{e_2} \cdots$.

The third innovation is the limit set, $B^* = \bigcap \{B_j : j \in J\}$ where the sequence $\{B_j : j \in J\}$ is obtained from the basic sequence of maximal e-transformations. If $n \in J$ and $b \in B_n$ but $b \notin B_{n+1}$ then b is said to be deleted from B_n by the maximal e_{n+1} transformation. In this terminology, B^* is the set of all elements of B^M which are not deleted by any transformation in the basic sequence of maximal e-transformations.

The introduction of maximal sets, not only replacing the original sets A, B, but also in each maximal e-transformation, provides greater ease in studying the composition of A + B because if $A^M + B^M \sim (A^M + B^M)^{(g)}$ for some $g \in J^+$, then $A^M \sim (A^M)^{(g)}$ and $B^M \sim (B^M)^{(g)}$ and

the basic sequence of maximal e-transformations is finite. Also, if A and B are maximal, the property of B being maximal is preserved under e-transformations, i.e., in the following diagram $\overline{B} = B'$.

$$\begin{tikzcd} B \arrow[dr, "e", "M"'] & \\ \overline{B} \arrow[r, "M"'] & A', B' \end{tikzcd}$$

The use of the basic sequence of maximal e-transformations provides for maximal sets at each stage of the basic sequence of maximal e-transformations and that a specific transformation be performed at each step rather than permitting an arbitrary selection.

The two cases, $B^* = \{0\}$ and $B^* \neq \{0\}$, serve to emphasize the importance of whether or not all of the positive elements of B^M are deleted by the basic sequence of maximal e-transformations.

If $B^* = \{0\}$, the number of transformations in the basic sequence of maximal e-transformations becomes crucial. If the basic sequence of maximal e-transformations is finite, a routine calculation yields $\delta(A + B) \geq \delta(A, B)$. When the basic sequence of maximal e-transformations is infinite, the same result is obtained by "translating" a subsequence of the basic sequence of maximal e-transformations and the associated pairs of derived sets. Thus, if $B^* = \{0\}$, then $\delta(A + B) \geq \delta(A, B)$, or in other words, if all positive elements of B^M are deleted by the basic sequence of maximal e-transformations then the asymptotic analog of Mann's Theorem is valid.

If $B^* \neq \{0\}$, then the elementary properties of $g(B^*)$ and the structure of residue classes, mod $g(B^*)$, combine to demonstrate that

the basic sequence of maximal e-transformations is finite and establishes the existence of $C = A^M + B^M$ such that $C \sim C$, $C \sim C^{(g(B^*))}$, and $\delta(C) \geq \delta(A^M, B^M) - 1/g$.

The analysis concludes by considering the situation when $\delta(A + B) < \delta(A, B)$. With $B^* \neq \{0\}$, it is established that there exists $g \in J^+$ such that $\delta(A + B) = \delta(A^{(g)}, B^{(g)}) - 1/g = \delta(A, B) - 1/g$ and $A + B \sim (A + B)^{(g)}$.

Since a new approach to Kneser's Theorem is being presented in this dissertation, several of the results are essentially those of Kneser. For some results, the new concepts which are introduced permit a new and simplified proof. For others, originality is not claimed for their proof, but rather for the manner in which they are employed within this new approach.

CHAPTER ONE

FUNDAMENTAL PROPERTIES OF THE BASIC SEQUENCE OF MAXIMAL e-TRANSFORMATIONS

This chapter will present the fundamental results concerning the basic sequence of maximal e-transformations and the corresponding pairs of derived sets. The notational conventions initiated in the Introduction will be continued and, important definitions from the Introduction will be recalled.

The maximal sets, A^M and B^M, associated with A and B are recalled and their basic properties and relationships with A, B, and A + B are stated in a corollary.

<u>Definition 1</u>: Let $A, B \subseteq J$. The <u>maximal sets associated with A and B</u>, <u>A^M</u> and <u>B^M</u> respectively, are defined by $B^M = \{x \varepsilon J : x + A \subseteq A + B\}$ and $A^M = \{x \varepsilon J : x + B^M \subseteq A + B\}$. If $A = A^M$, then <u>A is said to be maximal</u> and if $B = B^M$, then B is said to be maximal. The transformation taking A, B to A^M, B^M is called a <u>maximal transformation</u>, denoted $A, B \xrightarrow[M]{} A^M, B^M$.

<u>Corollary 2</u>: Let $A, B \subseteq J$. A^M and B^M are the largest supersets of A and B respectively, such that $A^M + B^M = A + B$. Further, $\delta(A) \leq \delta(A^M)$, $\delta(B) \leq \delta(B^M)$, $\delta(A, B) \leq \delta(A^M, B^M)$ and $\delta(A + B) = \delta(A^M + B^M)$.

A basic tool in asymptotic density analysis is the e-transformation. In this dissertation it will be used in composition with a

maximal transformation to give a maximal e-transformation. Following the definition of an e-transformation, two theorems present its basic properties. Finally, a lemma shows that when A and B are maximal, the maximality of B is preserved under an e-transformation.

<u>Definition 3</u>: Let $A, B \subseteq J$ and $e \varepsilon A$. An <u>e-transformation</u>, denoted $A, B \xrightarrow{e} A', B'$, is defined by $A' = A \cup (B + e)$ and $B' = B \cap (A - e)$.

<u>Theorem 4</u>: If $A, B \xrightarrow{e} A', B'$ is an e-transformation, then

I) $A \subseteq A'$, $B + e \subseteq A'$, and $B' \subseteq B$

II) $A' + B' \subseteq A + B$

III) if $y \geq x \geq e$, then $A(y) - A(x) + B(y - e) - B(x - e) = A'(y) - A'(x) + B'(y - e) - B'(x - e)$.

Proof:

Property I is an immediate consequence of the definition of an e-transformation.

To establish property II, let $c' \varepsilon A' + B'$. There exists $a' \varepsilon A'$ and $b' \varepsilon B'$ such that $c' = a' + b'$. $a' \varepsilon A'$ implies $a' \varepsilon A$ or $a' \varepsilon B + e$, and $b' \varepsilon B'$ implies $b' \varepsilon B$ and $b' \varepsilon A - e$. If $a' \varepsilon A$, then $a' + b' = c' \varepsilon A + B$ since $b' \varepsilon B$. If $a' \varepsilon B + e$, there exists $b \varepsilon B$ such that $a' = b + e$. Since $b' \varepsilon A - e$, there exists $a \varepsilon A$ such that $b' = a - e$. Now $c' = a' + b' = (b + e) + (a - e) = a + b \varepsilon A + B$. Therefore, $A' + B' \subseteq A + B$.

Let $y \geq x \geq e$. $A(y) - A(x)$ is the number of elements of $A \cap (x, y]$ and $A'(y) - A'(x)$ is the number of elements in $A' \cap (x, y]$. $A'(y) - A'(x) - [A(y) - A(x)]$ = the number of elements of $A' \cap (x, y]$ which are not elements of A = the number of elements of $(B + e) \cap (x, y]$

which are not elements of A = the number of elements of $B \cap (x - e, y - e]$ which are not elements of $B' \cap (x - e, y - e]$ = $B(y - e) - B(x - e) - [B'(y - e) - B'(x - e)]$.

Therefore, $A(y) - A(x) = (y - e) - (x - e) = A'(y) - A'(x) + B'(y - e) - B'(x - e)$.

Theorem 5 will demonstrate the invariance of the two-fold asymptotic density under an e-transformation.

<u>Theorem 5</u>: If $A, B \xrightarrow{e} A', B'$ is an e-transformation, then $\delta(A, B) = \delta(A', B')$.

Proof:

Let $y \geq e$. It should be noted that the two expressions $B(y) - B(y - e) + A(e)$ and $B'(y - e) - B'(y) - A'(e)$ are each bounded and that $\lim_{y\to\infty} \frac{B(y) - B(y - e) + A(e)}{y} = \lim_{y\to\infty} \frac{B'(y - e) - B'(y) - A'(e)}{y} = 0$.

$$\delta(A, B) = \liminf_{y\to\infty} \frac{A(y) + B(y)}{y}$$

$$= \liminf_{y\to\infty} \frac{A(y) - A(e) + B(y - e) + B(y) - B(y - e) + A(e)}{y}$$

$$= \liminf_{y\to\infty} \left(\frac{A(y) - A(e) + B(y - e) - B(0)}{y} + \frac{B(y) - B(y - e) + A(e)}{y} \right) \text{ (since } B(0) = 0)$$

$$= \liminf_{y\to\infty} \frac{A(y) - A(e) + B(y - e) - B(e - e)}{y}$$ (using the conclusion of the preceding paragraph)

$$= \liminf_{y\to\infty} \frac{A'(y) - A'(e) + B'(y - e) - B'(e - e)}{y}$$ (using property III of Theorem 4 with $y \geq x = e$)

$$= \liminf_{y\to\infty} \left(\frac{A'(y) + B'(y)}{y} + \frac{B'(y - e) - B'(y) - A'(e)}{y} \right)$$

$$= \liminf_{y\to\infty} \frac{A'(y) + B'(y)}{y} \quad \text{(using the conclusion of the preceding paragraph)}$$

$$= \delta(A', B')$$

<u>Lemma 6</u>: If A and B are both maximal and $A, B \xrightarrow{e} A', B'$, then B' is maximal.

Proof:

By definition, $B' \subseteq (B')^M$.

Assume $(B')^M \setminus B' \neq \emptyset$ and let $b' \varepsilon (B')^M \setminus B'$. Since $b' \varepsilon (B')^M$, $b' + A' \subseteq A' + B'$. $b' \not\varepsilon B'$ implies $b' \not\varepsilon A - e$; therefore, $b' + e \not\varepsilon A$. But $(b' + e) + B = b' + (B + e) = b' + A' \subseteq A' + B' \subseteq A + B$ which implies $b' + e \varepsilon A$ since A is maximal. We have reached a contradiction. Therefore, $(B')^M \setminus B' = \emptyset$ which implies $B' = (B')^M$.

We now define the concept of a maximal e-transformation. A corollary asserts the fundamental properties of this transformation. The corollary follows immediately from earlier results in this chapter.

<u>Definition 7:</u> Let $A, B \subseteq J$ and $e \varepsilon A$. A <u>maximal e-transformation</u>, denoted <u>$A, B \xrightarrow[M]{e} A', B'$</u>, is defined as the composition of an e-transformation followed by a maximal transformation, as shown in the following diagram.

$$\begin{tikzcd}
A, B \arrow[r, "e", "M"'] \arrow[d, "e"'] & A', B' \\
\bar{A}, \bar{B} \arrow[ur, "M"'] &
\end{tikzcd}$$

The sets <u>A', B' are said to be derived from A, B by the maximal e-transformation</u>, or simply <u>derived sets</u>. Note that $\bar{B} = B'$ by Lemma 6.

<u>Corollary 8</u>: If $A, B \xrightarrow[M]{e} A', B'$ is a maximal e-transformation, then

I) $A \subseteq A'$, $B' \subseteq B$ and $A' + B' \subseteq A + B$

II) $\delta(A, B) \leq \delta(A', B')$

<u>Definition 9</u>: Let $A, B \subseteq J$. A <u>sequence of maximal e-transformations</u> is constructed according to the following diagram:

$A, B \xrightarrow[M]{e_0} A_0, B_0 \xrightarrow[M]{e_1} A_1, B_1 \xrightarrow[M]{e_2} \cdots$, where $e_0 \epsilon A$ and $e_j \epsilon A_{j-1}$ for $j \epsilon J^+$. Such a sequence may be finite or infinite.

In view of Corollary 8, the following relationships exist between the derived sets generated by a sequence of maximal e-transformations:

I) $A \subseteq A_0 \subseteq A_1 \quad \cdots$

II) $B \supseteq B_0 \supseteq B_1 \quad \cdots$

III) $A + B \supseteq A_0 + B_0 \supseteq A_1 + B_1 \quad \cdots$

A particular sequence of maximal e-transformations, the basic sequence of maximal e-transformations, will be used exclusively throughout this dissertation.

<u>Definition 10</u>: Let $A, B \subseteq J$. The <u>basic sequence of maximal e-transformations</u> associated with A, B is a sequence of maximal e-transformations: $A^M, B^M \xrightarrow[M]{e_0} A_0, B_0 \xrightarrow[M]{e_1} A_1, B_1 \xrightarrow[M]{e_2} A_2, B_2 \xrightarrow[M]{e_3} \cdots$, where A^M, B^M are the maximal sets associated with A, B, $e_0 = 0$, and e_n, $n \epsilon J^+$ is selected as follows: if n is odd, then e_n is chosen as the smallest element of A_{n-1} which deletes the smallest possible element of B_{n-1} that can be deleted by a maximal e-transformation; if n is even, then e_n is selected as the smallest element of A_{n-1} which will delete an element of B_{n-1} by a maximal e-transformation.

Since the maximal e-transformations in the basic sequence are the only transformations to be used, the $A_0, A_1, A_2, \cdots$ and $B_0, B_1, B_2, \cdots$ will always refer to the derived sets generated by the basic sequence of maximal e-transformations.

The next theorem provides an insight into the composition of the derived sets obtained from the basic sequence of maximal e-transformations.

