Interpolatory Weighted-$\mathcal{H}_2$ Model Reduction

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Abstract

This paper introduces an interpolation framework for the weighted-$\mathcal{H}_2$ model reduction problem. We obtain a new representation of the weighted-$\mathcal{H}_2$ norm of SISO systems that provides new interpolatory first order necessary conditions for an optimal reduced-order model. The $\mathcal{H}_2$ norm representation also provides an error expression that motivates a new weighted-$\mathcal{H}_2$ model reduction algorithm. Several numerical examples illustrate the effectiveness of the proposed approach.

Key words: Model reduction, rational interpolation, feedback control, weighted model reduction, weighted-$\mathcal{H}_2$ approximation

1 Introduction

Consider a single input/single output (SISO) linear dynamical system with a realization

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \quad y(t) = \mathbf{c}^T\mathbf{x}(t) \tag{1}$$

for $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$. $\mathbf{x}(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, are respectively the *state*, *input*, and *output* of the system. The *transfer function* of this system is $G(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$. Following common usage, the underlying system will also be denoted by G. For many examples, the state-space dimension n is quite large, leading to untenable demands on computational resources. *Model reduction* attempts to address this by finding a reduced-order system of the form,

$$\mathbf{E}_r\,\dot{\mathbf{x}}_r(t) = \mathbf{A}_r\mathbf{x}_r(t) + \mathbf{b}_r u(t), \quad y_r(t) = \mathbf{c}_r^T\mathbf{x}_r(t) \tag{2}$$

with $G_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r$ for $\mathbf{E}_r$, $\mathbf{A}_r \in \mathbb{R}^{r\times r}$ and $\mathbf{b}_r$, $\mathbf{c}_r \in \mathbb{R}^r$ with $r \ll n$ such that $y_r(t) \approx y(t)$ over a large class of inputs $u(t)$. G_r is a low order, yet high fidelity, approximation to G. We construct G_r via state-space projection: Two matrices ("reduction bases") $\mathbf{V}_r$, $\mathbf{W}_r \in \mathbb{R}^{n\times r}$ are chosen. Then, system dynamics are approximated by $\mathbf{x}(t) \approx \mathbf{V}_r\mathbf{x}_r(t)$ and forcing a Petrov-Galerkin conditon ("orthogonal residuals") $\mathbf{W}_r^T(\mathbf{E}\mathbf{V}_r\dot{\mathbf{x}}_r(t) - \mathbf{A}\mathbf{V}_r\mathbf{x}_r(t) - \mathbf{b}\,u(t)) = \mathbf{0}$, together with the output equation $y_r(t) = \mathbf{c}^T\mathbf{V}_r\mathbf{x}_r(t)$ to produce

$$\begin{aligned} \mathbf{E}_r &= \mathbf{W}_r^T\mathbf{E}\mathbf{V}_r, \quad \mathbf{b}_r = \mathbf{W}_r^T\mathbf{b}, \\ \mathbf{A}_r &= \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r, \quad \text{and} \quad \mathbf{c}_r^T = \mathbf{c}^T\mathbf{V}_r. \end{aligned} \tag{3}$$

See [2,3] for more information.

1.1 Model Reduction by Interpolation

The reduction bases, $\mathbf{V}_r$ and $\mathbf{W}_r$, used in (3) will be chosen to force interpolation: $G_r(s)$ will interpolate $G(s)$ (possibly together with higher order derivatives) at selected interpolation points. This approach to rational interpolation has been considered in [20,21,5,8,7,3] and depends on the following result.

Theorem 1 *Given two sets of interpolation points $\{\sigma_k\}_{k=1}^r$ and $\{\zeta_k\}_{k=1}^r$, that are each closed under conjugation, and a dynamical system G as in (1), consider matrices $\mathbf{V}_r$ and $\mathbf{W}_r$ such that*

$$\begin{aligned} &\mathit{Range}(\mathbf{V}_r) = \mathit{span}\left\{(\sigma_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}\right\} \\ &\mathit{Range}(\mathbf{W}_r) = \mathit{span}\left\{(\zeta_i\mathbf{E} - \mathbf{A})^{-T}\mathbf{c}\right\} \end{aligned} \quad \textit{for } i = 1, \ldots, r. \tag{4}$$

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Then, $\mathbf{V}_r$ and $\mathbf{W}_r$ can be chosen to be real; $G_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r$ defined by (2)-(3) is a real dynamical system that satisfies $G(\sigma_k) = G_r(\sigma_k)$ and $G(\zeta_k) = G_r(\zeta_k)$ for $k = 1, \ldots, r$; and, if $\sigma_j = \zeta_j$ for some j, then $G'(\sigma_j) = G'_r(\sigma_j)$, as well where G' denotes the derivative of $G(s)$ with respect to s.

Theorem 1 can be generalized to higher-order derivative interpolation as well, see [20,21,5,8,7,3]. The subspaces of Theorem 1 are *rational Krylov subspaces* and so, interpolatory model reduction methods for SISO systems are sometimes referred to as *rational Krylov methods.*

1.2 Weighted Model Reduction

The $\mathcal{H}_\infty$ norm of a stable linear system associated with a transfer function, $G(s)$, is defined as $\|G\|_{\mathcal{H}_\infty} = \max_{\omega\in\mathbb{R}} |G(\imath\omega)|$. The $\mathcal{H}_2$ norm of G is defined as $\|G\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\imath\omega)|^2\, d\omega\right)^{1/2}$. The vector spaces of meromorphic functions that are analytic in the right halfplane, having either bounded $\mathcal{H}_\infty$ norm or bounded $\mathcal{H}_2$ norm will be denoted simply as $\mathcal{H}_\infty$ or $\mathcal{H}_2$, respectively. Let $W \in \mathcal{H}_\infty$ be given. The (W-)*weighted $\mathcal{H}_2$ norm* is defined as $\|G\|_{\mathcal{H}_2(W)} = \|G \cdot W\|_{\mathcal{H}_2}$.

