A STUDY OF THE RANGE OF VALIDITY FOR THE METHOD OF KRYLOFF AND BOGOLIUBOFF AS APPLIED TO A SATELLITE IN MOTION WITH A SPECIFIED CONSTANT THRUST

by

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LIST OF SYMBOLS

E	total specific energy, ft lbs/unit mass
h	specific angular momentum, lb ft sec/unit mass
p	semi-latus rectum, or conic parameter, ft
r	radial coordinate (distance from earth's center to satellite), ft
g	acceleration due to earth's gravity, ft/sec^2
μ	a gravitational constant (= GM), ft^3/sec^2
G	universal constant of gravitation, ft^4/lb sec^4
M	mass of the earth, slugs
ρ	dimensionless radii, $\frac{r(\text{mi.})}{3960\text{ mi.}}$
t	time coordinate, sec
u	transformation coordinate, ft^{-1}
V	velocity of satellite, ft/sec
$\overline{W}$	specific thrust acting on satellite, lb/unit mass
ϵ	orbital eccentricity
θ	angular coordinate of satellite (radians, degrees)
$K(k_1)$	complete elliptic integral of the first kind, modulus k_1
$E(k_1)$	complete elliptic integral of the second kind, modulus k_1

Subscripts

$(\ )_o$	refers to initial conditions

I. INTRODUCTION

For the past several years much of the world's attention has been focused on the so-called "Space Race" including the launching of man-made satellites, manned orbital flights and the much discussed project of putting a man on the moon.

This interest has given renewed emphasis to the mathematical theory of orbit mechanics. In this theory, one of the oldest problems that has plagued the investigator has been that associated with terms which are not cyclic in form and which usually infer a divergence in the solution, the often mentioned secular terms.

The motion of a satellite, under the influence of a central force field, has usually been solved by one of two basic methods. One method is that of "special perturbations" which requires a step-by-step numerical integration scheme of the differential equations of motion for its solution. The second procedure is that of "general perturbations"; this method considers the integration of differential equations by the use of series expansions.

Solutions to the orbital problem by these methods were either too laborious or too inaccurate, in the majority of cases, and consequently newer methods, such as that of Kryloff and Bogoliuboff[1] have been introduced to solve certain types of non-linear problems.

In the method of Kryloff and Bogoliuboff, the slowly varying derivatives are replaced by their "average" value, as determined for a period of the motion. In this averaging process the slowly varying dependent variables are assumed to be constants. Such a scheme gives the so-called "first approximation" to the solution.

A recent application of the method of Kryloff and Bogoliuboff, in solving for the motion of a satellite under a central force law, was suggested by Lass and Solloway[2]. The non-linear techniques of this method were employed to develop closed form analytical solutions for the case of a constant thrust applied normal to the orbital plane. Shapiro[3] has also used this method to describe a solution for the case of a constant tangential thrust.

The purpose of this thesis is to study the method of Kryloff and Bogoliuboff, as applied to the motion of a satellite, for various values of the specific thrust. In this investigation the thrusting cases considered are those of constant tangential and normal thrust. Too, in the present study it is intended that a limitation on the specific thrusting force can be defined which will describe a limit, or range of validity, for the method of Kryloff and Bogoliuboff. All of the work presented herein has been computed with the aid of the IBM 1620 Digital Computer.

II. MOTION OF A SATELLITE UNDER THE INFLUENCE OF A THRUST NORMAL TO THE PLANE OF MOTION

A satellite of mass m is in motion about a spherically symmetric attracting mass. It has a continuous thrusting action directed normal to the instantaneous plane of its motion. This thrusting action is assumed to be such that the loss in vehicle mass is negligible.

An inertial coordinate system (X,Y,Z), having an origin at the attracting center of the central force field, is chosen as a frame of reference. Consider now an origin, coincident with the inertial origin, for a moving axis system (x,y,z), oriented so that the satellite will always be located on the moving x axis; the plane of motion will be the xy plane, the z axis is coincident with the instantaneous angular momentum vector for the satellite.

Let the moving axis system have the general angular velocity $\bar{\omega} = \omega_x \bar{e}_x + \omega_y \bar{e}_y + \omega_z \bar{e}_z$ relative to the inertial axis system. Having chosen the instantaneous plane of motion for the satellite to be the xy plane then it follows that $\omega_y = 0$ and $\bar{\omega} = \omega_x \bar{e}_x + \omega_y \bar{e}_y$.

Denoting the specific thrust of the satellite as $\bar{W}$, then the thrust vector will be written as $\bar{W}\, \bar{e}_z$.

Since the satellite is located by the position vector $\bar{r} = r\bar{e}_x$, and is acted upon by the force of attraction $-\frac{\mu}{r^2}\, \bar{e}_x$, and a specific thrust $\bar{W}$, the equation of motion is:

$$\frac{d^2r}{dt^2} = \frac{-\mu}{r^2}\,\overline{e}_x + W\,\overline{e}_z \quad . \tag{1}$$

The velocity of the vehicle at any instant can be written as

$$\overline{V} = \overline{e}_x \frac{dr}{dt} + r\,\omega_z\,\overline{e}_y \tag{2}$$

and the acceleration is

$$\overline{a} = \left(\frac{d^2r}{dt^2} - r\,\omega_z^{\,2} \right) \overline{e}_x + \frac{1}{r}\frac{d}{dt}\left(r^2\,\omega_z\right) \overline{e}_y + r\,\omega_x\,\omega_z\,\overline{e}_z \tag{3}$$

Replacing the acceleration in Eq. (1) with the kinematic relation of Eq. (3), equating coefficients in the coordinate directions leads directly to the set of expressions:

$$\frac{d^2r}{dt^2} - r\,\omega_z^{\,2} = -\frac{\mu}{r^2} \tag{4a}$$

$$r^2\,\omega_z = \text{constant} = h \tag{4b}$$

$$r\,\omega_x\,\omega_z = \overline{W} \tag{4c}$$

The second equation of this set is recognized as the magnitude of the specific angular momentum vector, $|\overline{h}|$.

Taking account of the relations given in Eqs. (4), utilizing the definition of the time derivative of a unit vector, i.e. $\frac{d\overline{e}}{dt} = \overline{\omega} \times \overline{e}$ then the following set of equations is obtained:

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = - \frac{\mu}{r^2} \tag{5a}$$

$$\frac{d\bar{e}_x}{dt} = \omega_z \bar{e}_y = \frac{h}{r^2} \bar{e}_y \tag{5b}$$

$$\frac{d\bar{e}_y}{dt} = \omega_x \bar{e}_z - \omega_z \bar{e}_x = \frac{r\bar{W}}{h} \bar{e}_z - \frac{h}{r^2} \bar{e}_x \tag{5c}$$

$$\frac{d\bar{e}_z}{dt} = - \omega_x \bar{e}_y = - \frac{r\bar{W}}{h} \bar{e}_y \tag{5d}$$

This set of equations relates the unknowns of the problem ($\bar{e}_x$, $\bar{e}_y$, $\bar{e}_z$, and r). The solution to these equations will completely define the motion of the satellite. The constants of the problem are the central field strength (μ), the specific angular momentum of the satellite ($\bar{h}$), and the specific thrust ($\bar{W}$).

In order to solve the governing equations (5), one can follow the usual procedure of introducing a change of independent variable. By writing $h = r^2 \frac{d\theta}{dt}$ (6a)

and introducing a transform for the dependent variable, $u = 1/r$, then the time derivative operator can be written as

$$\frac{d}{dt} = hu^2 \frac{d}{d\theta} \tag{6b}$$

Using Eqs. (6), Eqs. (5) are rewritten in terms of the u, θ variables as:

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \tag{7a}$$

$$\frac{d\bar{e}_x}{d\theta} = \bar{e}_y \tag{7b}$$

$$\frac{d\bar{e}_y}{d\theta} = -\bar{e}_x + \frac{\bar{W}}{h^2 u^3}\bar{e}_z \tag{7c}$$

$$\frac{d\bar{e}_z}{d\theta} = -\frac{\bar{W}}{h^2 u^3}\bar{e}_y \tag{7d}$$

These expressions represent the governing equations in terms of the transformed variables u and θ. The solution to Eq. (7a) is the general expression for the path of motion

$$u = \frac{\mu}{h^2}\left(1 + \frac{Q\,h^2}{\mu}\cos\,(\theta + \emptyset)\right) \tag{8}$$

where Q is the "amplitude" and $\emptyset$ the "phase angle" for the solution to the homogeneous second order differential equation in $q = u - \frac{\mu}{h^2}$ (a transformation of the first equation of Eqs. (7)). The general form of Eq. (8), in the variables r and θ, is

$$r = \frac{p}{1 + \epsilon\cos\theta}$$

where p is the conic parameter and ϵ is the eccentricity of the conic section. Here $\emptyset$ has been chosen to be zero; this is done since θ

can be determined, to an additive constant, hence $\phi = 0$ can be assigned without loss in generality.

With $\bar{W} = 0$ and letting $\bar{e}_{x_o}$, $\bar{e}_{y_o}$, $\bar{e}_{z_o}$ be the initial known values for the moving unit vector system then it follows that (kinematically)

$$\bar{e}_x = \bar{e}_{x_o} \cos\theta + \bar{e}_{y_o} \sin\theta \tag{10a}$$

$$\bar{e}_y = - \bar{e}_{x_o} \sin\theta + \bar{e}_{y_o} \cos\theta \tag{10b}$$

and

$$\bar{e}_z = \bar{e}_{z_o} \tag{10c}$$

Here it is seen that the motion is planar, and that this plane is oriented so that its normal, $\bar{e}_{z_o}$, is fixed in inertial space. Hence the plane of motion is an invariant in space; the satellite follows a space fixed conical path, (see Eq. (9)).

<u>The Case of a Circular Orbit with Normal Thrusting.</u>

As a special case of the general normal thrust problem, consider a satellite in a circular orbit of radius r_o.

Let the initial and boundary conditions be given by the following:

$$\text{at} \quad t = 0 \quad \bar{e}_x = \bar{e}_{x_o} ; \ \bar{e}_y = \bar{e}_{y_o} ; \ \bar{e}_z = \bar{e}_{z_o}$$

$$\left(\frac{d\bar{e}_x}{dt}\right)_o = \omega_z \bar{e}_{y_o} ; \ \left(\frac{d\bar{e}_y}{dt}\right)_o = \frac{r_o \bar{W} \bar{e}_{z_o}}{h} - \frac{h}{r_o^2} \bar{e}_{x_o} ;$$

$$\left(\frac{d\bar{e}_z}{dt}\right)_o = - \omega_x \bar{e}_{y_o} ; \ r = r_o ; \ \frac{dr}{dt} = \frac{d^2 r}{dt^2} = 0 ; \ h = r_o^2 \frac{d\theta}{dt}$$

Solving Eqs. (5) with the above conditions the following results are easily obtained:

$$r_o = \frac{h^2}{\mu} \tag{11a}$$

$$\bar{e}_y = \bar{A} \sin \omega t + \bar{B} \cos \omega t \tag{11b}$$

where $\bar{A}$ and $\bar{B}$ are constant vectors of integration.

$$\bar{e}_x = \frac{\omega_z}{\omega} \left[- \bar{A} \cos \omega t + \bar{B} \sin \omega t \right] + \bar{C}_x \tag{11c}$$

$$\bar{e}_z = \frac{\omega_x}{\omega} \left[\bar{A} \cos \omega t - \bar{B} \sin \omega t \right] + \bar{C}_z \tag{11d}$$

Now the solution of Eqs. (7) gives the following results

$$\bar{e}_y = \bar{C} \sin \lambda \theta + \bar{D} \cos \lambda \theta \tag{12a}$$

$$\bar{e}_x = \frac{1}{\lambda} (- \bar{C} \cos \lambda \theta + \bar{D} \sin \lambda \theta) + \bar{K}_x \tag{12b}$$

$$\bar{e}_z = \frac{- \bar{W}}{h^2 u^3} \frac{1}{\lambda} (- \bar{C} \cos \lambda \theta + \bar{D} \sin \lambda \theta) + \bar{K}_z \tag{12c}$$

where $\lambda = \left[1 + \left(\frac{\bar{W}}{h^2 u^3} \right)^2 \right]^{1/2}$ and $\bar{C}$, $\bar{D}$, $\bar{K}_x$ and $\bar{K}_z$ are constant vectors of integration.

For this analysis the initial conditions are: at $\theta = 0$,

$\bar{e}_x = \bar{e}_{x_o}$; $\bar{e}_y = \bar{e}_{y_o}$; $\bar{e}_z = \bar{e}_{z_o}$ $r = r_o$, $\frac{dr}{d\theta} = \frac{d^2 r}{d\theta^2} = 0$, $\left(\frac{d\bar{e}_x}{d\theta} \right)_o = \bar{e}_{y_o}$;

$$\left(\frac{d\bar{e}_y}{d\theta} \right)_o = - \bar{e}_{x_o} + \frac{\bar{W}}{h^2 u^3} \bar{e}_{z_o} ; \left(\frac{d\bar{e}_z}{d\theta} \right)_o = - \frac{\bar{W}}{h^2 u^3} \bar{e}_{y_o}$$

The evaluation of the constants leads to the following equations:

$$\bar{e}_y = - \frac{1}{\lambda} \sin \lambda\, \theta\, \bar{e}_{x_o} + \cos \lambda\, \theta\, \bar{e}_{y_o} + \frac{1}{\lambda} \frac{\bar{W}}{h^2 u^3} \sin \lambda\, \theta\, \bar{e}_{z_o} \tag{13a}$$

$$\bar{e}_x = \left(\frac{1}{\lambda^2} \cos \lambda\, \theta + 1 - \frac{1}{\lambda^2} \right) \bar{e}_{x_o} + \frac{1}{\lambda} \sin \lambda\, \theta\, \bar{e}_{y_o} + \frac{\bar{W}}{h^2 u^3} \frac{1}{\lambda^2} (1 - \cos \lambda\, \theta)\, \bar{e}_{z_o} \tag{13b}$$

and

$$\bar{e}_z = \frac{\bar{W}}{h^2 u^3} \frac{1}{\lambda^2} (1 - \cos \lambda\, \theta)\, \bar{e}_{x_o} - \frac{\bar{W}}{h^2 u^3} \frac{1}{\lambda} \sin \lambda\, \theta\, \bar{e}_{y_o} + \left(\frac{\bar{W}}{h^2 u^3} \right) \frac{1}{\lambda^2} (\cos \lambda\, \theta - 1)\, \bar{e}_{z_o} + \bar{e}_{z_o} \tag{13c}$$

Since $\bar{r} = r\bar{e}_x$, the solution of Eq. (13b) will determine the position of the satellite. The use of Eq. (6b) will describe the position of the satellite at any time t.

The Elliptic Orbit with Normal Thrusting.

If the initial motion of the satellite is elliptical, and if $L = \frac{\bar{W}\, h^4}{\mu^3} = \frac{\bar{W}}{\mu/a^2} (1 - \epsilon^2)^2$ is sufficiently small, then the Kryloff-Bogoliuboff method is applicable.

Differentiating Eqs. (7) and performing the necessary substitutions the equations become

$$\frac{d^2\bar{e}_x}{d\theta^2} + \bar{e}_x = \frac{L}{(1 + \epsilon \cos \theta)^3} \bar{e}_z \tag{14a}$$

$$\frac{d\bar{e}_z}{d\theta} = - \frac{L}{(1 + \epsilon \cos \theta)^3} \frac{d\bar{e}_x}{d\theta} \tag{14b}$$

Assume solutions of the form

$$\bar{e}_x = \bar{A} \cos \theta + \bar{B} \sin \theta + L \bar{D} + L \sum_{n=2}^{\infty} (\bar{a}_n \cos n\theta + \bar{b}_n \sin n\theta) \tag{15a}$$

and

$$\bar{e}_z = \bar{C} + L \sum_{n=1}^{\infty} (\bar{c}_n \cos n\theta + \bar{d}_n \sin n\theta) \tag{15b}$$

where it is presumed that

$$\frac{d\bar{A}}{d\theta} = L\, \bar{f}(\bar{A}, \bar{B}, \bar{C}, \bar{D})$$

$$\frac{d\bar{B}}{d\theta} = L\, \bar{\beta}(\bar{A}, \bar{B}, \bar{C}, \bar{D})$$

$$\frac{d\bar{C}}{d\theta} = L\, \bar{\gamma}(\bar{A}, \bar{B}, \bar{C}, \bar{D})$$

and

$$\frac{d\bar{D}}{d\theta} = L\, \bar{\ell}(\bar{A}, \bar{B}, \bar{C}, \bar{D}) \tag{16}$$

The vectors $\bar{a}_n$, $\bar{b}_n$, $\bar{c}_n$, $\bar{d}_n$, $n = 2, 3, 4, \ldots$ are assumed to vary slowly so that $\frac{d\bar{a}_n}{d\theta} = L\, f_n$, etc.

