

# **Statistical Analysis of Electric Energy Markets with Large-Scale Renewable Generation Using Point Estimate Methods**

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## Abstract

The restructuring of the electric energy market and the proliferation of intermittent renewable-energy based power generation have introduced serious challenges to power system operation emanating from the uncertainties introduced to the system variables (electricity prices, congestion levels etc.). In order to economically operate the system and efficiently run the energy market, a statistical analysis of the system variables under uncertainty is needed. Such statistical analysis can be performed through an estimation of the statistical moments of these variables. In this thesis, the Point Estimate Methods (PEMs) are applied to the optimal power flow (OPF) problem to estimate the statistical moments of the locational marginal prices (LMPs) and total generation cost under system uncertainty. An extensive mathematical examination and risk analysis of existing PEMs are performed and a new PEM scheme is introduced. The applied PEMs consist of two schemes introduced by H.P. Hong, namely, the  $2n$  and  $2n + 1$  schemes, and a proposed combination between Hong's and M. E Harr's schemes. The accuracy of the applied PEMs in estimating the statistical moments of system LMPs is illustrated and the performance of the suggested combination of Harr's and Hong's PEMs is shown. Moreover, the risks of the application of Hong's  $2n$  scheme to the OPF problem are discussed by showing that it can potentially yield inaccurate LMP estimates or run into unfeasibility of the OPF problem. In addition, a new PEM configuration is also introduced. This configuration is derived from a PEM

introduced by E. Rosenblueth. It can accommodate asymmetry and correlation of input random variables in a more computationally efficient manner than its Rosenblueth's counterpart.

*To my beloved*  
*Father Jean Sanjab,*  
*Mother Afdokia Charbel*  
*and*  
*Brother Adon Sanjab*

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# **Chapter I: Introduction**

## **1.1. Motivation**

With the restructuring of the electric energy market and the proliferation of intermittent renewable-energy based power generation, new challenges have been introduced to the operation of the power system. These challenges emanate from the uncertainties introduced to the system variables, such as electricity prices, congestion levels etc. In order to operate the power system in an economical manner and efficiently run the energy market, statistical analyses of the uncertain system variables are needed. Such statistical analyses are based on the ability to estimate the statistical moments of these system variables. The objective of this thesis is to estimate the statistical moments of the system locational marginal prices (LMPs) and total generation cost under system uncertainty emanating from random generation, such as wind-based electrical energy and random loads.

In a nutshell, the problem of incorporating randomness in engineering models is as follows: Any engineering model relates output variables to input variables through a set of mathematical relations, which define the mathematical model. Hence, let us assume that a mathematical model is defined by the function  $\mathbf{H}(\mathbf{x})$  which takes a set of  $n$  input random variables  $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$  and generates a set of  $p$  output random variables  $\mathbf{y} = [y_1, y_2, \dots, y_p]^T$  such that  $\mathbf{y} = \mathbf{H}(\mathbf{x})$ . If each input variable,  $x_i$ , is random and each one of them follows a certain probability distribution, the output variables,  $\mathbf{y}$ , will also be random and will follow, most likely, different probability distributions. The complexity in this matter is to estimate the probability distributions of these output random variables. Throughout this thesis,  $\mathbf{H}(\mathbf{x})$  represents the optimal power flow (OPF) formulation, the input random variables are random wind power injection and random loads, and the output variables are the system LMPs and total generation cost. The most commonly used techniques in literature to estimate the statistics of output variables are Monte Carlo Simulation (MCS), analytical methods and approximate techniques [1-4]. Below, these techniques are briefly reviewed.

- a. Monte Carlo Simulation:

Monte Carlo Simulation is a widely used technique for dealing with system uncertainties in power system models [3, 5, 6]. Using MCS, a set of realizations of input random variables is generated and the deterministic solution routine is carried out to calculate a set of realizations of output variables. This process is repeated for a large number of iterations, where after each iteration, the statistical moments of the output variables are estimated. When these moments converge, the process is terminated [3, 7]. One of the main advantages of MCS is that it uses the deterministic routines to solve the probabilistic problems. However, MCS suffers from two main drawbacks. First, the process is computationally intense [1]. Second, it requires a full knowledge of the probability distributions of input random variables, which might not be always available [4]. MCS is usually used as a benchmark to test the performance of other methods [8].

#### b. Analytical Methods

Analytical methods are also used in literature to analytically obtain the expression of the probability distribution of output random variables from the distribution of input random variables. However, the mathematical relation between input variables and output variables might be subject to different equality and inequality constraints, as in the case of optimization problems, making the derivation of the expression of output variables distributions in terms of input variables distributions highly complex. Analytical methods usually include some mathematical simplifications to mitigate the mathematical complexity of the problem [1, 9].

#### c. Approximate Techniques

Approximate techniques estimate the statistical moments of the output variables using the statistical moments of the input variables. Many popular approximate techniques are used in literature [12-14]; however, the most widely used method amongst them is the point estimate method (PEM).

The PEM is introduced by Emilio Rosenblueth in [15] where the case of symmetric random variables is considered. Rosenblueth extends the proposed PEM to encompass asymmetric random variables in [16]. The PEM generates input variables samples, for which the solution routine is run, to generate output variables samples. However, each sample is given a specific weight such that the number of samples needed is much smaller than that of MCS [15, 16]. In a more elaborate way, the mathematical model is  $\mathbf{y} = \mathbf{H}(\mathbf{x})$  and the first few moments of the input

random variables,  $\mathbf{x}$ , are assumed to be known. The aim is to estimate the first few moments of the output variables  $\mathbf{y}$ . The PEM is composed of three steps:

1. The input random variables  $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$  are replaced by  $m$  points, known as probability concentrations, having appropriate weights,  $p_i$ , and locations  $\mathbf{x}_i = [x_{1,i}, x_{2,i}, \dots, x_{n,i}]^T$  for  $i = 1, \dots, m$ , such that the first few moments (mean, variance, skewness, kurtosis, etc.) of these weighted points match the first few moments of  $\mathbf{x}$ .
2. The deterministic solution routine,  $\mathbf{y}_i = H(\mathbf{x}_i)$ , is run for the  $m$  concentrations to generate a set of outputs  $\mathbf{y}_i = [y_{1,i}, y_{2,i}, \dots, y_{p,i}]^T$  for  $i = 1, \dots, m$ .
3. Each set of generated outputs  $\mathbf{y}_i$  has a weight  $p_i$ . Using these weighted realizations of the output random variables the output variables' statistical moments can be estimated.

M.E. Harr introduced another modified version of Rosenblueth PEM in [17]. The number of probability concentrations needed is less than that needed by Rosenblueth's method. The major drawback of Harr's method is that it only considers symmetrical input random variables. H.P Hong also introduced a modified version of Rosenblueth PEM in [18]. Hong's method also requires a lesser number of probability concentrations, as compared to Rosenblueth's method. However, Hong's method only considers uncorrelated input random variables. The main advantages of using point estimate methods can be summarized as follows:

1. PEMs use the deterministic routines for problem solutions, as in the case of MCS, but with low computational burden.
2. The mathematical complexity involved is much lower than that of analytical methods. In fact, there is no need to mathematically model the direct relation between the output variables and the input variables as is the case of analytical methods.
3. PEMs do not require a full knowledge of the probability distributions of the input random variables which might not be always available. In fact, they only require the knowledge of the first few moments of the input random variables.

In this thesis, PEMs are applied to the OPF problem to estimate the statistical moments of the system LMPs and total generation cost under system uncertainty. A literature review of LMPs statistical analysis is presented next.

## **1.2. Literature Review**

Many researchers have tried various methods for the estimation of LMPs statistical moments under system uncertainty. For instance, forecasting LMPs using analytical methods is proposed in [10] and [11] considering load uncertainty. In [19], the authors perform a statistical analysis of system LMPs under uncertainties introduced by wind energy in a competitive energy market environment using a Monte Carlo type of simulations. The authors use variance reduction techniques to reduce the entailed computational burden. In [20-22] the authors employ a point estimate method to the OPF problem. [20] employs Hong's  $2n$  PEM scheme to target the effects of the uncertainty in market behavior participants on the system LMPs. [21] employs Hong's  $2n + 1$  PEM scheme to the OPF problem incorporating random wind speeds. [22] investigates the optimal employment of storage units in the presence of wind energy using Hong's  $2n$  PEM scheme. The PEMs have gained a wide range of applications in power system research. A more detailed literature review of the application of PEMs in power system problems is provided in Chapter II.

In the following section, the contributions of this thesis are presented.

## **1.3. Contributions**

The contributions of this thesis are summarized in the following three subsections. A full description of these contributions and relevant explanations are available further in the thesis.

### **1.3.1. Statistical Analysis of LMPs Using PEMs**

In this thesis, the PEMs are applied to the OPF problem with a purpose of estimating the statistical moments of the system LMPs and total generation cost under uncertainty introduced by wind energy and random loads. PEMs have been applied to the OPF problem through [20-22]. However, the applied PEMs are restricted to the schemes introduced by H.P Hong. This thesis applies various PEMs with a purpose of showing the accuracy of the PEMs in estimating the statistical moments of the LMPs. The cases studied include systems with different wind integration levels and the tested PEM schemes consist of:

- a. Hong's  $2n$  PEM scheme.

- b. Hong's  $2n + 1$  PEM scheme.
- c. Hong's  $2n$  scheme with de-correlation.
- d. Hong's  $2n + 1$  scheme with de-correlation.
- e. Proposed combination of Harr and Hong's  $2n$  PEMs.
- f. Proposed combination of Harr and Hong's  $2n + 1$  PEMs.

The accuracy of the applied PEMs is analyzed by comparing the results to the results of Monte Carlo simulation. The results show that the proposed combination of Hong's and Harr's scheme can display a better performance than Hong's scheme, which is abundantly applied in literature.

### ***1.3.2. Mathematical Examination and Risk Analysis of PEMs***

PEMs are usually applied in power system literature without a full explanation of their mathematical background, restrictions and risks. In fact, not all PEMs are equivalent and some of them only apply to specific kind of problems but not to others. Moreover, some PEM schemes can, for some system conditions, lead completely inaccurate results. This thesis presents a full mathematical examination of the restrictions and applicability of widely used PEMs, namely, Rosenblueth's, Hong's and Harr's schemes. It also presents a complete derivation of mathematical proofs on which the work in the original papers where these PEMs are introduced [16-18] is based. Moreover, potential risks of applying Hong's  $2n$  scheme are investigated by showing that this scheme might yield inaccurate results or even cause non-feasibility in the OPF problem.

### ***1.3.3. Introduction of New Modified PEMs***

In this thesis two new PEM configurations are introduced. The first configuration consists of a combination between Harr's and Hong's PEMs. This scheme benefits from the advantages of each of these two schemes and can be applied to problems with correlated and uncorrelated input random variables. This scheme is tested and compared to other schemes. The second configuration is a modified version of Rosenblueth's PEM. It can accommodate various kinds of input random variables but uses a smaller number of probability concentrations as compared to

Rosenblueth's scheme. The second proposed PEM is not tested for the OPF problem since it still requires further analysis before being applied.

#### ***1.4. Chapter Organization***

This thesis is organized as follows: Chapter II presents an extensive literature review of the application of PEMs in power systems. Chapter III provides a review of the probabilistic optimal power flow problem. Chapter IV introduces Rosenblueth's and Hong's PEMs for systems with one input random variable. Chapter V introduces Rosenblueth's, Hong's and Harr's PEMs and the proposed combination of Hong's and Harr's PEMs for systems with multiple input random variables. Chapter VI presents the simulation results of the application of the PEM to the OPF problem. Chapter VII introduces the proposed modified PEM scheme. The conclusions and proposed future work are presented in Chapter VIII.



## ***Chapter II: Literature Review***

### ***2.1. Introduction***

This chapter presents a literature review of the implementation of point estimate methods in power systems. Through this review, one can see that the PEMs are applied to a variety of power system topics.

### ***2.2. Application of Point Estimate Methods in Power Systems***

With the increasing interest in modeling uncertainties in power system analysis the PEMs have gained a wide range of applications in power system research. [20-22] apply point estimate methods to the probabilistic optimal power flow (P-OPF) problem. [20] targets the effects of the uncertainty in market behavior participants on the system LMPs. In fact, the economic stress on market participants in electricity markets makes their market behavior uncertain. This uncertain behavior results in uncertainty in the LMPs. This paper applies Hong's  $2n$  PEM scheme to estimate the statistical properties of the LMPs by solving the P-OPF in the presence of market participants' uncertain bids and loads which are assumed to follow a normal probability distribution. The used PEM estimates the first three moments of the system LMPs through solving the OPF  $2n$  times for  $n$  input random variables. The results are compared to those obtained using MCS method. [21] employs Hong's  $2n + 1$  PEM scheme to estimate the first two statistical moments of the system voltages, branches power flows and wind power injections. The corresponding output variables' probability distributions are assumed to be Gaussian with parameters equal to the derived statistical moments. The accuracy of results is provided through a comparison with MCS. [22] investigates the optimal employment of storage units in the presence of wind energy. The probabilistic natures of wind and load are modeled through their corresponding probability distribution functions (p.d.f.) and the storage is modeled as either an uncertain load or generation. Hong's  $2n$  PEM is applied to the P-OPF in the presence of the mentioned uncertainties. [1], [4], and [23-25] apply a PEM technique to the Probabilistic Power Flow (PPF). [1] tests the application of four of Hong's PEM schemes ( $2n, 2n + 1, 3n$  and  $4n + 1$ ) to the PPF problem under load and generation uncertainty. [23] treats the PPF problem where the power injections and line parameters are modeled as random variables. Assuming the knowledge of the statistical properties of the random variables involved, the author estimates the

statistical properties of the power flow solution variables. A  $2n$  Hong's PEM scheme is used, and the results are compared to those obtained using MCS method. In [4], the probabilistic load flow in the presence of wind uncertainty is considered. The wind p.d.f. is not known and is not approximated. Only some of the wind statistics, showcased through the wind histogram, are extracted from wind data. The lack of knowledge of the full p.d.f. of wind speeds necessitates the use of the PEM. A  $2n + 1$  discrete PEM is employed to solve the PPF in the presence of wind energy. [24] deals with the PPF problem in the presence of uncertainties, resulting from uncertain loads and intermittent renewable energy resources connected to the grid, with a focus on reactive power violation at each generator bus. Hong's PEM is used to study the statistics of the uncertainties involved using the 3-point, 5-point and 7-point schemes. [25] deals with the study of the PPF problem including the uncertainty resulting from the high penetration level of wind energy. The loads are assumed to follow a normal distribution and the p.d.f. of wind speed is estimated using "short-term power prediction" through which wind speeds are modeled as correlated beta distributions. Hong's  $2n + 1$  PEM is used to calculate the mean of the distribution of the power flow in each branch which is dependent on uncertain power injections whose probability density functions have been forecasted. [26] applies Hong's PEM using the  $2n$ ,  $2n + 1$  and  $4n + 1$  schemes to deal with system uncertainties when performing a 3-phase load flow analysis of an unbalanced system. The uncertainties, resulting from variations in loads and grid configuration are considered and the results are compared to the ones obtained using MCS. [27] employs Hong's  $2n$  PEM scheme to estimate the statistics of the transfer capability under uncertainties in system parameters and power injections. The transfer capability is a function of many factors such as network parameters, generation and demand. The uncertainty in these parameters induces uncertainty in the value of transfer capability. Safety margins are considered in order to mitigate the risks of these uncertainties on the reliable operation of the system. The North American Electric Reliability Council (NERC) defines the Transfer Reliability Margin (TRM) to be a transmission capability safety margin to ensure a secure operation of the power system under transfer capability uncertainty. In other words TRM is defined as a volume of transfer capability that is reserved to cover the uncertainties in transfer capability calculations. To ensure reliability, the Transfer Reliability Margin should exceed the maximum "uncertainty" of the calculated transfer capability (resulting from uncertainty in system parameters). [27] defines a safety margin, which is the difference between the allocated

reliability margin and the transfer capability uncertainty. The safety margin is mainly used to assess and assign an adequate TRM. The probability of unreliability is the probability of having a negative safety margin. The results obtained through the use of the two PEM are compared to those obtained using MCS and the truncated Taylor series expansion.

[28] employs pattern recognition as a direct method for transient stability analysis. Rosenblueth's PEM is used as a way of calculating the statistics that are needed for the classification procedure of the pattern recognition method under study. A statistical analysis is needed because when a disturbance occurs, the initial accelerations of the generators in the system are assumed to be random variables. The PEM is used to study the statistics of the mechanical power of a generator which is a random variable by virtue of its dependence on the rotor acceleration of the machine which is also a random variable. [29] studies the power system stability problem in the presence of system uncertainties resulting from randomness in system parameters and operating conditions. The PEM is used as an estimation tool of the statistics of the random variables involved. Hong's  $2n$  PEM is used to estimate the statistics of the system eigenvalues. After estimating the mean and variance of the real part of the eigenvalues, and assuming that the real part follows a normal distribution, the probability of having a negative real part, i.e. probability of having a stable system, can be calculated.

[30, 31] employ a PEM technique for the solution of the probabilistic distribution feeder reconfiguration problem. [30] employs Hong's  $2n + 1$  PEM for the solution of a multi-objective distribution reconfiguration problem including uncertainties resulting from wind energy and loads. In fact, Hong's PEM is used to solve for the PPF under the mentioned system uncertainties. The multi-objective optimization problem objective functions include the total lost energy as well as the cost of the generated energy in addition to the total volume of emissions generated and the deviation in the bus voltages. In [31] the uncertainties in load and wind forecasts are represented using Hong's  $2n$  PEM scheme. The authors' purpose is to solve a distribution feeder reconfiguration problem which aims at reconfiguring a new system topology to reach an optimal configuration. The wind speeds are assumed to follow a Weibull distribution.

[32, 33] employ a PEM technique for the optimal allocation of shunt capacitors. Shunt capacitors are introduced to the system for many reasons which include decreasing the needless reactive power movement through the system, which results in decreasing the power loss in the

system and reducing the operational burden on the system's equipment, and enhancing the system's voltage profile. The optimal placement of shunt capacitors is an optimization problem solved to determine the optimal location of the capacitors as well as their quantity and capacity. This optimization problem needs also to deal with system uncertainties that result, for instance, from load forecasting errors and cost coefficient variation. These uncertainties are considered in [32] where Hong's  $2n$  PEM is used for the optimal capacitor placement problem. The optimization problem is a multi-objective problem aiming at minimizing the cost of total capacitor investment, total power losses cost, the expected power loss and voltage deviations at buses while the constraints include the power flow equations, commercially available capacitor sizes and voltage limits at the system buses. In [33], the probabilistic optimal capacitor placement problem is solved through reducing the optimization problem into a corresponding single-objective problem and using Hong's  $2n + 1$  PEM. The authors treat the case of an unbalanced 3-phase distribution system including uncertainties in load forecast.

[34] uses a PEM to incorporate system uncertainties in the price-based unit commitment problem used for an optimal bidding in the day-ahead market of the Virtual Power Plant (VPP). The studied uncertainties result from the integration of distributed generation, storage and uncertain load. The uncertainty in the power output of a distributed generation unit is high; hence, the offer price that the owner of these renewable resources will submit in a competitive market will be low because of the high risk incurred of not meeting the expectation. This discourages the owners of renewable energy generators from entering the competitive market. To mitigate the risk, a VPP is a combination of many distributed generators as well as storage units and flexible loads. The VPP can have a better profile and better incentives to enter the market with lower risks due to its generation profile robustness. In [34], the input random variables are assumed to be uncorrelated and Hong's  $2n$  PEM scheme is used.

[35, 36] employ a PEM scheme to estimate the correlation between different measurements. As stated in [35], phase voltages and currents are measured at substations. From these measurements, the power flow and power injections are calculated. These measurements are prone to errors that are usually modeled as independent Gaussian-distributed random variables. However, since the calculations of the power injections and power flows depend on the measurements, the errors in these quantities are not usually independent. This paper uses Hong's

PEM to estimate the correlation matrix that explores the dependency between the processed measurements. Most of the traditional state estimators use weighted least squares (WLS) to solve the state estimation problem. During the estimation process masking effects of the errors might occur subsequently preventing the bad data detection and elimination [36-38]. [36] proposes a simultaneous state estimation and bad data detection procedure through a reweighting process and considering measurements dependency in order to address the masking effects. The PEM is used to estimate the covariance matrix of these measurements.

### **2.3. Summary**

The increasing integration of intermittent generation has led to a growing uncertainty in the power system. For an economic and secure operation of the system, the statistical effects of this uncertainty need to be well investigated. Monte Carlo simulation is widely used in literature for this purpose. However, its application to real time power system operation can lack practicality because of the high computational burden it entails. Analytical methods are also widely used techniques in literature. However, the complexity of the models used in system operation makes the task of mathematically deriving the expressions of the probability distributions of output random variables highly challenging. The Point Estimate Method (PEM) is a computationally efficient technique to estimate the statistical properties of the system variables. Its computational efficiency and mathematical non-complexity has led to a wide application of the PEM in a variety of power system domains.

This thesis applies the PEM to the OPF problem to estimate the statistical properties of its output variables in the presence of uncertainty brought in by a high level of renewable generation. The OPF problem is widely used in power system operation to achieve an optimal dispatch of the system while respecting its security and operational limits. The next chapter provides a review of the fundamentals of the OPF problem.

## ***Chapter III: Review of Probabilistic Optimal Power Flow***

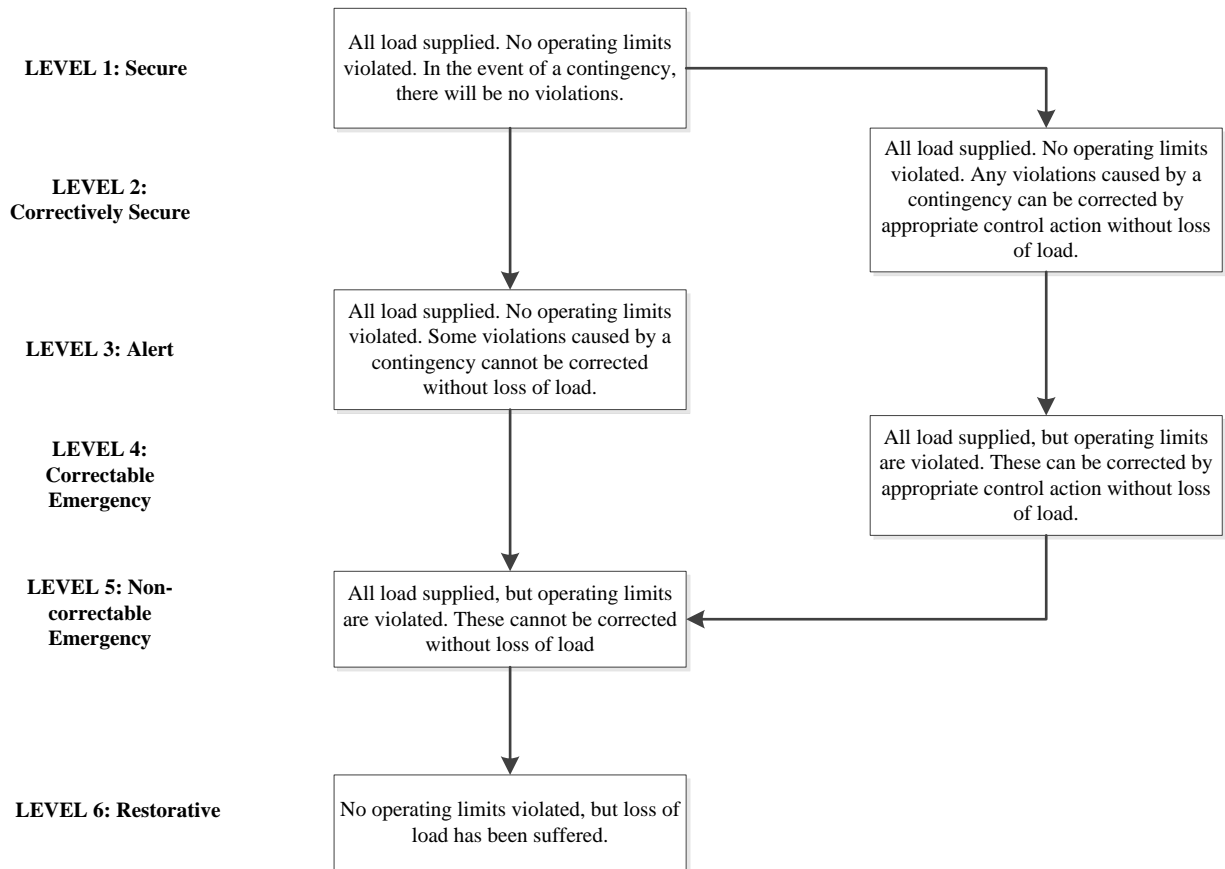
### ***3.1. Introduction***

In this chapter, a review of the P-OPF problem is introduced. First, an analysis of the original AC optimal power flow (AC OPF) problem formulation is given followed by the simplified DC optimal power flow (DC OPF) formulation discussing the differences between the two models. A quick review of the solution algorithms of the OPF problem is also included. Randomness is also considered in the OPF formulation to introduce the P-OPF.

The two terminologies, Probabilistic OPF and OPF, are highly interchangeable in literature. They mainly refer to the same formulation. The only difference between the two is that the P-OPF formulation has decision variables that are treated as random variables. Throughout this thesis, the probabilistic optimal power flow is used; however, it is referred to it as an optimal power flow with uncertain input variables. Hence, when optimal power flow is used throughout this thesis, it refers to the probabilistic optimal power flow.

Power system operation includes two usually conflicting objectives: optimality and security. Optimality is of great importance because of the economic benefits it provides and security is crucial to maintain the supply of electric power. A classification of static power system security levels is shown in Figure 3.1 where the transitions between different levels (represented through arrows) occur due to contingencies [51].

This thesis is focused on the optimality aspect of power systems. However, all the simulations and analyses performed can also be extended to models with higher security levels.



**Figure 3.1.** Power System Static Security Levels,

B. Stott, O. Alsac, and A.J. Monticelli, “Security analysis and optimization”, *Proceedings of the IEEE*, vol. 75, no. 12, pp. 1623-1644, Dec. 1987, Used under fair use, 2014.

### **3.2. AC Optimal Power Flow**

The AC OPF problem is first introduced by Carpentier in 1962 in the paper entitled “Contribution a l’Etude du Dispatching Economique” (English translation: Contributions to the Economic Dispatch Study) [39, 48]. This problem is elaborated on and completely formulated by Hermann Dommel and William Tinney in 1968 in [40]. Nowadays, the OPF problem is “one of the fundamental classes of static power system network calculations”. It is indeed at the heart of power system planning, operations and control as well as energy markets [41].

### 3.2.1. Problem Formulation

As defined by Carpentier, the OPF is the “determination of the complete state of a power system corresponding to the best operation within security constraints” [47]. The OPF is mainly an upgraded version of the economic dispatch problem including the network constraints, the security limits and power flow equations to the optimization formulation [39].

The OPF problem is an optimization problem aiming at minimizing an objective function subject to a set of equality and inequality constraints. The objective function is most commonly the total generation cost of the system which is the summation of the generation costs of all the generators contributing to the system. The objective function can also take some other forms as the system’s environmental cost function, total electric losses, etc. However, generation cost reduction is the most common objective of the OPF problem [39, 47]. The OPF total generation cost objective function is usually modeled by a polynomial (mainly a second or a third order) or by piecewise linear functions. The equality constraints of the OPF problem are mainly the power flow equations relating the power injections at each of the buses to the system states (i.e. complex bus voltages) [42]. The inequality constraints reflect the security limits of the system. They include, for instance, generators’ real and reactive power limits, bus voltages’ limits, line flow limits as well as transformer tap ratio limits, etc. These limits reflect the level of security of the system. When a higher level of security is needed some of these limits (most commonly the line flows and voltage magnitudes) should also apply for an  $n - 1$  or sometimes  $n - k$  security conditions. This formulation is referred to as security constrained optimal power flow (SCOPF) and dictates that post-contingency system conditions (line flows, voltages, etc.) remain within security limits [39, 47]. Increasing the security level of the system makes the optimization problem more constrained. An OPF problem is said to be “over-constrained” when the feasible region of the optimization problem is empty. In other words, the optimization problem does not have any feasible solution that meets all the set of constraints [47].

The most common OPF problem is mathematically formulated as follows [40]:

$$\text{Minimize } Total\ Generation\ Cost = \sum_{i=1}^n F_i(P_{G_i})$$

Subject to:



$$P_i(V, \theta) - P_{Net,i} = 0 \text{ for } i = 1, \dots, N \quad (3.1)$$

$$Q_i(V, \theta) - Q_{Net,i} = 0 \text{ for } i = 1, \dots, N \quad (3.2)$$

$$\mathbf{u}^{min} \leq \mathbf{u} \leq \mathbf{u}^{max} \quad (3.3)$$

$$h(\mathbf{x}, \mathbf{u}) \leq \mathbf{0} \quad (3.4)$$

where  $N$  is the number of buses in the system.

(3.1) and (3.2) represent the OPF equality constraints following the power flow equations which can be defined as follows:

$$V_i e^{-j\theta_i} \sum_{m=1}^N (G_{im} + jB_{im}) V_m e^{j\theta_m} = P_{Net,i} - jQ_{Net,i} \text{ for } i = 1, \dots, N \quad (3.5)$$

while  $P_{Net,i}$  and  $Q_{Net,i}$  are defined as follows:

$$P_{Net,i} = P_{G_i} - P_{L_i} \text{ for } i = 1, \dots, N \quad (3.6)$$

$$Q_{Net,i} = Q_{G_i} - Q_{L_i} \text{ for } i = 1, \dots, N \quad (3.7)$$

where,

$V_i$  is the voltage magnitude at bus  $i$

$\theta_i$  is the voltage angle at bus  $i$

$G_{im} + jB_{im}$  is the corresponding element of the admittance matrix

$P_{Net,i}$  and  $Q_{Net,i}$  are the net real and reactive power entering bus number  $i$ .

$P_{G_i}$  and  $Q_{G_i}$  are controllable real and reactive power entering bus  $i$  (usually generation)

$P_{L_i}$  and  $Q_{L_i}$  are the real and reactive power leaving bus  $i$  (usually load)

(3.3) shows the limits on control parameters such as voltage magnitudes at generation buses, transformer tap ratios, power output level of generators, etc.

(3.4) represents functional inequality constraints, which are limits on variables that depend on the control parameters  $\mathbf{u}$  such as upper and lower limits on voltage magnitude at load buses, limits on line flows, etc.

As previously mentioned, some of those constraints can also be included for  $n - 1$  security operation to reinforce the security of the generated solution. The OPF problem can also include binary variables reflecting the connection of shunt capacitors or inductors to the system.

One of the main outcomes of the OPF problem, which is heavily used in energy markets settlements, is the locational marginal prices. The LMPs are the incremental cost of electric energy at each node of the system, which reflects the sensitivity of the total cost to a change in load at a specific bus. From an optimization point of view the LMPs are the Lagrange multipliers of the real power flow equality constraints given in equation (3.1) [50].

### **3.2.2. Solution Algorithms**

By inspecting the OPF problem formulation, one can see that it is a non-linear programming problem by virtue of its non-linear constraints and objective function. Thus, non-linear programming (NLP) solution algorithms are usually used to solve the OPF problem. Moreover the OPF problem is a non-convex programming problem which makes its optimal solution harder to find [39, 41].

Over the years, many algorithms have been developed to solve the OPF problem. As a matter of fact, by formulating the problem as a NLP program, all well-known NLP algorithms and techniques have been implemented aiming to solve it [41, 43]. An extensive explanation and investigation of well-known NLP algorithms can be found in [46]. In the first paper in which the OPF problem is formulated [40], an iterative process is employed, where Newton's method is used to find a feasible power flow solution, and the gradient method (method of steepest descent) is used to minimize the objective function satisfying control variables inequality constraints. Functional inequality constraints are treated as soft constraints and are introduced to the objective function following the penalty method.

Linear programming techniques are also proposed to solve the OPF problem. These techniques require linearization of the OPF formulation. This linearized formulation is known as LP OPF [39]. [39] presents an overview of various applied algorithms and techniques for the solution of the OPF problem.

The OPF problem is considered to be one of the most challenging problems in power systems. It is indeed fundamental to the economic efficiency and reliability of the power system [41]. Because of the complexity of the original AC OPF problem the DC OPF is introduced. The DC OPF uses a DC power flow (DC PF) model instead of the AC power flow (AC PF) model in the OPF formulation [52]. Because of the potential saving in computation time that it introduces,

the DC OPF is widely used for large power systems operated by Regional Transmission Organizations (RTO) or Independent System Operators (ISO). This was illustrated through a survey conducted by the Federal Energy Regulatory Commission (FERC) in 2011 [45]. In this survey, a listing of the OPF solution approaches for various ISOs was published. The survey also pointed to the fact that using a DC OPF may result in suboptimal solutions [45]. RTOs usually apply a DC OPF model because the systems they operate can include thousands of lines and hundreds of variables that need to be controlled. For instance, PJM's network, which spans over fourteen states has around one thousand three hundred sixty five (1,365) interconnected generators with around six thousand (6,000) substations and fifty nine thousand and seven hundred fifty miles (59,750) of high voltage transmission lines [44].

Many DC power flow model versions are used in literature. Each of the applied versions is based on a set of simplifications and relies on a set of assumptions. [53] presents an extensive revision and analysis of various DC PF forms. The DC model is a MW-only power flow model that is computationally efficient because of its non-complexity and state-independence. It is widely used for applications where the effects of voltage levels and reactive power conditions are assumed to be minimal. The most classical DC power flow includes many approximations to the original AC power flow which are summarized as follows [52, 53]:

1. The branch losses are ignored.
2.  $\sin(\theta_i) \approx 0$  for  $i = 1, \dots, N$
3.  $V_i = 1$  for  $i = 1, \dots, N$

Through these approximations, the DC PF model is represented by a set of linear equations relating the buses real power injection to the bus voltage angles as shown in (3.8)

$$\mathbf{P} = \mathbf{B}\boldsymbol{\theta} \tag{3.8}$$

where  $\mathbf{P}$  and  $\boldsymbol{\theta}$  are respectively the bus real power injection vector and bus voltage angles vector while  $\mathbf{B}$  is the system's bus susceptance matrix. Such a linear set of equations has a unique solution that can be obtained without the need for iterations as follows:

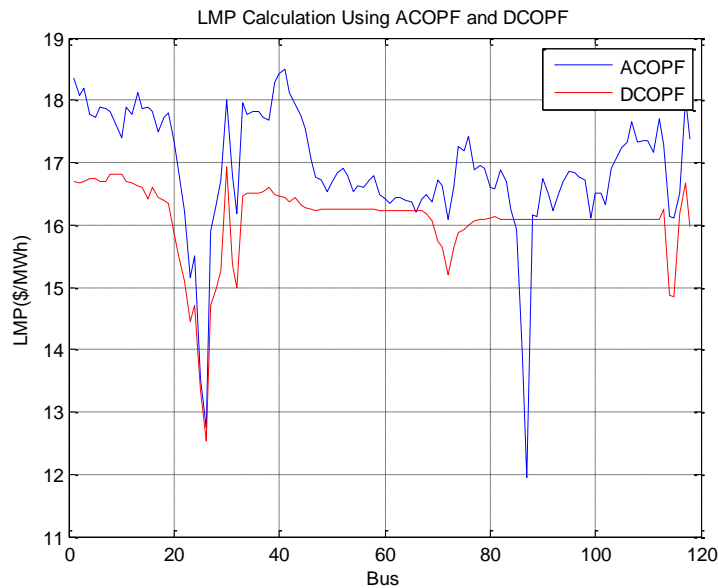
$$\boldsymbol{\theta} = \mathbf{B}^{-1}\mathbf{P} \tag{3.9}$$

The advantages of the DC PF over the AC PF are the followings:

1. Only real power injections equations are considered, which results in a smaller set of equations to solve.
2. No iterations are needed.
3. The susceptance matrix is state-independent and does not change unless the topology of the system changes.

The accuracy of the obtained DC PF results depends on the validity of the considered assumptions and the operating condition of the system [53].

The accuracy of using DC OPF instead of the original AC OPF for calculating LMPs is thoroughly discussed in [49]. The aforementioned reference investigates the difference in accuracy between the DC and AC OPF models in calculating the LMPs. It states that such approximation accuracy depends on the network under study. Most of our simulations throughout our research work are performed on the IEEE 118 – bus network. We have run an AC OPF and DC OPF for this system and compared the generated LMPs using the two models. The results are shown in Figure 3.2. This comparison was performed in order to investigate the accuracy of the DC OPF in calculating the system LMPs for the IEEE 118 – bus network.



**Figure 3.2.** LMPs Generated Using AC and DC OPF

Figure 3.2 shows that for the studied IEEE 118 – bus system, the DC OPF does not generate accurate LMPs as compared to the AC OPF. In fact, the average error introduced by the DC OPF

for LMPs calculation is equal to 6.74% and the maximum difference occurs for the LMP at bus 87 where the LMP generated using DC OPF differs from the LMP generated using AC OPF by 35%. Hence, the DC OPF formulation resulted in inaccurate LMPs calculation for the treated network, which necessitates the employment of the AC OPF formulation.

### ***3.3. Probabilistic Optimal Power Flow***

Uncertainties are naturally present in any engineering system which may result from either inaccuracy in the used model or from likely randomness of natural phenomena. Such uncertainties should be accounted for in system modeling to yield reliable results and accurate decision making.

The power flow and OPF models are originally designed as deterministic. In other words, all the variables included in the system are assumed to have a non-probabilistic nature. However, loads, generation, line parameters as well as network topology are all subject to uncertainties that may result from inaccurate forecasting, imprecise measurements, reliability characteristics or natural intermittency. Moreover, with the proliferation of renewable energy integration into the grid driven by newly developed energy policies, uncertainties' effects on the system became even greater. To cope with this probabilistic behavior of system variables, the probabilistic power flow is first introduced in [54]. In the probabilistic power flow model, the state variables as well as the output network quantities are modeled as random variables. A stochastic formulation is also first introduced to the optimal energy dispatch problem in [55]. [55] treats the optimal energy dispatch problem with uncertainties resulting from forecasting errors, measurements impreciseness and system configuration errors and studies the effects of such uncertainties on system operation and planning. A probabilistic optimal power flow model is also introduced in [12] where random variables are introduced to the original OPF formulation to accommodate system uncertainties. Hence, a probabilistic OPF problem has power injections, loads, system states, line flows, electric losses, LMPs, generation costs and currents modeled as random variables.

When introducing random variables to the OPF formulation, many widely used techniques referred to in Chapter I.1.1 can be applied to solve for the probabilistic OPF and to estimate the

probabilistic behavior of its output variables. These techniques include Monte Carlo simulation, analytical methods and approximate techniques.

Monte Carlo simulation is a widely used technique for dealing with system uncertainties in power system models [3, 5, 6]. Using MCS, a set of realizations of input random variables is generated and the deterministic solution routine is carried out to calculate a set of realizations of output variables. This process is repeated for a large number of iterations, where after each iteration, the statistical moments of the output variables are estimated. When these moments converge, the process is terminated [3, 7]. MCS is for instance applied to the OPF problem in [19], where the authors perform a statistical analysis of system LMPs under uncertainties introduced by wind energy in a competitive energy market environment. The main drawback behind Monte Carlo simulation is the large number of iterations it requires which might not be practical for large systems.

Analytical methods are also used in literature to analytically obtain the expression of the probability distribution of output random variables from the distribution of input random variables. However, the mathematical relation between input variables and output variables might be subject to different equality and inequality constraints, as in the case of the OPF problem. Thus, the derivation of the expression of output variables distributions in terms of input variables distributions can be highly complex. Analytical methods usually include some mathematical simplifications to mitigate the mathematical complexity of the problem [1, 9]. Analytical methods are, for instance, applied to the OPF problem in [10, 11] to forecast the LMPs considering load uncertainty.

Approximate techniques estimate the statistical moments of the output variables using the statistical moments of the input variables. Many well-known approximate techniques are used in literature [12-14]; however, the most widely used method amongst them is the point estimate method. The PEM serves to estimate the first few statistical moments of the output variables using the first few statistical moments of the input variables. In [20-22] the authors employ a point estimate method to the OPF problem. [20] employs Hong's  $2n$  PEM scheme to target the effects of the uncertainty in market behavior participants on the system LMPs. [21] employs Hong's  $2n + 1$  PEM scheme to the OPF problem incorporating random wind speeds. [22] investigates the optimal employment of storage units in the presence of wind energy using

Hong's  $2n$  PEM scheme. One of the main advantages of the PEM is that it requires a much smaller number of iterations than Monte Carlo simulation.

Throughout this thesis, various PEMs are applied to the OPF problem to estimate the statistical moments of the system LMPs and total generation cost.

### **3.4. Summary**

In this chapter, a review of the AC OPF and DC OPF formulations are presented including a discussion of the differences between the two formulations. Solution algorithms widely used for the OPF problem are also introduced and discussed. This chapter also illustrates the incorporation of randomness in the OPF formulation to form the P-OPF model. In the next chapter, a review of point estimate methods for problems with one input random variable is presented with the relevant mathematical derivations and analyses.

## ***Chapter IV: Point Estimate Method – Univariate Case***

### ***4.1. Introduction***

The point estimate method is used to estimate the first few moments of a random variable which is dependent, through a mathematical model, on other random variables whose corresponding first moments are known.

This chapter includes a full explanation of the point estimate methods introduced by Rosenblueth in [15] and [16] and by Hong in [18]. This chapter considers the univariate case where the output random variable depends only on one input random variable. In this chapter, all the mathematical proofs needed for the development of these methods are fully derived. These mathematical proofs form a basis of the work presented in [15, 16, 18].