We are interested in finding a reduced-order model G_r that minimizes a W-weighted $\mathcal{H}_2$ norm, i.e., that solves

$$\|G - G_r\|_{\mathcal{H}_2(W)} = \min_{\dim(\widetilde{G}_r)=r} \|G - \widetilde{G}_r\|_{\mathcal{H}_2(W)} \qquad (5)$$

The introduction of $W(s)$ allows one to penalize the error in certain frequency ranges more heavily than in others.

An illustrative example: controller reduction Consider a linear dynamical system, P (the *plant*) together with an associated stabilizing controller, G, that is connected to P in a feedback loop. Many control design methodologies, such as LQG and $\mathcal{H}_\infty$ methods, lead ultimately to controllers whose order is generically as high as the order of the plant, see [17,22] and references therein. Thus, high-order plants generally lead to high-order controllers. However, high-order controllers are usually undesirable in real-time applications due to complex hardware, degraded accuracy, and degraded computational speed. Thus, one prefers to use a reduced controller G_r to replace G. Requiring G_r to be a good approximation to G is often not enough in terms of closed-loop performance; plant dynamics need to be taken into account during the reduction process. This may be achieved through frequency weighting: Given a stabilizing controller G, if G has the same number of unstable poles as G_r and if $\left\|[G - G_r]P[I + PG]^{-1}\right\|_{\mathcal{H}_\infty} < 1$, then G_r will also be a stabilizing controller [1,22]. Hence the controller reduction problem may be formulated as finding a reduced-order controller G_r that minimizes or reduces the weighted error $\|(G - G_r)W\|_{\mathcal{H}_\infty}$ with $W(s) := P(s)(I + P(s)G(s))^{-1}$; i.e., controller reduction becomes an application of weighted model reduction. This approach has been considered in [17,1,14,10,6,19,12,18,16] and references therein, leading to variants of frequency-weighted balanced truncation. Conversely, the methods in [11] and [15] are tailored instead towards minimizing a weighted-$\mathcal{H}_2$ error as in (5).

2 Weighted-$\mathcal{H}_2$ model reduction

The methods proposed in [11] and [15] for approaching (5) require solving a sequence of large-scale Lyapunov or Riccati equations; they rapidly become computationally intractable as the system order, n, increases. We will approach (5) within an interpolatory model reduction framework requiring only the solution of (generally sparse) linear systems and no need for dense matrix computations or solution of large-scale Lyapunov or Riccati equations. Interpolatory approaches can be effectively applied even when n reaches the tens of thousands.

2.1 A representation of the weighted-$\mathcal{H}_2$ norm

Given transfer functions G, $H \in \mathcal{H}_2$, and $W \in \mathcal{H}_\infty$, define the weighted-$\mathcal{H}_2$ inner product as

$$\begin{aligned}\langle G,\ H\rangle_{\mathcal{H}_2(W)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G(\imath\omega)W(\imath\omega)}\,W(\imath\omega)H(\imath\omega)\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(-\imath\omega)W(-\imath\omega)W(\imath\omega)H(\imath\omega)\,d\omega,\end{aligned}$$

so that $\|G\|_{\mathcal{H}_2(W)} = \sqrt{\langle G,\ G\rangle_{\mathcal{H}_2(W)}}$. The following lemma gives a compact expression for the weighted-$\mathcal{H}_2$ inner product based on the poles and residues of $G(s)$, $H(s)$ and $W(s)$. By $\mathsf{res}[M(s), \pi]$, we denote the residue of $M(s)$ at $\pi \in \mathbb{C}$.

Lemma 2 *Suppose G, $H \in \mathcal{H}_2$ have poles denoted respectively as $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_m\}$, and suppose $W \in \mathcal{H}_\infty$ has poles denoted as $\{\gamma_1, \ldots, \gamma_p\}$. Assume that $H(s)$ and $W(s)$ have no common poles, and the poles of $W(s)$ are simple. Then*

$$\begin{aligned}\langle G,\ H\rangle_{\mathcal{H}_2(W)} &= \sum_{k=1}^{m} \mathsf{res}[G(-s)W(-s)W(s)H(s), \mu_k] \\ &\quad + \sum_{i=1}^{p} G(-\gamma_i)W(-\gamma_i)H(\gamma_i) \cdot \mathsf{res}[W(s), \gamma_i].\end{aligned}$$

Define $\chi_k = \mathsf{res}[G(-s)W(-s)W(s)H(s), \mu_k]$.

