ANALYTICITY PROPERTIES OF THE JOST FUNCTION

FOR SPHEROIDAL POTENTIALS

by

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I. INTRODUCTION

The significance of the Jost function[1] in scattering theory may be viewed on two levels. On one hand it has been useful in helping establish a relativistic theory of the S-matrix by allowing the conjectured analyticity properties of the S-matrix to be tested in potential scattering; there the Schrodinger theory provides an alternate reliable standard against which the plausibility of the arguments of the new theory are to be measured. On the other hand it has become an invaluable tool strictly in the domain of potential scattering where it has been instrumental in the study of bound and resonant states, and in the general analysis of low energy scattering data[2].

The research activity in the first instance has tapered off, especially after it was proved that Mandelstam's hypotheses[3] which are now at the center of attention in the S-matrix theory are satisfied for a large class of potentials. In the second instance, however, the Jost method is still enjoying continuing success and finding a variety of applications. These two areas of applications are naturally not disjoint, and the distinction is made here simply for the purpose of clearly stating the objective of the present investigation.

In order to give a just idea of its importance we will begin by recalling the context in which the Jost function made its appearance, review the relation between S-matrix and Jost function and outline the history of its successes in potential scattering theory.

The mathematical theory of scattering goes back to Lord Rayleigh[4] who was then studying the diffraction of sound waves. The notion of scattering amplitude which he introduced simplified considerably the problem of solving the Helmholz equation and provided an expression for the scattering cross-section which led directly to the experimental verification of the theory. The formulation was readily adapted to electromagnetic scattering and, when the quantum theory emerged as the "correct" way of describing the behavior of the microscopic world, the quantum scattering problem still consisted to a large extent in securing a reasonable expression for the scattering amplitude.

Subsequently, most of the research effort in this area concentrated on devising several more or less felicitous methods of approximating the scattering amplitude by taking the Schrodinger equation with a suitable potential as the starting point; the most notable of these has been the Born approximation which has had a replete history of successful applications in the field since then.

As experimental evidence for such phenomena as exchange, spin dependence and saturation of the nuclear force accumulated, the many potentials thought up to exhibit these features made an analytic treatment of scattering problems virtually impossible. Increasing time and effort were spent on phenomenological codification of low energy data with greater emphasis being placed on improving the reliability of numerical calculations.

The void created by this situation became more embarrassing as most attempts at developing a theory of strong interactions valid at high energies remained unsuccessful. If one had resigned oneself to live with the divergences of quantum electrodynamics because of the wealth of results it predicted that were eventually confirmed with unprecedented accuracy, any hope of dealing with hadronic interactions along similar lines remained unrewarded.

The S-matrix was then proposed as a possible way out of the dilemma. Suspecting the space-time formulation of physical processes as the source of the divergences of field theory, the advocators of the S-matrix sought a formulation of the laws of physics that would be closer to the experimental situation by doing away with the unobservable fields. Once the S-matrix of an interaction is known, all the observable quantities such as cross-section, polarization, lifetimes, bound states can be obtained from it. Furthermore, it offers a formalism that is free from the cumbersome machinery of renormalization, which gives it a certain degree of economy over field theoretical methods. Its successes, especially in hadronic physics which so far have remained impenetrable to theoretical treatment, have been spectacular enough to warrant an impressive amount of research on the subject itself and in related adjacent disciplines.[5]

The theory unfortunately suffers from the drawback that it cannot totally dispense with the theory of fields which it was originally destined to supersede. In order to determine the S-matrix, the analytic properties of its elements must be known as a function of the momenta; the formalism does not indicate how this information can be provided and one must go back to field theory to obtain it. In the case of two body interactions at sufficiently low energy, the Schrodinger equation is the source of this information, and the Jost function the vehicle through which it is extracted from the wave equation.

Jost attempted a partial verification of Heisenberg's prescriptions[6] by using them to obtain the stationary states of some well known potentials and comparing his newly found results to those already known from the Schrodinger method. In so doing he was led to define the function that now bears his name in order to resolve the ambiguity already pointed out by Ma and Kramers[7] concerning the poles of the S-matrix. He reached a definite criterium for distinguishing between the "true" and the "false" poles of the S-matrix and showed how the former are related to the bound states of the potential. [Jost actually called them "true" and "false" zeros (richtigen and falschen Nullstellen). This qui pro quo is due to the redefinition of the Jost function by Newton[8], which will be followed here for convenience in order to conform to current practice in the literature.] From this formulation, analyticity propeties were obtained which allowed for a test on potential scattering of the conclusions reached in field theory which had progressed far enough to produce analytic properties and dispersion relations for the total scattering amplitude.

It was thought for a while that the Jost function would not be of any great use for comparing the analyticity properties deduced from field theory and those obtained from the Schrodinger theory. Indeed the predictions of field theory related to the total scattering amplitude, whereas the Jost formalism depended upon a partial wave decomposition of the Schrodinger equation and provided analyticity properties only for the partial amplitudes. In the absence of a common domain of predictions, the full importance of the Jost function in that respect was temporarily obscurred. Then Mandelstam[9] conjectured his double dispersion relation which made definite predictions on the analyticity properties of the partial amplitudes. The interest in the Jost function was revived and it was used to show that Mandelstam's hypotheses hold true for the Yukawa potential and any superposition of Yukawa potentials with a suitable spectral function, which are strongly suggested by field theory. Blankenbecler et al's.[10] proof that the whole scattering amplitude could be reconstructed from dispersion relations and unitarity condition was the culminating point of the effort to arrive at a prediction of observables without solving the Schrodinger equation.

More akin to our present purpose are the application of the Jost function that are restricted essentially to potential scattering. We are now referring to the Jost function as an analytical tool that allows us to predict most of the relevant features of a potential without actually solving the Schrodinger equation. This course of action is essential for in most instances it is quite helpful to be able to extract most of the needed physical information even when a complete mathematical solution cannot be reached. The reverse process is also equally important, whereby, knowing the physical properties of a system, one tries to reconstruct the potential that will reproduce the observed phenomena.

Here the Jost approach is one of great simplicity for it does away with the customary dual treatments of bound and scattering states. These two sets are introduced as different but related aspects of the same phenomenon: the Jost function is directly related to the scattering amplitude which in turn predicts the cross-sections; on the other hand the bound and resonant state energies are obtained by continuing the Jost function into the complex momentum plane. In addition to, or maybe because of this simplicity, a physical picture emerges from the unified treatment, that emphasizes the global aspect of an interaction and puts in evidence the many features of two interacting particles, along with the mechanism responsible for the appearance of those features.

Yet the economy of the approach does not detract from its practicality, nor does it limit its applicability. Since the phase of the Jost function is directly related to the scattering phase shift, a knowledge of the Jost function leads to that of the phase shift and the customary phase shift analysis is readily applicable; in the low energy limit, the scattering length formula is recovered and the Breit-Wigner description of resonances can also be obtained by expanding the Jost function in the neighborhood of zeros lying in the negative imaginary half-plane and appropriately identifying the respective terms of the expansion. Many more recent methods[11] used for the identification of resonances in the KeV region use the Jost function as a point of departure; some other ones are but refinement of it.

Of much greater importance though are the predictions that the Jost function leads to as to the existence and the number of bound states of a potential. Levinson[12] was able to relate the number of bound states of fixed angular momentum to the corresponding zero energy phase shift, thereby making it possible to know the number of bound states from a purely phenomenological analysis of the observed cross-sections. Jost and Pais[13] extended the treatment to produce a condition for the existence of bound states dependent only on the magnitude of the first moment of the potential. Then Bargman[14] arrived at a more general statement on the number of bound states based solely on the magnitude of the moment of the potential, which included the Jost-Pais prediction as a special case.

The a priori knowledge of the number of bound states from the Bargman inequality is a remarkable prediction in itself; but it also has far reaching consequences for other aspects of the scattering problem. The Bargman inequality places an appreciable constraint on the potential, and in situations where the number of bound states is known experimentally--the deuteron for example is the only known proton-nuetron bound state--one may be guided by it in suggesting a plausible potential responsible for the interaction. Furthermore, in the application of variational methods where the validity of the calculation is very sensitive to the initial choice of trial function, it was shown that a knowledge of the upper limit on the number of bound states led to a reliable criterium in choosing the trial functions.[15]

However non-spherical potentials, and the simplest of them, the spheroidal potential, have received far from a comparable treatment, in spite of their importance. Spheroidal potentials are not uncommon in physics. They have been used to study the classical scattering of electromagnetic radiation and of sound waves by solid objects.[16] The scattering of electrons by diatomic molecules[17] has been successfully described by this type of potentials. Other examples of molecular interactions such as radiative transitions and photoionization[18] have been treated by means of spheroidal wave functions.

The present work is an attempt at continuing the effort toward a more thorough understanding of spheroidal potentials. Earlier[19] the short range discontinuous oblate spheroidal potential has been discussed by the method of phase shifts along with the application of the WKB approximation to the case of scattering at high energies. The particular case of the two center Coulomb problem has also been investigated[20] in the context of spheroidal potentials, with a method for dealing with the difficulties due to the long-range behavior of this potential. We will try here to extend the Jost function method to spheroidal potentials. A perturbation calculation approach is used based on the known results of the Jost method in spherical potential scattering. The Jost method in spherical potential scattering is first discussed; the physical meaning of the Jost function in different situations is examined along with mathematical properties that will be needed for the ensuing discussion on spheroidal potentials.

The concept of the Jost function is then extended to spheroidal potentials. The physical implications of the spheroidal Jost functions are compared to those of its spherical counterpart, and its domain of analyticity is studied.

II. THE SPHERICAL JOST FUNCTION

A. Angular momentum space representation of the S-matrix

The non-relativistic interaction of two isolated particles, 1 and 2 respectively, is described in configuration representation by the Schrodinger equation

$$\frac{-\hbar^2}{2}\left[\frac{1}{m_1}\nabla_1{}^2 + \frac{1}{m_2}\nabla_2{}^2\right]\Psi(\vec{r}_1, \vec{r}_2; t) \tag{II-1}$$

$$+ V(|\vec{r}_1 - \vec{r}_2|)\,\Psi(\vec{r}_1, \vec{r}_2; t) = i\hbar\frac{\partial}{\partial t}\Psi(\vec{r}_1, \vec{r}_2; t)$$

where m_1 and m_2 represent the respective masses of the particles, and $\vec{r}_1$ and $\vec{r}_2$ their positions with respect to an arbitrary frame of reference.

A good deal of simplification occurs if we go over the center of momentum system. Then Eq. (II-1) separates into two equations, one describing the free motion of the center of mass, the second describing the relative motion of the particles. If one of the particles is taken to be stationary and much more massive than the other, only the second equation needs to concern us:

$$\frac{-\hbar^2}{2\mu}\nabla^2\,\Psi(\vec{r},t) + V(\vec{r})\,\Psi(\vec{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) \tag{II-2}$$

with μ the reduced mass of the system and $\vec{r}$ the coordinate of the relative position of the particles. Furthermore for a potential which is time independent Eq. (II-2) admits a solution of the form

$$\Psi(\vec{r},t) = \psi(\vec{r})\, e^{-iEt/\hbar} \tag{II-3}$$

E being the total energy and $\psi(\vec{r})$ the solution of the Schrodinger time independent equation

$$\frac{-\hbar^2}{2\mu} \nabla^2 \psi(\vec{r}) + V(\vec{r})\, \psi(\vec{r}) = E\, \psi(\vec{r}) \tag{II-4}$$

This is the equation that we intend to study.

Before a solution of Eq. (II-4) is attempted, boundary conditions must be specified which describe the physical conditions under which a typical scattering experiment takes place. If the potential is short range, test particles located far away from the target will not be influenced by the potential. Therefore under the assumption that the incident particles arriving, say from the left of the target, are prepared in a state of definite momentum and do not have sufficient energy to excite the target, the detectors located to the far right of the target will observe a mixture of undeflected and of scattered particles, represented by the asymptotic wave function

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{} e^{i\vec{k}\cdot\vec{r}} + f(\theta)\, \frac{e^{ikr}}{r} \tag{II-5}$$

where k is the incident momentum, θ the angle of deflection after scattering, and $f(\theta)$ the scattering amplitude, which is related to the differential cross-section by

$$\sigma(\theta) = |f(\theta)|^2 \tag{II-5a}$$

The scattering amplitude is the quantity that we want to relate to the potential, for, knowing $f(\theta)$, we immediately know the asymptotic wave function also, and via the Jost method, can also obtain information on the bound states of the potential.

When the potential is spherically symmetric, it is convenient to make a partial wave decomposition of the total wave function such that the latter is expressed as a linear superposition of the form

$$\psi(\vec{r}) = \sum_{\ell=o} a_\ell \frac{\phi_\ell\,(k,r)}{kr} P_\ell(\cos\theta) \tag{II-6}$$

where a_ℓ is a constant, $P_\ell\,(\cos\theta)$ the Legendre polynomial, and $\phi_\ell\,(k,r)$ satisfies

$$\frac{d^2}{dr^2}\phi_\ell\,(k,r) + [\frac{2}{\hbar^2}\,(E - V(r)\,) - \frac{\ell(\ell+1)}{r^2}\,\phi_\ell\,(k,r) = o. \tag{II-7}$$

This decomposition is indeed suggested by the invariance of the angular momentum in the case of a central potential and it is natural to write the wave function as a sum of partial waves, each of which is an eigenstate of angular momentum of value $[\ell(\ell+1)]^{1/2}$. It must also be noted that the decomposition in partial waves, as expressed in Eq. (II-6), requires that $\phi_\ell(k,r)$ vanish at the origin in order that the wave function remain finite.