<u>Theorem 11</u>: Let $A, B \subseteq J$ and E be a finite non-empty subset of A^M. There exists $n \epsilon J$ such that $E + B_n \subseteq A_n$.

Proof:

Case One: The basic sequence of maximal e-transformations is finite.

Suppose $A^M, B^M \xrightarrow[M]{e_0} A_0, B_0 \xrightarrow[M]{e_1} \cdots \xrightarrow[M]{e_N} A_N, B_N$ is the basic sequence of maximal e-transformations. Since $E \subseteq A^M \subseteq A_N$ and no maximal e-transformations are possible on A_N, B_N, then $E + B_N \subseteq A_N$. Select $n = N$ and the desired result follows.

Case Two: The basic sequence of maximal e-transformations is infinite.

Suppose $A^M, B^M \xrightarrow[M]{e_0} A_0, B_0 \xrightarrow[M]{e_1} A_1, B_1 \xrightarrow[M]{e_2} \cdots$ is the basic sequence of maximal e-transformations. We consider the set $\{e_{2j} : j \epsilon J\}$ and show $e_{2j} \neq e_{2m}$ for $j \neq m$. Suppose $e_{2j} = e_{2m}$ with $m < j$. $e_{2m} + B_{2m-1} \subseteq A_{2m} \subseteq A_{2j-1}$ and $e_{2j} + B_{2j-1} \subseteq e_{2j} + B_{2m-1} = e_{2m} + B_{2m-1} \subseteq A_{2j-1}$. This contradicts the existence of an e_{2j} transformation. Therefore, $e_{2m} \neq e_{2j}$ when $j \neq m$.

Let a be the largest element of E. Since $\{e_{2j} : j \epsilon J\}$ is infinite and consists of distinct non-negative integers, there exists

$j \in J^+$ such that $e_{2j} > a$. The definition of e_{2j} implies $E + B_{2j-1} \subseteq A_{2j-1}$. Select $n = 2j - 1$ to complete Case Two.

The following theorem presents a basic inequality relating the number of elements in A, B and A + B.

<u>Theorem 12</u>: Let $A, B \subseteq J$ and $0 < \alpha \leq 1$. If $n \in J$ and $1 + A(x) + B(x) \geq \alpha(x + 1)$ for $x = 0, 1, 2, \cdots, n$, then $1 + (A + B)(x) \geq \alpha(x + 1)$ for $x = 0, 1, 2, \cdots, n$.

Proof:

Assume the desired result is false. Let n be the smallest non-negative integer for which the desired result does not hold. Choose a value of α, for the n selected, for which the desired result does not hold.

From the pairs of sets for which the desired result does not hold with this n and α, select a pair A, B for which B(n) is minimal. Since the only values of x under consideration are $x = 0, 1, 2, \cdots, n$, all elements of A and B which are greater than n will be removed from both sets.

Note: $1 + A(x) + B(x) \geq \alpha(x + 1)$, for $x = 0, 1, 2, \cdots, n$

$1 + (A + B)(x) \geq \alpha(x + 1)$, for $x = 0, 1, 2, \cdots, n - 1$

$1 + (A + B)(n) < \alpha(n + 1)$

Remark: A contradiction will be obtained by constructing a pair of sets, A' and B' for which $A' + B' \subseteq A + B$, $B'(n) < B(n)$, and $1 + A'(x) + B'(x) \geq \alpha(x + 1)$ for $x = 0, 1, 2, \cdots, n$. Since $A' + B' \subseteq A + B$, then $1 + (A' + B')(n) < \alpha(n + 1)$, which will contradict the selection of A and B. Because this construction is rather complicated some of the properties of the sets involved are established in a series of assertions.

Assertion One: $B(n) > 0$

If $B(n) = 0$, then B has no elements in the closed interval, $[1, n]$. Therefore, $1 + (A + B)(n) = 1 + A(n) = 1 + A(n) + B(n) \geq \alpha(n + 1)$, which contradicts the choice of A and B. Thus, $B(n)$ must be positive.

From Assertion One it follows that there exists at least one positive $b \varepsilon B$. If a is the largest element of A, then $a + b \notin A$. Let $a^* = \min \{a \varepsilon A : a + B \nsubseteq A\}$.

Define A' and B' by performing an a^*-transformation on A, B, $A, B \xrightarrow{a^*} A', B'$. Therefore, $A' = A \cup (B + a^*)$ and $B' = B \cap (A - a^*)$.

If $a^* = 0$, then $A(x) + B(x) = (A \cup B)(x) + (A \cap B)(x) = A'(x) + B'(x)$; therefore, $1 + A'(x) + B'(x) = 1 + A(x) + B(x) \geq \alpha(x - 1)$, for $x = 0, 1, 2, \cdots, n$. Property II of Theorem 4 insures $A' + B' \subseteq A + B$. The definition of a^* gives the existence of $b \varepsilon B$ such that $b \notin B'$, so $B'(n) < B(n)$. Now $1 + (A' + B')(n) \leq 1 + (A + B)(n) < \alpha(n + 1)$, using the note earlier in this proof. This contradicts the minimal property of B with respect to $B(n)$. Therefore, a^* must be non-zero and it may be assumed that $a^* > 0$.

Assertion Two: If $0 \leq r < a^*$ then I) $b + (A \cap [0, r]) \subseteq A$, for all $b \varepsilon B$ and II) if $1 + A(r) \geq \alpha(r + 1)$ then $A(b + r) - A(b - 1) \geq \alpha(r + 1)$ for all $b \varepsilon B$.

Property I follows from the definition of a^*.

Suppose $b \varepsilon B$ and $1 + A(r) \geq \alpha(r + 1)$. Property I of this assertion shows that the number of elements in $A \cap [b, b + r]$ is greater than or equal to $1 + A(r)$. Therefore, $A(b + r) - A(b - 1) \geq 1 + A(r) \geq \alpha(r + 1)$.

Assertion Three: If $0 \leq r < a^*$, then $1 + A(r) \geq \alpha(r + 1)$.

Assume the desired result is false and let r^* be the [illegible] value of r for which the result fails. With $\alpha \leq 1$ then $r^* \geq 1$. Let b denote the smallest positive element of B. Since $1 + A(r^*) + B(r^*) \geq \alpha(r^* + 1)$ and $1 + A(r^*) < \alpha(r^* + 1)$, then $1 \leq b \leq r^*$. Therefore $1 + A(b - 1) \geq \alpha b$ since $b - 1 \leq r^*$. This fact coupled with $1 + A(r^*) < \alpha(r^* + 1)$ implies $A(r^*) - A(b - 1) < \alpha(r^* - b + 1)$. Application of the second conclusion of Assertion Two with $r = r^* - b$ gives $A(r^*) - A(b - 1) \geq \alpha(r^* - b + 1)$. This is a contradiction and so the desired result must hold.

Assertion Four: $B'(n) < B(n)$

By the definition of a^*, there exists $b \in B$ such that $a^* + b \notin A$. Therefore, the a^*-transformation deletes b from B in forming B'.

Assertion Five: $1 + A'(x) + B'(x) \geq \alpha(x + 1)$ for $x = 0, 1, 2, \cdots, n$.

Since the only values of x being considered are $x = 0, 1, 2, \cdots, n$, the elements of B which will affect the computation are those for which $b + a^* > n$ or $b + a^* \notin A$.

Let $0 \leq m \leq n$. Our attention is now restricted to the closed interval, $[0, m]$, and any elements of A and B which exceed m are removed.

If $b \in B$, $0 \leq b \leq m$, and b is deleted from B by the a^*-transformation, then either $a^* + b > m$ or $a^* + b \leq m$ and $a^* + b \notin A$.

For each b which satisfies the second condition, $A'(m)$ equals $A(m)$ plus the number of such elements b, while $B'(m)$ equals $B(m)$

minus the number of such b. Therefore, $1 + A'(m) + B'(m) \geq$ $1 + A(m) + B(m) \geq \alpha(m + 1)$.

For any b which satisfies the first condition, $b + a^* \geq m + 1$ and $b \leq m$. Thus, $m - a^* + 1 \leq b \leq m$. Therefore, it will be sufficient to show $1 + A'(m) + B'(m) \geq \alpha(m + 1)$ even if all positive elements of B in the closed interval $[m - a^* + 1, m]$ are deleted by the a^*-transformation.

If $B \cap [m - a^* + 1, m] = \emptyset$, then no elements of B of the type under consideration are deleted from B by the a^*-transformation. In this case, as shown in the second preceding paragraph of this assertion, $1 + A'(m) + B'(m) \geq \alpha(m + 1)$.

If $B \cap [m - a^* + 1, m] \neq \emptyset$, let b^* be the smallest element of B such that $m - a^* + 1 \leq b^* \leq m$. Let $m = b^* + r$ where $0 \leq r < a^*$.

From the inductive hypothesis stated at the start of the proof, $1 + A(b^* - 1) + B(b^* - 1) \geq \alpha b^*$.

Applying Assertions Two and Three yields $A(b^* + r) - A(b^* - 1) \geq \alpha(r + 1)$.

Addition of these last two inequalities gives $1 + A(b^* + r) + B(b^* - 1) \geq \alpha(b^* + r + 1) = \alpha(m + 1)$.

The definition of b^* insures that $B(b^* - 1) = B(m - a^*)$.

Also note that $A'(m) = A(m) + B(m - a^*) - B'(m - a^*)$.

Therefore, $1 + A'(m) + B'(m) = 1 + A(m) + B(m - a^*) - B'(m - a^*) + B'(m) \geq 1 + A(m) + B(m - a^*) = 1 + A(m) + B(b^* - 1) \geq \alpha(m + 1)$.

Since m was arbitrary, $1 + A'(x) + B'(x) \geq \alpha(x + 1)$ for $x = 0, 1, 2, \cdots, n$ which establishes Assertion Five.

A' and B' are obtained from A, B by an a*-transformation; therefore, property II of Theorem 4 shows $A' + B' \subseteq A + B$. This fact, in conjunction with Assumptions Four and Five, shows that A' and B' have the necessary properties to complete the contradiction, establishing Theorem 12.

When A contains k consecutive integers, the next theorem shows that $\delta(A + B) \geq \frac{k}{k+1}\delta(A, B)$ or $A + B \sim J$.

<u>Theorem 13</u>: Let $A, B \subseteq J$. If A contains k consecutive integers then $A + B \sim J$ or $\delta(A + B) \geq \frac{k}{k+1}\delta(A, B)$.

Proof:

Let the k consecutive integers in A be $a, a + 1, \cdots, a + k - 1$. Define $A' = A - a$ and $B' = B$. Since $\delta(A + B) \geq \delta(A' + B')$ and $\delta(A, B) = \delta(A', B')$, it is sufficient to prove the result for A' and B'. Therefore, it may be assumed that the k consecutive integers in A are $0, 1, 2, \cdots, k - 1$.

Theorem 11 gives the existence of $j \varepsilon J$ such that $\{0, 1, \cdots, k - 1\} + B_j \subseteq A_j$. Because $A + B \supseteq A_j + B_j$, $\delta(A + B) \geq \delta(A_j + B_j)$ and $\delta(A_j, B_j) \geq \delta(A, B)$, it may be assumed without loss of generality that $\{0, 1, 2, \cdots, k - 1\} + B \subseteq A$.

If $\delta(A, B) = 0$, the desired result is immediate, so assume $\delta(A, B) > 0$.

Select $\gamma > 0$ such that $\gamma < \delta(A, B)$ and $\frac{\gamma k}{k+1} \leq 1$, i.e., $\gamma \leq 1 + \frac{1}{k}$.

There exists $x_0 \varepsilon J^+$ such that if $x \geq x_0$ then $A(x) + B(x) \geq \gamma x$. Choose x_0 to be the smallest such value. Therefore, $A(x_0 - 1) + B(x_0 - 1) \leq \gamma(x_0 - 1)$ and $A(x_0) + B(x_0) \geq \gamma x_0$. Combining inequalities

yields $A(x) - A(x_0 - 1) + B(x) - B(x_0 - 1) \geq \gamma(x - x_0 + 1)$ for all $x \geq x_0$.

In the preceding inequality, replace x by $x + x_0$ to yield

$$A(x + x_0) - A(x_0 - 1) + B(x + x_0) - B(x_0 - 1) \geq \gamma(x + 1),$$

for all $x \varepsilon J$. (I)

It follows from inequality I that $x_0 \varepsilon A \cup B$. Since $B \subseteq A$, then $x_0 \varepsilon A$.

A shift will now be performed on the sets A and B which will eventually permit the use of Theorem 12.

Let x_1 be the smallest element of B which is greater than or equal to x_0. Define $\overline{B} = (B - x_1) \cap J$ and $\overline{A} = (A - x_0) \cap J$.

$$\overline{A}(x) = A(x_0 + x) - A(x_0) = A(x_0 + X) - A(x_0 - 1) - 1$$

$$\overline{B}(x) = B(x + x_1) - B(x_1) = B(x + x_1) - B(x_1 - 1) - 1$$

$$\geq B(x_0 + x) - B(x_0 - 1) - 1$$

Therefore, $1 + \overline{A}(x) + \overline{B}(x) \geq 1 + A(x_0 + x) - A(x_0 - 1) - 1 + B(x_0 + x) - B(x_0 - 1) - 1$

$$\geq A(x_0 + x) - A(x_0 - 1) + B(x_0 + x) - B(x_0 - 1) - 1$$

$$\geq \gamma(x + 1) - 1, \text{ for all } x \varepsilon J \qquad \text{(II)}$$

This latter inequality follows from equation I.