- *If μ_k is a simple pole of $H(s)$, then*

$$\chi_k = G(-\mu_k)W(-\mu_k)W(\mu_k) \cdot \mathsf{res}[H(s), \mu_k].$$

- *If μ_k is a double pole of $H(s)$, then*

$$\begin{aligned}\chi_k =& G(-\mu_k)W(-\mu_k)W(\mu_k)\cdot \mathsf{res}[H(s),\mu_k]\\ &- h_{-2}(\mu_k)\cdot \frac{d}{ds}\left[G(s)W(s)W(-s)\right]\big|_{s=-\mu_k},\end{aligned}$$

where $h_{-2}(\mu_k) = \lim_{s\to\mu_k}(s-\mu_k)^2H(s)$.

Proof: $T(s) = G(-s)W(-s)W(s)H(s)$ has poles at

$$\{-\lambda_1,\ldots -\lambda_n\}\cup\{\pm\gamma_1,\ldots,\pm\gamma_p\}\cup\{\mu_1,\ldots,\mu_m\}.$$

For any $R>0$, define a semicircular contour in the left halfplane: $\Gamma_R = \{z\,|z=\imath\omega \text{ with } \omega\in[-R,R]\}\cup\left\{z\,\middle|z=R\,e^{\imath\theta} \text{ with } \theta\in[\frac{\pi}{2},\frac{3\pi}{2}]\right\}$. For R large enough, the region bounded by Γ_R contains $\{\gamma_1,\ldots,\gamma_p\}\cup\{\mu_1,\ldots,\mu_m\}$, constituting all the poles of $W(s)H(s)$, and hence all the stable poles of $G(-s)W(-s)W(s)H(s)$. Then, the Residue Theorem yields

$$\begin{aligned}\langle G,\ H\rangle_{\mathcal{H}_2(W)} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(-\imath\omega)W(-\imath\omega)W(\imath\omega)H(\imath\omega)\,d\omega\\ &= \lim_{R\to\infty}\frac{1}{2\pi\imath}\int_{\Gamma_R} G(-s)W(-s)W(s)H(s)\;ds\\ &= \sum_{k=1}^{m}\mathsf{res}[G(-s)W(-s)W(s)H(s),\mu_k]\\ &\qquad+\sum_{i=1}^{p}\mathsf{res}[G(-s)W(-s)W(s)H(s),\gamma_i].\end{aligned}$$

This leads to the first assertion. If μ_k is a simple pole for $H(s)$, then

$$\begin{aligned}&\mathsf{res}[G(-s)W(-s)W(s)H(s),\ \mu_k]\\ &=\lim_{s\to\mu_k}\left[(s-\mu_k)G(-s)W(-s)W(s)H(s)\right]\\ &= G(-\mu_k)W(-\mu_k)W(\mu_k)\lim_{s\to\mu_k}(s-\mu_k)H(s).\end{aligned}$$

Similarly, if μ_k is a double pole for $H(s)$, then it is also a double pole for $G(-s)W(-s)W(s)H(s)$ and

$$\begin{aligned}&\mathsf{res}[G(-s)W(-s)W(s)H(s),\ \mu_k]\\ &\quad=\lim_{s\to\mu_k}\frac{d}{ds}\left[(s-\mu_k)^2G(-s)W(-s)W(s)H(s)\right]\\ &\quad=\lim_{s\to\mu_k}G(-s)W(-s)W(s)\frac{d}{ds}\left[(s-\mu_k)^2H(s)\right]\\ &\qquad+\lim_{s\to\mu_k}(s-\mu_k)^2H(s)\frac{d}{ds}\left[G(-s)W(-s)W(s)\right]\\ &\quad= G(-\mu_k)W(-\mu_k)W(\mu_k)\cdot\mathsf{res}[H(s),\ \mu_k]\\ &\qquad- h_{-2}(\mu_k)\cdot\frac{d}{ds}\left[G(s)W(s)W(-s)\right]\big|_{s=-\mu_k}\ \Box\end{aligned}$$

Corollary 3 *If $G(s)$ and $W(s)$ in Lemma 2 each have simple poles, then*

$$\begin{aligned}\|G\|^2_{\mathcal{H}_2(W)} &= \sum_{k=1}^{n}G(-\lambda_k)W(-\lambda_k)W(\lambda_k)\cdot\mathsf{res}[G(s),\lambda_k]\\ &\quad+\sum_{k=1}^{p}G(-\gamma_k)W(-\gamma_k)G(\gamma_k)\cdot\mathsf{res}[W(s),\gamma_k].\end{aligned}\tag{6}$$

This new formula (6) for the weighted-$\mathcal{H}_2$ norm contains as a special case (with $W(s)=1$), a similar expression for the (unweighted) $\mathcal{H}_2$ norm introduced in [9].

Suppose $W\in\mathcal{H}_\infty$ has simple poles at $\{\gamma_1,\ldots,\gamma_p\}$ and define a linear mapping $\mathfrak{F}:\mathcal{H}_2\to\mathcal{H}_2$ by

$$\mathfrak{F}[G](s) = G(s)W(s)W(-s)+\sum_{k=1}^{p}G(-\gamma_k)W(-\gamma_k)\frac{\mathsf{res}[W(s),\gamma_k]}{s+\gamma_k}\tag{7}$$

Notice that $G(s)W(s)W(-s)$ has simple poles at $-\gamma_1,-\gamma_2,\ldots,-\gamma_p$, and

$$\begin{aligned}&\mathsf{res}[G(s)W(s)W(-s),-\gamma_k]\\ &\qquad= \lim_{s\to-\gamma_k}(s+\gamma_k)G(s)W(s)W(-s)\\ &\qquad= G(-\gamma_k)W(-\gamma_k)\lim_{s\to-\gamma_k}(s+\gamma_k)W(-s)\\ &\qquad= -G(-\gamma_k)W(-\gamma_k)\lim_{s\to\gamma_k}(s-\gamma_k)W(s)\\ &\qquad= -G(-\gamma_k)W(-\gamma_k)\cdot\mathsf{res}[W(s),\gamma_k].\end{aligned}$$

Thus $\mathfrak{F}[G](s)$ has poles only in the left half plane and indeed $\mathfrak{F}:\mathcal{H}_2\to\mathcal{H}_2$.