In the absence of a potential term, Eq. (II-7) is a well-known one and its solutions are the Ricatti-Bessel and Neumann functions or alternately the Ricatti-Hankel functions which are related to each other in the following manner:[21]

$$j_\ell(kr) = \frac{1}{2i}\,[h^+(kr) - h^-(kr)] \qquad \text{(II-8a)}$$

$$n_\ell(kr) = \frac{1}{2}\,[h^+(kr) + h^-(kr)] \qquad \text{(II-8b)}$$

$j_\ell(kr)$ and $n_\ell(kr)$ being the Ricatti-Bessel and Neumann functions respectively, and $h^\pm(kr)$ the Ricatti-Hankel functions. They behave asymptotically as follows

$$j_\ell(kr) \xrightarrow[r\to\infty]{} \sin(kr - \ell\pi/2) \qquad \text{(II-9a)}$$

$$n_\ell(kr) \xrightarrow[r\to\infty]{} \cos(kr - \ell\pi/2) \qquad \text{(II-9b)}$$

$$h^\pm(kr) \xrightarrow[r\to\infty]{} e^{\pm i(kr-\ell\pi/2)} \qquad \text{(II-9c)}$$

To take advantage of these considerations we restrict the discussion to potentials which vanish asymptotically faster than the centrifugal barrier term i.e.

$$\lim_{r\to\infty} V(r) \longrightarrow r^{-2-\varepsilon} \quad \varepsilon>o. \qquad \text{(II-10)}$$

The Coulomb potential is excluded from this class, however it can be solved exactly[22] and need not be subjected to the methods of approximation being considered here. If the potential in question satisfies Eq. (II-10), then the correct combination of free wave solutions that make up a partial wave solution to Eq. (II-7) consistent with the stipulated boundary conditions at the origin and at infinity is

$$\phi_\ell(k,r) \xrightarrow[r\to\infty]{} j_\ell(kr) + k\ a_\ell(k)\ h^+\ (kr) \tag{II-11}$$

or alternately

$$\phi_\ell(k,r) \xrightarrow[r\to\infty]{} N\ j_\ell(kr + \delta_\ell) \tag{II-12}$$

where δ_ℓ is the scattering phase shift caused by the potential and N a constant factor. It is useful to use Eqs. (II-8a) and (II-8b), and rewrite Eq. (II-11) as

$$\phi_\ell(k,r) \xrightarrow[r\to\infty]{} \frac{i}{2}\ [h^-\ (kr) - S_\ell(k)\ h^+(kr)] \tag{II-13}$$

which defines the one element scattering matrix $S_\ell(k)$ for the ℓth partial wave. It is easily established that

$$S_\ell(k) = e^{2i\delta\ell} \tag{II-14}$$

and

$$S_\ell(k) = 1 + 2ika_\ell(k). \tag{II-15}$$

B. Radial wave functions, S-matrix and Jost function

The essence of the present approach to scattering theory is to extract as much information from the S-matrix out of its analyticity properties as a function of momentum, as is available in the Schrodinger equation from which it is derived originally. To obtain the analyticity properties of $S_\ell(k)$ it is necessary to introduce additional radial wave equations to supplement the physical solution defined in Eq. (II-11) and which differ from it by the boundary conditions which define them.

First the two Jost solutions which asymptotically represent pure outgoing and incoming waves,

$$\chi_\ell^{\pm}(k,r) \xrightarrow[r\to\infty]{} h_\ell^{\pm}(kr). \qquad \text{(II-16)}$$

Neither one is regular at the origin nor are they proportional to the physical solution.

Then the regular solution is defined by the boundary condition that

$$U_\ell(k,r) \xrightarrow[r\to o]{} j_\ell(kr). \qquad \text{(II-17)}$$

Although the regular and the physical solutions vanish both at the origin, they differ from each other in that no mention is made of the asymptotic behavior of the former, and by the choice of normalizing constants at the origin, since Eq. (II-11) implies

$$\phi_\ell(k,r) \xrightarrow[r\to o]{} C\, j_\ell(kr) \qquad \text{(II-18)}$$

where C is a constant different from unity to be determined later.

$U_\ell(k,r)$ may be expressed in terms of the two linearly independent Jost solutions as

$$U_\ell(k,r) = A\, \chi_\ell^-(k,r) + B\, \chi_\ell^+(k,r) \qquad \text{(II-19)}$$

Since Eq. (II-7) contains only real parameters and the boundary condition defining the regular solution does not involve any complex quantity, the solution $U_\ell(k,r)$ itself is real. If in addition we recall that the function $h^\pm(kr)$ are complex conjugates of each other, it is then evident that $A=-B^*$. To conform to the traditional notation we introduce a factor i/2, make the k- and ℓ- dependence of the coefficients explicit, and write, instead of A and B,

$$U_\ell(k,r) = \frac{i}{2}\,[f_\ell(k)\ \chi_\ell^-(k,r) - f_\ell^*(k)\ \chi_\ell^+(k,r)] \qquad \text{(II-20a)}$$

with the asymptotic limit

$$U_\ell(k,r) \xrightarrow[r\to\infty]{} \frac{i}{2}\,[f_\ell(k)\ h_\ell^-(kr) - f_\ell^*(k)\ h^+(k,r)] \qquad \text{(II-20b)}$$

$f_\ell(k)$ is the Jost function. Physically $f_\ell(k)$ represents the incoming wave amplitude and $f_\ell^*(k)$ the outgoing wave amplitude.

Our next task is to relate the Jost function to the scattering matrix. To do so we rely upon the concept of the Wronskian of two solutions of a differential equation defined by

$$W[v(r),\ w(r)] = v(r)\,\frac{dw(r)}{d_r} - w(r)\,\frac{dv(r)}{d_r}\ , \tag{II-21}$$

From which it is evident that if, and only if, two solutions are linearly dependent, their Wronskian vanishes. We then use Eq. (II-6) to obtain some useful relations between the different solutions of the radial equation. First we observe that the Wronskian of any two solutions of Eq. (II-7) is independent of the coordinate r, and may therefore be evaluated at any arbitrary point that will suit our convenience. Using Eqs. (II-9) and (II-16) we establish the relation

$$W[\chi_\ell^+(k,r),\ \chi_\ell^-(k,r)] = W[\chi_\ell^+(k,\infty),\ \chi_\ell^-(k,\infty)] = -2ik \tag{II-22}$$

which confirms the linear independence of the two Jost solutions, and with the help of Eq. (II-20) the Jost function is shown to be equal to

$$f_\ell(k) = \frac{1}{k}\,W[\chi_\ell^+\ (k,r),\ U_\ell(k,r)]. \tag{II-23}$$

Furthermore, having determined that the physical and the regular solutions are linearly dependent, we expect their Wronskian to vanish. Indeed from Eqs. (II-13, II-18, II-20) the relation

$$W[U_\ell(k,r),\ \phi_\ell(k,r)] = o \tag{II-24}$$

will hold if and only if

$$S_\ell(k) = \frac{f_\ell^*(k)}{f_\ell(k)} \tag{II-25}$$

and

$$U_\ell(k,r) = f_\ell(k)\ \phi_\ell(k,r). \tag{II-26}$$

Eq. (II-25) tells us that the phase shift is the negative of the phase of the Jost function. The S-matrix, the partial amplitude, the phase shift and the Jost function may also all be tied together by the relations

$$a_\ell(k) = \frac{i}{2k}\left[\ \frac{f_\ell(k) - f_\ell^*(k)}{f_\ell(k)}\ \right] \tag{II-27a}$$

$$a_\ell(k) = \frac{i}{2k}\left[\ 1-S_\ell(k)\ \right] = \frac{1}{k}\, e^{i\delta_\ell}\ \sin\delta_\ell. \tag{II-27b}$$

We now have the S-matrix in terms of the Jost function and it is a simple matter to obtain the analyticity properties of the former in terms of those of the latter. On the other hand, thanks to Eq. (II-23) it is rather easy to determine the analyticity properties of the Jost function. This study will be reserved for a subsequent section. At this point we have gathered enough information to establish some interesting physical properties of the Jost function, namely the relation between bound states, resonances and the Jost function, and will illustrate them in the following section.

C. Bound states, resonant states, and Jost function

To the extent that k represents the magnitude of the momentum, its domain of validity is confined to positive values; only for such values does it have any significance as a physical quantity.

However the possibility of extracting further information from the Schrodinger equation and the whole essence of the Jost approach which leads to some interesting observations on the existence of bound states and of resonances depends on the process of allowing k to take on complex values. The variable k will therefore be considered as a mathematical parameter in the Schrodinger equation to be identified with the physical momentum when it is restricted to the real axis. In enlarging the domain of k to include complex values some modifications will have to be made to previous assertions which were originally based on the assumption that k is real.

The method of analytic continuation is the mathematical tool by means of which the process is carried out, and it is useful to recall at this point a theorem[23] which is fundamental to the method of analytic continuation and upon which we will rely frequently in the discussion that follows. Theorem 1: If two functions $f_1(z)$ and $f_2(z)$ are analytic on two distinct but overlapping regions R_1 and R_2 respectively, and if the equality $f_1(z) = f_2(z)$ holds on the intersection $R_1 \cap R_2$, then either one of these functions is the analytic continuation of the other; they represent together a simple function analytic as the union $R_1 \cup R_2$.

As a consequence of this theorem, we draw the important corollary that if a function $f_1(z)$ is defined on a line segment Γ_1 and if another function $f_2(z)$ is analytic on a region R_2 which includes Γ_1 such that $f_1(z) = f_2(z)$ on Γ_1, then $f_2(z)$ is the analytic continuation of $f_1(z)$ on R_2. This is the basis of the Schwartz reflection principle[24] according to which, if a function $f(z)$ is analytic in a region R which includes the real axis or some segment of it, and if that function is real for real values of z then

$$f(z) = [f(z^*)]^* \tag{II-28}$$

and $[f(z^*)]^*$ is the analytic continuation of $f(z)$ onto the region R made of points conjugate to those in region R.

After these preliminaries we are now able to examine the possibility of defining the S-matrix on the complex plane. It would seem at first that the S-matrix defined by

$$S_\ell(k) = \frac{f_\ell(k)^*}{f_\ell(k)} \tag{II-25}$$

is not an analytic function because $f_\ell(k)^*$ is in general not analytic even if $f_\ell(k)$ is analytic, and the ratio of the two functions would not therefore be defined. It may be shown however, that

$$f(k) = f(-k^*)^*, \tag{II-29}$$

indicating that $f(k)$ is real for pure imaginary values of k; we can therefore use the Schwartz reflection principle to express the S-matrix as

$$S\ (k) = \frac{[f_\ell(k^*)]^*}{f_\ell(k)} \tag{II-29a}$$

In this form the S-matrix appears as the ratio of two analytic functions, since $[f(k^*)]^*$ is an analytic function of k^* if $f(k)$ is analytic in k, and is therefore defined over their common region of analyticity, except maybe for some isolated singularities. A rigorous treatment of the exact region of analyticity will be reserved for the following section in order not to interrupt the present discussion; now that the idea of continuing the S-matrix onto the complex plane has been made plausible, we set out to establish the sought after connection between zeros of the Jost function and bound states and resonances.

Suppose that for some particular value of $k = k_o$ the Jost function vanishes and that the imaginary part of k_o is positive, then the regular solution in equation (II-20) takes the asymptotic form

$$U_\ell(k_o,r) \xrightarrow[r\to\infty]{} -\frac{i}{2} f_\ell(k_o)^* \; h^+ \; (k_o r). \tag{II-30}$$

But since the Hankel function behaves as

$$h_\ell^\pm \; (kr) \xrightarrow[r\to\infty]{} e^{\pm i(kr-\ell\pi/2)} \tag{II-9c}$$

the regular solution decreases exponentially at infinity, for, writing

$$k_o = k_R + ik_I \quad (k_I > 0) \tag{II-31}$$

and substituting into equation (II-30) we have the result

$$U_\ell(k_o,r) \xrightarrow[r\to\infty]{} -\frac{i}{2} f_\ell(k_o)^* \; e^{-k_I r} \quad e^{i(k_R r-\ell\pi/2)} \tag{II-32}$$

By definition this solution is regular at the origin and in view of the result in equation (II-32) represents a normalizable solution with eigenvalue

$$E = k_o^2 \tag{II-33}$$

which is the energy of a bound state.

Let us show now that the zeros of the Jost function in the upper half-plane must lie on the imaginary axis. The function $J_\ell(k_o,r)$ satisfies the equation

$$\frac{d^2}{dr^2} U_\ell(k,r) - [\frac{\ell(\ell+1)}{r^2} + V(r)]\, U_\ell(k,r) = -k^2 U_\ell(k,r). \tag{II-34a}$$

Taking the complex conjugate of this equation, remembering that ℓ and $V(r)$ are real quantities, we have

$$\frac{d^2}{dr^2} U_\ell^*(k,r) - [\frac{\ell(\ell+1)}{r^2} + V(r)]\, U_\ell^*(k,r) = -k^{2*}\, U_\ell^*(k,r) \tag{II-34b}$$

Multiplying equation (II-34a) by $U_\ell^*(k,r)$ and equation (II-34b) by $U_\ell(k,r)$, then subtracting the first result from the second, we obtain

$$U_\ell(k,r) \frac{d^2}{dr^2} U_\ell^*(k,r) - U_\ell^*(k,r) \frac{d^2}{dr^2} U_\ell(k,r) =$$

$$(k^2 - k^{2*})\, U_\ell^*(k,r)\, U_\ell(k,r) \tag{II-35}$$

which may be rewritten as

$$\frac{d}{dr}\,[U_\ell(k,r)\,\frac{d}{dr}\,U_\ell^*(k,r) - U_\ell^*(k,r)\,\frac{d}{dr}\,U_\ell(k,r) =$$

$$(k^2 - k^{2*})\; U_\ell^*(k,r)\; U_\ell(k,r) \tag{II-36}$$

Integrating both members of this equation betwen zero and infinity, the left side vanishes, since $U_\ell(k,r)$ is an eigenfunction, leaving

$$(k^2 - k^{2*}) \quad \int_o^\infty U_\ell^*(k,r)\; U_\ell(k,r)\; dr = 0 \tag{II-37}$$

indicating that

$$k^2 = k^{2*}. \tag{II-38}$$

k is therefore either real or pure imaginary. The first alternative is excluded by the initial hypothesis that the imaginary part of k is positive, thus proving the assertion that the zeros of f(k) in the upper half-plane lie on the imaginary axis and correspond to bound states of energy

$$E = -k_I^{\,2}\ . \tag{II-39}$$

The converse statement that the existence of bound states i.e. of normalizable solutions implies that the Jost function vanishes, is also true. Indeed if $U_\ell(k,r)$ is a normalizable solution with eigen value

$$E = -k_I^{\,2} \qquad\qquad (k_I > o) \tag{II-40}$$

it is evident that the Jost solution $\chi_\ell^+(k,r)$ which behaves asymptotically as

$$\chi_\ell^+(k,r) \xrightarrow[r\to\infty]{} \ell^{\,i(kr-\ell\pi/2)} \tag{II-16}$$

b

becomes a decreasing exponential when k lies on the positive imaginary axis. The regular solution $U_\ell(k,r)$ which is also a normalizable solution must then be proportional to $\chi_\ell^+(k,r)$; their Wronskian therefore vanishes and equation

$$f_\ell(k) = \frac{1}{k} W\,[\chi_\ell^+(k,r),\; U_\ell(k,r)]$$

shows that $f_\ell(k)$ indeed vanishes at $k = ik_I$.

The equivalence of bound states and zeros of the Jost function in the upper half-plane opens up two broad areas for further endeavors. On one hand, it may be postulated that this relation is more fundamental than the Schrodinger theory itself and is valid in a relativistic context, although the latter is not, because of its Lorentz non-invariance. One may then proceed from this premise to develop some theory of relativistic interactions and hope thereby to produce at least some predictions on observed phenomena. This is the contention of the proponents of the S-matrix theory.[25] On the other hand one may see this equivalence as an alternate method to the Schrodinger equation that may be used advantageously to compute the energy of bound states and make other interesting observations on their existence and this nature, whenever the Schrodinger equation would practically fail to lend itself to such treatments. The usefulness of the Jost function in this context and the many useful inferences that may be drawn from it will become even more evident as we explore the connection between the Jost function and resonances.