Now neglecting L^2 terms one can find from Eqs. (15), that

$$\frac{d\bar{e}_x}{d\theta} = -\bar{A}\sin\theta + \bar{B}\cos\theta + L\,\bar{f}\cos\theta + L\,\bar{\beta}\sin\theta - L\sum_{2}^{\infty} n(\bar{a}_n \sin n\theta - \bar{b}_n \cos n\theta) \tag{17a}$$

$$\frac{d^2\bar{e}_x}{d\theta^2} = -\bar{A}\cos\theta - \bar{B}\sin\theta - 2L\,\bar{f}\sin\theta + 2L\,\bar{\beta}\cos\theta - L\sum_{2}^{\infty} n^2(\bar{a}_n \cos n\theta + \bar{b}_n \sin n\theta) \tag{17b}$$

$$\frac{d\bar{e}_z}{d\theta} = L\,\bar{\gamma} - L\sum_{1}^{\infty} n(\bar{c}_n \sin n\theta - \bar{d}_n \cos n\theta) \tag{17c}$$

Substituting Eqs. (17) in Eq. (14) yields

$$L\,\bar{D} - 2L\,\bar{f}\sin\theta + 2L\,\bar{\beta}\cos\theta + L\sum_{2}^{\infty}(1 - n^2)(\bar{a}_n \cos n\theta + \bar{b}_n \sin n\theta) =$$

$$\frac{L\,\bar{C}}{(1 + \epsilon\cos\theta)^3} = L\bar{C}\left(\frac{\alpha_o}{2} + \sum_{n=1}^{\infty} \alpha_n \cos n\theta\right) \tag{18a}$$

$$L\,\overline{\gamma} - L\sum_{1}^{\infty} n(\overline{c}_n \sin n\theta - \overline{d}_n \cos n\theta) = \frac{L}{(1+\epsilon\cos\theta)^3}(\overline{A}\sin\theta - \overline{B}\cos\theta) \tag{18b}$$

with

$$\alpha_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos n\theta}{(1+\epsilon\cos\theta)^3}\,d\theta$$

and

$$\overline{\gamma} = -\frac{\overline{B}}{2\pi}\int_0^{2\pi} \frac{\cos\theta}{(1+\epsilon\cos\theta)^3}\,d\theta \tag{19}$$

The α_n, $n = 0, 1, 2, \ldots$ are the Fourier coefficients of $(1+\epsilon\cos\theta)^{-3}$, $\overline{\gamma}$ is the steady state component of $(\overline{A}\sin\theta - \overline{B}\cos\theta)/(1+\epsilon\cos\theta)^3$.

By equating coefficients in Eqs. (18) it is apparent that

$$\overline{D} = \frac{\alpha_o\,\overline{C}}{2} = \frac{2+\epsilon^2}{2}(1-\epsilon^2)^{-5/2}\;\overline{C}$$

$$\overline{f} = 0$$

$$\overline{\beta} = \frac{\alpha_1}{2}\overline{C} = -(3/2)\,\epsilon(1-\epsilon^2)^{-5/2}\;\overline{C}$$

$$\overline{b}_n = 0 \qquad\qquad n \geqq 2$$

$$\overline{a}_n = \frac{\overline{C}}{1-n^2}\,\alpha_n \qquad\qquad n \geqq 2$$

$$\overline{\gamma} = -(\alpha_1/2)\,\overline{B} = (3/2)\,\epsilon\,(1-\epsilon^2)^{-5/2}\;\overline{B} \tag{20}$$

so that

$$\frac{d\bar{A}}{d\theta} = 0$$

$$\frac{d\bar{B}}{d\theta} = \frac{\alpha_1}{2} L \bar{C}$$

$$\frac{d\bar{C}}{d\theta} = - \frac{\alpha_1}{2} L \bar{B} \tag{21}$$

whose solutions are

$$\bar{A} = \bar{A}_o$$

$$\bar{B} = \bar{B}_o \cos L \frac{\alpha_1}{2} \theta + \bar{C}_o \sin L \frac{\alpha_1}{2} \theta$$

$$\bar{C} = - \bar{B}_o \sin L \frac{\alpha_1}{2} \theta + \bar{C}_o \cos L \frac{\alpha_1}{2} \theta \tag{22}$$

The values of $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, $\bar{a}_n$ and $\bar{b}_n$, as given by Eqs. (20) and (22), determine the vector $\bar{e}_x$, and the motion is given by $\bar{r} = r\bar{e}_x$ with $r = \dfrac{p}{(1 + \epsilon \cos \theta)}$.

<u>Orbits of Small Eccentricity.</u>

For the case where $\epsilon << 1$, one can approximate u^{-3} by the expression $u^{-3} \approx \frac{h^6}{\mu^3} (1 - 3 \epsilon \cos \theta)$ which is obtained by the usual binomial expansion.

Equations (7) are now rewritten as

$$\frac{d\bar{e}_x}{d\theta} = \bar{e}_y$$

$$\frac{d\bar{e}_y}{d\theta} = - \bar{e}_x + L\,(1 - 3\,\epsilon \cos\,\theta)\,\bar{e}_z$$

$$\frac{d\bar{e}_z}{d\theta} = - L\,(1 - 3\,\epsilon \cos\,\theta)\,\bar{e}_y \tag{23}$$

where, again, neglecting ϵ^2 terms

$$L = \frac{\bar{W}\,h^4}{\mu^3} = \frac{\bar{W}\,\mu^2\,a^2}{\mu^3}\,(1 - \epsilon^2)^2 \approx \frac{\bar{W}}{\mu/a^2}$$

The solutions of Eqs. (23), for $\epsilon = 0$ are readily determined to be

$$\bar{e}_{x_1} = L\,\bar{A} + \bar{B}\,\sin\,\omega\,\theta + \bar{C}\,\cos\,\omega\,\theta$$

$$\bar{e}_{y_1} = \omega\,\bar{B}\,\cos\,\omega\,\theta - \omega\,\bar{C}\,\sin\,\omega\,\theta$$

$$\bar{e}_{z_1} = - L^2\,\bar{A} - L\,\bar{B}\,\sin\,\omega\,\theta - L\,\bar{C}\,\cos\,\omega\,\theta \tag{24}$$

where $\omega^2 = 1 + L^2$ and $\bar{A}$, $\bar{B}$, and $\bar{C}$ are arbitrary constant vectors. It is noted that the vectors $\bar{e}_{x_1}$, $\bar{e}_{y_1}$, $\bar{e}_{z_1}$ are essentially the solution for the circular case, analogous to the solution obtained previously.

Next, assume a solution to Eq. (23) of the form

$$\bar{e}_x = \bar{e}_{x_1} + \epsilon\, \bar{e}_{x_2}$$

$$\bar{e}_y = \bar{e}_{y_1} + \epsilon\, \bar{e}_{y_2}$$

$$\bar{e}_z = \bar{e}_{z_1} + \epsilon\, \bar{e}_{z_2} \qquad (25)$$

where ϵ is the (assumed) small eccentricity and the unit vectors $\bar{e}_{x_2}$, etc. are second approximations to $\bar{e}_x$, etc. with $\bar{e}_{x_1}$, $\bar{e}_{y_1}$, and $\bar{e}_{z_1}$ representing the first approximation (based on $\epsilon = 0$) to $\bar{e}_x$, $\bar{e}_y$ and $\bar{e}_z$. Here $\bar{e}_{x_2}$, $\bar{e}_{y_2}$, and $\bar{e}_{z_2}$ will estimate the influence of the departure from a circular orbit.

Substituting from Eqs. (24) and (25), and neglecting terms proportional to ϵ^2, then

$$\frac{d\bar{e}_{x_2}}{d\theta} = \bar{e}_{y_2}$$

$$\frac{d\bar{e}_{y_2}}{d\theta} = - \bar{e}_x + L\, \bar{e}_{z_2} - 3L \cos\theta\, (-L^2 \bar{A} - L \bar{B} \sin \omega\theta - L \bar{C} \cos \omega\theta)$$

$$\frac{d\bar{e}_{z_2}}{d\theta} = - L\, \bar{e}_{y_2} + 3\omega L \cos\theta\, (\bar{B} \cos\omega\theta - \bar{C} \sin\omega\theta)$$

$$(26)$$

The integration of Eqs. (26) yields

$$\bar{e}_x = L\,\bar{A} + \bar{B}\sin\omega\theta + \bar{C}\cos\omega\theta + 3/2\,L\,\epsilon\left\{B\left[\frac{\sin(\omega-1)\theta}{\omega-1} - \frac{\sin(\omega+1)\,\theta}{\omega+1}\right] +\right.$$

$$\bar{C}\left[\frac{\cos(\omega-1)\theta}{\omega-1} - \frac{\cos(\omega+1)\theta}{\omega+1}\right] - \frac{2\,\bar{A}}{L\,\omega^2}(\cos\omega\theta - \omega^2\cos\theta) +$$

$$\left.\frac{2\bar{C}}{\omega^2}\cos\omega\theta - \frac{2\bar{C}}{\omega^2}\;\frac{(2\,\omega^2 - 1)}{(\omega^2 - 1)} - \frac{2\,L\,\bar{A}}{\omega}\right\} \qquad (27a)$$

$$\bar{e}_y = \omega\,\bar{B}\cos\omega\theta - \omega\,\bar{C}\sin\omega\theta + \frac{3\,L\,\bar{A}}{\omega}\epsilon\,(\sin\omega\theta - \omega\sin\theta) +$$

$$3/2\,L^2\epsilon\left\{\bar{B}\;\left[\cos(\omega-1)\,\theta - \cos(\omega+1)\,\theta\right] +\right.$$

$$\left.\bar{C}\;\left[\sin(\omega+1)\,\theta - \sin(\omega-1)\,\theta\right] - \frac{2\bar{C}}{\omega}\sin\omega\theta\right\} \qquad (27b)$$

and

$$\bar{e}_z = - L^2 \bar{A} - L\bar{B} \sin \omega\theta - L\bar{C} \cos \omega\theta - \frac{3 L\bar{A} \epsilon}{\omega^2} (1 + \cos \omega\theta) +$$

$$\frac{3 L^2 \bar{C} \epsilon}{\omega} (1 + \cos \omega\theta) + 3 L^2 \bar{A} \epsilon (1 + \cos \theta) +$$

$$\frac{3}{2} \bar{C} \epsilon \frac{(3 L^2 + 1)}{L} + \frac{3}{2} L^2 \epsilon \left\{ \bar{B} \left[\frac{\sin (\omega-1) \theta}{\omega - 1} - \frac{\sin (\omega+1) \theta}{\omega + 1} \right] + \right.$$

$$\bar{C} \left[\frac{\cos (\omega-1) \theta}{\omega - 1} - \frac{\cos (\omega+1) \theta}{\omega + 1} \right] + \frac{\bar{B}}{L^2} \left[\frac{\sin (\omega+1) \theta}{\omega + 1} + \frac{\sin (\omega-1) \theta}{\omega - 1} \right] +$$

$$\left. \frac{\bar{C}}{L^2} \left[\frac{\cos (\omega+1) \theta}{\omega + 1} - \frac{\cos (\omega-1) \theta}{\omega - 1} \right] \right\} \tag{27c}$$

By definition

$$\bar{e}_{x_2}(0) = \bar{e}_{y_2}(0) = \bar{e}_{z_2}(0) = 0$$

hence to evaluate constants $\bar{A}$, $\bar{B}$, and $\bar{C}$ recall that at $\theta = 0$ $\bar{e}_x = \bar{e}_{x_o}$, $\bar{e}_y = \bar{e}_{y_o}$, $\bar{e}_z = \bar{e}_{z_o}$, thus

$$\bar{e}_{x_o} = L\bar{A} + \bar{C}; \quad \bar{e}_{y_o} = \omega \bar{B}$$

$$\bar{e}_{z_o} = - L^2\bar{A} - L\bar{C}$$

III. GOVERNING EQUATIONS FOR SMALL CONSTANT TANGENTIAL THRUST

The equations of motion here are developed analogous to those for the normal thrust case; too, the same assumptions regarding spherical symmetry and negligible loss of vehicle mass are employed. The major distinction between the two cases is that the thrust vector is now tangent to the flight path and in the plane of satellite motion; whereas in the previous case the thrust vector was normal to the radius vector and also normal to the instantaneous plane of motion.

The component equations of motion, in transformed coordinates, are

$$\frac{d^2u}{d\theta^2} + u = \frac{u}{h^2} \tag{28}$$

$$\frac{d(h^2)}{d\theta} = \frac{2\overline{W}}{u^2\sqrt{u^2 + \left(\frac{du}{d\theta}\right)^2}} \;. \tag{29}$$

Case of $\overline{W} = 0$.

Now, for the case of $\overline{W} = 0$ it follows that

$$\frac{d(h^2)}{d\theta} = 0 \quad \text{or} \quad h^2 = \text{constant} \tag{30}$$

and

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} = \text{constant} \quad . \tag{31}$$

Introducing the variable $q = u - \mu/h^2$ then Eq. (31) becomes

$$\frac{d^2q}{d\theta^2} + q = 0 \tag{32}$$

which has the solution

$$q = K_1 \cos(\theta + K_2) \quad , \tag{33}$$

or in terms of the variable u,

$$u = \frac{\mu}{h^2} + K_1 \cos(\theta + K_2) \tag{34}$$

where K_1 and K_2 are scalar constants of integration.

<u>Case of $\overline{W} \neq 0$.</u>

If $\overline{W} \neq 0$ the solution form is the same as that for $\overline{W} = 0$; i.e.

$$u = \frac{\mu}{h^2} + K_1 \cos(\theta + K_2) \tag{35}$$

except that in this instance h, K_1 and K_2 are all functions of the variable θ.

In the case of $\overline{W} = 0$, it is apparent that

$$\frac{du}{d\theta} = - K_1 \sin(\theta + K_2) \tag{36}$$

while for $\overline{W} \neq 0$

$$\frac{du}{d\theta} = -\frac{\mu}{h^4}\frac{d(h^2)}{d\theta} + \frac{dK_1}{d\theta}\cos(\theta + K_2) - K_1 \sin(\theta + K_2)\left(1 + \frac{dK_2}{d\theta}\right) \tag{37}$$

Comparing Eqs. (36) and (37) above, a requirement for them to match is that

$$-\frac{\mu}{h^4}\frac{d(h^2)}{d\theta} + \frac{dK_1}{d\theta}\cos(\theta + K_2) - K_1\frac{dK_2}{d\theta}\sin(\theta + K_2) = 0 \tag{38}$$

or substituting Eq. (29) into (38)

$$-\frac{\mu}{h^4}\frac{2\overline{W}}{u^2\sqrt{u^2 + \left(\frac{du}{d\theta}\right)^2}} + \frac{dK_1}{d\theta}\cos(\theta + K_2) - K_1\frac{dK_2}{d\theta}\sin(\theta + K_2) = 0 \tag{39}$$

then

$$\frac{d^2u}{d\theta^2} = -\frac{dK_1}{d\theta}\sin(\theta + K_2) - K_1\cos(\theta + K_2)\left[1 + \frac{dK_2}{d\theta}\right] \tag{40}$$

Now substituting for the derivative forms and for u, the equations for the radial motion become

$$-\frac{dK_1}{d\theta}\sin(\theta + K_2) - K_1\frac{dK_2}{d\theta}\cos(\theta + K_2) = 0 \tag{41}$$

Solving Eqs. (39) and (41) simultaneously one finds that

$$\frac{dK_2}{d\theta} = - \frac{\mu}{K_1\ h^4} \quad \frac{2\overline{W}\ \sin\ (\theta + K_2)}{u^2 \sqrt{u^2 + (\frac{du}{d\theta})^2}} \tag{42}$$

$$\frac{dK_1}{d\theta} = \frac{\mu}{h^4} \quad \frac{2\overline{W}\ \cos\ (\theta + K_2)}{u^2 \sqrt{u^2 + (\frac{du}{d\theta})^2}} \tag{43}$$

Noting that

$$u^2 + (\frac{du}{d\theta})^2 = \frac{K_1}{\epsilon} \left[2u + \frac{K_1}{\epsilon} (\epsilon^2 - 1)\right] \tag{44}$$

where

$$\epsilon = \frac{K_1\ h^2}{\mu}$$

then

$$u^2 + (\frac{du}{d\theta})^2 = \frac{K_1}{\epsilon}\left[\frac{2\mu}{h^2} + 2\ K_1\ \cos\ (\theta + K_2) + \frac{K_1}{\epsilon} (\epsilon^2 - 1)\right] \tag{45}$$

or

$$u^2 + (\frac{du}{d\theta})^2 = (\frac{K_1}{\epsilon})^2 \left[1 + 2\ \epsilon\ \cos\ (\theta + K_2) + \epsilon^2\right] \tag{46}$$

thus

$$\left[u^2 + (\frac{du}{d\theta})^2\right]^{1/2} = \frac{K_1}{\epsilon} \left[1 + \epsilon^2 + 2\ \epsilon\ \cos\ (\theta + K_2)\right]^{1/2}. \tag{47}$$

Now

$$u = \frac{\mu}{h^2}\left[1 + \frac{K_1\ h^2}{\mu} \cos\ (\theta + K_2)\right] = \frac{K_1}{\epsilon}\left[1 + \epsilon\ \cos\ (\theta + K_2)\right] \tag{48}$$

and the substitution of Eqs. (47) and (48) into (42) and (43) yields

$$\frac{dK_1}{d\theta} = \left(\frac{\epsilon}{h\,K_1}\right)^2 \frac{2\overline{W}\cos(\theta + K_2)}{\left[1 + \epsilon\cos(\theta + K_2)\right]^2 \left[1 + \epsilon^2 + 2\epsilon\cos(\theta + K_2)\right]^{1/2}} \tag{49}$$

$$\frac{dK_2}{d\theta} = -\left(\frac{\epsilon}{K_1\,h}\right)^2 \frac{1}{K_1} \frac{2\overline{W}\sin(\theta + K_2)}{\left[1 + \epsilon\cos(\theta + K_2)\right]^2 \left[1 + \epsilon^2 + 2\epsilon\cos(\theta + K_2)\right]^{1/2}} \tag{50}$$

To determine the average values of the variables h^2, K_1 and K_2, one notes that the derivatives with respect to θ are proportional to $\overline{W}$; thus for small $\overline{W}$ these should vary slowly over one period of motion. Designate $(\theta + K_2) = \sigma$, then the average value of the derivative is obtained by treating h^2, K_1 and K_2 as constants for the integration with respect to σ, or

$$\left\langle \frac{d(h^2)}{d\theta} \right\rangle = \frac{2\overline{W}\,\epsilon^3}{2\pi\,K_1^{\,3}} \int_0^{2\pi} \frac{d\sigma}{(1 + \epsilon\cos\sigma)^2 (1 + \epsilon^2 + 2\epsilon\cos\sigma)^{1/2}} \tag{51}$$

where the sign $\langle\,\rangle$ is employed to designate an "average" valued quantity.