Corollary 4 *Suppose G and W are stable with poles $\{\lambda_1,\ldots,\lambda_n\}$ and $\{\gamma_1,\ldots,\gamma_p\}$, respectively. Choose μ arbitrarily in the left half plane distinct from these points. Then for $F(s)=\mathfrak{F}[G](s)$, $\left\langle G,\ \frac{1}{s-\mu}\right\rangle_{\mathcal{H}_2(W)} = F(-\mu)$ and $\left\langle G,\ \frac{1}{(s-\mu)^2}\right\rangle_{\mathcal{H}_2(W)} = -F'(-\mu)$.*

Proof: By Lemma 2,

$$\begin{aligned}\left\langle G,\ \frac{1}{s-\mu}\right\rangle_{\mathcal{H}_2(W)} &= G(-\mu)W(-\mu)W(\mu)\\ &\quad+\sum_{k=1}^{p}G(-\gamma_k)W(-\gamma_k)\frac{\mathsf{res}[W(s),\gamma_k]}{\gamma_k-\mu} = F(-\mu),\ \text{and}\end{aligned}$$

$$\begin{aligned}\left\langle G,\ \frac{1}{(s-\mu)^2}\right\rangle_{\mathcal{H}_2(W)} &= -\frac{d}{ds}\left[G(s)W(s)W(-s)\right]\big|_{s=-\mu}\\ &\quad+\sum_{k=1}^{p}G(-\gamma_k)W(-\gamma_k)\frac{\mathsf{res}[W(s),\gamma_k]}{(\gamma_k-\mu)^2} = -F'(-\mu).\ \Box\end{aligned}$$

2.2 Weighted-$\mathcal{H}_2$ optimality conditions

Consider the problem of finding a reduced order system, G_r, that solves (5). The feasible set for (5) is nonconvex, so finding a true (global) minimizer is generally intractable. Nonetheless, we are able to obtain descriptive *necessary conditions* for G_r to satisfy (5).

Theorem 5 *If G_r has simple poles, $\{\hat{\lambda}_1, \ldots, \hat{\lambda}_r\}$, and solves (5), then G_r must satisfy: for $k = 1, \ldots, r$,*

$$F_r(-\hat{\lambda}_k) = F(-\hat{\lambda}_k) \quad \text{and} \quad F_r'(-\hat{\lambda}_k) = F'(-\hat{\lambda}_k) \qquad (8)$$

where $F = \mathfrak{F}[G]$ and $F_r = \mathfrak{F}[G_r]$ are defined from (7).

Proof: Suppose by way of contradiction that, for some $\mu \in \{\hat{\lambda}_1, \ldots, \hat{\lambda}_r\}$, $\left\langle G - G_r,\ \frac{1}{s-\mu} \right\rangle_{\mathcal{H}_2(W)} = \alpha_0 \neq 0$. By hypothesis, G_r can be represented as $G_r(s) = \sum_{i=1}^{r} \frac{\hat{\varphi}_i}{s-\hat{\lambda}_i}$ and for some index k, $\mu = \hat{\lambda}_k$. Define $\vartheta_0 = \arg(\alpha_0)$ and with $\varepsilon > 0$, define

$$\widetilde{G}_r^{(\varepsilon)}(s) = \frac{\hat{\varphi}_k + \varepsilon\, e^{-\imath\vartheta_0}}{s-\mu} + \sum_{i \neq k} \frac{\hat{\varphi}_i}{s - \hat{\lambda}_i}.$$

Then

$$\|G_r - \widetilde{G}_r^{(\varepsilon)}\|_{\mathcal{H}_2(W)} = \left\| \frac{-\varepsilon\, e^{-\imath\vartheta_0}}{s-\mu} \right\|_{\mathcal{H}_2(W)} \leq \|W\|_{\mathcal{H}_\infty} \frac{\varepsilon}{\sqrt{2|\mathsf{Re}(\mu)|}}$$

so that $\|G_r(s) - \widetilde{G}_r^{(\varepsilon)}(s)\|_{\mathcal{H}_2(W)} = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$. Since G_r solves (5),

$$\begin{aligned}
\|G - G_r\|^2_{\mathcal{H}_2(W)} &\leq \|G - \widetilde{G}_r^{(\varepsilon)}\|^2_{\mathcal{H}_2(W)} \\
&\leq \|(G - G_r) + (G_r - \widetilde{G}_r^{(\varepsilon)})\|^2_{\mathcal{H}_2(W)} \\
&\leq \|G - G_r\|^2_{\mathcal{H}_2(W)} + 2\mathsf{Re}\left\langle G - G_r,\ G_r - \widetilde{G}_r^{(\varepsilon)} \right\rangle_{\mathcal{H}_2(W)} \\
&\quad + \|G_r - \widetilde{G}_r^{(\varepsilon)}\|^2_{\mathcal{H}_2(W)}.
\end{aligned}$$

Thus,
$0 \leq 2\,\mathsf{Re}\left\langle G - G_r,\ G_r - \widetilde{G}_r^{(\varepsilon)} \right\rangle_{\mathcal{H}_2(W)} + \|G_r - \widetilde{G}_r^{(\varepsilon)}\|^2_{\mathcal{H}_2(W)}$.
This implies first that $0 \leq -\varepsilon|\alpha_0| + \mathcal{O}(\varepsilon^2)$, which then leads to a contradiction, $\alpha_0 = 0$.