Resonances are among some of the most frequently occurring phenomena in physics and manifest themselves particularly in nuclear and particle physics, at low as well as at high energies. One may justifiably expect that some insight into the mechanism responsible for their production will provide at the same time a better understanding of the interactions that occur in nature in general. Resonances manifest themselves physically as peaks and dips in cross-sections; we will try therefore to see how these sudden variations in cross-sections may be explained in terms of the analyticity properties of the S-matrix.

In the upper half-plane, the Jost function may vanish only on the imaginary axis. No such restriction exists in the lower half-plane; we consider therefore a possible zero of the Jost function located at

$$k_o = k_R - ik_I \qquad (k_I > o). \tag{II-41}$$

It must be noted that resonances occur in pairs, for in view of equation (II-29), a zero at k_o implies that f(k) also vanishes at $-k_o^*$, so that resonances lie symmetrically with respect to the imaginary axis. Although this zero is not necessarily simple, we assume that it is, for this restriction does not alter the nature of the discussion, but merely curtails unessential mathematical manipulations, so as to arrive more directly at the intended results. In the vicinity of k_o, the Jost function may then be expressed as

$$f_\ell(k) \simeq \left(\frac{df_\ell(k)}{dk}\right)_{k_o} (k-k_o). \tag{II-42}$$

To obtain the momentum dependence of the phase shift, we substitute equation (II-42) into equation (II-29a) and use equation (II-14); then

$$\ell^{2i\delta_\ell(k)} \simeq C\,\frac{k-k_R-ik_I}{k-k_R+ik_I} \tag{II-43}$$

where C is a constant term independent of k. The left member of equation (II-43) has modulus one. So does the term in k in the right member. The constant term therefore has modulus one and is conveniently rewritten as

$$C = e^{2i\delta_{BG}} \tag{II-44}$$

Upon substitution of this result into equation (II-43) and after some elementary trigonometric manipulations, equation (II-43) now reads

$$\tan[\delta_\ell(k) - \delta_{BG}] = \frac{k_I}{k_R-k} \tag{II-44a}$$

or

$$\delta_\ell(k) = \delta_{BG} + \tan^{-1}\frac{k_I}{k_R-k} \tag{II-44b}$$

The first term in the right member of equation (II-44b) is commonly referred to as the background phase shift, and the second as the resonant phase shift.

Before going further it is necessary to decide upon a rigorous definition of a resonance because of the many alternate definitions which have been advanced, some of which do not characterize a true resonance. We adopt that of Newton[26] which identifies a resonance as the rapid increase of the phase shift by π over a narrow momentum region, as the resonant phase shift goes through the point $\pi/2$. Examination of equation (II-44b) reveals that the phase shift does indeed vary between δ_{BG} and $\delta_{BG} + \pi$, provided that the inverse tangent function is appropriately defined. In addition it also indicates that the phase shift increases by $\pi/2$ as k varies from $k_R - k_I$ to $k_R + k_I$. The width of the momentum interval $2k_I$ over which an appreciable variation in phase shift takes place, is narrower the closer k_o lies to the real axis. If k_o sits very far from the real axis, noticeable effects may not take place and we must therefore accept the fact that a zero of the Jost function in the lower half-plane does not always correspond to an observable phenomenon. However, the immense success obtained in studying resonances by this method warrant retaining it until a better approach is proposed.

Cross-sections are experimentally given in terms of energy; it is therefore natural to recast the Jost function and the phase shift as functions of energy. Since two points $\pm k$ correspond to the same point on the E-plane, an ambiguity arises as to which k-value it meant when the scattering functions are expressed as functions of energy. To circumvent the difficulty, the E-plane is divided into two Riemann sheets which coincide along a branch cut from zero to infinity. The first sheet, which is referred to as the physical sheet contains images of points in the k-plane such that $[0 \leq \arg k \leq \pi]$. The second, or unphysical sheet contains those for which $[\pi < \arg k < 2\pi]$. At the seam of the two sheets the pair of points with $[\arg k = o]$ and $[\arg k = \pi]$ coincide; we therefore choose arbitrarily that points on the positive real E-axis refer to those of the k-plane with $[\arg k = o]$. With these conventions everything is unambiguously defined as the E-plane. Bound states lie on the negative real E-axis of the physical sheet and resonances sit on the unphysical sheet.

So no confusion is likely to arise, let $f_\ell(E)$ and $\delta_\ell(E)$ represent the Jost function and the phase shift respectively as functions of energy. The existence of a zero of $f_\ell(E)$ at a point

$$E_o = E_R - i\Gamma/2 \tag{II-45}$$

allows that in the vicinity of E_o, $f_\ell(E)$ be approximately expressed as

$$f_\ell(E) \simeq \alpha(E-E_o) \tag{II-46}$$

and the phase shift as

$$e^{2i[\delta_\ell(E)-d]} = \frac{E-E_R-i\Gamma/2}{E-E_R+i\Gamma/2} \; . \tag{II-47}$$

From this expression one obtains

$$\sin^2 [\delta_\ell(E) - \alpha] = \frac{(\Gamma/2)^2}{(E-E_R)^2 + (\Gamma/2)^2} \tag{II-48}$$

The partial cross-sections are given by equation (II-27b),

$$\sigma_\ell(E) \; \alpha \; |a_\ell(E)|^2 = \frac{1}{E} \sin^2\delta_\ell(E) \; . \tag{II-49}$$

The shape of the partial cross-sections in the neighborhood of a resonance may be anticipated by examining equation (II-48). The left side may be expanded to read

$$\sin^2 \delta_\ell(E) \cos 2\alpha - \frac{1}{2}\sin 2\delta_\ell(E)\sin 2\alpha + \sin^2\alpha =$$

$$\frac{(\Gamma/2)^2}{(E-E_R)^2 + (\Gamma/2)^2} \; . \tag{II-50}$$

We may now study the shape of the partial cross-sections for different values of α. When α=o, we have the simplest case of a resonance, known as a pure Breit-Wigner which is a Lorentzian of width Γ, peaked at $E = E_R$.

For $\alpha=\pi/2$, equation (II-50) reduces to

$$\sin^2\delta_\ell(E) = 1 - \frac{(\Gamma/2)^2}{(E-E_R)^2 + (\Gamma/2)^2} \tag{II-51}$$

which is an inverted Lorentzian. The curves dip to a minimum value of zero at $E = E_R$ with a width of Γ. When $\alpha=\pi/4$, equation (II-50) becomes

$$1 - \sin 2\delta_\ell(E) = \frac{2(\Gamma/2)^2}{(E-E_R)^2 + (\Gamma/2)^2} \tag{II-52}$$

which after some lengthy manipulations may be reduced to

$$\sin^2\delta_\ell(E) = \frac{1}{2} - \frac{(E-E_R)\Gamma/2}{(E-E_R)^2 + (\Gamma/2)^2} \, . \tag{II-53}$$

The curve exhibits a peak at $E=E_R-\Gamma/2$, goes through a point of inflexion at $E=E_R$ then dips to a minimum at $E=E_R+\Gamma/2$.

The presence of a resonance manifests itself also in the total cross-section which is the sum of the partial cross-sections. If the other partial phase shifts do not vary appreciably, the total cross-section is dominated by the resonating partial wave and it becomes easy to detect the presence of a resonance of angular momentum ℓ. In practice, however, the situation is not quite that simple and usually requires more involved analysis to probe the existence of a resonance.

D. Analyticity properties of the spherical Jost function

The preceding sections have served to demonstrate how the Jost function may be used advantageously to study various aspects of scattering phenomena, particularly when the momentum variable is allowed to take on complex values. Naturally the possibility that the Jost function and the other scattering parameters exist away from the real axis was implied throughout the discussion without further elaboration on the conditions, if any, under which the procedure was acceptable. In instances where functions originally defined for real values of the momentum would obviously not be analytic for complex values of the momentum, appropriate modifications based mostly on the Schwartz reflection principle were resorted to in order that they may still be defined on a domain enlarged to include complex values. But the precise regions into which they may be continued were not explored even though various physical properties are associated with very specific regions of the complex plane. We will now complete the discussion on the spherical Jost function and make up for these omissions with a mathematical study of the analyticity properties of the Jost function that will lead to restrictions on the class of potentials for which a Jost approach is possible. The task will be made easier by choosing integral representations for the radial wave functions discussed in the preceding sections and for the Jost function.

If equation (II-7) is rewritten as

$$\frac{d^2}{dr^2}\,\phi_\ell(k,r) - \left[\frac{\ell(\ell+1)}{r^2} - k^2\right]\phi_\ell(k,r) = \gamma\; V(r)\phi_\ell(k,r) \tag{II-54}$$

to bring in evidence the perturbation term $\gamma V(r)\phi_\ell(k,r)$, in which γ measures the strength of the potential, the Green function method may be used to obtain integral representations of the different solutions of equations (II-54), provided that boundary conditions are specified in addition.

The regular solution is defined uniquely by the boundary condition at the origin that

$$U_\ell(k,r) \xrightarrow[r\to o]{} j_\ell(kr) \ .$$

It may therefore be written as

$$U_\ell(k,r) = j_\ell(kr) + \gamma \int_o^\infty dr^1 \, G_\ell(k,r;r') \, V(r') \, U_\ell(k,r'); \tag{II-55}$$

$G_\ell(k,r;r')$ is the Green function which is known to satisfy the equation

$$\frac{d^2}{dr^2} G_\ell(k,r;r') - [\frac{\ell(\ell+1)}{r^2} - k^2] \, G_\ell(k,r;r') = \delta(r-r'). \tag{II-56}$$

This implies that $G_\ell(k,r;r')$ is continuous at $r=r'$, and that its first derivative has a jump of one across the same point, such that

$$\lim_{\varepsilon\to o} \frac{d}{dr} G_\ell(k,r;r') \Bigg|_{r=r'-\varepsilon}^{r=r'+\varepsilon} = 1. \tag{II-57}$$

These conditions determine the Green function which may be shown to be

$$G_\ell(k,r;r') \begin{cases} \frac{1}{k} [j_\ell(kr) \, n_\ell(kr') - j_\ell(kr') \, n_\ell(kr)] & r' \leq r \\ 0 & r' \geq r \end{cases} \tag{II-58}$$

A similar line of reasoning is followed to obtain an integral representation of the Jost solutions. They are defined by the boundary condition

$$\chi_\ell^\pm(k,r) \xrightarrow[r\to\infty]{} h_\ell^\pm(kr),$$

and therefore may be represented as

$$\chi_\ell^\pm(k,r) = h_\ell^\pm(kr) + \int_0^\infty dr'\ g_\ell(k,r;r')\ V(r')\ \chi_\ell^\pm(kr') \ , \tag{II-59}$$

with the Green function now given by

$$g_\ell(k,r;r') = \begin{cases} 0 & r' \leq r, \\ \frac{1}{k}\,[j_\ell(kr)n_\ell(kr') - j_\ell(kr')\ n_\ell(kr)] & r' \geq r. \end{cases} \tag{II-60}$$

We shall need to determine bounds on the various Ricatti-Bessel functions in order to obtain iterative solutions to equations (II-55) and (II-59) respectively. It is useful to recall that the Ricatti-Bessel functions are solutions of a regular differential equation, the only singularities of which lie at zero and at infinity. We may then expect them to be continuous and finite at points other than these. In the neighborhood of the singularities the Ricatti-Bessel function behaves as

$$j_\ell(kr) \xrightarrow[r\to o]{} (kr)^{\ell+1} \tag{II-61a}$$

and $$j_\ell(kr) \xrightarrow[r\to\infty]{} \sin\,(kr-\ell\pi/2). \tag{II-61b}$$

Consistent with these observations, the following bound is established for r real and positive, and all k,

$$|j_\ell(kr)| < C\, e^{|\mathrm{Im}k|r} \left[\frac{|k|r}{1+|k|r}\right]^{\ell+1} . \tag{II-61c}$$

The bound

$$|n_\ell(kr)| \leq C\, e^{|\mathrm{Im}k|r} \left[\frac{|k|r}{1+|k|r}\right]^{-\ell} \tag{II-62a}$$

is also established if we recall the behavior of the Ricatti-Newmann function at zero and at infinity,

$$n_\ell(kr) \xrightarrow[r\to o]{} (kr)^{-\ell} \tag{II-62b}$$

and
$$n_\ell(kr) \xrightarrow[r\to\infty]{} \cos\,(kr-\ell\pi/2). \tag{II-62c}$$

And finally we have for the Ricatti-Hankel function

$$h^+_\ell(kr) \xrightarrow[r\to o]{} (kr)^{-\ell} \tag{II-63a}$$

$$h^+_\ell(kr) \xrightarrow[r\to\infty]{} {}^{i(kr-\ell\pi/2)} \tag{II-63b}$$

with the corresponding bound

$$|h^+_\ell(kr)| \leq C\, \ell^{-\mathrm{Im}k\ r} \left[\frac{|k|r}{1+|k|r}\right]^{-\ell} . \tag{II-63c}$$

Equations (II-61c), (II-62c), and (II-63c) in turn determine bounds on the Green functions defined in equations (II-58) and (II-60), and remembering that $r' \leq r$ in Eq. (II-58) we have

$$|G_\ell(k,r;r')| \leq C \frac{e^{|Imk|(r-r')}}{|k|} \left[\frac{|k|r}{1+|k|r}\right]^{\ell+1} \left[\frac{|k|r'}{1+|k|r'}\right]^{-\ell} . \tag{II-64}$$

For $r' \leq r$ in equation (II-60) we obtain

$$|g_\ell(k,r;r')| \leq C \frac{e^{|Imk|(r'-r)}}{|k|} \left[\frac{|k|r'}{1+|k|r'}\right]^{\ell+1} \left[\frac{|k|r}{1+|k|r}\right]^{-\ell} . \tag{II-65}$$

We now solve equation (II-55) by successive approximations:

$$U_\ell(k,r) = \sum_{m=o}^{\infty} \gamma^m U_\ell^{(m)}(k,r) \tag{II-66a}$$

where

$$U_\ell^{(o)}(k,r) = j_\ell(kr) \tag{II-66b}$$

and

$$U_\ell^{(m)}(k,r) = \int_o^r dr' \, G_\ell(k,r;r') \, V(r') \, U_\ell^{(m-1)}(k,r'), \tag{II-66c}$$

or

$$U_\ell^{(m)}(k,r) = \int_o^r dr_m \cdots \int_o^{r_2} dr_1 \, G_\ell(k,r;r_m) \, V(r_m) \cdots G_\ell(k,r_2;r_1) \, V(r_1) \, j_\ell(kr_1). \tag{II-66d}$$

If we use equations (II-64) and (II-61c), the m^{th} term of the iterative solution is bounded by

$$|U_\ell^{(m)}(k,r)| \le C^{M+1} \left[\frac{r|k|}{1+|k|r}\right]^{\ell+1} e^{|\mathrm{Im}k|r} \int_0^r dr_m \ldots \int_0^{r_2} dr_1$$

$$\frac{r_m}{1+|k|r_m} V(r_m) \ldots \frac{r_1}{1+|k|r_1} V(r_1) \quad \text{(II-67)}$$