It will now be demonstrated in four cases that $1 + \overline{A}(x) + \overline{B}(x) \geq \frac{k}{k + 1} \gamma(x + 1)$ for all $x \varepsilon J$.

Case One: $0 \leq x < x_1 - x_0$.

Now $x_0 \leq x + x_0 < x_1$.

The definition of x_1 insures that $B(x + x_0) - B(x_0 - 1) = 0$. Therefore, $1 + \bar{A}(x) + \bar{B}(x) \geq 1 + \bar{A}(x) = A(x + x_0) - A(x_0 - 1) + B(x + x_0) - B(x_0 - 1) \geq \gamma(x + 1)$ by inequality I. It then follows that $1 + \bar{A}(x) + \bar{B}(x) \geq \frac{k}{k + 1}\gamma(x + 1)$.

Case Two: $x_1 - x_0 \leq x < x_1 - x_0 + k$

Since $x_1 \varepsilon B$ and $\{0, 1, 2, \cdots, k - 1\} + B \subseteq A$, then $\{x_1, x_1 + 1, \cdots, x_1 + k - 1\} \subseteq A$. Therefore, $\{x_1 - x_0, x_1 - x_0 + 1, \cdots, x_1 - x_0 + k - 1\} \subseteq \bar{A}$. Now $\bar{A}(x) - \bar{A}(x_1 - x_0 - 1) = x - x_1 + x_0 + 1$. (<u>III</u>)

Therefore, $1 + \bar{A}(x) + \bar{B}(x) = 1 + \bar{A}(x_1 - x_0 - 1) + \bar{B}(x_1 - x_0 - 1) + \bar{A}(x) - \bar{A}(x_1 - x_0 - 1) + \bar{B}(x) - \bar{B}(x_1 - x_0 - 1) \geq \frac{k\gamma}{k + 1}(x_1 - x_0) + x - x_1 + x_0 + 1$, by Case One, equation III, and the fact that $\bar{B}(x) - \bar{B}(x_1 - x_0 - 1)$ is non-negative. Thus, $1 + \bar{A}(x) + \bar{B}(x) \geq \frac{k\gamma}{k + 1}(x_1 - x_0) + \frac{k\gamma}{k + 1}(x - x_1 + x_0 + 1) = \frac{k\gamma}{k + 1}(x + 1)$.

Case Three: $x_1 - x_0 + k \leq x < \frac{k + 1}{\gamma} - 1$

$\bar{A}(x) \geq k - 1$ since $\{x_1 - x_0, x_1 - x_0 + 1, \cdots, x_1 - x_0 + k - 1\} \subseteq \bar{A}$. Therefore, $1 + \bar{A}(x) + \bar{B}(x) \geq k$. But $x + 1 < \frac{k + 1}{\gamma}$ which implies $k > \frac{k\gamma}{k + 1}(x + 1)$. Thus $1 + \bar{A}(x) + \bar{B}(x) \geq \frac{k\gamma}{k + 1}(x + 1)$.

Case Four: $x \geq \frac{k + 1}{\gamma} - 1$

Now $\frac{\gamma(x + 1)}{k + 1} \geq 1$ implying $\frac{\gamma(x + 1)}{k + 1} - 1 \geq 0$. Therefore, $\gamma(x + 1)\cdot\left(1 - \frac{k}{k + 1}\right) - 1 = \frac{\gamma(x + 1)}{k + 1} - 1 \geq 0$ which implies $\gamma(x + 1) - 1 \geq \frac{k\gamma}{k + 1}(x + 1)$.

Thus, $1 + \bar{A}(x) + \bar{B}(x) \geq \gamma(x + 1) - 1 \geq \frac{k\gamma(x + 1)}{k + 1}$ using inequality II.

With these four cases it now follows that $1 + \bar{A}(x) + \bar{B}(x) \geq \frac{k\gamma(x + 1)}{k + 1}$, for all $x \varepsilon J$.

By Theorem 12, $1 + (\bar{A} + \bar{B})(x) \geq \frac{k\gamma(x + 1)}{k + 1}$, for all $x \varepsilon J$.

Assume that $\frac{k}{k + 1}\delta(A, B) > 1$. Select $\gamma = \frac{k + 1}{k}$. Then $\frac{k\gamma}{k + 1} = 1$ and $0 < \gamma < \delta(A, B)$. $1 + (\bar{A} + \bar{B})(x) \geq$ [illegible] which implies $(\bar{A} + \bar{B})(x) \geq$ x for all $x \varepsilon J$. Therefore, $(\bar{A} + \bar{B})(x) =$ [illegible] for all $x \varepsilon J$, so $\bar{A} + \bar{B} = J$. But $A + B \supseteq (\bar{A} + x_0) + (\bar{B} + x_1) = (\bar{A} + \bar{B}) + (x_0 + x_1) = J + (x_0 + x_1)$. Therefore, $A + B \sim J$.

Now consider the case when $\frac{k\delta(A, B)}{k + 1} \leq 1$. Let $\varepsilon > 0$ and select $\gamma = \gamma(\varepsilon)$ such that $\delta(A, B) - \varepsilon < \gamma < \delta(A, B)$. Then $\frac{k\gamma}{k + 1} > (\delta(A, B) - \varepsilon)\frac{k}{k + 1}$. Therefore, $1 + (\bar{A} + \bar{B})(x) \geq \frac{k\gamma(x + 1)}{k + 1}$ and $\delta(A + B) = \delta(\bar{A} + \bar{B}) \geq \frac{k(\delta(A, B) - \varepsilon)}{k + 1}$. Thus, $\delta(A + B) \geq \frac{k(\delta(A, B) - \varepsilon)}{k + 1}$ and since this inequality holds for every $\varepsilon > 0$, then $\delta(A + B) \geq \frac{k\delta(A, B)}{k + 1}$.

Concluding Chapter One are the definitions and basic properties of the two set functions, f and g. These functions will play a crucial role in the analysis of the derived sets obtained from the basic sequence of maximal e-transformations.

<u>Definition 14</u>: Let $C \subseteq J$, $C \neq \emptyset$, and $C \neq \{0\}$. <u>f(c)</u> $= \min\{|c - c'| : c, c' \varepsilon C, c \neq c'\}$, the minimum positive difference between distinct elements of C and <u>g(C)</u> is the greatest common divisor of the elements of C.

Of particular interest are the sequences $\{f(B_j) : j \varepsilon J\}$ and $\{g(B_j) : j \varepsilon J\}$. The basic properties of these sequences are the subject of the next corollary.

<u>Corollary 15</u>: The sequences $\{f(B_j) : j \varepsilon J\}$ and $\{g(B_j) : j \varepsilon J\}$ satisfy:

I) both are non-decreasing sequences of positive integers

II) $f(B_j) \geq g(B_j)$, for all $j \varepsilon J$.

III) if $\{f(B_j) : j\varepsilon J\}$ is bounded then there exists $k\varepsilon J^+$ such that the subsequences $\{f(B_j) : j \geq k\}$ and $\{g(B_j) : j \geq k\}$ are both constant.

IV) if $\{f(B_j) : j\varepsilon J\}$ is unbounded, then $\delta(A + B) \geq \delta(A, B)$.

Proof:

The first two properties are immediate consequences of the definitions of f and g and the facts that $B_0 \supseteq B_1 \supseteq B_2 \quad \cdots$ and $g(B_j) \Big| |b - b'|$ for all b, $b'\varepsilon B_j$ with $b \neq b'$ and $j\varepsilon J$.

Property III follows directly from properties I and II.

To establish property IV, let $j\varepsilon J$ and $x\varepsilon J^+$. Obviously $B_j(x) \leq x/f(B_j)$. Therefore, $(A_j + B_j)(x) \geq A_j(x) \geq A_j(x) + B_j(x) - x/f(B_j)$. Now $\delta(A_j + B_j) = \liminf_{x\to\infty} \frac{(A_j + B_j)(x)}{x} \geq \liminf_{x\to\infty} \left(\frac{A_j(x) + B_j(x)}{x} - \frac{1}{f(B_j)} \right) = \delta(A_j, B_j) - 1/f(B_j)$. Consequently, $\delta(A + B) \geq \delta(A_j + B_j) \geq \delta(A_j, B_j) - 1/f(B_j) \geq \delta(A, B) - 1/f(B_j)$, for each $j\varepsilon J$. Since $f(B_j) \to \infty$ as $j \to \infty$, it follows that $\delta(A + B) \geq \delta(A, B)$.

CHAPTER TWO

B^* AND KNESER'S THEOREM

The presentation of Kneser's Theorem given in this chapter utilizes three new concepts: the maximal e-transformation, the basic sequence of maximal e-transformations, and the limit set, B^*. The fundamental properties of the first two concepts were given in Chapter One. The limit set B^* will be recalled now.

<u>Definition 1</u>: Let $A, B \subseteq J$. $\underline{B^*} = \bigcap \{B_j : j \in J\}$. B^* is the set of all elements of B^M which are not deleted by any transformation in the basic sequence of maximal e-transformations.

In this new approach to Kneser's Theorem, the composition of B^* is considered. Two cases arise: (I) $B^* = \{0\}$, i.e., all non-zero elements are deleted from B^M and (II) $B^* \neq \{0\}$, i.e., there are non-zero elements of B^M which are not deleted from B^M.

The next theorem, by a simple calculation, will show that $\delta(A + B) \geq \delta(A, B)$ when $B^* = \{0\}$ and the basic sequence of maximal e-transformations contains a finite number of maximal e-transformations.

<u>Theorem 2</u>: If $B^* = \{0\}$ and the basic sequence of maximal e-transformations is finite, then $\delta(A + B) \geq \delta(A, B)$.

Proof:

Suppose $A^M, B^M \xrightarrow[M]{e_0} A_0, B_0 \xrightarrow[M]{e_1} \cdots \xrightarrow[M]{e_n} A_n, B_n$ is the basic sequence of maximal e-transformations. Since $B^* = B_n = \{0\}$, $\delta(A_n + B_n) = \delta(A_n) = \delta(A_n, B_n)$.

Therefore, $\delta(A + B) = \delta(A^M + B^M) \geq \delta(A_n + B_n) = \delta(A_n, B_n) \geq \delta(A^M, B^M) \geq \delta(A, B)$.

For use in the next theorem, two lemmas concerned with the properties of the greatest common divisor of a non-empty set $A \subseteq J$, $A \neq \{0\}$, are presented.

<u>Lemma 3</u>: Let $A \subseteq J$ with at least one non-zero element in A. Then $g(A) = \min \{g(C) : C \subseteq A$, C is finite and contains at least one non-zero element$\}$.

Proof:

Let $d = \min \{g(C) : C \subseteq A$, C is finite and contains at least one non-zero element$\}$. Let $C \subseteq A$ such that C is finite, contains a non-zero element, and $d = g(C)$. $g(A)$ divides each element of C, hence $g(A) \leq d$.

We now show that d is a divisor of each element of A which implies $d \leq g(A)$.

Assume there exists $a \in A$ such that $d \nmid a$. Let $D = \{a\} \cup C$, where C is given in the first paragraph of this proof. $D \subseteq A$, D is finite and contains a non-zero element. Since $C \subseteq D$, $C \neq D$ because $a \notin C$, and $d \nmid a$, it follows that $g(C) = d > g(D)$. This contradicts the definition of d. Therefore, d must be a divisor of every element of A and $d \leq g(A)$.

Therefore, $d = g(A)$.

<u>Lemma 4</u>: Let $A \subseteq J$ with at least one non-zero element in A. If the elements of A are arranged in natural order, $a_1 < a_2 < \cdots$, then there exists an initial segment, $a_1 < a_2 < \cdots < a_n$, such that $g(A) = \gcd\{a_1, a_2, \cdots, a_n\}$ and $g(A) \neq \gcd\{a_1, a_2, \cdots, a_{n-1}\}$.

Proof:

Consider the non-empty collection of all finite subsets of A whose greatest common divisor equals g(A).

Let e be the smallest positive integer that can be the largest element of one of these subsets. Select one of these subsets, D, whose largest element is e; therefore, $D = \{d_1 < d_2 < \cdots < d_m = e\}$.

Consider the initial segment of A, $a_1 < a_2 < \cdots < a_n = d_m = e$. Now $\gcd\{a_1, a_2, \cdots, a_n\} \leq g(D) \leq g(A)$ because $D \subseteq \{a_1, a_2, \cdots, a_n\}$. But $\{a_1, a_2, \cdots, a_n\} \subseteq A$ so $g(A) \leq \gcd\{a_1, a_2, \cdots, a_n\}$. Therefore, $\gcd\{a_1, a_2, \cdots, a_n\} = g(A)$.