To show the next assertion, suppose that for some $\mu \in \{\hat{\lambda}_1, \ldots, \hat{\lambda}_r\}$, $\left\langle G - G_r,\ \frac{1}{(s-\mu)^2} \right\rangle_{\mathcal{H}_2(W)} = \alpha_1 \neq 0$. Then for some k, $\mu = \hat{\lambda}_k$ and we define $\vartheta_1 = \arg(\hat{\varphi}_k \cdot \alpha_1)$. For $\varepsilon > 0$ sufficiently small, define

$$\widetilde{G}_r^{(\varepsilon)}(s) = \frac{\hat{\varphi}_k}{s - (\mu + \varepsilon\, e^{-\imath\vartheta_1})} + \sum_{i \neq k} \frac{\hat{\varphi}_i}{s - \hat{\lambda}_i}$$

As $\varepsilon \to 0$, we have

$$\|G_r - \widetilde{G}_r^{(\varepsilon)}\|_{\mathcal{H}_2(W)} = \left\| \frac{-\varepsilon\, \hat{\varphi}_k\, e^{-\imath\vartheta_1}}{(s-\mu)^2 - \varepsilon\, e^{-\imath\vartheta_1}} \right\|_{\mathcal{H}_2(W)} = \mathcal{O}(\varepsilon)$$

Following a similar argument as before, we find that $0 \leq -\varepsilon|\hat{\varphi}_k \cdot \alpha_1| + \mathcal{O}(\varepsilon^2)$ as $\varepsilon \to 0$, which leads to a contradiction, $\alpha_1 = 0$. □

The interpolation conditions described in (8) give first order necessary conditions for G_r to solve the optimal weighted-$\mathcal{H}_2$ model reduction problem (5). Note that for $W(s) = 1$, one obtains $F(s) = G(s)$ and $F_r(s) = G_r(s)$; thus (8) contains the interpolatory $\mathcal{H}_2$ optimality conditions of [9] for the unweighted problem as a special case. Unfortunately, there does not appear to be a straightforward generalization of the corresponding computational approach that was described in [9] for the optimal (unweighted) $\mathcal{H}_2$ model reduction problem. Instead, we consider a different systematic approach to this problem motivated by an expression for the weighted-$\mathcal{H}_2$ error.

2.3 *A weighted-$\mathcal{H}_2$ error expression*

The weighted-$\mathcal{H}_2$ norm expression in Corollary 3 leads immediately to an expression for the weighted-$\mathcal{H}_2$ error that forms the basis for our computational approach.

Corollary 6 *Suppose that G, G_r and W are stable with simple poles $\{\lambda_i\}_{i=1}^n$, $\left\{\hat{\lambda}_j\right\}_{j=1}^r$, and $\{\gamma_k\}_{k=1}^p$, respectively, and that there are no common poles. Define residues: $\phi_i := \mathsf{res}[G(s), \lambda_i]$; $\hat{\phi}_j := \mathsf{res}[G_r(s), \hat{\lambda}_j]$; and $\psi_k := \mathsf{res}[W(s), \gamma_k]$. The weighted-$\mathcal{H}_2$ error is given by*

$$\begin{aligned}
\|G - G_r\|^2_{\mathcal{H}_2(W)} &= \sum_{i=1}^{n} (G(-\lambda_i) - G_r(-\lambda_i)) W(-\lambda_i) W(\lambda_i) \cdot \phi_i \\
&+ \sum_{j=1}^{r} (G_r(-\hat{\lambda}_j) - G(-\hat{\lambda}_j)) W(-\hat{\lambda}_j) W(\hat{\lambda}_j) \cdot \hat{\phi}_j \qquad (9) \\
&+ \sum_{k=1}^{p} (G(-\gamma_k) - G_r(-\gamma_k)) W(-\gamma_k) (G(\gamma_k) - G_r(\gamma_k)) \cdot \psi_k
\end{aligned}$$

One may recover the (unweighted) $\mathcal{H}_2$ error expression of [9] as a special case by taking $W(s) = 1$. Notice that the weighted error depends on the mismatch of G and G_r at the reflected full system poles $\{-\lambda_i\}$, reflected reduced poles $\{-\hat{\lambda}_j\}$, and reflected weight poles $\{-\gamma_k\}$.