Through the point estimate methods, the distributions of the input random variables are replaced by a number of probability concentrations at specific locations such that the first few moments of these probability concentrations match, up to a certain order, the first few moments of the input random variables. Those probability concentrations are used by the system model as inputs to generate output variables realizations. Using these output realizations and their corresponding weights the statistical moments of the output random variables can be estimated. This process is based on the assumption that the first few moments of the input random variables are known. Rosenblueth introduces a two point estimate method (2PEM) where each input random variable,  $x$ , is replaced by two probability concentrations,  $p_1$  and  $p_2$ , at two locations  $x_1$  and  $x_2$ . Hong studies a more general case using Taylor series expansion. According to Hong's method, an input random variable is replaced by a set of probability concentrations  $p_i$  at specific locations  $x_i$  such that the expected value of the Taylor series expansion of the output variable around the mean value of the input variable matches the weighted average (where the weights are  $p_i$ ) of the Taylor series expansion of the output variable around the mean value of the input variable evaluated at the generated probability concentrations locations (i.e.  $x_i$ ).

The application of the point estimate methods to the multivariate case is introduced and studies in the next chapter.



## 4.2. Review of Emilio Rosenblueth's PEM Approach

Emilio Rosenblueth's PEM is introduced in [15] and [16]. A full description of Rosenblueth's method (for the univariate case) including all the relevant mathematical proofs and explanations is presented next.

Let  $Y = h(X)$  where  $X$  is a random variable. We assume that the moments of  $X$  are known.

The moment of order  $i$  of a random variable  $X$  is defined as follows:

$$M_i(X) = \int_{-\infty}^{+\infty} x^i p_X(x) dx \quad (4.1)$$

$p_X(x)$  is the p.d.f. of  $X$  at the realization  $x$ .

The central moment of order  $i$  of a random variable  $X$  is defined as follows:

$$M'_i(X) = \int_{-\infty}^{+\infty} (x - \bar{X})^i p_X(x) dx \quad (4.2)$$

where,  $\bar{X}$  is the mean (expectation) of  $X$ .

The order zero central moment of  $X$  is:

$$M'_0(X) = \int_{-\infty}^{+\infty} (x - \bar{X})^0 p_X(x) dx = \int_{-\infty}^{+\infty} p_X(x) dx = 1 \quad (4.3)$$

The first order central moment of  $X$  is:

$$M'_1(X) = \int_{-\infty}^{+\infty} (x - \bar{X})^1 p_X(x) dx = \int_{-\infty}^{+\infty} x p_X(x) dx - \bar{X} \int_{-\infty}^{+\infty} p_X(x) dx = \bar{X} - \bar{X} = 0 \quad (4.4)$$

The second order central moment of  $X$  is:

$$M'_2(X) = \int_{-\infty}^{+\infty} (x - \bar{X})^2 p_X(x) dx = \sigma_X^2 \quad (4.5)$$

where,  $\sigma_X$  is the standard deviation of  $X$  and  $\sigma_X^2$  is the variance of  $X$ .

The third order central mean of  $X$  is:

$$M'_3(X) = \int_{-\infty}^{+\infty} (x - \bar{X})^3 p_X(x) dx = \nu \sigma_X^3 \quad (4.6)$$

$M'_3(X)$  is known to be the skewness of  $X$ .  $\nu$  is the skewness coefficient of  $X$  and is defined to be the ratio of the third central moment of  $X$  to the standard deviation of  $X$  raised to the power three

$$\nu = \frac{\mu_{3,X}}{\sigma_X^3} = \frac{M'_3(X)}{\sigma_X^3} \quad (4.7)$$

In the 2PEM, developed by Rosenblueth, the p.d.f. of  $X$  is estimated by two probability concentrations at two locations such that their first moments match those of the p.d.f. of  $X$ . For the  $m$  point estimate method, the p.d.f. of  $X$  is estimated by  $m$  probability concentrations at  $m$  locations such that their first central moments match those of the p.d.f. of  $X$ . The 2PEM case ( $m = 2$ ) is first considered. The estimate function of the p.d.f. of  $X$  is:

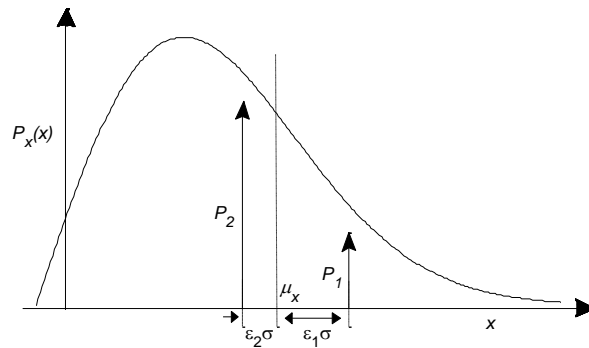
$$p_X(x) = p_1\delta(x - x_1) + p_2\delta(x - x_2) \quad (4.8)$$

$p_X(x)$  has two probability concentrations  $p_1$  and  $p_2$  located at  $x = x_1$  and  $x = x_2$  as shown in Figure 4.1.

We define the quantity  $\xi$  to be:

$$\xi_i = \frac{|x_i - \bar{X}|}{\sigma} \text{ for } i = 1, 2. \quad (4.9)$$

As previously mentioned, the estimates of  $X$  should have moments that match the real first moments of  $X$ . Here we are interested in matching up to the third moment.



**Figure 4.1.** Estimate of p.d.f. – 2PEM,

E. Rosenblueth, “Two-point estimates in probability”, *Appl. Math. Model.*, vol. 5, no. 5, pp. 329-335, Oct. 1981, Used under fair use, 2014.

The zeroth moment of  $X$  with  $p_X(x) = p_1\delta(x - x_1) + p_2\delta(x - x_2)$  is given by:

$$\int_{-\infty}^{+\infty} (x - \bar{X})^0 p_X(x) dx = \int_{-\infty}^{+\infty} [p_1 \delta(x - x_1) + p_2 \delta(x - x_2)] dx = p_1 + p_2 \quad (4.10)$$

The exact zeroth moment of  $X$  is equal to 1 as previously proven; hence, for the estimate to match the exact zeroth moment one should have:

$$p_1 + p_2 = 1 \quad (4.11)$$

The first moment of  $X$  with  $p_X(x) = p_1 \delta(x - x_1) + p_2 \delta(x - x_2)$  is given by:

$$\begin{aligned} \int_{-\infty}^{+\infty} (x - \bar{X})^1 p_X(x) dx &= \int_{-\infty}^{+\infty} (x - \bar{X})^1 (p_1 \delta(x - x_1) + p_2 \delta(x - x_2)) dx = \\ \int_{-\infty}^{+\infty} (x - \bar{X})^1 (p_1 \delta(x - x_1)) dx &+ \int_{-\infty}^{+\infty} (x - \bar{X})^1 (p_2 \delta(x - x_2)) dx = p_1(x_1 - \bar{X}) + \\ p_2(x_2 - \bar{X}) \end{aligned} \quad (4.12)$$

Knowing that  $\xi_i = \frac{|x_i - \bar{X}|}{\sigma}$ ; then,  $\xi_1 = \frac{x_1 - \bar{X}}{\sigma_X}$  and  $\xi_2 = \frac{\bar{X} - x_2}{\sigma_X}$ . This is due to the fact that  $x_1 \geq \bar{X}$  and  $x_2 \leq \bar{X}$  (refer to Figure 4.1).

Hence, the first moment of  $X$  with  $p_X(x) = p_1 \delta(x - x_1) + p_2 \delta(x - x_2)$  is:

$$p_1(x_1 - \bar{X}) + p_2(x_2 - \bar{X}) = p_1 \xi_1 \sigma_X - p_2 \xi_2 \sigma_X \quad (4.13)$$

The real (i.e. exact) first moment of  $X$  is equal to 0 as previously proven; hence, for the estimate to match the real first moment one should have:

$$p_1 \xi_1 \sigma_X - p_2 \xi_2 \sigma_X = 0 \text{ or } p_1 \xi_1 - p_2 \xi_2 = 0 \quad (4.14)$$

The second moment of  $X$  with  $p_X(x) = p_1 \delta(x - x_1) + p_2 \delta(x - x_2)$  is given by:

$$\begin{aligned} \int_{-\infty}^{+\infty} (x - \bar{X})^2 p_X(x) dx &= \int_{-\infty}^{+\infty} (x - \bar{X})^2 (p_1 \delta(x - x_1) + p_2 \delta(x - x_2)) dx = \\ \int_{-\infty}^{+\infty} (x - \bar{X})^2 (p_1 \delta(x - x_1)) dx &+ \int_{-\infty}^{+\infty} (x - \bar{X})^2 (p_2 \delta(x - x_2)) dx = p_1(x_1 - \bar{X})^2 + \\ p_2(x_2 - \bar{X})^2 \end{aligned} \quad (4.15)$$

Since  $\xi_1 = \frac{x_1 - \bar{X}}{\sigma_X}$  and  $\xi_2 = \frac{\bar{X} - x_2}{\sigma_X}$ , then the second moment of  $X$  with  $p_X(x) = p_1 \delta(x - x_1) + p_2 \delta(x - x_2)$  is:

$$p_1(x_1 - \bar{X})^2 + p_2(x_2 - \bar{X})^2 = p_1\xi_1^2\sigma_X^2 + p_2\xi_2^2\sigma_X^2 \quad (4.16)$$

The real second moment of  $X$  is equal to  $\sigma_X^2$  as previously proven; hence, for the estimate to match the real second moment one should have:

$$p_1\xi_1^2\sigma_X^2 + p_2\xi_2^2\sigma_X^2 = \sigma_X^2 \text{ or } p_1\xi_1^2 + p_2\xi_2^2 = 1 \quad (4.17)$$

The third moment of  $X$  with  $p_X(x) = p_1\delta(x - x_1) + p_2\delta(x - x_2)$  is given by:

$$\begin{aligned} \int_{-\infty}^{+\infty} (x - \bar{X})^3 p_X(x) dx &= \int_{-\infty}^{+\infty} (x - \bar{X})^3 (p_1\delta(x - x_1) + p_2\delta(x - x_2)) dx = \\ \int_{-\infty}^{+\infty} (x - \bar{X})^3 (p_1\delta(x - x_1)) dx &+ \int_{-\infty}^{+\infty} (x - \bar{X})^3 (p_2\delta(x - x_2)) dx = p_1(x_1 - \bar{X})^3 + \\ p_2(x_2 - \bar{X})^3 \end{aligned} \quad (4.18)$$

Since  $\xi_1 = \frac{x_1 - \bar{X}}{\sigma_X}$  and  $\xi_2 = \frac{\bar{X} - x_2}{\sigma_X}$  then, the third moment of  $X$  with  $p_X(x) = p_1\delta(x - x_1) + p_2\delta(x - x_2)$  is:

$$p_1(x_1 - \bar{X})^3 + p_2(x_2 - \bar{X})^3 = p_1\xi_1^3\sigma_X^3 - p_2\xi_2^3\sigma_X^3 \quad (4.19)$$

The real third moment of  $X$  is equal to  $\nu\sigma_X^3$  as previously proven; hence, for the estimate to match the real third moment one should have:

$$p_1\xi_1^3\sigma_X^3 - p_2\xi_2^3\sigma_X^3 = \nu\sigma_X^3 \text{ or } p_1\xi_1^3 - p_2\xi_2^3 = \nu \quad (4.20)$$

Hence, the four equations that need to be simultaneously solved for the weighted point estimates to match up to the third moment of  $X$  are:

$$p_1 + p_2 = 1 \quad (4.21)$$

$$p_1\xi_1 - p_2\xi_2 = 0 \quad (4.22)$$

$$p_1\xi_1^2 + p_2\xi_2^2 = 1 \quad (4.23)$$

$$p_1\xi_1^3 - p_2\xi_2^3 = \nu \quad (4.24)$$

Solving this set of four equations, (4.21) to (4.24), with four unknowns, namely,  $p_1, p_2, \xi_1$  and  $\xi_2$  generates the locations at which the two probability concentrations should be placed and their corresponding weights/probabilities. The solution is presented through equations (4.37) – (4.40).

The solution can be derived as follows:

$$(4.22) \text{ implies that } \xi_1^2 p_1^2 - \xi_2^2 p_2^2 = 0$$

$$(4.21) \text{ implies that } p_2 = 1 - p_1$$

Replacing  $p_2$  by  $1 - p_1$  in the other three equations one gets:

$$\xi_1^2 p_1^2 - \xi_2^2 (1 - p_1)^2 = 0 \rightarrow \xi_1^2 p_1^2 - \xi_2^2 (1 + p_1^2 - 2p_1) = 0 \quad (4.25)$$

$$\xi_1^2 p_1 + \xi_2^2 (1 - p_1) = 1 \quad (4.26)$$

$$\xi_1^3 p_1 - \xi_2^3 (1 - p_1) = \nu \quad (4.27)$$

Multiplying (4.26) by  $p_1$  results in:

$$\xi_1^2 p_1^2 + \xi_2^2 p_1 (1 - p_1) = p_1 \text{ which yields } \xi_1^2 p_1^2 + \xi_2^2 p_1 - \xi_2^2 p_1^2 = p_1 \quad (4.28)$$

Subtracting (4.25) from (4.28) yields:

$$-\xi_2^2 p_1 + \xi_2^2 = p_1 \Rightarrow \begin{cases} p_1 = \frac{\xi_2^2}{1 + \xi_2^2} \quad (I) \\ \xi_2^2 = \frac{p_1}{1 - p_1} \quad (II) \end{cases}$$

However, as shown in equation (4.22),  $p_1 \xi_1 = p_2 \xi_2$ ; then, replacing  $p_2$  by  $1 - p_1$  one gets:

$$p_1 \xi_1 = \xi_2 (1 - p_1) \Rightarrow \frac{\xi_1}{\xi_2} = \frac{(1 - p_1)}{p_1} \quad (4.29)$$

Replacing solution (II) in equation (4.29) yields:

$$\frac{\xi_1}{\xi_2} = \frac{(1 - p_1)}{p_1} = \frac{1}{\xi_2^2} \Rightarrow \xi_1 = \frac{1}{\xi_2} \quad (4.30)$$

Equation (4.24) implies that

$$\begin{aligned} \nu &= p_1 \xi_1^3 - p_2 \xi_2^3 = \xi_1^3 p_1 - \xi_2^3 (1 - p_1) \\ &\Rightarrow \frac{\nu}{p_1} = \xi_1^3 - \xi_2^3 \frac{1 - p_1}{p_1} \\ &\Rightarrow \frac{\nu}{p_1} = \xi_1^3 - \xi_2^3 \frac{1}{\xi_2^2} \end{aligned}$$

$$\Rightarrow \frac{\nu}{p_1} = \xi_1^3 - \xi_2 \quad (4.31)$$

Replacing  $p_1$  by  $\frac{\xi_2^2}{1+\xi_2^2}$  and  $\xi_1$  by  $\frac{1}{\xi_2}$  yields:

$$\begin{aligned} \frac{\nu}{\frac{\xi_2^2}{1+\xi_2^2}} &= \frac{1}{\xi_2^3} - \xi_2 \Rightarrow \frac{\nu(1+\xi_2^2)}{\xi_2^2} = \frac{1-\xi_2^4}{\xi_2^3} \\ \Rightarrow \nu(1+\xi_2^2)\xi_2 &= 1-\xi_2^4 \\ \Rightarrow \nu &= \frac{1-\xi_2^4}{(1+\xi_2^2)\xi_2} \\ \Rightarrow \nu &= \frac{(1+\xi_2^2)(1-\xi_2^2)}{(1+\xi_2^2)\xi_2} = \frac{(1-\xi_2^2)}{\xi_2} \\ \Rightarrow \nu &= \frac{(1-\xi_2^2)}{\xi_2} \\ \Rightarrow \nu\xi_2 &= 1-\xi_2^2 \\ \Rightarrow \xi_2^2 + \nu\xi_2 - 1 &= 0 \end{aligned} \quad (4.32)$$

One can solve for  $\xi_2$  by finding the roots of the second order equation  $\xi_2^2 + \nu\xi_2 - 1 = 0$ .

$$\Delta = \nu^2 + 4$$

$$\xi_{2,1} = \frac{-\nu - \sqrt{\nu^2 + 4}}{2} \text{ (this solution is neglected since } \xi_2 \text{ should be non-negative, } \xi_i = \frac{|x_i - \bar{X}|}{\sigma} \text{)}$$

$$\xi_{2,2} = \frac{-\nu + \sqrt{\nu^2 + 4}}{2} \quad (4.33)$$

Hence,

$$\xi_2 = -\frac{\nu}{2} + \sqrt{\left(\frac{\nu}{2}\right)^2 + 1} \quad (4.34)$$

Knowing that  $\xi_1 = \frac{1}{\xi_2}$ ,

$$\xi_1 = \frac{1}{-\frac{\nu}{2} + \sqrt{\left(\frac{\nu}{2}\right)^2 + 1}} = \frac{\frac{\nu}{2} + \sqrt{\left(\frac{\nu}{2}\right)^2 + 1}}{\left(-\frac{\nu}{2} + \sqrt{\left(\frac{\nu}{2}\right)^2 + 1}\right)\left(\frac{\nu}{2} + \sqrt{\left(\frac{\nu}{2}\right)^2 + 1}\right)} = \frac{\frac{\nu}{2} + \sqrt{\left(\frac{\nu}{2}\right)^2 + 1}}{\frac{\nu^2}{4} + \frac{\nu^2}{4} + 1} = \frac{\frac{\nu}{2} + \sqrt{\left(\frac{\nu}{2}\right)^2 + 1}}{1} = \frac{\nu}{2} + \sqrt{\left(\frac{\nu}{2}\right)^2 + 1} \quad (4.35)$$

When it comes to  $p_1$ , we know from solution (I) that  $p_1 = \frac{\xi_2^2}{1+\xi_2^2}$ ; hence,

$$p_1 = \frac{\xi_2}{\frac{1}{\xi_2} + \xi_2} = \frac{\xi_2}{\xi_1 + \xi_2} \text{ since } \xi_1 = \frac{1}{\xi_2} \quad (4.36)$$

Thus, the solution to the set of equations (4.21) to (4.24) is as follows:

$$\xi_1 = \frac{\nu}{2} + \sqrt{\left(\frac{\nu}{2}\right)^2 + 1} \quad (4.37)$$

$$\xi_2 = -\frac{\nu}{2} + \sqrt{\left(\frac{\nu}{2}\right)^2 + 1} = \xi_1 - \nu \quad (4.38)$$

$$p_1 = \frac{\xi_2}{\xi_1 + \xi_2} \quad (4.39)$$

$$p_2 = 1 - p_1 \quad (4.40)$$

The first three moments of  $Y = h(X)$  (i.e. the output random variable) are estimated next.

The first central moment of  $Y$  is equal to zero (as previously shown through equation (4.4), the first central moment of a random variable is equal to zero) and is as follows:

$$M'_1(Y) = (y_1 - \bar{Y})p_1 + (y_2 - \bar{Y})p_2 \quad (4.41)$$

Thus,

$$(y_1 - \bar{Y})p_1 + (y_2 - \bar{Y})p_2 = 0 \quad (4.42)$$

The second central moment of  $Y$  is equal to  $\sigma_Y^2$  (as previously shown through equation (4.5), the second central moment of a random variable is equal to its variance) and is as follows:

$$M'_2(Y) = (y_1 - \bar{Y})^2 p_1 + (y_2 - \bar{Y})^2 p_2 \quad (4.43)$$

Thus,

$$(y_1 - \bar{Y})^2 p_1 + (y_2 - \bar{Y})^2 p_2 = \sigma_Y^2 \quad (4.44)$$

The third central moment of  $Y$  is equal to  $\nu_Y \sigma_Y^3$  (as previously shown through equation (4.6), the third central moment of a random variable is equal to its skewness) and is as follows:

$$M'_3(Y) = (y_1 - \bar{Y})^3 p_1 + (y_2 - \bar{Y})^3 p_2 \quad (4.45)$$

Thus,

$$(y_1 - \bar{Y})^3 p_1 + (y_2 - \bar{Y})^3 p_2 = v_Y \sigma_Y^3 \quad (4.46)$$

Hence, the three equations from which the statistical moments of  $Y$  can be obtained are:

$$(y_1 - \bar{Y})p_1 + (y_2 - \bar{Y})p_2 = 0 \quad (4.47)$$

$$(y_1 - \bar{Y})^2 p_1 + (y_2 - \bar{Y})^2 p_2 = \sigma_Y^2 \quad (4.48)$$

$$(y_1 - \bar{Y})^3 p_1 + (y_2 - \bar{Y})^3 p_2 = v_Y \sigma_Y^3 \quad (4.49)$$

It is desired to solve for  $\bar{Y}$ ,  $\sigma_Y$  and  $v_Y$ .  $y_1 = h(x_1)$  and  $y_2 = h(x_2)$  are known quantities. The values of  $p_1$  and  $p_2$  were previously calculated, so they are known. Hence, equations (4.47) – (4.49) form a set of three equations with three unknowns.

The associated solution is illustrated through equations (4.51), (4.52) and (4.55). The derivation of the solution is done as follows:

$$\begin{aligned} (y_1 - \bar{Y})p_1 + (y_2 - \bar{Y})p_2 &= 0 \\ \Rightarrow y_1 p_1 - \bar{Y} p_1 + y_2 p_2 - \bar{Y} p_2 &= 0 \\ \Rightarrow \bar{Y}(p_1 + p_2) &= y_1 p_1 + y_2 p_2 \end{aligned} \quad (4.50)$$

On the other hand, it is already shown that  $p_1 + p_2 = 1$ ; then,

$$\bar{Y} = y_1 p_1 + y_2 p_2 \quad (4.51)$$

The expression of  $\sigma_Y$  can be derived as follows:

As already notice,  $p_2 = 1 - p_1$  and  $p_1 = 1 - p_2$ . Moreover,  $(y_1 - \bar{Y})^2 p_1 + (y_2 - \bar{Y})^2 p_2 = \sigma_Y^2$  and  $\bar{Y} = y_1 p_1 + y_2 p_2$ ; then,

$$\begin{aligned} (y_1 - \bar{Y})^2 p_1 + (y_2 - \bar{Y})^2 p_2 &= \sigma_Y^2 \\ \Rightarrow (y_1 - (y_1 p_1 + y_2 p_2))^2 p_1 + (y_2 - (y_1 p_1 + y_2 p_2))^2 p_2 &= \sigma_Y^2 \\ \Rightarrow (y_1(1 - p_1) - y_2 p_2)^2 p_1 + (y_2(1 - p_2) - y_1 p_1)^2 p_2 &= \sigma_Y^2 \end{aligned}$$



$$\Rightarrow (y_1 p_2 - y_2 p_2)^2 p_1 + (y_2 p_1 - y_1 p_1)^2 p_2 = \sigma_Y^2$$

$$\Rightarrow p_2^2 p_1 (y_1 - y_2)^2 + p_1^2 p_2 (y_2 - y_1)^2 = \sigma_Y^2$$

$$\Rightarrow (y_1 - y_2)^2 (p_2^2 p_1 + p_1^2 p_2) = \sigma_Y^2$$

$$\Rightarrow (y_1 - y_2)^2 (p_1 p_2 (p_2 + p_1)) = \sigma_Y^2$$

$$\Rightarrow (y_1 - y_2)^2 p_1 p_2 = \sigma_Y^2$$

Then,

$$\sigma_Y = \sqrt{p_1 p_2} |y_1 - y_2| \quad (4.52)$$

The expression of  $\nu_Y \sigma_Y$  can be obtained as follows:

From equation (4.49) we know that  $\nu_Y \sigma_Y^3 = (y_1 - \bar{Y})^3 p_1 + (y_2 - \bar{Y})^3 p_2$ ; then,

$$\nu_Y \sigma_Y = \frac{\nu_Y \sigma_Y^3}{\sigma_Y^2} = \frac{(y_1 - \bar{Y})^3 p_1 + (y_2 - \bar{Y})^3 p_2}{\sigma_Y^2} \quad (4.53)$$

On the other hand, it is already shown that  $\sigma_Y = \sqrt{p_1 p_2} |y_1 - y_2|$ ; then,

$$\sigma_Y^2 = p_1 p_2 (y_1 - y_2)^2 \quad (4.54)$$

Thus,

$$\begin{aligned} \nu_Y \sigma_Y &= \frac{(y_1 - \bar{Y})^3 p_1 + (y_2 - \bar{Y})^3 p_2}{p_1 p_2 (y_1 - y_2)^2} \\ &= \frac{(y_1 - \bar{Y})^3}{p_2 (y_1 - y_2)^2} + \frac{(y_2 - \bar{Y})^3}{p_1 (y_1 - y_2)^2} \\ &= \frac{(y_1 - (y_1 p_1 + y_2 p_2))^3}{p_2 (y_1 - y_2)^2} + \frac{(y_2 - (y_1 p_1 + y_2 p_2))^3}{p_1 (y_1 - y_2)^2} \\ &= \frac{(y_1(1-p_1) - y_2 p_2)^3}{p_2 (y_1 - y_2)^2} + \frac{(y_2(1-p_2) - y_1 p_1)^3}{p_1 (y_1 - y_2)^2} \\ &= \frac{(y_1 p_2 - y_2 p_2)^3}{p_2 (y_1 - y_2)^2} + \frac{(y_2 p_1 - y_1 p_1)^3}{p_1 (y_1 - y_2)^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{p_2^3(y_1-y_2)^3}{p_2(y_1-y_2)^2} + \frac{p_1^3(y_2-y_1)^3}{p_1(y_1-y_2)^2} \\
&= p_2^2(y_1 - y_2) - p_1^2(y_1 - y_2) \\
&= (y_1 - y_2)(p_2^2 - p_1^2) \\
&= (y_1 - y_2)(p_2 - p_1)(p_2 + p_1) \\
&= (y_1 - y_2)(p_2 - p_1)
\end{aligned}$$

$$\Rightarrow v_Y \sigma_Y = (y_1 - y_2)(p_2 - p_1) \quad (4.55)$$

Hence,

$$\bar{Y} \cong y_1 p_1 + y_2 p_2 \quad (4.56)$$

$$\sigma_Y \cong \sqrt{p_1 p_2} |y_1 - y_2| \quad (4.57)$$

$$v_Y \sigma_Y \cong (y_1 - y_2)(p_2 - p_1) \quad (4.58)$$

### 4.3. Review of H.P. Hong's PEM Approach

Hong's approach based on Taylor series expansion is introduced in [18]. Hong's approach matches that of Rosenblueth, for the univariate case, in estimating the statistical moments of the output variables. As a matter of fact, it analyzes the claim saying that by approximating the input random variable by a new set of concentrations and evaluating the output function for this set of concentrations one is in fact getting a good approximation of the statistical moments of the output variables as well.

As previously mentioned,  $Y$  is defined to be a random variable that depends on a random variable  $X$  such that  $Y = h(X)$ . The p.d.f. of  $X$  is denoted by  $f_X(x)$  and the mean, standard deviation and coefficient of variation of  $X$  are referred to by  $\mu_X$ ,  $\sigma_X$  and  $v_X$  respectively.

The  $i^{th}$  order central moment of  $X$  is given by the following:

$$M'_i(X) = \int_{-\infty}^{+\infty} (x - \mu_X)^i f_X(x) dx \quad (4.59)$$

Throughout [18] the following notation is used:

$$\lambda_{X,i} = M'_i(X)/\sigma_X^i \quad (4.60)$$

$$\lambda_{X,0} = \int_{-\infty}^{+\infty} (x - \mu_X)^0 f_X(x) dx = \int_{-\infty}^{+\infty} f_X(x) dx = 1$$

Then,

$$\lambda_{X,0} = 1 \quad (4.61)$$

$$\lambda_{X,1} = \left( \int_{-\infty}^{+\infty} (x - \mu_X)^1 f_X(x) dx \right) / \sigma_X^1$$

$$\text{However, } \int_{-\infty}^{+\infty} (x - \mu_X)^1 f_X(x) dx = \int_{-\infty}^{+\infty} x f_X(x) dx - \mu_X \int_{-\infty}^{+\infty} f_X(x) dx = \mu_X - \mu_X = 0$$

Thus,

$$\lambda_{X,1} = 0 \quad (4.62)$$

$$\lambda_{X,2} = \left( \int_{-\infty}^{+\infty} (x - \mu_X)^2 f_X(x) dx \right) / \sigma_X^2$$

$$\text{However, } \int_{-\infty}^{+\infty} (x - \mu_X)^2 f_X(x) dx = \sigma_X^2$$

Then,

$$\lambda_{X,2} = 1 \quad (4.63)$$

$$\lambda_{X,3} = \text{coefficient of skewness of } X$$

$$\lambda_{X,4} = \text{coefficient of kurtosis of } X$$

The Taylor series expansion of  $h(x)$  around  $\mu_X$  is given by the following:

$$h(x) = h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) (x - \mu_X)^i \quad (4.64)$$

$h^{(i)}$  is the  $i^{\text{th}}$  derivative of  $h(x)$ .

The expected value of  $Y$  is as follows:

$$\begin{aligned} E[Y] &= E[h(x)] = \int_{-\infty}^{+\infty} \left( h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) (x - \mu_X)^i \right) f_X(x) dx \\ &= h(\mu_X) \int_{-\infty}^{+\infty} f_X(x) dx + \int_{-\infty}^{+\infty} \left( \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) (x - \mu_X)^i f_X(x) dx \right) \end{aligned}$$

$$\begin{aligned}
&= h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) M'_i(X) \\
&= h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) \lambda_{X,i} \sigma_X^i \\
\Rightarrow E[Y] &= h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) \lambda_{X,i} \sigma_X^i \tag{4.65}
\end{aligned}$$

As was previously mentioned, two probability concentrations  $p_1$  and  $p_2$  are included at locations  $x_1$  and  $x_2$  in a manner where this set of probability concentrations has the same moments as the original input variable  $X$  as shown in Figure 4.1. The locations are assigned as follows:

$$x_i = \mu_X + \xi_i \sigma_X. \tag{4.66}$$

$\xi_i$ , for  $i = 1, 2$ , are two constants, and their values are calculated through the derivations. These constants are referred to as concentrations coefficients.

[18] suggests to calculate  $p_1 h(x_1) + p_2 h(x_2)$  and match it to  $E[Y]$ . Hence, the main difference between Rosenblueth's explanation and that of Hong is as follows: the purpose of both is to estimate the statistical moments of the output function. Rosenblueth picks the probability concentrations such that their moments match those of the input random variable. Rosenblueth then evaluates the output function for the chosen points (probability concentrations) and estimates the statistics of the output variables accordingly. Hong, on the other hand, picks the probability concentrations locations (i.e.  $x_1$  and  $x_2$ ) such that the expected value of the Taylor series expansion of the output variable around the mean value of the input variable matches the weighted average of the Taylor series expansion of the output variable around the mean value of the input variable evaluated at the generated probability concentrations locations (i.e.  $x_1$  and  $x_2$ ). It is shown next that for the univariate case, these two approaches are similar.

As already mentioned, the Taylor series expansion of  $h(x)$  around  $\mu_X$  is given by  $h(x) = h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) (x - \mu_X)^i$

Then,

$$p_1 h(x_1) = p_1 h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) p_1 (x_1 - \mu_X)^i \tag{4.67}$$

and

$$p_2 h(x_2) = p_2 h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) p_2 (x_2 - \mu_X)^i \quad (4.68)$$

Hence,

$$\begin{aligned} p_1 h(x_1) + p_2 h(x_2) &= (p_1 + p_2) h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) p_1 (x_1 - \mu_X)^i + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) p_2 (x_2 - \mu_X)^i \\ &= (p_1 + p_2) h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) [p_1 (x_1 - \mu_X)^i + p_2 (x_2 - \mu_X)^i] \\ &= (p_1 + p_2) h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) [p_1 (\xi_1 \sigma_X)^i + p_2 (\xi_2 \sigma_X)^i] \\ &= (p_1 + p_2) h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) [p_1 \xi_1^i + p_2 \xi_2^i] \sigma_X^i \\ \Rightarrow p_1 h(x_1) + p_2 h(x_2) &= (p_1 + p_2) h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) [p_1 \xi_1^i + p_2 \xi_2^i] \sigma_X^i \end{aligned} \quad (4.69)$$

[18] suggests to approximate  $E[Y]$  by  $p_1 h(x_1) + p_2 h(x_2)$  by matching the first four terms of each. The first four terms of  $E[Y]$  are:

$$h(\mu_X) + \frac{1}{1!} h'(\mu_X) \lambda_{X,1} \sigma_X^1 + \frac{1}{2!} h''(\mu_X) \lambda_{X,2} \sigma_X^2 + \frac{1}{3!} h'''(\mu_X) \lambda_{X,3} \sigma_X^3$$

The first four terms of  $p_1 h(x_1) + p_2 h(x_2)$  are:

$$(p_1 + p_2) h(\mu_X) + \frac{1}{1!} h'(\mu_X) [p_1 \xi_1^1 + p_2 \xi_2^1] \sigma_X^1 + \frac{1}{2!} h''(\mu_X) [p_1 \xi_1^2 + p_2 \xi_2^2] \sigma_X^2 + \frac{1}{3!} h'''(\mu_X) [p_1 \xi_1^3 + p_2 \xi_2^3] \sigma_X^3$$

Then,

$$p_1 + p_2 = 1 = \lambda_{X,0} \quad (4.70)$$

$$\lambda_{X,1} = M'_1(X) / \sigma_X^1 = p_1 \xi_1^1 + p_2 \xi_2^1 \quad (4.71)$$

$$\lambda_{X,2} = M'_2(X) / \sigma_X^2 = p_1 \xi_1^2 + p_2 \xi_2^2 \quad (4.72)$$

$$\lambda_{X,3} = M'_3(X) / \sigma_X^3 = p_1 \xi_1^3 + p_2 \xi_2^3 \quad (4.73)$$

Hence, for  $i = 0, \dots, 3$

$$\sum_{j=1}^k p_j \xi_j^i = \frac{M'_i(X)}{\sigma_X^i} \quad (\text{where } k = 1,2) \quad (4.74)$$

These are four equations with four unknowns which are  $p_1, p_2, \xi_1$  and  $\xi_2$ .

This set of equations is equivalent to the one obtained by Rosenblueth. This can be clearly observed by comparing equations (4.21) - (4.24) to equations (4.70) – (4.71) while bearing in mind that  $\lambda_{X,0} = 1, \lambda_{X,1} = 0, \lambda_{X,2} = 1, \lambda_{X,3} = \nu$  and that for Rosenblueth's approach  $\xi_1 = \frac{x_1 - \mu_X}{\sigma_X}$  and  $\xi_2 = \frac{\mu_X - x_2}{\sigma_X}$  while for Hong's approach  $\xi_i = \frac{x_i - \mu_X}{\sigma_X}$  for  $i = 1,2$ .

Hence, the sets of equations to be solved using Rosenblueth's approach and Hong's approach, for the univariate case, are completely similar. As a result, by matching up to the third moment of the input random variable one is performing a third order Taylor series approximation of the mean of the output variable.

Solving this system of equations one gets:

$$\xi_j = \frac{\lambda_{X,3}}{2} + (-1)^{3-j} \sqrt{1 + (\lambda_{X,3/2})^2} \quad j = 1,2 \quad (4.75)$$

$$p_j = (-1)^j \frac{\xi_{3-j}}{\zeta} \quad j = 1,2 \quad (4.76)$$

$$\text{where } \zeta = \xi_1 - \xi_2 = 2 \sqrt{1 + \left(\frac{\lambda_{X,3}}{2}\right)^2} \quad (4.77)$$

It is already known that

$$p_1 h(x_1) + p_2 h(x_2) = (p_1 + p_2) h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) [p_1 \xi_1^i + p_2 \xi_2^i] \sigma_X^i$$

On the other hand,

$$p_1 + p_2 = 1 \text{ and } \sum_{j=1}^k p_j \xi_j^i = \frac{M'_i(X)}{\sigma_X^i} \text{ for } i = 1, \dots, 3 \text{ then,}$$

$$p_1 h(x_1) + p_2 h(x_2) = h(\mu_X) + \sum_{i=1}^3 \frac{1}{i!} h^{(i)}(\mu_X) \lambda_{X,i} \sigma_X^i + \sum_{i=4}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) [p_1 \xi_1^i + p_2 \xi_2^i] \sigma_X^i$$

Hence,

$$h(\mu_X) + \sum_{i=1}^3 \frac{1}{i!} h^{(i)}(\mu_X) \lambda_{X,i} \sigma_X^i = p_1 h(x_1) + p_2 h(x_2) - \sum_{i=4}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) [p_1 \xi_1^i + p_2 \xi_2^i] \sigma_X^i$$

We have already derived the expectation of  $Y$  to be

$$\mu_Y = E[Y] = h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) \lambda_{X,i} \sigma_X^i$$

Then,

$$\begin{aligned} \mu_Y = E[Y] &= h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) \lambda_{X,i} \sigma_X^i = \\ &h(\mu_X) + \sum_{i=1}^3 \frac{1}{i!} h^{(i)}(\mu_X) \lambda_{X,i} \sigma_X^i + \sum_{i=4}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) \lambda_{X,i} \sigma_X^i \end{aligned}$$

However, we already know that

$$h(\mu_X) + \sum_{i=1}^3 \frac{1}{i!} h^{(i)}(\mu_X) \lambda_{X,i} \sigma_X^i = p_1 h(x_1) + p_2 h(x_2) - \sum_{i=4}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) [p_1 \xi_1^i + p_2 \xi_2^i] \sigma_X^i$$

Then,

$$\begin{aligned} \mu_Y &= E[Y] \\ &= h(\mu_X) + \sum_{i=1}^3 \frac{1}{i!} h^{(i)}(\mu_X) \lambda_{X,i} \sigma_X^i + \sum_{i=4}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) \lambda_{X,i} \sigma_X^i \\ &= p_1 h(x_1) + p_2 h(x_2) - \sum_{i=4}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) [p_1 \xi_1^i + p_2 \xi_2^i] \sigma_X^i + \sum_{i=4}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) \lambda_{X,i} \sigma_X^i \\ &= p_1 h(x_1) + p_2 h(x_2) + \sum_{i=4}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) [\lambda_{X,i} - (p_1 \xi_1^i + p_2 \xi_2^i)] \sigma_X^i \\ \mu_Z &= p_1 h(x_1) + p_2 h(x_2) + \sum_{i=4}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) [\lambda_{X,i} - (p_1 \xi_1^i + p_2 \xi_2^i)] \sigma_X^i \end{aligned} \tag{4.78}$$

Hence, one can conclude that

$$\mu_Y \cong p_1 h(x_1) + p_2 h(x_2) \tag{4.79}$$

is a third (i.e.  $2m - 1$ ) order approximation. In fact, for  $i \geq 4$  this equation does not hold.

Note: if  $h(x)$  is a polynomial of order three or less, the 2PEM gives an exact estimation of the mean of  $Y$ .