Since $a_1 = 0$, if $n = 2$ it follows that a_2 cannot be deleted from $\{a_1, a_2\}$ and have $\gcd\{a_1\} = g(A)$. Now assume $n > 2$.

If a_n is removed from the initial segment $\{a_1, a_2, \cdots, a_n\}$ to give a new initial segment $\{a_1, a_2, \cdots, a_{n-1}\}$, then $\gcd\{a_1, a_2, \cdots, a_{n-1}\} > g(A)$ otherwise we would contradict the definition of D. Therefore, $\gcd\{a_1, a_2, \cdots, a_{n-1}\} \neq g(A)$.

Theorem 5 combines the two preceding lemmas with the basic properties of residue classes, mod $g(B^*)$, to demonstrate that when $B^* \neq \{0\}$, the basic sequence of maximal e-transformations will be finite and there exists $C \subseteq A + B$ such that $0 \in C$, $C \sim C^{g(B^*)}$, and $\delta(C) \geq \delta(A, B) - 1/g(B^*)$. The existence of such a set C will be valuable in the analysis when $\delta(A + B) < \delta(A, B)$.

<u>Theorem 5</u>: Let $A, B \subseteq J$. If $B^* \neq \{0\}$ and $g = g(B^*)$, then

1) the basic sequence of maximal e-transformations is finite

2) there exists $C \subseteq A + B$ such that $0 \varepsilon C$, $C \sim C^{(g)}$, and $\delta(C) \geq \delta(A, B) - 1/g$.

Proof:

By lemma 4, there exists an initial segment of positive elements of B^*, $b_1 < b_2 < \cdots < b_n$, such that $g(B^*) = \gcd\{b_1, b_2, \cdots, b_n\}$ and b_n cannot be removed from this initial segment and have the last equality hold.

Since the sequence of sets, $\{B_j : j \varepsilon J\}$, is a decreasing sequence there exists an even integer N_1 such that $B_{N_1} \cap (0, b_n] = \{b_1, b_2, \cdots, b_n\}$.

Select N_2 to be an even integer, $N_2 \geq N_1$, such that A_{N_2} has a maximal number of residue classes, mod g, represented in it. The existence of N_2 follows from the fact that the sequence $\{A_j : j \varepsilon J\}$ is increasing.

$\{b_1, b_2, \cdots, b_n\} + A_{N_2} \subseteq A_{N_2}$ since b_j cannot be deleted from B_{N_2} for $j = 1, 2, \cdots, n$. This fact coupled with $B_{N_2} \subseteq A_{N_2}$ implies $a + c_1 b_1 + \cdots + c_n b_n \varepsilon A_{N_2}$ for any $a \varepsilon A$ and $c_j \varepsilon J$ for $j = 1, 2, \cdots, n$.

Divide $b_1, b_2, \cdots, b_n$ by g obtaining $b_j = \alpha_j g$ for $j = 1, 2, \cdots, n$. A theorem of Frobenius (cf. Brauer and Shockley, "On a problem of Frobenius," Journal für die reine und angwarte Mathematik, 1962) asserts the existence of $K \varepsilon J^+$ such that every positive integer greater than K can be represented as a linear combination of $\alpha_1, \alpha_2, \cdots, \alpha_n$ with positive coefficients.

We will now show that $A_{N_2} \sim A_{N_2}{}^{(g)}$. From each residue class, mod g, which has a representative in A_{N_2}, select the smallest positive

representative in A_{N_2}. Denote these representatives by $a_1 < a_2 < \cdots < a_r$ and define $M = a_r + Kg$. Let $a \varepsilon A_{N_2}$ and $\alpha \varepsilon J^+$ such that $a + \alpha g > M$. $a + \alpha g = a_j + \beta g$ for some j, $\beta \varepsilon J^+$ with $\beta > K$. Since $\beta > K$, there exists $c_1, c_2, \cdots, c_n \varepsilon J^+$ such that $\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$, using the theorem of Frobenius. Therefore, $a + \alpha g = a_j + \beta g = a_j + (c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n)g = a_j + c_1(\alpha_1 g) + c_2(\alpha_2 g) + \cdots + c_n(\alpha_n g) = a_j + c_1 b_1 + c_2 b_2 + \cdots + c_n b_n \varepsilon A_{N_2}$. Hence, $A_{N_2} \sim A_{N_2}^{(g)}$.

Since $A_{N_2} \sim A_{N_2}^{(g)}$, there exists $p \varepsilon J^+$ such that $A_{N_2} \cap [p, \infty) = A_{N_2}^{(g)} \cap [p, \infty)$. Select $N \geq N_2$ such that B_N has a minimal intersection with the closed interval, $[0, p]$.

We now show that $B_N = B^*$. By definition, $B^* \subseteq B_N$. Assume there exists $b \varepsilon B_N \setminus B^*$. There exists $N_3 \geq N$ such that $b \varepsilon B_{N_3}$ and $b \not\varepsilon B_{N_3+1}$. Since $B_{N_3} \subseteq B_N$, $B_{N_3} \cap [0, p] = B_N \cap [0, p]$, by the choice of N. Because $N_3 \geq N_2$, $A_{N_3}^{(g)} \cap [p, \infty) = A_{N_2}^{(g)} \cap [p, \infty) = A_{N_2} \cap [p, \infty) \subseteq A_{N_3} \cap [p, \infty)$: therefore, $A_{N_3}^{(g)} \cap [p, \infty) = A_{N_3} \cap [p, \infty)$. There exists $a \varepsilon A_{N_3}$ such that $a + b \not\varepsilon A_{N_3}$, but $a + b \varepsilon A_{N_3+1}$. Since $b > p$, $a + b > p$ and $a + b$ must belong to some residue class, mod g, already represented in A_{N_3}. Therefore, $a + b \varepsilon A_{N_3}$ since $N_3 \geq N$. This is a contradiction. Therefore, $B_N \setminus B^* = \emptyset$, which gives $B_N = B^*$.

Since $B_N = B^*$, the basic sequence of maximal e-transformations is finite.

The maximal property of A_{N_2} with respect to residue classes, mod g, and $A_{N_2} \subseteq A_N$ imply that $A_N \sim A_N^{(g)}$. Therefore, $A_N + B_N \sim (A_N + B_N)^{(g)}$.

$B_N(x) \leq x/g$ implies $B_N(x) - x/g \leq 0$; therefore, $\delta(A_N) = \liminf_{x\to\infty} \frac{A_N(x)}{x} \geq \liminf_{x\to\infty} \frac{A_N(x) + B_N(x) - x/g}{x} = \delta(A_N, B_N) - 1/g$.

Select $C = A_N + B_N$. Since $A_N + B_N \subset A + B$, C is a subset of $A + B$. $0 \varepsilon A_N$ and $0 \varepsilon B_N$ implies $0 \varepsilon C$. Since $A_N + B_N \sim (A_N + B_N)^{(g)}$, $C \sim C^{(g)}$. $\delta(C) = \delta(A_N + B_N) = \delta(A_N{}^{(g)}) = \delta(A_N) \geq \delta(A_N, B_N) - 1/g \geq \delta(A, B) - 1/g$.

The situation when $B^* = \{0\}$ and the basic sequence of maximal e-transformations is infinite now comes under consideration.

In the analysis of this situation, a translation of the derived sets associated with the basic sequence of maximal e-transformations is a crucial step. The basic map, Λ, is introduced to effect this translation and three lemmas present the fundamental properties of Λ.

<u>Definition 6</u>: Let $A \subseteq J$, $g \varepsilon J^+$ and $0 = r_0 < r_1 < \cdots < r_{h-1}$ be the smallest representatives in A of the residue classes, mod g, which make up $A^{(g)}$. If I denotes the set of integers, then the transformation $\underline{\Lambda}$: $A^{(g)} \to I$ defined by $\Lambda(r_j + \alpha g) = j + \alpha h$ where $\alpha \varepsilon I$ and $j = 0, 1, \cdots, h - 1$, is the <u>basic map from $A^{(g)}$ to I</u>.

Λ is one to one and if R_Λ denotes the range of Λ, then Λ^{-1} : $R_\Lambda{}^{(h)} \to A^{(g)}$ is the basic map from $R_\Lambda{}^{(h)}$ onto $A^{(g)}$.

For notational convenience in dealing with Λ, when $x \varepsilon A^{(g)}$, $\Lambda(x)$ will be denoted $\hat{x}$ and if $C \subseteq A^{(g)}$, then $\Lambda(C)$ will be denoted by $\hat{C}$.

<u>Lemma 7</u>: Let $0 \varepsilon B \subseteq A \subseteq J$ and $g = g(B)$. Let h be the number of residue classes, mod g, with representatives in A and Λ be the basic map from $A^{(g)}$ to I. Then:

(1) $h = g(\hat{B})$

(2) if $C \subseteq B$, $C \neq \{0\}$, $C \neq \emptyset$, then $\frac{1}{g}\, g(C) = \frac{1}{h}\, g(\hat{C})$

(3) $B(xg) = \hat{B}(xh)$, $x \varepsilon J$

(4) $A(xg) = \hat{A}(xh) + O(1)$, $x \varepsilon J$ and $O(1)$ denotes a bounded function of x

(5) if $a \varepsilon A^{(g)}$ and $x \equiv 0$, mod g, then $\Omega(a + x) = \Omega(a) + \Omega(x)$, i.e., $\widehat{a + x} = \hat{a} + \hat{x}$ provided $x \equiv 0$, mod g

(6) if $a \equiv r_j$, mod g, and $x \equiv r_j$, mod g, for some j, then $\Omega(a - x) = \Omega(a) - \Omega(x)$, i.e., $\widehat{a - x} = \hat{a} - \hat{x}$ provided $a \equiv x$, mod g

(7) $\Omega(A + B) = \Omega(A) + \Omega(B)$, i.e., $\widehat{A + B} = \hat{A} + \hat{B}$

Proof:

(1) Since $B = \{0 + gk : k \varepsilon K\}$ where $K \subseteq J$ and $g(K) = 1$, $\hat{B} = \{0 + hk : k \varepsilon K\}$. By Lemma 3, $g(\hat{B}) = \min \{g(X) : X \subseteq \hat{B}, X \neq \{0\}$, X is finite$\} = \min \{g(h \cdot Y) : Y \subseteq K, Y \neq \{0\}$, Y is finite$\} = h \cdot \min \{g(Y) : Y \subseteq K, Y \neq \{0\}$, Y is finite$\} = h \cdot g(K) = h$.

(2) Let $C \subseteq B$, $C \neq \emptyset$ and $C \neq \{0\}$. There exists $F \subseteq J$ such that $C = \{xg : x \varepsilon F\} = g \cdot F$. Therefore $g(C) = g \cdot g(F)$. $\Omega(C) = \Omega(g \cdot F) = h \cdot F = \hat{C}$. Thus $g(\hat{C}) = h \cdot g(F)$ and $\frac{1}{g} g(C) = \frac{1}{h} g(\hat{C})$.

(3) There exists $E \subseteq J$ such that $B = \{xg : x \varepsilon E\} = g \cdot E$. $B(xg) = (g \cdot E)(xg) = E(x)$. $\hat{B} = \Omega(B) = \Omega(g \cdot E) = h \cdot E$ so $\hat{B}(xh) = (h \cdot E)(xh) = E(x)$. Therefore $B(xg) = \hat{B}(xh)$.

(4) $A(xh)$ is the number of elements in $A \cap (0, xh]$. Since Ω is one to one, this is the same as the number of elements in $\Omega^{-1}(A \cap (0, xh])$ which is the same as the number of elements in $(\{r_0 + \alpha g : \quad = 1, 2, \cdots, x\} \cup \bigcup_{j=1}^{h-1} \{r_j + \alpha g : \alpha = 0, 1, 2, \cdots, x - 1\}) \cap A$. The number of elements in $\{r_j + \alpha g : \alpha = 0, 1, 2, \cdots, x - 1\}$ which are between xg and $r_j + (x - 1)g$ is no greater than

$|1 - r_j/g|$, for each j, $j = 1, 2, \cdots, h - 1$. Therefore, $A(xg) = \hat{A}(xh) + O(1)$.

(5) Let $a = r_j + \alpha g$ and $x = \beta g$ where $\alpha, \beta \varepsilon J$. $\Lambda(a + x) = \Lambda(r_j + (\alpha + \beta)g) = j + (\alpha + \beta)h = (j + \alpha h) + \beta h = \Lambda(a) + \Lambda(x)$.

(6) Let $a = r_j + \alpha g$ and $x = r_j + \beta g$, where $\alpha, \beta \varepsilon J$. $\Lambda(a - x) = \Lambda((\alpha - \beta)g) = (\alpha - \beta)h = (j + \alpha h) - (j + \beta h) = \Lambda(a) - \Lambda(x)$.

(7) This follows immediately from the fact that each element of B is congruent to zero, mod g, and property five of this lemma.

<u>Lemma 8</u>: Let $A, B \subseteq J$ and $g = g(B)$. Then $\delta(A, B) = \liminf_{x\to\infty} \frac{A(xg) + B(xg)}{xg}$ and $\delta(A + B) = \liminf_{x\to\infty} \frac{(A + B)(xg)}{xg}$.