2.4 *An algorithm for the weighted-$\mathcal{H}_2$ model reduction problem: W-IRKA*

In order to reduce the weighted error, one may eliminate some terms in the error expression, by forcing *interpolation* at selected (mirrored) poles. Since r is required to be much smaller than n, there is not enough degrees of freedom to force interpolation at all the terms in the first and third components of the weighted-$\mathcal{H}_2$ error. However, the second term, i.e. the mismatch at $\hat{\lambda}_j$, can be completely eliminated by enforcing $G(-\hat{\lambda}_j) = G_r(-\hat{\lambda}_j)$ for $j = 1, \ldots, r$. Hence, as in the unweighted $\mathcal{H}_2$ problem, the mirror images of the reduced-order poles play a crucial role. This motivates an algorithm with iterative rational Krylov steps to enforce the desired interpolation property as outlined in Algorithm 1 below. However, a crucial difference from the unweighted $\mathcal{H}_2$ problem is that we will not enforce interpolation of $G'(s)$ at

Algorithm 1. Weighted Iterative Rational Krylov Algorithm (W-IRKA)

Given $G(s) = \mathbf{c}^T(s\mathbf{E} - A)^{-1}\mathbf{b}$ and $W(s) = \mathbf{c}_w^T(s\mathbf{E}_w - \mathbf{A}_w)^{-1}\mathbf{b}_w$, reduction order $r = \nu + \varpi$ with $\nu, \varpi \geq 0$, let $\{\lambda_i\}_{i=1}^{\nu}$ denote the ν dominant poles of G and $\{\gamma_k\}_{k=1}^{\varpi}$ the ϖ dominant poles of W.

(1) Make an initial interpolation point selection: $\zeta_i = -\lambda_i$ for $i = 1, \ldots, \nu$, $\zeta_{j+\nu} = -\gamma_j$ for $j = 1, \ldots, \varpi$; $\sigma_k = \zeta_k$ for $k = 1, \ldots, r$;

(2) Construct bases, $\mathbf{V}_r$ and $\mathbf{W}_r$, that satisfy (4).

(3) Repeat, while (relative change in $\{\sigma_i\} >$ tol)
   (a) $\mathbf{A}_r = \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r$ and $\mathbf{E}_r = \mathbf{W}_r^T\mathbf{E}\mathbf{V}_r$
   (b) Solve the eigenvalue problem $\mathbf{A}_r\mathbf{x}_j = \hat{\lambda}_j\mathbf{E}_r\mathbf{x}_j$ and assign $\sigma_j \longleftarrow -\hat{\lambda}_j$ for $j = 1, \ldots, r$.
   (c) Update $\mathbf{V}_r$ so that Range$(\mathbf{V}_r)$ = span$\{(\sigma_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}, \cdots, (\sigma_r\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}\}$.

(4) $\mathbf{A}_r = \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r$, $\mathbf{E}_r = \mathbf{W}_r^T\mathbf{E}\mathbf{V}_r$, $\mathbf{b}_r = \mathbf{W}_r^T\mathbf{b}$, and $\mathbf{c}_r^T = \mathbf{c}^T\mathbf{V}_r$

these points; instead we use the remaining r degrees of freedom to reflect the weight information $W(s)$ and also to eliminate terms from the first component of the error term. The error expression (9) shows that interpolation errors are multiplied by the residues ϕ_i and ψ_k. Hence, we use the remaining r variables to eliminate terms in the first and third components of the error expression corresponding to the dominant residues ϕ_k and ψ_k. Note that in several cases, such as in the controller reduction problem, the state-space dimension of the weight will be of the same order as that of G, $\mathcal{O}(p) \approx \mathcal{O}(n)$. We measure dominance in a relative sense; i.e., normalized by the largest (in amplitude) ϕ_k and ψ_k in every set. More details on this selection process can be found in Section 3 where several examples are used to illustrate these concepts. Note that one *never* needs to compute a full eigenvalue decomposition to obtain the residues of $G(s)$ and $W(s)$. Since only a small subset of poles is needed, one could use, for example, the *dominant pole algorithm* proposed by Rommes [13] which computes effectively those eigenvalues that correspond to the dominant residues without requiring a full eigenvalue decomposition.

Upon convergence of Algorithm 1, $\sigma_j = -\hat{\lambda}_j$ for $j = 1, \ldots, r$; G_r interpolates G at these points, and the second sum in (9) is eliminated. $\mathbf{W}_r$ is unchanged throughout, so G_r interpolates G at r (aggregated) dominant poles of G and W, eliminating ν and ϖ terms from the first and third sums in (9), respectively. Examples in Section 3 illustrate the effectiveness of this approach.

3 Numerical examples

We provide two examples related to controller reduction. $\Phi^{(N)}$ and $\Psi^{(N)}$ denote the set of normalized residues of $G(s)$ and $W(s)$, respectively.

3.1 A building model

The plant, P, is linearized a model for the Los Angeles University Hospital, and has order 48; see [4] for details. An LQG-based controller, G, of the same order, $n = 48$, is designed to dampen oscillations in the impulse response. The ten highest normalized residues of $G(s)$ and of $W(s)$ are:

$$\Phi^{(N)} = [\ 1.0000\ 1.0000\ 0.0286\ 0.0286\ 0.0088,\ 0.0088\ 0.0080\ 0.0080\ 0.0060\ 0.0060\ ]$$

$$\Psi^{(N)} = [\ 1.0000\ 1.0000\ 0.8416\ 0.8416\ 0.3935,\ 0.3935\ 0.2646\ 0.2646\ 0.0951\ 0.0951\ ]$$

There is a significant drop in $\Phi^{(N)}$ values after the second entry, so we take the first two residues of G as dominant. $\Psi^{(N)}$ remains at roughly the same order until the 9$^{\text{th}}$ entry. Thus, we choose $\nu = 2$; and $\varpi = r - \nu = r - 2$ for a given reduction order, r. To illustrate the effect of this dominant pole selection, we apply **W-IRKA**, varying ν from 0 to r. Tables 1 below lists the resulting weighted-$\mathcal{H}_2$ errors for three cases: $r = 12$, $r = 14$, and $r = 16$.