To simplify the expression appearing above, we will demonstrate the following theorem that will help us dispose of the series of integration appearing in equation (II-67), Theorem 2:

$$\int_0^r dr_m \ldots \int_0^{r_2} dr_1 \prod_{i=1}^{m} f(r_i) = \frac{1}{m!} \left[\int_0^r dr' f(r')\right]^m \quad \text{(II-68)}$$

Proof: Let us first demonstrate the case m=2, from which the general case for arbitrary m will follow. Consider the integral

$$I = \int_0^r dr_2 \int_0^{r_2} dr_1 \; f(r_1)\; f(r_2) \;, \quad r \le r_2 < r_1 \le o \quad \text{(II-69)}$$

Changing the order of integration, equation (II-69) becomes

$$I = \int_0^r dr_1 \int_{r_1}^r dr_2 \; f(r_1)\; f(r_2) \quad \text{(II-70)}$$

After interchanging the labels r_1 and r_2, equation (II-70) reads

$$I = \int_0^r dr_2 \int_{r_2}^r dr_1 \; f(r_1)\; f(r_2) \;. \quad \text{(II-71)}$$

If we add equation (II-69) and (II-71) we obtain

$$I = \frac{1}{2} \int_o^r dr_2 \left[\int_{r_2}^{r} + \int_o^{r_2} dr_1 \, f(r_1) \, f(r_2) \right] , \qquad \text{(II-72)}$$

or

$$I = \frac{1}{2} \left[\int_o^r dr' \, f(r') \right]^2 . \qquad \text{(II-72a)}$$

If instead of two, we have m variables, there are m! possible permutations, and following a similar procedure equation (II-68) is proven. When this result is applied to equation (II-67), the latter becomes

$$|U_\ell^{(m)} (k,r)| \leq C \left[\frac{|k|r}{1+|k|r}\right]^{\ell+1} e^{|Imk|r} \frac{C^m}{m!} \left[\int_o^r dr' \frac{r'}{1+|k|r'} V(r')\right] \qquad \text{(II-73)}$$

Referring back to equation (II-66a), it is then established that

$$|U_\ell(k,r)| \leq C \left[\frac{|k|r}{1+|k|r}\right]^{\ell+1} e^{|Imk|r} \exp \left[C \int_o^r dr' \frac{r'}{1+|k|r'} V(r')\right) \qquad \text{(II-74)}$$

$U_\ell(k,r)$ is bounded, provided that the integral

$$\int_o^r dr' \frac{r'}{1+|k|r'} V(r') \qquad \text{(II-75)}$$

exists, which requires that V(r) be less singular than $\frac{1}{r^2}$ at the origin. So the infinite series in equation (II-66a) converges. Furthermore the term in $U_\ell^{(o)}$ (k,r) is analytic in k for fixed r, so is the Green function of equation (II-58). Therefore each term of the series is analytic in k and $U_\ell(k,r)$ is an entire function of k.

The same kind of reasoning may be followed to determine the region of analyticity of the function $f_\ell^+(k,r)$. For the sake of brevity only the main equations and the essential points of the argument will be given here. For $r' > r$ a bound may be imposed on the Green function of equation (II-60) by using the limits set in equations (II-61c) and (II-62c). The result is that

$$|g_\ell(k,r;r')| \le C\, e^{|\mathrm{Im}k|(r'-r)} \left[\frac{|k|r}{1+|k|r}\right]^{-\ell} \left[\frac{|k|r'}{1+|k|r'}\right]^{\ell+1} . \tag{II-76}$$

If an iterative approximation to $f_\ell(k,r)$ is attempted such that

$$\chi_\ell^+ (k,r) = \sum_{m=o}^{\infty} \gamma^m \chi_\ell^{(m)} (k,r) \tag{II-77}$$

with

$$\chi_\ell^{(m)} (k,r) = \int_o^\infty dr'\ g_\ell(k,r;r')\ V(r')\ \chi_\ell^{(m-1)} (k,r') \tag{II-78}$$

and

$$\chi_\ell^{(o)} (k,r) = h_\ell^+ (kr)$$

then $\chi_\ell(k,r)$ is shown to be bounded by

$$|\chi_\ell(k,r)| \le C\, e^{\ \mathrm{Imkr}} \left[\frac{|k|r}{1+|k|r}\right]^{-\ell} \exp\ [C \int_r^\infty dr'\ V(r')\ \frac{r'}{1+|k|r'}\ e^{(|\mathrm{Im}k|-\ \mathrm{Im}k)r'} \tag{II-79}$$

and therefore converges for $r > \varepsilon$, $(\varepsilon > o)$, and any closed region of the complex k-plane, excluding k=o, provided that the integral

$$\int_o^\infty dr' \; r'V(r')e^{(|\mathrm{Im}k|-\mathrm{Im}k)r'} \tag{II-80}$$

is defined. Equation (II-80) indicates that the asymptotic form of the potential determines the region of definition of $f_\ell(k,r)$. Three major cases are of importance. First, if the potential vanishes beyond a certain point r=a, then $f_\ell(k,r)$ is continuous or the whole k-plane provided that at the origin V(r) is no more singular than $+\frac{1}{r^2}$. On the other hand if

$$\int_o^\infty dr \; r \; V(r) < \infty \tag{II-81}$$

then $f_\ell(k,r)$ is continuous for $I_m k \geq o$, except for k=o. Finally if V(r) is bounded exponentially at infinity such that

$$\int_o^\infty dr \; r \; |V(r)| \; e^{\mu r} < \infty, \tag{II-82}$$

then the region of continuity is the half-plane $I_m k \geq -\mu/2$, except the point k=o.

These arguments prove that $f_\ell(k,r)$ is continuous in a certain region, but do not suffice to show analyticity in that region; at the upper limit, the integral in equation (II-78) is improper and one cannot therefore conclude that each term of the series (II-77) is analytic in k. One may however differentiate $f_\ell(k,r)$ with respect to k, since the series is convergent and show that the first derivative indeed converges, thereby proving analyticity of $f_\ell(k,r)$ in either $I_m k \geq o$ or $I_m k \geq -\mu/2$, depending on whether the potential satisfies conditions (II-81) or (II-82) respectively. The condition for analyticity is then

$$\int_o^\infty dr\ r^2\ V(r) < \infty \tag{II-83}$$

for the region $I_m k \leq o$, and

$$\int_o^\infty dr\ r^2\ V(r)\ e^{\mu r} < \infty \tag{II-84}$$

for the region $I_m k \leq \mu/2$.

Let us mention finally that the boundaries $I_m k \leq o$ or $I_m k \leq \mu/2$ do not constitute natural boundaries, that is there exists at least one singularity on the boundary but not every point on the boundary is a singularity.[27] Indeed if V(r) is regular in a region

$$-\pi/2 \leq -\sigma < \arg r < \sigma \leq \pi/2 \tag{II-84a}$$

and such that the function

$$V_\phi(x) = e^{2i\phi}\ V(xe^{i\phi}) \tag{II-84b}$$

satisfies equation (II-83) and (II-81), one may, in equation (II-54) make the substitutions $r = xe^{i\phi}$ and $k = ke^{-i\phi}$ and define

$$\chi_{\phi}(K,x) = \chi_{\ell}^{+}(Ke^{-i\phi}, xe^{i\phi}). \tag{II-85}$$

Then $\chi\phi$ satisfies the differential equation (II-54) with the potential now replaced by the new function in equation (II-84). One may then repeat the preceding arguments and establish that $\chi\phi\,(k,x)$ is regular in the upper half of the complex K-plane, for $x = |x|e^{-i\phi}$. For r real the equality

$$\chi_{\ell}^{+}(k,r) = \chi_{\phi}^{+}(ke^{i\phi}, re^{-i\phi}) \tag{II-86}$$

holds, and $f_{\ell}(k,r)$ is a regular function of k for

$$-\sigma < \arg k < \sigma + \pi.$$

In particular if V(r) can be continued up to the imaginary axis, the singularities of $f_{\ell}(k,r)$ are then confined to the imaginary axis.

In conclusion if the potential is analytic in the sense of equations (II-84a) and (II-84b), then $f_{\ell}(k,r)$ is analytic on the entire k-plane except along the imaginary axis from zero to $-i\infty$; and if the potential decreases exponentially at infinity, the singularities are confined to segment of the imaginary axis from $-i\mu/2$ to $-i\infty$.

We are now in a position to obtain an integral expression for the Jost function. In the integral representation of the regular function

$$U_{\ell}(k,r) = j_{\ell}(k,r) + \gamma \int_{o}^{r} dr'\, G_{\ell}(k,r;r')\, V(r')\, U_{\ell}(k,r') \tag{II-87}$$

we use the identity

$$j_\ell(kr) = \frac{i}{2}\,[h_\ell^-(kr) - h_\ell^+(kr)]$$

and takes the limit $r \to \infty$. Then

$$U_\ell(k,r) \xrightarrow[r\to\infty]{} \frac{i}{2}\left\{ h_\ell^-(kr)\ [1 + \frac{\gamma}{k}\int_o^\infty dr' h_\ell^+(kr')V(r')U_\ell(k,r')]\right.$$

$$\left. - h_\ell^+(kr)\ [1 + \frac{\gamma}{k}\int_o^\infty dr'\ h_\ell^+(kr')V(r')U_\ell(k,r')]\right\} \qquad \text{(II-88)}$$

A comparison with equation (II-20) leads to the identification of the Jost function as

$$f_\ell(k) = 1 + \frac{\gamma}{k}\int_o^\infty dr\ h_\ell^+\ (kr)\ V(r)\ U_\ell(kr) \qquad \text{(II-89)}$$

For potentials satisfying (II-75) the integral converges in the upper k-plane, including the real axis and diverges exponentially in the lower k-plane. With an exponentially decreasing potential, as in (II-82), the region of convergence extends to $I_m k \geq -\mu/2$. Then if equations (II-75) and (II-82) are satisfied, the Jost function is analytic in $I_m k \geq o$ and $I_m k \geq -\mu/2$ respectively.

III. ANALYTICITY PROPERTIES OF THE SPHEROIDAL JOST FUNCTION

A. The Schrodinger Equation in Spheroidal Coordinates

Though far from exhaustive, the preceding chapter still contains all the essential ingredients from which the mathematical properties of the Jost function and their physical implications may be derived for a spherically symmetric potential. The salient point of the discussion is that the Jost function is a unifying element which contributes to a more aesthetic treatment of the two particle problem and simultaneously sheds some light on the physical processes of bound and resonant states. All this comes about after letting the real, physical momentum variable take on imaginary values and observing the corresponding behavior of the Jost function and of other related physical parameters as we move about different regions of the complex plane. In so doing we are led to examine the analyticity properties of the Jost function, without which the validity of the process of analytic continuation would remain questionable. Finally we have seen that although the nature of the existing singularities varies with individual potentials, we can, for a large class of potentials, draw conclusions about the regions of analyticity which are general enough to insure that the intended program may be carried out.

We now set about the task of extending this program to spheroidal potentials, or rather of adapting it, since the transition from spherical to spheroidal potentials will be accompanied by increasing mathematical difficulties which will necessarily cause that some properties of the spherical Jost function either be drastically altered or, in some instances, become completely invalid when we consider spheroidal potentials. This will be done best by representing the Schrodinger equation in spheroidal coordinates[28], which by definition are related to the cartesian coordinates by

$$x = \frac{d}{2}\,[(1-\eta^2)\ (\xi^2-1)]^{\frac{1}{2}} \cos\phi \tag{III-1a}$$

$$y = \frac{d}{2}\,[(1-\eta^2)\ (\xi^2-1)]^{\frac{1}{2}} \sin\phi \tag{III-1b}$$

$$z = \frac{d}{2}\eta\xi \tag{III-1c}$$

such that the new variables ξ, η, ϕ are defined over the domain

$$1 \leq \xi < \infty\ ,\quad -1 \leq \eta \leq 1\ ,\quad o \leq \phi \leq 2\pi. \tag{III-2a}$$

The surface ξ = constant is an ellipse of revolution about the z-axis, the interfocal distance of which is measured by the parameter d. As we let d approach zero, we reach the limits

$$\frac{1}{2}d\xi \longrightarrow r \text{ and } \eta \longrightarrow \cos\theta \tag{III-2b}$$

and the spheroidal coordinate system reduces to the familiar spherical one.

The Laplacian is easily expressed in terms of the new coordinates and with that the Schrodinger time independent equation

$$\frac{-\hbar^2}{2\mu}\nabla^2\,\psi(\vec{r}) + V(\vec{r})\ \psi(\vec{r}) = \frac{\hbar^2 k^2}{2\mu}\,\psi(\vec{r}) \tag{III-3}$$

becomes

$$[\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}+\left(\frac{1}{1-\eta^2}-\frac{1}{\xi^2-1}\right)\frac{\partial}{\partial\phi^2}$$

$$+(\tfrac{1}{2}kd)^2\ (\xi^2-1)-\frac{\mu d^2}{2\hbar^2}(\xi^2-\eta^2)\ V(\xi,\eta,\phi)]\ \psi(\xi,\eta,\phi)=0 \tag{III-4}$$

We are interested in spheroidal potentials, that is potentials for which the last term inside the brackets is a function of the radial coordinate only, i.e.

$$\frac{kd^2}{2\hbar^2}(\xi^2-\eta^2)\ V(\xi,\eta,\phi)\equiv U(\xi)\ . \tag{III-5}$$

For such potentials equation (III-4) becomes separable into two equations: one of them defines the spheroidal angle function; the other will be referred to as the radial equation. For mathematical simplicity the azimuthal variable may be eliminated by conveniently choosing the direction of incidence along the z-axis. The first equation then reads

$$\frac{d}{d\eta}(1-\eta^2)\frac{d}{d\eta}S_n\ (c,\eta)+[\lambda_n(c^2)-c^2\eta^2]\ S_n(c,\eta)=o \tag{III-6}$$

where the definition

$$c=\tfrac{1}{2}kd \tag{III-6a}$$

has been introduced, and $\lambda_n(c^2)$ is an eigenvalue which is appropriately expressed as a power series of the form[29]

$$\lambda_n(c^2) = \sum_{i=o}^{\infty} \ell_{2i}^n \, c^{2i} \; ; \tag{III-7a}$$

its first two coefficients are respectively equal to

$$\ell_o^n = n(n+1) \tag{III-7b}$$

$$\ell_2^n = \frac{1}{2}\,[1 + \frac{1}{(2_n-1)(2_n+3)}] \; . \tag{III-7c}$$

Equation (III-6) is independent of the potential and need not concern us any further.