Proof:

Let $d = \liminf_{x\to\infty} \frac{A(xg) + B(xg)}{xg}$. Now $d \geq \delta(A, B)$.

Let $\varepsilon > 0$. There exists $K \varepsilon J^+$ such that if $xg > K$, then $\frac{A(xg) + B(xg)}{xg} > d - \varepsilon$.

Let $x \varepsilon J^+$ such that $x > K + g$. There exists α_x, depending on x, such that $(\alpha_x - 1)g \leq x < \alpha_x g$. $A(x) + B(x) = A(\alpha_x g) + B(\alpha_x g) + O(1)$, so $\frac{A(x) + B(x)}{x} \geq \frac{A(\alpha_x g) + B(\alpha_x g) + O(1)}{\alpha_x g}$.

From this last inequality, $\delta(A, B) = \liminf_{x\to\infty} \frac{A(x) + B(x)}{x} \geq \liminf_{x\to\infty} \frac{A(\alpha_x g) + B(\alpha_x g) + O(1)}{\alpha_x g} \geq d$.

Therefore, $\liminf_{x\to\infty} \frac{A(xg) + B(xg)}{xg} = d = \delta(A, B)$.

The second assertion follows in a similar manner.

<u>Lemma 9</u>: Let $0 \varepsilon B \subseteq A \subseteq J$, $g = g(B)$, and Λ be the basic map from $A^{(g)}$ to I. Then:

(1) $g(B) = \frac{g}{h} g(\hat{B})$ and $f(B) = \frac{g}{h} f(\hat{B})$

(2) $\delta(A, B) = \frac{h}{g} \delta(\hat{A}, \hat{B})$

(3) $\delta(A + B) = \frac{h}{g}\,\delta(\hat{A} + \hat{B})$

(4) $A + B \sim (A + B)^{(g)}$ if and only if $\hat{A} + \hat{B} \sim C$

Proof:

(1) The first equality follows directly from properties 1 and 2 of Lemma 7 with $C = B$. $g \mid f(B)$, so $f(B) = \alpha g$. Therefore, $f(\hat{B}) = \alpha h$ and combining the last two equalities gives $f(B) = \alpha g = \frac{g}{h} f(\hat{B})$.

(2) $\delta(A, B) = \liminf_{x\to\infty} \frac{A(xg) + B(xg)}{xg}$, by Lemma 8

$= \liminf_{x\to\infty} \frac{\hat{A}(xh) + \hat{B}(xh) + O(1)}{xg}$, by properties 3 and 4 of Lemma 7

$= \frac{h}{g} \liminf_{x\to\infty} \frac{\hat{A}(xh) + \hat{B}(xh) + O(1)}{xh}$

$= \frac{h}{g}\,\delta(\hat{A}, \hat{B})$

(3) For $j = 0, 1, 2, \cdots, h - 1$, let $[r_j]$, mod g, denote the residue class, mod g, with representative r_j and define E_j by $A \cap ([r_j], \text{mod } g) = \{r_j + xg : x \in E_j\} = r_j + g \cdot E_j$. Therefore, $A = \bigcup_{j=0}^{h-1} (r_j + g \cdot E_j)$. Let E be defined by $B = \{xg : x \in E\} = g \cdot E$.

$$A + B = \bigcup_{j=0}^{h-1} (r_j + g \cdot E_j) + g \cdot E = \bigcup_{j=0}^{h-1} (r_j + g \cdot (E_j + E)).$$

Define $T_j = E_j + E$, for $j = 0, 1, 2, \cdots, h - 1$ and the last equality becomes $A + B = \bigcup_{j=0}^{h-1} (r_j + g \cdot T_j)$.

The number of elements of $A + B$ which are between xg and $r_j + xg$ is bounded, independent of x. Therefore, $(A + B)(xg) = \sum_{j=0}^{h-1} (r_j + g \cdot T_j)(xg) = \sum_{j=0}^{h-1} T_j(x) + O(1)$. By a similar argument $(\hat{A} + \hat{B})(xh) = \sum_{j=0}^{h-1} T_j(x) + O(1)$. Therefore, $(A + B)(xg) = (\hat{A} + \hat{B})(xh) + O(1)$.

$$\delta(A + B) = \liminf_{x\to\infty} \frac{(A + B)^{(xg)}}{xg}, \text{ by Lemma 8}$$

$$= \liminf_{x\to\infty} \frac{(\hat{A} + \hat{B})^{(xh)} + O(1)}{xg}$$

$$= \frac{h}{g} \liminf_{x\to\infty} \frac{(\hat{A} + \hat{B})^{(xh)} + O(1)}{xh}$$

$$= \frac{h}{g}\, \delta(\hat{A} + \hat{B})$$

(4) Using the notation from the proof of property three, $A + B = \bigcup_{j=0}^{h-1} (r_j + g \cdot T_j)$ and $\hat{A} + \hat{B} = \bigcup_{j=0}^{h-1} (j + h \cdot T_j)$.

$A + B \sim (A + B)^{(g)}$ if and only if $T_j \sim J$ for $j = 0, 1, 2, \cdots, h - 1$ and this occurs if and only if $\hat{A} + \hat{B} \sim J$.

A sequence of pairs of sets, $\{\hat{A}_j, \hat{B}_j : j\varepsilon J\}$, will now be constructed by Ω from pairs of derived sets obtained from the basic sequence of maximal e-transformations.

If $\{f(B_j) : j\varepsilon J\}$ is unbounded, then $\delta(A + B) \geq \delta(A, B)$ by part IV of Corollary 15, Chapter One. Assume that $\{f(B_j) : j\varepsilon J\}$ is bounded and part III of the same corollary gives the existence of $N_1 \varepsilon J^+$ such that $\{f(B_j) : j \geq N_1\}$ and $\{g(B_j) : j \geq N_1\}$ are both constant. If $j \geq N_1$, then $A_j^{(g)}$ and $A_{N_1}^{(g)}$ will contain exactly the same residue classes, mod g, where $g = g(B_j)$, $j \geq N_1$.

Let $0 = r_0 < r_1 < \cdots < r_{h-1}$ be the minimal representatives in A_{N_1} of the residue classes, mod g, in $A_{N_1}^{(g)}$. There exists $N \geq N_1$ such that $B_N \cap (0, r_{h-1}] = \emptyset$ because $B^* = \{0\}$. $\{f(B_j) : j \geq N\}$ and $\{g(B_j) : j \geq N\}$ are both constant, and if $j, k \geq N$, then $A_j^{(g)}$ and $A_k^{(g)}$ contain exactly the same residue classes, mod g; furthermore, the minimal representatives in A_j and A_k of these residue classes, mod g, are identical.

The sequence $\{\hat{A}_j, \hat{B}_j : j\varepsilon J\}$ is defined by: $\hat{A}_j = \Omega(A_{N+j})$ and $\hat{B}_j = \Omega(B_{N+j})$ for all $j\varepsilon J$. The diagram below depicts this construction.

$$\begin{tikzcd}
A_N, B_N \arrow[r, "e_{N+1}", "M"'] \arrow[d, leftrightarrow, "\Omega"'] & A_{N+1}, B_{N+1} \arrow[r, "e_{N+2}", "M"'] \arrow[d, leftrightarrow, "\Omega"'] & A_{N+2}, B_{N+2} \arrow[r, "e_{N+3}", "M"'] \arrow[d, leftrightarrow, "\Omega"'] & \cdots \\
\hat{A}_0, \hat{B}_0 & \hat{A}_1, \hat{B}_1 & \hat{A}_2, \hat{B}_2 & \cdots
\end{tikzcd}$$

The following properties of $\{\hat{A}_j, \hat{B}_j : j\varepsilon J\}$ are immediate consequences of the properties of Ω, the basic sequence of maximal e-transformations, the sequences $\{f(B_j) : j \geq N\}$ and $\{g(B_j) : j \geq N\}$ being constant, and the assumption that $B^* = \{0\}$ with the basic sequence of maximal e-transformations infinite.

<u>Corollary 10:</u> The sets $\hat{A}_j$ and $\hat{B}_j$, $j\varepsilon J$ have the following properties:

(1) $\hat{A}_0 \subseteq \hat{A}_1 \subseteq \hat{A}_2 \quad \cdots$

(2) $\hat{B}_0 \supseteq \hat{B}_1 \supseteq \hat{B}_2 \quad \cdots$

(3) $\hat{A}_0 + \hat{B}_0 \supseteq \hat{A}_1 + \hat{B}_1 \quad \cdots$

(4) $\delta(\hat{A}_0, \hat{B}_0) \leq \delta(\hat{A}_1, \hat{B}_1) \leq \delta(\hat{A}_2, \hat{B}_2) = \cdots$

(5) the sequences $\{f(\hat{B}_j) : j\varepsilon J\}$ and $\{g(\hat{B}_j) : j\varepsilon J\}$ are both constant

(6) $\{0, 1, 2, \cdots, h-1\} \subseteq \hat{A}_0$

(7) if $b\varepsilon B_j$ then $b \equiv 0$, mod g, for $j \geq N$

(8) if $\hat{B}^* = \bigcap\{\hat{B}_j : j\varepsilon J\}$, then $\hat{B}^* = \{0\}$

The next two theorems present additional properties possessed by the sequence $\{\hat{A}_j, \hat{B}_j : j\varepsilon J\}$.

<u>Theorem 11</u>: If $\hat{E}$ is a finite non-empty subset of $\hat{A}_j$ for some $j \varepsilon J$, then there exists $k \varepsilon J^+$, $k > j$ such that $\hat{E} + \hat{B}_k \subseteq \hat{A}_k$.

Proof:

$E = \Lambda^{-1}(E)$ is a finite non-empty subset of A_{N+j}.

The infinite sequence $\{e_{2j} : j \varepsilon J\}$ is composed of distinct non-negative integers. If $e_{2m} = e_{2n}$ with $m < n$, then $e_{2n} + B_{2n-1} \subseteq e_{2m} + B_{2m-1} \subseteq A_{2m} \subseteq A_{2n-1}$, which contradicts the existence of a maximal e_{2n} transformation.

If a denotes the largest element of E, then there exists $i \varepsilon J^+$, $i > N/2$ such that $e_{2i} > a$. The definition of e_{2i} insures that $E + B_{2i-1} \subseteq A_{2i-1}$.

Select $k = 2i + j - N$ so k is positive and $k > j$. $E + B_{2i-1} \subseteq A_{2i-1} \subseteq A_{2i+j}$ implies $E + B_{2i+j} \subseteq E + B_{2i-1} \subseteq A_{2i+j}$. Therefore, $E + B_{N+k} \subseteq A_{N+k}$. $\hat{E} + \hat{B}_k = \widehat{E + B_k} = \Lambda(E + B_{N+k}) \subseteq \Lambda(A_{N+k}) = \hat{A}_k$.

<u>Theorem 12</u>: If $m \varepsilon J^+$, then there exists $k \varepsilon J^+$ such that $\hat{A}_k$ contains mh consecutive integers where $h = g(\hat{B}_0)$.

Proof:

We consider residue classes, mod f, where f is the constant value of the sequence $\{f(\hat{B}_j) : j \varepsilon J\}$. Select $j \varepsilon J$ such that as few such residue classes, mod f, as possible are represented in $\hat{B}_j$. Let $0 = b_1 < b_2 < \cdots < b_t < f$ be distinct representatives of those residue classes, mod f, which contain at least one element of B_j. If $n \geq j$ then the number of residue classes, mod f, represented in $\hat{B}_n$ is the same as the number represented in $\hat{B}_j$ and since $\hat{B}_n \subseteq \hat{B}_j$, these will be the same residue classes, mod f, as represented by b_1, b_2, $\cdots$, b_t.

$h = g(\hat{B}_j)$ implies there exists a finite subset of $\hat{B}_j$, say $\{y_1, y_2, \cdots, y_n\}$ and integers $\alpha_1, \alpha_2, \cdots, \alpha_n$, such that $h = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n$. Reducing the numbers, mod f, gives the existence of $\beta_1, \beta_2, \cdots, \beta_t \varepsilon J$ such that $h \equiv \beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_t b_t \equiv c_1 + c_2 + \cdots + c_s$, mod f, where $s = \beta_1 + \beta_2 + \cdots + \beta_t$ and the set $\{c_1, c_2, \cdots, c_s\}$ consists of the elements $b_1, b_2, \cdots, b_t$ appearing with multiplicities $\beta_1, \beta_2, \cdots, \beta_t$ respectively.

Let G be any finite non-empty subset of $\hat{A}_0$. A lengthy argument will show that there exists $z \varepsilon J^+$ and $k \varepsilon J^+$ such that $z + G \subseteq \hat{A}_k$ and $(z + h) + G \subseteq \hat{A}_k$.

By Theorem 11, there exists $m' > j$ such that $G + \hat{B}_{m'} \subseteq \hat{A}_{m'}$. The residue class, mod f, with representative c_1 is represented in $\hat{B}_{m'}$ which implies there exists $x_1 \varepsilon \hat{B}_{m'}$ such that $x_1 \equiv c_1$, mod f, and $x_1 + G \subseteq \hat{A}_{m'}$.