$r = 12$:

ν/ϖ	12/0	10/2	8/4	6/6	4/8	2/10	0/12
	1.4021	1.1433	0.6548	0.6863	0.3576	0.2181	0.2853

$r = 14$:

ν/ϖ	14/0	12/2	10/4	8/6	6/8	4/10	2/12	0/14
	1.4734	1.3436	0.6477	0.3019	0.1538	0.1425	0.1351	0.2224

$r = 16$:

ν/ϖ	16/0	14/2	12/4	10/6	8/8
	1.4206	1.1934	0.7258	0.2898	0.1917
ν/ϖ	6/10	4/12	2/14	0/16	
	0.1221	0.1154	0.1309	0.1388	

Table 1
Weighted-$\mathcal{H}_2$ error as ν and ϖ vary

The weighted-$\mathcal{H}_2$ error decreases as we take more dominant poles of $W(s)$ over those of $G(s)$; suggesting the importance of the residues of $W(s)$ in the error expression (9). Choosing ν=2 is the best choice for most cases. Tables 1 illustrate that while the weighted error initially decreases as ν decreases, it starts increasing when $\nu < 2$, justifying the choice $\nu = 2$. For the case of $r = 16$, similar observations hold Although $\nu = 2$ is not the optimal choice when $r = 16$, the error for $\nu = 2$ is nearly smallest, making $\nu = 2$ still a very good candidate for **W-IRKA**. These numerical results support the idea of choosing ν and ϖ according to the decay of the normalized residues. Even though this choice seems to yield small weighted errors, there may be variations that are even better. The residues are multiplied by quantities such as $W(-\lambda_i)W(\lambda_i)$, so one might consider incorporating these multiplied quantities as well.

A satisfactory reduced-order controller should not only approximate the full-order controller, but also provide the same closed-loop behavior as the original controller. Let T and T_r denote the full-order and reduced-order closed-loop systems, respectively: T corresponds to the feedback connection of P with G; and T_r to the feedback connection of P with G_r. Figure 1-(a) depicts the amplitude Bode plots of G and G_r for $r = 14$ obtained with $\nu = 2$. G_r is an accurate match to G. Figure 1-(b) shows that the reduced-closed loop behavior T_r almost exactly replicates T.

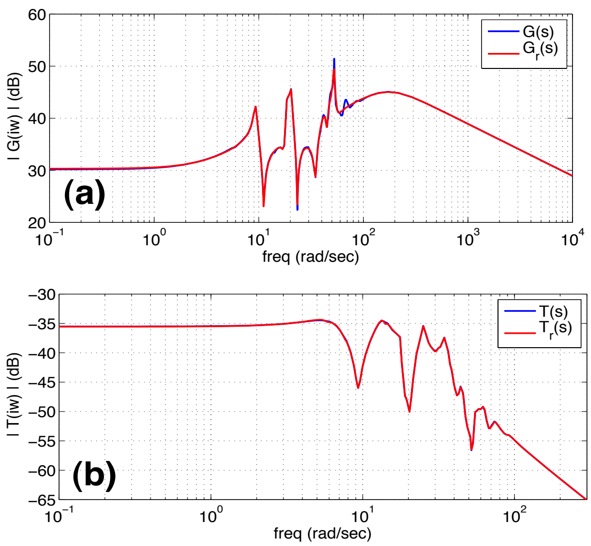

Fig. 1. Bode Plots (a) Full and reduced controller (b) Full and reduced closed-loop system

We now compare **W-IRKA** with Frequency Weighted Balanced Truncation (**FWBT**) and **IRKA** of [9] for the (unweighted) $\mathcal{H}_2$ problem. Comparison with **IRKA** is included to illustrate the importance of including weighting in the $\mathcal{H}_2$-based model reduction process. We vary the reduction order from $r = 10$ to $r = 20$ in increments of 2, and compute weighted $\mathcal{H}_\infty$ and $\mathcal{H}_2$ errors for each case. We use $\nu = 2$ for all cases even though it might not the best choice for **W-IRKA**. Results are listed in Table 2. Note that for every r value, **W-IRKA** outperforms **FWBT** with respect to the weighted-$\mathcal{H}_2$ norm. This might be anticipated since **W-IRKA** is designed to reduce the $\mathcal{H}_2$ error. But **W-IRKA** outperforms **FWBT** with respect to the weighted-$\mathcal{H}_\infty$ norm as well in all except the $r = 18$ case. This is significant since balanced truncation approaches generally yield small $\mathcal{H}_\infty$ norms. This behavior is similar to the behavior of **IRKA** where one often observes that **IRKA** consistently yields satisfactory $\mathcal{H}_\infty$ approximants as well [9]. Note that for $r = 10$, the reduced-order controller due to **FWBT** fails to produce a stable closed-loop system. Table 2 also illustrates that **W-IRKA** significantly outperforms **IRKA** in terms of the weighted error norms. This is what we have expected since unlike **W-IRKA**, **IRKA** is tailored towards the unweighted $\mathcal{H}_2$ model reduction problem. This becomes clearer after inspecting Table 3, which shows that, in terms of the unweighted error $\|G-G_r\|_{\mathcal{H}_\infty}$, **IRKA** outperforms **W-IRKA**. Thus, while G_r from **IRKA** is a better approximation to G in an open-loop sense, once the weight is taken into consideration, **W-IRKA** does what it is designed for, leading to a smaller weighted error.