In a similar manner, the order of approximation of  $E[Y^2] \cong p_1 (h(x_1))^2 + p_2 (h(x_2))^2$  is explored. The following analysis forms the basis of the work in [18].

$$E[Y^2] = E[h^2(X)]$$

$$\begin{aligned} &= \int_{-\infty}^{+\infty} \left( h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X)(x - \mu_X)^i \right)^2 f_X(x) dx \\ &= \int_{-\infty}^{+\infty} \left( h^2(\mu_X) + \left( \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X)(x - \mu_X)^i \right)^2 + 2 \sum_{i=1}^{\infty} \frac{1}{i!} h(\mu_X) h^{(i)}(\mu_X)(x - \mu_X)^i \right) f_X(x) dx \\ &= h^2(\mu_X) + 2 \sum_{i=1}^{\infty} \frac{1}{i!} h(\mu_X) h^{(i)}(\mu_X) M'_i(x) + \underbrace{\int_{-\infty}^{+\infty} \left( \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X)(x - \mu_X)^i \right)^2 f_X(x) dx}_b \end{aligned}$$

$$\Rightarrow E[Y^2] = h^2(\mu_X) + 2 \sum_{i=1}^{\infty} \frac{1}{i!} h(\mu_X) h^{(i)}(\mu_X) M'_i(x) + b \quad (4.80)$$

Using Taylor series expansion of  $h(x)$  around  $\mu_x$ ,  $h(x) = h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X)(x - \mu_X)^i$ , one obtains:

$$\begin{aligned} p_1 h^2(x_1) &= p_1 \left( h(\mu_X) + \sum_{i=1}^{\infty} \frac{1}{i!} (x_1 - \mu_X)^i h^{(i)}(\mu_X) \right)^2 \\ &= p_1 \left( h^2(\mu_X) + \left( \sum_{i=1}^{\infty} \frac{1}{i!} (x_1 - \mu_X)^i h^{(i)}(\mu_X) \right)^2 + 2 \sum_{i=1}^{\infty} \frac{1}{i!} (x_1 - \mu_X)^i h(\mu_X) h^{(i)}(\mu_X) \right) \end{aligned}$$

Hence,

$$p_1 h^2(x_1) = \quad (4.81)$$

$$p_1 \left( h^2(\mu_X) + \left( \sum_{i=1}^{\infty} \frac{1}{i!} (x_1 - \mu_X)^i h^{(i)}(\mu_X) \right)^2 + 2 \sum_{i=1}^{\infty} \frac{1}{i!} (x_1 - \mu_X)^i h(\mu_X) h^{(i)}(\mu_X) \right)$$

$$p_2 h^2(x_2) = \quad (4.82)$$

$$p_2 \left( h^2(\mu_X) + \left( \sum_{i=1}^{\infty} \frac{1}{i!} (x_2 - \mu_X)^i h^{(i)}(\mu_X) \right)^2 + 2 \sum_{i=1}^{\infty} \frac{1}{i!} (x_2 - \mu_X)^i h(\mu_X) h^{(i)}(\mu_X) \right)$$

Using equations (4.81) and (4.82)

$$p_1 h^2(x_1) + p_2 h^2(x_2) =$$

$$\begin{aligned} &(p_1 + p_2)(h^2(\mu_X)) + p_1 \left( \sum_{i=1}^{\infty} \frac{1}{i!} (x_1 - \mu_X)^i h^{(i)}(\mu_X) \right)^2 + p_2 \left( \sum_{i=1}^{\infty} \frac{1}{i!} (x_2 - \mu_X)^i h^{(i)}(\mu_X) \right)^2 + \\ &2 \sum_{i=1}^{\infty} \frac{1}{i!} h(\mu_X) h^{(i)}(\mu_X) (p_1 (x_1 - \mu_X)^i + p_2 (x_2 - \mu_X)^i) \end{aligned}$$



$$\begin{aligned}
& p_1 h^2(x_1) + p_2 h^2(x_2) = \\
& (p_1 + p_2)(h^2(\mu_x)) + \underbrace{p_1 \left( \sum_{i=1}^{\infty} \frac{1}{i!} (x_1 - \mu_x)^i h^{(i)}(\mu_x) \right)^2 + p_2 \left( \sum_{i=1}^{\infty} \frac{1}{i!} (x_2 - \mu_x)^i h^{(i)}(\mu_x) \right)^2}_{a} + \\
& 2 \sum_{i=1}^{\infty} \frac{1}{i!} h(\mu_x) h^{(i)}(\mu_x) (p_1 \xi_1^i + p_2 \xi_2^i) \sigma^i \\
& p_1 h^2(x_1) + p_2 h^2(x_2) = (p_1 + p_2)(h^2(\mu_x)) + 2 \sum_{i=1}^3 \frac{1}{i!} h(\mu_x) h^{(i)}(\mu_x) (p_1 \xi_1^i + p_2 \xi_2^i) \sigma^i + \\
& 2 \sum_{i=4}^{\infty} \frac{1}{i!} h(\mu_x) h^{(i)}(\mu_x) (p_1 \xi_1^i + p_2 \xi_2^i) \sigma^i + a
\end{aligned}$$

Then,

$$\begin{aligned}
& (p_1 + p_2)(h^2(\mu_x)) + 2 \sum_{i=1}^3 \frac{1}{i!} h(\mu_x) h^{(i)}(\mu_x) (p_1 \xi_1^i + p_2 \xi_2^i) \sigma^i = p_1 h^2(x_1) + \\
& p_2 h^2(x_2) - 2 \sum_{i=4}^{\infty} \frac{1}{i!} h(\mu_x) h^{(i)}(\mu_x) (p_1 \xi_1^i + p_2 \xi_2^i) \sigma_X^i - a
\end{aligned} \tag{4.83}$$

Let us return to the expression of  $E[Y^2]$ .

$$E[Y^2] = h^2(\mu_x) + 2 \sum_{i=1}^3 \frac{1}{i!} h(\mu_x) h^{(i)}(\mu_x) \lambda_{x,i} \sigma_x^i + 2 \sum_{i=4}^{\infty} \frac{1}{i!} h(\mu_x) h^{(i)}(\mu_x) \lambda_{x,i} \sigma_x^i + b$$

On the other hand, we already know that  $\sum_{j=1}^k p_j \xi_j^i = \lambda_{x,i} = \frac{M'_i(X)}{\sigma_x^i}$

Hence, we can replace  $h^2(\mu_x) + 2 \sum_{i=1}^3 \frac{1}{i!} h(\mu_x) h^{(i)}(\mu_x) \lambda_{x,i} \sigma_x^i$  by  $(p_1 + p_2)(h^2(\mu_x)) + 2 \sum_{i=1}^3 \frac{1}{i!} h(\mu_x) h^{(i)}(\mu_x) (p_1 \xi_1^i + p_2 \xi_2^i) \sigma_X^i$  in the expression of  $E[Y^2]$ .

We have just proven that

$$\begin{aligned}
& (p_1 + p_2)(h^2(\mu_x)) + 2 \sum_{i=1}^3 \frac{1}{i!} h(\mu_x) h^{(i)}(\mu_x) (p_1 \xi_1^i + p_2 \xi_2^i) \sigma_X^i = p_1 h^2(x_1) + p_2 h^2(x_2) - \\
& 2 \sum_{i=4}^{\infty} \frac{1}{i!} h(\mu_x) h^{(i)}(\mu_x) (p_1 \xi_1^i + p_2 \xi_2^i) \sigma_X^i - a
\end{aligned}$$

Then,

$$\begin{aligned}
& E[Y^2] = \\
& p_1 h^2(x_1) + p_2 h^2(x_2) + 2 \sum_{i=4}^{\infty} \frac{1}{i!} h(\mu_x) h^{(i)}(\mu_x) \lambda_{x,i} \sigma_x^i - 2 \sum_{i=4}^{\infty} \frac{1}{i!} h(\mu_x) h^{(i)}(\mu_x) (p_1 \xi_1^i + \\
& p_2 \xi_2^i) \sigma_X^i + b - a
\end{aligned}$$

$$E[Y^2] = p_1 h^2(x_1) + p_2 h^2(x_2) + 2 \sum_{i=4}^{\infty} \frac{1}{i!} h(\mu_X) h^{(i)}(\mu_X) (\lambda_{x,i} - p_1 \xi_1^i - p_2 \xi_2^i) \sigma_x^i + b - \quad (4.84)$$

$a$

The term  $b - a$  is analyzed as follows.

$$b = \int_{-\infty}^{+\infty} \left( \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) (x - \mu_X)^i \right)^2 f_X(x) dx$$

$$\left( \sum_{i=1}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) (x - \mu_X)^i \right)^2 = \left( \sum_{i=1}^3 \frac{1}{i!} h^{(i)}(\mu_X) (x - \mu_X)^i + \sum_{i=4}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) (x - \mu_X)^i \right)^2$$

We are trying to prove that  $E(Y^2) \cong p_1 (h(x_1))^2 + p_2 (h(x_2))^2$  is a third order approximation; hence, we can neglect the terms  $\sum_{i=4}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) (x - \mu_X)^i$ .

$$\text{On the other hand, } (a + b + c)^2 = ((a + b) + c)^2 = (a + b)^2 + c^2 + 2(a + b)c = a^2 + b^2 + 2ab + c^2 + 2ac + 2bc = a^2 + b^2 + c^2 + 2(ab + ac + bc)$$

Then,

$$\left( \sum_{i=1}^3 \frac{1}{i!} h^{(i)}(\mu_X) (x - \mu_X)^i \right)^2 = \left( h'(\mu_X)(x - \mu_X) + \frac{1}{2} h''(\mu_X)(x - \mu_X)^2 + \frac{1}{6} h'''(\mu_X)(x - \mu_X)^3 \right)^2$$

$$\begin{aligned} \left( \sum_{i=1}^3 \frac{1}{i!} h^{(i)}(\mu_X) (x - \mu_X)^i \right)^2 &= \left( h'(\mu_X)(x - \mu_X) \right)^2 + \left( \frac{1}{2} h''(\mu_X)(x - \mu_X)^2 \right)^2 + \left( \frac{1}{6} h'''(\mu_X)(x - \mu_X)^3 \right)^2 \\ &+ h'(\mu_X) h''(\mu_X) (x - \mu_X)(x - \mu_X)^2 + \frac{1}{3} h'(\mu_X) h'''(\mu_X) (x - \mu_X)(x - \mu_X)^3 + \\ &\frac{1}{6} h''(\mu_X) h'''(\mu_X) (x - \mu_X)^2 (x - \mu_X)^3 \end{aligned}$$

$$\underbrace{\left( \sum_{i=1}^3 \frac{1}{i!} h^{(i)}(\mu_X) (x - \mu_X)^i \right)^2}_c =$$

$$\begin{aligned} &(h'(\mu_X))^2 (x - \mu_X)^2 + \frac{1}{4} (h''(\mu_X))^2 (x - \mu_X)^4 + \frac{1}{36} (h'''(\mu_X))^2 (x - \mu_X)^6 + h'(\mu_X) h''(\mu_X) (x - \mu_X)^3 \\ &+ \frac{1}{3} h'(\mu_X) h'''(\mu_X) (x - \mu_X)^4 + \frac{1}{6} h''(\mu_X) h'''(\mu_X) (x - \mu_X)^5 \end{aligned}$$

$$b = \int_{-\infty}^{+\infty} c f_X(x) dx = (h'(\mu_X))^2 M_2'(x) + \frac{1}{4} (h''(\mu_X))^2 M_4'(x) + \frac{1}{36} (h'''(\mu_X))^2 M_6'(x) +$$

$$h'(\mu_X) h''(\mu_X) M_3'(x) + \frac{1}{3} h'(\mu_X) h'''(\mu_X) M_4'(x) + \frac{1}{6} h''(\mu_X) h'''(\mu_X) M_5'(x)$$

$$a = p_1 \underbrace{\left( \sum_{i=1}^{\infty} \frac{1}{i!} (x_1 - \mu_X)^i h^{(i)}(\mu_X) \right)^2}_d + p_2 \underbrace{\left( \sum_{i=1}^{\infty} \frac{1}{i!} (x_2 - \mu_X)^i h^{(i)}(\mu_X) \right)^2}_e$$

$$d = \left( \sum_{i=1}^3 \frac{1}{i!} (x_1 - \mu_X)^i h^{(i)}(\mu_X) + \sum_{i=4}^{\infty} \frac{1}{i!} (x_1 - \mu_X)^i h^{(i)}(\mu_X) \right)^2 \cong \left( \sum_{i=1}^3 \frac{1}{i!} (x_1 - \mu_X)^i h^{(i)}(\mu_X) \right)^2$$

$$e = \left( \sum_{i=1}^3 \frac{1}{i!} (x_2 - \mu_X)^i h^{(i)}(\mu_X) + \sum_{i=4}^{\infty} \frac{1}{i!} (x_2 - \mu_X)^i h^{(i)}(\mu_X) \right)^2 \cong \left( \sum_{i=1}^3 \frac{1}{i!} (x_2 - \mu_X)^i h^{(i)}(\mu_X) \right)^2$$

$$d = \left( (x_1 - \mu_X) h'(\mu_X) + \frac{1}{2} (x_1 - \mu_X)^2 h''(\mu_X) + \frac{1}{6} (x_1 - \mu_X)^3 h'''(\mu_X) \right)^2 = (h'(\mu_X))^2 (x_1 - \mu_X)^2 + \frac{1}{4} (h''(\mu_X))^2 (x_1 - \mu_X)^4 + \frac{1}{36} (h'''(\mu_X))^2 (x_1 - \mu_X)^6 + h'(\mu_X) h''(\mu_X) (x_1 - \mu_X)^3 + \frac{1}{3} h'(\mu_X) h''(\mu_X) (x_1 - \mu_X)^4 + \frac{1}{6} h''(\mu_X) h'''(\mu_X) (x_1 - \mu_X)^5$$

$$e = (h'(\mu_X))^2 (x_2 - \mu_X)^2 + \frac{1}{4} (h''(\mu_X))^2 (x_2 - \mu_X)^4 + \frac{1}{36} (h'''(\mu_X))^2 (x_2 - \mu_X)^6 + h'(\mu_X) h''(\mu_X) (x_2 - \mu_X)^3 + \frac{1}{3} h'(\mu_X) h''(\mu_X) (x_2 - \mu_X)^4 + \frac{1}{6} h''(\mu_X) h'''(\mu_X) (x_2 - \mu_X)^5$$

$$a = p_1 d + p_2 e = (h'(\mu_X))^2 (p_1 (x_1 - \mu_X)^2 + p_2 (x_2 - \mu_X)^2) + \frac{1}{4} (h''(\mu_X))^2 (p_1 (x_1 - \mu_X)^4 + p_2 (x_2 - \mu_X)^4) + \frac{1}{36} (h'''(\mu_X))^2 (p_1 (x_1 - \mu_X)^6 + p_2 (x_2 - \mu_X)^6) + h'(\mu_X) h''(\mu_X) (p_1 (x_1 - \mu_X)^3 + p_2 (x_2 - \mu_X)^3) + \frac{1}{3} h'(\mu_X) h''(\mu_X) (p_1 (x_1 - \mu_X)^4 + p_2 (x_2 - \mu_X)^4) + \frac{1}{6} h''(\mu_X) h'''(\mu_X) (p_1 (x_1 - \mu_X)^5 + p_2 (x_2 - \mu_X)^5)$$

$$a = p_1 d + p_2 e = (h'(\mu_X))^2 (p_1 \xi_1^2 + p_2 \xi_2^2) \sigma_X^2 + h'(\mu_X) h''(\mu_X) (p_1 \xi_1^3 + p_2 \xi_2^3) \sigma_X^3 + \frac{1}{4} (h''(\mu_X))^2 (p_1 \xi_1^4 + p_2 \xi_2^4) \sigma_X^4 + \frac{1}{3} h'(\mu_X) h''(\mu_X) (p_1 \xi_1^4 + p_2 \xi_2^4) \sigma_X^4 + \frac{1}{6} h''(\mu_X) h'''(\mu_X) (p_1 \xi_1^5 + p_2 \xi_2^5) \sigma_X^5 + \frac{1}{36} (h'''(\mu_X))^2 (p_1 \xi_1^6 + p_2 \xi_2^6) \sigma_X^6$$

$$a = \tag{4.85}$$

$$\begin{aligned} & (h'(\mu_X))^2 (p_1 \xi_1^2 + p_2 \xi_2^2) \sigma_X^2 + h'(\mu_X) h''(\mu_X) (p_1 \xi_1^3 + p_2 \xi_2^3) \sigma_X^3 + \left( \frac{1}{4} (h''(\mu_X))^2 (p_1 \xi_1^4 + \right. \\ & \left. p_2 \xi_2^4) + \frac{1}{3} h'(\mu_X) h''(\mu_X) (p_1 \xi_1^4 + p_2 \xi_2^4) \right) \sigma_X^4 + \frac{1}{6} h''(\mu_X) h'''(\mu_X) (p_1 \xi_1^5 + p_2 \xi_2^5) \sigma_X^5 + \\ & \left. \frac{1}{36} (h'''(\mu_X))^2 (p_1 \xi_1^6 + p_2 \xi_2^6) \sigma_X^6 \right. \end{aligned}$$

We have already proven that  $b = \int_{-\infty}^{+\infty} cf(x)dx = (h'(\mu_x))^2 M'_2(x) + \frac{1}{4}(h''(\mu_x))^2 M'_4(x) + \frac{1}{36}(h'''(\mu_x))^2 M'_6(x) + h'(\mu_x)h''(\mu_x)M'_3(x) + \frac{1}{3}h'(\mu_x)h'''(\mu_x)M'_4(x) + \frac{1}{6}h''(\mu_x)h'''(\mu_x)M'_5(x)$

$$b = (h'(\mu_x))^2 M'_2(x) + h'(\mu_x)h''(\mu_x)M'_3(x) + \frac{1}{4}(h''(\mu_x))^2 M'_4(x) + \frac{1}{3}h'(\mu_x)h'''(\mu_x)M'_4(x) + \frac{1}{6}h''(\mu_x)h'''(\mu_x)M'_5(x) + \frac{1}{36}(h'''(\mu_x))^2 M'_6(x)$$

$$b = (h'(\mu_x))^2 \lambda_{x,2} \sigma_X^2 + h'(\mu_x)h''(\mu_x) \lambda_{x,3} \sigma_X^3 + \frac{1}{4}(h''(\mu_x))^2 \lambda_{x,4} \sigma_X^4 + \frac{1}{3}h'(\mu_x)h'''(\mu_x) \lambda_{x,4} \sigma_X^4 + \frac{1}{6}h''(\mu_x)h'''(\mu_x) \lambda_{x,5} \sigma_X^5 + \frac{1}{36}(h'''(\mu_x))^2 \lambda_{x,6} \sigma_X^6$$

$$b = \tag{4.86}$$

$$(h'(\mu_x))^2 \lambda_{x,2} \sigma_X^2 + h'(\mu_x)h''(\mu_x) \lambda_{x,3} \sigma_X^3 + \left(\frac{1}{4}(h''(\mu_x))^2 \lambda_{x,4} + \frac{1}{3}h'(\mu_x)h'''(\mu_x) \lambda_{x,4}\right) \sigma_X^4 + \frac{1}{6}h''(\mu_x)h'''(\mu_x) \lambda_{x,5} \sigma_X^5 + \frac{1}{36}(h'''(\mu_x))^2 \lambda_{x,6} \sigma_X^6$$

Then,

$$b - a = (h'(\mu_x))^2 \underbrace{(\lambda_{x,2} - p_1 \xi_1^2 - p_2 \xi_2^2)}_{=0} \sigma_X^2 + h'(\mu_x)h''(\mu_x) \underbrace{(\lambda_{x,3} - p_1 \xi_1^3 - p_2 \xi_2^3)}_{=0} \sigma_X^3 + \left(\frac{1}{4}(h''(\mu_x))^2 (\lambda_{x,4} - p_1 \xi_1^4 - p_2 \xi_2^4) + \frac{1}{3}h'(\mu_x)h'''(\mu_x) (\lambda_{x,4} - p_1 \xi_1^4 - p_2 \xi_2^4)\right) \sigma_X^4 + \frac{1}{6}h''(\mu_x)h'''(\mu_x) (\lambda_{x,5} - p_1 \xi_1^5 - p_2 \xi_2^5) \sigma_X^5 + \frac{1}{36}(h'''(\mu_x))^2 (\lambda_{x,6} - p_1 \xi_1^6 - p_2 \xi_2^6) \sigma_X^6$$

$$b - a = \left(\frac{1}{4}(h''(\mu_x))^2 (\lambda_{x,4} - p_1 \xi_1^4 - p_2 \xi_2^4) + \frac{1}{3}h'(\mu_x)h'''(\mu_x) (\lambda_{x,4} - p_1 \xi_1^4 - p_2 \xi_2^4)\right) \sigma_X^4 + \frac{1}{6}h''(\mu_x)h'''(\mu_x) (\lambda_{x,5} - p_1 \xi_1^5 - p_2 \xi_2^5) \sigma_X^5 + \frac{1}{36}(h'''(\mu_x))^2 (\lambda_{x,6} - p_1 \xi_1^6 - p_2 \xi_2^6) \sigma_X^6 \tag{4.87}$$

As previously proven,  $E[Y^2] = p_1 h^2(x_1) + p_2 h^2(x_2) + 2 \sum_{i=4}^{\infty} \frac{1}{i!} h(\mu_X) h^{(i)}(\mu_X) (\lambda_{x,i} - p_1 \xi_1^i - p_2 \xi_2^i) \sigma_X^i + b - a$  For  $E[Y^2] \cong p_1 h^2(x_1) + p_2 h^2(x_2)$  to be a third order approximation,  $b - a$  should be equal to zero. Then,  $\sum_{j=1}^k p_j \xi_j^i = \lambda_{x,i} = \frac{M'_i(X)}{\sigma_X^i}$  needs to be applicable for  $i = 1, \dots, 6$ . By taking two concentrations, it is applicable for  $i = 1, \dots, 3$ . By taking three

concentrations,  $\sum_{j=1}^k p_j \xi_j^i = \lambda_{x,i} = \frac{M'_i(x)}{\sigma_x^i}$  is applicable for  $i = 1, \dots, 5$ . By taking four concentrations  $\sum_{j=1}^k p_j \xi_j^i = \lambda_{x,i} = \frac{M'_i(x)}{\sigma_x^i}$  is applicable for  $i = 1, \dots, 7$ . Hence, for  $m = 4$  a third (i.e.  $m - 1$ ) order Taylor series approximation of  $E[Y^2]$  can be obtained.

Hence, for  $E[Y^2] \cong p_1 h^2(x_1) + p_2 h^2(x_2)$  to be a third order approximation more concentrations are needed to satisfy  $\sum_{j=1}^k p_j \xi_j^i = \lambda_{x,i} = \frac{M'_i(x)}{\sigma_x^i}$  for  $i = 1, \dots, 6$ . By increasing the number of concentrations the  $\mu_Y \cong p_1 h(x_1) + p_2 h(x_2)$  approximation will also have a higher order.

By taking  $E[Y^2] \cong p_1 h^2(x_1) + p_2 h^2(x_2)$ , the expressions of the variance and standard deviation of  $Y$  can be obtained as follows:

$$\begin{aligned}
\sigma_Y^2 &= E(Y^2) - \mu_Y^2 \\
&= p_1 (h(x_1))^2 + p_2 (h(x_2))^2 - (p_1 h(x_1) + p_2 h(x_2))^2 \\
&= p_1 (h(x_1))^2 + p_2 (h(x_2))^2 - p_1^2 (h(x_1))^2 - p_2^2 (h(x_2))^2 - 2p_1 h(x_1) p_2 h(x_2) \\
&= (h(x_1))^2 (p_1 - p_1^2) + (h(x_2))^2 (p_2 - p_2^2) - 2p_1 h(x_1) p_2 h(x_2) \\
&= (h(x_1))^2 p_1 (1 - p_1) + (h(x_2))^2 p_2 (1 - p_2) - 2p_1 h(x_1) p_2 h(x_2) \\
\Rightarrow \sigma_Y^2 &= (h(x_1))^2 p_1 (1 - p_1) + (h(x_2))^2 p_2 (1 - p_2) - 2p_1 h(x_1) p_2 h(x_2) \tag{4.88}
\end{aligned}$$

On the other hand,  $p_1 + p_2 = 1$ ; hence,  $1 - p_1 = p_2$  and  $1 - p_2 = p_1$ ; then,

$$\begin{aligned}
\sigma_Y^2 &= (h(x_1))^2 p_1 p_2 + (h(x_2))^2 p_1 p_2 - 2p_1 h(x_1) p_2 h(x_2) \\
&= p_1 p_2 \left( (h(x_1))^2 + (h(x_2))^2 - 2h(x_1)h(x_2) \right) \\
&= p_1 p_2 (h(x_1) - h(x_2))^2 \\
\Rightarrow \sigma_Y^2 &= p_1 p_2 (h(x_1) - h(x_2))^2 \tag{4.89}
\end{aligned}$$

Thus,

$$\sigma_Y = \sqrt{p_1 p_2} |h(x_1) - h(x_2)| \quad (4.90)$$

which matches the result obtained using Roseblueth approach.

[18] shows that if a three point estimate method is used (i.e.  $m = 3$ ):

$$p_i = \frac{\xi_j \xi_k + 1}{(\xi_j - \xi_i)(\xi_k - \xi_i)} \text{ for } i, j, k = 1, 2, 3 \text{ and } i \neq j \neq k \neq i \quad (4.91)$$

and  $\xi_i$  are the solution of the following equation:

$$d_3 \xi^3 + d_2 \xi^2 + d_1 \xi + d_0 = 0 \quad (4.92)$$

where

$$d_3 = \lambda_{X,4} - (1 + \lambda_{X,3}^2)$$

$$d_2 = \lambda_{X,3}(\lambda_{X,4} + 1) - \lambda_{X,5}$$

$$d_1 = \lambda_{X,3}(\lambda_{X,5} - \lambda_{X,3}) + \lambda_{X,4}(1 - \lambda_{X,4})$$

$$d_0 = \lambda_{X,5} - \lambda_{X,3}(2\lambda_{X,4} - \lambda_{X,3}^2)$$

In general, to match the first  $2m - 1$  moments of  $X$   $m$  probability concentrations are needed. As a matter of fact, for the previously derived case two probability concentrations were used ( $m = 2$ ) which generated the ability to match the first three moments of  $X$  ( $3 = 2 \times 2 - 1$ ). If one takes  $m = 3$ , but places one of the concentrations at the mean of the input random variable, then he/she can match up to the fourth moment of the input random variable.

For  $m$  probability concentrations (at the locations  $x_i = \mu_X + \xi_i \sigma_X$  for  $i = 1, \dots, m$ ), meeting the set of simultaneous equations  $\sum_{j=1}^m p_j \xi_j^i = \frac{M'_i(X)}{\sigma_X^i}$ , the mean of  $Y$  is expressed as

$$\mu_Y = \sum_{j=1}^m p_j h(x_j) + \sum_{i=2}^{\infty} \frac{1}{i!} h^{(i)}(\mu_X) [\lambda_{X,i} - \sum_{j=1}^m p_j \xi_j^i] \sigma_X^i \quad (4.93)$$

$$\mu_Y \cong \sum_{j=1}^m p_j h(x_j) \quad (4.94)$$

which is an approximation of order  $2m - 1$ . Higher order moments' approximation can also be done in a similar manner but will have a lower order of approximation.

In general, for  $m$  probability concentrations, the PEM leads to an exact approximation of the mean of the output function if  $Y = h(X)$  is of order  $2m - 1$  or less. Moreover, the PEM provides an exact approximation of the standard deviation of the output function if  $Y = h(X)$  is of order  $m - 1$  or less.

#### **4.4. Summary**

In this chapter, a review of two PEM approaches, the first introduced by E. Rosenblueth and the second by H.P. Hong, for dealing with problems with one input random variable is presented. This chapter serves as an introduction to the next chapter where the application of point estimate methods to problems with multiple input random variables is considered. In fact, the P-OPF problem includes multiple input variables that are modeled as random variables. Hence, the PEM needs to be extended to the multivariate case before being applied to the P-OPF model.

## ***Chapter V: Point Estimate Method – Multivariate Case***

### ***5.1. Introduction***

In the previous chapter, Rosenblueth's and Hong's PEMs are explored and the mathematical concepts behind their application to the univariate case are derived and explained. In this chapter, a more general case is treated; the multivariate case. In the multivariate case, the system's output variables depend on multiple input variables. This chapter explores and derives the mathematical concepts behind three different point estimate methods, namely, Rosenblueth's, Hong's and Harr's methods for the multivariate case. Moreover, this chapter includes a proposed combination of Hong's and Harr's PEMs that can treat problems with uncorrelated and correlated input random variables in a computationally efficient manner. At the last section of this chapter, the methodology of implementing the different PEMs and MCS to the OPF problem to estimate the LMPs and total generation cost statistical moments is provided.

### ***5.2. Review of Emilio Rosenblueth's PEM Approach***

Emilo Rosenblueth's PEM for the multivariate case is introduced in [15] and [16]. This section includes a full description of Rosenblueth's method for the multivariate case including all the relevant mathematical proofs and explanations.

As explained in Chapter IV, in the Two Points Estimate Method developed by Rosenblueth the input random variables are each estimated by two probability concentrations at two locations such that their first moments match those of the input random variables. Hence, to find the appropriate locations and weights of those probability concentrations, a set of simultaneous equations needs to be solved which dictates that the first few central moments of the probability concentrations match the first few central moments of the original input variables. In Chapter IV, for the univariate case, those equations are given through equations (4.21) – (4.24).

With multiple random variables, the number of equations to be solved simultaneously, so that the assigned probability concentrations have a distribution that has moments similar to those of the distribution of the input random variables, increases drastically. In the next explanation, the following notation is considered:



$n$ : the number of random variables

$i$ : the order of the moment of the probability distribution

$N_i$ : the number of moments of order  $i$  to be satisfied

The number of equations to be solved simultaneously for the probability concentrations to have moments similar to the moments of the input random variables is as shown in Table 5.1.

Table 5.1. Number of Conditions to Be Met for the Desired Moments to Be Matched, E. Rosenblueth, “Two-point estimates in probability”, *Appl. Math. Model.*, vol. 5, no. 5, pp. 329-335, Oct. 1981, Used under fair use, 2014.

n	1	2	3	4	5	6	7
$N_0$	1	1	1	1	1	1	1
$N_0 + N_1$	2	3	4	5	6	7	8
$N_0 + N_1 + N_2$	3	6	10	15	21	28	36
$N_0 + N_1 + N_2 + N_3$	4	10	20	35	56	84	120

Table 5.1 shows, for example, that for one random variable if one needs to meet up to the third moment of the input random variable he/she then needs to solve four simultaneous equations. This was indeed proven through the derivations in the case of a single random variable in Chapter IV. For two input random variables, if one needs to meet the first three moments of these input random variables; then, solving ten simultaneous equations is required. This shows how much the complexity increases when the number of input variables increases. The required number of equations to be solved simultaneously can be obtained as explained next.

For multivariate probability distributions  $p(x_1, x_2, x_3 \dots)$ , the multivariate moment  $j, k, \dots$  is defined as follows:

$$\mu_{j,k,\dots} = E \left[ (X_1 - E(X_1))^j (X_2 - E(X_2))^k \dots \right] \quad (5.1)$$

Hence, for example, the covariance is nothing but the central moment of two variables with order 1,1 ( $cov(X_1, X_2) = \mu_{1,1}$ )

An explanation of the significance of the number  $N_i$  is presented next.

Considering the case of  $n = 2$  i.e. two random variables:

$N_0 = 1$  which corresponds to  $\mu_{0,0}$ .

$N_1 = 2$  which corresponds to  $\mu_{1,0}$  and  $\mu_{0,1}$ . That's why in Table 5.1, one can see that  $N_0 + N_1 = 3$ .

$N_2 = 3$  which corresponds to  $\mu_{2,0}$ ,  $\mu_{0,2}$  and  $\mu_{1,1}$ . That's why in Table 5.1 one can see that  $N_0 + N_1 + N_2 = 6$ .

$N_3 = 4$  which corresponds to  $\mu_{3,0}$ ,  $\mu_{0,3}$ ,  $\mu_{2,1}$  and  $\mu_{1,2}$ . That's why in Table 5.1, one can see that  $N_0 + N_1 + N_2 + N_3 = 10$ .

For the case of  $n = 3$  i.e. three random variables:

$N_0 = 1$  which corresponds to  $\mu_{0,0,0}$

$N_1 = 3$  which corresponds to  $\mu_{1,0,0}$ ,  $\mu_{0,1,0}$  and  $\mu_{0,0,1}$ . That's why in Table 5.1, one can see that  $N_0 + N_1 = 4$ ; and so on and so forth.

Hence, for the chosen probability concentrations to match up to the first  $k$  moments of the input random variables, the number of equations to be solved simultaneously is equal to  $\sum_{i=0}^K N_i$  equations. It can be proven (the result is given in [16]) that for  $n$  random variables:

$$\sum_{i=0}^K N_i = \frac{(k+n)!}{k!n!} \quad (5.2)$$

This is the number of equations that needs to be solved simultaneously so that the chosen probability concentrations match up to the  $k^{th}$  moment of the  $n$  input variables joint distribution (where  $n$  is the number of input random variables). The unknowns in this set of equations are the locations and weights of the corresponding probability concentrations.

Indeed, for every probability concentration, one should find its associated location, or the specific realization for each of the input random variables (i.e. the values of  $x_1, x_2, \dots, x_n$ ), and weight  $p$ . Hence, every probability concentration corresponds to  $n + 1$  unknowns or parameters. Thus, the total number of parameters to be determined, i.e. the total number of unknowns, is

$(n + 1) \times \text{number of concentrations}$ . In other words, the number of equations that can be solved using these parameters is equal to  $(n + 1) \times \text{number of concentrations}$ .

Hence, as previously discussed, the number of equations that need to be solved simultaneously is equal to  $\sum_{i=0}^K N_i$ . This is indeed the number of equations resulting from one's will to match, up to a certain order, the central moments of the input probability distribution(s). Since for every concentration, the corresponding  $n + 1$  unknowns/parameters govern  $n + 1$  equations, then the total number of parameters given by  $(n + 1) \times \text{number of concentrations}$  should be greater than the number of conditions to be met i.e. the number of equations to be solved simultaneously given by equation (5.2); In other words, the total number of unknowns/parameters should be greater than the total number of equations that need to be met since by having an enough number of unknowns/parameters one can tune these unknowns so that all the required equations are satisfied.

In mathematical terms, the number of concentrations needed should follow

$$(n + 1) \times \text{number of concentrations} \geq \sum_{i=0}^K N_i \quad (5.3)$$

Thus,

$$\text{number of concentrations} \geq \sum_{i=0}^K \frac{N_i}{n+1} \quad (5.4)$$

The total number of unknowns generates flexibility in solving the set of simultaneous equations since by changing these unknowns one can match the associated set of equations. This is why these unknowns are looked upon as parameters to be tuned more than as unknowns to be solved for.

The number of unknowns directly results from the number of probability concentrations that one chooses to allocate to each random variable in the system. For instance, for the univariate case treated in Chapter IV, to match up to the third moment of the random variable, solving simultaneously four equations given by equations (4.21)-(4.24) is required. On the other hand, the associated number of unknowns is defined as  $(n + 1) \times \text{number of concentrations}$ . Hence, in the univariate case,  $n = 1$  and the number of concentrations is chosen to be equal to two, i.e. the random variable is replaced by two probability concentrations at two locations. Thus, the total number of unknowns/parameters is equal to four which is enough to solve the

four equations. Has one chosen to have only one probability concentration, the number of unknowns would have been equal to two and he/she would not have been able to write the four equations necessary to meet the first three central moments of the input random variable. In other words, using only one probability concentration results in matching only the equations of the zeroth and first central moments of the input random variable; indeed, using only one probability concentration at one location one cannot match the variance of the input random variable since having only one point the variance is always equal to zero. The same logic applies for higher order central moments.

Hence, the minimum number of concentrations required to meet the required conditions (insuring that the chosen concentrations' moments match the input variable's first  $k$  moments) is the ceiling of  $\sum_{i=0}^K \frac{N_i}{n+1}$ . Knowing that  $\sum_{i=0}^K N_i = \frac{(k+n)!}{k!n!}$  from equation (5.2), the smallest number of concentrations is then

$$ceiling\left(\sum_{i=0}^K \frac{N_i}{n+1}\right) = ceiling\left(\frac{(k+n)!}{k!n!(n+1)}\right) = ceiling\left(\frac{(k+n)!}{k!(n+1)!}\right) \quad (5.5)$$

For  $k = 3$ , we get:

$$ceiling\left(\frac{(k+n)!}{k!(n+1)!}\right) = ceiling\left(\frac{(3+n)!}{3!(n+1)!}\right) = ceiling\left(\frac{(3+n)(2+n)}{3!}\right)$$

This value is consistent with the results obtained in Table 5.2.

The number of redundant parameters if the smallest number of concentrations which is  $ceiling\left(\sum_{i=0}^K \frac{N_i}{n+1}\right)$  is taken, will be equal to  $ceiling\left(\sum_{i=0}^K \frac{N_i}{n+1}\right) \times (n+1) - \sum_{i=0}^K N_i$ . If one uses  $2^n$  probability concentrations for  $n$  random variables, he/she will have  $2^n \times (n+1) - \sum_{i=0}^K N_i$  redundant parameters. A redundant parameter is a parameter which value can be assigned prior to the solution of the simultaneous equations. Table 5.2 shows the number of concentrations and the number of redundant parameters for  $k = 3$  (i.e. match up to the third central moment of the input random variables).

The first column, (1), is the number of random variables.

The second column, (2), is the number of required equations to be met in order to match up to the third moment i.e.  $\sum_{i=0}^3 N_i = \frac{(3+n)!}{3!n!}$

The third column, (3), is the minimum number of concentrations needed  $\left(\frac{N_0+N_1+N_2+N_3}{n+1}\right) = \frac{\sum_{i=0}^3 N_i}{n+1}$

The fourth column, (4), is the ceiling of the third.

The fifth column, (5), is the number of redundant parameters if the minimum number of concentrations (i.e. number in (4)) is chosen. This is equal to "(4)"  $\times$  (n + 1) – "(2)"; where, "(2)" and "(4)" are the corresponding values in the second and fourth columns respectively.

The sixth column, (6), is the number of redundant parameters if  $2^n$  concentrations are used. This number is equal to  $2^n \times (n + 1) - "(2)"$ .

Table 5.2. Number of Concentrations and Number of Redundant Parameters, E. Rosenblueth, "Two-point estimates in probability", *Appl. Math. Model.*, vol. 5, no. 5, pp. 329-335, Oct. 1981, Used under fair use, 2014.

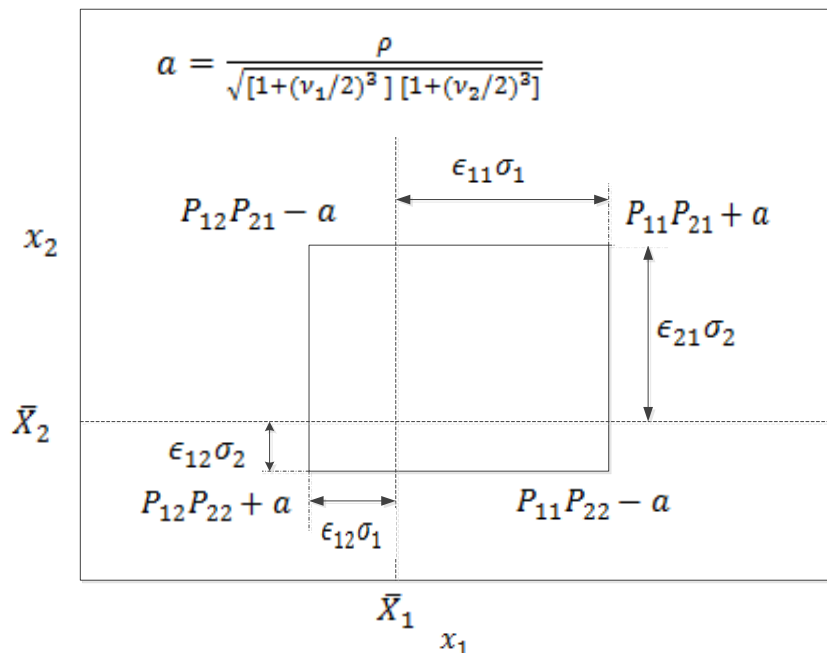
# of Random Variables	# of Required Equations	Minimum Concentrations Needed	Ceiling of Min. Concentrations Needed	Redundant Parameters for Min. Number of Concentrations	Redundant Parameters for $2^n$ Concentrations
1	4	2	2	0	0
2	10	3.3	4	2	2
3	20	5	5	0	12
4	35	7	7	0	45
5	56	9.3	10	4	136
6	84	12	12	0	364
7	120	15	15	0	904

As can be seen in Table 5.1 the number of equations that needs to be solved increases dramatically with the increase in the number of random variables. For instance, taking the case of two random variables with the need of matching up to the third moment of the input random variable requires the solution of ten simultaneous equations. Taking four probability concentrations at four locations results in twelve unknowns i.e  $4 \times (n + 1) = 4 \times (2 + 1) = 12$  (the unknowns are the coordinates  $X_1$  and  $X_2$  as well as the probability magnitude at each of the four points). Having twelve parameters and ten equations; then, one can pick arbitrary values for

two out of the twelve unknowns and solve for the others (two of the parameters are then redundant in this case).

The simultaneous equations to be solved are nonlinear equations. When the number of equations grows the solution of this set of equations becomes complicated. Because of the fast increase in the number of equations with the increase in the number of random variables, another approach can be used. This approach is based on taking a large number of concentrations creating a large number of redundant parameters. Some of the parameters are given specific values and the equations are solved for the remaining parameters. The logic behind this approach lies in the plan of increasing the number of parameters and fixing some of them so that the set of equations to be solved becomes linear.

In Rosenblueth's approach, a large number of concentrations is used ( $2^n$  for  $n$  random variables). However, the coordinates of the location of these concentrations are chosen in a manner to form a rectangle, prism or a hypoprism as shown in Figure 5.1. Hence, most of the coordinates are picked (except of two of them) but the probability concentrations' weights need to be calculated.



**Figure 5.1.** Concentrations for a Function of Two Variables,

E. Rosenblueth, "Two-point estimates in probability", *Appl. Math. Model.*, vol. 5, no. 5, pp. 329-335, Oct. 1981, Used under fair use, 2014.