The radial equation reads

$$\frac{d}{d\xi}(\xi^2-1)\,\frac{d}{d\xi}\,T_n(c,\xi) - [\lambda_n(c^2) - c^2\xi^2 + U(\xi)]\,T_n(c,\xi) = o. \tag{III-8}$$

Our purpose is to treat the spheroidal problem as a perturbation from the spherical case in such a manner that the parameter d introduced in equations (III-2) measure the perturbation strength. It is therefore necessary to bring equation (III-8) into a form where the similarities with the radial spherical equation can readily be seen and the form of the perturbation term brought in evidence. An initial step in this direction is made by performing the succession of transformations defined by

$$x = c\xi \;\; , \;\; c \leq x < \infty \tag{III-9a}$$

$$T_n(c,x) = \frac{\psi_n(c,x)}{x} \; . \tag{III-9b}$$

Equation (III-8) is now reduced to

$$(x^2-c^2)\frac{d^2}{dx^2}\psi_n(k,x) + \frac{2c^2}{x}\frac{d}{dx}\psi_n(k,x)$$

$$-[\lambda_n'(c^2) - x^2 + \frac{2c^2}{x^2} + U'(x)]\ \psi_n(k,x) = 0. \qquad \text{(III-10)}$$

Without danger of confusion we will remove the "primes" which appear in equation (III-10) and write λ_n, $U(x)$ in place of λ_n', $U'(x)$ respectively in order to avoid a cumbersome notation.

If we let c approach zero, the two foci of the equipotential spheroids coincide and become spheres centered at the origin; equation (III-10) then becomes

$$\frac{d^2}{dx^2}\psi_n(k,x) - [\frac{n(n+1)}{x^2} + \overline{V}(x) - k^2]\ \psi_n(k,x) = 0$$

with

$$\overline{V}(x) = \frac{U(x)}{x^2}\ . \qquad \text{(III-11a)}$$

This suggests isolating the term in c and rewriting equation (III-10) as

$$\frac{d^2}{dx^2}\psi_n(c,x) - [\frac{n(n+1)}{x^2} + \overline{V}(x) - k^2]\ \psi_n(c,x) =$$

$$c^2\frac{1}{x^2}[\frac{d^2}{dx^2} - \frac{2}{x}\frac{d}{dx} + \frac{2}{x^2} + \overline{\lambda}(c^2)]\ \psi_n(c,x) \qquad \text{(III-12)}$$

with

$$\bar{\lambda}_n(c^2) = \sum_{i=1}^{\infty} \ell^n_{2i} \, c^{2i} \, . \qquad \text{(III-12a)}$$

In the remainder of the discussion equation (III-12) with c set equal to zero will be referred to as the unperturbed equation and the term

$$c^2 A(x)\psi_n(k,x) \equiv c^2 \frac{1}{x^2} \left[\frac{d^2}{dx^2} - \frac{2}{x}\frac{d}{dx} + \frac{2}{x^2}\bar{\lambda}(c^2)\right] \psi_n(k,x) \qquad \text{(III-13)}$$

as the perturbation term. All the essential properties of the unperturbed equation which are related to the spherical Jost function, are known from the discussion in the preceding chapter. It is therefore natural to proceed from there on and treat the right member of equation (III-12) as a perturbation that generates an additional term to be superposed on the spherical Jost function, and represents the effect of the aspherical nature of the potential. This method of presentation of the spheroidal Jost function entails that we first represent the S-matrix in spheroidal coordinates.

B. Jost Function and S-matrix in Spheroidal Coordinate Representation

Starting from equation (III-10)

$$(x^2-c^2)\frac{d^2}{dx^2}\psi_n(c,x) + \frac{2c^2}{x}\frac{d}{dx}\psi_n(c,x)$$

$$-[\lambda_n(c^2) - x^2 + \frac{2c^2}{x^2} + U(x)]\,\psi_n(c,x) = 0 \qquad \text{(III-10)}$$

We wish to derive a "spheroidal Jost function" that will have properties analogous to those of its spherical counterpart and will be suitable for the purpose of characterizing spheroidal potentials. The initial step of the derivation is to study the singularities of equation (III-10).

There are two regular singularities, at x=c and x=-c respectively. The second of these need not concern us here, being outside of the range of the variable x,

$$c \leq x < \infty \; . \tag{III-9b}$$

There is also an irregular singularity at infinity.

We restrict the discussion to short range potentials which allow the existence of pure free waves at infinity; in the context of equation (III-10), such a potential is one that does not grow so fast as the eigenvalue term, which would correspond to the centrifugal barrier in the spherical case. This means that

$$\ell im \; U(x) \xrightarrow[x\to\infty]{} x^{-\varepsilon} \; , \; \varepsilon > o \; . \tag{III-14}$$

Equation (III-11a) indicates that potentials satisfying this requirement are restricted to the asymptotic form

$$V(x) \xrightarrow[x\to\infty]{} x^{-2-\varepsilon} \; , \; \varepsilon > o \; . \tag{III-14a}$$

If U(x) satisfies equation (III-14), equation (III-10) reduces to

$$\frac{d^2}{dx^2} \psi_n(c,x) + \psi_n(k,x) = 0 \tag{III-15}$$

in the limit of large x. Two solutions of equation (III-15) are

$$\psi_n^{\pm}(c,x) = e^{\pm ix} ,$$

which have the same asymptotic behavior as the Ricatti-Hankel functions. Let us therefore define formally the spheroidal Jost solutions $X_n^{\pm}(c,x)$ with boundary conditions at infinity, by

$$X_n^{\pm}(c,x) \xrightarrow[x\to\infty]{} h_n^{\pm}(x) . \qquad \text{(III-16)}$$

The behavior of the solutions at x = c may be inferred directly from equation (III-10). In the limit $x \to c$, equation (III-10) reduces approximately to

$$(x-c)\,\frac{d^2}{dx^2}\psi_n(c,x) + \frac{d}{dx}\psi_n(c,x) - \frac{1}{2c}[\;\;{}_n(c^2-c^2) + U(c)-2]\psi_n$$

or

$$(x-c)\,\frac{d^2}{dx^2}\,\psi_n(k,x) + \frac{d}{dx}\psi_n(k,x) + \alpha(c)\psi_n(k,x) = 0 \qquad \text{(III-17)}$$

where $\alpha(c)$ is a function of c, independent of x, provided that U(x) is not singular at x = c.

The transformation

$$x\;\;c = \frac{y^2}{2\alpha(c)} \qquad \text{(III-18)}$$

brings equation (III-17) into the form

$$y^2 \frac{d^2\psi}{dy^2} + y \frac{d\psi}{dy} + y^2\psi \tag{III-19}$$

This is Bessel's equation of order zero. Two independent solutions to equation (III-19) are the Bessel and Newmann functions, $J_o(y)$ and $Y_o(y)$ respectively. Their behavior at the origin is given by

$$J_o(y) \xrightarrow[y\to o]{} 1 \tag{III-20a}$$

and

$$Y_o(y) \xrightarrow[y\to o]{} \ln y \ . \tag{III-20b}$$

We could define the boundary conditions for the regular solution as

$$\Phi_n(c,x) \Big|_{x=c} = \text{real constant} \tag{III-21}$$

The regular solution thus defined may be expressed as a linear combination of the two spheroidal Jost solutions in equation (III-16) by

$$\Phi_n(c,x) = \frac{i}{2}[F_n^+(c)\ X_n^-(c,x) + F_n^-(c)\ X_n^+(c,x)] \ , \tag{III-22a}$$

and asymptotically it becomes

$$\Phi_n(c,x) \xrightarrow[x\to\infty]{} \frac{i}{2}[F_n^+(c)\ h_n^-(x) + F_n^-(c)\ h_n^+(x)] \ . \tag{III-22b}$$

If we take into account that

$$[h_n^{\pm}(x)\ ^*] = h_n^{-}(x) \tag{III-23}$$

for real x, and that the regular solution must be real because of the absence of any complex quantities in equation (III-10) and of the reality of the boundary conditions (III-21), it is then seen that

$$F_n^-(c) = -F_n^*(c) \; ; \qquad \text{(III-24)}$$

equation (III-22a) now reads

$$\Phi_n(c,x) = \frac{i}{2}[F_n(c)\; X_n^-(c,x) - F_n^*(c)\; X^+(c,x)] \; . \qquad \text{(III-25)}$$

$F_n(c)$ defines the spheroidal Jost function.

An explicit definition in terms of regular and Jost solutions will become necessary and to arrive at it we need to make some remarks concerning the Wronskian of any two solutions of equation (III-10).

If two functions $v_1(c,x)$ and $w_2(c,x)$ are solutions of equation, they satisfy

$$(x^2-c^2)\,\frac{d^2}{dx^2}\,v(c,x) + \frac{2c^2}{x}\,\frac{d}{dx}\,v(c,x)$$

$$-[\lambda_n(c^2) - x^2 + \frac{2c^2}{x^2} + U(x)]\;v(c,x) = 0 \qquad \text{(III-26a)}$$

and

$$(x^2-c^2)\,\frac{d^2}{dx^2}\,w(c,x) + \frac{2c^2}{x}\,\frac{d}{dx}\,w(c,x)$$

$$-[\lambda_n(c^2) - x^2 + \frac{2c^2}{x^2} + U(x)]\;w(c,x) = 0 \qquad \text{(II-26b)}$$

Multiplying equation (III-26a) by w(c,x) and (III-26b) by v(x), subtracting the second result from the first we obtain

$$(x^2-c^2)\ [v(c,x)\ \frac{d^2}{dx}\ w(c,x)\ -\ w(c,x)\ \frac{d^2v(c,x)}{dx^2}] \qquad \text{(III-27)}$$

$$+\ \frac{2c^2}{x}\ [v(c,x)\ \frac{d}{dx}\ w(c,x)\ -\ w(c,x)\ \frac{d}{dx}\ v(c,x)]\ =\ 0$$

The expression in the first bracket is identified with the first derivative of the Wronskian, which we label W for short,

$$W\ \equiv\ v(c,x)\ \frac{dw(c,x)}{dx}\ -\ w(c,x)\ \frac{dv(c,x)}{dx}\ . \qquad \text{(III-28)}$$

Equation (III-27) is a differential equation in W as a function of x,

$$\frac{dW}{dx} + \frac{2c^2}{x(x^2-c^2)}\ W\ =\ 0 \qquad \text{(III-29)}$$

It may be put in a more transparent form by multiplying it though by $\frac{x^2-c^2}{x^2}$ so that (III-29) becomes

$$\left(\frac{x^2-c^2}{x^2}\right)\ \frac{dW}{dx} + \frac{2c^2}{x^3}\ W\ =\ 0\ ,$$

with the obvious result that

$$[\frac{x^2-c^2}{x^2}]\ W[v(c,x),\ w(c,x)]\ =\ \text{constant}. \qquad \text{(III-30)}$$

It is evident that W[v,w] = constant when c=o, which conforms to the results of the spherical radial function. The arbitrary constant appearing in the right member of equation (III-30) is easily determined in the special case when v(c,x) and w(c,x) are replaced by the two Jost solutions respectively. We have

$$\left[\frac{x^2-c^2}{x^2}\right] W[X_n^+(c,x),\ X_n^-(c,x)] = \text{constant.} \tag{III-31a}$$

In the limit where x goes to infinity, equation (III-31a) becomes

$$\lim_{x\to\infty} \left[\frac{x^2-c^2}{x^2}\right] W[X_n^+(c,x),\ X_n^-(c,x)] = W[e^{i(kx-\ell\pi/2)}, e^{-i(kx-\ell\pi/2)}] = -2ic$$

and equation (III-31a) in turn becomes

$$\left[\frac{x^2-c^2}{x^2}\right] W[X^+(c,x),\ X^-(c,x)] = -2ic\ . \tag{III-31b}$$

Finally taking the Wronskian of $X_n^+(c,x)$ with the regular solution in equation (III-25) yields the useful expression for the spheroidal Jost function

$$F_n(c) = \frac{1}{c} \left[\frac{x^2-c^2}{x^2}\right] W[X_n^+(c,x),\ \Phi_n(c,x)]\ . \tag{III-32}$$

The preceding introduction to the spheroidal Jost function which so far has remained strictly mathematical, does not supply direct information about phenomena that may be tested experimentally. Equation (III-32) for example, a defining relation for the Jost function, hardly anticipates what the scattering cross-section for a spheroidal potential will look like or at what energies resonant and bound states are expected to be formed. Notions of this kind are contained in parameters such as phase shifts and scattering amplitude which are themselves directly obtainable after analysis of scattering events. The next step in the discussion therefore will be to establish relations between the Jost function and those parameters so that the Jost function becomes the central source of most of the information that one may wish to have on the interactions of two particles. Hopefully one would then be able to interpret irregularities in cross-sections as peculiarities in the behavior of the Jost function, hoping thereby to assign a physical interpretation to dips and peaks or any other odd appearances in data plots.

Regardless of the structure or the symmetry properties of a potential, provided that it is short range, the wave function of particles scattered by it will, at large distances from the scattering center, be composed of a plane wave term to account for those particles which have gone through unscattered and of a second term representing the deflection of the scattered ones. The asymptotic form of the wave function is then

$$\Psi(\vec{r}) \xrightarrow[r\to\infty]{} e^{ik\cdot r} + f(\theta)\,\frac{e^{ikr}}{r} \,. \tag{III-33}$$

In general the scattering amplitude would be a function of an azimuthal coordinate also. Here, we impart rotational symmetry to the problem by choosing impacting particles traveling along the axis of symmetry of a spheroid, and will dispense with the azimuthal angle. To further take advantage of the symmetry properties of the potential we expand the total wave function in a convenient set of basis eigenfunctions adapted to the symmetry. $\Psi(\vec{r})$ is therefore expressed in terms of the prolate spheroidal angle functions $S_{om}(c,\eta)$ and the radial eigenfunctions $T_o(c,\xi)$. The symbols c, η, ξ refer to parameters in the spheroidal coordinate system defined in equation (III-1). The subscript zero reflects the absence of an azimuthal coordinate due to the rotational symmetry of the problem. The total wave function is then written as

$$\Psi(\vec{r}) = 2 \sum_{n=o}^{\infty} i^n \frac{1}{N_{on}(c)} S_{on}(c,1) S_{on}(c,\eta)\ T_{on}(c,\xi) \qquad \text{(III-34)}$$

where $N_{on}(c)$ is the normalization constant[30] for the angle functions,

$$\int_o^{2\pi} S_{mn}(c,\eta)\ S_{mn'}(c,\eta)\ d\eta = N_{mn}(c)\ \delta_{mn'} \qquad \text{(III-35)}$$

$T_{on}(c,\xi)$ satisfies equation (III-8).