The form of (51) is that of an elliptic integral which can be evaluated as

$$\left\langle \frac{d(h^2)}{d\theta} \right\rangle = \frac{2\bar{W}\ \epsilon^3}{\pi\ K_1^{\ 3}\ (1 - \epsilon)^2(1 + \epsilon)^3} \left[(3 + \epsilon^2)\ K - (1 + \epsilon^2)\ E \right] \tag{52}$$

where
$$K = K(k) = K\left(\frac{2\sqrt{\epsilon}}{1+\epsilon}\right) \quad \text{and} \quad E = E(k) = E\left(\frac{2\sqrt{\epsilon}}{1+\epsilon}\right) .$$

Now the average value of

$$\frac{dK_1}{d\theta} = \left(\frac{\epsilon}{h\ K_1}\right)^2 \frac{2\bar{W} \cos\sigma}{(1 + \epsilon\cos\sigma)^2(1 + \epsilon^2 + 2\epsilon\cos\sigma)^{1/2}} \tag{53}$$

can also be evaluated by the use of elliptic integral transforms. It is easily shown that

$$\left\langle \frac{dK_1}{d\theta} \right\rangle = \frac{2\bar{W}\ \epsilon}{\pi(h\ K_1^{\ 2})(1 + \epsilon)^3(1 - \epsilon^2)} \left[(1 + \epsilon)^2\ E(k) - (1 + 3\epsilon^2)\ K(k) \right] \tag{54}$$

where
$$E = E(k) \quad \text{and} \quad K = K(k), \quad \text{with } k^2 = \left(\frac{2\sqrt{\epsilon}}{1+\epsilon}\right)^2$$

Now using the Gauss transformation,

$$E(k_1) = \frac{E(k) + k'\ K(k)}{1 + k'} \tag{55}$$

$$K(k_1) = \frac{(1 + k')\ K(k)}{2} \tag{56}$$

where
$$k_1 = \frac{1 - k'}{1 + k'} ; \qquad k' = (1 - k^2)^{1/2}$$

or

$$k' = \frac{1 - \epsilon}{1 + \epsilon} \quad ; \qquad k_1 = \epsilon$$

and changing modulus, it follows that

$$E(k_1) = \frac{E(k) + k' \, K(k)}{1 + k'} \tag{57}$$

or

$$E(k) = \frac{2}{1 + \epsilon} \, E(k_1) - \frac{1 - \epsilon}{1 + \epsilon} \, K(k) \quad . \tag{58}$$

Likewise

$$K(k) = (1 + k_1) \, K(k_1) = (1 + \epsilon) \, K(k_1) \tag{59}$$

thus

$$E(k) = \frac{2}{1 + \epsilon} \, E(k_1) - (1 - \epsilon) \, K(k_1) \tag{60}$$

and

$$K(k) = (1 + \epsilon) \, K(k_1) \tag{61}$$

Now changing the modulus from k to k_1 it is apparent that

$$\left\langle \frac{dK_1}{d\theta} \right\rangle = \frac{4\bar{W} \, \epsilon}{\pi (h \, K_1)^2 (1 + \epsilon)^2 (1 - \epsilon^2)} \left[E(k_1) - (1 + \epsilon^2) \, K(k_1) \right] \tag{62}$$

where $k_1 = \epsilon$, for $\epsilon < 1$.

Too, the average value of $\frac{d(h^2)}{d\theta}$ becomes

$$\left\langle \frac{d(h^2)}{d\theta} \right\rangle = \frac{4\bar{W} \quad \epsilon^3}{\pi \, K_1{}^3 (1 + \epsilon)^2 (1 - \epsilon)^2} \left[2K(k_1) - E(k_1) \right] \tag{63}$$

To evaluate the change in eccentricity over the orbit it is recalled that

$$\epsilon = \frac{K_1 h^2}{\mu} \quad ,$$

thus

$$\frac{d\epsilon}{d\theta} = \frac{K_1}{\mu} \frac{d(h^2)}{d\theta} + \frac{h^2}{\mu} \frac{dK_1}{d\theta} \tag{64}$$

and the averaging process applied here leads to the average value of $\frac{d\epsilon}{d\theta}$ in terms of the previous determined values. Making the necessary substitutions and simplifying, it is found that

$$\left\langle \frac{d\epsilon}{d\theta} \right\rangle = \frac{4\overline{W}\ \epsilon}{\pi\ \mu\ K_1^{\ 2}(1 + \epsilon)(1 - \epsilon)} \left[E(k_1) - K(k_1) \right] \tag{65}$$

Dividing Eq. (65) by Eq. (63) and dropping the average value signs, then

$$\frac{d\epsilon}{d(h^2)} = \frac{1 - \epsilon^2}{\epsilon\ h^2} \left[\frac{E(k_1) - K(k_1)}{2K(k_1) - E(k_1)} \right] \tag{66}$$

or

$$\frac{d(h^2)}{h^2} = \frac{d(1 - \epsilon^2)}{1 - \epsilon^2} - \frac{\epsilon\ E(k_1)\ d\epsilon}{(1 - \epsilon^2)\left[K(k_1) - E(k_1) \right]} \quad ; \tag{67}$$

but

$$\frac{d}{d\epsilon}\left[K(k_1) - E(k_1) \right] = \frac{\epsilon\ E(k_1)}{1 - \epsilon^2} \tag{68}$$

thus

$$\frac{1 - \epsilon^2}{K(k_1) - E(k_1)} = Q\ h^2 \tag{69}$$

where Q is a constant of integration that can be evaluated in accordance with the initial conditions.

Dividing Eq. (62) by Eq. (63), and again dropping average value signs, it is found that

$$\frac{1}{K_1}\ \frac{dK_1}{d\epsilon} = \frac{E(k_1) - (1+\epsilon^2)\ K(k_1)}{\epsilon(1-\epsilon^2)(E(k_1) - K(k_1))} \tag{70}$$

where $\epsilon = \dfrac{K_1\ h^2}{\mu}$ and $\dfrac{1-\epsilon^2}{K(k_1) - E(k_1)} = Q\ h^2$. (70a)

On substituting Eq. (70a) into Eq. (70) and integrating it is found that

$$K_1 = \frac{\mu\ \epsilon\ Q}{1-\epsilon^2}\left[K(k_1) - E(k_1)\right] \tag{71}$$

Since $h^2\ K_1 = \epsilon\mu$ Eq. (71) yields

$$\frac{K_1}{Q\mu} = \frac{\epsilon}{1-\epsilon^2}\left[K(\epsilon) - E(\epsilon)\right] \tag{72}$$

and from Eq. (65), after substituting Eqs. (69) and (72),

$$\frac{d\theta}{d\epsilon} = \frac{-\pi\ Q^2\ \mu^3}{4\overline{W}}\ \frac{\epsilon}{1-\epsilon^2}\ \left[K(\epsilon) - E(\epsilon)\right]\ . \tag{73}$$

Using the following approximation[4] for $K(\epsilon) - E(\epsilon)$, namely

$$K(\epsilon) - E(\epsilon) = 0.3862944 - 0.3510428\ \eta - 0.0352516\ \eta^2 -$$

$$(0.5 - 0.1239049\ \eta - 0.0123767\ \eta^2)\ \ln \eta = f(\eta) \qquad (74)$$

where $\eta = 1 - \epsilon^2$

then

$$\int_\epsilon \frac{\epsilon \left[K(\epsilon) - E(\epsilon)\right] d\epsilon}{1 - \epsilon^2} = \frac{1}{2} \int_\eta \frac{f(\eta)}{\eta}\, d\eta \qquad (75)$$

Performing the above integration and substituting into Eq. (73) yields

$$\theta - \psi = \frac{\pi\, Q^2\, \mu^3}{8\, \overline{W}} \Big[(- 0.4956677 + 0.5163877\ \epsilon^2 - 0.02072\ \epsilon^4) +$$

$$(0.5163877 - 0.1362815\ \epsilon^2 + 0.0061884\ \epsilon^4)\ \ln (1 - \epsilon^2) -$$

$$\frac{1}{4} \ln^2 (1 - \epsilon^2)\Big] \qquad (76)$$

where ψ is a constant of integration employed to give the proper polar angle position initially.

Equations (76) and (72) are next substituted into Eq. (34)

$$u = \frac{\mu}{h^2} (1 + \epsilon \cos \theta) \qquad (77)$$

where θ is found from Eq. (76), and ψ is the constant of integration.

The solution to the exact equations (Eqs. (28) and (29)) will next be compared to the solution obtained by the approximate method (Eq. (77)).

IV. NUMERICAL PROCEDURE

An examination of Eqs. (28) and (29) will reveal their non-linearity in u. Equation (28) and the corresponding expression for angular momentum were reduced to a third order differential equation. This was solved by the Runge-Kutta method of numerical integration. This provides what is called here "the exact solution"; the corresponding approximate solution was obtained by the solution of Eq. (77), employing the proper variables as defined from the evaluation of the averaged derivatives.

The specific thrusting parameter $\bar{W}$ and the satellite velocity were varied from one orbit to another in this investigation. An arbitrary initial perigee radius of 300 miles was selected; however, this could easily be varied as a fundamental parameter. The satellite velocities, chosen as initial values (at perigee), were 18,000 mph and 21,000 mph respectively. The value of 18,000 mph was chosen to observe the various thrusting effects on a slightly elliptical orbit, while the value of 21,000 mph was chosen to observe these effects on a more definitely elliptical orbit. Incidentally, the velocity of escape, at the perigee altitude, is approximately 24,000 mph.

Solutions were obtained by letting the specific thrusting parameter ($\bar{W}$) assume the values of 0.005, 0.01, 0.1, 0.5 and 1.0 for each selected value of the satellite perigee velocity, (V). Results of the calculations are plotted and compared in Figures 1 through 27.

The increment of arc used in the Runge-Kutta method was set at $2\pi/360$. This increment was tested by halving the interval of integration. The ensuing calculations evidenced no noticeable change in the eighth place of the mantissa which is indicative of a calculation procedure accurate to $(2\pi/360)^4$.

For the normal thrust case, the solution of Eqs. (7) was not possible to obtain on the 1620 computer due to their vector form. Equations (13) can be solved however, and their results show that a change in $\bar{W}$ does not affect the components of $\bar{e}_x$ in the $\bar{e}_{x_o}$ or $\bar{e}_{y_o}$ directions; but the component in the $\bar{e}_{z_o}$ direction is influenced by a change in this parameter.

V. DISCUSSION OF RESULTS

The orbits for the various specific thrusting forces are plotted on various scales to facilitate readibility.

The constant of integration K_2, in Eq. (34) was arbitrarily chosen to be zero. The selection of this value introduces no loss of generality since it enters as an additive arbitrary constant initially.

For the case of a tangential thrusting situation, Figures 1 through 27, the exact and approximate solutions indicate an advancing line of apsides. Although the distance to apogee in the exact and approximate solutions are not identical (see Figures 1, 4, 16, and 19) the line of apsides does coincide for each individual case. The maximum deviation between the radii, for the exact and approximate solutions in Figure 1, amounts to 46.7 miles or 0.98 percent of the exact solution value and occurs at $\theta = 94^{o}$. In Figure 4, the maximum radial deviation is 9 miles or 0.19 percent of the exact solution and occurs at a value of $\theta = 90^{o}$. In Figure 16 the lines of apsides are coincident and the largest deviation between the radii is 97 miles or 0.7 percent of the exact solution value, and occurs at apogee. In Figure 19 this maximum deviation is 106 miles, or 0.8 percent of the exact solution, and this too occurs at apogee.

Figure 7 shows a maximum deviation of 89.6 miles, or 1.85 percent of the exact solution, which occurs at $\theta = 92^{o}$. Figure 10 has a maximum deviation between radii of 450 miles or 9.2 percent of the exact solution value, occurring at $\theta = 94^{o}$. In Figure 22, 643 miles is the maximum deviation between radii which is observed at $\theta = 232^{o}$. This is 5.54 percent of the exact solution value at that point. In Figure 13, the exact solution shows the satellite reaching escape conditions at $\theta = 240^{o}$ at a radius of 9,186.7 miles. The approximate solution has an eccentricity which decreases to zero at $\theta = 313^{o}$, reaching a radius of 46,522 miles. The exact solution, Figure 25, indicates that the satellite escapes at $\theta = 150^{o}$, having reached a radius of 13,706.2 miles. The approximate solution, however, has its eccentricity diminishing to zero at a θ of 196.2^{o}, and attaining a radius of 61,565.5 miles. Figure 13 corresponds to a "1 g" acceleration with an initial perigee velocity of 18,000 mph.

The graphs of the specific energy (E) versus the specific angular momentum (h), and the eccentricity (ϵ) versus the semi-latus rectum (p), are marked to denote the corresponding values of θ, along the orbital path, as measured from perigee.

Recalling that the eccentricity (ϵ) defines the shape of the orbit and the parameter (p) defines the size, then a small eccentricity means a nearly circular orbit. As ϵ increases the ellipticity of the orbit increases, and an increasing value of p implies that the physical size of the orbit is increasing.

It is observed that in the graphs of E versus h, the exact and approximate solutions vary greatly for small values of $\overline{W}$. However, as $\overline{W}$ is increased the plot for these two compared solutions more closely coincide with one another. This would tend to indicate that the method of solution is becoming more accurate as the value of $\overline{W}$ is increased. On the other hand, on closer examination of the plots where $\overline{W}$ is small it is noted that although the graphs deviate in an odd fashion; the change between these graphs is relatively small. Hence, in the case of $\overline{W}$ comparatively large, the influence is to reduce the local variation in the dynamic parameters while not actually reducing the percentage variance between the exact and approximate answers.

VI. CONCLUSIONS

If a thrusting vehicle can be assumed to have a constant mass, and if the oblateness of the earth and its atmosphere can be neglected, then the following conclusions are offered:

(1) The method of Kryloff and Bogoliuboff is a valid approach for the solution to the problem of satellite motion with a constant thrust, provided that the thrust is kept small.

(2) As the initial orbit increases in ellipticity the validity of the method of Kryloff and Bogoliuboff decreases with an increase in specific thrust.

(3) The equation (Eq. (33)) for $\bar{i}$, in the reference by Lass and Solloway[2], is inconsistent with the corresponding equation (Eq. (27a) developed in this study.

(4) Summarily, the range of validity of the method of Kryloff and Bogoliuboff was found to be dependent on both $\overline{W}$ and V. The limit of validity varies with the initial ellipticity of the orbit. Depending upon the degree of accuracy required, the solutions obtained by the use of the method of Kryloff and Bogoliuboff are valid until an "escape" occurs.