Now there exists $m'' > m'$ such that $(x_1 + G) + \hat{B}_{m''} \subseteq \hat{A}_{m''}$. As in the preceding paragraph, there exists $x_2 \varepsilon \hat{B}_{m''}$ such that $x_2 \equiv c_2$, mod f, and $x_1 + x_2 + G \subseteq \hat{A}_{m''}$.

The process is continued until an i, $i > j$, is found such that $\hat{B}_i$ contains elements $x_1, x_2, \cdots, x_s$ such that $x_1 \equiv c_1$, $x_2 \equiv c_2$, $\cdots$, $x_s \equiv c_s$, mod f, and $(x_1 + x_2 + \cdots + x_s) + G \subseteq \hat{A}_i$.

Since $x_1 + x_2 + \cdots + x_s \equiv c_1 + c_2 + \cdots + c_s \equiv h$, mod f, there exists an integer r such that $x_1 + x_2 + \cdots + x_s = h + rf$.

$G \subseteq \hat{A}_i$ and $(h + rf) + G \subseteq \hat{A}_i$ implies $G \cup ((h + rf) + G) \subseteq \hat{A}_i$. For the remainder of this proof the latter inclusion will be referred to as <u>Statement One</u>.

By Theorem 11, there exists $p > i$ such that $(G \cup ((h + rf) + G)) + \hat{B}_p \subseteq \hat{A}_p$. Since $f(\hat{B}_p) = f$, there exists $y \varepsilon \hat{B}_p$ such that $y + f \varepsilon \hat{B}_p$. Therefore, $y + (G \cup ((h + rf) + G)) \subseteq \hat{A}_p$ and $(y + f) + (G \cup ((h + rf) + G)) \subseteq \hat{A}_p$. From these two inclusions, $(y + G) \cup ((y + h + rf) + G) \cup ((y + f) + G) \cup ((y + f + h + rf) + G) \subseteq \hat{A}_p$.

Three cases now present themselves.

Case One: $r = 0$

Select $z = y$ and $k = p$. Then $z + G$ and $(z + h) + G$ are both subsets of $\hat{A}_k$, giving the desired result.

Case Two: $r > 0$

$((y + f) + G) \cup ((y + rf + h) + G) \subseteq \hat{A}_p$. Define $G' = (y + f) + G$. Therefore $G' \cup ((h + (r - 1)f) + G') \subseteq \hat{A}_p$. This last inclusion is of the same form as Statement One with the value of r reduced by one. Repeating the procedure which follows Statement One r times and letting $G'' = (y_1 + y_2 + \cdots y_r + rf) + G$ yields $G'' \cup (h + G'') \subseteq \hat{A}_k$ for some $k > p$. Select $z = y_1 + y_2 + \cdots + y_r + rf$ to give $z + G$ and $(z + h) + G$ as subsets of $\hat{A}_k$.

Case Three: $r < 0$

$(y + G) \cup ((y + f + h + rf) + G) \subseteq \hat{A}_p$. Define $G' = y + G$. Therefore $G' \cup ((h + (r + 1)f) + G') \subseteq \hat{A}_p$. This latter inclusion is of the same form as Statement One with the value of r increased by one. Proceeding as in the preceding case will yield the desired result.

With these three cases, it follows that there exists $z \varepsilon J^+$ and $k \varepsilon J^+$ such that $z + G \subseteq \hat{A}_k$ and $(z + h) + G \subseteq \hat{A}_k$.

Let $G = \{0, 1, 2, \cdots, h - 1\}$. By part 6 of corollary 10, $G \subseteq \hat{A}_0$. There exists $z \varepsilon J^+$ and $m' \varepsilon J^+$ such that $G' = (z - 3) \cup ((z + h) + G) = \{z, z + 1, \cdots, z + h - 1, z + h, \cdots z + 2h - 1\} \subseteq \hat{A}_{m'}$. G' contains $2h$ consecutive integers.

Using the same technique on G' yields $m'' \varepsilon J^+$ such that $\hat{A}_{m''}$ contains $3h$ consecutive integers.

Repeating this procedure $m - 1$ times gives $k \varepsilon J^+$ such that $\hat{A}_k$ contains mh integers.

<u>Lemma 13</u>: If $\hat{A}_0 + \hat{B}_0 \sim J$, then $A_N + B_N \sim (A_N + B_N)^{(g)}$. Consequently, $A_N \sim A_N^{(g)}$, $B_N \sim B_N^{(g)}$ and the sequence of maximal e-transformations, $A_N, B_N \xrightarrow[M]{e_{N+1}} A_{N+1}, B_{N+1} \xrightarrow[M]{e_{N+2}} \cdots$ is finite.

Proof:

The first assertion follows from property four of Lemma 9.

Suppose $A_N + B_N \sim (A_N + B_N)^{(g)}$. There exists $K \varepsilon J^+$ such that if $x \varepsilon A_N + B_N$ and $\alpha \varepsilon J$ with $x + \alpha g \geq K$, then $x + \alpha g \varepsilon A_N + B_N$.

Let $S = \{x \varepsilon J : x \equiv 0, \text{ mod } g, \text{ and } x \geq K\}$. Let $a \varepsilon A_N$ and $s = \alpha g \varepsilon S$. $a + s = a + \alpha g \geq K$, so $a + \alpha g \varepsilon A_N + B_N$; therefore, $S \subseteq B_N$. Since all elements of B_N are congruent to zero, mod g, $B_N \sim B_N^{(g)}$.

Let $a \varepsilon A_N$ and $\alpha \varepsilon J$ such that $a + \alpha g \geq K$. If $b \varepsilon B_N$, then $b = \gamma g$ for some $\gamma \varepsilon J$. Therefore, $a + \alpha g + b = (a + b) + \alpha g \geq K$ so $a + b + \alpha g \varepsilon A_N + B_N$. Therefore, $a + \alpha g \varepsilon A_N$ due to maximality and $A_N \sim A_N^{(g)}$.

Since no new residue classes, mod g, are introduced by the basic sequence of maximal e-transformations after A_N, B_N and those residue classes, mod g, in A_N have only a finite number of elements missing, only a finite number of maximal e-transformations will be possible after A_N, B_N.

The next theorem shows that if $B^* = \{0\}$ then $\delta(A + B) \geq \delta(A, B)$. Therefore, the asymptotic analog of Mann's Theorem holds in this situation.

<u>Theorem 14</u>: If $B^* = \{0\}$, then $\delta(A + B) \geq \delta(A, B)$.

Proof:

If the basic sequence of maximal e-transformations is finite, then Theorem 2 gives $\delta(A + B) \geq \delta(A, B)$.

If the basic sequence of maximal e-transformations is infinite and the sequence $\{f(B_j) : j \epsilon J\}$ is unbounded, property IV of Corollary 15 in Chapter One shows $\delta(A + B) \geq \delta(A, B)$.

If the sequence $\{f(B_j) : j \epsilon J\}$ is bounded and the basic sequence of maximal e-transformations is infinite, then consider the sequence $\{\hat{A}_j, \hat{B}_j : j \epsilon J\}$.

Let $k \epsilon J^+$. There exists $i = i(k)$ such that $\hat{A}_i$ will contain kh consecutive integers. By Theorem 13 in Chapter One, $\hat{A}_i + \hat{B}_i \sim J$ or $\delta(\hat{A}_i + \hat{B}_i) \geq \frac{kh}{kh + 1} \delta(\hat{A}_i, \hat{B}_i)$. If $\hat{A}_i + \hat{B}_i \sim J$ then $\hat{A}_0 + \hat{B}_0 \sim J$ and Lemma 13 shows that the basic sequence of maximal e-transformations is finite which is impossible. Therefore, $\hat{A}_i + \hat{B}_i \not\sim J$.

$$\delta(\hat{A}_0 + \hat{B}_0) \geq \delta(\hat{A}_i + \hat{B}_i) \geq \frac{kh}{kh + 1} \delta(\hat{A}_i, \hat{B}_i) \geq \frac{kh}{kh + 1} \delta(\hat{A}_0, \hat{B}_0)$$

for every $k \epsilon J^+$ which implies $\delta(\hat{A}_0 + \hat{B}_0) \geq \delta(\hat{A}_0, \hat{B}_0)$. By Lemma 9, $\delta(A_N + B_N) \geq \delta(A_N, B_N)$; therefore, $\delta(A + B) \geq \delta(A, B)$.

Theorem 14 shows that if $\delta(A + B) < \delta(A, B)$, then $B^* \neq \{0\}$. However, it is possible that $\delta(A + B) \geq \delta(A, B)$ when $B^* \neq \{0\}$, as the following example shows.

Example: $B^* \neq \{0\}$ and $\delta(A + B) \geq \delta(A, B)$

For this example only, [a] will denote the set of non-negative integers which are congruent to a, mod 4.

$S = \{9 + 2j(j + 5) : j \in J\} = \{9, 21, 39, 57, 81, 109, \ldots\} =$ $\{9, 9 + 12, 9 + 12 + 16, 9 + 12 + 16 + 20, \ldots\}$. S is formed by selecting elements, beginning with 9, which are congruent to one, mod 4, and building increasing gaps between consecutive elements.

Let $A = [0] \cup \{1\}$ and $B = [0] \cup S$.

$A + B = [0] \cup [1] \cup (S + 1)$ and an immediate calculation shows A and B are maximal and $\delta(A + B) = \delta(A, B)$.

A maximal zero transformation is performed as shown below.

$$\begin{tikzcd} A, B \arrow[r, "M"'] \arrow[d] & A_0, B_0 \\ A', B' \arrow[ur, "M"'] & \end{tikzcd}$$

$A' = A \cup B = [0] \cup \{1\} \cup S$

$B' = A \cap B = [0]$

$A' + B' = [0] \cup [1]$

Therefore, $A_0 = [0] \cup [1]$ and $B_0 = [0]$.

No further transformations are possible so $B^* = B_0 = [0] \neq \{0\}$.

Before an analysis of the situation when $\delta(A + B) < \delta(A, B)$, three lemmas which describe sets that are asymptotically equal to unions of residue classes will be given.

<u>Lemma 15</u>: Let $A \subseteq J$ and $m, g \in J^+$ such that $m|g$. If $A \sim A^{(m)}$ then $A \sim A^{(g)}$.

<u>Lemma 16</u>: Let $A, B \subseteq J$ with $A \sim A^{(k)}$ and $B \sim B^{(m)}$ where $k, m \in J^+$. If h is the least common multiple of k and m, then $A \cup B \sim (A \cup B)^{(h)}$.

<u>Lemma 17</u>: Let $A \subseteq J$ and $m, k \in J^+$. If $d = (m, k)$, then $A^{(d)} = (A^{(m)})^{(k)} = (A^{(k)})^{(m)}$.

Proof:

$A^{(d)} = \{a + \alpha d : a \in A, \alpha \in I\}$, I is the set of integers

$A^{(m)} = \{a + \alpha m : a \in A, \alpha \in I\}$

$(A^{(m)})^{(k)} = \{x + \alpha k : x \in A^{(m)}, \alpha \in I\} = \{a + \beta m + \alpha k : a \in A, \alpha, \beta \in I\}$

$(A^{(k)})^{(m)} = \{x + \beta m : x \in A^{(k)}, \beta \in I\} = \{a + \beta m + \alpha k : a \in A, \alpha, \beta \in I\}$

$(A^{(m)})^{(k)} = (A^{(k)})^{(m)}$.

Let $a + \alpha d \in A^{(d)}$. There exist $s, t \in I$ such that $d = ms + kt$. $a + \alpha d = a + \alpha(ms + kt) = a + (\alpha s)m + (\alpha t)k \in (A^{(m)})^{(k)}$. Therefore $A^{(d)} \subseteq (A^{(m)})^{(k)}$.

Let $a + \alpha k + \beta m \in (A^{(m)})^{(k)}$. There exist $s, t \in I$ such that $k = sd$ and $m = td$. $a + \alpha k + \beta m = a + \alpha(sd) + \beta(td) = a + (\alpha s + \beta t)d \in A^{(d)}$. Therefore, $(A^{(m)})^{(k)} \subseteq A^{(d)}$.

Therefore, $A^{(d)} = (A^{(m)})^{(k)} = (A^{(k)})^{(m)}$.

The next theorem, which details the structure of $A + B$ when $\delta(A + B) < \delta(A, B)$, is a generalized form of the second conclusion of Theorem 5. This theorem will play a crucial role in the analysis of the situation when $\delta(A + B) < \delta(A, B)$ because a corollary to this theorem will show that in this situation there exists $g \in J^+$ such that $A + B \sim (A + B)^{(g)}$.

<u>Theorem 18</u>: Let $A, B \subseteq J$ with $\delta(A + B) < \delta(A, B)$. If F is a finite non-empty subset of $A + B$, then there exists $C \subseteq A + B$ and $g \in J^+$ such that $F \subseteq C$, $C \sim C^{(g)}$ and $\delta(C) \geq \delta(A, B) - 1/g$.

Proof:

Let n be the number of elements in F. The proof will be by induction on n.

Suppose $n = 1$ and $F = \{c_1\}$. Since $c_1 \in A + B$, there exists $a \in A$ and $b \in B$ such that $c_1 = a + b$. Let A' and B' denote the non-negative elements of $A - a$ and $B - b$ respectively.