This number of unknowns is enough to match the zeroth, first and second moment of the p.d.f. of each of the  $x_i$ 's given by  $\int_{-\infty}^{+\infty} (x_i - \bar{X}_i)^2 p_{x_i}(x_i) dx_i$

When the input random variables are non-skewed [15] and [59] state that Rosenblueth PEM is as follows:

Let  $y = y(x_1, x_2, x_3, \dots, x_n)$  be the output random variable which depends on  $n$  input random variables  $x_1, x_2, x_3, \dots, x_n$ . For the general multivariate case, the output function is evaluated  $2^n$  times when the output function depends on  $n$  input random variables. The concentrations are located at  $\bar{x}_i \pm s_i$  for each of the random variables. Hence, each random variable is replaced by two concentrations; that's why the total number of concentrations is  $2^n$ . The output function is evaluated as follows:

$$y_{\pm\pm\pm\dots} = y(\bar{x}_1 \pm s_1, \bar{x}_2 \pm s_2, \bar{x}_3 \pm s_3, \dots, \bar{x}_n \pm s_n) \quad (5.6)$$

The weights associated with these locations are as follows (for  $n = 3$ ):

$$p_{+++} = p_{---} = \left(\frac{1}{2}\right)^3 (1 + \rho_{12} + \rho_{23} + \rho_{31}) \quad (5.7)$$

$$p_{++-} = p_{--+} = \left(\frac{1}{2}\right)^3 (1 + \rho_{12} - \rho_{23} - \rho_{31}) \quad (5.8)$$

$$p_{+-+} = p_{-+-} = \left(\frac{1}{2}\right)^3 (1 - \rho_{12} - \rho_{23} + \rho_{31}) \quad (5.9)$$

$$p_{+--} = p_{-++} = \left(\frac{1}{2}\right)^3 (1 - \rho_{12} + \rho_{23} - \rho_{31}) \quad (5.10)$$

Next, a proof that the locations and weights given in equations (5.6)-(5.10) do match the input variables central moments is derived. The case of three random variables,  $x_1, x_2$  and  $x_3$  whose means and standards deviations respectively are  $(\mu_1, \sigma_1), (\mu_2, \sigma_2)$  and  $(\mu_3, \sigma_3)$ , is considered. The correlation between random variables  $x_i$  and  $x_j$  is denoted by  $\rho_{ij}$ . Following Rosenblueth 2PEM, eight probability concentrations defined as shown in Table 5.3 are considered. The equations that need to be solved simultaneously matching the central moments of the concentrations in Table 5.3 to the exact input random variables central moments given in equation (5.1) are as follows:

Table 5.3. Rosenblueth's 2PEM Concentrations Locations & Weights for Three Input Variables

LOCATION	PROBABILITY
$(\mu_1 + \sigma_1, \mu_2 + \sigma_2, \mu_3 + \sigma_3)$	$P_{+++}$
$(\mu_1 + \sigma_1, \mu_2 + \sigma_2, \mu_3 - \sigma_3)$	$P_{++-}$
$(\mu_1 + \sigma_1, \mu_2 - \sigma_2, \mu_3 + \sigma_3)$	$P_{+-+}$
$(\mu_1 + \sigma_1, \mu_2 - \sigma_2, \mu_3 - \sigma_3)$	$P_{+--}$
$(\mu_1 - \sigma_1, \mu_2 + \sigma_2, \mu_3 + \sigma_3)$	$P_{-++}$
$(\mu_1 - \sigma_1, \mu_2 + \sigma_2, \mu_3 - \sigma_3)$	$P_{-+-}$
$(\mu_1 - \sigma_1, \mu_2 - \sigma_2, \mu_3 + \sigma_3)$	$P_{--+}$
$(\mu_1 - \sigma_1, \mu_2 - \sigma_2, \mu_3 - \sigma_3)$	$P_{---}$

- $\mu_{000}$

$$(P_{+++}) + (P_{++-}) + (P_{+-+}) + (P_{+--}) + (P_{-++}) + (P_{-+-}) + (P_{--+}) + (P_{---}) = 1$$

- $\mu_{001}$

$$(\mu_3 + \sigma_3 - \mu_3)P_{+++} + (\mu_3 - \sigma_3 - \mu_3)P_{++-} + (\mu_3 + \sigma_3 - \mu_3)P_{+-+} + (\mu_3 - \sigma_3 - \mu_3)P_{+--} +$$

$$(\mu_3 + \sigma_3 - \mu_3)P_{-++} + (\mu_3 - \sigma_3 - \mu_3)P_{-+-} + (\mu_3 + \sigma_3 - \mu_3)P_{--+} + (\mu_3 - \sigma_3 - \mu_3)P_{---} = 0$$

$$\Rightarrow \sigma_3(P_{+++}) - \sigma_3(P_{++-}) + \sigma_3(P_{+-+}) - \sigma_3(P_{+--}) + \sigma_3(P_{-++}) - \sigma_3(P_{-+-}) + \sigma_3(P_{--+}) -$$

$$\sigma_3(P_{---}) = 0$$

- $\mu_{010}$

$$\sigma_2(P_{+++}) + \sigma_2(P_{++-}) - \sigma_2(P_{+-+}) - \sigma_2(P_{+--}) + \sigma_2(P_{-++}) + \sigma_2(P_{-+-}) - \sigma_2(P_{--+}) -$$

$$\sigma_2(P_{---}) = 0$$

- $\mu_{100}$

$$\sigma_1(P_{+++}) + \sigma_1(P_{++-}) + \sigma_1(P_{+-+}) + \sigma_1(P_{+--}) - \sigma_1(P_{-++}) - \sigma_1(P_{-+-}) - \sigma_1(P_{--+}) -$$

$$\sigma_1(P_{---}) = 0$$

- $\mu_{002}$



$$(\mu_3 + \sigma_3 - \mu_3)^2 P_{+++} + (\mu_3 - \sigma_3 - \mu_3)^2 P_{++-} + (\mu_3 + \sigma_3 - \mu_3)^2 P_{+-+} + (\mu_3 - \sigma_3 - \mu_3)^2 P_{+--} + (\mu_3 + \sigma_3 - \mu_3)^2 P_{-++} + (\mu_3 - \sigma_3 - \mu_3)^2 P_{-+-} + (\mu_3 + \sigma_3 - \mu_3)^2 P_{--+} + (\mu_3 - \sigma_3 - \mu_3)^2 P_{---} = \sigma_3^2$$

$$\Rightarrow \sigma_3^2(P_{+++}) + \sigma_3^2(P_{++-}) + \sigma_3^2(P_{+-+}) + \sigma_3^2(P_{+--}) + \sigma_3^2(P_{-++}) + \sigma_3^2(P_{-+-}) + \sigma_3^2(P_{--+}) + \sigma_3^2(P_{---}) = \sigma_3^2$$

$$\Rightarrow (P_{+++}) + (P_{++-}) + (P_{+-+}) + (P_{+--}) + (P_{-++}) + (P_{-+-}) + (P_{--+}) + (P_{---}) = 1$$

- $\mu_{020}$

$$(P_{+++}) + (P_{++-}) + (P_{+-+}) + (P_{+--}) + (P_{-++}) + (P_{-+-}) + (P_{--+}) + (P_{---}) = 1$$

- $\mu_{200}$

$$(P_{+++}) + (P_{++-}) + (P_{+-+}) + (P_{+--}) + (P_{-++}) + (P_{-+-}) + (P_{--+}) + (P_{---}) = 1$$

- $\mu_{011}$

$$(\mu_2 + \sigma_2 - \mu_2)(\mu_3 + \sigma_3 - \mu_3)P_{+++} + (\mu_2 + \sigma_2 - \mu_2)(\mu_3 - \sigma_3 - \mu_3)P_{++-} + (\mu_2 - \sigma_2 - \mu_2)(\mu_3 + \sigma_3 - \mu_3)P_{+-+} + (\mu_2 - \sigma_2 - \mu_2)(\mu_3 - \sigma_3 - \mu_3)P_{+--} + (\mu_2 + \sigma_2 - \mu_2)(\mu_3 + \sigma_3 - \mu_3)P_{-++} + (\mu_2 + \sigma_2 - \mu_2)(\mu_3 - \sigma_3 - \mu_3)P_{-+-} + (\mu_2 - \sigma_2 - \mu_2)(\mu_3 + \sigma_3 - \mu_3)P_{--+} + (\mu_2 - \sigma_2 - \mu_2)(\mu_3 - \sigma_3 - \mu_3)P_{---} = cov(x_2, x_3)$$

$$\Rightarrow \sigma_2\sigma_3P_{+++} - \sigma_2\sigma_3P_{++-} - \sigma_2\sigma_3P_{+-+} + \sigma_2\sigma_3P_{+--} + \sigma_2\sigma_3P_{-++} - \sigma_2\sigma_3P_{-+-} - \sigma_2\sigma_3P_{--+} + \sigma_2\sigma_3P_{---} = cov(x_2, x_3)$$

$$\Rightarrow (P_{+++}) - (P_{++-}) - (P_{+-+}) + (P_{+--}) + (P_{-++}) - (P_{-+-}) - (P_{--+}) + (P_{---}) = \rho_{23}$$

- $\mu_{101}$

$$\sigma_1\sigma_3P_{+++} - \sigma_1\sigma_3P_{++-} + \sigma_1\sigma_3P_{+-+} - \sigma_1\sigma_3P_{+--} - \sigma_1\sigma_3P_{-++} + \sigma_1\sigma_3P_{-+-} - \sigma_1\sigma_3P_{--+} + \sigma_1\sigma_3P_{---} = cov(x_1, x_3)$$

$$\Rightarrow (P_{+++}) - (P_{++-}) + (P_{+-+}) - (P_{+--}) - (P_{-++}) + (P_{-+-}) - (P_{--+}) + (P_{---}) = \rho_{13}$$

- $\mu_{110}$

$$\sigma_1\sigma_2P_{+++} + \sigma_1\sigma_2P_{++-} - \sigma_1\sigma_2P_{+-+} - \sigma_1\sigma_2P_{+--} - \sigma_1\sigma_2P_{-++} - \sigma_1\sigma_2P_{-+-} + \sigma_1\sigma_2P_{--+} + \sigma_1\sigma_2P_{---} = cov(x_1, x_2)$$

$$\Rightarrow (P_{+++}) + (P_{++-}) - (P_{+-+}) - (P_{+--}) - (P_{-++}) - (P_{-+-}) + (P_{--+}) + (P_{---}) = \rho_{12}$$

Hence, the associated set of equations is the following:

$$(P_{+++}) + (P_{++-}) + (P_{+-+}) + (P_{+--}) + (P_{-++}) + (P_{-+-}) + (P_{--+}) + (P_{---}) = 1 \quad (5.11)$$

$$(P_{+++}) - (P_{++-}) + (P_{+-+}) - (P_{+--}) + (P_{-++}) - (P_{-+-}) + (P_{--+}) - (P_{---}) = 0 \quad (5.12)$$

$$(P_{+++}) + (P_{++-}) - (P_{+-+}) - (P_{+--}) + (P_{-++}) + (P_{-+-}) - (P_{--+}) - (P_{---}) = 0 \quad (5.13)$$

$$(P_{+++}) + (P_{++-}) + (P_{+-+}) + (P_{+--}) - (P_{-++}) - (P_{-+-}) - (P_{--+}) - (P_{---}) = 0 \quad (5.14)$$

$$(P_{+++}) + (P_{++-}) + (P_{+-+}) + (P_{+--}) + (P_{-++}) + (P_{-+-}) + (P_{--+}) + (P_{---}) = 1 \quad (5.15)$$

$$(P_{+++}) + (P_{++-}) + (P_{+-+}) + (P_{+--}) + (P_{-++}) + (P_{-+-}) + (P_{--+}) + (P_{---}) = 1 \quad (5.16)$$

$$(P_{+++}) + (P_{++-}) + (P_{+-+}) + (P_{+--}) + (P_{-++}) + (P_{-+-}) + (P_{--+}) + (P_{---}) = 1 \quad (5.17)$$

$$(P_{+++}) - (P_{++-}) - (P_{+-+}) + (P_{+--}) + (P_{-++}) - (P_{-+-}) - (P_{--+}) + (P_{---}) = \rho_{23} \quad (5.18)$$

$$(P_{+++}) - (P_{++-}) + (P_{+-+}) - (P_{+--}) - (P_{-++}) + (P_{-+-}) - (P_{--+}) + (P_{---}) = \rho_{13} \quad (5.19)$$

$$(P_{+++}) + (P_{++-}) - (P_{+-+}) - (P_{+--}) - (P_{-++}) - (P_{-+-}) + (P_{--+}) + (P_{---}) = \rho_{12} \quad (5.20)$$

By solving this set of equations one obtains the solutions given in equations (5.7) – (5.10).  $\rho_{ij}$  is the correlation coefficient between the random variables  $x_i$  and  $x_j$ . The sign of the coefficient multiplying  $\rho_{ij}$  in the expression of the weights given in equations (5.7) – (5.10) depends on the multiplication of the sign of  $i$  and  $j$  in  $p_{ijk}$ . For instance, for the case of  $p_{+-+}$  and  $p_{-+-}$ : the sign of the coefficient that multiplies  $\rho_{12}$  in the expression of the weights  $p_{+-+}$  and  $p_{-+-}$  given in equations (5.9) is the sign of  $(+) \times (-) = (-) \times (+) = -$ , the sign that multiplies  $\rho_{23}$  is the sign of  $(-) \times (+) = (+) \times (-) = -$  and the sign that multiplies  $\rho_{31}$  is the sign of  $(+) \times (+) = (-) \times (-) = +$ .

Hence,

$$p_{+-+} = p_{-+-} = \left(\frac{1}{2}\right)^3(1 - \rho_{12} - \rho_{23} + \rho_{31}) .$$

This applies to the general case of  $n$  random variables. The total number of concentrations is  $2^n$ , and the total number of correlation coefficients is  $\frac{n(n-1)}{2}$  which is the number of

combinations of 2 out of  $n$ . The sign that multiplies each of the correlation coefficients in the expression of the weights is obtained using the rule described. The right hand side coefficient of the expression of the probability concentrations' weights is  $\left(\frac{1}{2}\right)^n$ . Each of the weights is a probability; hence,  $0 \leq p \leq 1$  and  $\sum p_{\pm\pm\pm\dots} = 1$ .

The  $q^{th}$  order expectation of the output random variable, for  $n = 3$  for example, is calculated as follows:

$$E[y^q] = p_{+++}y_{+++}^q + p_{++-}y_{++-}^q + \dots + p_{---}y_{---}^q \quad (5.21)$$

The expectation of  $y$  is obtained by replacing  $q$  by 1. The variance of  $y$  is obtained using the following formula:

$$s_y^2 = E[y^2] - (E[y])^2 \quad (5.22)$$

### 5.3. Review of H.P Hong's PEM Approach

In this section, Hong's approach, dealing with the multivariate case where the output random variable  $Z$  depends on  $n$  input random variables  $X_1, X_2, \dots, X_n$  such that  $Z = h(X_1, X_2, \dots, X_n)$ , is considered. Hong's PEM is introduced in [18]. The mean, standard deviation and coefficient of variation of  $X_k$  are respectively given by  $\mu_k, \sigma_k$  and  $\nu_k$ . The correlation coefficient between any two random variables  $X_i$  and  $X_j$  is denoted by  $\rho_{ij}$ . Hong's method was primarily developed to incorporate non-correlated input random variables ( $\rho_{ij} = 0$ ). Hong suggests using a matrix transformation when having correlated input random variables to transform them into a set of uncorrelated random variables before applying the suggested PEM schemes.

Following Hong's method, the locations of the concentrations are chosen to be as follows:

$$location_{k,i} = l_{k,i} = (\mu_1, \mu_2, \dots, x_{k,i}, \dots, \mu_{n-1}, \mu_n) \quad (5.23)$$

As previously explained, the PEM probability concentrations are located at specific locations. Each location is defined by realizations of the input random variables. In other words, for the case of  $n$  input random variables, one can think of the problem as having an  $n -$  dimensional space. Each point in the  $n -$  dimensional space has  $n$  coordinates. Each of these  $n$  coordinates refers to a specific value of each input random variable. Each probability

concentration generated using any PEM is located at a location in this  $n$  – dimensional space and has a specific weight. Hence, each concentration location is nothing but a set of realizations of each of the input random variables. For Hong's PEM, each concentration is located at a location where its coordinates are equal to the mean values of the corresponding input random variables except for one of the coordinates which is placed at  $x_{k,i}$  as shown in equations (5.23) and (5.24). The associated probability concentration weight for each location,  $l_{k,i} = (\mu_1, \mu_2, \dots, x_{k,i}, \dots, \mu_{n-1}, \mu_n)$ , is given by  $p_{ki}$ .

$$x_{k,i} = \mu_k + \xi_{k,i} \sigma_k \text{ for } i = 1, 2, \dots, m, k = 1, 2, \dots, n \quad (5.24)$$

$k$  denotes the index of the random variable.

$i$  denotes the index of the probability concentrations .

$\xi_{k,i}$  are the concentration coefficients defined in Chapter IV where

$$\xi_{k,i} = \frac{x_{k,i} - \mu_k}{\sigma_k} \text{ for } i = 1, 2, \dots, m, k = 1, 2, \dots, n \quad (5.25)$$

Hong allocates  $m$  probability concentrations for each of the  $n$  random variable. Thus, the total number of points (i.e. probability concentrations) is equal to  $m \times n$ .

Following the same procedure as in the univariate case, the multivariate Taylor series expansion of  $h(\underline{X})$  is written. The case of two input random variables is considered next such that

$$Z = h(\mathbf{X}) = h(x, y). \quad (5.26)$$

The multivariate Taylor series expansion of  $Z = h(\mathbf{X})$  around the mean value of  $\mathbf{X}$ ,  $\bar{\mathbf{X}} = \mu_{\mathbf{X}} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$ , can be written as:

$$\begin{aligned} Z = h(\mathbf{X}) &= h(\bar{\mathbf{X}}) + \sum_{i=1}^n (X_i - \bar{X}_i) h_i(\bar{\mathbf{X}}) + \frac{1}{2!} \sum_{i=1}^n \sum_{j=1}^n (X_i - \bar{X}_i)(X_j - \bar{X}_j) h_{ij}(\bar{\mathbf{X}}) + \\ &\frac{1}{3!} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n (X_i - \bar{X}_i)(X_j - \bar{X}_j)(X_k - \bar{X}_k) f_{ijk}(\bar{\mathbf{X}}) + \dots \\ Z = h(\mu_x, \mu_y) &+ (x - \mu_x) h_x(\mu_x, \mu_y) + (y - \mu_y) h_y(\mu_x, \mu_y) + \frac{1}{2!} [(x - \mu_x)(y - \\ \mu_y) h_{xy}(\mu_x, \mu_y) &+ (x - \mu_x)^2 h_{xx}(\mu_x, \mu_y) + (y - \mu_y)(x - \mu_x) h_{yx}(\mu_x, \mu_y) + \end{aligned} \quad (5.27)$$

$$(y - \mu_x)^2 h_{yy}(\mu_x, \mu_y)] + \frac{1}{3!} [(x - \mu_x)^3 h_{xxx}(\mu_x, \mu_y) + (y - \mu_y)^3 h_{yyy}(\mu_x, \mu_y) + 3(x - \mu_x)^2 (y - \mu_y) h_{xxy}(\mu_x, \mu_y) + 3(x - \mu_x)(y - \mu_y)^2 h_{yyx}(\mu_x, \mu_y)]$$

where  $h_x$  is the partial derivative of  $h(x, y)$  with respect to  $x$  and  $h_y$  is the partial derivative of  $h(x, y)$  with respect to  $y$ .

Two probability concentrations for each random variable is taken where the points are located at the following locations:

$$l_{x,1} = (x_1, \mu_y) \quad (5.28)$$

$$l_{x,2} = (x_2, \mu_y) \quad (5.29)$$

$$l_{y,1} = (\mu_x, y_1) \quad (5.30)$$

$$l_{y,2} = (\mu_x, y_2) \quad (5.31)$$

The third order Taylor series expansion of  $h(\mathbf{X})$  around  $\mu_x$  and  $\mu_y$  at the four locations  $l_{x,1}, l_{x,2}, l_{y,1}$  and  $l_{y,2}$ , described through equations (5.28)-(5.31), can be derived as shown in equations (5.32)-(5.35)

$$h(x_1, \mu_y) = h(\mu_x, \mu_y) + (x_1 - \mu_x)h_x(\mu_x, \mu_y) + \frac{1}{2!}(x_1 - \mu_x)^2 h_{xx}(\mu_x, \mu_y) + \frac{1}{3!}(x_1 - \mu_x)^3 h_{xxx}(\mu_x, \mu_y) \quad (5.32)$$

$$h(x_2, \mu_y) = h(\mu_x, \mu_y) + (x_2 - \mu_x)h_x(\mu_x, \mu_y) + \frac{1}{2!}(x_2 - \mu_x)^2 h_{xx}(\mu_x, \mu_y) + \frac{1}{3!}(x_2 - \mu_x)^3 h_{xxx}(\mu_x, \mu_y) \quad (5.33)$$

$$h(\mu_x, y_1) = h(\mu_x, \mu_y) + (y_1 - \mu_y)h_y(\mu_x, \mu_y) + \frac{1}{2!}(y_1 - \mu_y)^2 h_{yy}(\mu_x, \mu_y) + \frac{1}{3!}(y_1 - \mu_y)^3 h_{yyy}(\mu_x, \mu_y) \quad (5.34)$$

$$h(\mu_x, y_2) = h(\mu_x, \mu_y) + (y_2 - \mu_y)h_y(\mu_x, \mu_y) + \frac{1}{2!}(y_2 - \mu_y)^2 h_{yy}(\mu_x, \mu_y) + \frac{1}{3!}(y_2 - \mu_y)^3 h_{yyy}(\mu_x, \mu_y) \quad (5.35)$$

$p_{x1}, p_{x2}, p_{y1}$  and  $p_{y2}$  are the respective weights of  $h(x_1, \mu_y), h(x_2, \mu_y), h(\mu_x, y_1), h(\mu_x, y_2)$ .

The weighted sum of  $h(x_1, \mu_y), h(x_2, \mu_y), h(\mu_x, y_1), h(\mu_x, y_2)$  can then be derived as follows:

$$\begin{aligned}
& p_{x1}h(x_1, \mu_y) + p_{x2}h(x_2, \mu_y) + p_{y1}h(\mu_x, y_1) + p_{y2}h(\mu_x, y_2) = h(\mu_x, \mu_y)(p_{x1} + \quad (5.36) \\
& p_{x2} + p_{y1} + p_{y2}) + h_x(\mu_x, \mu_y)(p_{x1}(x_1 - \mu_x) + p_{x2}(x_2 - \mu_x)) + \\
& h_y(\mu_x, \mu_y)(p_{y1}(y_1 - \mu_y) + p_{y2}(y_2 - \mu_y)) + \frac{1}{2!}h_{xx}(\mu_x, \mu_y)(p_{x1}(x_1 - \mu_x)^2 + \\
& p_{x2}(x_2 - \mu_x)^2) + \frac{1}{2!}h_{yy}(\mu_x, \mu_y)(p_{y1}(y_1 - \mu_y)^2 + p_{y2}(y_2 - \mu_y)^2) + \\
& \frac{1}{3!}h_{xxx}(\mu_x, \mu_y)(p_{x1}(x_1 - \mu_x)^3 + p_{x2}(x_2 - \mu_x)^3) + \frac{1}{3!}h_{yyy}(\mu_x, \mu_y)(p_{y1}(y_1 - \mu_y)^3 + \\
& p_{y2}(y_2 - \mu_y)^3)
\end{aligned}$$

The joint probability distribution of  $x$  and  $y$  is denoted by  $f(x, y)$ . The expectation of the Taylor series expansion of  $Z$  around  $\mu_X = (\mu_x, \mu_y)$  can be derived as follows:

$$\begin{aligned}
E(Z) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\mathbf{X})f(x, y)dxdy = \\
& \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\mu_x, \mu_y)f(x, y)dxdy + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_x(\mu_x, \mu_y)(x - \mu_x)f(x, y)dxdy + \\
& \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_y(\mu_x, \mu_y)(y - \mu_y)f(x, y)dxdy + \\
& \frac{1}{2!}2[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \mu_x)(y - \mu_y)h_{xy}(\mu_x, \mu_y)f(x, y)dxdy] + \\
& \frac{1}{2!} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \mu_x)^2 h_{xx}(\mu_x, \mu_y)f(x, y)dxdy + \\
& \frac{1}{2!} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (y - \mu_y)^2 h_{yy}(\mu_x, \mu_y)f(x, y)dxdy + \\
& \frac{1}{3!} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \mu_x)^3 h_{xxx}(\mu_x, \mu_y)f(x, y)dxdy + \\
& \frac{1}{3!} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (y - \mu_y)^3 h_{yyy}(\mu_x, \mu_y)f(x, y)dxdy + \\
& \frac{1}{3!}3 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \mu_x)^2 (y - \mu_y)h_{xxy}(\mu_x, \mu_y)f(x, y)dxdy + \frac{1}{3!}3 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \mu_x)(y - \\
& \mu_y)^2 h_{yyx}(\mu_x, \mu_y)f(x, y)dxdy
\end{aligned}$$

If  $X$  and  $Y$  are assumed to be uncorrelated,

$$\begin{aligned}
E(Z) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\mathbf{X})f(x, y)dxdy = \\
& h(\mu_x, \mu_y) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)dxdy + h_x(\mu_x, \mu_y) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \mu_x) f(x, y)dxdy + \\
& h_y(\mu_x, \mu_y) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (y - \mu_y) f(x, y)dxdy + \frac{1}{2!} h_{xx}(\mu_x, \mu_y) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \mu_x)^2 f(x, y)dxdy + \\
& \frac{1}{2!} h_{yy}(\mu_x, \mu_y) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (y - \mu_y)^2 f(x, y)dxdy +
\end{aligned}$$

$$\frac{1}{3!} h_{xxx}(\mu_x, \mu_y) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \mu_x)^3 f(x, y) dx dy +$$

$$\frac{1}{3!} h_{yyy}(\mu_x, \mu_y) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (y - \mu_y)^3 f(x, y) dx dy$$

However,  $\int_{-\infty}^{+\infty} f(x, y) dx = f(y)$  and

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x) f(x, y) dx dy = \int_{-\infty}^{+\infty} g(x) \int_{-\infty}^{+\infty} f(x, y) dx dy = \int_{-\infty}^{+\infty} g(x) f(x) dx$$

If this property is used,  $E(Z)$  reduces to the following:

$$E(Z) = h(\mu_x, \mu_y) + h_x(\mu_x, \mu_y) M'_{1,x} + h_y(\mu_x, \mu_y) M'_{1,y} + \frac{1}{2} h_{xx}(\mu_x, \mu_y) M'_{2,x} +$$

$$\frac{1}{2} h_{yy}(\mu_x, \mu_y) M'_{2,y} + \frac{1}{3!} h_{xxx}(\mu_x, \mu_y) M'_{3,x} + \frac{1}{3!} h_{yyy}(\mu_x, \mu_y) M'_{3,y}$$

Matching it to the expression of  $p_{x1}h(x_1, \mu_y) + p_{x2}h(x_2, \mu_y) + p_{y1}h(\mu_x, y_1) + p_{y2}h(\mu_x, y_2)$  given in equation (5.36) one gets the following set of equations:

$$p_{x1} + p_{x2} + p_{y1} + p_{y2} = 1 \quad (5.38)$$

$$p_{x1}(x_1 - \mu_x) + p_{x2}(x_2 - \mu_x) = M'_{1,x} \Rightarrow p_{x1}\xi_{x,1}\sigma_x + p_{x2}\xi_{x,2}\sigma_x = \lambda_{x1}\sigma_x \quad (5.39)$$

$$p_{y1}(y_1 - \mu_y) + p_{y2}(y_2 - \mu_y) = M'_{1,y} \Rightarrow p_{y1}\xi_{y,1}\sigma_y + p_{y2}\xi_{y,2}\sigma_y = \lambda_{y1}\sigma_y \quad (5.40)$$

$$p_{x1}(x_1 - \mu_x)^2 + p_{x2}(x_2 - \mu_x)^2 = M'_{2,x} \Rightarrow p_{x1}\xi_{x,1}^2\sigma_x^2 + p_{x2}\xi_{x,2}^2\sigma_x^2 = \lambda_{x2}\sigma_x^2 \quad (5.41)$$

$$p_{y1}(y_1 - \mu_y)^2 + p_{y2}(y_2 - \mu_y)^2 = M'_{2,y} \Rightarrow p_{y1}\xi_{y,1}^2\sigma_y^2 + p_{y2}\xi_{y,2}^2\sigma_y^2 = \lambda_{y2}\sigma_y^2 \quad (5.42)$$

$$p_{x1}(x_1 - \mu_x)^3 + p_{x2}(x_2 - \mu_x)^3 = M'_{3,x} \Rightarrow p_{x1}\xi_{x,1}^3\sigma_x^3 + p_{x2}\xi_{x,2}^3\sigma_x^3 = \lambda_{x3}\sigma_x^3 \quad (5.43)$$

$$p_{y1}(y_1 - \mu_y)^3 + p_{y2}(y_2 - \mu_y)^3 = M'_{3,y} \Rightarrow p_{y1}\xi_{y,1}^3\sigma_y^3 + p_{y2}\xi_{y,2}^3\sigma_y^3 = \lambda_{y3}\sigma_y^3 \quad (5.44)$$

which reduces to the following set of equations:

$$p_{x1}\xi_{x,1} + p_{x2}\xi_{x,2} = \lambda_{x1} \quad (5.45)$$

$$p_{x1}\xi_{x,1}^2 + p_{x2}\xi_{x,2}^2 = \lambda_{x2} \quad (5.46)$$

$$p_{x1}\xi_{x,1}^3 + p_{x2}\xi_{x,2}^3 = \lambda_{x3} \quad (5.47)$$

and

$$p_{y1}\xi_{y,1} + p_{y2}\xi_{y,2} = \lambda_{y1} \quad (5.48)$$

$$p_{y1}\xi_{y,1}^2 + p_{y2}\xi_{y,2}^2 = \lambda_{y2} \quad (5.49)$$

$$p_{y1}\xi_{y,1}^3 + p_{y2}\xi_{y,2}^3 = \lambda_{y3} \quad (5.50)$$

Hence, the equations that need to be solved simultaneously for the general case of  $n$  variables are:

$$\sum_{i=1}^m p_{k,i} (\xi_{k,i})^j = \lambda_{k,j} \text{ for } j = 1, 2, \dots, 2m - 1 \text{ and } k = 1, 2, \dots, n \quad (5.51)$$

In addition to having:

$$\sum_{k=1}^n \sum_{i=1}^m p_{k,i} = 1 \quad (5.52)$$

Accordingly, for each random variable the following set of equations applies.

$$\sum_{i=1}^m p_{k,i} (\xi_{k,i})^j = \lambda_{k,j} \text{ for } j = 1, \dots, 2m - 1 \text{ (these are } 2m - 1 \text{ equations)}$$

$$\sum_{i=1}^m p_{k,i} = \frac{1}{n} \text{ (1 equation)}$$

As a result, Hong's PEM necessitates the solution of  $2m$  simultaneous equations for each random variable with  $2m$  unknowns. The solution of this set of equations results in a third order Taylor series approximation of the mean of the output variable  $Z$ .

In the general case, taking  $m$  probability concentrations to replace each of the input random variables,

$$E(Z) \cong \sum_{k=1}^n \sum_{i=1}^m p_{k,i} (h(\mu_1, \mu_2, \dots, x_{k,i}, \dots, \mu_{n-1}, \mu_n)) \quad (5.53)$$

is a  $2m - 1$  order Taylor series approximation of  $E(Z)$  and

$$E(Z^2) \cong \sum_{k=1}^n \sum_{i=1}^m p_{k,i} \left( h(\mu_1, \mu_2, \dots, x_{k,i}, \dots, \mu_{n-1}, \mu_n) \right)^2 \quad (5.54)$$

is an  $m - 1$  order approximation of  $E(Z^2)$ .

Equations (5.51) and (5.52) were also derived for the univariate case, developed in Chapter IV, where  $m = 2$  and  $\sum_{i=1}^m p_i (\xi_i)^j = \lambda_j$ , for  $j = 1, \dots, 3 = 2m - 1$ . This shows the similarity between the multivariate and univariate cases which is due to the un-correlation of the input random variables assumption. Because of this similarity, the proofs derived for the orders of approximation of the output variable's mean and standard deviation when replacing the input variable by  $m$  probability concentrations are also valid for the multivariate case.



$\lambda_{k,j}$  is expressed as follows:

$$\lambda_{k,j} = \frac{j\text{th moment about the mean of } X_K}{(\sigma_k)^j} = \frac{M_{j,k}}{\sigma_k^j} \quad (5.55)$$

$$\text{where } M_{j,k} = \int_{-\infty}^{+\infty} (x_k - \mu_k)^j f_{X,k} dX_k$$

According to [1] the set of equations given in equations (5.51) and (5.52) can be solved analytically only for the cases where  $m = 2$  or  $m = 3$  with one of the concentration located at the respective random variable's mean. For higher number of concentrations numerical solution algorithms are described in [58].

For the case where two probability concentrations are considered for each random variable (i.e.  $m = 2$ ) the solution to the set of  $2m$  equations for each of the random variables is as follows:

$$\xi_{k,i} = \frac{\lambda_{k,3}}{2} + (-1)^{3-i} \sqrt{n + \left(\frac{\lambda_{k,3}}{2}\right)^2}, \text{ for } i = 1,2 \text{ and } k = 1, \dots, n \quad (5.56)$$

$$p_{k,i} = \frac{1}{n} (-1)^i \frac{\xi_{k,3-i}}{\zeta_K} \quad (5.57)$$

where

$$\zeta_K = 2\sqrt{n + \left(\frac{\lambda_{k,3}}{2}\right)^2} \quad (5.58)$$

[1] points out the following issue. As can be seen in the expression of  $\xi_{k,i}$  given in equation (5.56), as the number of random variables in the system,  $n$ , increases,  $\xi_{k,i}$  increases and hence  $x_{k,i} = \mu_k + \xi_{k,i}\sigma_k$  drifts away from the mean of  $x_k$ , i.e.  $\mu_k$ . Hence, if the number of random variables is significantly big,  $x_{k,i}$  might fall in a location where the corresponding random variable is not defined. This issue is addressed in [57].

If the skewness of all the random variables is null, i.e.  $\lambda_{k,3} = 0$ , equations (5.56) – (5.58) can be written as follows:

$$\xi_{k,i} = (-1)^{3-i} \sqrt{n}, \text{ for } i = 1,2 \text{ and } k = 1, \dots, n \quad (5.59)$$

$$\zeta_K = 2\sqrt{n} \quad (5.60)$$

$$p_{k,i} = \frac{1}{n} (-1)^i \frac{(-1)^{3-(3-i)} \sqrt{n}}{2\sqrt{n}} = \frac{1}{2n} (-1)^i (-1)^i = \frac{1}{2n} \quad (5.61)$$

Hence, the probability concentrations are equal at all the locations ( $p_{k,i} = \frac{1}{2n}$  for  $i = 1, 2$  and  $k = 1, 2, \dots, n$ )

When  $m = 3$  and one of the three concentrations is fixed at the corresponding random variable's mean (this method allows matching up to the fourth moment of the marginal p.d.f. of the corresponding variable), the obtained solution is as follows:

$$\xi_{k,i} = \frac{\lambda_{k,3}}{2} + (-1)^{3-i} \sqrt{\lambda_{k,4} - 3 \frac{\lambda_{k,3}^2}{4}} \quad i = 1, 2 \quad (5.62)$$

$$\xi_{k,3} = 0 \quad (5.63)$$

$$p_{k,i} = \frac{(-1)^{3-i}}{\xi_{k,i}(\xi_{k,1} - \xi_{k,2})}, \quad i = 1, 2 \quad (5.64)$$

$$p_{k,3} = \frac{1}{n} - p_{k,1} - p_{k,2} = \frac{1}{n} - \frac{1}{\lambda_{k,4} - \lambda_{k,3}^2} \quad (5.65)$$

Since one out of the three concentrations for each random variable is taken to be at its mean; then, out of the total  $3 \times n$  concentrations, there is  $n$  that are in fact the same point which coordinates are the means of each of the random variables  $(\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n)$ . This is the case, since the concentration location for each of the concentrations is  $(\mu_1, \mu_2, \dots, x_{k,i}, \dots, \mu_{n-1}, \mu_n)$  but, as mentioned, one of the concentrations is taken to be located at the mean of the respective random variable; hence,  $x_{k,i} = \mu_k$ . The sum of the probabilities (weights) of this common point is:

$$p_0 = \sum_{k=1}^n p_{k,3} = \sum_{k=1}^n \frac{1}{n} - \frac{1}{\lambda_{k,4} - \lambda_{k,3}^2} = 1 - \sum_{k=1}^n \frac{1}{\lambda_{k,4} - \lambda_{k,3}^2} \quad (5.66)$$

Hence, this PEM scheme is in fact a  $2n + 1$  concentrations scheme instead of  $3n$ . As can be seen from the expression of  $\xi_{k,i}$  given in equation (5.62), the location of the probability concentration, in this case, does not depend on the number of random variables involved as opposed to the  $2n$  case. [1] states that this is the case for all the  $m \times n + 1$  concentration schemes. The aforementioned reference also states that the  $2n + 1$  scheme is more

accurate than its  $2n$  counterpart since it takes into consideration the kurtosis,  $\lambda_{k,4}$ , of the input random variables. On the other hand  $\lambda_{k,4} - 3 \frac{\lambda_{k,3}^2}{4}$  can be negative and hence  $\xi_{k,i}$  can be complex.

The  $j^{th}$  moment of  $Z$  denoted by  $E[Z^j]$  can be approximated as follows:

$$E[Z^j] = p_0 \left( h(\mu_1, \mu_2, \dots, \mu_i, \dots, \mu_{n-1}, \mu_n) \right)^j + \sum_{k=1}^n \sum_{i=1}^m p_{k,i} \left( h(\mu_1, \mu_2, \dots, x_{k,i}, \dots, \mu_{n-1}, \mu_n) \right)^j \quad (5.67)$$

If  $m = 3$  without placing one of the concentrations for each random variable at its mean; then,  $\xi_{k,i}$  for  $i = 1, 2, 3$  can be obtained by solving:

$$d_3 \xi^3 + d_2 \xi^2 + d_1 \xi + d_0 = 0 \quad (5.68)$$

where, for  $k = 1, \dots, n$ ,

$$d_3 = \frac{(\lambda_{k,4} - (1 + \lambda_{k,3}^2))}{n} \quad (5.69)$$

$$d_2 = \lambda_{k,3} \left( \lambda_{k,4} + \frac{1}{n} \right) - \frac{\lambda_{k,5}}{n} \quad (5.70)$$

$$d_1 = \lambda_{k,3} \left( \frac{\lambda_{k,5}}{n} - \lambda_{k,3} \right) + \lambda_{k,4} \left( 1 - \frac{\lambda_{k,4}}{n} \right) \quad (5.71)$$

$$d_0 = \lambda_{k,5} - \lambda_{k,3} (2\lambda_{k,4} - \lambda_{k,3}^2) \quad (5.72)$$

and the concentrations weights are equal to:

$$p_{k,i} = \frac{\xi_{k,j} \xi_{k,i}}{(\xi_{k,j} - \xi_{k,i})(\xi_{k,i} - \xi_{k,l})} \text{ where: } i, j, l = 1, 2, 3, i \neq j \neq l \neq i \quad (5.73)$$

The output variables moments can then be approximated as follows:

$$E[Z^j] \cong \sum_{k=1}^n \sum_{i=1}^m p_{k,i} \left( h(\mu_1, \mu_2, \dots, \mu_i, \dots, \mu_{n-1}, \mu_n) \right)^j \quad (5.74)$$

One should note that Hong's method is not a probability distribution transformation since nothing prevents the concentrations  $p_{k,i}$  from being non-positive. This method is a weighing method.

Hong's method can be used instead of that of Rosenblueth when the number of variables is large since this method needs to evaluate  $Z = h(X)$   $m \times n$  times instead of  $2^n$  times as per Rosenblueth's 2PEM.

### **Transforming correlated random variables into uncorrelated ones**

Hong's method assumed that the input random variables are uncorrelated i.e.  $\rho_{ij} = 0$ . However, if they are correlated, i.e.  $\rho_{ij} \neq 0$ , Hong advises on the usage of a matrix transformation such as a rotational transformation which is based on the eigenvectors of the covariance matrix to transform the correlated random variables  $X_i$  into uncorrelated ones  $X'_i$  after which Hong's PEM can be used [18]. This method maps the input random variables into a new coordinate system in which they are uncorrelated. Harr's PEM presented in [17] is based on the mapping of the input random variables into a new coordinate system whose basis are the eigenvectors of the input random variables correlation matrix. [21] uses a de-correlation mechanism based on the singular value decomposition of the covariance matrix to transform correlated input random variables into uncorrelated ones.

Next, a full explanation of the process of transforming a set of correlated variables into uncorrelated ones is presented. The explanation is based on the content of [56].

A linear transformation can be performed to a matrix of random variables. This transformation is considered as a mapping of these random variables to a new coordinate system. This process creates another matrix of random variables as follows:

$$Y = LX \tag{5.75}$$

which results in the following:  $Y_j = \sum_{k=1}^n L_{jk}X_k$

If  $X$  is a vector of correlated normally distributed random variables; then,  $Y$  is also a vector of correlated random variables such that:

$$\vec{v} = L\vec{u} \tag{5.76}$$

$$C' = LCL^T \tag{5.77}$$

where  $\vec{v}$  is the mean vector of  $Y$ ,  $\vec{u}$  is the mean vector of  $X$ ,  $C'$  is the covariance matrix of  $Y$  and  $C$  is the covariance matrix of  $X$

Considering the case where

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

and

$$C = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

Then,

$$C' = LCL^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sigma_1^2 + \rho\sigma_1\sigma_2 & \rho\sigma_1\sigma_2 + \sigma_2^2 \\ \sigma_1^2 - \rho\sigma_1\sigma_2 & \rho\sigma_1\sigma_2 - \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$C' = \frac{1}{2} \begin{bmatrix} \sigma_1^2 + \rho\sigma_1\sigma_2 + \rho\sigma_1\sigma_2 + \sigma_2^2 & \sigma_1^2 + \rho\sigma_1\sigma_2 - \rho\sigma_1\sigma_2 - \sigma_2^2 \\ \sigma_1^2 - \rho\sigma_1\sigma_2 + \rho\sigma_1\sigma_2 - \sigma_2^2 & \sigma_1^2 - \rho\sigma_1\sigma_2 - \rho\sigma_1\sigma_2 + \sigma_2^2 \end{bmatrix} =$$

$$\frac{1}{2} \begin{bmatrix} \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2 & \sigma_1^2 - \sigma_2^2 \\ \sigma_1^2 - \sigma_2^2 & \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 \end{bmatrix}$$

For the case where  $\sigma_1 = \sigma_2 = \sigma$ :

$$C' = \frac{1}{2} \begin{bmatrix} 2\sigma^2 + 2\rho\sigma^2 & 0 \\ 0 & 2\sigma^2 - 2\rho\sigma^2 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 + \rho & 0 \\ 0 & 1 - \rho \end{bmatrix}$$

Hence,  $Y_1$  and  $Y_2$  are uncorrelated in this case since  $\rho_{Y_1, Y_2} = 0$ .

$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$  is nothing but a  $45^\circ$  rotation in space.

The logic described to transform a set of correlated random variables to a set of uncorrelated ones is based on the following logic:

A rotation is nothing but a linear transformation where the transformation matrix is a square and orthogonal matrix i.e.  $L^T = L^{-1}$ . Let the matrix  $K$  be defined as follows:

$$K = L^T \tag{5.78}$$

One can note that  $K$  is also orthogonal i.e.  $K^T = K^{-1}$ .