VII. ACKNOWLEDGMENTS

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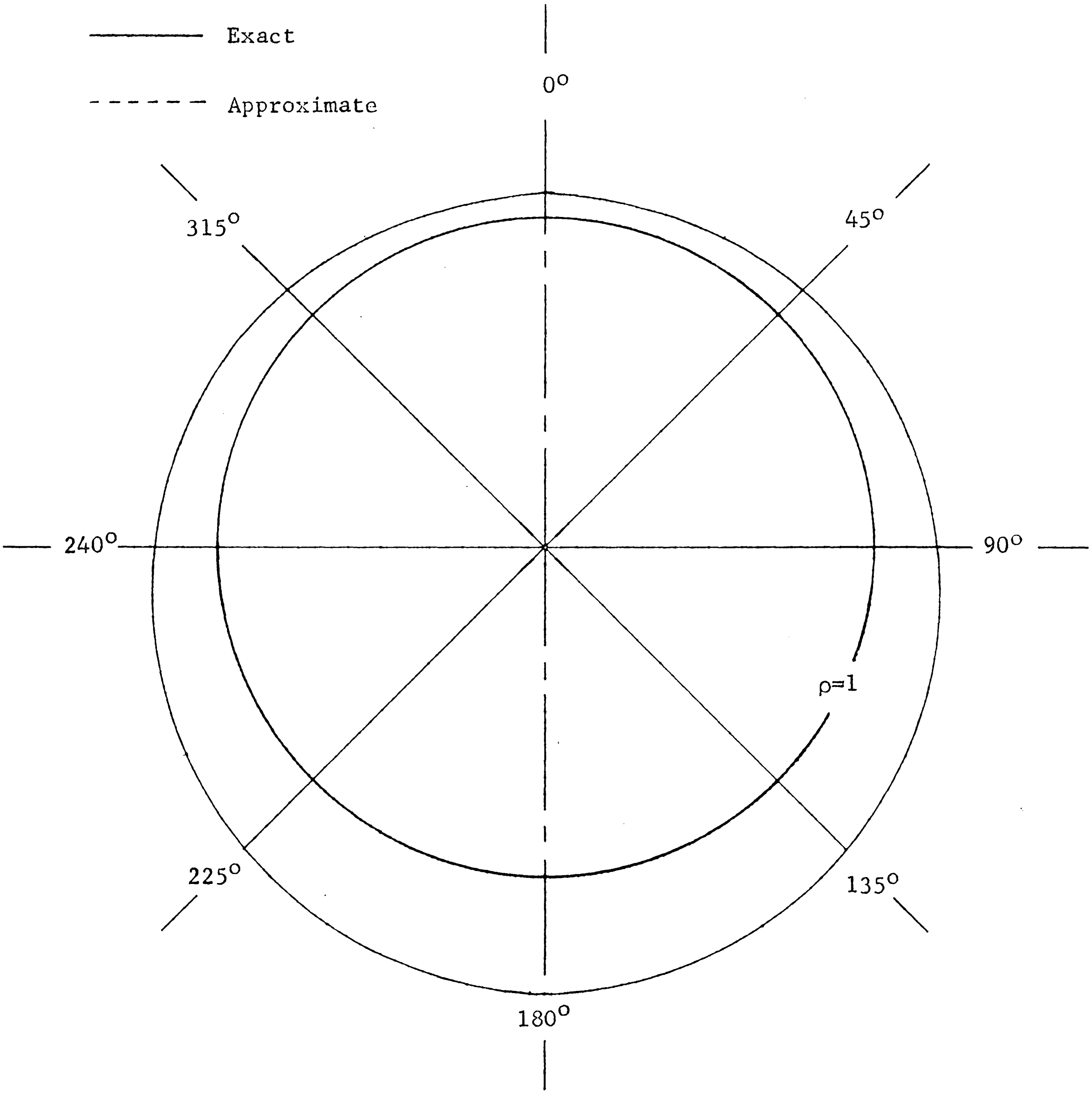
Due to the proximity of the exact and approximate solutions only one is shown.