A real potential cannot act as a sink nor as a source of particles. The wave function is therefore expected to have unit amplitude and the most general asymptotic form of the radial function to meet that demand is

$$T_{on}(c,\xi) \xrightarrow[c\xi\to\infty]{} A_n[R_{on}^{(1)}(c,\xi)\cos\delta_n + R_{on}^{(2)}(c,\xi)\sin\delta_n]\ ; \qquad \text{(III-36)}$$

δ_n is real and represents the spheroidal phase shift; $R_{on}^{(1)}(c,\xi)$ and $R_{on}^{(2)}(c,\xi)$ are solutions of the free radial equation. Using the asymptotic forms

$$R_{on}^{(1)}(c\xi) \xrightarrow[c\xi\to\infty]{} \frac{1}{c\xi} \sin\left(c\xi - \frac{n\pi}{2}\right) \qquad \text{(III-37a)}$$

and

$$R_{on}^{(2)}(c\xi) \xrightarrow[c\xi\to\infty]{} \frac{1}{c\xi} \cos(c\xi - n\pi/2) , \qquad \text{(III-37b)}$$

$T_{on}(c,\xi)$ may be written as

$$T_{on}(c,\xi) \xrightarrow[c\xi\to\infty]{} \frac{A_n}{c\xi} \sin\left(c\xi - \frac{n\pi}{2} + \delta_n\right) . \qquad \text{(III-38)}$$

If the plane wave term is decomposed in terms of spheroidal functions, it becomes

$$e^{i\vec{k}\cdot\vec{r}} = 2 \sum_{n=o}^{\infty} i^n \frac{1}{N_{on}(c)} S_{on}(c,1)\, S_{on}(c,\eta)\, R_{on}^{(1)}(c,\xi) , \qquad \text{(III-39a)}$$

and asymptotically

$$e^{i\vec{k}\cdot\vec{r}} \xrightarrow[r\to\infty]{} 2 \sum_{n=o}^{\infty} \frac{i^n}{N_{on}(c)} S_{on}(c,1)\, S_{on}(c,\eta) \frac{\sin(c\xi - n\pi/2)}{c\xi} \qquad \text{(III-39b)}$$

Substituting equation (III-39b) into equation (III-33) and equating the two asymptotic expressions (III-33) and (III-34) for $\Psi(\vec{r})$, we obtain

$$\left[\sum_{n=o}^{\infty} \frac{2i^n}{N_{on}(c)} S_{on}(c,1)\, S_{on}(c,\eta)\, e^{-in\pi/2} + 2ikf(\theta)\right] e^{ic\xi}$$

$$+ \left[\sum_{n=o}^{\infty} \frac{2i^n}{N_{on}(c)} S_{on}(c,1)\, S_{on}(c,\eta)\, e^{2i\delta_n} A_n\right] e^{-ic\xi}$$

$$= \left[\sum_{n=o}^{\infty} \frac{2i^n}{N_{on}(c)} S_{on}(c,1)\, S_{on}(c,\eta)\, e^{in\pi/2} \right] e^{ic\xi}$$

$$+ \left[\sum_{n=o}^{\infty} \frac{2i^n}{N_{on}(c)} S_{on}(c,1)\, S_{on}(c,\eta)\, e^{-i\delta_n} A_n\, e^{in\pi/2} \right] e^{-ic\xi} \; . \tag{III-40}$$

The functions $e^{ic\xi}$ and $e^{-ic\xi}$ are linearly independent; their respective coefficients in both members of equation (III-40) must be equal. We now have

$$\sum_{n=o}^{\infty} \left[\frac{2i^n}{N_{on}(c)} S_{on}(c,1)\, S_{on}(c,\eta)\, e^{-in\pi/2} + 2ikf(\theta)\right]$$

$$= \sum_{n=o}^{\infty} \left[\frac{2i^n}{N_{on}(c)} S_{on}(c,1)\, S_{on}(c,\eta)\, e^{2i\delta_n} A_n\right] , \tag{III-41a}$$

and

$$\sum_{n=o}^{\infty} \left[\frac{2i^n}{N_{on}(c)} S_{on}(c,1)\, S_{on}(c,\eta)\, e^{in\pi/2}\right]$$

$$= \sum_{n=o}^{\infty} \left[\frac{2i^n}{N_{on}(c)} S_{on}(c,1)\, S_{on}(c,\eta)\, e^{-i\delta_n} A_n\, e^{in\pi/2}\right] . \tag{III-41b}$$

Using the orthogonality relation of the angle function (III-35), we deduce from equation (III-41a) that

$$A_n = e^{i\delta_n} \; . \tag{III-42}$$

Equation (III-41b) yields for the scattering amplitude

$$f(\theta) = \frac{1}{ik} \sum_{n=o}^{\infty} \frac{1}{N_{on}(c)} S_{on}(c,1)\, S_{on}(c,\eta)\, (e^{2i\delta_n} - 1), \tag{III-43}$$

from which we define the partial wave amplitude

$$a_n(k) = \frac{1}{2ik} (e^{2i\delta_n} - 1) = \frac{1}{2ik} [S_n(k) - 1] \ . \qquad \text{(III-44a)}$$

$$S_n(k) = e^{2i\delta_n} \qquad \text{(III-44b)}$$

is the spheroidal S-matrix.

The complete asymptotic wave function (III-33) may now be written as

$$\Psi(\vec{r}) \xrightarrow[r\to\infty]{} \sum_{n=o}^{\infty} \frac{2i^n}{N_{on}(c)} S_{on}(c,1)\ S_{on}(c,\eta)\ [R_{on}^{(1)}\ (c,\xi) + ka_n(k)R_{on}^{(3)}(c\xi)]$$

$$\text{(III-45)}$$

where we have used

$$R_{on}^{(3)}(c\xi) = R_{on}^{(1)}(c\xi) + i\ R_{on}^{(2)}(c\xi) \qquad \text{(III-46a)}$$

and $R_{on}^{(3)}$ is given asymptotically by

$$R_{on}^{(3)}(c,\xi) \xrightarrow[c\xi\to\infty]{} e^{i(c\xi - n\pi/2)} \qquad \text{(III-46b)}$$

Asymptotically the radial wave function takes the form

$$T_{on}(c,\xi) \xrightarrow[c\xi\to\infty]{} \frac{1}{c\xi}\frac{i}{2}\ [h_n^-(c,\xi) - (1 + 2ika_n\ (k))\ h_n^+(c\xi)], \qquad \text{(III-47)}$$

and

$$\psi_n(c,\xi) = (c\xi)\ T_{on}(c,\xi) \xrightarrow[c\xi\to\infty]{} \frac{i}{2}\ [h_n^-(c\xi) - S_n(k)\ h_n^+(c\xi)]. \text{(III-48)}$$

The physical wave function and the regular function being both regular at x = c may differ only by a constant factor. Consequently their Wronskian vanishes. Comparing equations (III-48) and (III-25), the Wronskian of the two solutions vanishes if and only if the physical and the regular solutions are related by

$$\psi_n(c,\xi) = \frac{\Phi_n(c,\xi)}{F_n(c)} , \qquad \text{(III-48)}$$

and the Jost function and S-matrix by

$$S_n(k) = \frac{F_n^*(k)}{F_n(k)} . \qquad \text{(III-49)}$$

In anticipation that it will be necessary to define the S-matrix for complex values of k, it must be said that as it appears in equation (III-49), this cannot be done yet. For even if $F_n(k)$ is analytic in k, $F_n(k)^*$ is not. For real values of k, the relation

$$F_n^*(k^*) = F_n(k)^* \qquad \text{(III-50)}$$

holds. Equation (III-49) may then be expressed as

$$S_n(k) = \frac{[F_n(k^*)]^*}{F_n(k)} . \qquad \text{(III-51)}$$

In this present form the S-matrix may be analytically continued, for if $F_n(k)$ is analytic on a domain R, $F_n[(k^*)]^*$ is then analytic over the conjugate domain R^*.

The phase shift may also be written in terms of the Jost function by using equation (III-44b). We have the relation

$$e^{2i\delta_n} = \frac{F_n^*(k^*)}{F_n(k)} \tag{III-52}$$

from which it is easily seen that

$$\tan\delta_n = \frac{1}{i}\left[\frac{F_n^*(k^*)-F_n(k)}{F_n^*(k^*)+F_n(k)}\right] . \tag{III-53}$$

An expression for the partial amplitudes is also obtained by substituting equation (III-52) into equation (III-44a)

$$a_n(k) = \frac{1}{2ik}\left[\frac{F_n^*(k^*)-F_n(k)}{F_n(k)}\right] . \tag{III-54}$$

With the preceding equations, the physical parameters have been tied in with the Jost function. They thus make it possible to determine experimentally whether certain assumptions on the form of potentials governing non-relativistic interactions are valid. In the laboratory, cross-sections are the most readily accessible measurements and, in addition, the graphic displays which they yield make it desirable to express them in terms of theoretical parameters of interest. In the present case the bearing of the Jost function on the total cross-section may be seen by using the expression for the scattering amplitude (III-43). The total cross-section is related to the scattering amplitude by

$$\sigma(k) = \int_{\Omega} |f(\eta)|^2 \, d\Omega . \tag{III-55}$$

with the orthogonality relation (III-35) satisfied by the spheroidal angle functions $S_n(c,\eta)$, one obtains the simple expression

$$\sigma(k) = \sum_{n=o}^{\infty} 2\pi \; |S_{on}(c,1)|^2 \; | \; a_n(k)|^2 \; . \qquad \text{(III-56)}$$

The respective terms of the infinite sum are the partial cross-sections, and $a_n(k)$ the partial amplitude which is given by

$$a_n(k) = \frac{1}{k} \, e^{i\delta_n} \sin\delta_n \; . \qquad \text{(III-57)}$$

This last expression and equation (III-53) make the relation to the Jost function evident.

C. Bound States And Resonances of Spheroidal Potentials

Of the many questions that present themselves in scattering theory and in the two body problem in general, two are of primary interest: the formation of bound states and of resonances. Even if their very nature is not fully grasped, one would like at least to have a phenomenological understanding of these phenomena to the extent that such questions may be answered as to, given a potential, do bound states or resonances exist, and if they exist, of what energies are they expected to be seen. One would wish that the mathematical apparatus designed to answer these questions would be broad enough to incorporate these two types of events under a single phenomenological treatment that explains one of them in terms of the other. For spherical potentials the Jost function method fulfills that wish to a good degree of satisfaction. Bound states are identified with zeros of the Jost function on the positive imaginary momentum axis and resonant states with zeros in the lower momentum half-plane. This suggests indeed the analogy that resonances are bound states with complex energies and therefore decay in time. We will now show that the spheroidal Jost function answers the same questions in the same manner and that the analogy holds when spheroidal potentials are involved.

Looking at equation (III-25), it is evident that if for some value k_o of k such that

$$k_o = k_R + ik_I \ , \ k_I > 0 \ , \tag{III-58}$$

the function $F_n(k)$ vanishes, the regular solution reduces to

$$\Phi_n(c,\xi) = - \frac{i}{2}F_n^*(k) \ X(c,\xi) \ . \tag{III-59}$$

A glance at the asymptotic form (III-22b) confirms that equation (III-59) then becomes

$$\Phi_n(c,\xi) \xrightarrow[c\xi\to\infty]{} - \frac{i}{2}F_n^*(k) \ e^{ik_R x} \ e^{-k_I x} \ . \tag{III-60}$$

Since $\Phi_n(c,x)$ is regular at x = c and is exponentially bounded at infinity, it is therefore a normalizable solution, i.e. a true eigenfunction with eigenvalue

$$E = k_o^2 \ . \tag{III-61}$$

That the zeros of the spheroidal Jost function in the upper half of the k-plane must lie on the imaginary axis, may be proved very simply. In fact we will adopt a proof that does not depend on the separability of the Schrodinger equation and is therefore valid for any real static potential since it is merely a consequence of the hermiticity of the Hamiltonian. The unseparated Schrodinger equation reads

$$\nabla^2\Psi(\vec{r}) - V(\vec{r})\Psi(\vec{r}) = -k^2\Psi(\vec{r}) \ , \qquad \text{(III-62a)}$$

and its complex conjugate

$$\nabla^2\Psi^*(\vec{r}) - V(\vec{r})\Psi(\vec{r}) = -k^{2*}\Psi(\vec{r})^* \ . \qquad \text{(III-62b)}$$

Multiplying equation (III-62a) by $\Psi(\vec{r})^*$ and (III-62b) by $\Psi(\vec{r})$, subtracting the first result from the second we have

$$\Psi(\vec{r})\nabla^2\Psi^*(\vec{r}) - \Psi^*(\vec{r})\nabla^2\Psi(\vec{r}) = (k^2-k^{2*})\ \Psi(\vec{r})\Psi^*(\vec{r}) \ . \qquad \text{(III-63)}$$

The left member is first transformed into the divergence of a gradient; then integrating equation (III-63) over a sphere the radius of which tends to infinity, we have

$$\int_V \nabla\ [\Psi^*(\vec{r})\nabla\Psi^*(\vec{r}) - \Psi^*(\vec{r})\nabla\Psi(\vec{r})]dv = (k^2-k^{2*}) \int_V \Psi^*(\vec{r})\Psi(\vec{r})dv \qquad \text{(III-64)}$$

The divergence theorem is used to change the volume integral in the left member into an integral over a surface at infinity

$$\int_A [\Psi(\vec{r})\nabla\Psi^*(\vec{r}) - \Psi^*(\vec{r})\nabla\Psi(\vec{r})] \cdot d\vec{s} = (k^2-k^{2*}) \int_V \Psi^*(\vec{r})\Psi(\vec{r})dv \ . \qquad \text{(III-65)}$$

The left member vanishes since the wave function vanishes on the surface A. The relation

$$k^2 = k^{2*} \qquad \text{(III-66)}$$

must hold, which implies that k is either real or pure imaginary. k cannot be real because of the original hypothesis (III-58). We conclude therefore that k must lie on the imaginary axis.

The equivalence of bound states and zeros of the Jost function in the upper half of the k-plane is finally established by proving the converse statement that if $\Phi_n(k,\xi)$ is a true eigenfunction with eigenvalue

$$k = ik_I \ , \qquad \text{(III-67)}$$

the Jost function vanishes. The truth of this statement follows directly from equation (III-32)

$$F_n(k) = \frac{1}{k}\left[\frac{x^2-c^2}{x^2}\right] W[X_n^+(c,x),\ \Phi_n(c,x)] \qquad \text{(III-32)}$$

and the asymptotic form of the Jost solution (III-16)

$$X_n^+(c,\xi) \xrightarrow[c\xi\to\infty]{} e^{i(kr-\ell\pi/2)} \ . \qquad \text{(III-67)}$$

For if $k = ik_I$, the Jost solution is exponentially bounded and may differ from the regular solution only by a constant factor. Consequently their Wronskian vanishes; so does the Jost function, as indicated by equation (III-32). We may therefore conclude that for spheroidal as well as for spherical potentials, bound state energies correspond to zeros of the Jost function along the positive imaginary k-axis.

Zeros of the Jost function located in the lower half of the k-plane are associated with the presence of resonances. As was the case with spherical potentials, it is also true that the correspondence between zeros below the real axis and resonances is not quite so exact as that between zeros in the upper half plane and bound states. A zero in the lower half-plane will manifest itself by the presence of a peak or a dip or a combination of the two, to the extent that it lies sufficiently close to the real axis. If it is too far away from the real axis, its presence is no longer detectable. The following discussion which parallels that for spherical potentials illustrates the point.