Thus, the mean and covariance matrix of the rotated set of random variables are:

$$\vec{v} = L\vec{u} \quad (5.79)$$

and

$$C' = LCL^T = (L^{-1})^{-1}CL^T = (K)^{-1}CK = K^TCK \quad (5.80)$$

Since the aim is to have uncorrelated random variables, when it comes to the new rotated set of random variables, the new covariance matrix should be a diagonal matrix as follows:

$$C' = K^TCK = \text{diag}[\sigma_1^2, \sigma_2^2, \dots, \sigma_m^2]$$

$C$  is a covariance matrix; hence, it is a symmetric and positive definite matrix. Thus,  $C$  can be written as:

$$C = U \Lambda U^{-1} \quad (5.81)$$

where

$U = [u_1, u_2, \dots, u_m]$  is the matrix of eigenvectors

and

$\Lambda = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_m]$  with all  $\lambda_i > 0$  for  $i = 1, \dots, m$ .

$\lambda_i$ 's are the corresponding eigenvalues of the covariance matrix.

$C$  can take the form of equation (5.80) because of the following:

Let  $P$  be the matrix whose columns are the eigenvectors of  $C$ ; then,

$$P = [v_1, v_2, \dots, v_k] = \begin{bmatrix} v_{11} & v_{21} & \dots & v_{k1} \\ v_{12} & v_{22} & \dots & v_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1k} & v_{2k} & \dots & v_{kk} \end{bmatrix}$$

Let  $D$  be a diagonal matrix whose elements are the eigenvalues of  $C$ ; then,

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_k \end{bmatrix}$$

Hence,

$$CP = C[v_1, v_2, \dots, v_k] = [Cv_1, Cv_2, \dots, Cv_k] = [\lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_k v_k] =$$

$$\begin{bmatrix} \lambda_1 v_{11} & \lambda_2 v_{21} & \dots & \lambda_k v_{k1} \\ \lambda_1 v_{12} & \lambda_2 v_{22} & \dots & \lambda_k v_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 v_{1k} & \lambda_2 v_{2k} & \dots & \lambda_k v_{kk} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{21} & \dots & v_{k1} \\ v_{12} & v_{22} & \dots & v_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1k} & v_{2k} & \dots & v_{kk} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_k \end{bmatrix} = PD$$

Then,

$$CP = PD \Rightarrow C = PDP^{-1}$$

$$\text{Or } C = U\Lambda U^{-1} \Rightarrow \Lambda = U^{-1}CU$$

Thus,

$$\Lambda = U^{-1}CU$$

$$C' = K^TCK$$

Thus, if we want to have  $C' = \Lambda = U^{-1}CU$  we need to set

$$U = K \tag{5.82}$$

However, as per equation (5.77)

$$K = L^T \Rightarrow L = K^T = U^T \tag{5.83}$$

Thus, using a transformation matrix,  $L = U^T$ , the transformed set of random variables will have a covariance matrix given by:

$$C' = \Lambda = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_m] \tag{5.84}$$

resulting in a set of uncorrelated random variables.

#### **5.4. Review of M.E. Harr's PEM Approach**

Harr's method is introduced in [17] and is based on the correlation between the input random variables. When dealing with one input random variable Harr's method is similar to that of Rosenblueth and Hong when the input random variable is taken to be non-skewed. Harr's method can be used to treat the case of correlated input random variables. However, it necessitates non-skewness of the input random variables. In other words, it only treats

symmetrical input random variables. Harr's method is based on a mapping of the input random variables to a new coordinate system based on the eigenvectors of the correlation matrix of the input variables. In the new coordinate system the input random variables are uncorrelated.

Hence, the main idea behind Harr's method is to transform a set of correlated random variables into uncorrelated ones through a change of coordinate systems then apply a method similar to Rosenblueth's while relying on the assumption that the skewness of each of the input random variables is equal to zero. This is similar to performing random variables de-correlation followed by Hong's method as described in the previous section of this chapter. However, Harr's PEM de-correlation process is embedded within the concentrations locations calculation. Hence, no extra matrix transformation is required. The similarity between Hong's and Harr's PEMs is explored later in this chapter.

The eigenvalues and eigenvectors of the correlation matrix  $K$  can be calculated as follows:

$$(\mathbf{K} - \lambda_i \mathbf{I})\mathbf{e}_i = 0 \quad (5.85)$$

One should note that the sum of the eigenvalues of the correlation matrix is always equal to its trace which is equal to the total number of input random variables denoted by  $n$ .

The logic behind Harr's method is to diagonalize the input random variables correlation matrix through a transformation based on its eigenvectors to generate a set of uncorrelated random variables. This process is based on a coordinate system transformation as explained next. Harr's PEM method is explained through an example. This example is directly obtained from [17].

Assume that we have three input random variables  $x_1, x_2$  and  $x_3$ . We assume that we have the corresponding means, standard deviations and correlation matrix of these input random variables. The means of  $x_1, x_2$  and  $x_3$  are respectively equal to 7.25, 7.67 and 6.92 while their corresponding standard deviations are equal to 2.73, 3.60 and 3.82.

The correlation matrix  $\mathbf{K}$  is given by the following:

$$\mathbf{K} = \begin{bmatrix} 1 & 0.89 & 0.75 \\ 0.89 & 1 & 0.89 \\ 0.75 & 0.89 & 1 \end{bmatrix}$$



The eigenvalues corresponding to this correlation matrix are:

$$\lambda_1 = 2.69, \lambda_2 = 0.25 \text{ and } \lambda_3 = 0.06$$

The corresponding eigenvectors are:  $\mathbf{e}_1 = \begin{pmatrix} 0.57 \\ 0.6 \\ 0.57 \end{pmatrix}$ ,  $\mathbf{e}_2 = \begin{pmatrix} 0.71 \\ -0.01 \\ -0.7 \end{pmatrix}$  and  $\mathbf{e}_3 = \begin{pmatrix} 0.42 \\ -0.8 \\ 0.43 \end{pmatrix}$

The matrix  $E$  is the eigenvectors matrix which columns are the eigenvectors of the correlation matrix; hence,

$$\mathbf{E} = \begin{bmatrix} 0.57 & 0.71 & 0.42 \\ 0.6 & -0.01 & -0.8 \\ 0.57 & -0.7 & 0.43 \end{bmatrix}$$

As previously mentioned, Harr's method is based on a change of coordinate system. In the new coordinate system, the eigenvectors of the correlation matrix form a basis of the system. The new coordinate system is called standardized coordinate system. The origin of the system is at a point with coordinates equal to the means of the  $n$  input random variables i.e *origin* =  $(\mu_1, \mu_2, \dots, \mu_n)$ . For our treated example, the origin of the new coordinate system is located at the point with coordinates  $(\mu_1, \mu_2, \mu_3) = (7.25, 7.67, 6.92)$  expressed in terms of the original coordinate system. The axis of the new coordinate system are labeled  $ds_1, ds_2$  and  $ds_3$  and have directions such as the elements of the eigenvectors matrix,  $E$ , constitute direction cosines between the components of the three eigenvectors and the three axis of the standardized coordinate system. For instance, axis  $ds_1$  has a direction such as it makes a  $\cos^{-1}(0.57) = 55.25^\circ$  angle with  $e_1$ , a  $\cos^{-1}(0.71) = 44.76^\circ$  angle with  $e_2$  and  $\cos^{-1}(0.42) = 65.16^\circ$  angle with  $e_3$ . The same logic applies to  $ds_2$  and  $ds_3$ . The old and new coordinate systems with their corresponding axis are shown in Figure 5.2. In this new coordinate system, the correlation matrix can be represented by a hypersphere of radius  $\sqrt{n}$  centered at the new coordinate system origin. Each eigenvector originates from the new standardized coordinate system's origin following its corresponding direction. Each eigenvector crosses the correlation matrix hypersphere in two points. These points are used as the locations of Harr's PEM probability concentrations. This logic can also be seen in Figure 5.2. Hence, each eigenvalue results in two concentrations. The output function is run for these two concentrations to generate two realizations of the output variables. Hence, these two output variables realizations correspond to the same eigenvalue. The

average of these two output variables realizations is calculated to obtain one average output realization for the corresponding eigenvalue. The weight of this output realization is equal to the ratio of its corresponding eigenvalue to the summation of all the eigenvalues of the correlation matrix as shown in equation (5.86)

$$p_i = \frac{\lambda_i}{\sum_{i=1}^n \lambda_i} \tag{5.86}$$

However, as we previously noted, the summation of the eigenvalues of a correlation matrix is equal to the trace of the correlation matrix which is equal to the number of random variables. Hence, equation (5.87) can be written as:

$$p_i = \frac{\lambda_i}{\sum_{i=1}^n \lambda_i} = \frac{\lambda_i}{n} \tag{5.87}$$

The magnitude of the eigenvalues, reflects the scatter or variance of each random variable. For simplification purposes, one can neglect the probability concentrations corresponding to small eigenvalues because their effect on the system and on the estimation process is minimal.

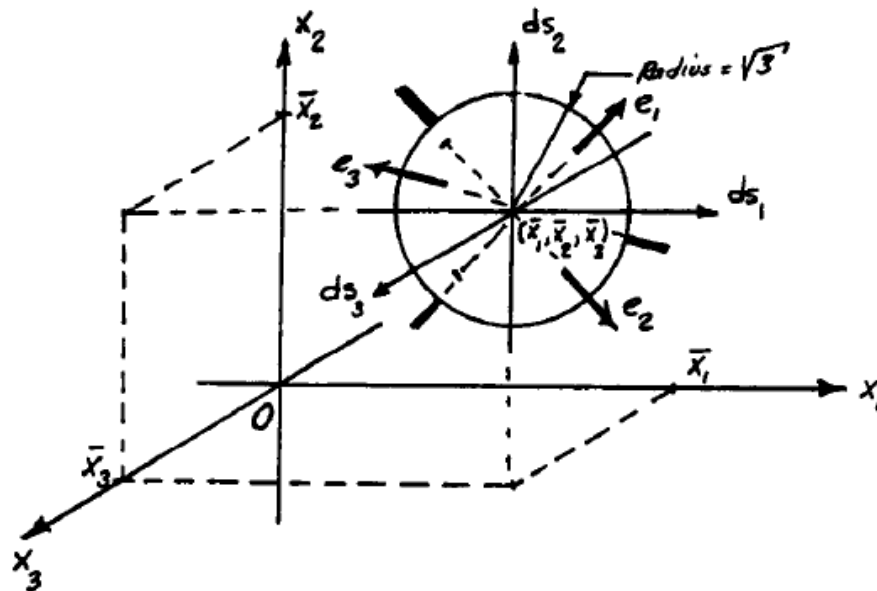


Figure 5.2. Harr's PEM for  $n = 3$ ,

M. E. Harr, "Probabilistic estimates for multivariate analysis," Appl. Math. Model., vol. 13, no. 5, pp. 313–318, 1989, Used under fair use, 2014.

Harr's method steps can be summarized as follows:

The correlation matrix is an  $n \times n$  matrix and is denoted by  $K$ .

1. We calculate the eigenvectors and eigenvalues of the correlation matrix following equation (5.85).
2. We calculate the locations of the  $2n$  points used by Harr's method according to the following formulas:

$$\mathbf{x}_{i1} = \bar{\mathbf{x}} + \sqrt{n} \times \boldsymbol{\sigma}_i \cdot \times |\mathbf{e}_i| \quad (5.88)$$

$$\mathbf{x}_{i2} = \bar{\mathbf{x}} - \sqrt{n} \times \boldsymbol{\sigma}_i \cdot \times |\mathbf{e}_i| \quad (5.89)$$

where " $\cdot \times$ " is an element by element multiplication and not a matrix multiplication.  $\mathbf{e}_i$  is the  $i^{th}$  eigenvector.

These locations are in fact the intersection of the eigenvectors originating from the new coordinate system origin with the correlation matrix hypersphere scaled by the random variable's corresponding standard deviation  $\sigma_i$  as previously explained and shown in Figure 5.2.

3. We calculate the weight of each couple of points (i.e. probability concentrations) corresponding to the same eigenvalue as given in equation (5.87) i.e.  $p_i = \frac{\lambda_i}{n}$  for  $i = 1, \dots, n$
4. We solve for the output random variable(s) for each of the  $2n$  points. The output variable for each of the concentrations is denoted by  $y_t$  for  $t = 1, \dots, 2n$ .
5. Each eigenvalue generates two points. After calculating the output function of all the points, we calculate the average of each two output variables realizations corresponding to the pair of points (i.e. probability concentrations) generated by the same eigenvector. This average output variable realization is denoted by  $y_i$  and is equal to:

$$y_i = \frac{y_t + y_{t+1}}{2} \text{ for } i = 1, \dots, n \text{ and } t = 1, \dots, 2n \quad (5.90)$$

where  $y_t$  and  $y_{t+1}$  correspond to the same eigenvalue  $\lambda_i$

6. The output variables moments can be calculated as:

$$E[y^q] = \sum_{i=1}^n p_i y_i^q \quad (5.91)$$

For the case of one random variable, Harr's method is similar to Rosenblueth's and Hong's methods when the skewness of the input random variable is taken to be equal to zero. This is due to the fact that using Harr's, the random variable is estimated by two points using the formulas  $x = \bar{x} + \sqrt{n} \times \sigma \times e$  and  $x = \bar{x} - \sqrt{n} \times \sigma \times e$ . For the case of one random variable,  $n = 1$  and the correlation matrix is a  $1 \times 1$  matrix with 1 as the only element. Hence, its eigenvalue is equal to 1 and eigenvector is equal to 1 as well. Then, using the previously given formulas, the two concentrations will be located at  $x_i = \bar{x}_i + \sigma_i$  and  $x_i = \bar{x}_i - \sigma_i$  which is nothing but the results presented by Rosenblueth and Hong.

### 5.5. Proposed Combination of Hong's and Harr's PEMs

Through the derivations of Hong's and Harr's methods one can see the similarity between Harr's method and Hong's method with input random variables de-correlation mechanism. In fact with both methods, the input random variables are de-correlated through a matrix transformation resulting in a change of coordinate systems. Using Hong's de-correlation process, the input random variables are multiplied by a transformation matrix which is equivalent to a mapping of these new variables into a new coordinate system. Using Harr's method the same change of coordinate systems is done but it is embedded in the process and no additional matrix transformation is required.

If one takes a closer look at Hong's PEM concentration locations, one can see through equations (5.59)-(5.61) that using Hong's  $2n$  scheme for non-skewed input random variables, the probability concentrations are placed at

$$x_{k,1} = \mu_k + \sqrt{n} \times \sigma_k \quad (5.92)$$

$$x_{k,2} = \mu_k - \sqrt{n} \times \sigma_k \quad (5.93)$$

The concentrations have weights expressed as

$$p_{k,i} = \frac{1}{n} (-1)^i \frac{(-1)^{3-(3-i)\sqrt{n}}}{2\sqrt{n}} = \frac{1}{2n} (-1)^i (-1)^i = \frac{1}{2n} \quad (5.94)$$

This is due to the fact that

$$\xi_{k,1} = \sqrt{n} \quad (5.95)$$

$$\xi_{k,1} = -\sqrt{n} \quad (5.96)$$

As can be seen from equations (5.91) and (5.92) using Hong's  $2n$  for non-skewed input random variables results in probability concentration locations that are somehow similar to those of Harr expressed in equations (5.88) and (5.89).

On the other hand, if the input random variables are uncorrelated, the correlation matrix will be a diagonal matrix resulting in unit eigenvectors with one element equal to 1 and the remaining elements equal to zeros. Hence, by examining equations (5.88) and (5.89) Harr's probability concentrations will be placed at:

$$x_{k,1} = \bar{x} + \sqrt{n} \times \sigma_k \text{ for } k = 1, \dots, n \quad (5.97)$$

$$x_{k,2} = \bar{x} - \sqrt{n} \times \sigma_k \text{ for } k = 1, \dots, n \quad (5.98)$$

These locations are completely similar to the locations generated through Hong's method with non-skewed input variables expressed in equations (5.92) and (5.93). This process shows the similarity between Harr's and Hong's PEMs.

In this chapter, we suggest a new point estimate scheme which combines Hong's and Harr's PEMs. This scheme can be applied to systems with a mixture of correlated and uncorrelated input random variables. Accordingly, Harr's PEM is employed for correlated non-skewed input random variables while Hong's method is applied to uncorrelated input random variables. In the case where the correlated random variables are skewed, a de-correlation mechanism can be applied followed by an employment of Hong's PEM.

The combination of the two PEMs is applied following Hong's PEM sequential logic. Hong's PEM sequential logic states that, during each iteration, only one random variable is replaced by a new location expressed as  $x_{k,i} = \mu_k + \xi_{k,i}\sigma_k$  while the other random variables are placed at their mean values. This is done sequentially for every input random variable. Hence, with every iteration the random variables have the following values:  $l_{k,i} = (\mu_1, \mu_2, \dots, x_{k,i}, \dots, \mu_{n-1}, \mu_n)$ . This process is thoroughly explained in the section of this chapter dedicated to exploring Hong's multivariate PEM (Section 5.3).

Because of the similarity between Hong's and Harr's PEMs that we just explored, one way of combining them is to follow the same sequential order as the one followed by Hong's PEM. In other words, we first employ Hong's PEM to treat uncorrelated input random variables while we fix the correlated random variables at their mean values. Then, we employ Harr's PEM to treat correlated symmetric input random variables while the uncorrelated input random variables are fixed at their mean values. When it comes to the weights, the weights of each of the generated Hong's probability concentrations are scaled by the ratio of the number of uncorrelated random variables to the total number of input random variables as shown in equation (5.99) while the weights of each of the generated Harr's PEM probability concentrations are scaled by the ratio of the number of correlated input random variables to the total number of input random variables as shown in equation (5.100). In this manner, the summation of all the weights will still add up to 1.

$$p_{k,i} = \left[ \frac{1}{n} (-1)^i \frac{\xi_{k,3-i}}{2 \sqrt{n + \left(\frac{\lambda_{k,3}}{2}\right)^2}} \right] \times \frac{\text{number of uncorrelated random variables}}{\text{Total number of random variables}} \quad (5.99)$$

$$p_i = \left[ \frac{\lambda_i}{\sum_{i=1}^n \lambda_i} \right] \times \frac{\text{number of correlated random variables}}{\text{Total number of random variables}} \quad (5.100)$$

where  $\lambda_i$  is the  $i^{th}$  eigenvalue of the correlation matrix associated with the correlated input random variables.

In equations (5.99) and (5.100)  $n$  refers only to respectively the number of uncorrelated and correlated random variables in the system. In fact when sequentially Hong's and Harr's PEMs are employed, when the uncorrelated random variables are treated, the correlated ones are assumed to be deterministic with values set to their mean values. When treating correlated random variables the uncorrelated random variables are taken to be deterministic and set at their mean values. Hence, when Hong's PEM is employed, the total number of random variables in the system,  $n$ , is taken to be equal to only the total number of uncorrelated random variables and when Harr's method is employed,  $n$  is taken to be equal to the total number of correlated random variables. Hence, in equations (5.99), (5.100), (5.88), (5.89), (5.56), (5.57) and (5.58)  $n$  is treated accordingly.

Table 5.4. Concentrations Locations and Weights of Hong's and Harr's Combination PEM

LOCATION	WEIGHT
$(\mu_1 + \xi_{1,1}\sigma_1, \mu_2, \mu_3, \mu_4)$	$\frac{1}{2} \frac{-\xi_{1,2}}{\sqrt{2 + \left(\frac{\lambda_{1,3}}{2}\right)^2}} \times \frac{2}{4}$
$(\mu_1 + \xi_{1,2}\sigma_1, \mu_2, \mu_3, \mu_4)$	$\frac{1}{2} \frac{\xi_{1,1}}{\sqrt{2 + \left(\frac{\lambda_{1,3}}{2}\right)^2}} \times \frac{2}{4}$
$(\mu_1, \mu_2 + \xi_{2,1}\sigma_2, \mu_3, \mu_4)$	$\frac{1}{2} \frac{-\xi_{2,2}}{\sqrt{2 + \left(\frac{\lambda_{2,3}}{2}\right)^2}} \times \frac{2}{4}$
$(\mu_1, \mu_2 + \xi_{2,2}\sigma_2, \mu_3, \mu_4)$	$\frac{1}{2} \frac{\xi_{2,1}}{\sqrt{2 + \left(\frac{\lambda_{2,3}}{2}\right)^2}} \times \frac{2}{4}$
$(\mu_1, \mu_2, \mu_3 + \sqrt{2} \times \sigma_3 \times  E_{1,1} , \mu_4 + \sqrt{2} \times \sigma_4 \times  E_{2,1} )$	$\frac{\lambda_1}{\lambda_1 + \lambda_2} \times \frac{2}{4}$
$(\mu_1, \mu_2, \mu_3 - \sqrt{2} \times \sigma_3 \times  E_{1,1} , \mu_4 - \sqrt{2} \times \sigma_4 \times  E_{2,1} )$	
$(\mu_1, \mu_2, \mu_3 + \sqrt{2} \times \sigma_3 \times  E_{1,2} , \mu_4 + \sqrt{2} \times \sigma_4 \times  E_{2,2} )$	$\frac{\lambda_2}{\lambda_1 + \lambda_2} \times \frac{2}{4}$
$(\mu_1, \mu_2, \mu_3 - \sqrt{2} \times \sigma_3 \times  E_{1,2} , \mu_4 - \sqrt{2} \times \sigma_4 \times  E_{2,2} )$	

For instance, if we have four input random variables given by  $x_1, x_2, x_3$  and  $x_4$  where  $x_3$  and  $x_4$  are correlated and symmetric while the others are uncorrelated, using the proposed PEM combining Hong's and Harr's PEMs, the probability concentration locations and weights will be as shown in Table 5.4. The eigenvectors matrix of the correlation matrix of  $x_3$  and  $x_4$  is given by

$\mathbf{E} = \begin{bmatrix} E_{1,1} & E_{1,2} \\ E_{2,1} & E_{2,2} \end{bmatrix}$ , where  $\mathbf{e}_1 = \begin{bmatrix} E_{1,1} \\ E_{2,1} \end{bmatrix}$  and  $\mathbf{e}_2 = \begin{bmatrix} E_{1,2} \\ E_{2,2} \end{bmatrix}$  are the two eigenvectors, and the corresponding eigenvalues are denoted by  $\lambda_1$  and  $\lambda_2$ .

One should note that in the proposed PEM scheme combining Hong's and Harr's PEMs any of Hong's schemes can be used and the method is not limited to the  $2n$  or  $2n + 1$  schemes.

## **5.6. PEM Implementation Procedure**

In this section the procedures taken to apply each of the point estimate methods to the OPF problem incorporating random wind energy and probabilistic load aiming at estimating the statistical moments of the system LMPs and total generation cost are described.

### **5.6.1. Hong's $2n$ Scheme**

The procedure of applying Hong's  $2n$  PEM to the OPF problem for estimating LMPs and total generation cost statistical moments under uncertainties introduced by wind energy is as shown next (demand is assumed to be deterministic in this case).

Having the wind speed distributions and the wind turbines power models Monte Carlo simulation is applied to estimate the statistics of the wind farms' output powers namely, the mean, standard deviation, skewness and kurtosis, which are needed for Hong's PEM. One can note here that a PEM could have been used to estimate the statistics of the wind farms power outputs. However, MCS is used in this case since estimating the output power's statistics is not a computationally cumbersome process in contrast to estimating OPF output variables' statistics. Accordingly, reducing the number of iterations needed does not show a significant reduction in computation time. The wind farms output powers are considered to be input random variables to our OPF problem. Hence after obtaining the first few moments of these input random variables, Hong's  $2n$  scheme PEM can be applied to obtain the corresponding probability concentrations weights and locations as shown in equations (5.56) and (5.57) along with equation (5.23). Each concentration is fed to the OPF problem to generate one realization of the system LMPs and total generation cost. This process is repeated for all  $n \times m$  concentrations after which  $n \times m$  weighted LMPs and total generation cost realizations are available. From these realizations the LMPs and total generation cost statistics can be obtained as shown in equation (5.74).

### **5.6.2. Hong's $2n + 1$ Scheme**

The procedure of applying Hong's  $2n + 1$  PEM to the OPF problem for estimating LMPs and total generation cost statistical moments under uncertainties introduced by wind energy (demand is assumed to be deterministic in this case) is similar to that of Hong's  $2n$ . The only



difference is in the generated probability concentrations. Hong's  $2n + 1$  probability concentrations locations and weights are generated using equations (5.62) – (5.65) along with equation (5.23).

### **5.6.3. Harr's Point Estimate Method**

Harr's PEM is applied to model the random correlated normally distributed loads to include them in the P-OPF problem. The procedure of applying Harr's PEM is as shown next. First, the loads correlation matrix is obtained. The eigenvalues and eigenvectors of the correlation matrix are then calculated. Using equations (5.88) and (5.89) the probability concentrations locations can be calculated. The OPF problem is solved for each of these concentrations to generate LMPs and total generation cost realizations. The average of each two LMPs or two total generation cost realizations pertaining to the same eigenvalue is calculated to form a new set of LMPs and total generation cost realizations. The weights of these realizations is calculated as shown in equation (5.87) and the LMPs and total generation cost statistical moments can be estimated using equation (5.91).

### **5.6.4. Proposed Combination of Hong's and Harr's PEMs**

As previously noted, Hong's approach is used to model the probabilistic wind farms power outputs whereas Harr's approach is used to model the probabilistic loads. The two methods are combined as explained in Chapter V.5.5. The application of this proposed combination to the OPF problem incorporating random wind and loads for the estimation of the statistical moments of the system LMPs and total generation cost is as shown next.

While Hong's approach is used to model the WFs' probabilistic output power, the loads are kept at their mean values. The OPF deterministic routine is run for Hong's concentrations to generate associated LMPs and total generation cost for each run. These output realizations are weighted by Hong's associated weights scaled by the ratio of the number of wind farms to the total number of random variables in the system i.e.  $p_{k,i} * \left( \frac{\#WFs}{\#WFs + \#Loads} \right)$ .

While Harr's approach is used to model the random normally distributed loads, the wind farms' power outputs are kept at their mean values. The OPF deterministic routine is run for

Harr's concentrations to generate associated LMPs and total generation cost for each run. These output realizations are weighted by Harr's associated weights scaled by the ratio of the number of loads to the total number of random variables in the system i.e.  $p_i * \left( \frac{\#Loads}{\#WFS + \#Loads} \right)$ .

From these realizations the LMPs and total generation cost statistics can be obtained as shown in equations (5.74) and (5.91) using the scaled weights given in equations (5.99) and (5.100).

### **5.6.5. Hong's PEM with De-correlation**

The process of transforming correlated random variables into uncorrelated random variables followed by an application of Hong's method is first discussed through Hong's original work [18]. Hong, in fact, proposed multiplying the set of correlated random variables by the inverse of the matrix of eigenvectors of their corresponding covariance matrix. In our case, we did not exactly use this approach since our covariance matrix is highly sparse because of the non-correlation between many loads in the system which led to an associated matrix of eigenvectors that is ill conditioned; hence, making the usage of its inverse in de-correlating the set of input random variables highly unreliable. As an alternative to this method, we used a transformation technique based on the singular value decomposition of the random variables covariance matrix described in [21]. The process is based on mapping the random variables into a new set of uncorrelated random variables which can be used for Hong's PEM. We have also included MCS in the process to estimate the mean, standard deviation, skewness and kurtosis of the different uncorrelated loads after being transformed into an uncorrelated set. These first few moments are used by Hong's PEM approach. The process follows the following steps:

1. Generate a set of random variables realizations (load realizations). Each set of realizations forms a column vector with a realization for each of the random variables. Each vector is labeled as  $\mathbf{z}_j = [x_1, x_2, \dots, x_n]^T$  where  $j$  is the index of the realization number. The set of all realizations is then expressed in matrix form as  $\mathbf{Z} = [\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_v]$  where  $v$  is the number of realizations generated and  $n$  is the number of random variables.  $\mathbf{Z}$  is an  $n \times v$  matrix.
2. Calculate a mean corrected realizations vector by subtracting the mean value from each corresponding realization:  $\mathbf{z}_j^M = \mathbf{z}_j - \boldsymbol{\mu}_z$ .  $\boldsymbol{\mu}_z$  is a column vector of random variables means

such that  $\boldsymbol{\mu}_z = [\mu_1, \mu_2, \dots, \mu_n]^T$  and  $\mathbf{z}_j^M$  is a mean-corrected column vector of random variables realizations.

3. Apply a Singular Value Decomposition (SVD) to the covariance matrix  $C$ . The SVD of  $C$  is such as  $C = U\Sigma V^*$ .  $C$  is an  $n \times n$  covariance matrix where  $n$  is the number of correlated random variables. In our case  $C$  is a  $99 \times 99$  matrix where 99 is the number of loads in the system.
4. Apply a matrix transformation to transform each set of mean corrected realizations into a set of uncorrelated realizations. The process is as follows:  $\mathbf{z}_j^U = \mathbf{U}^T \mathbf{z}_j^M$ .  $\mathbf{z}_j^U$  is a column vector formed of mean corrected uncorrelated realizations of the input random variables.
5. Steps 1 to 4 are repeated for a large number of times as part of a MCS process to produce an estimate of the mean, standard deviation, skewness and kurtosis of the transformed uncorrelated loads.
6. After obtaining the first few moments of the de-correlated input random variables, we apply Hong's PEM to generate the associated weights  $p_{k,i}$  and concentrations coefficients  $\xi_{k,i}$  where  $k$  is the random variable index and  $i$  is the associated concentration index.
7. Generate the associated locations. Each set of locations forms a column vector containing associated values of each random variable i.e.  $location(k, i) = l_{k,i} = [\mu_1^c, \mu_2^c, \dots, x_{k,i}, \dots, \mu_n^c]^T$  where  $\mu_k^c$  is the mean of the mean corrected random variable  $k$ . Hence,  $\mu_k^c = 0$ .  $x_{k,i} = \mu_k^c + \xi_{k,i} \sigma_k^c = \xi_{k,i} \sigma_k^c$  where  $\sigma_k^c$  is the standard deviation of the mean corrected random variable  $k$ .
8. Perform an inverse transformation to remap the locations to the original physical set reference frame as follows:  $l_{k,i}^O = \boldsymbol{\mu} + \mathbf{U} \times l_{k,i}$ .  $l_{k,i}^O$  is a column vector containing random variables locations in the original frame. This back transformation is done for the following reason. Having a set of transformed uncorrelated random variables we apply Hong's PEM to generate the associated concentration locations and weights. These locations are however derived from the transformed set which does not physically describe our system. Hence, we apply an inverse transformation, using the inverse of the first transformation matrix, to remap the generated locations to the original basis.
9. Run a deterministic OPF for each set of locations  $l_i^O$  and study the statistical moments of the system LMPs and total generation cost.

This process is applied to deal with random correlated loads using Hong's PEM which can only consider uncorrelated input random variables. During this process the wind farms' power outputs are kept at their mean levels. Then, after incorporating the loads, Hong's PEM is applied to model wind farms while the loads are kept at their mean values. The generated LMPs and total generated cost are weighted accordingly to estimate their moments, namely, mean and standard deviation in a similar manner as described in section 5.6.4 of this chapter where only Hong's PEM is applied instead of a combination between Hong's and Harr's.

### **5.6.6. Monte Carlo Simulation**

As previously noted, Monte Carlo simulation techniques are used to generate results which are considered to be benchmark to which the PEMs' results are compared. Using MCS, one realization for each of the random variables in the system, namely, wind speeds and loads, is generated and fed to the deterministic OPF solution routine. The OPF problem is solved for each one set of realizations to produce one set of output realizations, namely, LMPs and total generation cost. This process is repeated multiple times during which the convergence of the estimates of the mean, standard deviation, etc., of the output quantities is examined. When the output variables' statistical moments estimates converge the process is stopped and the converged moments are considered to be an estimation of the LMPs and total generation cost statistical moments.

## **5.7. Summary**

In this chapter, detailed explanations of Rosenbleuth's, Hong's and Harr's PEM approaches for problems with multiple input random variables are presented. Moreover, a proposed combination of Hong's and Harr's PEMs is introduced and the process of implementing the various PEMs to the OPF problem in order to estimate the statistical moments of the system LMPs and total generation cost is illustrated.

In the next chapter, the various PEM schemes are applied to the OPF problem to estimate the statistical moments of the LMPs and total generation cost in the presence of large scale wind integration. The performances of the applied methods are illustrated by comparing their generated results to those of Monte Carlo simulations. Moreover, the risks entailed with the application of Hong's  $2n$  scheme are investigated. Lastly, an extended analysis is presented

showing how the statistical estimation obtained through the PEMs can be used to perform a statistical analysis of the power system.

## ***Chapter VI: Illustrative Results***

### ***6.1. Introduction***

In this chapter, the application of various point estimate methods to estimate the statistical moments of the Probabilistic OPF's LMPs and total generation cost in the presence of intermittent wind energy and random loads is investigated. Through the simulations, the OPF takes as inputs both deterministic and probabilistic quantities denoted by  $\mathbf{X}$  while the output probabilistic random variables are denoted by  $\mathbf{Y}$  such that  $\mathbf{Y} = OPF(\mathbf{X})$ . The deterministic inputs are the line parameters, system configuration and topology, and generator cost functions. The probabilistic input variables are the wind speeds at different locations and the system loads. The output variables,  $\mathbf{Y}$ , of the OPF problem are the system states (i.e. bus voltage magnitude and angles), power injection at each bus (i.e. real and reactive power contributions of all generating units), line flows, electrical losses, LMPs, total generation cost... The aim of the applied PEMs is to estimate the statistical moments of the output variables  $\mathbf{Y}$  in a computationally efficient and accurate manner. The main interest lies in the estimation of system LMPs and total generation cost statistical moments.

Widely used PEM schemes are applied as well as a new PEM scheme consisting of a combination between Harr's and Hong's PEMs proposed in Chapter V.5.5. The results obtained are used to assess and compare the performance of the applied PEMs and to investigate potential risks that they might entail. The simulations are performed using MATPOWER [64]. The accuracy of the employed point estimate methods is assessed through their ability to estimate the statistics, namely, the means and standard deviations, of the system LMPs and total generation cost. The assessment criteria are based on the system LMPs since the locational marginal prices are considered to be the main outcome of the OPF problem and are largely used in electric energy markets applications on the wholesale market level as explained in Chapter III. The simulations are grouped in three sets as follows:

1. Simulation Set 1: In this set, the WSCC 9 – bus system (representing a portion of the Western System Coordinating Council) is considered with one wind farm installed at bus 4 of the system. Hong's  $2n$  and  $2n + 1$  schemes are employed and the results are compared to those obtained using Monte Carlo simulation. This simple case study is

considered to get an insight of the accuracy and computational efficiency of the point estimate method and to thoroughly illustrate, in a step by step manner, the implementation of Hong's PEMs including all the calculation details.

2. Simulation Set 2: In this set three different groups of simulations split according to the level of wind integration in the system are considered. The IEEE 118 – bus system is used (version including line flow limits and appropriate for OPF analysis) [65] and the load is considered to be deterministic at this stage. Since random wind energy is the only random input variable treated in this case, Hong's  $2n$  and  $2n + 1$  schemes are applied and the results are compared to those obtained using Monte Carlo simulation. The wind integration levels considered are the followings:
  - a. Simulation 1 – 10% Wind integration: The total rated capacity of the integrated wind energy represents 10% of the system's total real load.
  - b. Simulation 2 – 20% Wind integration: The total rated capacity of the integrated wind energy represents 20% of the system's total real load.
  - c. Simulation 3 – 30% Wind integration: The total rated capacity of the integrated wind energy represents 30% of the system's total real load.
3. Simulation Set 3: This set of simulation cases investigates the application of various point estimate methods to the estimation of the system LMPs and total generation cost statistics with the incorporation of uncorrelated probabilistic wind energy and correlated randomly distributed loads. The uncorrelated wind and correlated loads investigate the application of PEMs to systems with a mixture of correlated and uncorrelated random variables. The modified IEEE 118 – bus system that is used in Simulation set 2 is also used in this set of simulations. Hong's  $2n$  and Hong's  $2n + 1$  with a de-correlation mechanism explained in section 5.6.5 of Chapter V are applied as well as the proposed combination of Harr's and Hong's PEM schemes explained in sections 5.5 and 5.6.4 of Chapter V and the results are compared to those obtained using MCS. This set of simulations is also used to perform a comparison between the performances of the various applied PEMs. Two different groups of simulations, are considered, split according to the level of wind integration in the system as follows:
  - a. Simulation 1 – 20% Wind integration: The total rated capacity of the integrated wind energy represents 20% of the system's total real load.

- b. Simulation 2 – 30% Wind integration: The total rated capacity of the integrated wind energy represents 30% of the system’s total real load.

As explained in Chapter V, new PEM schemes consisting of a combination of traditionally used PEM schemes are introduced. This helps benefiting from the advantages that each of the PEMs provides and combine these advantages together in one scheme. The PEM schemes that are tested are the following:

- i. Employ Hong’s  $2n$  scheme PEM: Since Hong’s approach only deals with uncorrelated random variables, the correlated normally distributed loads are transformed, through a linear transformation, into a set of uncorrelated random variables; then, Hong’s  $2n$  scheme is employed.
- ii. Employ Hong’s  $2n + 1$  scheme PEM: As previously mentioned Hong’s approach only deals with uncorrelated random variables. Thus, the correlated normally distributed loads are transformed, through a linear transformation, into a set of uncorrelated random variables; then, Hong’s  $2n + 1$  scheme is employed.
- iii. Employ the proposed combination of Harr’s and Hong’s PEMs: Employ Hong’s  $2n$  scheme to treat the uncorrelated wind energy and Harr’s approach to deal with the correlated load. Since the wind farm power outputs are assumed to be uncorrelated, Hong’s approach represents a good fit for their case. On the other hand, the loads are assumed to be correlated and to follow a normal distribution. Harr’s approach deals with correlated random variables but only symmetrical ones. Since the normal distribution is a symmetrical probability distribution, Harr’s approach would be a good fit to treat the case of random loads.
- iv. Employ the proposed combination of Harr’s and Hong’s PEMs: Employ Hong’s  $2n + 1$  scheme to treat the uncorrelated wind farms and Harr’s approach to deal with the correlated load. Hong’s approach efficiently treats uncorrelated input random variables, as the case of the wind farms’ power outputs, while Harr’s approach efficiently treats correlated symmetric input random variables as the case of the random loads.



## 6.2. Modeling the Wind Power

First the location and size of the used wind farms need to be assigned. The wind farms are considered to be placed at load buses where no load is connected and are modeled as negative loads with a zero cost function and constant power factor. Modeling wind farms as negative loads with constant power factors is widely used in literature for steady state power system analysis. In our case, we assume that the wind farms have a constant 0.85 lagging power factor [60]. Modeling wind farms as negative loads is similar to modeling them as generators with zero cost functions. This is a realistic assumption that necessitates the full usage of the output power of the wind, throughout the OPF optimization process, whenever it is available. The wind farms are also considered to be placed at geographical locations that are assumed to be apart in a way that the wind speeds at these different locations are assumed to be uncorrelated. This assumption is also widely used by power system researchers [61].

The wind speeds at each of the associated locations are assumed to follow a Weibull distribution. This assumption is widely used in power system literature to model wind speed probability distribution [6, 31, 66]. The wind turbine power model is the model that relates each wind speed to its corresponding wind turbine output power. The power model used in the simulation is a cubic power model and is as follows [67].

$$P_m = \begin{cases} 0, & V_w \leq V_{cut-in} \text{ or } V_w \geq V_{cut-off} \\ 0.5\rho A_{wt} C_p V_w^3, & V_{cut-in} < V_w \leq V_{rated} \\ P_{rated}, & V_{rated} < V_w \leq V_{cut-off} \end{cases} \quad (6.1)$$

where

$P_m$ : mechanical power output of the wind turbine

$\rho$ : air density

$C_p$ : performance coefficient

$\lambda$ : is the tip speed ratio  $v_t/v_w$

$A_{wt} = \pi R^2$  surface area covered by the turbine rotor ( $R$ : radius from hub to tip)

$V_w$ : wind speed

$\beta$ : blade pitch angle

$V_{cut-in}$ : speed below which the turbine doesn't rotate properly to generate power

$V_{cut-off}$ : wind speed for which the turbine is shut down for security reasons

$V_{rated}$ : wind speed for which the output power of the wind turbine is the rated power

The wind turbine electrical power output is the mechanical power output multiplied by the electrical efficiency. In the simulations the electrical efficiency is taken to be equal to one, for simplicity, since the electrical efficiency of the wind turbine does not affect the preciseness of the PEM results.

In this thesis, three scenarios of wind energy integration levels are considered such that the total rated integrated wind power capacity respectively represents 10%, 20% and 30% of the total real power base load. Such a large integration of wind capacity helps accurately assessing the performance of the applied PEMs since the included random quantities in the system have large magnitudes and do significantly affect the system outputs. High wind levels penetration also helps making an appropriate study of the effects of wind energy integration on the power system operation and the amount of uncertainty that wind energy introduces on the system's LMPs, bus voltages, line flows, electric losses, total generation cost...

An online research is conducted on commercially available wind turbines that are used with some modifications throughout the simulations. The used wind turbine (WT) models' parameters are introduced next. Air density is assumed to be equal to  $\rho = 1.225 \text{ kg/m}^3$  at all the wind farms locations. The power coefficient denoted by  $C_p$  for the used wind turbines is not given. An appropriate value of  $C_p$  is assumed to match the assumption that the WT output power follows the model introduced in (6.1):

- WT1 based on model available in [62]: WT1 is an asynchronous 3,600 KW rated power wind turbine. Its parameters are considered to be as follows:

Table 6.1. WT1 Characteristics

PARAMETERS	VALUE
Diameter	120 m
Swept Area	11,300 $m^2$
$V_{cu,in}$	4 m/s
$V_r$	12 m/s
$V_{cu,off}$	25 m/s

The power coefficient denoted by  $C_p$  of this wind turbine is not given and is assumed to be equal to 0.301 to match the assumption that the output power follows the previously mentioned output power model given in equation (6.1).