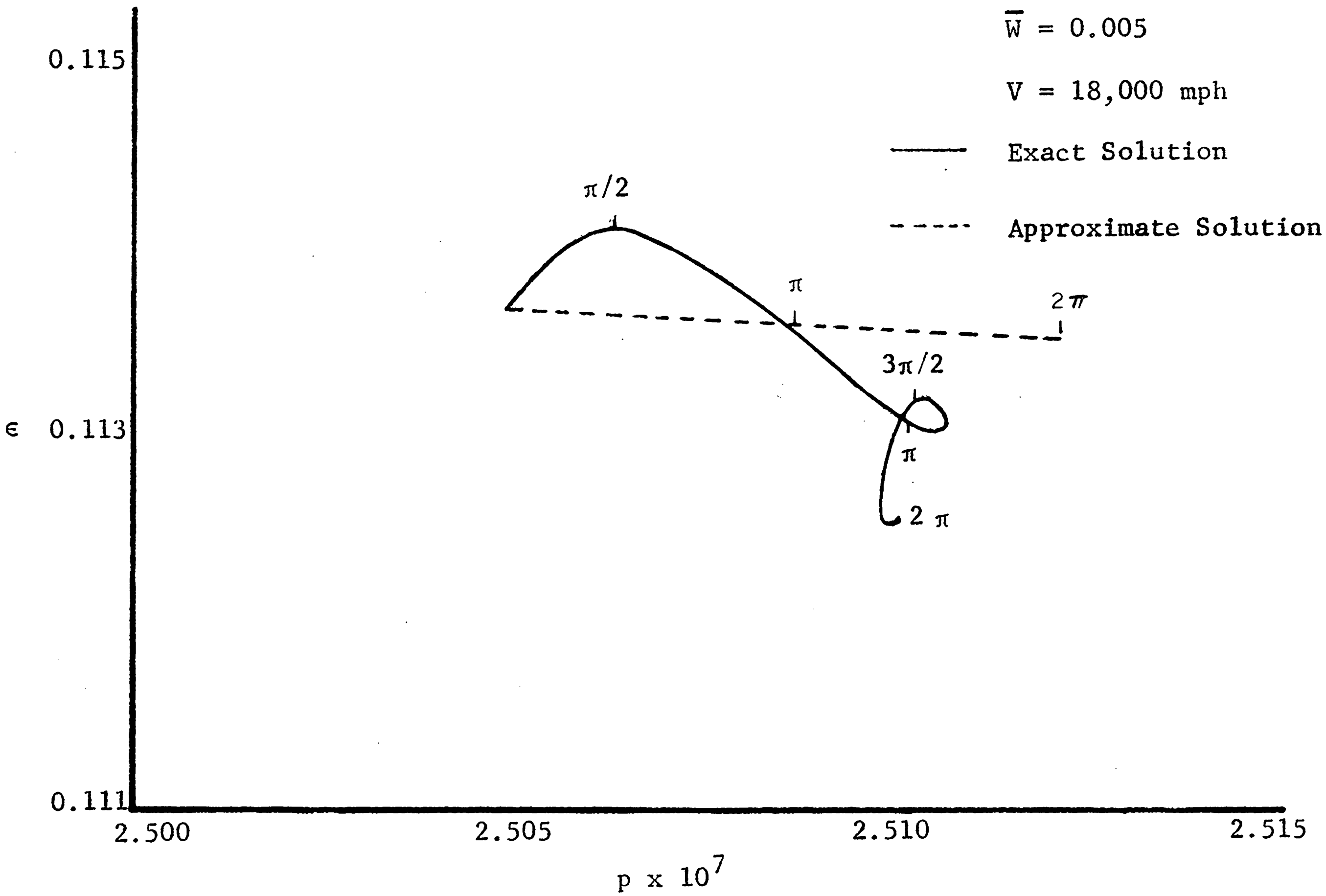
ρ vs. θ

$\overline{W} = 0.005$

V = 18,000 mph

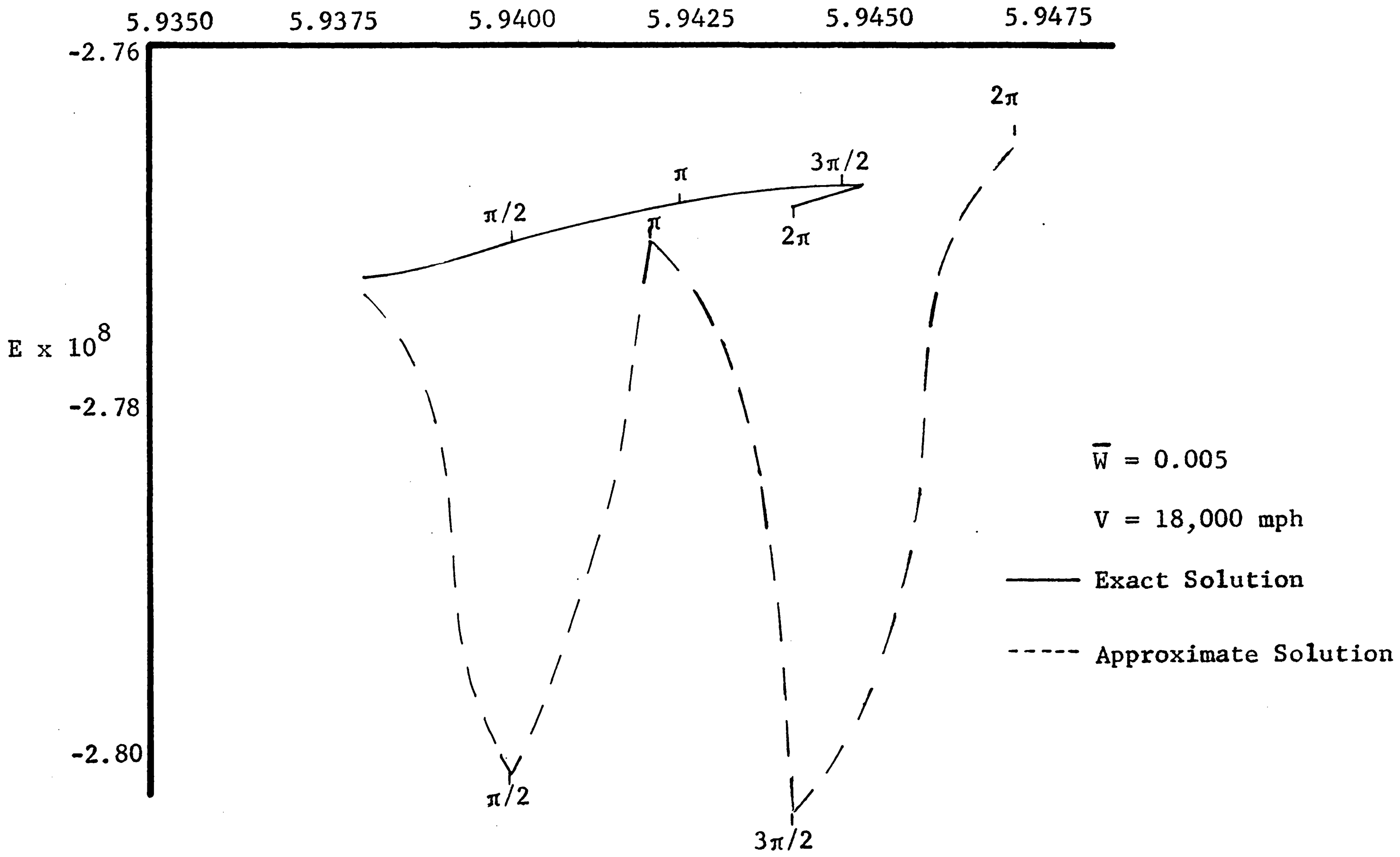
Tangential Thrust

FIGURE 1



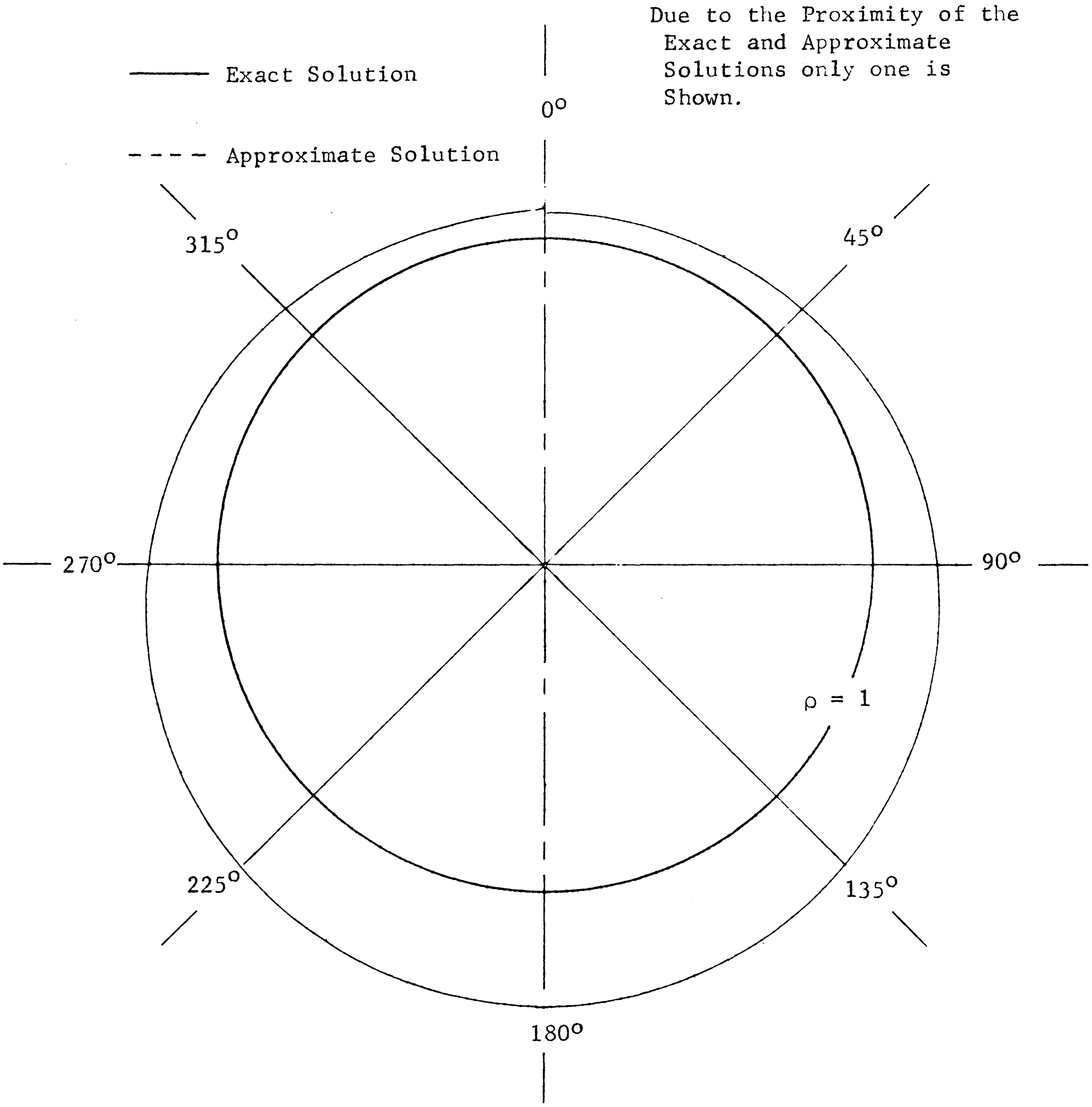
Eccentricity vs. p

FIGURE 2



Energy vs. Angular Momentum, Tangential Thrust

FIGURE 3



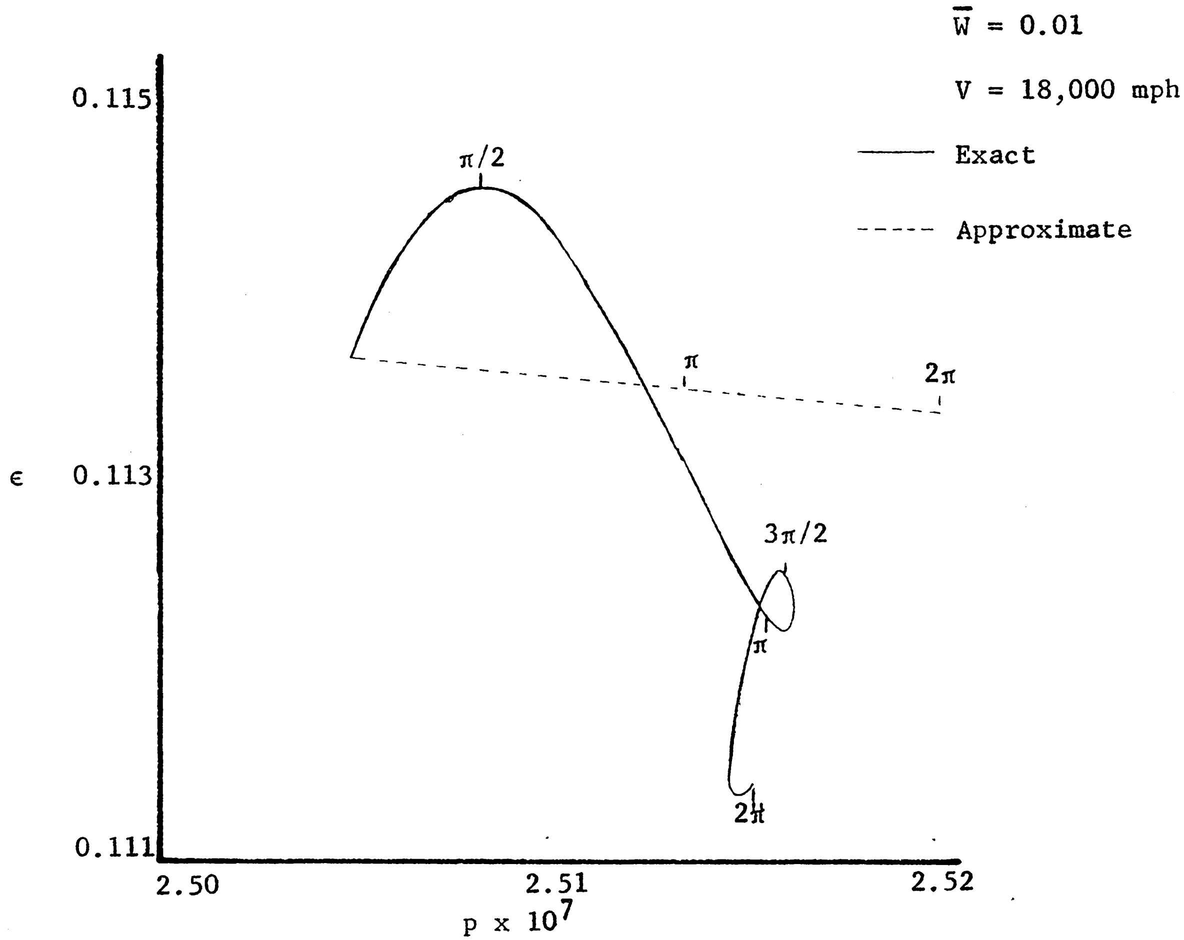
ρ vs. θ

$\overline{W} = 0.01$

V = 18,000 mph

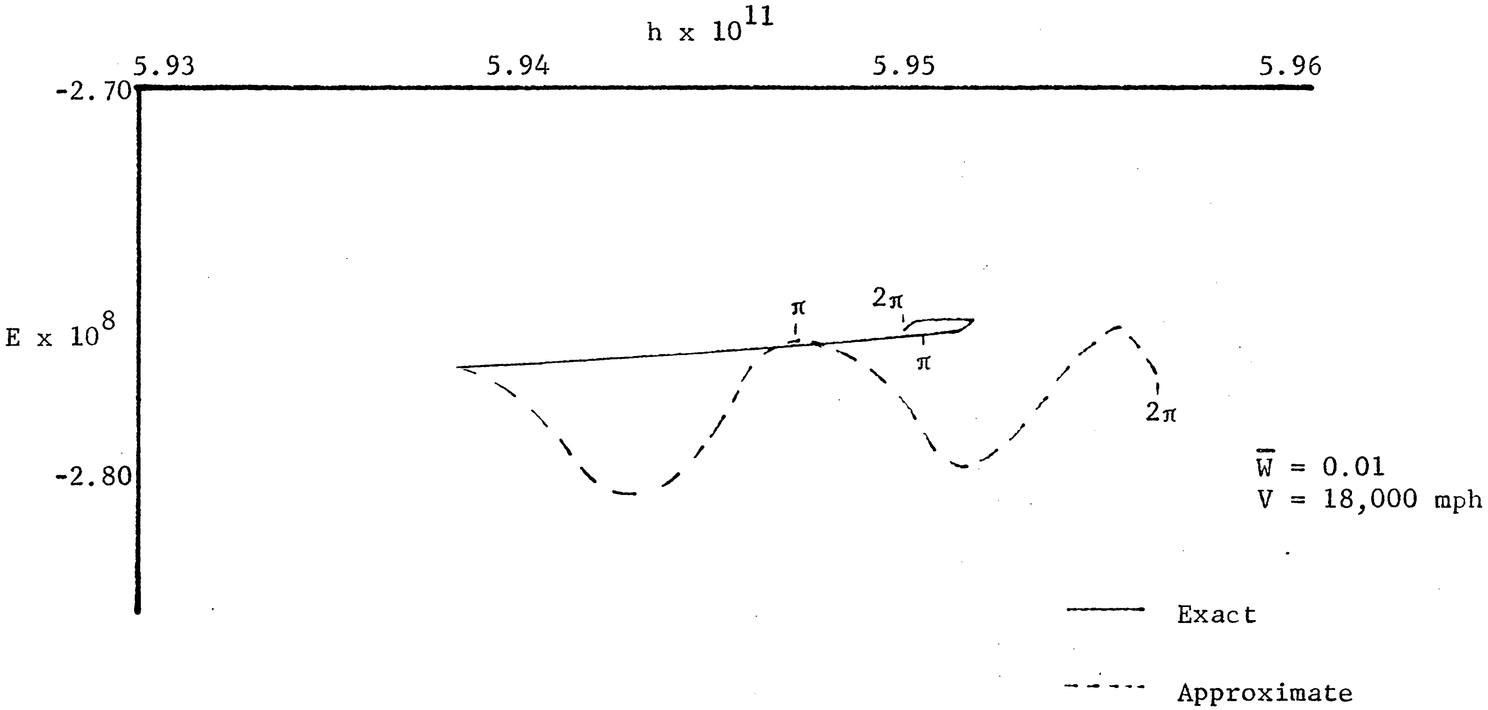
Tangential Thrust

FIGURE 4



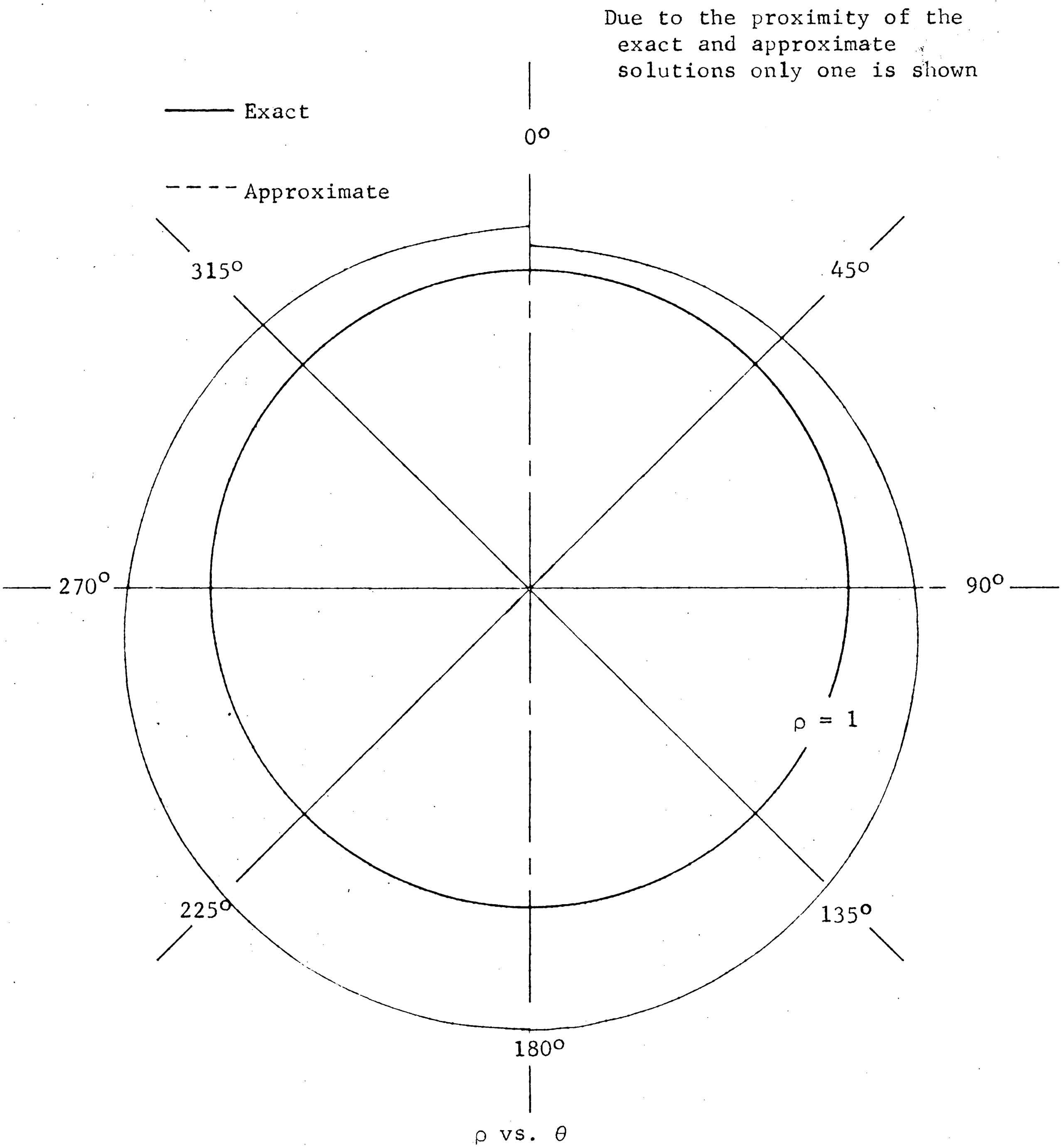
Eccentricity vs. p

FIGURE 5