Observe that $\delta(A' + B') \leq \delta(A + B)$. This follows from the fact that $A' + B' + c_1 \subseteq A + B$.

Since $A'(x) + B'(x) = A(x + a) - A(a) + B(x + b) - B(b) = A(x) + B(x) + O(1)$, $\delta(A', B') = \delta(A, B)$.

Therefore, $\delta(A' + B') < \delta(A', B')$ and if $A'_0, B'_0, A'_1, B'_1, \cdots$ denotes the sequence of derived sets obtained from the basic sequence of maximal e-transformations on A', B', then Theorem 14 insures $(B')^* = \bigcap \{B'_j : j \in J\} \neq \{0\}$.

The second conclusion of Theorem 5, applied to A', B', asserts the existence of $C' \subseteq A' + B'$ and $g \in J^+$ such that $0 \in C'$, $C' \sim C'^{(g)}$, and $\delta(C') \geq \delta(A', B') - 1/g$.

Define $C = C' + c_1$. $c_1 \in C$ since $0 \in C'$. $C \subseteq A + B$, $C \sim C^{(g)}$ and $\delta(C) = \delta(C') \geq \delta(A', B') - 1/g = \delta(A, B) - 1/g$.

Therefore, the desired result holds when $n = 1$.

Assume the result holds when F contains $n - 1$ elements where $n > 1$.

Let $F = \{c_1, c_2, \cdots, c_n\} \subseteq A + B$.

By the inductive hypothesis there exists $E_1 \subseteq A + B$ and $p \in J^+$ such that $\{c_1, c_2, \cdots, c_{n-1}\} \subseteq E_1$, $E_1 \sim E_1^{(p)}$ and $\delta(E_1) \geq$

$\delta(A, B) - 1/p$. From the case where $n = 1$, there exists $E_2 \not\subseteq A + B$ and $m \epsilon J^+$ such that $c_n \epsilon E_2$, $E_2 \sim E_2^{(m)}$, and $\delta(E_2) \geq \delta(A, B) - 1/m$.

With $C = E_1 \cup E_2$, we have $F \subseteq C$.

If $E_1 \subseteq E_2^{(m)}$ or $E_2 \subseteq E_1^{(p)}$, the result is immediate.

Assume $E_1^{(p)} \cap E_2^{(m)}$ is a proper subset of $E_1^{(p)}$ and also a proper subset of $E_2^{(m)}$.

Let g be the least common multiple of p and m. $C \sim E_1 \cup E_2 \sim E_1^{(p)} \cup E_2^{(m)} = (E_1 \cup E_2)^{(g)}$ by Lemma 16. This implies $C \sim C^{(g)}$.

It remains to show $\delta(C) \geq \delta(A, B) - 1/g$ when:

(1) $C \sim E_1^{(p)} \cup E_2^{(m)}$

(2) $\delta(E_1^{(p)}) \geq \delta(A, B) - 1/p$, $\delta(E_2^{(m)}) \geq \delta(A, B) - 1/m$ <u>Conditions I</u>

(3) $E_1^{(p)} \cap E_2^{(m)}$ is a proper subset of both $E_1^{(p)}$ and $E_2^{(m)}$.

Let $x_1, x_2, \cdots, x_r$ be distinct representatives of the residue classes, mod p, which are contained in $E_1^{(p)}$ and $y_1, y_2, \cdots, y_s$ be distinct representatives of the residue classes, mod m, which are contained in $E_2^{(m)}$. $1 \leq r \leq p - 1$ and $1 \leq s \leq m - 1$, otherwise $E_1^{(p)}$ or $E_2^{(m)}$ would be equal to J.

$$\delta(C) = \delta(E_1^{(p)} \cup E_2^{(m)}) = \delta(E_1^{(p)}) + \delta(E_2^{(m)}) - \delta(E_1^{(p)} \cap E_2^{(m)}).$$

If $\delta(E_1^{(p)} \cap E_2^{(m)}) = 0$, then $\delta(E_1^{(p)}) + \delta(E_2^{(m)}) \geq \delta(A, B) - \frac{1}{p} + \frac{s}{m}$ and $\delta(E_1^{(p)}) + \delta(E_2^{(m)}) \geq \delta(A, B) - \frac{1}{m} + \frac{r}{p}$. $\delta(C) = \max\left\{\delta(A, B) - \frac{1}{p} + \frac{s}{m}, \delta(A, B) - \frac{1}{m} + \frac{r}{p}\right\} = \delta(A, B) + \max\left\{\frac{s}{m} - \frac{1}{p}, \frac{r}{p} - \frac{1}{m}\right\}$. $0 \leq s - 1 + r - 1 \leq p(s - 1) + m(r - 1) = ps - p + mr - m$. Therefore, $0 \leq \frac{ps - m}{pm} + \frac{mr - p}{pm} = \left(\frac{s}{m} - \frac{1}{p}\right) + \left(\frac{r}{p} - \frac{1}{m}\right)$. The latter inequality implies that

$\max\left\{\frac{s}{m} - \frac{1}{p}, \frac{r}{p} - \frac{1}{m}\right\} \geq 0$. Therefore, $\delta(C) > \delta(A, B)$ so $\delta(A + B) \geq \delta(A, B)$ which is contradictory to the hypothesis of the theorem. Therefore, $\delta(E_1^{(p)} \cap E_2^{(m)}) > 0$.

Let $d = (p, m)$. There exists $p_1, m_1 \in J^+$ such that $p = p_1 d$ and $m = m_1 d$. Therefore $g = p_1 m = m_1 p$.

Define $X_1, X_2, \cdots X_d$ as subsets of $E_1^{(p)}$ formed by placing the residue class, mod p, which contains x_j in X_t if and only if $x_j \equiv t$, mod d. These subsets of $E_1^{(p)}$ are disjoint. In a similar manner, $Y_1, Y_2, \cdots, Y_d$, disjoint subsets of $E_2^{(m)}$, are defined by placing the residue class, mod m, which contains y_j in Y_t if and only if $y_j \equiv t$, mod d.

Let r_t denote the number of residue classes, mod p, which are contained in X_t and s_t denote the number of residue classes, mod m, which are contained in Y_t for $t = 1, 2, \cdots, d$. For each t, $0 \leq r_t \leq p_1$ and $0 \leq s_t \leq m_1$.

If $t \neq u$, then $X_t \cap Y_u = \emptyset$. If $x \in X_t \cap Y_u$, then $x \equiv x_i$, mod p, and $x_i \equiv t$, mod d, for some i and $x \equiv y_j$, mod m, and $y_j \equiv u$, mod d, for some j. These congruences imply $t \equiv u$, mod d, so $t = u$, which is a contradiction.

We now consider $X_t \cap Y_t$. If $r_t = 0$ or $s_t = 0$, then $X_t \cap Y_t = \emptyset$ and $\delta(X_t \cap Y_t) = 0 = r_t s_t / g$. Assume $r_t \neq 0$ and $s_t \neq 0$. Let $x_1, x_2, \cdots, x_{r_t}$ be distinct representatives of all residue classes, mod p, which are contained in X_t and $y_1, y_2, \cdots, y_{s_t}$ be distinct representatives of all residue classes, mod m, which are contained in Y_t. $x_j \equiv y_i \equiv t$, mod d, for $j = 1, 2, \cdots, r_t$ and $i = 1, 2, \cdots, s_t$.

$w \in X_t \cap Y_t$ if and only if $w \equiv x_j$, mod p, and $w \equiv y_k$, mod m, for some j and some k. This system of congruences has a solution if and only if $d \mid (x_j - y_k)$ which does occur since $x_j \equiv y_k$, mod d. The solution of this system of congruences is unique up to congruence, mod g. Therefore, $X_t \cap Y_t$ will consist of $r_t s_t$ residue classes, mod g, which implies $\delta(X_t \cap Y_t) = r_t s_t / g$.

$$\delta(E_1^{(p)} \cap E_2^{(m)}) = \sum_{i,j} \delta(X_i \cap Y_j) = \sum_{j=1}^{d} \delta(X_j \cap Y_j) = \sum_{j=1}^{d} \frac{r_j s_j}{g}.$$

$$\delta(E_1^{(p)} \cup E_2^{(m)}) = \delta(E_1^{(p)}) + \delta(E_2^{(m)}) - \delta(E_1^{(p)} \cap E_2^{(m)})$$

$$= \delta(E_1^{(p)}) + \delta(E_2^{(m)}) - \sum_{j=1}^{d} \frac{r_j s_j}{g}$$

$$= \delta(E_1^{(p)}) + \frac{s}{m} - \sum_{j=1}^{d} \frac{r_j s_j}{g}$$

$$= \delta(E_1^{(p)}) + \frac{1}{m} \sum_{j=1}^{d} s_j - \sum_{j=1}^{d} \frac{r_j s_j}{g}$$

$$= \delta(E_1^{(p)}) + \frac{p_1}{g} \sum_{j=1}^{d} s_j - \sum_{j=1}^{d} \frac{r_j s_j}{g}$$

$$= \delta(E_1^{(p)}) + \frac{1}{g} \sum_{j=1}^{d} (p_1 - r_j) s_j$$

$$= \delta(E_1^{(p)}) + \sum_{j=1}^{d} \frac{(p_1 - r_j) s_j}{g}$$

A similar argument gives $\delta(E_1^{(p)} \cup \delta E_2^{(m)}) = \delta(E_2^{(m)}) + \sum_{j=1}^{d} \frac{(m_1 - s_j) r_j}{g}$.

$\delta(E_1^{(p)} \cap E_2^{(m)}) = \sum_{j=1}^{d} \frac{r_j s_j}{g} > 0$ implies there exists j_0, $1 \leq j_0 \leq d$, such that $r_{j_0} s_{j_0} \geq 1$.

Two cases appear: Case One; $r_{j_0} = p_1$ and $s_{j_0} = m_1$ and Case Two; $r_{j_0} \neq p_1$ or $s_{j_0} \neq m_1$.

We will now demonstrate that Case One can be reduced to Case Two.

If Case One holds then $E_1^{(p)}$ and $E_2^{(m)}$ contain at least one common residue class, mod d. Let t be the number of residue classes, mod d, which are contained in $E_1^{(p)} \cap E_2^{(m)}$. Define X' to be $E_1^{(p)}$ with the residue classes, mod d, which are in $E_1^{(p)} \cap E_2^{(m)}$ removed and Y' to be $E_2^{(m)}$ with residue classes, mod d, contained in $E_1^{(p)} \cap E_2^{(m)}$ removed. Let T be the collection of all residue classes, mod d, which are contained in $E_1^{(p)} \cap E_2^{(m)}$. $E_1^{(p)} \cup E_2^{(m)} = X' \cup Y' \cup T$ and $\delta(E_1^{(p)} \cup E_2^{(m)}) = \delta(X' \cup Y') + \delta(T) = \delta(X' \cup Y') - t/d$. $\delta(X') = \delta(E_1^{(p)}) - t/d \geq (\delta(A, B) - t/d) - 1/p$ and $\delta(Y') \geq (\delta(A, B) - t/d) - 1/m$.

X' and Y' satisfy conditions which are essentially the same as those of $E_1^{(p)}$ and $E_2^{(m)}$ in Conditions I. If the sets $X_1, X_2, \cdots, X_d$ and $Y_1, Y_2, \cdots, Y_d$ are constructed for X' and Y', the definitions of X', Y', and T will insure that X_t will not contain p_1 residue classes, mod d, or Y_t will not contain m_1 residue classes, mod d, for $t = 1, 2, \cdots, d$. This reduces Case One to the situation of Case Two.

Case Two: If $r_{j_0} = p_1$ and $s_{j_0} < m_1$, then $\delta(E_1^{(p)} \cup E_2^{(m)}) = \delta(E_2^{(m)}) + \frac{1}{g} \sum_{j=1}^{d} (m_1 - s_j) r_j \geq \delta(A, B) - \frac{1}{m} + \frac{1}{g} (m_1 - s_{j_0}) r_{j_0} \geq \delta(A, B) - \frac{1}{m} + \frac{r_{j_0}}{g} = \delta(A, B) - \frac{1}{m} + \frac{p_1}{g} = \delta(A, B) - \frac{1}{m} + \frac{1}{m} = \delta(A, B)$. Therefore, $\delta(C) \geq \delta(A, B)$ which implies $\delta(A + B) \geq \delta(A, B)$, a contradiction of

the theorem's hypothesis. Thus, it is impossible to have $r_{j_0} = p_1$ and $s_{j_0} < m_1$. A similar argument shows that it is impossible to have $r_{j_0} < p_1$ and $s_{j_0} = m_1$.