- WT2 based on model available in [63]: This is an asynchronous 3,000 KW rated power output wind turbine. Its parameters are as follows:

Table 6.2. WT2 Characteristics

PARAMETER	VALUE
Diameter	110 m
Swept Area	9469 $m^2$
$V_{cu,in}$	3 m/s
$V_r$	11.5 m/s
$V_{cu,off}$	25 m/s

The power coefficient denoted by  $C_p$  of this wind turbine is not given and is assumed to be equal to 0.3401 to match the assumption that the output power follows the model given in (6.1).

Every wind farm is formed of a set of identical wind turbines. Information about the integrated wind farms size and capacity, for the different wind energy integration levels, is available in Table 6.3. The wind speeds at each of the ten locations are assumed to follow a Weibull distribution with scale parameter ( $\alpha$ ) and shape parameter ( $\beta$ ) defined as follows:

- Bus 5:  $\alpha = 12, \beta = 2.5$
- Bus 9:  $\alpha = 11.5, \beta = 1.75$
- Bus 30:  $\alpha = 11, \beta = 2$
- Bus 37:  $\alpha = 11.8, \beta = 2.2$
- Bus 38:  $\alpha = 10.8, \beta = 3$
- Bus 63:  $\alpha = 11, \beta = 4$
- Bus 64:  $\alpha = 10.9, \beta = 3$
- Bus 68:  $\alpha = 12, \beta = 5$
- Bus 71:  $\alpha = 12, \beta = 2.5$
- Bus 81:  $\alpha = 11.5, \beta = 2$

Table 6.3. Wind Farm Characteristics

Wind energy integration level	Bus	WT used	Rated wind speed (m/s)	Weibull distribution parameters for wind speed ( $\alpha, \beta$ )	Number of wind turbines used	Rated power of each wind turbine (MW)	Total wind farm rated power (MW)	Total wind capacity integration (MW)
10%	5	WT1	12	(12, 2.5)	22	3.6	79.2	426
	37	WT1	12	(11.8, 2.2)	26	3.6	93.6	
	64	WT2	11.5	(10.9, 3)	28	3	84	
	71	WT1	12	(12, 2.5)	22	3.6	79.2	
	81	WT2	11.5	(11.5, 2)	30	3	90	
20%	5	WT1	12	(12, 2.5)	18	3.6	64.8	828
	9	WT1	12	(11.5, 1.75)	28	3.6	100.8	
	30	WT2	11.5	(11, 2)	28	3	84	
	37	WT1	12	(11.8, 2.2)	28	3.6	100.8	
	38	WT2	11.5	(10.8, 3)	28	3	84	
	63	WT2	11.5	(11, 4)	20	3	60	
	64	WT2	11.5	(10.9, 3)	28	3	84	
	68	WT1	12	(12, 5)	28	3.6	100.8	
	71	WT1	12	(12, 2.5)	18	3.6	64.8	
	81	WT2	11.5	(11.5, 2)	28	3	84	
30%	5	WT1	12	(12, 2.5)	28	3.6	100.8	1278
	9	WT1	12	(11.5, 1.75)	40	3.6	144	
	30	WT2	11.5	(11, 2)	40	3	120	
	37	WT1	12	(11.8, 2.2)	40	3.6	144	
	38	WT2	11.5	(10.8, 3)	40	3	120	
	63	WT2	11.5	(11, 4)	40	3	120	
	64	WT2	11.5	(10.9, 3)	40	3	120	
	68	WT1	12	(12, 5)	40	3.6	144	
	71	WT1	12	(12, 2.5)	32	3.6	115.2	
81	WT2	11.5	(11.5, 2)	50	3	150		

The Weibull distribution of each wind speed is chosen so that the scale parameter of the wind speed is around the rated speed of the wind turbine installed at the corresponding location. It is chosen this way using reverse engineering logic of the process of installing a wind turbine at a location such that its rated speed is equal to the scale parameter of the wind at the associated location.

### 6.3. Modeling the Load

The active load is assumed to follow a normal distribution with mean equal to the base load real power and standard deviation equal to a percentage of the mean. Modeling loads as normally distributed random variables is widely common in literature [60]. By examining the IEEE 118 – bus network provided in Figure 6.1, one can see that it is split in three different regions.

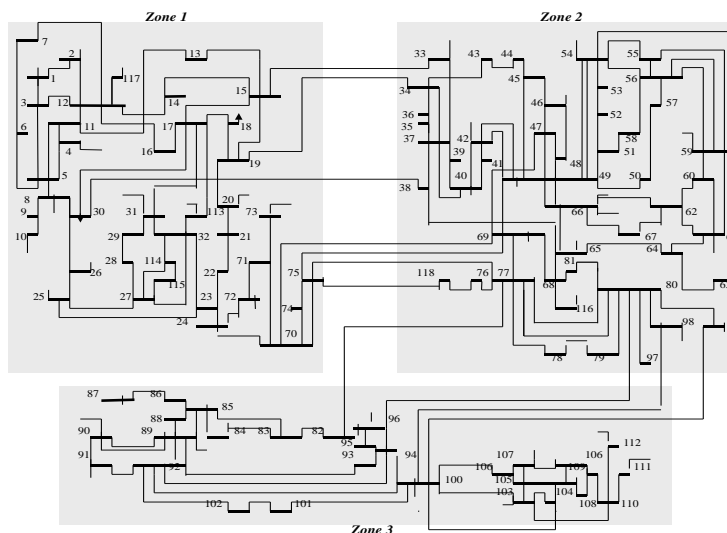


Figure 6.1. IEEE 118 – bus network

The standard deviations of the real parts of the loads in zones 1,2 and 3 are assumed to be respectively equal to 10%, 5% and 15% of their corresponding mean values. For instance, the load at bus 2 is equal to  $S_2 = 20 + j9$ . The real part of the load is probabilistically modeled to follow a normal distribution with mean equal to 20 and standard deviation equal to 10% of 20 i.e.  $P_2 \sim N(20,2)$ . For the imaginary part of the load, i.e. the reactive power, the load is assumed to preserve its power factor when modeled as a random variable. For instance, taking the case of the load at bus 2, the base load power factor is equal to  $pf_2 = \cos(\theta_2) = \cos(\tan^{-1} \frac{9}{20}) = \cos(24.23) = 0.912$  lagging. Hence, when modeled as a random variable the imaginary part of the load at bus 2 is taken to be equal to  $Q_2 = |S| \sin(\theta_2) = \frac{P_2}{\cos(\theta_2)} \sin(\theta_2) = P_2 \tan(\theta_2) = \tan(24.23) P_2 = 0.4104 P_2$ .

When it comes to modeling the correlation between the loads, the correlation coefficient which reflects the correlation between two random variables is assigned based on the loads

geographic proximity. Loads located in different zones are assumed to be uncorrelated. Each of the zones 1, 2 and 3 are split in three areas as illustrated in Table 6.4.

Table 6.4. Defined Geographical Areas

ZONE	AREA	BUSES INCLUDED
1	1-1	1,2,3,4,5,6,7,8,9,10,11,12,30,117
	1-2	13,14,15,16,17,18,19
	1-3	20,21,22,23,24,25,26,27,28,29,31,32,70,71,72,73,74,75,113,114,115
2	2-1	33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48
	2-2	49,50,51,52,53,54,55,56,57,58,59,60,61,62,64,66,67
	2-3	63,65,68,69,76,77,78,79,80,81,97,98,99,116,118
3	3-1	82,83,84,85,86,87,88,89,90,91,92,93,94,95,96
	3-2	101,102
	3-3	100, 103,104,105,106,107,108,109,110,111,112

The correlation coefficients between the loads located in each of the three areas of zone 1 are assigned as shown in Table 6.5.

Table 6.5. Correlation Coefficients between the Loads in the Three Areas of Zone 1

AREA	1-1	1-2	1-3
1-1	0.8	0.5	0.2
1-2	0.5	0.8	0.5
1-3	0.2	0.5	0.8

Table 6.5 is interpreted as follows:

- Correlation coefficients between the loads located inside Area 1-1 are assumed to be equal to 0.8. The same applies for the correlation coefficients between the loads located respectively inside Area 1-2 and Area 1-3.
- Correlation coefficients between loads located in Area 1-1 and Area 1-2 are assumed to be equal to 0.5.
- Correlation coefficients between loads located in Area 1-2 and Area 1-3 are assumed to be equal to 0.5.

- Correlation coefficients between loads located in Area 1-1 and Area 1-3 are assumed to be equal to 0.2.

In a similar manner, the correlation coefficients of the loads located in the three areas of zones 2 and 3 are assigned as illustrated in Table 6.6 and Table 6.7 respectively.

Table 6.6. Correlation Coefficients between the Loads in the Three Areas of Zone 2

AREA	2-1	2-2	2-3
2-1	0.8	0.5	0.2
2-2	0.5	0.8	0.5
2-3	0.2	0.5	0.8

Table 6.7. Correlation Coefficients between the Loads in the Three Areas of Zone 3

AREA	3-1	3-2	3-3
3-1	0.8	0.5	0.2
3-2	0.5	0.8	0.2
3-3	0.2	0.2	0.8

To model the loads we first create a vector of means and a vector of standard deviations containing respectively the means and standard deviations of the real power of the loads. Next, the correlation matrix is defined using the correlation coefficients defined in Table 6.5, Table 6.6 and Table 6.7. To generate correlated normally distributed random variables using MATLAB, we need to define the covariance matrix. By definition, the covariance matrix of a random vector

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{bmatrix} \text{ is given by:}$$

$$\text{cov} = \tag{6.2}$$

$$\begin{bmatrix} E[(Y_1 - \mu_1)(Y_1 - \mu_1)] & E[(Y_1 - \mu_1)(Y_2 - \mu_2)] & \dots & E[(Y_1 - \mu_1)(Y_m - \mu_m)] \\ E[(Y_2 - \mu_2)(Y_1 - \mu_1)] & E[(Y_2 - \mu_2)(Y_2 - \mu_2)] & \dots & E[(Y_2 - \mu_2)(Y_m - \mu_m)] \\ \vdots & \vdots & \vdots & \vdots \\ E[(Y_m - \mu_m)(Y_1 - \mu_1)] & E[(Y_m - \mu_m)(Y_2 - \mu_2)] & \dots & E[(Y_m - \mu_m)(Y_m - \mu_m)] \end{bmatrix}$$

The covariance matrix is obtained using the corresponding correlation coefficients and respective standard deviations as follows:

The correlation coefficient of  $Y_1$  and  $Y_2$  is given by:

$$\rho_{Y_1, Y_2} = \frac{E[(Y_1 - \mu_1)(Y_2 - \mu_2)]}{\sigma_{Y_1} \sigma_{Y_2}} \quad (6.3)$$

Hence,  $E[(Y_1 - \mu_1)(Y_2 - \mu_2)]$  can be expressed as follows:

$$E[(Y_1 - \mu_1)(Y_2 - \mu_2)] = \rho_{Y_1, Y_2} \sigma_{Y_1} \sigma_{Y_2} \quad (6.4)$$

The covariance matrix can then be written as follows:

$$\text{covariance} = \begin{bmatrix} \rho_{Y_1, Y_1} \sigma_{Y_1} \sigma_{Y_1} & \rho_{Y_1, Y_2} \sigma_{Y_1} \sigma_{Y_2} & \dots & \rho_{Y_1, Y_m} \sigma_{Y_1} \sigma_{Y_m} \\ \rho_{Y_1, Y_2} \sigma_{Y_1} \sigma_{Y_2} & \rho_{Y_2, Y_2} \sigma_{Y_2} \sigma_{Y_2} & \dots & \rho_{Y_2, Y_m} \sigma_{Y_2} \sigma_{Y_m} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{Y_1, Y_m} \sigma_{Y_1} \sigma_{Y_m} & \rho_{Y_2, Y_m} \sigma_{Y_2} \sigma_{Y_m} & \dots & \rho_{Y_m, Y_m} \sigma_{Y_m} \sigma_{Y_m} \end{bmatrix} \quad (6.5)$$

## 6.4. Simulation Results

In this section the simulation results are provided with discussion and accuracy analyses. The simulations are split in three different sets as previously explained in section 6.1 of this chapter.

In the first set, the WSCC 9-bus network is used including a wind farm at bus 4. Hong's  $2n$  and  $2n + 1$  PEM schemes are applied to the OPF problem. The second and third simulation sets consider the IEEE 118-bus network. In the second simulation set, three different scenarios of wind integration levels are considered: 10%, 20% and 30%. The load is assumed to be deterministic and Hong's  $2n$  and  $2n + 1$  schemes are applied to the OPF problem. In the third simulation set, two different scenarios of wind integration levels are considered: 20% and 30%. The loads are modeled as correlated random variables. Various PEM schemes are applied to the OPF problem including Hong's  $2n$  and  $2n + 1$  schemes with de-correlation mechanism in addition to the proposed combination of Harr's and Hong's  $2n$  as well as Harr's and Hong's  $2n + 1$  schemes.

### 6.4.1. Simulation Set 1

As previously mentioned, this simulation set considers the WSCC 9 – bus system with one wind farm installed at bus 4 of the system. The wind speed at bus 4 is assumed to follow a Weibull distribution with scale and shape parameters equal to 10 and 2 respectively.



The wind farm is taken to be composed of twenty five identical wind turbines. Each of the turbines has parameters illustrated in Table 6.8.

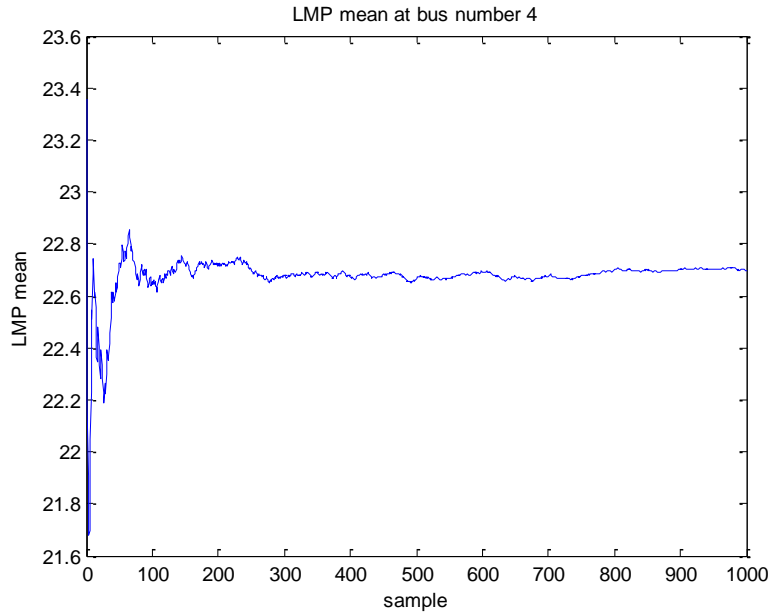
Table 6.8. WT Parameters

PARAMETER	VALUE
$\rho$	1.2235
$R$	45
$C_p$	0.473
$V_{c-in}$	3
$V_{c-off}$	25
$V_{rated}$	10.28
$P_{rated}$	2

In this simple simulation, wind power output is modeled as a negative power injection with unity power factor. Hong's  $2n$  and  $2n + 1$  schemes are used to estimate the means and standard deviations of the system LMPs and total generation cost and the results are compared to those obtained using Monte Carlo simulation. In this simulation, the employed wind turbines are assumed to have a unity power factor implying that they neither consume nor produce reactive power. This is just a simple simulation set to illustrate the application of Hong's PEM schemes and to get an insight about their statistical moments' estimation accuracy as compared to MCS.

a. Monte Carlo Simulation:

Monte Carlo simulation results are considered to be benchmark results to which the PEM results are compared. We run a 1000-sample Monte Carlo simulation and we check the convergence of the LMPs' moments. Figure 6.2 shows the convergence of the mean of the LMP at bus 4. As one can see, the mean of the LMP at this bus needs around 350 runs for it to converge. By examining all the means and standard deviations of all the LMPs at all the buses, one can see that 350 runs are enough for all the output quantities to converge.



**Figure 6.2.** Convergence of the Mean of the LMP at Bus 4 Using MCS

b. Hong’s PEM

Next, Hong’s method, using the  $2n$  and  $2n + 1$  schemes, is applied. In this case there is only one input random variable which is the output power of the wind farm. First the statistics of this output power, knowing the distribution of the wind speed at that specific location, should be estimated using the wind turbine output power model. Then, the output power distribution is replaced by two concentrations at two locations for the  $2n$  scheme and by three concentrations at three locations for the  $2n + 1$  scheme. The OPF problem is then solved for these locations and the corresponding LMPs and total cost are calculated. The means of the LMPs and total cost are estimated through a weighted sum of the obtained LMPs and total generation cost results where the weights are the associated probability concentration magnitudes. The estimation results obtained using Hong’s PEM are compared to those obtained using MCS.

Let us first start by estimating the mean, standard, deviation, skewness and kurtosis of the wind farm output power. To do this, a 1000-sample Monte Carlo simulation is used. For each sample a realization of the wind speed is generated, following the wind speed Weibull distribution, and this wind speed is fed to the wind turbine power model to generate an output power sample. The results of the 1000-sample MCS when it comes to estimating the first few moments of the output power of the wind farm are presented in Table 6.9.

Table 6.9. WF Output Power First Few Moments

QUANTITY	VALUE
Mean	27.2637
Standard Deviation	19.8548
Skewness	-0.0123
Kurtosis	1.3314

i. Hong's  $2n$  Scheme:

Knowing the values of  $\mu_x$ ,  $\sigma_x$  and  $\lambda_{X,3}$ , the probability concentrations and their locations for Hong's  $2n$  scheme can be calculated.

For a single variable function and for the  $2n$  case:

$$\xi_j = \frac{\lambda_{X,3}}{2} + (-1)^{3-j} \sqrt{1 + (\lambda_{X,3/2})^2} \quad j = 1,2 \quad (6.6)$$

$$p_j = (-1)^j \frac{\xi_{3-j}}{\zeta} \quad j = 1,2 \quad (6.7)$$

where

$$\zeta = \xi_1 - \xi_2 = 2\sqrt{1 + \left(\frac{\lambda_{X,3}}{2}\right)^2} \quad (6.8)$$

Thus,

$$\xi_1 = \frac{-0.0123}{2} + \sqrt{1 + \left(\frac{-0.0123}{2}\right)^2} = 0.9939$$

$$\xi_2 = \frac{-0.0123}{2} - \sqrt{1 + \left(\frac{-0.0123}{2}\right)^2} = -1.0062$$

$$\zeta = \xi_1 - \xi_2 = 0.99387 + 1.0062 = 2$$

The locations of the concentrations are defined as follows:

$$x_i = \mu_x + \xi_i \sigma_x \quad (6.9)$$

Then,

$$x_1 = 27.2637 + 0.9939 \times 19.8548 = 46.9974$$

$$x_2 = 27.2637 - 1.0062 \times 19.8548 = 7.2858$$

The probability concentration magnitudes are:

$$p_1 = -\frac{\xi_2}{\zeta} = -\frac{-1.0062}{2} = 0.5031$$

$$p_2 = \frac{\xi_1}{\zeta} = \frac{0.9939}{2} = 0.4969$$

Next, the deterministic OPF problem is solved twice: the first time the output power of the wind farm at bus 4 is taken to be 46.9974 MW and the second time to be 7.2858 MW. The generated LMPs resulting from taking the wind farm power output to be equal to 46.9974 MW and 7.2858 MW are respectively shown in Table 6.10 and Table 6.11.

Table 6.10. LMPs for the First Concentration – Hong’s  $2n$

BUS NUMBER	LMP
1	21.2123
2	20.8668
3	20.9023
4	21.2128
5	21.5030
6	20.9023
7	21.0571
8	20.8668
9	21.4879

The associated total generation cost is equal to 4216.8 \$/h

Table 6.11. LMPs for the Second Concentration – Hong’s  $2n$

BUS NUMBER	LMP
1	24.2005
2	23.5435
3	23.5839
4	24.2007
5	24.4523
6	23.5839
7	23.7583
8	23.5435
9	24.4496

The associated total generation cost is equal to 5118.3 \$/h.

Hence, using the  $2n$  scheme the mean of the LMPs at the different buses of the system can be estimated as follows:

$$E[LMP] = p_1 LMP_1 + p_2 LMP_2 \quad (6.10)$$

$$E[LMP] = 0.5031 \times \begin{bmatrix} 21.2123 \\ 20.8668 \\ 20.9023 \\ 21.2128 \\ 21.5030 \\ 20.9023 \\ 21.0571 \\ 20.8668 \\ 21.4879 \end{bmatrix} + 0.4969 \times \begin{bmatrix} 24.2005 \\ 23.5435 \\ 23.5839 \\ 24.2007 \\ 24.4523 \\ 23.5839 \\ 23.7583 \\ 23.5435 \\ 24.4496 \end{bmatrix} = \begin{bmatrix} 22.6971 \\ 22.1969 \\ 22.2348 \\ 22.6975 \\ 22.9685 \\ 22.2348 \\ 22.3993 \\ 22.1969 \\ 22.9596 \end{bmatrix}$$

The expected total cost using the 2PEM is estimated as follows:

$$E[Total\ Cost] = p_1 TotalCost_1 + p_2 TotalCost_2 \quad (6.11)$$

$$p_1 TotalCost_1 + p_2 TotalCost_2 = 0.5031 \times 4216.8 + 0.4969 \times 5118.3 = 4664.76\ \$/h.$$

The second moment of the LMPs can also be estimated as follows:

$$E[LMP^2] = 0.5031 \times \begin{bmatrix} 21.2123^2 \\ 20.8668^2 \\ 20.9023^2 \\ 21.2128^2 \\ 21.5030^2 \\ 20.9023^2 \\ 21.0571^2 \\ 20.8668^2 \\ 21.4879^2 \end{bmatrix} + 0.4969 \times \begin{bmatrix} 24.2005^2 \\ 23.5435^2 \\ 23.5839^2 \\ 24.2007^2 \\ 24.4523^2 \\ 23.5839^2 \\ 23.7583^2 \\ 23.5435^2 \\ 24.4496^2 \end{bmatrix} = \begin{bmatrix} 517.3923 \\ 494.4914 \\ 496.1834 \\ 517.4077 \\ 529.7268 \\ 496.1834 \\ 503.5539 \\ 494.4914 \\ 529.3346 \end{bmatrix}$$

$$E[Total - Cost^2] = 0.5031 \times 4216.8^2 + 0.4969 \times 5118.3^2 = 21963110.23$$

The variance of the LMPs and total generation cost can be estimated using the following expression:

$$variance(Z) = E[Z^2] - (E[Z])^2 \quad (6.12)$$

Then,

$$\text{variance}(LMPs) = \begin{bmatrix} 517.3923 \\ 494.4914 \\ 496.1834 \\ 517.4077 \\ 529.7268 \\ 496.1834 \\ 503.5539 \\ 494.4914 \\ 529.3346 \end{bmatrix} - \begin{bmatrix} 22.6971^2 \\ 22.1969^2 \\ 22.2348^2 \\ 22.6975^2 \\ 22.9685^2 \\ 22.2348^2 \\ 22.3993^2 \\ 22.1969^2 \\ 22.9596^2 \end{bmatrix} = \begin{bmatrix} 2.2322 \\ 1.7911 \\ 1.7977 \\ 2.2318 \\ 2.1745 \\ 1.7977 \\ 1.8241 \\ 1.7911 \\ 2.1928 \end{bmatrix}$$

$$\text{Standard Deviation LMPs} = \begin{bmatrix} 1.4941 \\ 1.3383 \\ 1.3408 \\ 1.4939 \\ 1.4746 \\ 1.3408 \\ 1.3506 \\ 1.3383 \\ 1.4808 \end{bmatrix}$$

$$\text{variance}(\text{Total Cost}) = 21963110.23 - 4664.76^2 = 203124.37$$

$$\text{Standard Deviation (Total Cost)} = \sqrt{203124.37} = 450.69 \text{ \$/h}$$

ii. Hong's  $2n + 1$  Scheme

Knowing the values of  $\mu_x$ ,  $\sigma_x$ ,  $\lambda_{x,3}$  and  $\lambda_{x,4}$ , the probability concentrations and their locations for Hong's  $2n + 1$  scheme can be calculated.

For a single variable function and for the  $2n + 1$  case:

$$\xi_j = \frac{\lambda_{x,3}}{2} + (-1)^{3-j} \sqrt{\lambda_4 - 3(\lambda_{x,3/2})^2} \quad j = 1,2 \quad (6.13)$$

$$\xi_3 = 0 \text{ (since the third point is located at the mean).} \quad (6.14)$$

$$p_j = \frac{(-1)^{3-j}}{\xi_j(\xi_1 - \xi_2)} \quad j = 1,2 \quad (6.15)$$

$$p_3 = \frac{1}{n} - \frac{1}{\lambda_4 - \lambda_3^2} \quad (6.16)$$

$$\xi_1 = \frac{-0.0123}{2} + \sqrt{1.3314 - 3 \times \left(\frac{-0.0123}{2}\right)^2} = 1.1477$$

$$\xi_2 = \frac{-0.0123}{2} - \sqrt{1.3314 - 3 \times \left(\frac{-0.0123}{2}\right)^2} = -1.1600$$

The locations of the concentrations are defined as follows:

$$x_i = \mu_X + \xi_i \sigma_X \quad (6.17)$$

then,

$$x_1 = 27.2637 + 1.1477 \times 19.8548 = 50.0511$$

$$x_2 = 27.2637 - 1.16 \times 19.8548 = 4.2321$$

$$x_3 = 27.2637$$

The probability concentration magnitudes are:

$$p_1 = \frac{1}{1.1477 (1.1477+1.16)} = 0.3776$$

$$p_2 = \frac{-1}{-1.16 (1.1477+1.16)} = 0.3736$$

$$p_3 = \frac{1}{1} - \frac{1}{1.3314 - (-0.0123)^2} = 0.2488$$

Next, the deterministic OPF problem is solved three times: the first time the output power of the wind farm at bus 4 is taken to be 50.0511 MW, the second time to be 4.2321 MW and the third time to be 27.2637 MW.

The LMPs resulting from taking the wind farm power output to be equal to 50.0511 MW are shown in Table 6.12.

Table 6.12. LMPs for the First Concentration – Hong's  $2n + 1$

BUS NUMBER	LMP
1	20.9852
2	20.6609
3	20.6961
4	20.9857
5	21.2781
6	20.6961
7	20.8494
8	20.6609
9	21.2623

The associated total generation cost is equal to 4152.4\$/h

The LMPs resulting from taking the wind farm power output to be equal to 4.2321 MW are shown in Table 6.13.

The LMPs resulting from taking the wind farm power output to be equal to 27.2637 MW are shown in Table 6.14.

Table 6.13. LMPs for the Second Concentration – Hong’s  $2n + 1$

BUS NUMBER	LMP
1	24.4329
2	23.7492
3	23.7901
4	24.4331
5	24.6810
6	23.7901
7	23.9660
8	23.7492
9	24.6794

The associated total generation cost is equal to 5192.6 \$/h.

Table 6.14. LMPs for the Third Concentration – Hong’s  $2n + 1$

BUS NUMBER	LMP
1	22.6892
2	22.1969
3	22.2349
4	22.6896
5	22.9628
6	22.2349
7	22.3994
8	22.1969
9	22.9534

The associated total generation cost is equal to 4650 \$/h.

Hence, using the  $2n + 1$  scheme, the mean of the LMPs at the different buses of the system can be calculated as follows:

$$E[LMP] = p_1LMP_1 + p_2LMP_2 + p_2LMP_3 \quad (6.18)$$



$$E[LMP] = \begin{bmatrix} 22.6972 \\ 22.1969 \\ 22.2349 \\ 22.6976 \\ 22.9686 \\ 22.2349 \\ 22.3994 \\ 22.1969 \\ 22.9597 \end{bmatrix}$$

The expected total cost using the  $2n + 1$  scheme can be estimated as follows:

$$E[Total Cost] = p_1 TotalCost_1 + p_2 TotalCost_2 + p_3 TotalCost_3 \quad (6.19)$$

$$E[Total Cost] = 0.3776 \times 4152.4 + 0.3736 \times 5192.6 + 0.2488 \times 4650 = 4664.82 \text{ \$/h}$$

The second moment of the LMPs can also be estimated as follows:

$$E[LMP^2] = \begin{bmatrix} 517.3958 \\ 494.491 \\ 496.1872 \\ 517.4119 \\ 529.7301 \\ 496.1872 \\ 503.5572 \\ 494.491 \\ 529.3394 \end{bmatrix}$$

$$E[Total - Cost^2] = 0.3776 \times 4152.4^2 + 0.3736 \times 5192.6^2 + 0.2488 \times 4650^2 = 21963830.17$$

The variance of the LMPs and total generation cost can be estimated using the following expression:

$$variance(Z) = E[Z^2] - (E[Z])^2 \quad (6.20)$$

Then,

$$variance(LMPs) = \begin{bmatrix} 2.2322 \\ 1.7911 \\ 1.7977 \\ 2.2318 \\ 2.1746 \\ 1.7977 \\ 1.8240 \\ 1.7911 \\ 2.1928 \end{bmatrix}$$

$$\text{Standard Deviation LMPs} = \begin{bmatrix} 1.4941 \\ 1.3383 \\ 1.3408 \\ 1.4939 \\ 1.4746 \\ 1.3408 \\ 1.3506 \\ 1.3383 \\ 1.4808 \end{bmatrix}$$

$$\text{variance}(\text{Total Cost}) = 21963830.17 - 4664.82^2 = 203284.5376$$

$$\text{Standard Deviation}(\text{Total Cost}) = \sqrt{203284.5376} = 450.87 \text{ \$/h}$$

c. Results comparison

In this section the accuracy of Hong's  $2n$  and  $2n + 1$  schemes in estimating the means and standard deviations of the system LMPs and total generation cost is analyzed by comparing the results obtained through these two PEMs to the results obtained using MCS.

Table 6.15 and Table 6.16 show a comparison between the results obtained using Hong's  $2n$  and  $2n + 1$  PEM schemes respectively and the results obtained through MCS when it comes to estimating the means of the system LMPs and total generation cost under uncertainty introduced by the integration of intermittent wind energy at Bus 4. The errors are calculated as follows:

$$\%e = \frac{(\text{PEM result}) - (\text{MCS result})}{\text{MCS result}} \quad (6.21)$$

Table 6.15. LMPs' Means Estimation Using Hong's  $2n$  PEM and MCS

BUS	LMP PEM	LMP MCS	% ERROR
1	22.6971	22.6972	5.05E-05
2	22.1969	22.1969	8.7E-05
3	22.2348	22.2349	-0.00021
4	22.6975	22.6975	0.000275
5	22.9685	22.9686	-8.2E-05
6	22.2348	22.2349	-0.00021
7	22.3993	22.3994	-2.7E-05
8	22.1969	22.1969	8.7E-05
9	22.9596	22.9596	0.000187
	Cost PEM	Cost MCS	%error
	4664.76	4664.8	-0.00086

Table 6.16. LMPs' Means Estimation Using Hong's  $2n + 1$  PEM and MCS

BUS	LMP PEM	LMP MCS	% ERROR
1	22.6972	22.6972	6.84E-05
2	22.1969	22.1969	-0.00025
3	22.2349	22.2349	-0.00013
4	22.6976	22.6975	0.000346
5	22.9686	22.9686	-0.0001
6	22.2349	22.2349	-0.00013
7	22.3994	22.3994	7.12E-06
8	22.1969	22.1969	-0.00025
9	22.9597	22.9596	0.000322

Cost PEM	Cost MCS	%error
4664.82	4664.8	0.000429

In a similar manner, Table 6.17 shows a comparison between the LMPs and total generation cost standard deviations estimation generated using Hong's PEMs and MCS.

As can be seen, the  $2n$  and  $2n + 1$  schemes give results that completely match those obtained using MCS. However, MCS requires the solution of the OPF problem three hundred and fifty (350) times whereas the  $2n$  scheme requires solving the OPF problem two times and the  $2n + 1$  scheme requires solving the OPF problem three times.

Table 6.17. Standard Deviation using MCS,  $2n$  and  $2n + 1$  Schemes

BUS	MCS	2n	%e (2n)	2n+1	%e (2n+1)
1	1.494	1.4941	0.004802	1.4941	0.003868
2	1.3383	1.3383	0.001845	1.3383	0.000428
3	1.3408	1.3408	-0.00189	1.3408	-0.0018
4	1.4939	1.4939	0.001456	1.4939	0.001855
5	1.4746	1.4746	0.0015	1.4746	0.002744
6	1.3408	1.3408	-0.00189	1.3408	-0.0018
7	1.3506	1.3506	-0.00189	1.3506	-0.00226
8	1.3383	1.3383	0.001845	1.3383	0.000428
9	1.4808	1.4808	0.001485	1.4808	-0.00036
cost	450.8295	450.69	-0.03094	450.87	0.006765

### 6.4.2. Simulation Set 2

As previously discussed at the beginning of this chapter, this set of simulations considers three different groups of simulations split according to the level of wind integration in the system. The previously mentioned IEEE 118-bus system is used and the load is considered to be deterministic at this stage. Hong's  $2n$  and  $2n + 1$  schemes are used to estimate the means and standard deviations (or coefficients of variation) of the system LMPs and total generation cost and the results are compared to those obtained using Monte Carlo simulation. The wind farm locations, capacity and wind speed characteristics are all shown in Table 6.3 for each level of wind integration, namely, 10%, 20% and 30%.

a. 10% Wind Integration:

Monte Carlo simulation is first run to obtain an estimation of the wind farms output power statistics, the results are presented in Table 6.18. These results are used by Hong's  $2n$  and  $2n + 1$  PEM schemes.

After estimating the wind farms output powers' first few moments, Hong's  $2n$  and  $2n + 1$  schemes and MCS are applied to the probabilistic OPF problem resulting in an estimation of the mean and standard deviation of the system LMPs and total generation cost. Figure 6.3 shows the LMP Means at the different buses obtained through the three methods, namely, MCS, Hong's  $2n$  and Hong's  $2n + 1$ . Figure 6.4 shows the LMPs coefficients of variations at the different buses estimated using MCS and Hong's  $2n$  and  $2n + 1$  PEM schemes.

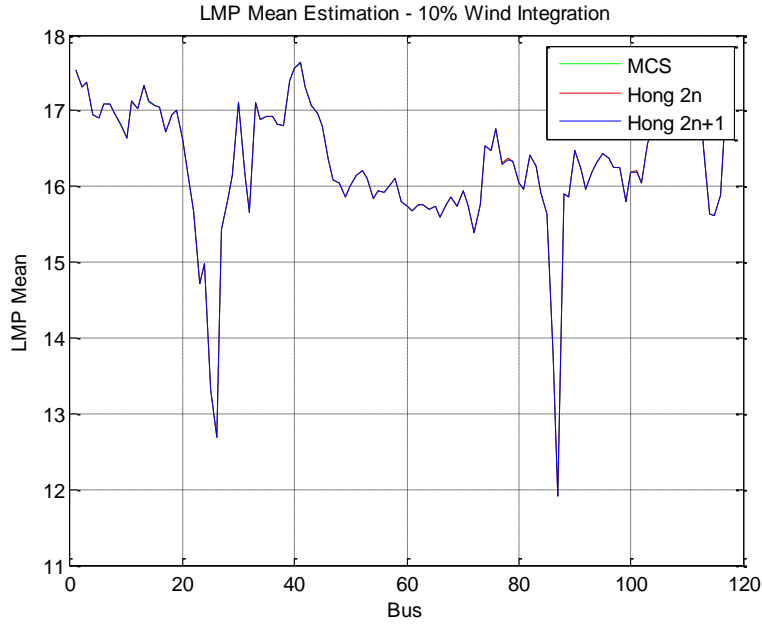
Table 6.18. WF Output Power First Few Moments – 10% Integration

WF AT BUS	MEAN	STANDARD DEVIATION	SKEWNESS	KURTOSIS
5	48.0276	29.6474	-0.2487	1.4606
37	53.6098	36.498	-0.1387	1.366
64	49.5579	30.5221	-0.1511	1.4599
71	46.6461	30.4397	-0.1975	1.405
81	49.9837	36.1234	-0.0787	1.312

The coefficient of variation is nothing but the ratio of the standard deviation to the mean of a random variable. It is usually expressed in percentage form as follows:

$$\% \text{ Coefficient of variation } (x) = \frac{\sigma_x}{\mu_x} \times 100 \quad (6.22)$$

The coefficient of variation is estimated in this case since it shows how much do the LMPs at different buses show variations with respect to their mean values.



**Figure 6.3.** LMP Mean Estimation Using MCS, Hong's  $2n$  Scheme and Hong's  $2n + 1$  Scheme

The error between the quantity estimated using a point estimate method and that estimated using MCS is expressed as follows:

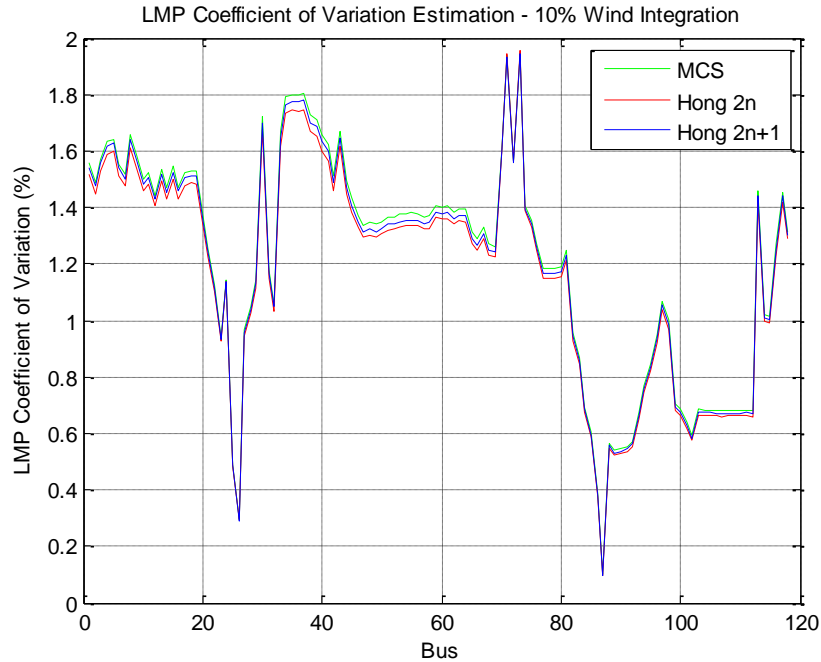
$$\%e = \frac{PEM - MCS}{MCS} \times 100 \quad (6.23)$$

In order to get general information of the accuracy of any PEM in estimating a statistical moment of the system LMPs the Root-Mean-Square Error (RMSE) is used which is defined as follows:

$$RMSE = \sqrt{\frac{1}{n} \sum_{i=1}^n error_i^2} \quad (6.24)$$

where  $error_i$  is taken to be the percentage error in our case expressed in (6.23).

The RMSE of the LMP means and coefficients of variation is shown in Table 6.19.



**Figure 6.4.** LMP Coefficient of Variation Estimation Using MCS, Hong’s  $2n$  Scheme and Hong’s  $2n + 1$  Scheme

**Table 6.19.** RMSE – LMP Mean and Coefficient of Variation (%)

LMP (%)	HONG $2n$	HONG $2n + 1$
Mean	0.01533	0.01366
Coefficient of Variation	0.03658	0.01727

The errors in estimating the total generation cost mean and coefficient of variation with respect to MCS are shown in Table 6.20.

MCS needs around 800 runs for the LMPs means and standard deviations to converge. Hence, MCS necessitates the solution of the deterministic OPF 800 times. Whereas, Hong’s  $2n$  approach requires the solution of the deterministic OPF  $2 \times n = 2 \times 5 = 10$  times and Hong’s  $2n + 1$  scheme requires the solution of the deterministic OPF  $2n + 1 = 2 \times 5 + 1 = 11$  times. This clearly shows the superior computational efficiency of the PEMs as compared to MCS.

**Table 6.20.** Errors in Total Generation Cost Mean and Coefficient of Variation Estimation (%)

TOTAL GENERATION COST (%)	HONG $2n$	HONG $2n + 1$
MEAN	0.02500	0.01000
COEFFICIENT OF VARIATION	-0.06851	-0.06754

*Results Interpretation:*

By examining Figure 6.3 one can see the accuracy of Hong’s both schemes in estimating the system LMPs means. In fact, the three graphs entirely overlap which reflects the preciseness of Hong’s LMP means estimates. This preciseness is also shown in Table 6.19 where the RMSE of Hong’s both schemes with respect to MCS are negligible. This is also the case when it comes to estimating the mean and coefficient of variation of the system total generation cost which is illustrated in Table 6.20. By examining Figure 6.4 one can see the accuracy of Hong’s  $2n$  and  $2n + 1$  schemes in estimating the LMPs coefficient of variations. It can be also observed that Hong’s  $2n + 1$  scheme shows a slightly higher accuracy. The difference in accuracy can also be observed through Table 6.19.