Energy vs. Angular Momentum, Tangential Thrust

FIGURE 6

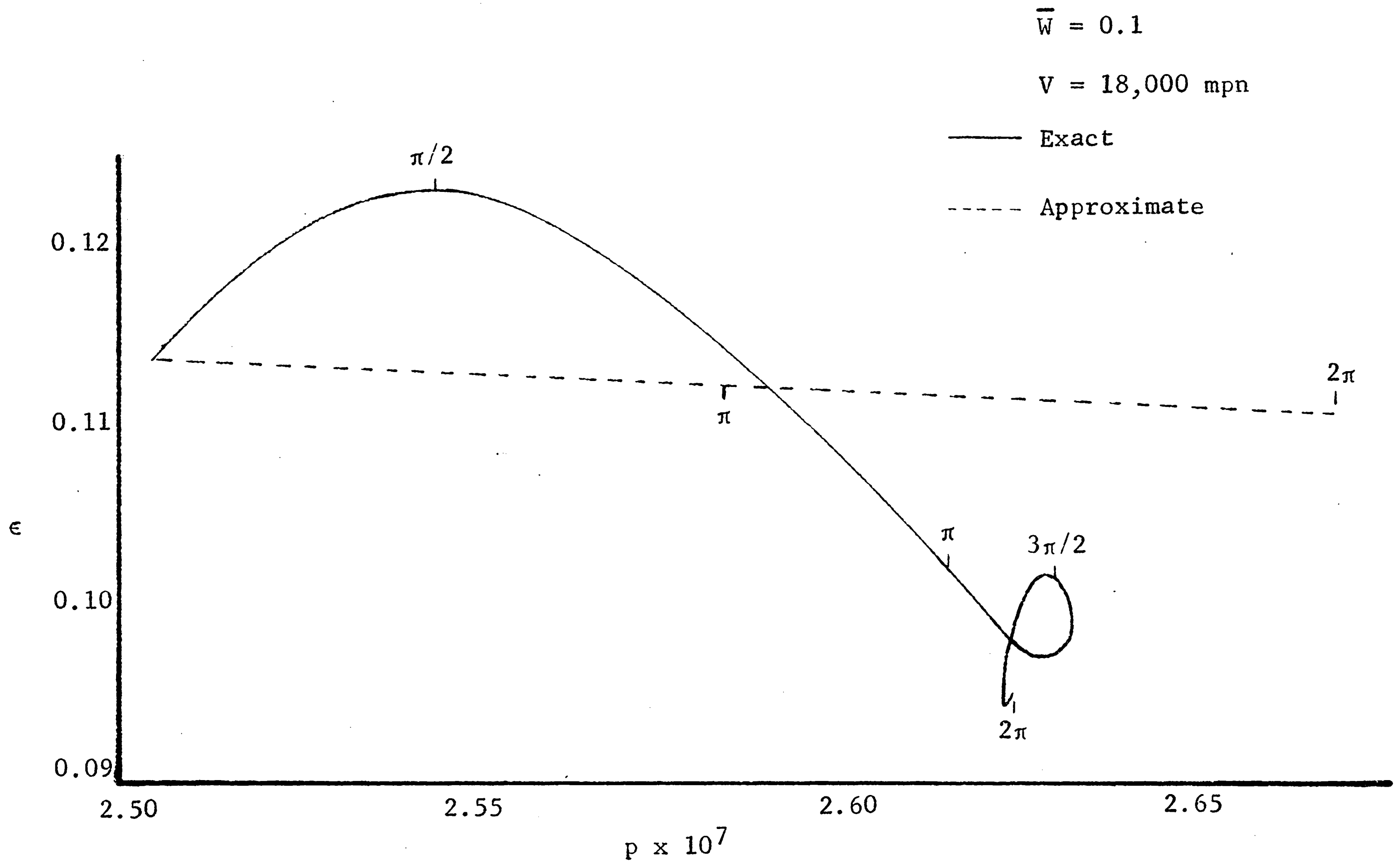


ρ vs. θ

$\overline{W} = 0.1$, V = 18,000 mph

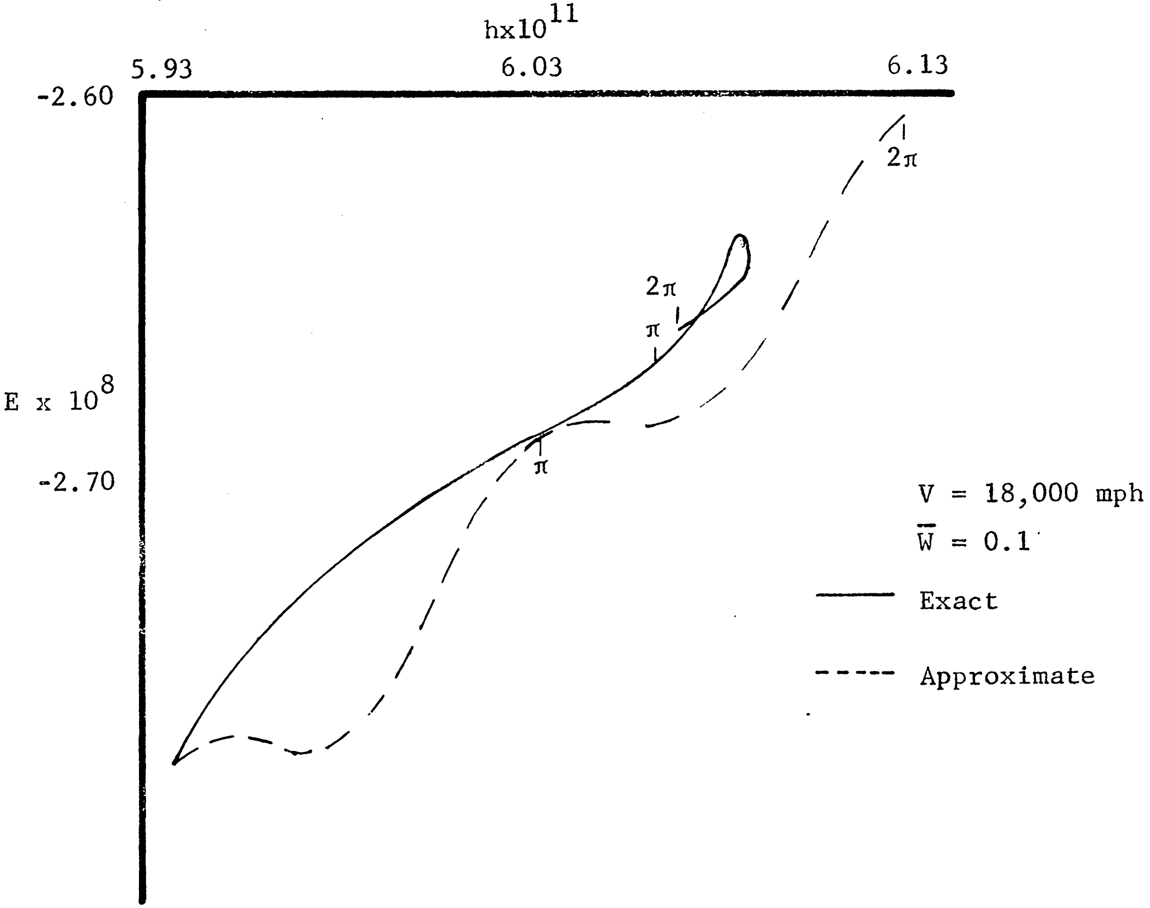
Tangential Thrust

FIGURE 7



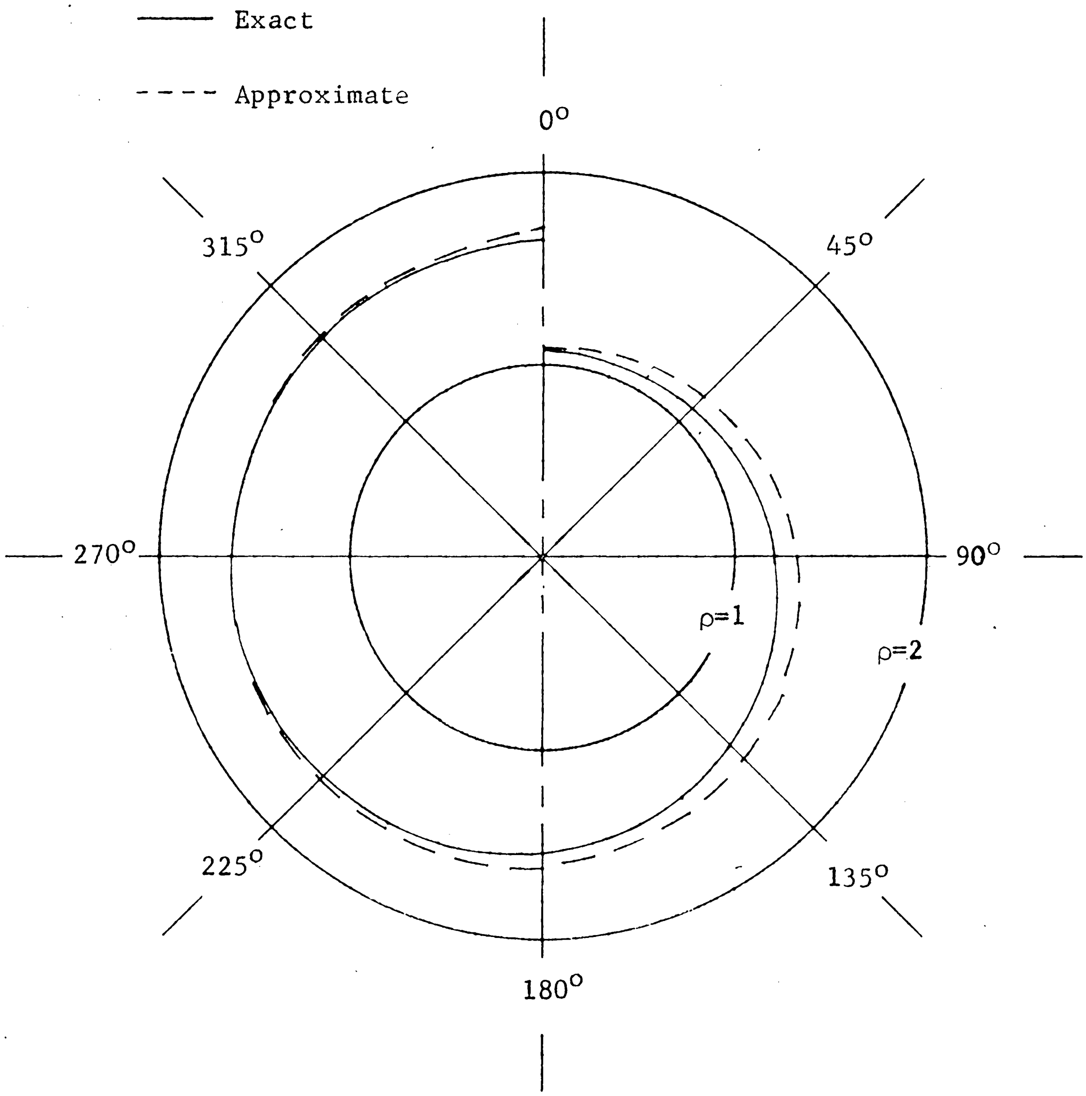
Eccentricity vs. p

FIGURE 8



Energy vs. Angular Momentum, Tangential Thrust

FIGURE 9

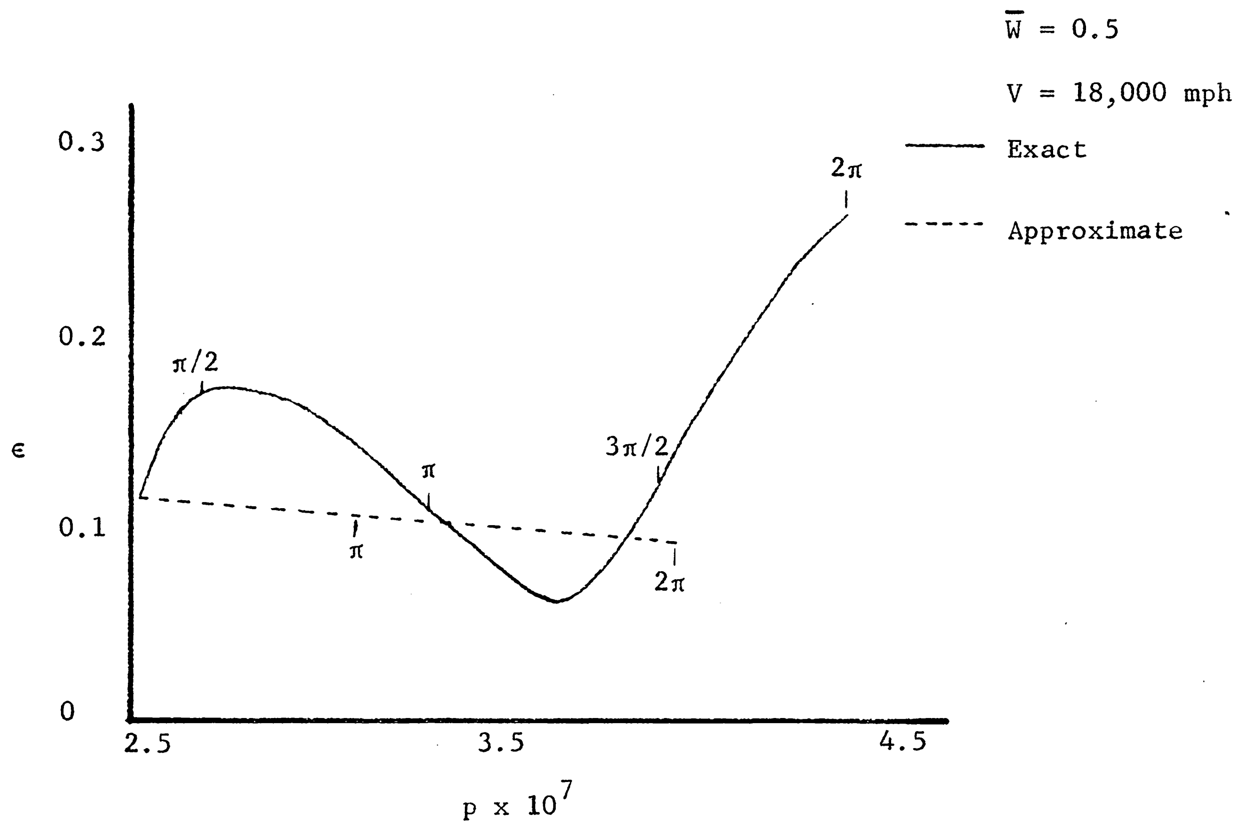


ρ vs. θ

$\overline{W} = 0.5$, V = 18,000 mph

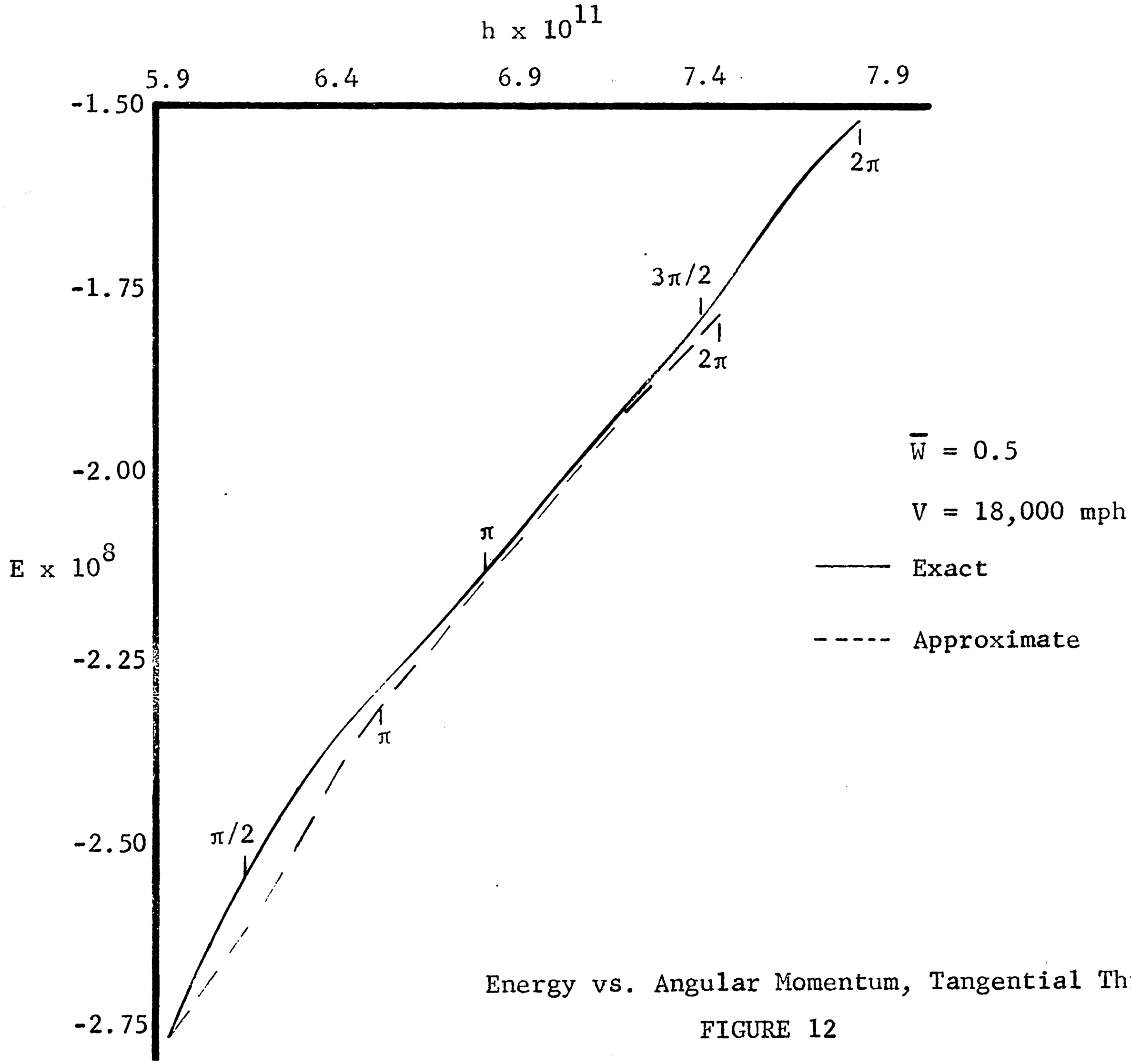
Tangential Thrust

FIGURE 10



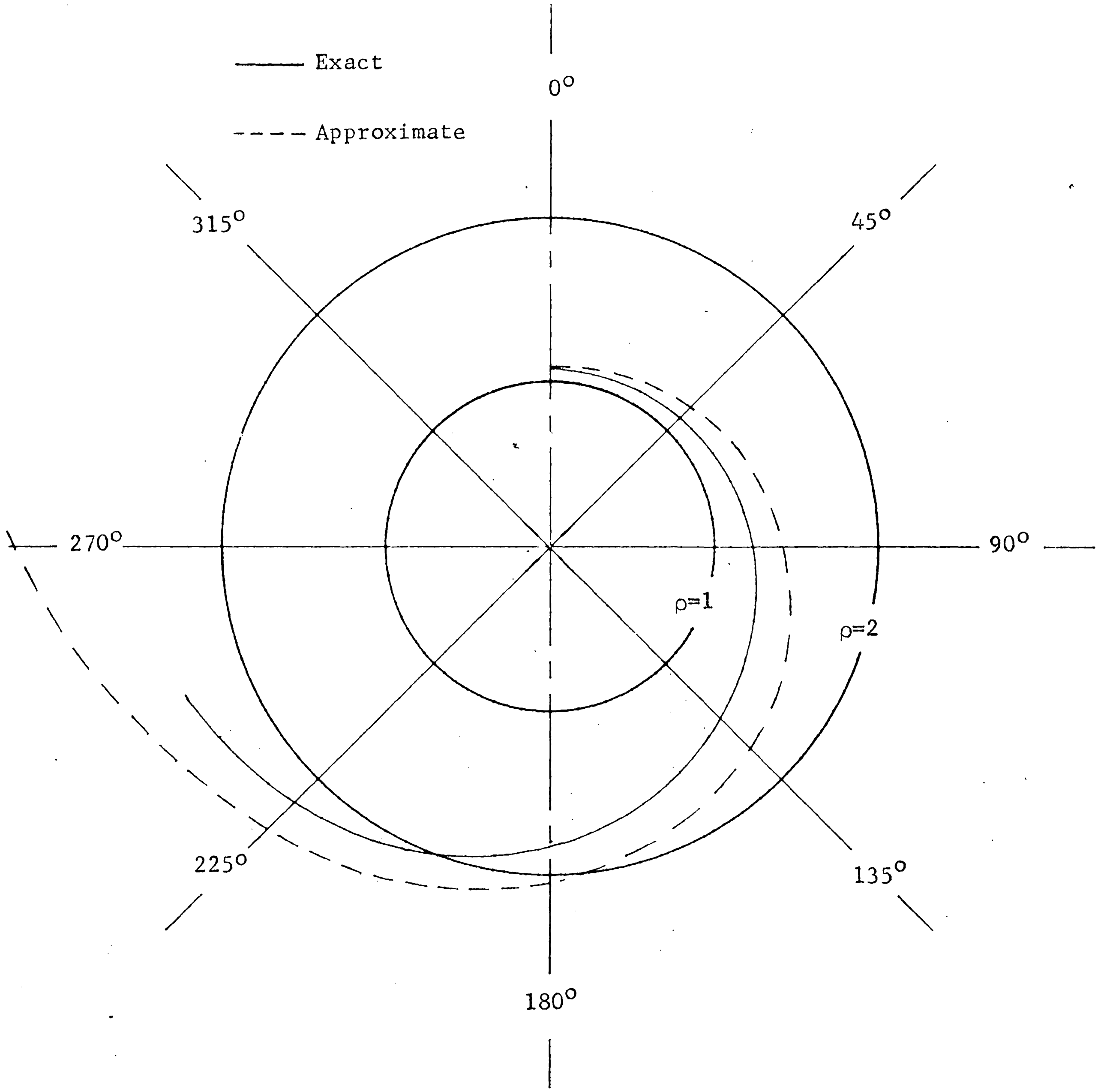