Indeed if we use equation (III-52) to represent the phase shift in the neighborhood of a zero of the Jost function on the second sheet of the energy plane at

$$E = E_R - i\Gamma/2 \ ,$$

we obtain a sequence of relations that parallel those for the spherical phase shift. We have

$$e^{i[\delta_n(E) - \alpha]} = \frac{E-E_R-i\Gamma/2}{E-E_R-i\Gamma/2} \ , \tag{III-68}$$

from which the following expression may be obtained:

$$\sin^2[\delta_\ell(E) - \alpha] = \frac{(\Gamma/2)^2}{(E-E_R)+(\Gamma/2)^2} \tag{III-68a}$$

To follow the terminology of spherical scattering theory, we may call α the spheroidal background phase shift. The shape of the cross-section at a resonance will depend on the magnitude of the background phase shift. It is useless to repeat here the discussion that follows equation (II-49) for the arguments are identical. For the special values α = o and $\pi/2$, the cross-section is characterized by a peak and a dip at $E = E_R$ respectively. For $\alpha=\pi/4$, we have the more complicated pattern of a peak followed by a dip. For other values of the background phase shift, the patterns become less symmetric.

D. Integral Representation of Spheroidal Jost Function

All the physical properties associated with the Jost function depend on the possibility of continuing it into the complex momentum plane. To study the existence of bound states we need to know that the Jost function is defined in the upper half-plane, and information on the formation of resonances may be had only if the domain of analyticity includes at least a portion of the lower half-plane. We need now to justify the premise of analytic continuability of the Jost function, otherwise the conclusions reached earlier concerning physical states of a spheroidal potential will no longer hold.

The method consists first in transforming the spheroidal radial equation (III-8)

$$(\xi^2-1)\,\frac{d^2}{d\xi^2}\,T_n(c,\xi) + 2\xi\,\frac{d}{d\xi}\,T_n(c,\xi) - [\lambda_n(c^2)-c^2\xi^2+U(\xi)]T_n(c,\xi)=0 \tag{III-8}$$

into an integral equation. The perturbation term is chosen so that the equation for the unperturbed wave function is the one encountered in spherical scattering. If we recall from equation (III-7a) that the eigenvalue term may be expanded as

$$\lambda_n(c^2) = n(n+1) + \lambda_n^{'}(c^2) \tag{III-7a}$$

equation (III-8) is appropriately rewritten as

$$\xi^2\frac{d^2}{d\xi^2}T_n(c,\xi) + 2\xi\frac{d}{d\xi}T_n(c,\xi) - [n(n+1) - c^2\xi^2 + U(\xi)]\;T_n(c,\xi)$$
$$= [\frac{d^2}{d\xi^2} + \lambda_n(c^2)]\;T_n(c,\xi) \tag{III-69}$$

The transformation

$$T_n(c,\xi) = \frac{\psi_n(c,\xi)}{\xi} \qquad \text{(III-70)}$$

turns the left member of equation (III-69) into the suggestive form

$$\frac{d^2}{d\xi^2}\,\psi_n(c,\xi) - \left[\frac{\ell(\ell+1)}{\xi^2} - c^2 + \frac{U(\xi)}{\xi}\right]\psi_n(c,\xi) \qquad \text{(III-71)}$$

$$= \frac{1}{\xi^2}\left[\frac{d^2}{d\xi^2} - \frac{2}{\xi}\frac{d}{d\xi} + \frac{2}{\xi^2} + \lambda_n'(c^2)\right]\psi_n(c,\xi)\ .$$

The left member of equation (III-71) is indeed comparable to equation (II-7) and will be identified as the unperturbed wave equation:

$$\frac{d^2}{d\xi^2}\,\psi_n(c,\xi) - \left[\frac{\ell(\ell+1)}{\xi^2} - c^2 + \frac{U(\xi)}{\xi^2}\right]\psi_n(c,\xi) = 0\ . \qquad \text{(III-72)}$$

The perturbation term is represented by

$$\frac{1}{\xi^2}\left[\frac{d^2}{d\xi^2} - \frac{2}{\xi}\frac{d}{d\xi} + \frac{2}{\xi^2} + \lambda_n'(c^2)\right]\psi_n(c,\xi) \equiv A_n(c,\xi)\,\psi_n(c,\xi)\ . \qquad \text{(III-73)}$$

In order that equation (III-72) be adaptable to the treatment followed in chapter one, the term $\frac{U(\xi)}{\xi^2}$ needs to be restricted by constraints analogous to those imposed on the spherical potential in equations (II-81) and (II-83), namely

$$\int \frac{U(\xi)}{\xi}\,d\xi < \infty \qquad \text{(III-74)}$$

and

$$\int U(\xi)\, d\xi < \infty \qquad \text{(III-75)}$$

$\frac{U(\xi)}{\xi^2}$ will be designated by $\overline{V}(\xi)$ for convenience. If these are true, we recall from equations (II-59), (II-79), and (II-86) that two solutions, with boundary conditions defined at infinity, exist which behave as

$$\chi_n^{\pm}(c,\xi) \xrightarrow[\xi\to\infty]{} h_n^{\pm}(c\xi) \qquad \text{(III-16)}$$

and is bounded by

$$|\chi_n^{\pm}(c,\xi)| \leq C\ e^{\pm \mathrm{Im}c\,\xi}\ \left[\frac{|c|\xi}{1+|c|\xi}\right]^{-\ell} \exp\left[\ C\int_{\xi}^{\infty} d\xi'\overline{V}(\xi')\frac{\xi'}{1+|c|\xi}\ e^{(|\mathrm{Im}c|\pm\mathrm{Im}c)(\xi'-\xi)}\right] \qquad \text{(III-75)}$$

That these two solutions are linearly independent and therefore form a fundamental system of solutions is evidenced by their non-vanishing Wronskian. Since the Wronskian of any two solutions of equation (III-72) is independent of ξ, we have readily

$$W[\chi_n^{+}(c,\xi),\ \chi_n^{-}(c,\xi)] = W[h_n^{+}(c\xi),\ h_n^{-}(c,\xi)] = -2ic \qquad \text{(III-76)}$$

$$(c\neq o).$$

The complete equation (III-71) has a regular solution at $\xi=1$, as shown in equation (III-20). The existence of the fundamental set of solutions $\chi_n^{\pm}(c,\xi)$ to the unperturbed equation, suggest looking for a regular solution to equation (III-71) which is regular at $\xi=1$, of the form

$$\Phi_n(c,\xi) = \alpha_n(c,\xi)\ \chi_n^+(c,\xi) + \beta_n(c,\xi)\ \chi_n^-(c,\xi) \tag{III-77}$$

with the additional requirement that

$$\alpha_n'(c,\xi)\ \chi_n^+(c,\xi) + \beta_n'(c,\xi)\ \chi_n^-(c,\xi) = 0\ . \tag{III-78}$$

Equations (III-77, 78) also yield

$$\Phi_n'(c,\xi) = \alpha_n'(c,\xi)\ \chi_n^+\ (c,\xi) + \beta_n'(c,\xi)\ \chi_n^-(c,\xi)\ . \tag{III-78a}$$

The primes indicate derivatives with respect to ξ, and $\alpha_n(c,\xi)$ and $\beta_n(c,\xi)$ are unknown functions to be determined. Furthermore, since $\Phi_n(c,\xi)$ is regular at $\xi=1$, we attempt to find a solution that satisfies

$$\Phi_n(c,\xi) \xrightarrow[\xi\to\infty]{} U_n(c,\xi)$$

Evaluating the second derivative of equation (III-77) consistent with requirement (III-78), and substituting the result into equation (III-71), we obtain

$$\alpha_n'(c,\xi)\ \chi_n^{+'}(c,\xi) + \beta_n'(c,\xi)\ \chi_n^{-'}(c,\xi) = A_n(c,\xi)\ \Phi_n(c,\xi) \tag{III-80}$$

Equations (III-78) and (III-80) constitute a system where $\alpha_n'(c,\xi)$ and $\beta_n'(c,\xi)$ are the unknowns. Solving for $\alpha_n'(c,\xi)$ and $\beta_n'(c,\xi)$,

$$\alpha_n'(c,\xi) = \frac{\chi_n^-(c,\xi)A_n(c,\xi)\Phi_n(c,\xi)}{\chi_n^-(c,\xi)\chi_n^{+'}(c,\xi) - \chi_n^{-'}(c,\xi)\chi_n^+(c,\xi)} \tag{III-81}$$

$$\beta_n'(c,\xi) = \frac{\chi_n^+(c,\xi)A_n(c,\xi)\Phi_n(c,\xi)}{\chi_n^+(c,\xi)\chi^{-'}(c,\xi)-\chi_n^{+'}(c,\xi)\chi_n^-(c,\xi)} \,. \tag{III-82}$$

Using equation (III-76), they become

$$\alpha_n'(c,\xi) = \frac{\chi_n^-(c,\xi)A_n(c,\xi)\Phi_n(c,\xi)}{2ic} \tag{III-83}$$

and

$$\beta_n'(c,\xi) = -\frac{\chi_n^+(c,\xi)A_n(c,\xi)\Phi_n(c,\xi)}{2ic} \,. \tag{III-84}$$

If, according to equation (II-20a) we write

$$U_n(c,\xi) = \frac{i}{2}[f_n(c)\chi_n^-(c,\xi) - f_n^*(c)\chi_n^+(c,\xi)] \tag{III-85}$$

equation (III-85) and the boundary condition (III-79) can be read to imply

$$\lim_{\xi\to 1} \alpha_n(c,\xi) = -\frac{i}{2}f_n^*(c) \tag{III-86}$$

$$\lim_{\xi\to 1} \beta_n(c,\xi) = \frac{i}{2}f_n(c) \,. \tag{III-87}$$

This enables us to solve equations (III-83) and (III-84):

$$\alpha_n(c,\xi) = -\frac{i}{2}f_n^*(c) + \frac{1}{2ic}\int_1^\xi d\xi'\chi_n^-(c,\xi')A_n(c,\xi')\Phi_n(c,\xi') \tag{III-88}$$

and

$$\beta_n(c,\xi) = \frac{i}{2}f_n(c) - \frac{1}{2ic}\int_1^\xi d\xi'\chi_n^+(c,\xi')A_n(c,\xi')\Phi_n(c,\xi') . \tag{III-89}$$

We finally obtain an integral equation for $\Phi_n(c,\xi)$ by substituting these results into (III-77):

$$\Phi_n(c,\xi) = \frac{i}{2}[f(c)\chi_n^-(c,\xi) - f^*(c)\chi_n^+(c,\xi)] \tag{III-90}$$

$$+ \frac{i}{2k}\int_1^\xi d\xi' \; [\chi_n^-(c,\xi)\chi_n^+(c,\xi') - \chi_n^-(c,\xi')\chi_n^+(c,\xi)]A_n(c,\xi')\Phi_n(c')$$

If $\Phi_n(c,\xi)$ is expanded in terms of the spheroidal Jost function according to (III-22a)

$$\Phi_n(c,\xi) = \frac{i}{2}[F_n(c)\chi^-(c,\xi)-F_n^*(c)\chi^+(c,\xi) \; , \tag{III-91}$$

we recover the spheroidal Jost function by taking the limit $\xi\to\infty$ on both sides of equation (III-90):

$$F_n(c) = f_n(c) + \frac{1}{c}\int_1^\infty d\xi'\chi_n^+(c,\xi')A_n(c,\xi')\Phi_n(c,\xi') \; . \tag{III-92}$$

To determine the domain of analyticity of $F_n(c)$ we need first to study that of $\Phi_n(c,\xi)$ in equation (III-90):

$$\Phi_n(c,\xi)=U_n(c,\xi) + \frac{i}{2c}\int_1^\xi d\xi'[\chi_n^-(c,\xi)\chi_n^+(c,\xi')-\chi_n^-(c,\xi')\chi_n^+(c,\xi)]A_n(c,\xi')\Phi_n(c,\xi') \tag{III-90}$$

The presence of differential operators in the term $A_n(c,\xi')$ presents slight difficulties that will be circumvented by operating on both sides of the equation with $A_n(c,\xi)$. Caution must be exercised, however, in differentiating under the integral sign because of the variable upper limit. Evaluating the first and second derivatives of equation (III-90) we get

$$\Phi_n'(c,\xi)=U_n'(c,\xi)+\frac{i}{2c}\int_1^{\xi} d\xi'\,[\chi_n^{-'}(c,\xi)\chi_n^{+}(c,\xi')-\chi_n^{+'}(c,\xi)\chi_n^{-}(c,\xi')]A_n(c,\xi')\Phi_n(c,\xi') \qquad \text{(III-93)}$$

$$\Phi_n''(c,\xi)=U_n''(c,\xi)+\frac{i}{2c}\int_1^{\xi} d\xi'\,[\chi^{-''}_n(c,\xi)\chi_n^{+}(c,\xi')-\chi_n^{+''}(c,\xi)\chi_n^{-}(c,\xi')]A_n(c,\xi')\Phi_n(c,\xi')$$

$$+ A_n(c,\xi)\Phi_n(c,\xi) \; . \qquad \text{(III-94)}$$

With the definition of the operator $A_n(c,\xi)$ given by equation (III-73), we now obtain

$$A_n(c,\xi)\Phi_n(c,\xi) = A_n(c,\xi)\; U_n(c,\xi) + \frac{1}{\xi^2} A_n(c,\xi)\Phi_n(c,\xi)$$

$$+ \frac{i}{2c}\int_1^{\xi} d\xi'\,[A_n(c,\xi)\chi_n^{-}(c,\xi)\chi_n^{+}(c,\xi')-A_n(c,\xi)\chi_n^{+}(c,\xi)\chi_n^{-}(c,\xi')]$$

$$\times A_n(c,\xi')\Phi_n(c,\xi') \; . \qquad \text{(III-95)}$$

With the more compact notation

$$\Phi_n^{A}(c,\xi) = A_n(c,\xi)\Phi_n(c,\xi) \qquad \text{(III-96)}$$

$$U_n^{A}(c,\xi) = A_n(c,\xi)U_n(c,\xi) \qquad \text{(III-97)}$$

$$\chi_n^{\pm A}(c,\xi) = A_n(c,\xi)\Phi_n(c,\xi) \qquad \text{(III-98)}$$

$$G_n^{A}(c,\xi;\xi') = \chi_n^{-A}(c,\xi)\chi_n^{+}(c,\xi') - \chi_n^{+A}(c,\xi)\chi_n^{-}(c,\xi') \; , \qquad \text{(III-99)}$$

equation (III-95) becomes

$$\Phi_n^A(c,\xi) = \frac{\xi^2}{\xi^2-1} U_n^A(c,\xi) + \frac{i}{2c} \int_1^{\xi} \frac{\xi^2}{\xi^2-1} G_n^A(c,\xi;\xi') \Phi_n^A(c,\xi')\, d\xi' \qquad \text{(III-100)}$$

It is important to observe that the function $\Phi_n(c,\xi)$ and consequently $\Phi_n^A(c,\xi)$ is regular at $\xi=1$. In that case the terms

$$\frac{\xi^2}{\xi^2-1}\, U_n^A(c,\xi) \quad \text{and} \quad \frac{\xi^2}{\xi^2-1}\, G_n^A(c,\xi;\xi')$$

must therefore remain finite at the point $\xi=1$. These remarks must be kept in mind when later we set upper bounds on various terms of equation (III-100).