Thus, both PEM schemes showed accurate estimation of the LMPs’ and total generation cost’s means and standard deviations (coefficients of variation) while significantly reducing the computational burden required by MCS. The reduction is of the order of  $\frac{800-10}{800} \times 100 = 98.75\%$

b. 20% Wind Integration:

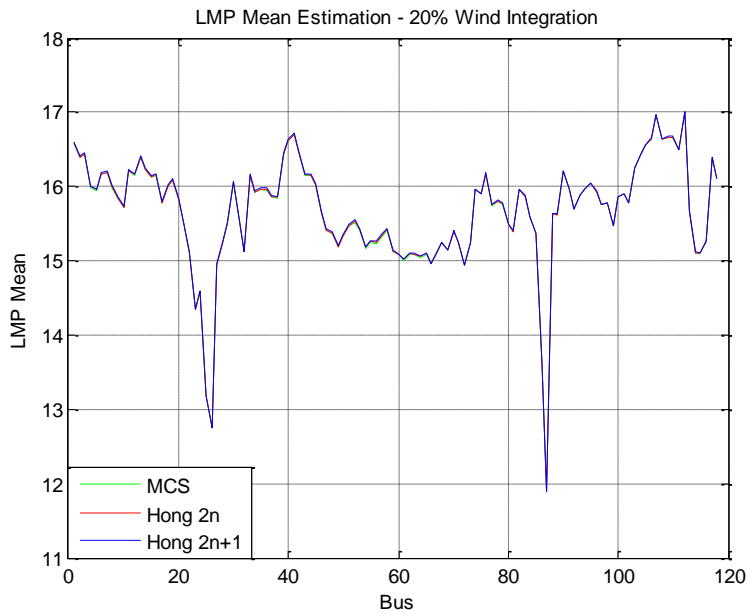
Monte Carlo simulation is first run to obtain an estimation of the wind farms output power statistical moments. The results are presented in Table 6.21.

Table 6.21. WF Output Power First Few Moments – 20% Integration

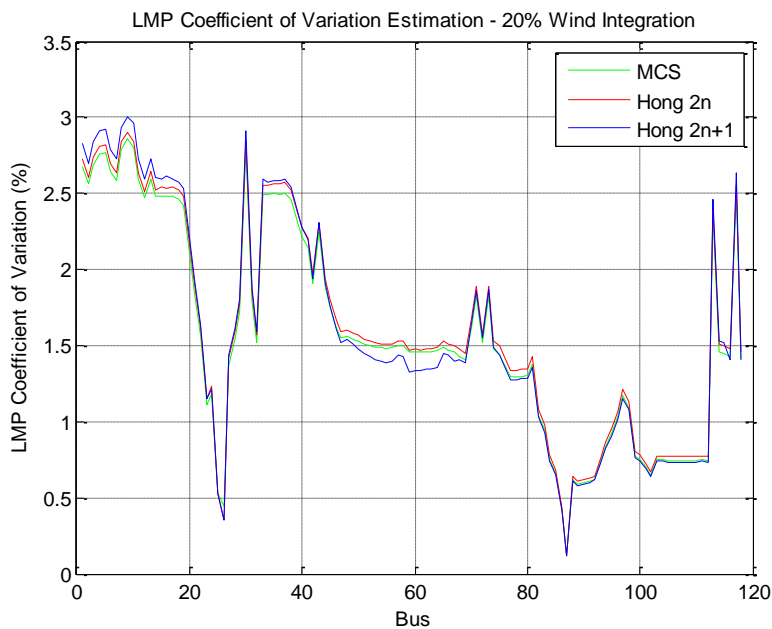
WF AT BUS	MEAN	STANDARD DEVIATION	SKEWNESS	KURTOSIS
5	39.2301	24.3354	-0.2515	1.4641
9	50.8613	41.7243	0.0826	1.295
30	45.2547	33.5009	-0.0182	1.3252
37	58.3859	39.2809	-0.1748	1.3894
38	48.1232	30.1341	-0.0698	1.479
63	38.4503	19.7951	-0.3132	1.6438
64	48.7383	30.2209	-0.1017	1.4651
68	72.6925	30.3692	-0.6656	2.0694
71	39.5775	24.3311	-0.2658	1.4642
81	46.807	33.5974	-0.074	1.311

These results are used by Hong’s  $2n$  and  $2n + 1$  PEM schemes. After estimating the wind farms output powers’ first few moments Hong’s  $2n$  and  $2n + 1$  schemes and MCS are applied to

the probabilistic OPF problem resulting in an estimation of the means and standard deviations of the system LMPs and total generation cost. Figure 6.5 shows the LMP means at the different buses obtained through the three methods, namely, MCS, Hong's  $2n$  and Hong's  $2n + 1$ . Figure 6.6 shows the LMPs coefficients of variations at the different buses estimated using MCS as well as Hong's  $2n$  and  $2n + 1$  PEM schemes.



**Figure 6.5.** LMP Mean Estimation Using MCS, Hong's  $2n$  Scheme and Hong's  $2n + 1$  Scheme



**Figure 6.6.** LMP Coefficient of Variation Estimation Using MCS, Hong's  $2n$  Scheme and Hong's  $2n + 1$  Scheme



The RMSE of the LMP means and coefficients of variation are shown in Table 6.22. Moreover, the errors in the estimation of the total generation cost mean and coefficient of variation with respect to MCS results are shown in Table 6.23.

Table 6.22. RMSE – LMP Mean and Coefficient of Variation (%)

LMP (%)	HONG 2n	HONG 2n+1
MEAN	0.04317	0.1056
COEFFICIENT OF VARIATION	0.04517	0.07696

Table 6.23. Errors in Total generation Cost Mean and Coefficient of Variation Estimation (%)

TOTAL GENERATION COST (%)	HONG 2n	HONG 2n+1
MEAN	-0.03738	-0.01602
COEFFICIENT OF VARIATION	0.09500	0.1038

MCS needs around 800 runs for the LMPs means and standard deviations to converge. Hence, MCS necessitates the solution of the deterministic OPF 800 times. Whereas, Hong's  $2n$  approach requires the solution of the deterministic OPF  $2 \times n = 2 \times 10 = 20$  times and Hong's  $2n + 1$  scheme requires the solution of the deterministic OPF  $2n + 1 = 2 \times 10 + 1 = 21$  times. This clearly shows the superior computational efficiency of the PEMs as compared to MCS.

*Results Interpretation:*

By examining Figure 6.5, one can see the accuracy of Hong's both schemes in estimating the system LMP means. In fact, the three graphs entirely overlap which reflects the preciseness of the obtained LMPs means' estimates. This preciseness is also shown in Table 6.22 where the RMSE of Hong's both schemes with respect to MCS are negligible. This is also the case when it comes to estimating the mean and coefficient of variation of the system's total generation cost which is illustrated in Table 6.23. By examining Figure 6.6, one can see the accuracy of Hong's  $2n$  and  $2n + 1$  schemes in approximating the LMPs coefficient of variations. It can be also observed that in this case Hong's  $2n$  scheme shows a slightly higher accuracy. The difference in accuracy can also be observed through Table 6.23.

Thus, both PEM schemes showed accurate estimation of the LMPs' and total generation cost's means and standard deviations (coefficients of variation) while significantly reducing the computational burden required by MCS. The reduction is of the order of  $\frac{800-20}{800} \times 100 = 97.5\%$

c. 30% Wind Integration:

Monte Carlo simulation is first run to obtain an estimation of the wind farms output power statistics, the results are presented in Table 6.24. These results are used by Hong's  $2n$  and  $2n + 1$  PEM schemes.

Table 6.24. WF Output Power First Few Moments – 30% Integration

WF AT BUS	MEAN	STANDARD DEVIATION	SKEWNESS	KURTOSIS
5	60.262	38.9268	-0.2333	1.3964
9	71.2307	58.9774	0.1471	1.3199
30	64.3164	47.5898	0.0102	1.3228
37	80.8896	56.6091	-0.1016	1.3524
38	69.8822	43.4228	-0.1209	1.4697
63	75.9087	40.2336	-0.2884	1.6266
64	71.0521	42.6771	-0.1503	1.4991
68	103.3604	42.1959	-0.642	2.1321
71	70.3287	44.025	-0.3003	1.4573
81	85.3668	59.9879	-0.1342	1.3267

After estimating the wind farms output powers' first few moments Hong's  $2n$  and  $2n + 1$  schemes and MCS are applied to the OPF problem resulting in an estimation of the means and standard deviations of the LMPs and total generation cost. Figure 6.7 shows the LMP means at the different buses obtained through the three methods, namely, MCS, Hong's  $2n$  and Hong's  $2n + 1$ . Figure 6.8 shows the LMPs coefficients of variations at the different buses estimated using MCS as well as using Hong's  $2n$  and  $2n + 1$  PEM schemes.

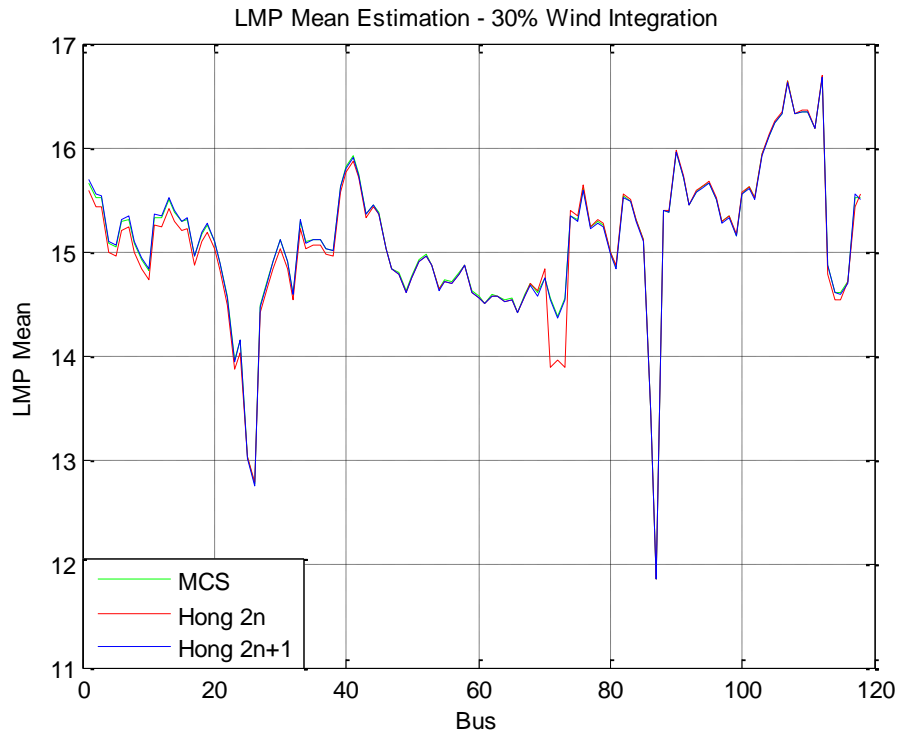


Figure 6.7. LMP Mean Estimation Using MCS, Hong's  $2n$  Scheme and Hong's  $2n + 1$  Scheme

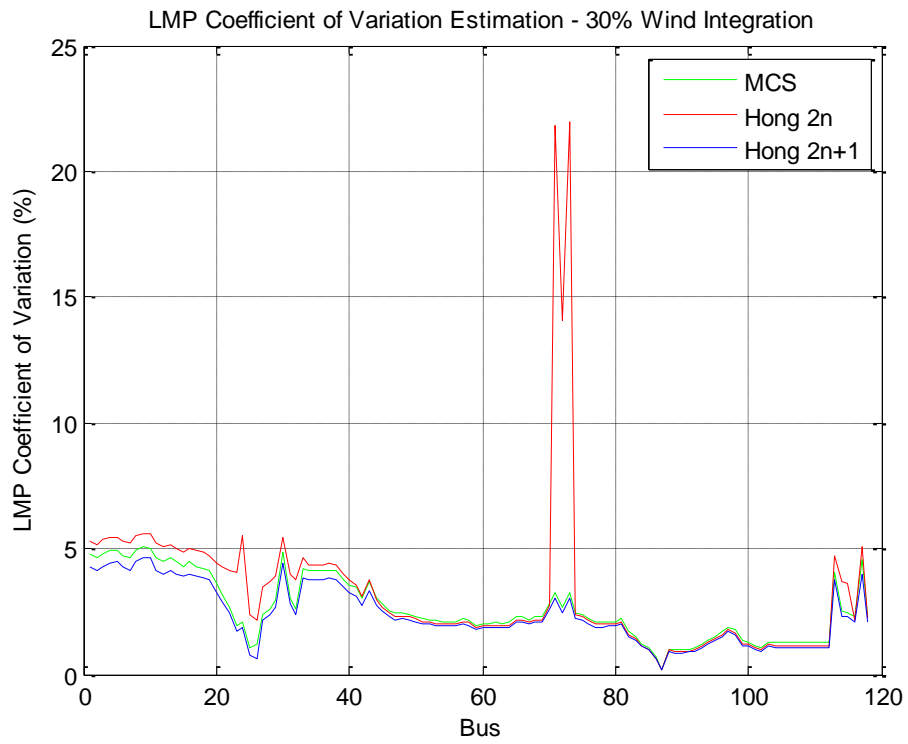


Figure 6.8. LMP Coefficient of Variation Estimation Using MCS, Hong's  $2n$  Scheme and Hong's  $2n + 1$  Scheme

The Root-Mean-Square Error of the LMPs means and coefficients of variation is shown in Table 6.25. In addition, the errors in estimating the total generation cost mean and coefficient of variation with respect to MCS are shown in Table 6.26.

Table 6.25. RMSE – LMP Mean and Coefficient of Variation (%)

LMP (%)	HONG 2n	HONG 2n+1
MEAN	0.7195	0.08945
COEFFICIENT OF VARIATION	2.7093	0.2827

Table 6.26. Errors in Total Generation Cost Mean and Coefficient of Variation Estimation (%)

TOTAL GENERATION COST (%)	HONG 2n	HONG 2n+1
MEAN	-0.08995	-0.04976
COEFFICIENT OF VARIATION	-0.09104	-0.09929

MCS needs around 800 runs for the LMPs means and standard deviations to converge. Hence, MCS necessitates the solution of the deterministic OPF 800 times. Whereas, Hong's  $2n$  approach requires the solution of the deterministic OPF  $2 \times n = 2 \times 10 = 20$  times and Hong's  $2n + 1$  scheme requires the solution of the deterministic OPF  $2n + 1 = 2 \times 10 + 1 = 21$  times. This clearly shows the superior computational efficiency of the PEMs as compared to MCS.

*Results Interpretation:*

As one can see from Figure 6.7, Hong's  $2n + 1$  scheme shows a high preciseness in estimating the LMP means as compared to MCS results. In fact, the LMP means estimated through Hong's  $2n + 1$  PEM and MCS completely overlap. This is not the case for Hong's  $2n$  scheme in opposition to the previous two studied levels of wind integration, namely, 10% and 20%. In fact, Hong's  $2n$  scheme was able to accurately estimate the LMP means for all buses except for buses 71, 72 and 73. This can be explained as follows:

The concentration locations for Hong's  $2n$  scheme depend on the number of incorporated input random variables. In fact, the concentration coefficient for Hong's  $2n$  scheme is given by:

$$\xi_{k,i} = \frac{\lambda_{k,3}}{2} + (-1)^{3-i} \sqrt{n + \left(\frac{\lambda_{k,3}}{2}\right)^2}, \text{ for } i = 1, 2 \text{ and } k = 1, \dots, n \quad (6.25)$$

Hence, when  $n$  is large,  $\xi_{k,i}$  tends to have a larger value. However, the concentration locations are defined as follows:

$$l_{k,i} = (\mu_1, \mu_2, \dots, x_{k,i}, \dots, \mu_{n-1}, \mu_n) \text{ for } i = 1, 2, \dots, m, k = 1, 2, \dots, n \quad (6.26)$$

where,

$$x_{k,i} = \mu_k + \xi_{k,i} \sigma_k \text{ for } i = 1, 2, \dots, m, k = 1, 2, \dots, n \quad (6.27)$$

Hence, when  $\xi_{k,i}$  gets larger, the concentration location,  $x_{k,i}$ , tends to drift away from its mean value. It might drift away to a point where the optimization problem would not have any feasible solutions and fail to converge (this case is illustrated in Simulation Set 3).

This is not the case for Hong's  $2n + 1$  scheme. In fact, for Hong's  $2n + 1$  PEM the concentration coefficients are given by:

$$\xi_{k,i} = \frac{\lambda_{k,3}}{2} + (-1)^{3-i} \sqrt{\lambda_{k,4} - 3 \frac{\lambda_{k,3}^2}{4}} \quad i = 1, 2 \quad (6.28)$$

Thus, the concentration coefficient is not dependent on the number of input random variables involved. As a result, the concentration locations tend to be more and more confined around their corresponding mean values reducing the risk of jeopardizing the feasibility of the optimization problem. One should also note that Hong's  $2n + 1$  scheme includes the input random variables fourth central moment, i.e. the kurtosis, into the concentrations calculation. Hong's  $2n$  included up to only the third central moments, i.e. skewness, of the input random variables. This higher order of approximation yields a better degree of accuracy in output variables moment's estimation.

In our case, through the application of Hong's  $2n$  PEM, the associated locations for one of the iterations assigned a big wind farm contribution at bus 71 (203.09 MW) which made the branch connecting buses 70 and 71 congested (line 70 – 71 thermal limit is 175 MVA) preventing the cheap energy produced by the wind farm at bus 71 from reaching buses 70, 74, 75 and 69. On the other hand, buses 71, 72 and 73 were benefiting from the cheap wind energy generated at bus 71 and ended up with very low LMPs that are equal to 1.0954 \$/MWh, 5.7578\$/MWh and 1.0264 \$/MWh respectively. Such low LMP levels have decreased the

average LMPs at buses 71, 72 and 73 as can be seen through Figure 6.7. The resulting congestion did not occur through Monte Carlo simulations, which implies that such congestion is not realistic. It is indeed just caused by Hong's  $2n$  scheme.

On the other hand, Hong's  $2n + 1$  scheme concentrations did not assign large contributions, which are significantly away from their associated mean values, to the system wind farms which decreased the likelihood of causing any unrealistic congestion in the system branches.

Next, a mathematical explanation of the cause of the congestion caused by Hong's  $2n$  scheme is provided. Accordingly, the 17<sup>th</sup> iteration of Hong's  $2n$  and  $2n + 1$  schemes is investigated. In this iteration, all the wind farms have an output equal to their mean output power shown in Table 6.24 except for the wind farm at bus 71 which has an output power equal to  $x_{9,1} = \mu_9 + \xi_{9,1}\sigma_9$  where  $\mu_9$  is the output power mean of the wind farm at bus 71,  $\sigma_9$  is the output power standard deviation of the wind farm at bus 71 and  $\xi_{9,1}$  is the associated concentration coefficient. For the case of Hong's  $2n$  PEM:

$$\xi_{9,1} = \frac{\lambda_{9,3}}{2} + (-1)^{3-1} \sqrt{n + \left(\frac{\lambda_{9,3}}{2}\right)^2}$$

For the case of Hong's  $2n + 1$  PEM:

$$\xi_{9,1} = \frac{\lambda_{9,3}}{2} + (-1)^{3-i} \sqrt{\lambda_{9,4} - 3 \frac{\lambda_{9,3}^2}{4}}$$

$n$  is the number of random variables in the system which is equal to the number of wind farms installed which in our case is equal to 10.  $\lambda_{9,3}$  and  $\lambda_{9,4}$  are respectively the skewness and kurtosis of the output power of the wind farm at bus 71. The mean, standard deviation, skewness and kurtosis of the output power of the wind farm at bus 71 are shown in Table 6.24. Replacing the different quantities by their corresponding values one gets:

For Hong's  $2n$  PEM:

$$\xi_{9,1} = \frac{\lambda_{9,3}}{2} + (-1)^{3-1} \sqrt{n + \left(\frac{\lambda_{9,3}}{2}\right)^2} = \left(-\frac{0.3003}{2}\right) + \sqrt{10 + \left(\frac{-0.3003}{2}\right)^2} = 3.0157$$

For Hong's  $2n + 1$  PEM:

$$\xi_{9,1} = \frac{\lambda_{9,3}}{2} + (-1)^{3-i} \sqrt{\lambda_{9,4} - 3 \frac{\lambda_{9,3}^2}{4}} = \left( -\frac{0.3003}{2} \right) + \sqrt{1.4573 - 3 \frac{(-0.3003)^2}{4}} = 1.0287$$

It can be noticed that the concentration coefficient of Hong's  $2n$  scheme is almost three times bigger than that of Hong's  $2n + 1$  scheme. This results in a concentration location magnitude that is significantly large.

For Hong's  $2n$  PEM:

$$x_{9,1} = \mu_9 + \xi_{9,1} \sigma_9 = 70.3287 + 3.0157 \times 44.025 = 203.0949$$

For Hong's  $2n + 1$  PEM:

$$x_{9,1} = \mu_9 + \xi_{9,1} \sigma_9 = 70.3287 + 1.0287 \times 44.025 = 115.6172$$

Hence, following Hong's  $2n$  scheme, the wind farm output power at bus 71 is 203.0949 MW for this iteration. This large wind farm contribution has caused an "unrealistic" congestion in the branch connecting buses 70 and 71 which has a 175 MVA line capacity. Hong's  $2n + 1$  scheme did not cause such congestion and produced results that matched those of MCS.

The accuracy of Hong's  $2n + 1$  in estimating the LMP and total generation cost means is also illustrated in Table 6.25 and Table 6.26.

Hong's  $2n + 1$  scheme also shows a high accuracy in estimating the coefficient of variation of the system LMPs and total generation cost. This is clearly shown through Figure 6.8 and Table 6.25 and Table 6.26. Hong's  $2n$  did not produce good LMPs coefficients of variation estimates for buses 71, 72 and 73 as shown in Figure 6.8.

Hong's  $2n + 1$  scheme has proven to be highly accurate when it comes to estimating the means and coefficients of variations of the system LMPs and total generation cost. Hong's  $2n + 1$  PEM is highly computationally efficient requiring 21 runs of the deterministic OPF as compared to 800 runs for MCS. Hence, Hong's  $2n + 1$  PEM introduced a  $\frac{800-21}{800} \times 100 = 97.375\%$  decrease in computational burden as compared to MCS.

d. Simulation Set 2 Results Interpretation:

Table 6.27 shows a summary of the results obtained throughout Simulation Set 2. It shows the accuracy of both Hong's  $2n$  and  $2n + 1$  schemes in estimating the means and coefficients of variation of the system LMPs and total generation cost for 10% and 20% levels of wind integration. Hong's  $2n + 1$  shows a high accuracy as compared to MCS in estimating the LMPs and total generation cost mean and standard deviation for the 30% wind integration case as well. On the overall, Hong's  $2n + 1$  showed a superior performance when compared to Hong's  $2n$ . This superior performance of the  $2n + 1$  scheme is addressed in [60] for the probabilistic power flow case. In our work, this conclusion is also successfully extended and tested for the probabilistic optimal power flow case.

Table 6.27. Summary of Simulation Set 2 Results – A Comparison of the Estimation Errors of Hong's  $2n$  and  $2n + 1$  Schemes for Different Wind Integration Levels

PARAMETER	INTEGRATION LEVEL	LMP (%)		TOTAL GENERATION COST (%)	
		HONG $2n$	HONG $2n+1$	HONG $2n$	HONG $2n+1$
MEAN	10%	0.01533	0.01366	0.025	0.0100
	20%	0.04317	0.1056	-0.0374	-0.01602
	30%	0.7195	0.08945	-0.09	-0.04976
COEF. VAR.	10%	0.03658	0.01727	-0.0685	-0.06754
	20%	0.04517	0.07696	0.095	0.1038
	30%	2.7093	0.2827	-0.091	-0.09929

As one can also notice, the accuracy of Hong's  $2n + 1$  PEM was not affected neither by the number of random variables (number of wind farms) nor by their magnitudes (wind integration level). It indeed preserved very low RMSE levels as compared to MCS results.

### 6.4.3. Simulation Set 3

As previously mentioned this set of simulation cases investigates the application of various point estimate methods to estimate the statistical moments of the system LMPs and total generation cost with the incorporation of uncorrelated intermittent wind energy and correlated randomly distributed loads. Two levels of wind integration are considered, namely, 20% and 30%. The wind farms locations and capacity as well as the wind speeds distributions for the two different integration levels are shown in Table 6.3. In this simulation set, the load is considered



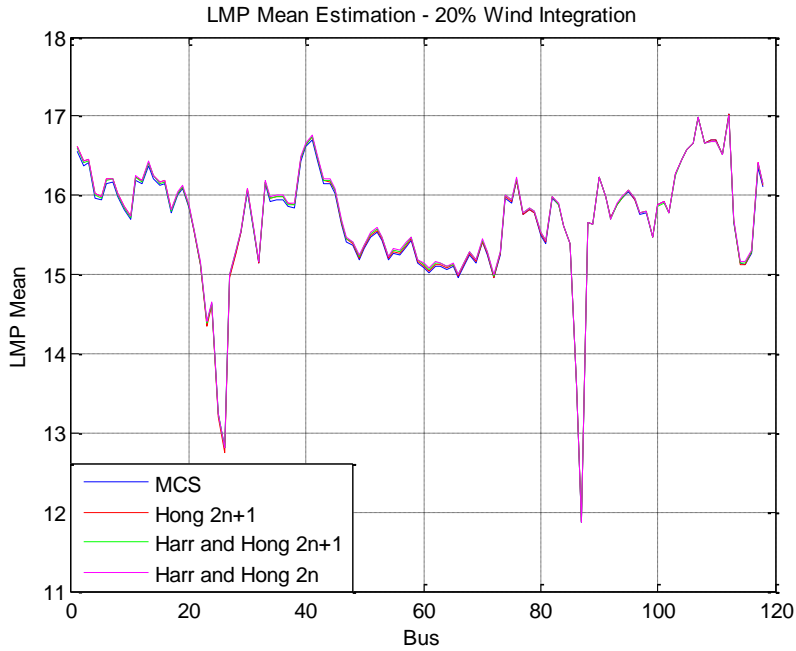
to have a probabilistic nature and to follow a normal distribution with specific means and standard deviations as described in section 6.3 of this chapter.

The PEM schemes that are tested are the following:

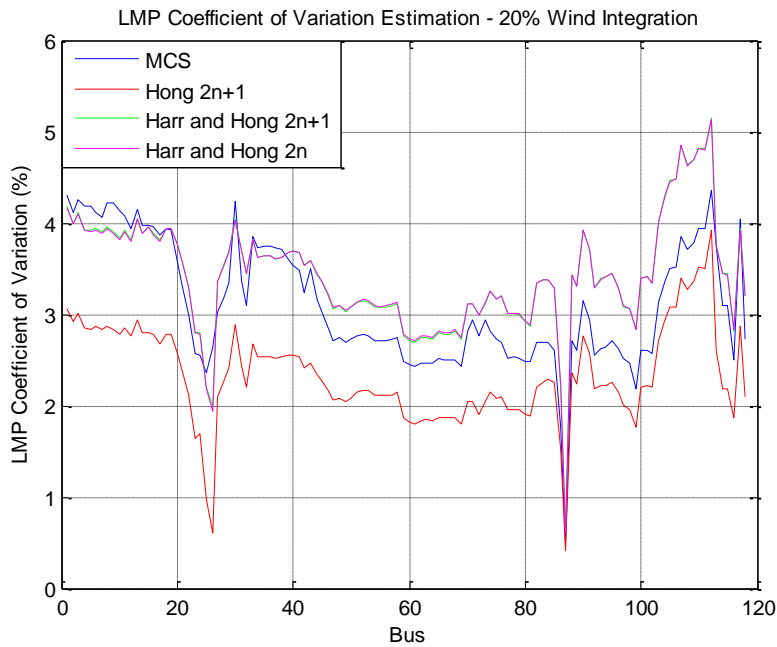
- ✓ Apply a rotational matrix transformation based on the Singular Value Decomposition of the load covariance matrix to end up with a set of uncorrelated random variables after which Hong's  $2n$  scheme can be applied. The number of loads in the system is equal to ninety nine (99). Ten (10) wind farms are also included. Hence, the total number of probability concentrations needed in this case is  $2 \times 10 + 2 \times 99 = 20 + 198 = 218$ .
- ✓ Apply a rotational matrix transformation based on the Singular Value Decomposition of the load covariance matrix to end up with a set of uncorrelated random variables after which Hong's  $2n + 1$  scheme can be applied. The total number of concentrations needed in this case is equal to  $(2 \times 10 + 1) + (2 \times 99 + 1) = 220$ .
- ✓ Employ Hong's  $2n$  scheme to treat the uncorrelated wind farms and Harr's approach to include the random loads. The total number of concentrations needed in this case is equal to  $2 \times 10 + 2 \times 99 = 218$ .
- ✓ Employ Hong's  $2n + 1$  scheme to treat the uncorrelated wind farms and Harr's approach to include the random loads. The total number of concentrations needed in this case is equal to  $(2 \times 10 + 1) + (2 \times 99) = 21 + 198 = 219$ .

a. 20% Wind Integration

First, the 20% wind integration level is considered. Figure 6.9 illustrates the estimation of the system LMPs' means using MCS, Hong's  $2n + 1$  PEM with a de-correlation mechanism, a proposed combination between Harr's and Hong's  $2n + 1$  PEMs and a proposed combination between Harr's and Hong's  $2n$  PEMs.



**Figure 6.9.** LMP Mean Estimation Using Various PEMs and MCS



**Figure 6.10.** LMP Coefficient of Variation Estimation Using Various PEMs and MCS

Figure 6.10 illustrates the application of the mentioned PEM schemes to estimating the system LMPs' coefficients of variation. Table 6.28 and Table 6.29 illustrate the RMSE of the different PEM schemes in estimating the means and coefficients of variations of respectively the system LMPs and total generation cost as compared to Monte Carlo simulation.

Table 6.28. RMSE – LMP Mean and Coefficient of Variation (%)

LMP (%)	HONG $2n + 1$	HARR + HONG $2n + 1$	HARR + HONG $2n$
MEAN	0.1778	0.1900	0.2589
COEFFICIENT OF VARIATION	0.8545	0.4658	0.4684

Table 6.29. Errors in Total Generation Cost Mean and Coefficient of Variation Estimation (%)

TOTAL GENERATION COST (%)	HONG $2n + 1$	HARR + HONG $2n + 1$	HARR + HONG $2n$
MEAN	0.09079	0.1246	0.2296
COEFFICIENT OF VARIATION	-1.0760	1.0157	1.0134

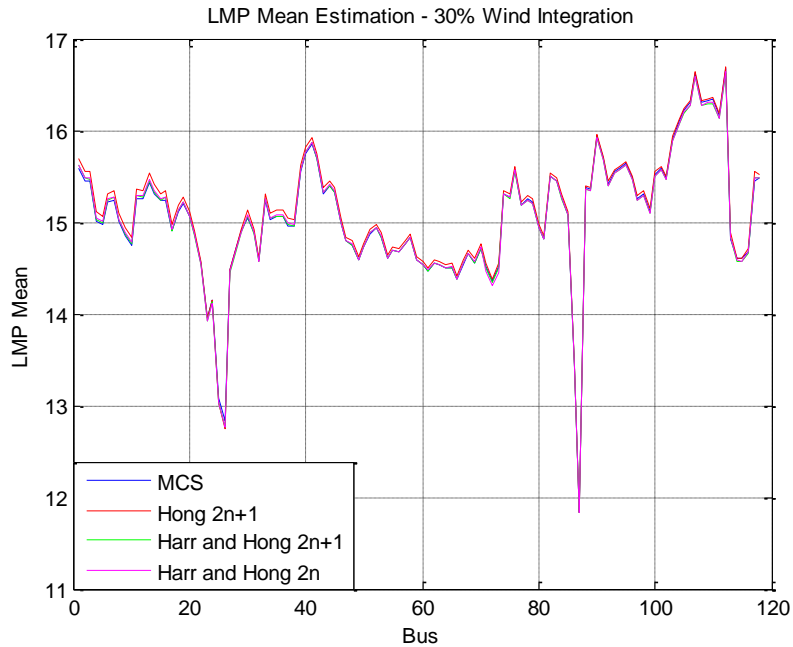
*Results Interpretation:*

As one can notice by examining Figure 6.9, the three different PEM schemes generated accurate estimates of the system LMPs means as compared to Monte Carlo simulation. By examining Table 6.28 and Table 6.29 one can see that the three schemes have achieved negligible RMSE which also proves their accuracy. When it comes to estimating the LMPs coefficients of variation, the three schemes have also generated acceptably accurate results with a superiority of the combination between Harr’s and Hong’s PEMs over the application of only Hong’s  $2n + 1$  PEM with a de-correlation mechanism. These conclusions are extracted from Figure 6.10 and Table 6.28 and Table 6.29.

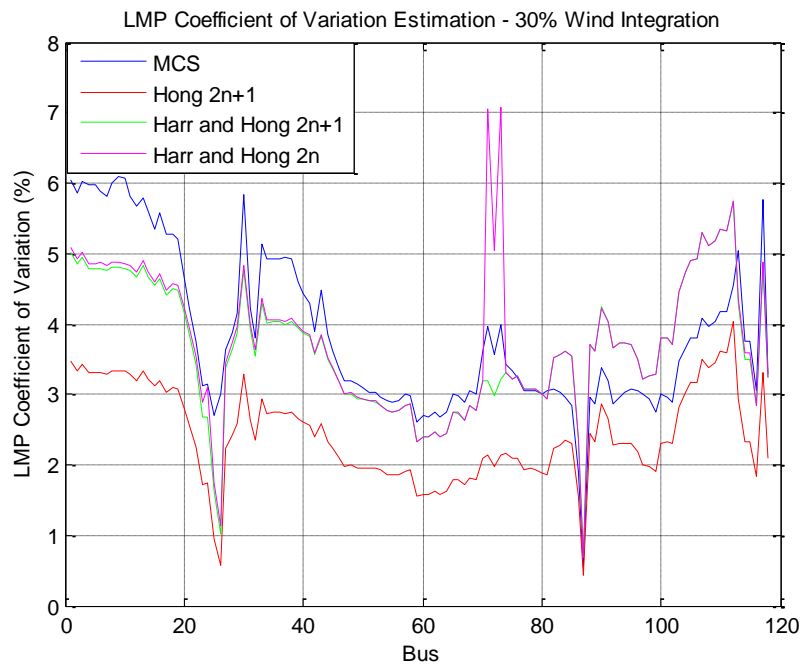
Hence, the studied PEM schemes have achieved a significantly high accuracy while requiring 220, 218 and 219 runs of the deterministic OPF as compared to 1000 runs for MCS. Hence, the studied PEM schemes have achieved a  $\frac{1000-219}{1000} \times 100 = 78.1\%$  decrease in computational burden with significantly high estimation accuracy as compared to MCS.

b. 30% Wind Integration

In this section, the 30% wind integration level is considered. Figure 6.11 illustrates the estimation of the system LMPs’ means using MCS, Hong’s  $2n + 1$  PEM with a de-correlation mechanism, a proposed combination between Harr’s and Hong’s  $2n + 1$  PEMs and a proposed combination between Harr’s and Hong’s  $2n$  PEMs. Figure 6.12 illustrates the application of the mentioned PEM schemes to estimating the system LMPs’ coefficients of variation.



**Figure 6.11.** LMP Mean Estimation Using Various PEMs and MCS



**Figure 6.12.** LMP Coefficient of Variation Estimation Using Various PEMs and MCS

Table 6.30 and Table 6.31 illustrate the RMSE of the different PEM schemes in estimating the means and coefficients of variation of respectively the system LMPs and total generation cost as compared to Monte Carlo simulation.

Table 6.30. RMSE – LMP Mean and Coefficient of Variation (%)

LMP (%)	HONG $2n + 1$	HARR + HONG $2n + 1$	HARR + HONG $2n$
MEAN	0.3635	0.1329	0.1583
COEFFICIENT OF VARIATION	1.5689	0.7185	0.7998

Table 6.31. Errors in Total Generation Cost Mean and Coefficient of Variation Estimation (%)

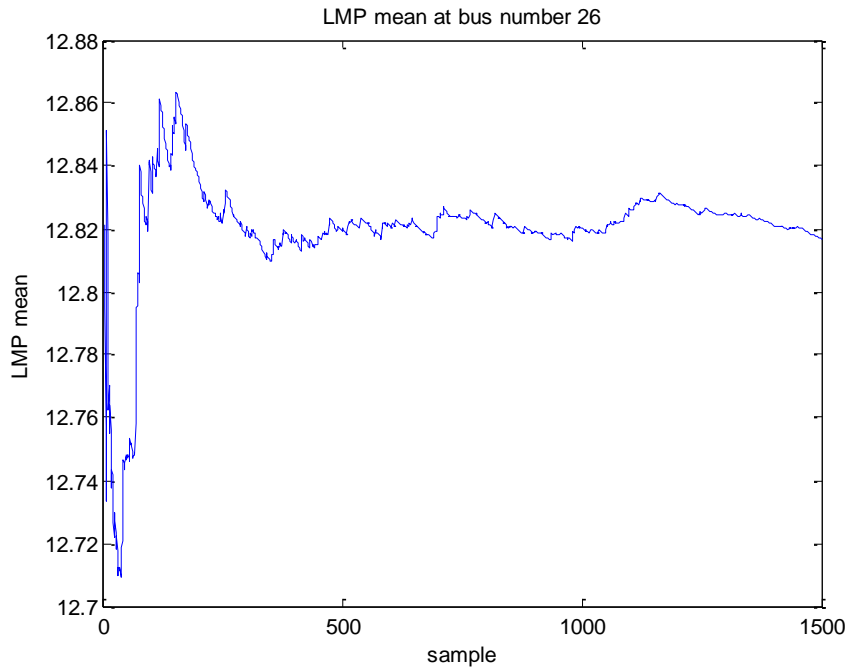
TOTAL GENERATION COST (%)	HONG $2n + 1$	HARR + HONG $2n + 1$	HARR + HONG $2n$
MEAN	0.2551	0.09975	0.1343
COEFFICIENT OF VARIATION	-1.6018	0.5380	0.5365

*Results Interpretation:*

By examining Figure 6.11 and Table 6.30 one can see that the three schemes have accurately estimated the system LMPs' means. In the PEM scheme combining Harr's and Hong's  $2n$  PEMs, Hong's  $2n$  did not show the inaccuracy that it showed in Simulation Set 2 with 30% wind integration shown in Figure 6.6. That is due to the fact that the low LMPs that are generated in one of the iterations discussed before did not have a large effect on the average because of the large population sample. In fact, in the Simulation Set 2, the population sample is equal to twenty (20); hence, one low LMP value can largely affect the average. Whereas, in the Simulation Set 3 case, the population sample is  $99 \times 2 + 20 = 118$ ; hence, one low LMP value would not have a large effect on the average value. However, the inaccuracy of Hong's  $2n$  scheme still shows in estimating the LMPs coefficients of variations shown in Figure 6.12 similarly to the Simulation Set 2 case shown in Figure 6.7.

The combination of Harr's and Hong's  $2n + 1$  PEM shows a superior accuracy as compared to the other schemes. This can be clearly observed through Table 6.30 and Table 6.31. In fact, the combination of Harr's and Hong's  $2n + 1$  PEMs estimated, with high accuracy, the system LMPs and total generation costs means and coefficients of variation while requiring the solution of the deterministic OPF two hundred and nineteen (219) times. Hong's  $2n + 1$  with de-correlation mechanism required the solution of the deterministic OPF two hundred and twenty (220) times but it also required some computational complexity throughout the de-correlation mechanism. MCS required one thousand two hundred (1200) runs of the deterministic OPF to

reach converging LMPs means and coefficients of variations. This is clearly shown in Figure 6.13 which illustrates the convergence of Bus 26 LMP mean using MCS.



**Figure 6.13.** Bus 26 LMP Mean Estimation Convergence using MCS

Hence, Harr’s and Hong’s  $2n + 1$  PEMs combination achieved significantly high accurate results while reducing the computational burden by  $\frac{1200-219}{1200} \times 100 = 81.75\%$  as compared to MCS.

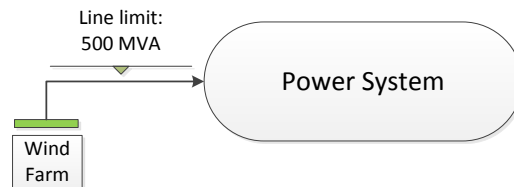
*Hong’s  $2n$  Performance:*

In the previous Simulation Set 3 analysis, we did not show the results obtained using Hong’s  $2n$  scheme with a de-correlation mechanism. This is due to the fact that this scheme was not able to converge for both wind integration levels for some of its iterations. The reason for this non-convergence is as previously pointed to: Hong’s  $2n$  concentrations locations depend on the number of random variables in the system. Hence, when the number of variables increases, the locations tend to drift away from their mean values. In our case, the number of random variables is relatively big, equal to ninety nine (99). For one of the locations, the loads are low and the reactive power is relatively high because of the connected wind farms. Thus, the optimization problem did not have any feasible solutions, in the given operation conditions, that can abide by

the voltage limits and hence the optimization problem did not converge. The tendency of assigning power injections that are far away from their mean values using Hong's  $2n$  PEM can lead to non-feasible OPF solutions because of many reasons as:

1. Total load less than the total generators minimum power output.
2. Total load greater than the total generators maximum output power.
3. Highly congested lines.
4. Voltage limits violation...

Let us assume, for instance, that the system shown in Figure 6.14 is considered. This example illustrates the risks of the application of Hong's  $2n$  scheme.