Eccentricity vs. p

FIGURE 11



Energy vs. Angular Momentum, Tangential Thrust

FIGURE 12

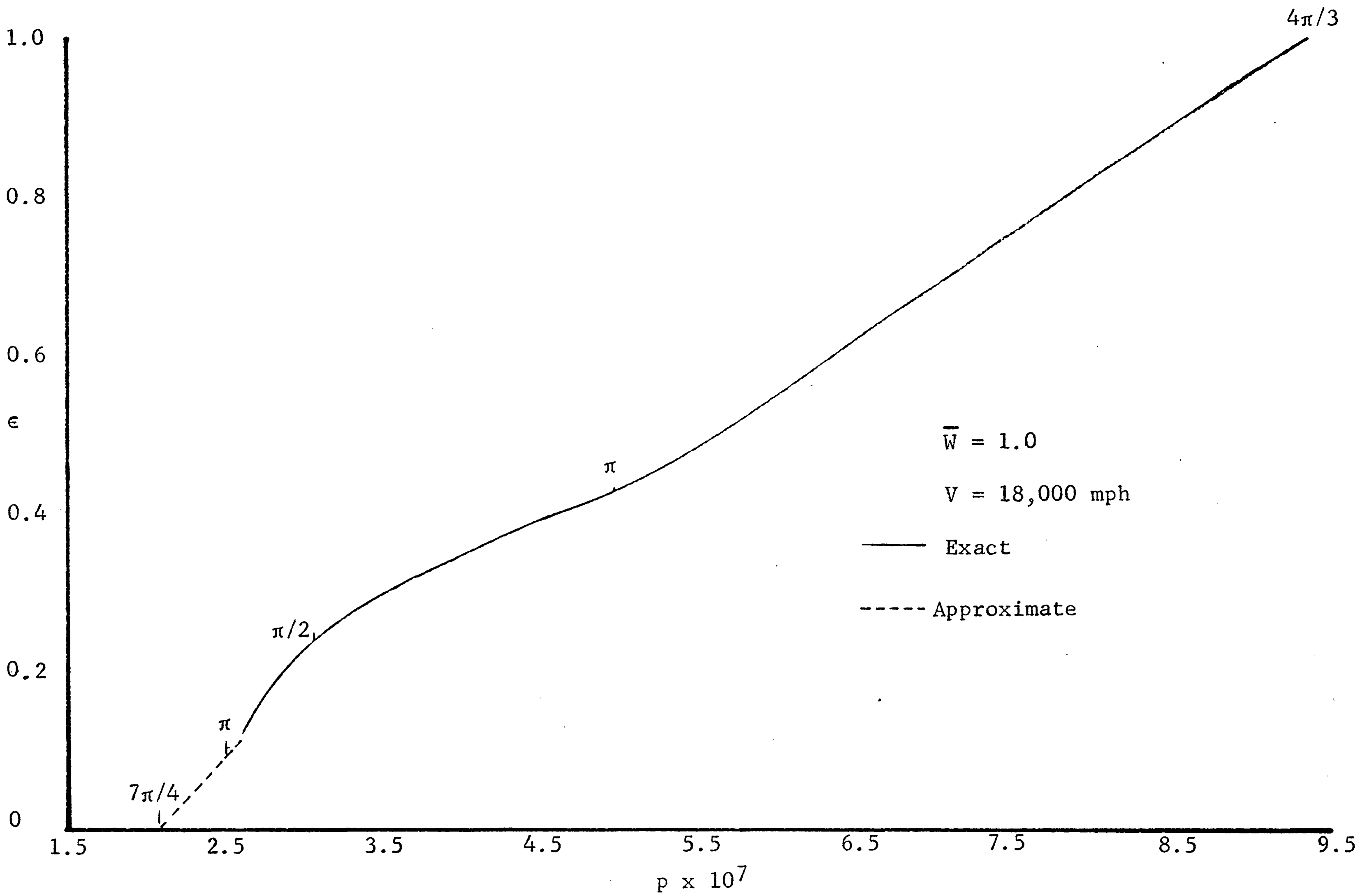


$\rho = \theta$

$\overline{W} = 1.0$, V = 18,000 mph

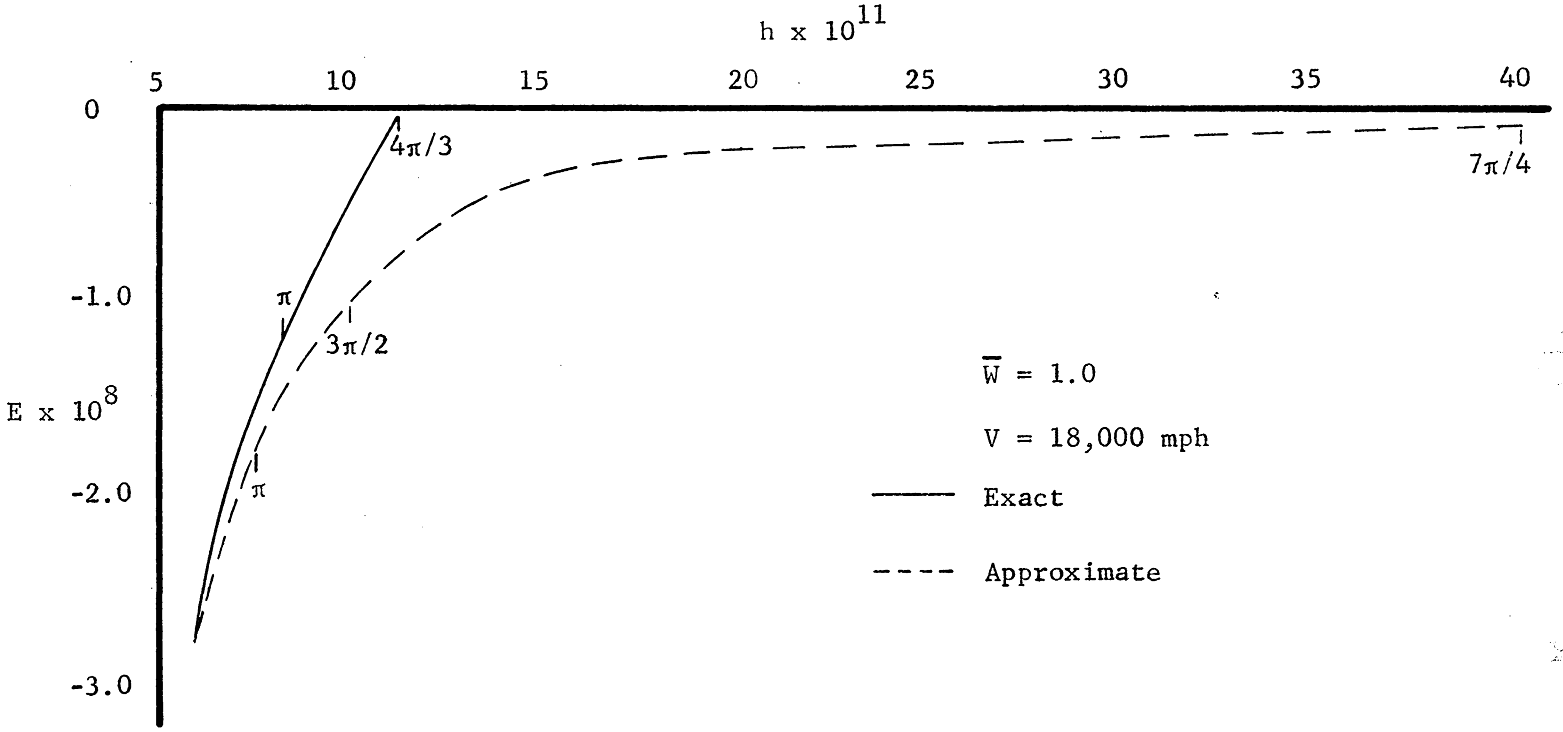
Tangential Thrust

FIGURE 13



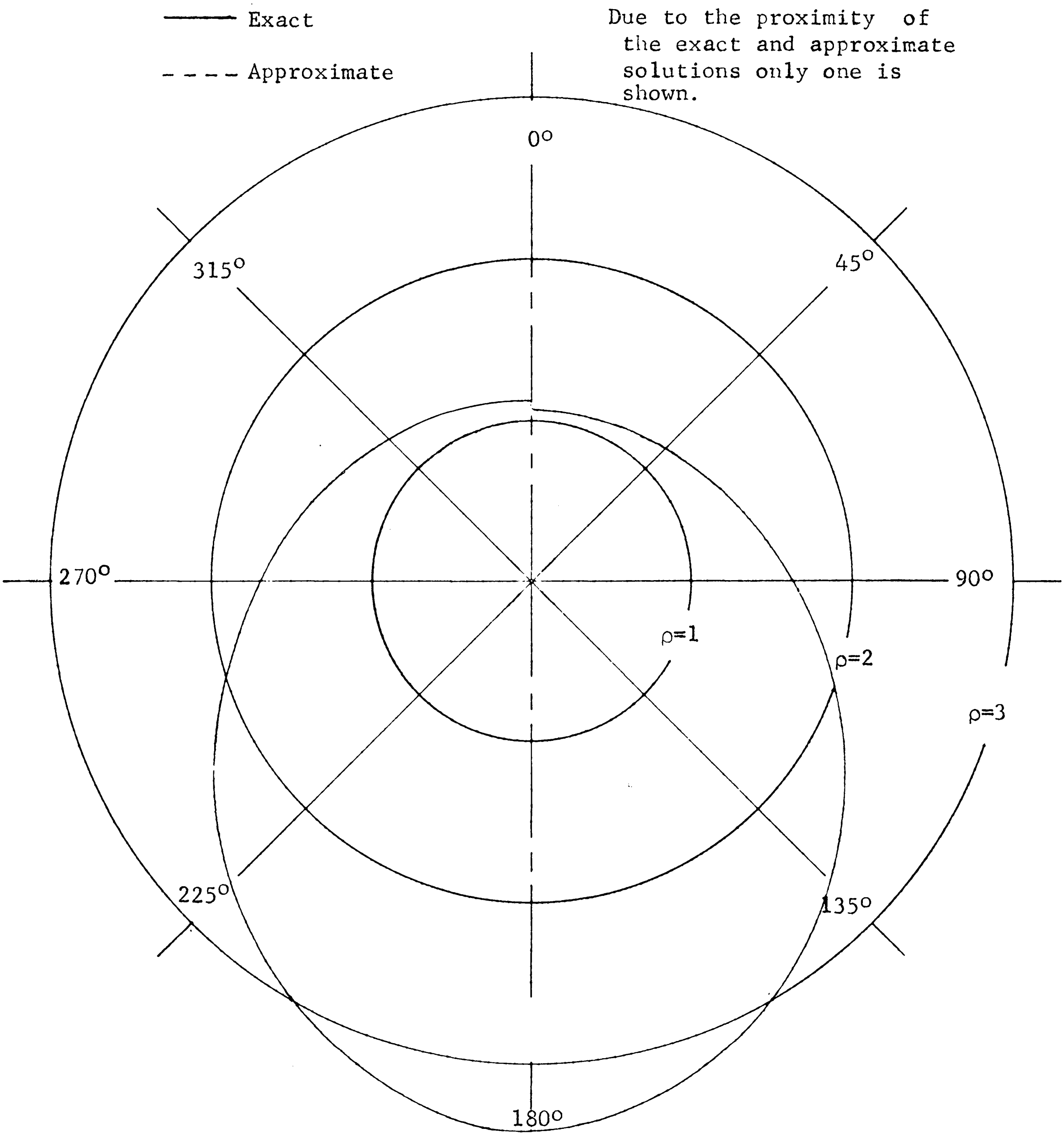
Eccentricity vs. p

FIGURE 14



Energy vs. Angular Momentum, Tangential Thrust

FIGURE 15

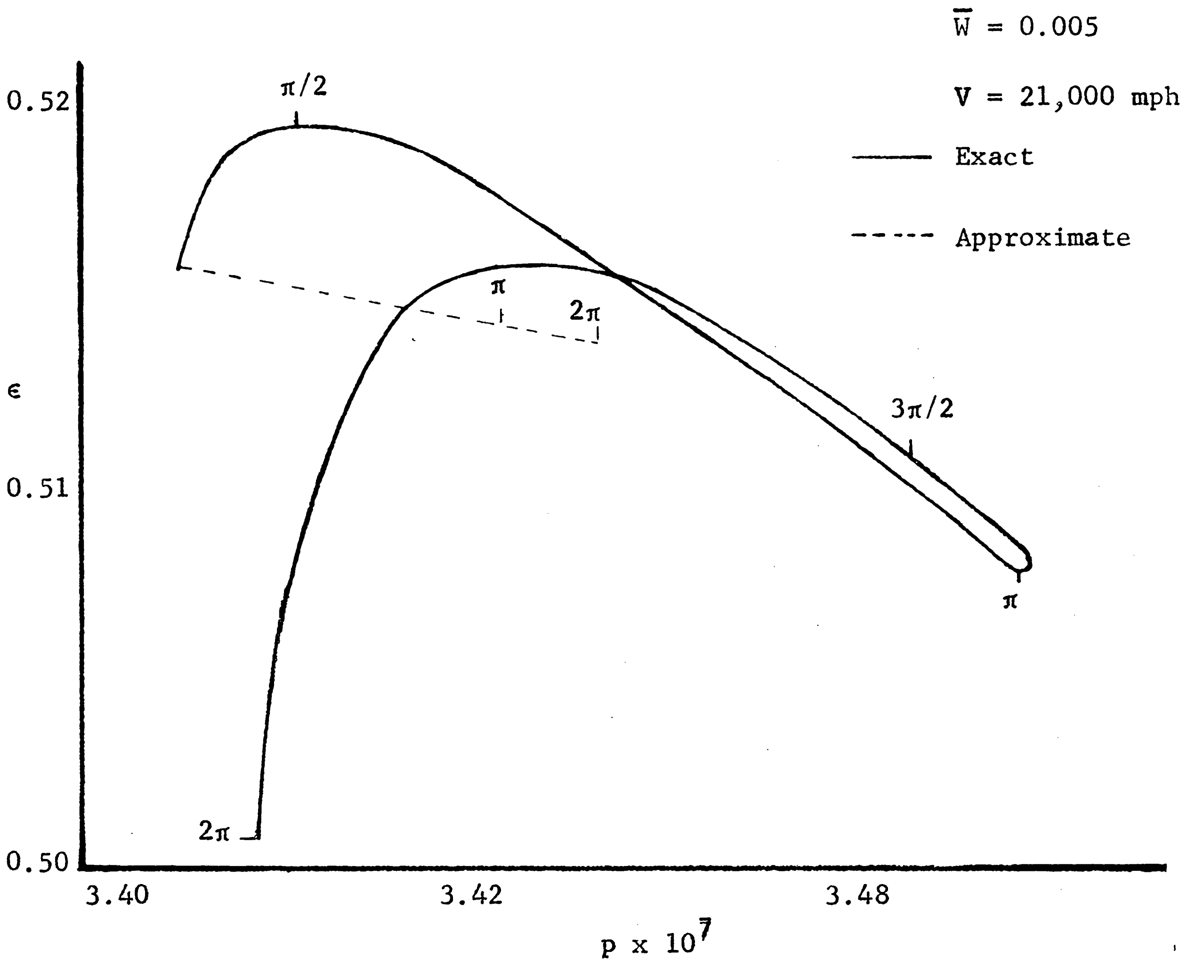


ρ vs. θ

$\overline{W} = 0.005$, $V = 21{,}000$ mph

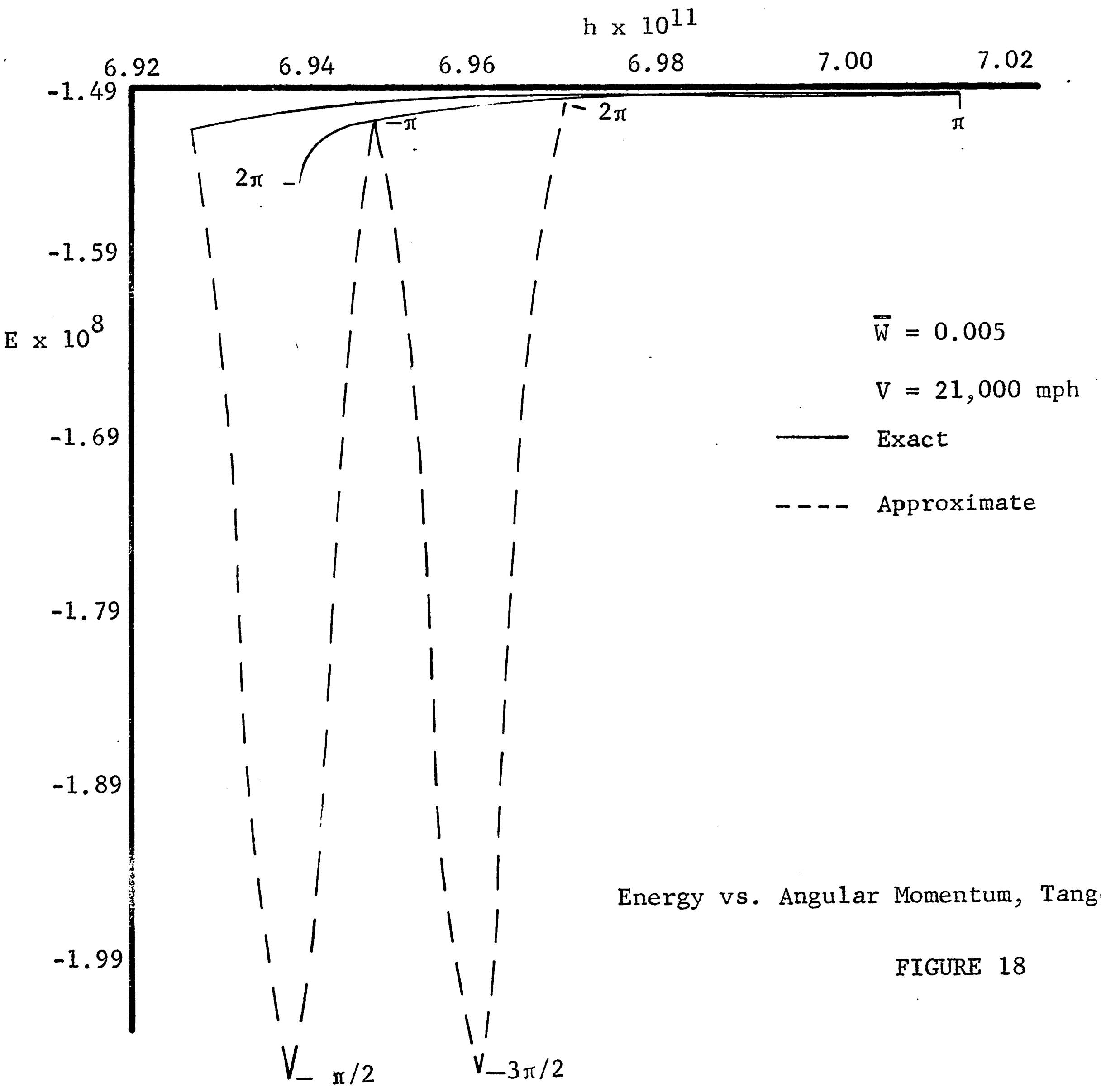