We wish ultimately to establish a domain of analyticity for $\Phi_n^A(c,\xi)$ as a function of k. To do so we need now to know the region of analyticity of $U_n^A(c,\xi)$ and $\chi_n^{\pm A}(c,\xi)$. The following section will deal with these intermediate questions before we return to discuss the integral equation (III-100).

E. Analyticity Properties of the Spheroidal Jost Function

The analytic properties of the solutions $\chi_n^{\pm A}(c,\xi)$ are readily established by considering equation (II-59)

$$\chi_n^{\pm}(c,\xi) = h_n^{\pm}(c\xi) + \int_{\xi}^{\infty} d\xi'\ g_n(c,\xi;\xi')\ \overline{V}(\xi')\ \chi_n^{\pm}(c,\xi') \qquad \text{(II-59)}$$

This equation yields the first and second derivatives with respect to ξ:

$$\chi_n^{\pm\prime}(c,\xi) = h_n^{\pm\prime}(c\xi) + \int_{\xi}^{\infty} d\xi'\ g_n'(c,\xi;\xi')\ \overline{V}(\xi')\ \chi_n^{\pm}(c,\xi') \qquad \text{(III-101)}$$

and

$$\chi_n^{\pm''}(c,\xi) = h_n^{\pm''}(c\xi) + \int_\xi^\infty d\xi' \; g_n''(c,\xi;\xi')\overline{V}(\xi') \; \chi_n^{\pm}(c,\xi')$$

$$- V(\xi) \; \chi_n^{\pm}(c,\xi) \; . \tag{III-102}$$

It has been demonstrated in section D of chapter two that $\chi_n^{\pm}(c,\xi)$ are analytic in c at least when $I_n c>o$ for $\chi_n^{+}(c,\xi)$, and $I_n c<o$ for $\chi_n^{-}(c,\xi)$. The Hankel functions are analytic functions of their arguments. Consequently the Green function

$$g_n(c,\xi;\xi') \; \frac{i}{2k}[h_n^{+}(c\xi)h_n^{-}(c\xi') - h_n^{-}(c\xi) \; h_n^{+}(c\xi')] \tag{III-103}$$

is analytic; $\chi_n^{\pm'}(c,\xi)$ and $\chi_n^{\pm''}(c,\xi)$ are therefore also analytic in the same regions as the functions themselves. Meixner and Schafke[31] have shown that the term $\lambda_n'(c^2)$ contained in the operator $A_n(c,\xi)$ is also analytic in the finite c plane; we can then conclude that for

$$\chi_n^{\pm A}(c,\xi) \equiv \frac{1}{\xi^2} \; [\frac{d^2}{d\xi^2} - \frac{2}{\xi}\frac{d}{d\xi} + \frac{2}{\xi^2} + \lambda_n'(c^2)] \; \chi_n^{\pm}(c,\xi) \tag{III-104}$$

$\chi_n^{+A}(c,\xi)$ is analytic in $I_m k>o$, and $\chi_n^{-A}(c,\xi)$ in $I_m k<o$. Examining the form of the Green function $G_n^A(c,\xi;\xi')$ in equation (III-99) that would leave no region of analyticity if we content ourselves with the weakest assumptions (III-74) and (III-75) on the form $\overline{V}(\xi)$. If, however, we make the stronger restriction that $\overline{V}(\xi)$ is a Yukawa potential, then $\chi_n^{+A}(c,\xi)$ is analytic in the entire plane,

except on a segment of the imaginary axis between $-i\mu/2$ and $-i\infty$. Conversely $\chi_n^{-A}(c,\xi)$ is then analytic on the entire plane except on the positive imaginary axis betwen $i\mu/2$ and $i\infty$. Or if the potential vanishes beyond a finite distance, then $\chi_n^{+A}(c,\xi)$ are analytic over the whole plane. Therefore we are forced, for the remainder of the discussion to stick to these two special classes of potentials.

To obtain an iterative solution to equation (III-100) it is convenient to use the auxiliary functions

$$U_n^A(c,\xi) = \frac{i}{2}\,[f_n(c)\chi_n^{-A}(c,\xi) - f_n^*(c^*)\chi_n^{+A}(c,\xi)] \qquad \text{(III-105a)}$$

and

$$v_n^A(c,\xi) = \frac{i}{2}\,[f_n(c)\chi_n^{-A}(c,\xi) + f_n^*(c^*)\chi_n^{+A}(c,\xi)]\ . \qquad \text{(III-105b)}$$

Written in terms of these functions, equation (III-100) becomes

$$\Phi_n^A(c,\xi) = \frac{\xi^2}{\xi^2-1}\,U_n^A(c,\xi) + \frac{i}{2cf(c)f^*(c')}\int_1^\xi d\xi'\,\frac{\xi^2}{\xi^2-1}[U_n^A(c,\xi)v_n(c,\xi')$$

$$-v_n^A(c,\xi)\,U_n(c,\xi')]\,A_n(c,\xi')\Phi_n(c,\xi') \qquad \text{(III-106)}$$

Recalling that

$$\chi_n^{\pm}(c,\xi) = h_n^{\pm}(c\xi) + \int_\xi^\infty d\xi'\,g_n(c,\xi;\,\xi')\,\overline{V}(\xi')\chi_n^{\pm}(c,\xi') \quad \text{(III-107)}$$

and $$U_n(c,\xi) = j_n(c\xi) - \int_0^\xi d\xi'\,g_n(c,\xi;\xi')\,\overline{V}(\xi')\,U_n(c,\xi') \quad \text{(III-108)}$$

where $g_n(c,\xi;\xi')$ is given by equation (III-103), and using the definitions (III-96) to (III-99), the following bounds are obtained:

$$|U_n^A(c,\xi)| \le M\ [|f_n(c)|+|f_n^*(c^*)|]\ [1+|\lambda_n(c^2)|]\ [\frac{|c|\xi}{1+|c|\xi}]^{\ell-1}\ \frac{e^{|\mathrm{Im}c|\xi}}{\xi^2} \tag{III-109}$$

and

$$|v_n^A(c,\xi)| \le M\ [|f_n(c)|+|f_n^*(c^*)|]\ [1+|\lambda_n(c^2)|]\ [\frac{|c|\xi}{1+|c|\xi}]^{-\ell-2}\ \frac{e^{|\mathrm{Im}c|\xi}}{\xi^2}\ . \tag{III-110}$$

We introduce the simpler notation

$$N(c) \equiv M[|f_n(c)|+|f_n^*(c^*)|]\ [1+|\lambda_n(c^2)|]\ . \tag{III-111}$$

Then for $\xi'<\xi$, the kernel of equation (III-106) is bounded by

$$|G_n^A(c,\xi;\xi')| \le \frac{N(c)}{2|cf(c)f^*(c^*)|}\ [\frac{|c|\xi}{1+|c|\xi}]^{\ell+1}\ [\frac{|c|\xi'}{1+|c|\xi'}]^{-\ell}\ e^{|\mathrm{Im}c|(\xi-\xi')} \tag{III-112}$$

We can proceed now with the iteration. Let

$$\Phi_n^A(c,\xi) = \sum_{i=o}^{\infty} \Phi_n^{A(i)}(c,\xi) \tag{III-113}$$

where

$$\Phi_n^{A(o)} = \frac{\xi^2}{\xi^2-1}\ U_n^A(c,\xi) \tag{III-114}$$

and

$$\Phi_n^{A(i)}(c,\xi) = \int_1^{\xi} d\xi_i \cdot \cdot \cdot \int_1^{\xi_2} d\xi_1 G_n^A(c,\xi;\xi_i) \cdot \cdot \cdot G_n^A(c,\xi_2\xi_1)$$

$$\times \; \Phi_n^{A(o)}(c,\xi_1) \; . \qquad \text{(III-115)}$$

Using the bounds given in equations (III-109) and (III-112) the i^{th} term of the series is bounded by

$$|\Phi_n^{A(i)}(c,\xi)| \le N(c) \left[\frac{|c|\xi}{1+|c|\xi}\right]^{\ell+1} \frac{e^{|Imc|\xi}}{\xi^2}$$

$$\times \left[\frac{N(c)}{|2cf(c)f^*(c^*)|}\right]^i \int_1^{\xi} d\xi_i \cdot \cdot \cdot \int_1^{\xi_2} d\xi_1 \frac{|c|\xi_i}{\xi_i^2[1+|c|\xi_i]} \cdot \cdot \cdot \frac{|c|\xi_1}{\xi_1^2[1+|c|\xi_1]}$$

$$\text{(III-116)}$$

or, using equation (II-72a)

$$|\Phi_n^{A(i)}(c,\xi)| \le N(c) \left[\frac{|c|\xi}{1+|c|\xi}\right]^{\ell+1} \frac{e^{|Imc|\xi}}{\xi^2} \qquad \text{(III-117)}$$

$$\times \frac{1}{i!} \left[\frac{N(c)}{2|f(c)f^*(c^*)|}\right]^i \left[\int_1^{\xi} \frac{d\xi'|c|\xi'}{\xi'^2[1+|c|\xi']}\right]^i$$

The whole series is therefore bounded by

$$|\Phi_n^A(c,\xi)| \le N(c) \left[\frac{|c|\xi}{1+|c|\xi}\right]^{\ell+1} \frac{e^{|Imc|\xi}}{\xi^2} e^{P(\xi)} \qquad \text{(III-118)}$$

where

$$P(\xi) \equiv \frac{N(c)}{2|f(c)f^*(c^*)|} \int_1^{\xi} d\xi' \frac{|c|\xi'}{\xi'^2[1+|c|\xi']} \qquad \text{(III-119)}$$

Since the series is bounded, its analyticity properties depend on those of its individual terms, which in turn are functions of the "potential term" $\overline{V}(\xi)$. If $\overline{V}(\xi)$ is of the Yukawa form, then the spherical regular solution $U_n(c,\xi)$ and $U_n^A(c,\xi)$ is analytic in c for the whole finite plane. The Green function $G_n^A(c,\xi;\xi')$ is analytic on the whole finite c-plane except for two segments of the imaginary axis extending for $-i\infty$ to $-i\mu/2$ and from $i\mu/2$ to $i\infty$ respectively. Each form of the series is, therefore, analytic on this region and so is the whole series. In the case of a truncated potential, both the Green function and the corresponding spherical regular solution are analytic on the whole finite c-plane. The regular spheroidal solution is, therefore, analytic on the same region since each term of the convergent series is analytic on that region.

With the bound obtained in equation (III-118) the analyticity properties are established readily by substitution into equation (III-92). The region of analyticity of the Jost function is further diminished from that of the regular solution by the convergent condition required on the integral in equation (III-92). Using equation (III-118), we have

$$|F_n(c) - f_n(c)| \leq \frac{Q(c)}{|c|} \int_1^\infty d\xi' \left[\frac{|c|\xi}{1+|c|\xi'}\right]^{-\ell} e^{-\mathrm{Im}c\xi'}$$

$$\times \left[\frac{|c|\xi'}{1+|c|\xi'}\right]^{\ell+1} \frac{e^{|\mathrm{Im}c|\xi'}}{\xi'^2}$$

The upper limit of integration requires that $\mathrm{Im}c > o$ for the integral to converge. When $\mathrm{Im}c = o$, the integral still converges because of the $1/\xi^2$ factor.

We may finally conclude that if $\overline{V}(\xi)$ is a Yukawa potential, the spheroidal Jost function is analytic in the upper half of the imaginary momentum plane, except on the imaginary axis from $i\mu/2$ to $i\infty$ and possibly a pole at $c=o$. For a truncated potential, the spheroidal Jost function is analytic in the finite upper half-plane. In both cases, it is continuous along the real axis. The spheroidal perturbation in both cases has contributed to the disappearance of the lower half-plane from the domain of analyticity of the corresponding spherical Jost function.

IV. CONCLUSION

We have attempted in this study to establish the analyticity properties of the Jost function of a spheroidal potential, by considering the corresponding spherical problem and analyzing the effect caused by the perturbing spheroidal geometry. The results obtained divide themselves in three categories. In the first case, if the spherical potential is required only to have a finite first and second moment, the spheroidal Jost function cannot be continued into the complex plane. We can merely say in this case that it is continuous on the real axis. This is in marked contrast to the purely spherical case where the Jost function is analytic on the entire finite upper half-plane. Under the stronger constraint of a spherical Yukawa potential, the domain of analyticity of the spheroidal Jost function is that of the spherical Jost function less the entire lower half-plane and the positive imaginary axis from $i\infty$ to $i\mu/2$. If the potential is restricted even further to the case of a truncated potential, then the spheroidal Jost function is analytic on the entire finite upper half-plane and continuous on the real axis, which means that the lower half-plane has been lost from the purely spherical case.

In the context of the S-matrix theory these observations imply that the spheroidal structure of the potential eliminates the possibility of continuing the S-matrix, and the scattering amplitude off the real axis, for either one of these quantities is expressed as the ratio of the complex conjugate of the Jost function to the function itself. Since the domain of analyticity of the function is the conjugate of the domain of analyticity of the conjugate function, only the real axis makes up the intersection of these two domains.

Analytic properties of the Jost function in the upper half-plane alone can, however, give information on bound states of the potential. For example, since a regular function can only have a finite number of zeros within its domain of regularity, we may infer that for a truncated spheroidal potential has a finite number of bound states corresponding to each partial wave. In the case of a Yukawa potential, the number of bound states within the energy range from $-\mu^2/4$ to zero is finite. For the truncated potential we must exclude the point at infinity since the presence of the eigenvalue term $\lambda_n(c^2)$ prevents the Jost function from being analytic at that point and an accumulation of bound states is still possible there. Because of the presence of the eigenvalue term dependent on c^2, we cannot for example prove the spheroidal equivalent of Levinson's theorem for it is not possible to have the logarithm of the Jost function vanish on a semi-circle on the upper-half plane embracing all the bound states. In this respect again the spheroidal structure takes away many interesting properties associated with spherical potentials.

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ANALYTICITY PROPERTIES OF THE JOST FUNCTION

FOR SPHEROIDAL POTENTIALS

by

James Roger Malebranche

(ABSTRACT)

The Jost function method is extended to study scattering phenomena produced by potentials having spheroidal symmetry. The spheroidal radial functions are constructed by taking the spherical wave functions as bases.

The role of the Jost function in spherical potential scattering is reviewed. The relation between zeros of the Jost function and the formation of bound and resonant states is then established for spheroidal potentials. The domain of analyticity of the spheroidal Jost function is studied for three classes of basis functions: those belonging to spherical potentials having only a bounded first and second moments, those belonging to a Yukawa potential, and those belonging to truncated potentials.