**Figure 6.14.** Hong's  $2n$  PEM Risks Illustration

We assume that the line connecting the wind farm to the power system has a 500 MVA thermal limit. Let us assume that the rated power output of the wind farm is 352.94 MVA corresponding to a 300 MW rated real power and 0.85 power factor. Using MCS, the wind farm power output would never exceed 352.84 MVA. Using Hong's  $2n + 1$  the WF power output will always have values that are close to the mean which will not likely violate the line thermal limit. On the other hand, if we assume that the power system has other connected wind farms or any other kinds of input random variables making the total number of random variables in the system large; then, if we apply Hong's  $2n$  PEM, the wind farm power output might have a value of 600 MVA. Since we are modeling the wind farm's power output as a negative load, the optimization problem has a binding equality constraint that necessitates that the wind farm injects all its power output. Since the line connecting the wind farm to the system has a thermal limit of 500 MVA the wind farm cannot inject all its power unless the line limit is violated. Line limit violation is impossible because of the line limits inequality constraints that are part of the OPF formulation. Hence, the optimization problem does not have a feasible solution and the solution algorithm would not converge.

c. Simulation Set 3 Results Interpretation:

Table 6.32 shows a summary of the results obtained throughout Simulation Set 3. It shows the accuracy of the different tested PEM schemes illustrating the high performance of the suggested combination of Harr’s and Hong’s  $2n + 1$  PEM scheme and its invulnerability to the wind integration level.

Table 6.32. Summary of Simulation Set 3 Results – A Comparison of the Estimation Errors of the Different Tested PEM Schemes for Different Wind Integration Levels

PARAMETER	INTEGRATION LEVEL	LMP (%)			TOTAL GENERATION COST (%)		
		HONG $2n + 1$	HARR & HONG $2n + 1$	HARR & HONG $2n$	HONG $2n + 1$	HARR & HONG $2n + 1$	HARR & HONG $2n$
Mean	20%	0.1778	0.1900	0.2589	0.0908	0.1246	0.2296
	30%	0.3635	0.1329	0.1583	0.2551	0.0998	0.1343
Coefficient of Variation	20%	0.8545	0.4658	0.4684	-1.0760	1.0157	1.0134
	30%	1.5689	0.7185	0.7998	-1.6018	0.5380	0.5365

### 6.5. Extended Analysis

The PEMs applicability can extend beyond the statistical moments estimation of LMPs and total generation cost. In fact, using PEMs one can estimate the statistical moments of any of the OPF output variables, namely, system states, line flows, power injection, electric losses, etc. For instance, PEMs can be used to estimate the statistical moments of the system’s voltage magnitudes under uncertainty introduced by large scale wind generation. Using these statistical moments one can investigate the amount of uncertainty that is reflected on the system voltages and the effects of the incorporation of wind energy on the system’s voltage profile. Moreover, the PEMs can be also used to estimate the statistics of the power flows in each branch of the system under uncertainty introduced by wind energy. This helps in investigating the effects of the introduced wind energy on the system’s congestion levels. Table 6.33 illustrates the estimation of the means of the power flow in each branch under 30% wind integration level from which the average capacity usage of the associated branches is calculated. The means of the capacity usage of each line with wind integration is compared to the line capacity usage with no wind integration. This helps studying how the congestion level of each of the branches has changed with the introduction of wind energy. One can see for instance that wind integration reduced the power flows in branches 9, 96 and 104 while it increased the power flows, on the



other hand, in branches 119 and 127. Using PEMs the standard deviation of the power flows can also be estimated to analyze the scatter of the power flows around their corresponding means.

Table 6.33. Effects of Wind Energy Integration on System Congestion Levels

BRANCH	FROM BUS	TO BUS	CAPACITY USAGE IN CASE OF NO WIND INTEGRATION	AVERAGE CAPACITY WITH 30% WIND
9	9	10	45%	31%
96	38	65	35%	21%
104	65	68	70%	67%
119	69	77	41%	61%
127	81	80	27%	50%

## 6.6. Conclusions

In this chapter, various PEMs are applied to the OPF problem in order to estimate the statistical moments of system LMPs and total generation cost. The generated results illustrate the accuracy of the various applied PEMs, the risks of the usage of Hong's  $2n$  scheme and the potential superior performance of the proposed combination of Hong's and Harr's PEMs as compared to Hong's  $2n + 1$  scheme which is the most used scheme in literature. The various simulations also show the reduction in computational burden introduced by the PEMs as compared to MCS which consists the main advantage of using the PEMs. An extended analysis is also presented showing that the application of PEM is not limited to estimating the statistical moments of LMPs and total generation cost but can also extend to the estimation of any of the system variables' statistics.

In the next chapter, a modified PEM is proposed based on Rosenblueth's PEM. Rosenblueth's approach is the most general PEM approach designed to be applied to problems with various types of input random variables. For instance, Hong's method can only be applied to problems with uncorrelated input random variables while Harr's method can only be applied to problems with symmetric input variables. Rosenblueth's PEM, on the other hand, can incorporate correlated as well as skewed input variables. The main drawback of Rosenblueth's approach is that it requires a relatively large number of iterations (i.e.  $2^n$  for  $n$  input variables) which made it

impractical for the OPF problem. The proposed method can be applied to problems with correlated and skewed input variables while requiring a smaller number of iterations as compared to Rosenblueth's PEM.

## **Chapter VII: A New Point Estimate Method**

### **7.1. Introduction**

In this chapter, a new modified PEM is introduced. This PEM is based on the logic introduced by Rosenblueth which aims at taking a large number of probability concentrations which makes the solution of the simultaneous equations described in Chapter V.5.2 easier to solve. The proposed PEM requires the usage of  $\frac{(k+n)!}{k!n!}$  probability concentrations as compared to  $2^n$  for the case of Rosenblueth's PEM.  $k$  is the order up to which the input variables' statistical moments are matched and  $n$  is the number of input random variables. A full explanation of the proposed method is presented next.

### **7.2. Proposed Modified PEM**

As explained in the previous chapters, the point estimate methods' process is as follows:

We consider a system that has  $n$  input random variables and  $p$  output random variables described by the model  $\mathbf{Y} = H(\mathbf{X})$  where  $\mathbf{X} = [X_1, X_2, \dots, X_n]^T$  is the set of input random variables and  $\mathbf{Y} = [Y_1, Y_2, \dots, Y_p]^T$  is the set of output random variables. The output random variables depend on the input random variables and their dependency is based on the system model represented by the output function  $H$ . The PEM aims at estimating the statistical moments of the output random variables,  $\mathbf{Y}$ , using the statistical moments of the input random variables  $\mathbf{X}$ . The process is as follows, the PEM generates  $m$  probability concentrations,  $p_i$ , at specific locations,  $\mathbf{X}_i = [X_{1,i}, X_{2,i}, \dots, X_{n,i}]^T$ , such that these  $\mathbf{X}_i$ , for  $i = 1, \dots, m$ , with weights  $p_i$ , for  $i = 1, \dots, m$ , have first few moments that match the exact first few moments of the input random variables,  $\mathbf{X}$ , up to a certain order. Sequentially, each of these points,  $\mathbf{X}_i$ , are fed to the output function,  $H(\mathbf{X}_i)$ , generating a set of outputs  $\mathbf{Y}_i = [Y_{1,i}, Y_{2,i}, \dots, Y_{p,i}]^T$ . This process is repeated for all the generated probability concentrations (i.e. from  $i = 1, \dots, m$ ). The outputs,  $\mathbf{Y}_i$ , for  $i = 1, \dots, m$ , have the same weights,  $p_i$ , as their corresponding inputs  $\mathbf{X}_i$ . From the weighted outputs,  $\mathbf{Y}_i$ , with weights,  $p_i$ , the statistical moments of the output random variables,  $\mathbf{Y}$ , can be estimated.

Throughout Chapter V, we have explored and investigated the point estimate method proposed by Rosenblueth for the case of multiple input random variables. We have shown that

for the generated probability concentrations central moments to match the moments of the input random variables up to order  $k$  the number of simultaneous equations that needs to be solved is equal to

$$\sum_{i=0}^K N_i = \frac{(k+n)!}{k!n!} \quad (7.1)$$

where  $n$  is the number of input random variables.

We have also shown in Chapter V that in order to match up to  $k^{th}$  central moment of the input random variables the number of probability concentrations needed is equal to

$$\text{number of concentrations} = m \geq \frac{(k+n)!}{k!(n+1)!} \quad (7.2)$$

We have also focused on the fact that each probability concentration results in  $n + 1$  unknowns or parameters which are the  $n$  locations  $[X_{1,i}, X_{2,i}, \dots, X_{n,i}]$  and the associated weights  $p_i$ . These parameters need to be tuned to meet the  $\frac{(k+n)!}{k!n!}$  simultaneous equations that result in a match between the probability concentrations central moments and the exact central moments of the input random variables. To solve for the  $\frac{(k+n)!}{k!n!}$  equations,  $\frac{(k+n)!}{k!n!}$  parameters are needed. Thus, all extra parameters are redundant parameters that can be fixed at defined values.

Through the analysis in Chapter V, we explained that in the case of multiple input random variables, the number of equations to be solved simultaneously, so that the picked probability concentrations have a distribution that has moments similar to those of the distribution of the input random variables, increases drastically as shown in Table 5.1. This can be explained through equation (7.1) that shows the increase in the number of equations with the increase in  $n$ . Since the involved equations are non-linear, their simultaneous solution for a large number of equations is complicated. The proposed scheme by Rosenblueth is to take an abundant number of probability concentrations,  $m = 2^n$ , and to fix some of the locations and solve for the remaining locations and weights. This method aims at making the solution of the simultaneous equations easier by fixing some of the parameters (that are redundant) and solving for the others in a way that makes the solution of the set of simultaneous equations simple.

The main disadvantage of Rosenblueth's method, is that it requires a large number of probability concentrations,  $m = 2^n$ , which results in a need to evaluate the output function  $2^n$  times. If the output function is a complex function that requires a relatively large period of time to be solved, running it for  $2^n$  times might be impractical.

The proposed PEM is derived from Rosenblueth's approach. It uses the same logic proposed by Rosenblueth suggesting taking a large number of concentrations and fixing some of the locations so that the simultaneous set of equations becomes simple to solve. However, it requires  $m = \frac{(k+n)!}{k!n!}$  concentrations instead of  $2^n$  which reduces the computational burden of Rosenblueth's PEM for the case where  $n$  is large. The proposed method generates then  $\frac{(k+n)!}{k!n!}$  probability concentrations, fixes their corresponding locations and solves only for the weights,  $p_i$ , such that the simultaneous set of equations becomes a linear function of the unknowns which is relatively easy to solve. The proposed method can incorporate correlated and skewed random variables; hence, it does not suffer from the downsides of Harr's and Hong's approaches. Moreover, the suggested method relates the number of concentrations needed to the desired order up to which the central moments of the input random variables needs to be matched,  $k$ . The proposed method is fully explained in next.

Let us consider the case of two input random variables on which the output random variable depends. The input random variables are denoted by  $X$  and  $Y$  while the output random variable is denoted by  $Z$  such as  $Z = H(X, Y)$ . The joint distribution of  $X$  and  $Y$  is denoted by  $f_{XY}(X, Y)$  while the marginal distributions of  $X$  and  $Y$  are respectively denoted by  $f_X(X)$  and  $f_Y(Y)$ . For the case of two jointly distributed random variables,  $X$  and  $Y$ , the following applies:

$$\int_{-\infty}^{+\infty} f_{XY}(x, y) dy = f_X(x) \quad (7.3)$$

$$E[X] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x f_{XY}(x, y) dy dx = \int_{-\infty}^{+\infty} x f_X(x) dx \quad (7.4)$$

$$E[Y] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y f_{XY}(x, y) dx dy = \int_{-\infty}^{+\infty} y f_Y(y) dy \quad (7.5)$$

$$\sigma_X^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \mu_X)^2 f_{XY}(x, y) dy dx = \int_{-\infty}^{+\infty} (x - \mu_X)^2 f_X(x) dx \quad (7.6)$$

$$\sigma_Y^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (y - \mu_Y)^2 f_{XY}(x, y) dx dy = \int_{-\infty}^{+\infty} (y - \mu_Y)^2 f_Y(y) dy \quad (7.7)$$

In our analysis, let us assume that we need to meet up to the third joint central moment of the input random variables probability density function (i.e.  $k = 3$ ). For a multivariate probability distribution  $f_{XY\dots}(x, y, \dots)$ , the multivariate moment  $j, k, \dots$  is defined as follows:

$$\mu_{j,k,\dots} = E \left[ (X - E(X))^j (Y - E(Y))^k \dots \right] \quad (7.8)$$

The joint central moments up to order three are as follows:

- Zeroth joint central moment:

$$\mu_{0,0} = E \left[ (X - E(X))^0 (Y - E(Y))^0 \right] = 1 \quad (7.9)$$

- First joint central moments:

$$\mu_{0,1} = E \left[ (X - E(X))^0 (Y - E(Y))^1 \right] = E \left[ (Y - E(Y))^1 \right] = 0 \quad (7.10)$$

$$\mu_{1,0} = E \left[ (X - E(X))^1 (Y - E(Y))^0 \right] = E \left[ (X - E(X))^1 \right] = 0 \quad (7.11)$$

- Second joint central moments:

$$\mu_{2,0} = E \left[ (X - E(X))^2 (Y - E(Y))^0 \right] = E \left[ (X - E(X))^2 \right] = \sigma_X^2 \quad (7.12)$$

$$\mu_{0,2} = E \left[ (X - E(X))^0 (Y - E(Y))^2 \right] = E \left[ (Y - E(Y))^2 \right] = \sigma_Y^2 \quad (7.13)$$

$$\mu_{1,1} = E \left[ (X - E(X))^1 (Y - E(Y))^1 \right] = \text{cov}(X, Y) \quad (7.14)$$

- Third joint central moments

$$\mu_{3,0} = E \left[ (X - E(X))^3 (Y - E(Y))^0 \right] = E \left[ (X - E(X))^3 \right] = \nu_X \sigma_X^3 \quad (7.15)$$

$$\mu_{0,3} = E \left[ (X - E(X))^0 (Y - E(Y))^3 \right] = E \left[ (Y - E(Y))^3 \right] = \nu_Y \sigma_Y^3 \quad (7.16)$$

$$\mu_{2,1} = E \left[ (X - E(X))^2 (Y - E(Y))^1 \right] = \quad (7.17)$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^2 (Y - E(Y))^1 f_{XY}(x, y) dx dy$$

$$\mu_{1,2} = E \left[ (X - E(X))^1 (Y - E(Y))^2 \right] = \quad (7.18)$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^1 (Y - E(Y))^2 f_{XY}(x, y) dx dy$$

There are ten equations to be solved simultaneously in order to match the input random variables central moments up to the third order. Thus, at least ten parameters are needed to be able to solve for those equations.

Following equation (7.2)

$$\text{number of concentrations needed} = m \geq \frac{(k+n)!}{k!(n+1)!}$$

In our case,  $k = 3$  since we need to meet up to the third central moment and  $n = 2$  since we have two input random variables  $X$  and  $Y$ .

Hence,

$$m \geq \frac{(k+n)!}{k!(n+1)!} = \frac{(3+2)!}{3!(2+1)!} = \frac{5!}{3!3!} = 3.333 \quad (7.19)$$

Thus,

$$m = 4 \quad (7.20)$$

Hence, four concentrations are considered which correspond to  $4(n + 1) = 4(2 + 1) = 12$  parameters since each concentration is associated with  $(n + 1)$  unknowns.

The concentrations are placed at the following points, i.e. realizations of the random variables  $X$  and  $Y$ :  $(x_1, y_1), (x_2, y_2), (x_3, y_3)$  and  $(x_4, y_4)$ . The probability distribution used is defined to be as follows:

$$P_{X,Y}(X, Y) = p_1 \delta(x - x_1, y - y_1) + p_2 \delta(x - x_2, y - y_2) + p_3 \delta(x - x_3, y - y_3) + p_4 \delta(x - x_4, y - y_4) \quad (7.21)$$

The moments of the new probability distribution,  $P_{X,Y}(X, Y)$ , are as follows:

- Zeroth central moment

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^0 (Y - E(Y))^0 P_{X,Y}(X, Y) dXdY = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p_1 \delta(x - x_1, y - y_1) + p_2 \delta(x - x_2, y - y_2) + p_3 \delta(x - x_3, y - y_3) + p_4 \delta(x - x_4, y - y_4) dxdy = p_1 + p_2 + p_3 + p_4 \quad (7.22)$$

This expression should match that of  $\mu_{0,0}$ .

Thus, the first equation that needs to be met is:

$$p_1 + p_2 + p_3 + p_4 = 1. \quad (7.23)$$

- First central moments

- $\mu_{0,1}$

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^0 (Y - E(Y))^1 P_{X,Y}(X,Y) dXdY &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (Y - E(Y))^1 (p_1 \delta(x - x_1, y - y_1) + p_2 \delta(x - x_2, y - y_2) + p_3 \delta(x - x_3, y - y_3) + p_4 \delta(x - x_4, y - y_4)) dXdY = \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (Y - E(Y))^1 p_1 \delta(x - x_1, y - y_1) + (Y - E(Y))^1 p_2 \delta(x - x_2, y - y_2) + (Y - E(Y))^1 p_3 \delta(x - x_3, y - y_3) + (Y - E(Y))^1 p_4 \delta(x - x_4, y - y_4) dXdY \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^0 (Y - E(Y))^1 P_{X,Y}(X,Y) dXdY &= (y_1 - E(Y))p_1 + (y_2 - E(Y))p_2 + (y_3 - E(Y))p_3 + (y_4 - E(Y))p_4 \end{aligned} \quad (7.24)$$

This expression should match that of  $\mu_{0,1}$ . Thus, the second equation that needs to be met is:

$$(y_1 - E(Y))p_1 + (y_2 - E(Y))p_2 + (y_3 - E(Y))p_3 + (y_4 - E(Y))p_4 = 0. \quad (7.25)$$

- $\mu_{1,0}$

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^1 (Y - E(Y))^0 P_{X,Y}(X,Y) dXdY &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^1 (p_1 \delta(x - x_1, y - y_1) + p_2 \delta(x - x_2, y - y_2) + p_3 \delta(x - x_3, y - y_3) + p_4 \delta(x - x_4, y - y_4)) dXdY = \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^1 p_1 \delta(x - x_1, y - y_1) + (X - E(X))^1 p_2 \delta(x - x_2, y - y_2) + (X - E(X))^1 p_3 \delta(x - x_3, y - y_3) + (X - E(X))^1 p_4 \delta(x - x_4, y - y_4) dXdY \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^1 (Y - E(Y))^0 P_{X,Y}(X,Y) dXdY &= (x_1 - E(X))p_1 + (x_2 - E(X))p_2 + (x_3 - E(X))p_3 + (x_4 - E(X))p_4 \end{aligned} \quad (7.26)$$

This expression should match that of  $\mu_{1,0}$ . Thus, the third equation that needs to be met is:

$$(x_1 - E(X))p_1 + (x_2 - E(X))p_2 + (x_3 - E(X))p_3 + (x_4 - E(X))p_4 = 0 \quad (7.27)$$



- Second central moments

- $\mu_{2,0}$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^2 (Y - E(Y))^0 P_{X,Y}(X,Y) dXdY = \quad (7.28)$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^2 (p_1 \delta(x - x_1, y - y_1) + p_2 \delta(x - x_2, y - y_2) + p_3 \delta(x - x_3, y - y_3) + p_4 \delta(x - x_4, y - y_4)) dXdY = (x_1 - E(X))^2 p_1 + (x_2 - E(X))^2 p_2 + (x_3 - E(X))^2 p_3 + (x_4 - E(X))^2 p_4$$

This expression should match that of  $\mu_{2,0}$ . Thus, the fourth equation that needs to be met is:

$$(x_1 - E(X))^2 p_1 + (x_2 - E(X))^2 p_2 + (x_3 - E(X))^2 p_3 + (x_4 - E(X))^2 p_4 = \sigma_X^2 \quad (7.29)$$

- $\mu_{0,2}$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^0 (Y - E(Y))^2 P_{X,Y}(X,Y) dXdY = \quad (7.30)$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (Y - E(Y))^2 (p_1 \delta(x - x_1, y - y_1) + p_2 \delta(x - x_2, y - y_2) + p_3 \delta(x - x_3, y - y_3) + p_4 \delta(x - x_4, y - y_4)) dXdY = (y_1 - E(Y))^2 p_1 + (y_2 - E(Y))^2 p_2 + (y_3 - E(Y))^2 p_3 + (y_4 - E(Y))^2 p_4$$

This expression should match that of  $\mu_{0,2}$ . Thus, the fifth equation that needs to be met is:

$$(y_1 - E(Y))^2 p_1 + (y_2 - E(Y))^2 p_2 + (y_3 - E(Y))^2 p_3 + (y_4 - E(Y))^2 p_4 = \sigma_Y^2 \quad (7.31)$$

- $\mu_{1,1}$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^1 (Y - E(Y))^1 P_{X,Y}(X,Y) dXdY = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^1 (Y - E(Y))^1 (p_1 \delta(x - x_1, y - y_1) + p_2 \delta(x - x_2, y - y_2) + p_3 \delta(x - x_3, y - y_3) + p_4 \delta(x - x_4, y - y_4)) dXdY = (x_1 - E(X))(y_1 - E(Y))p_1 + (x_2 - E(X))(y_2 - E(Y))p_2 + (x_3 - E(X))(y_3 - E(Y))p_3 + (x_4 - E(X))(y_4 - E(Y))p_4 \quad (7.32)$$

This expression should match that of  $\mu_{1,1}$ . Thus, the sixth equation that needs to be met is:

$$(x_1 - E(X))(y_1 - E(Y))p_1 + (x_2 - E(X))(y_2 - E(Y))p_2 + (x_3 - E(X))(y_3 - E(Y))p_3 + (x_4 - E(X))(y_4 - E(Y))p_4 = \sigma_{XY} \quad (7.33)$$

$$E(Y)p_3 + (x_4 - E(X))(y_4 - E(Y))p_4 = cov(X, Y)$$

- Third central moments

- $\mu_{3,0}$

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^3 (Y - E(Y))^0 P_{X,Y}(X, Y) dXdY = & \quad (7.34) \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^3 (p_1 \delta(x - x_1, y - y_1) + p_2 \delta(x - x_2, y - y_2) + p_3 \delta(x - \\ x_3, y - y_3) + p_4 \delta(x - x_4, y - y_4)) dXdY = & (x_1 - E(X))^3 p_1 + (x_2 - E(X))^3 p_2 + \\ (x_3 - E(X))^3 p_3 + (x_4 - E(X))^3 p_4 & \end{aligned}$$

This expression should match that of  $\mu_{3,0}$ . Thus, the seventh equation that needs to be met is:

$$(x_1 - E(X))^3 p_1 + (x_2 - E(X))^3 p_2 + (x_3 - E(X))^3 p_3 + (x_4 - E(X))^3 p_4 = v_x \sigma_x^3 \quad (7.35)$$

- $\mu_{0,3}$

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^0 (Y - E(Y))^3 P_{X,Y}(X, Y) dXdY = & \quad (7.36) \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (Y - E(Y))^3 (p_1 \delta(x - x_1, y - y_1) + p_2 \delta(x - x_2, y - y_2) + p_3 \delta(x - \\ x_3, y - y_3) + p_4 \delta(x - x_4, y - y_4)) dXdY = & (y_1 - E(Y))^3 p_1 + (y_2 - E(Y))^3 p_2 + \\ (y_3 - E(Y))^3 p_3 + (y_4 - E(Y))^3 p_4 & \end{aligned}$$

This expression should match that of  $\mu_{0,3}$ . Thus, the eighth equation that needs to be met is:

$$(y_1 - E(Y))^3 p_1 + (y_2 - E(Y))^3 p_2 + (y_3 - E(Y))^3 p_3 + (y_4 - E(Y))^3 p_4 = v_y \sigma_y^3 \quad (7.37)$$

- $\mu_{2,1}$

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^2 (Y - E(Y))^1 P_{X,Y}(X, Y) dXdY = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^2 (Y - \\ E(Y))^1 (p_1 \delta(x - x_1, y - y_1) + p_2 \delta(x - x_2, y - y_2) + p_3 \delta(x - x_3, y - y_3) + \\ p_4 \delta(x - x_4, y - y_4)) dXdY = & (x_1 - E(X))^2 (y_1 - E(Y)) p_1 + (x_2 - E(X))^2 (y_2 - \\ E(Y)) p_2 + (x_3 - E(X))^2 (y_3 - E(Y)) p_3 + (x_4 - E(X))^2 (y_4 - E(Y)) p_4 & \quad (7.38) \end{aligned}$$

This expression should match that of  $\mu_{2,1}$ . Thus, the ninth equation that needs to be met is:

$$(x_1 - E(X))^2(y_1 - E(Y))p_1 + (x_2 - E(X))^2(y_2 - E(Y))p_2 + (x_3 - E(X))^2(y_3 - E(Y))p_3 + (x_4 - E(X))^2(y_4 - E(Y))p_4 = \mu_{2,1} \quad (7.39)$$

○  $\mu_{1,2}$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^1 (Y - E(Y))^2 P_{X,Y}(X,Y) dXdY = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (X - E(X))^1 (Y - E(Y))^2 (p_1 \delta(x - x_1, y - y_1) + p_2 \delta(x - x_2, y - y_2) + p_3 \delta(x - x_3, y - y_3) + p_4 \delta(x - x_4, y - y_4)) dXdY = (x_1 - E(X))(y_1 - E(Y))^2 p_1 + (x_2 - E(X))(y_2 - E(Y))^2 p_2 + (x_3 - E(X))(y_3 - E(Y))^2 p_3 + (x_4 - E(X))(y_4 - E(Y))^2 p_4 \quad (7.40)$$

This expression should match that of  $\mu_{1,2}$ . Thus, the tenth equation that needs to be met is:

$$(x_1 - E(X))(y_1 - E(Y))^2 p_1 + (x_2 - E(X))(y_2 - E(Y))^2 p_2 + (x_3 - E(X))(y_3 - E(Y))^2 p_3 + (x_4 - E(X))(y_4 - E(Y))^2 p_4 = \mu_{1,2} \quad (7.41)$$

Hence, the set of equations that needs to be solved simultaneously is as follows:

$$p_1 + p_2 + p_3 + p_4 = 1 \quad (7.42)$$

$$(x_1 - E(X))p_1 + (x_2 - E(X))p_2 + (x_3 - E(X))p_3 + (x_4 - E(X))p_4 = 0 \quad (7.43)$$

$$(y_1 - E(Y))p_1 + (y_2 - E(Y))p_2 + (y_3 - E(Y))p_3 + (y_4 - E(Y))p_4 = 0 \quad (7.44)$$

$$(x_1 - E(X))^2 p_1 + (x_2 - E(X))^2 p_2 + (x_3 - E(X))^2 p_3 + (x_4 - E(X))^2 p_4 = \sigma_x^2 \quad (7.45)$$

$$(y_1 - E(Y))^2 p_1 + (y_2 - E(Y))^2 p_2 + (y_3 - E(Y))^2 p_3 + (y_4 - E(Y))^2 p_4 = \sigma_y^2 \quad (7.46)$$

$$(x_1 - E(X))(y_1 - E(Y))p_1 + (x_2 - E(X))(y_2 - E(Y))p_2 + (x_3 - E(X))(y_3 - E(Y))p_3 + (x_4 - E(X))(y_4 - E(Y))p_4 = cov(X, Y) \quad (7.47)$$

$$(x_1 - E(X))^3 p_1 + (x_2 - E(X))^3 p_2 + (x_3 - E(X))^3 p_3 + (x_4 - E(X))^3 p_4 = \nu_x \sigma_x^3 \quad (7.48)$$

$$(y_1 - E(Y))^3 p_1 + (y_2 - E(Y))^3 p_2 + (y_3 - E(Y))^3 p_3 + (y_4 - E(Y))^3 p_4 = \nu_y \sigma_y^3 \quad (7.49)$$

$$(x_1 - E(X))^2 (y_1 - E(Y))p_1 + (x_2 - E(X))^2 (y_2 - E(Y))p_2 + (x_3 - E(X))^2 (y_3 - E(Y))p_3 + (x_4 - E(X))^2 (y_4 - E(Y))p_4 = \mu_{2,1} \quad (7.50)$$

$$(x_1 - E(X))(y_1 - E(Y))^2 p_1 + (x_2 - E(X))(y_2 - E(Y))^2 p_2 + (x_3 - E(X))(y_3 - E(Y))^2 p_3 + (x_4 - E(X))(y_4 - E(Y))^2 p_4 = \mu_{1,2} \quad (7.51)$$

$$E(Y)^2 p_3 + (x_4 - E(X))(y_4 - E(Y))^2 p_4 = \mu_{1,2}$$

This set of equations is formed of ten equations with twelve unknowns, i.e. parameters. These parameters are the following:  $x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, p_1, p_2, p_3$  and  $p_4$ . As can be seen, this set of equations is a set of non-linear equations which solution is not easy to obtain.

Rosenblueth proposes to use a large number of concentrations and fix some of the concentration locations and solve for the other locations and the concentrations magnitudes. This technique reduces the complexity of obtaining the solution. Rosenblueth proposed the usage of  $2^n$  probability concentrations for  $n$  input random variables.

The new proposed PEM suggests the usage of a number of concentrations equal to the number of equations that needs to be solved simultaneously given in equation (7.1) i.e.  $m = \frac{(k+n)!}{k!n!}$ . In this process, the locations of the concentrations are assigned so that one needs to solve only for the concentrations magnitudes. Thus, the obtained set of equations is a set of linear equations to be solved which solution is easy to find.

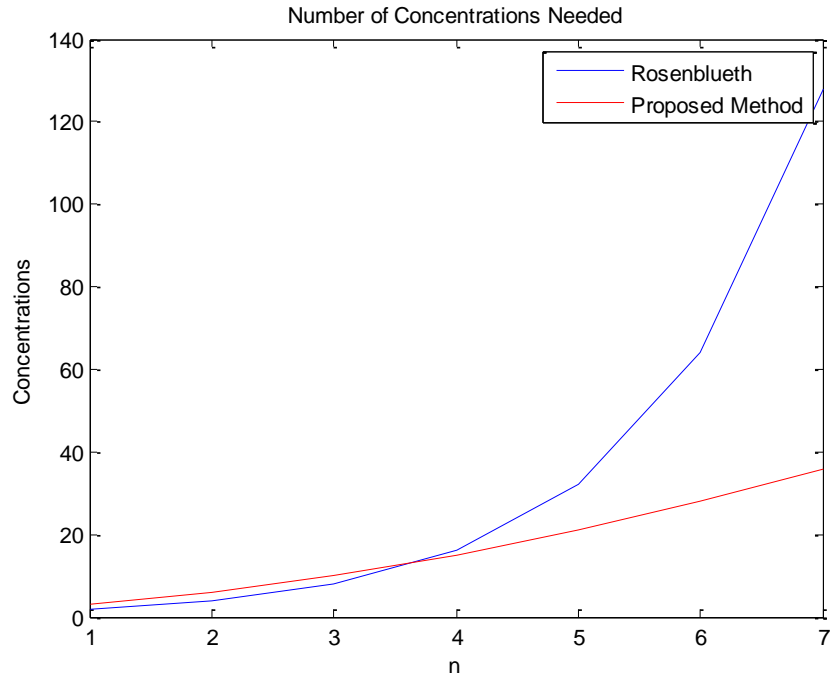
Table 7.1 gives a comparison between the number of concentrations needed using Rosenblueth's PEM and the proposed PEM.

Table 7.1. Number of Concentrations Needed for Rosenblueth's Method & Proposed Method

POINT ESTIMATE METHOD	ROSENBLUETH	PROPOSED
NUMBER OF CONCENTRATIONS	$2^n$	$\frac{(k+n)!}{k!n!}$

With  $2^n$  concentrations, using Rosenblueth's method, the generated probability concentrations meet the zeroth and first central moments as well as the second order marginal central moments of the input random variables [16].

Figure 7.1 shows a comparison between the numbers of concentrations needed by the proposed PEM compared to Rosenblueth's PEM. As can be seen in Figure 7.1, for a number of input random variables greater than four, the number of concentrations needed by the proposed method is less than the number of concentrations needed by Rosenblueth 2PEM. For a higher number of random variables, the difference between the numbers of concentrations needed for the two methods increases significantly.



**Figure 7.1.** Number of Concentrations Needed by the Proposed Method Compared to Rosenblueth’s 2PEM

Table 7.2 shows a comparison between the numbers of concentrations needed by the proposed method and by Rosenblueth 2PEM for a specific number of random variables.

**Table 7.2.** Number of Concentrations Needed for Rosenblueth’s 2PEM and the Proposed Method

NUMBER OF CONCENTRATIONS NEEDED		
INPUT RANDOM VARIABLES (n)	PROPOSED METHOD $\frac{(k+n)!}{k!n!}$ (k = 2)	ROSENBLUETH 2PEM $2^n$
2	6	4
3	10	8
4	15	16
5	21	32
6	28	64
7	36	128
8	45	256

The number of concentrations needed is important since it indicates the number of times the problem needs to be solved, i.e. the number of times the output function needs to be evaluated. For instance, if the number of concentrations is ten, which means that there is ten points, or ten sets of realizations, this implies that the output function should be evaluated, or the problem should be solved, ten times. If the number of concentrations is significantly large, the

computation time of the point estimate method might reach or surpass that of Monte Carlo simulation.

Let us take the case of two input random variables where we plan to meet up to the second moment of their joint probability distribution. Hence, we need to meet the first six equations of the previously derived ten equations, namely, equations (7.42) – (7.47). Indeed, if we are planning to meet up to the second order,  $k = 2$ , then the number of equations that needs to be solved simultaneously is equal to, following equation (7.1):

$$\sum_{i=0}^2 N_i = \frac{(2+n)!}{2!n!} \quad (7.52)$$

In our case we have  $n = 2$ ; then, the number of equations that need to be solved simultaneously is:

$$\sum_{i=0}^2 N_i = \frac{(2+2)!}{2!2!} = 6. \quad (7.53)$$

If six probability concentrations are used, where all the locations are fixed and the probability concentrations need to be determined, we get a set of linear equations to be solved simultaneously where the concentrations weights,  $p_i$ , are the only unknowns. The probability distribution for six probability concentrations is as follows:

$$P_{X,Y}(X,Y) = p_1\delta(x - x_1, y - y_1) + p_2\delta(x - x_2, y - y_2) + p_3\delta(x - x_3, y - y_3) + \quad (7.54)$$

$$p_4\delta(x - x_4, y - y_4) + p_5\delta(x - x_5, y - y_5) + p_6\delta(x - x_6, y - y_6).$$

The set of equations that need to be solved to match up to the second central moment of the input random variables can be derived in a similar manner to equations (7.22) – (7.33) using the distribution function given in equation (7.54). The set of simultaneous equations is as follows:

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1 \quad (7.55)$$

$$(x_1 - E(X))p_1 + (x_2 - E(X))p_2 + (x_3 - E(X))p_3 + (x_4 - E(X))p_4 + \quad (7.56)$$

$$(x_5 - E(X))p_5 + (x_6 - E(X))p_6 = 0$$

$$(y_1 - E(Y))p_1 + (y_2 - E(Y))p_2 + (y_3 - E(Y))p_3 + (y_4 - E(Y))p_4 + \quad (7.57)$$

$$(y_5 - E(Y))p_5 + (y_6 - E(Y))p_6 = 0$$

$$(x_1 - E(X))^2 p_1 + (x_2 - E(X))^2 p_2 + (x_3 - E(X))^2 p_3 + (x_4 - E(X))^2 p_4 + \quad (7.58)$$

$$(x_5 - E(X))^2 p_5 + (x_6 - E(X))^2 p_6 = \sigma_X^2$$

$$(y_1 - E(Y))^2 p_1 + (y_2 - E(Y))^2 p_2 + (y_3 - E(Y))^2 p_3 + (y_4 - E(Y))^2 p_4 + \quad (7.59)$$

$$(y_5 - E(Y))^2 p_5 + (y_6 - E(Y))^2 p_6 = \sigma_Y^2$$

$$(x_1 - E(X))(y_1 - E(Y))p_1 + (x_2 - E(X))(y_2 - E(Y))p_2 + (x_3 - E(X))(y_3 - E(Y))p_3 + (x_4 - E(X))(y_4 - E(Y))p_4 = cov(X, Y) \quad (7.60)$$

By fixing all the locations of the concentrations, the unknowns that we need to solve for are then  $p_1, p_2, p_3, p_4, p_5$  and  $p_6$ .

The set of equations can be expressed in matrix form as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ (x_1 - E(X)) & (x_2 - E(X)) & (x_3 - E(X)) & (x_4 - E(X)) & (x_5 - E(X)) & (x_6 - E(X)) \\ (y_1 - E(Y)) & (y_2 - E(Y)) & (y_3 - E(Y)) & (y_4 - E(Y)) & (y_5 - E(Y)) & (y_6 - E(Y)) \\ (x_1 - E(X))^2 & (x_2 - E(X))^2 & (x_3 - E(X))^2 & (x_4 - E(X))^2 & (x_5 - E(X))^2 & (x_6 - E(X))^2 \\ (y_1 - E(Y))^2 & (y_2 - E(Y))^2 & (y_3 - E(Y))^2 & (y_4 - E(Y))^2 & (y_5 - E(Y))^2 & (y_6 - E(Y))^2 \\ (x_1 - E(X))(y_1 - E(Y)) & (x_2 - E(X))(y_2 - E(Y)) & (x_3 - E(X))(y_3 - E(Y)) & (x_4 - E(X))(y_4 - E(Y)) & (x_5 - E(X))(y_5 - E(Y)) & (x_6 - E(X))(y_6 - E(Y)) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \sigma_X^2 \\ \sigma_Y^2 \\ cov(X, Y) \end{bmatrix}$$

Then, the concentrations magnitudes (i.e. weights) can be obtained as follows:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix} = T^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \sigma_X^2 \\ \sigma_Y^2 \\ cov(X, Y) \end{bmatrix} \quad (7.61)$$

The statistical moments of the output variables are then calculated as:

$$E[Z^q] = \sum_{i=1}^m p_i Z_i^q \quad (7.62)$$

where

$$Z_i = H(x_i, y_i) \quad (7.63)$$

The challenging task, is to create a pattern for the point placement that always results in a well-conditioned matrix  $T$ .

### **7.3. Conclusions**

In this chapter, a new modified PEM is introduced. This PEM is a modified version of the PEM introduced by Rosenblueth where it can incorporate correlated as well as asymmetric input random variables using  $\frac{(k+n)!}{k!n!}$  probability concentrations as compared to  $2^n$  for the case of Rosenblueth's PEM (with  $k$  being the order up to which the central moments of the input variables are matched and  $n$  is the number of input random variables). The next chapter presents the summary of the thesis including a set of conclusions as well as suggested future research work.



## ***Chapter VIII: Conclusions and Future Work***

### ***8.1. Conclusions***

The proliferation of intermittent renewable-energy based power generation introduces new challenges to the operation of power systems. These challenges are due to the uncertainties introduced to the system variables, such as electricity prices, congestion levels etc. Statistical analyses of the uncertain system variables are necessary to maintain the operational optimality, reliability and the security of the grid. Such statistical analyses are based on the ability to estimate the statistical moments of these system variables. The objective of this thesis is to estimate the statistical moments of the system locational marginal prices (LMPs) and total generation cost under system uncertainty emanating from random generation, such as wind-based electrical energy and random loads. In this thesis, the Point Estimate Methods are applied to the optimal power flow problem to estimate the statistical moments of the locational marginal prices and total generation cost under system uncertainty.

We have reviewed three well-known PEMs introduced by Rosenblueth, Hong, and Harr. We investigated the mathematical properties of these PEMs and derived mathematical proofs that illustrate the relevance of their applications. Moreover, we have introduced two new point estimate method schemes. The first one is a scheme combining both those presented by Hong and Harr used to treat correlated and non-correlated input random variables in a computationally efficient manner. The second proposed scheme is an original scheme derived from the PEM proposed by Rosenblueth. This proposed scheme can be applied to problems with various types of random variables as in the case of Rosenblueth's PEM but in a more computationally efficient manner.

We investigated the application of various PEMs to the probabilistic optimal power flow problem in the presence of large scale wind integration. The PEMs are used to estimate the statistical properties, such as mean and standard deviation, of the system LMPs and the total generation cost under uncertainty introduced by wind energy and demand. The performances of the applied methods are illustrated by comparing their generated results to those of Monte Carlo simulations. The various PEMs are also compared and the comparison indicates that the proposed scheme combining Hong's and Harr's schemes shows better performance than the

commonly used Hong  $2n + 1$  scheme. The risks of the application of Hong's  $2n$  scheme are also explored demonstrating that this scheme might lead inaccurate results or might even not converge because of non-feasibility of the OPF problem for a large number of input random variables. Lastly, an extended analysis is presented showing how the statistical estimation obtained through the PEMs can be used to perform a statistical analysis of the power system.

## **8.2. Future Work**

We believe that the challenges listed below have potential for future work:

1. The applied PEMs assume that the statistical moments of input variables are available and assumed to be known and do not include any restrictions on the potential availability of these estimated moments. A proposed future work is to study forecasting techniques used in literature to estimate the statistical properties of the input variables such as:
  - a. Probability distribution of the wind speed
  - b. Probability distribution of random loads
  - c. Correlation of random loads
2. Only one time snapshot of a power network has been studied and the time factor has been disregarded. A proposed future work is to apply the developed work for multiple correlated time instants where time series analysis can be used to estimate/forecast the input random variables and the OPF problem can be solved for different time periods where the solution of the OPF, for instance, at time  $t$  depends on the OPF solution at time  $t - 1$ .
3. The presented research work aims at estimating the statistical moments of the OPF output variables. For instance, we have estimated the LMPs' means and standard deviations subject to uncertainties introduced by large scale wind integration and random loads. A proposed future work is to study the relevance of the estimated statistical moments. In other words, how much information do we obtain about the statistical behavior of the LMPs through estimating their means and standard deviations? Are the means and standard deviations sufficient to describe the probability distribution of the LMPs or is estimating the higher order statistical moments necessary?

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