Assume $1 \leq r_{j_0} < p_1$ and $1 \leq s_{j_0} < m_1$. $\delta(E_1^{(p)} \cup E_2^{(m)}) \geq \delta(E_1^{(p)}) + \frac{1}{g} s_{j_0}(p_1 - r_{j_0})$. Therefore,

$$\delta(E_1^{(p)} \cup E_2^{(m)}) \geq \delta(A, B) - \frac{1}{p} + \left(\frac{1}{g} - \frac{1}{g}\right) + \frac{1}{g} s_{j_0}(p_1 - r_{j_0})$$

$$\geq \delta(A, B) - \frac{1}{g} + \left[\frac{1}{g} - \frac{1}{p_1} + \frac{1}{g} s_{j_0}(p_1 - r_{j_0})\right].$$

A similar argument will give $\delta(E_1^{(p)} \cup E_2^{(m)}) \geq \delta(A, B) - \frac{1}{g} + \left[\frac{1}{g} - \frac{1}{m_1} + \frac{1}{g} r_{j_0}(m_1 - s_{j_0})\right]$. Combining these inequalities gives $\delta(E_1^{(p)} \cup E_2^{(m)}) \geq \delta(A, B) - \frac{1}{g} + \max\left\{\frac{1}{g} - \frac{1}{p_1} + \frac{1}{g} s_{j_0}(p_1 - r_{j_0}), \frac{1}{g} - \frac{1}{m_1} + \frac{1}{g} r_{j_0}(m_1 - s_{j_0})\right\}$.

$$0 = (r_{j_0} + 1)(s_{j_0} - 1) + (s_{j_0} + 1)(r_{j_0} - 1) - 2(r_{j_0}s_{j_0} - 1)$$

$$\leq p_1(s_{j_0} - 1) + m_1(r_{j_0} - 1) - 2(r_{j_0}s_{j_0} - 1)$$

$$= s_{j_0}(p_1 - r_{j_0}) + r_{j_0}(m_1 - s_{j_0}) - p_1 - m_1 + 2.$$

Division by g gives

$$0 \leq \frac{s_{j_0}}{g}(p_1 - r_{j_0}) + \frac{r_{j_0}}{g}(m_1 - s_{j_0}) - \frac{p_1}{g} - \frac{m_1}{g} + \frac{2}{g}$$

$$= \frac{s_{j_0}}{g}(p_1 - r_{j_0}) + \frac{r_{j_0}}{g}(m_1 - s_{j_0}) - \frac{1}{m_1} - \frac{1}{p_1} + \frac{2}{g}.$$

Regrouping yields

$$0 \leq \left(\frac{1}{g} - \frac{1}{p_1} + \frac{s_{j_0}}{g}(p_1 - r_{j_0})\right) + \left(\frac{1}{g} - \frac{1}{m_1} + \frac{r_{j_0}}{g}(m_1 - s_{j_0})\right)$$

which establishes that the maximum of those two expressions is non-negative.

Therefore, $\delta(E_1^{(p)} \cup E_2^{(p)}) \geq \delta(A, B) - 1/g$ which implies $\delta(C) \geq \delta(A, B) - 1/g$.

Therefore, $\{c_1, c_2, \cdots, c_n\} \subseteq C$, $C \sim C^{(g)}$ and $\delta(C) \geq \delta(A, B) - 1/g$, so the desired result holds for n.

Theorem 18 now follows by induction.

<u>Corollary 19</u>: Let $A, B \subseteq J$ with $\delta(A + B) < \delta(A, B)$, then there exists $g \varepsilon J^+$ such that $A + B \sim (A + B)^{(g)}$.

Proof:

For each $x \varepsilon A + B$, by the preceding theorem, there exists $C_x \subseteq A + B$ and $k = k(x) \varepsilon J^+$ such that $x \varepsilon C_x$, $C_x \sim C_x^{(k(x))}$, and $\delta(C_x) \geq \delta(A, B) - 1/k(x)$.

The set $\{k(x) : x \varepsilon A + B\}$ must be bounded, otherwise $\delta(A + B) \geq \delta(A, B)$ which is a contradiction. Therefore, $\{k(x) : x \varepsilon A + B\}$ must be finite.

Select g as the least common multiple of $\{k(x) : x \varepsilon A + B\}$. Since $C_x \sim C^{(k(x))}$, Lemma 15 gives $C_x \sim C_x^{(g)}$ for each $x \varepsilon A + B$.

Let $x_1, x_2, \cdots, x_j$ be incongruent representatives in A + B of all residue classes, mod g, which have a non-empty intersection with A + B. In view of the definition of C_{x_i}, it follows that $x_i + \alpha g \varepsilon A + B$ for $i = 1, 2, \cdots, j$ and all sufficiently large α.

Therefore, $A + B \sim (A + B)^{(g)}$.

The final theorem, an equivalent form of Kneser's Theorem, describes the situation when the asymptotic analog of Mann's Theorem fails to hold.

<u>Theorem 20</u>: (Kneser) Let $A, B \subseteq J$ with $\delta(A + B) < \delta(A, B)$, then there exists $g \in J^+$ such that $A + B \sim (A + B)^{(g)}$ and $\delta(A + B) = \delta((A + B)^{(g)}) \geq \delta(A^{(g)}, B^{(g)}) - 1/g \geq \delta(A, B) - 1/g$.

Proof:

By Corollary 19, there exists $h \in J^+$ such that $A + B \sim (A + B)^{(h)}$. Select g as the smallest positive integer for which $A + B \sim (A + B)^{(g)}$.

$(A + B)^{(g)} = A^{(g)} + B^{(g)}$.

If $\delta(A^{(g)} + B^{(g)}) \geq \delta(A^{(g)}, B^{(g)})$, then $\delta(A + B) = \delta((A + B)^{(g)}) = \delta(A^{(g)} + B^{(g)}) \geq \delta(A^{(g)}, B^{(g)}) \geq \delta(A, B)$ which is a contradiction. Therefore, $\delta(A^{(g)} + B^{(g)}) < \delta(A^{(g)}, B^{(g)})$.

Let $c_1, c_2, \cdots, c_j$ be incongruent representatives in $A^{(g)} + B^{(g)}$ of all residue classes, mod g, which are contained in $A^{(g)} + B^{(g)}$. By Theorem 18, there exists $C \subseteq A^{(g)} + B^{(g)}$ and $h \in J^+$ such that $\{c_1, c_2, \cdots, c_j\} \subseteq C$, $C \sim C^{(h)}$, and $\delta(C) \geq \delta(A^{(g)}, B^{(g)}) - 1/h$.

Let $d = (g, h)$. We will now show $d = g$ and consequently $g \leq h$. By Lemma 17, $(A + B)^{(d)} = ((A + B)^{(g)})^{(h)} = (C^{(g)})^{(h)} = (C^{(h)})^{(g)} \sim C^{(g)} \sim A + B$. Therefore $A + B \sim (A + B)^{(d)}$ which implies $d = g$ due to the minimal character of g.

$\delta(A + B) = \delta((A + B)^{(g)}) = \delta(C^{(g)}) \geq \delta(C) \geq \delta(A^{(g)}, B^{(g)}) - 1/h \geq \delta(A^{(g)}, B^{(g)}) - 1/g \geq \delta(A, B) - 1/g$.

CHAPTER THREE

CONCLUSION

The approach to Kneser's Theorem presented in this dissertation contrasts with Kneser's methods. While Kneser attacked the general situation with an arbitrary sequence of e-transformations, the approach here begins with replacing A, B by the associated maximal sets, A^M, B^M, and exclusively uses the basic sequence of maximal e-transformations. Kneser reduced the general situation through several special cases. By introducing the limit set B^*, the new approach emphasizes the importance of the non-zero elements of B^M which are deleted by the basic sequence of maximal e-transformations.

It might appear that the introduction of maximal sets and maximal e-transformations would complicate the analysis of A, B, and A + B; however, $A^M + B^M = A + B$, $\delta(A^M, B^M) \geq \delta(A, B)$, and the basic properties of these two concepts demonstrate that A^M, B^M is a superior point from which to initiate the analysis.

In analyzing the asymptotic analog of Mann's Theorem, a natural consideration is whether or not all of the non-zero elements of B^M are deleted by the basic sequence of maximal e-transformations; namely, is $B^* = \{0\}$ or $B^* \neq \{0\}$? With this criterion, the new approach considers two cases.

When $B^* = \{0\}$, the new approach shows $\delta(A + B) \geq \delta(A, B)$. In this case, the role of maximal sets and maximal e-transformations is

evident when the basic sequence of maximal e-transformations is infinite and the sequence $\{f(B_j) : j\varepsilon J\}$ is bounded. With N sufficiently large, $\{f(B_{N+j}) : j\varepsilon J\}$ and $\{g(B_{N+j}) : j\varepsilon J\}$ are both constant and the minimal representatives in A_j, $j \geq N$, of residue classes, mod $g(B_N)$, are invariant. Then the subsequence $\{A_{N+j}, B_{N+j} : j\varepsilon J\}$ is "translated" by the fundamental map, Ω, to produce the sequence $\{\hat{A}_j, \hat{B}_j : j\varepsilon J\}$. The maximality of A_N, B_N and an infinite basic sequence of maximal e-transformations implies $A_N + B_N \not\sim (A_N + B_N)^{(h)}$ for any $h\varepsilon J^+$ so $\delta(A_N + B_N) \geq \delta(A_N, B_N)$; therefore, $\delta(A + B) \geq \delta(A, B)$. The crucial fact in this analysis is that if A, B are maximal and $A + B \sim (A + B)^{(h)}$ for some $h\varepsilon J^+$, then the basic sequence of maximal e-transformations is finite.

When $\delta(A + B) < \delta(A, B)$, $B^* \neq \{0\}$, and it becomes important to acquire information about the structure of A + B. In this circumstance, the new approach makes its greatest contribution to the analysis. By analyzing the derived sets from the basic sequence of maximal e-transformations, using the elementary properties of $g(B^*)$ and the residue classes, mod $g(B^*)$, it is shown initially that the basic sequence of maximal e-transformations is finite. Secondly, and very crucial, the existence of $C \subseteq A + B$ such that $0\varepsilon C$, $C \sim C^{(g(B^*))}$, and $\delta(C) \geq \delta(A, B) - 1/g(B^*)$ is established. The existence of C is vital in the analysis to establish the existence of $g\varepsilon J^+$ such that $A + B \sim (A + B)^{(g)}$ and $\delta(A + B) = \delta((A + B)^{(g)}) \geq \delta(A^{(g)}, B^{(g)}) - 1/g \geq \delta(A, B) - 1/g$.

In summary, this new approach exploits the properties of maximal sets by shifting to the maximal sets associated with A, B and preserves

these properties for derived sets by using maximal e-transformations. By exclusive use of the basic sequence of maximal e-transformations the limit set B^* is introduced as the collection of all non-zero elements of B^M not deleted by the basic sequence of maximal e-transformations. Consideration of whether or not $B^* = \{0\}$ produces a more direct analysis than Kneser's. This is shown most dramatically by establishing that $B^* \neq \{0\}$ is a necessary condition for $\delta(A + B) < \delta(A, B)$ and then using only the elementary properties of greatest common divisor and residue classes to obtain vital results concerning the structure of $A + B$ when $\delta(A + B) < \delta(A, B)$. These results lead directly to the existence of $g \in J^+$ such that $A + B \sim (A + B)^{(g)}$ and $\delta(A + B) = \delta(A, B) - 1/g$ when $\delta(A + B) < \delta(A, B)$.

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A NEW APPROACH TO KNESER'S THEOREM ON ASYMPTOTIC DENSITY

by

John B. Lane

(ABSTRACT)

A new approach to Kneser's Theorem, which achieves a simplification of the analysis through the introduction of maximal sets, the basic sequence of maximal e-transformations, and the limit set, B^*, is presented.

For two sets of non-negative integers, A and B, with $0 \in A \cap B$, the maximal sets, A^M and B^M, are the largest supersets of A and B, respectively, such that $A^M + B^M = A + B$. By shifting from A and B to A^M and B^M to initiate the analysis, the maximal properties of A^M and B^M are exploited to simplify the analysis.

A maximal e-transformation is a Kneser e-transformation in which the image sets are maximized in order to preserve the properties of maximal sets. The basic sequence of maximal e-transformation is a specific sequence of maximal e-transformations which is exclusively used throughout the analysis.

B^* is the set of all non-negative elements of B^M which are not deleted by any transformation in the basic sequence of maximal e-transformations. Whether or not $B^* = \{0\}$ divides the analysis into two cases.

One significant result is that $B^* = \{0\}$ implies $\delta(A + B) \geq \delta(A, B)$ where $\delta(A + B)$ is the asymptotic density of $A + B$ and $\delta(A, B)$ is the two-fold asymptotic density of A and B.

The second major result describes the structure of $A + B$ when $\delta(A + B) < \delta(A, B)$. With $B^* \neq \{0\}$ it is shown, using only elementary properties of greatest common divisor and residue classes, that there exists $C \subseteq A + B$, $0 \in C$, such that $\delta(C) \geq \delta(A, B) - 1/g$ where g is the greatest common divisor of B^* and C is asymptotically equal to $C^{(g)}$, the union of all residue classes, mod g, which have a representative in C. The existence of C provides the crucial step in obtaining an equivalent form of Kneser's Theorem: If A and B are two subsets of non-negative integers, $0 \in A \cap B$, and $\delta(A + B) < \delta(A, B)$, then there exists a positive integer g such that $A + B$ is asymptotically equal to $(A + B)^{(g)}$ and $\delta(A + B) = \delta((A + B)^{(g)}) \geq \delta(A^{(g)}, B^{(g)}) - 1/g \geq \delta(A, B) - 1/g$.