r	10	12	14	16	18	20
FWBT	1.409	0.5286	0.0723	0.0811	0.0498	0.0830
W-IRKA	0.9175	0.1562	0.0723	0.0721	0.0722	0.0516
IRKA	1.4032	0.4837	0.1335	0.0987	0.1194	0.1293

$\|G - G_r\|_{\mathcal{H}_\infty(W)}$

r	10	12	14	16	18	20
FWBT	2.1080	1.1723	0.1415	0.1386	0.1214	0.1310
W-IRKA	0.6677	0.2180	0.1351	0.1309	0.1028	0.0956
IRKA	1.9540	0.8620	0.2102	0.1286	0.2066	0.2317

$\|G - G_r\|_{\mathcal{H}_2(W)}$

Table 2
Comparison of **W-IRKA**, **FWBT**, and **IRKA**

r	10	12	14	16	18	20
W-IRKA	0.8152	0.2750	0.3679	0.4078	0.1274	0.0518
IRKA	0.1062	0.1168	0.0478	0.0513	0.0123	0.0082

$\|G - G_r\|_{\mathcal{H}_\infty} \,/\, \|G\|_{\mathcal{H}_\infty}$

Table 3
Comparison of **W-IRKA** and **IRKA**: Unweigthed error

3.2 International Space Station 12A Module

The plant, P, is a model for the International Space Station 12A Module with dimension 1412. It is lightly damped and its impulse response exhibits long-lasting oscillations. A state-feedback, full-order, observer-based controller of order $n = 1412$ is designed to dampen these oscillations. The decay rate of the first 50 normalized residues $\Phi^{(N)}$ and $\Psi^{(N)}$ are shown in Figure 2-(a). While there is almost a two order-of-magnitude drop in $\Phi^{(N)}$ between the third and fourth components, $\Psi^{(N)}$ continues to stay significant. Hence, we take $\nu = 3$ and reduce order from $n = 1412$ to $r = 60$ using **W-IRKA**. For comparison, we also apply **FWBT**. We denote the resulting reduced-order closed-loop systems due to **W-IRKA** and **FWBT** by T_r and T_{fwbt}, respectively. Note that T_{fwbt} was unstable for $r = 60$. Indeed, $r = 88$ is the *smallest* order **FWBT**-derived reduced controller that lead to a stable closed-system. All **FWBT**-derived G_r are stable; however for $r < 88$ when G_r is connected to P, the resulting T_{fwbt} is unstable. Hence, we compare below the $r = 60$ case for **W-IRKA** with the $r = 88$ case for **FWBT**. In Figure 2-(b), we plot the absolute value of the errors in the impulse responses due to both methods. **W-IRKA** outperforms **FWBT** even with a lower-order controller. We also simulate both T and T_r for a sinusoidal input of $u(t) = \cos(2t)$. Results 2-(c) illustrate the superior performance of **W-IRKA** even more clearly.

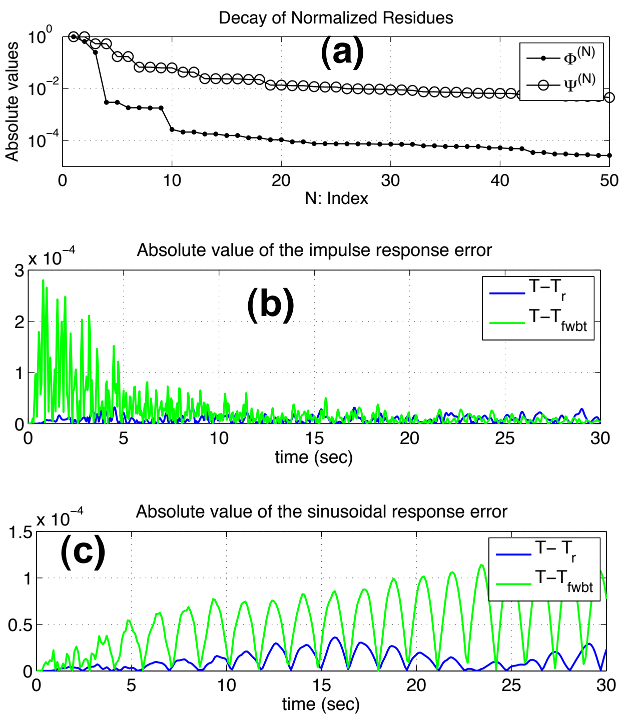

Fig. 2. (a) Decay of the normalized residues (b)-(c) Comparison of **W-IRKA** and **FWBT** using closed-loop responses.

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5 Conclusions

We have presented new formulae for the weighted-$\mathcal{H}_2$ inner product and norm that explicitly reveal the contribution of poles and residues both of the full-order model and of the weight. One of the major consequences of this new representation are new interpolatory optimality conditions for weighted-$\mathcal{H}_2$ approximation. Based on derived weighted-$\mathcal{H}_2$ error expressions, we have introduced an approach for producing high-fidelity weighted-$\mathcal{H}_2$ reduced models. The effectiveness of this approach has been illustrated with two examples.

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