Tangential Thrust

FIGURE 16



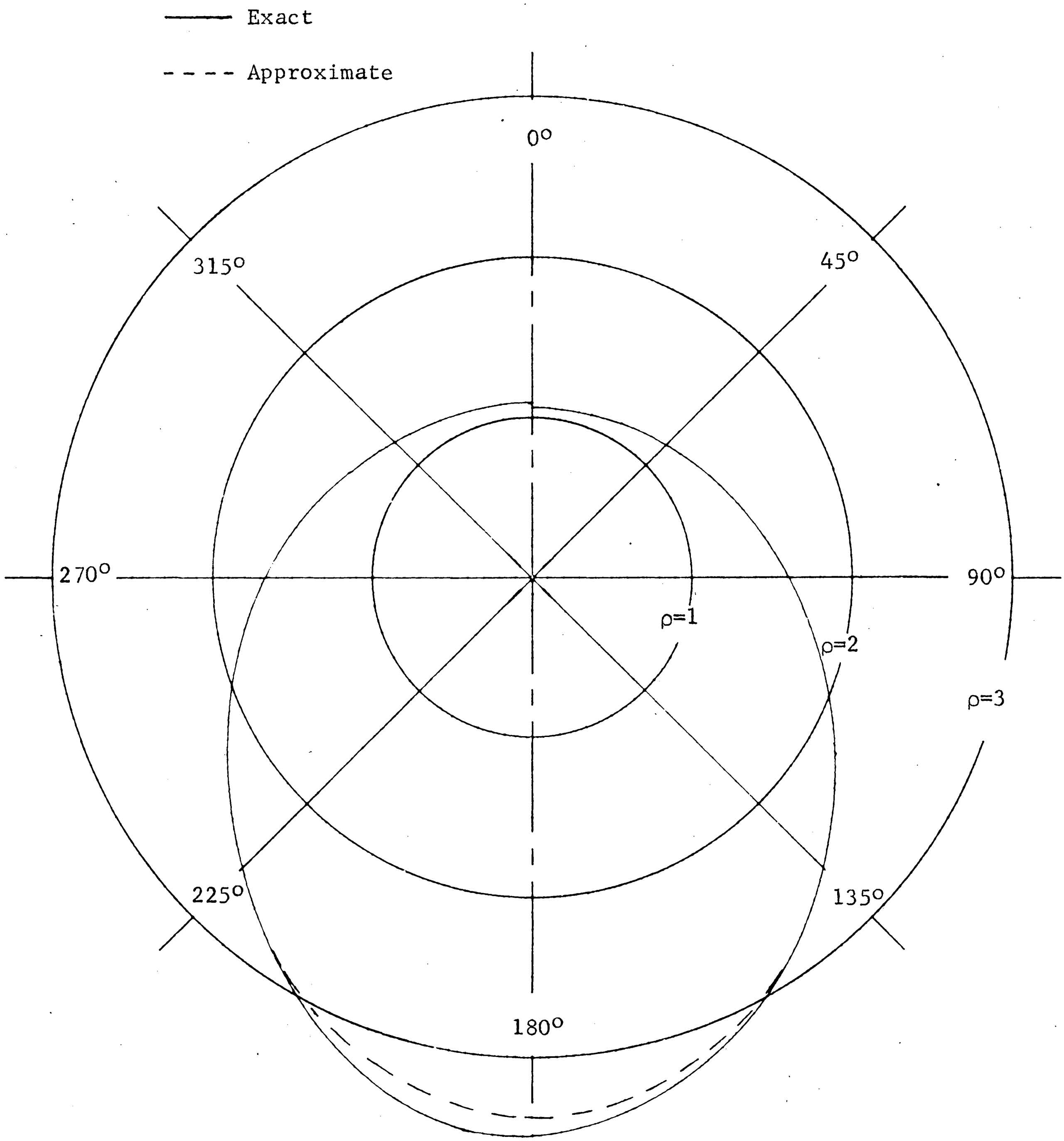
Eccentricity vs. p

FIGURE 17



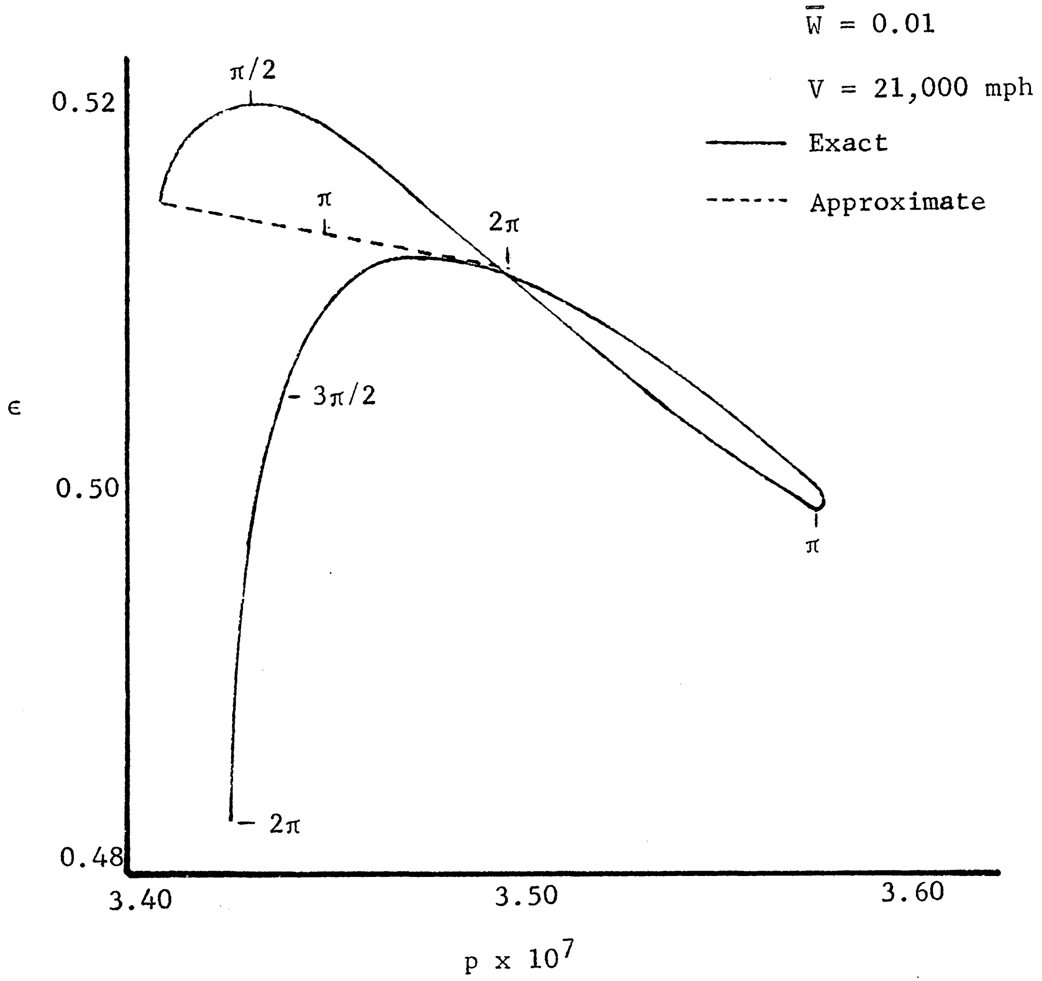
Energy vs. Angular Momentum, Tangential Thrust

FIGURE 18



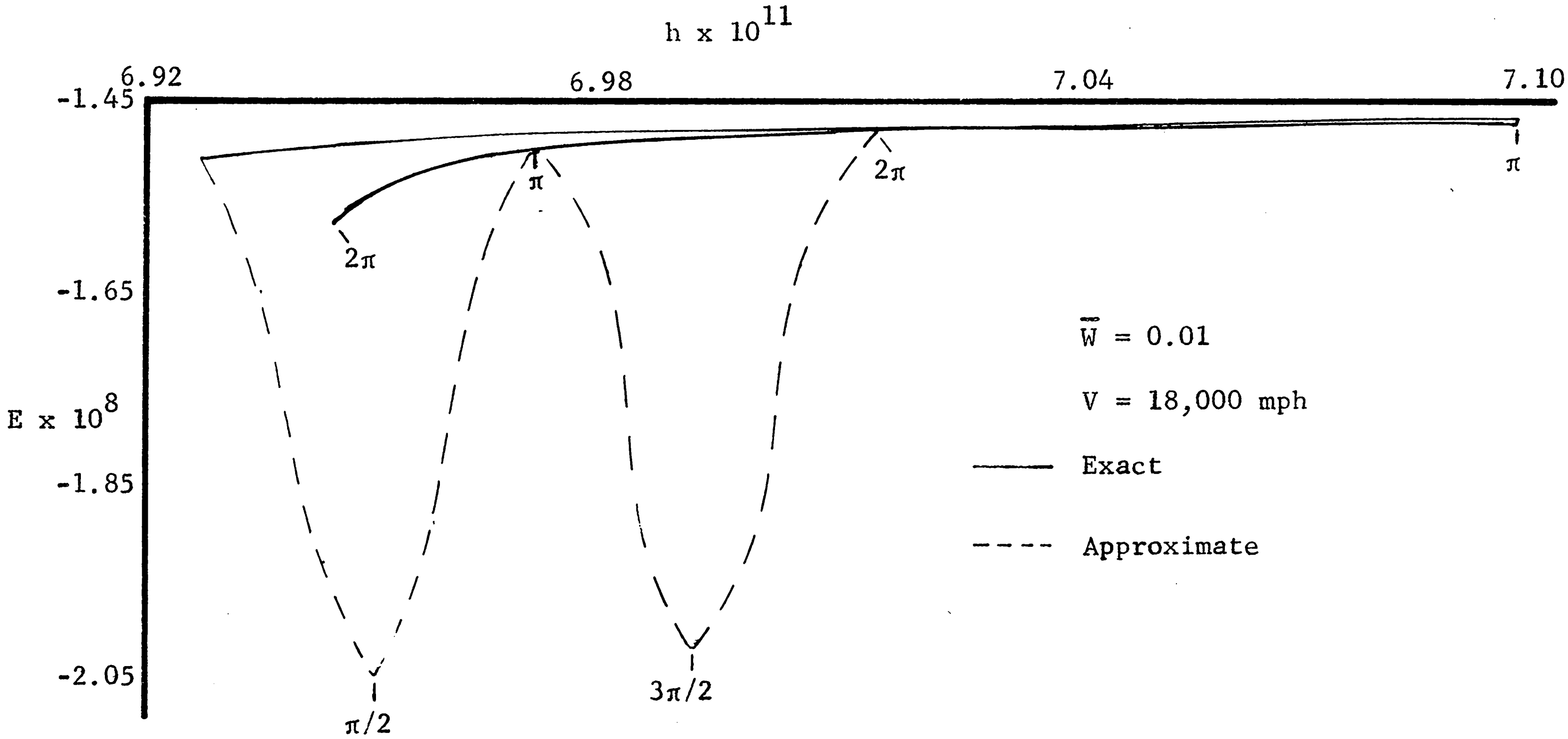
ρ vs. θ
$\overline{W} = 0.01$, V = 21,000 mph
Tangential Thrust

FIGURE 19



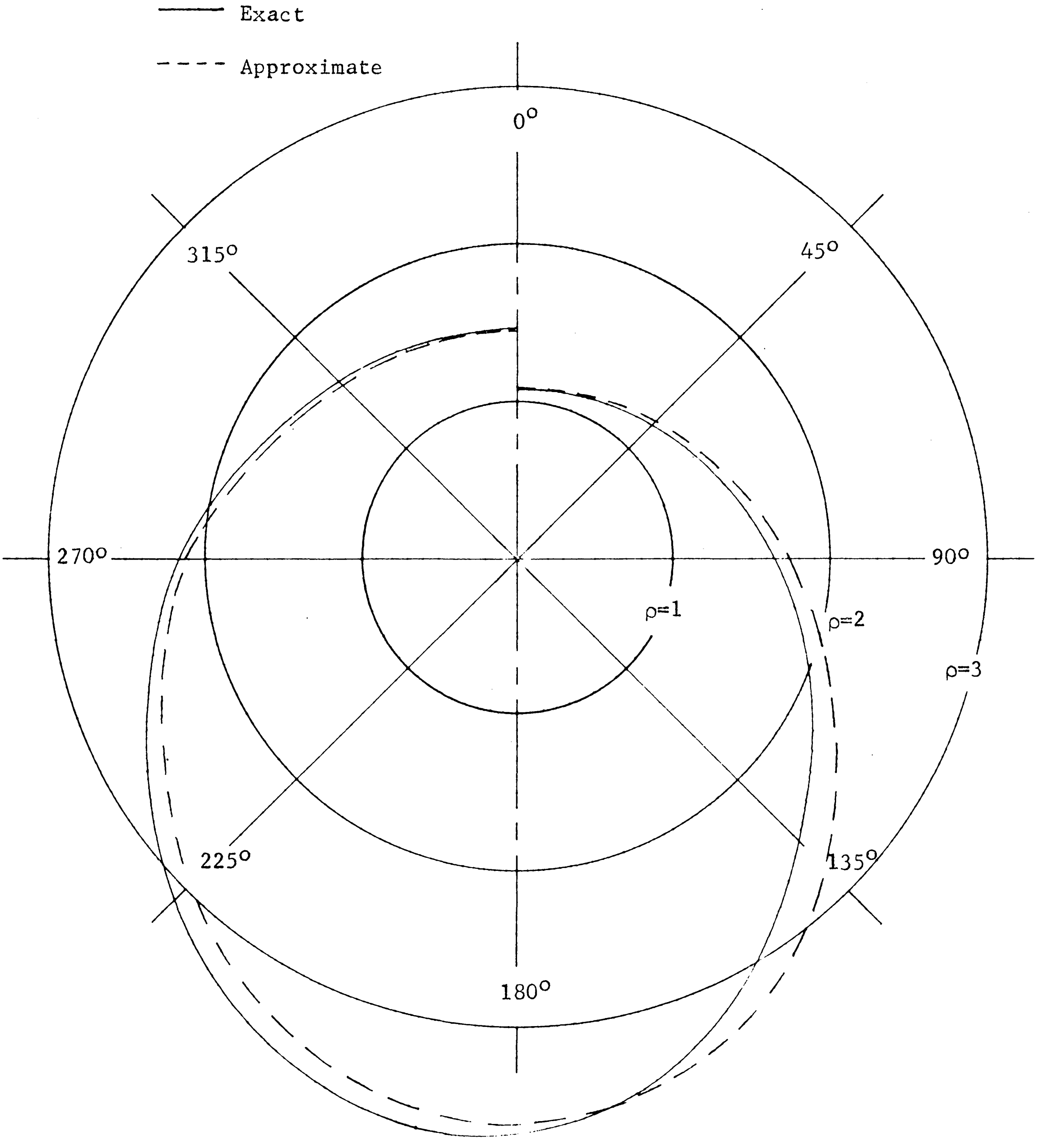
Eccentricity vs. p

FIGURE 20



Energy vs. Angular Momentum, Tangential Thrust

FIGURE 21

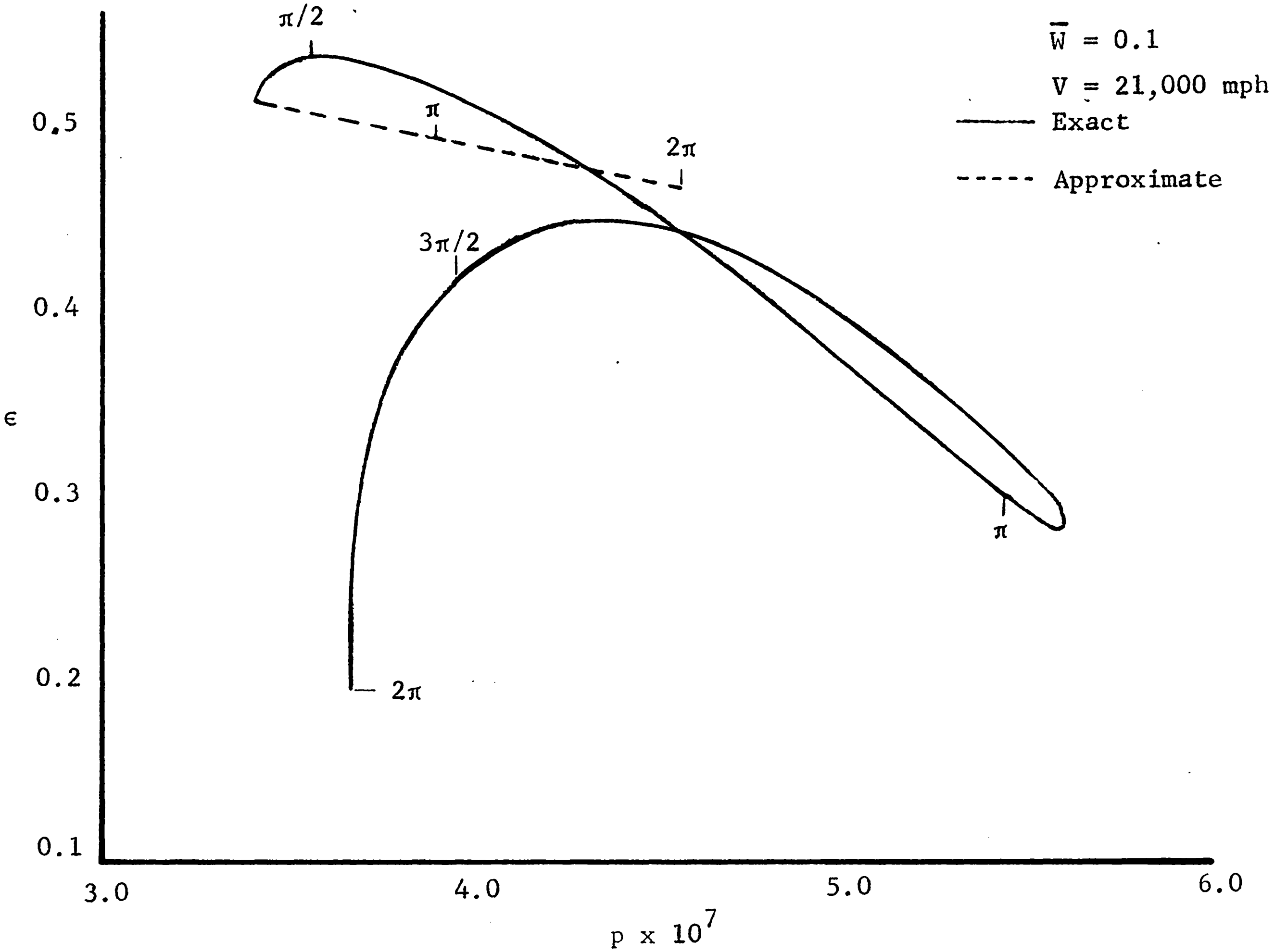


ρ vs. θ

$\overline{W}$ = 0.1, V = 21,000 mph

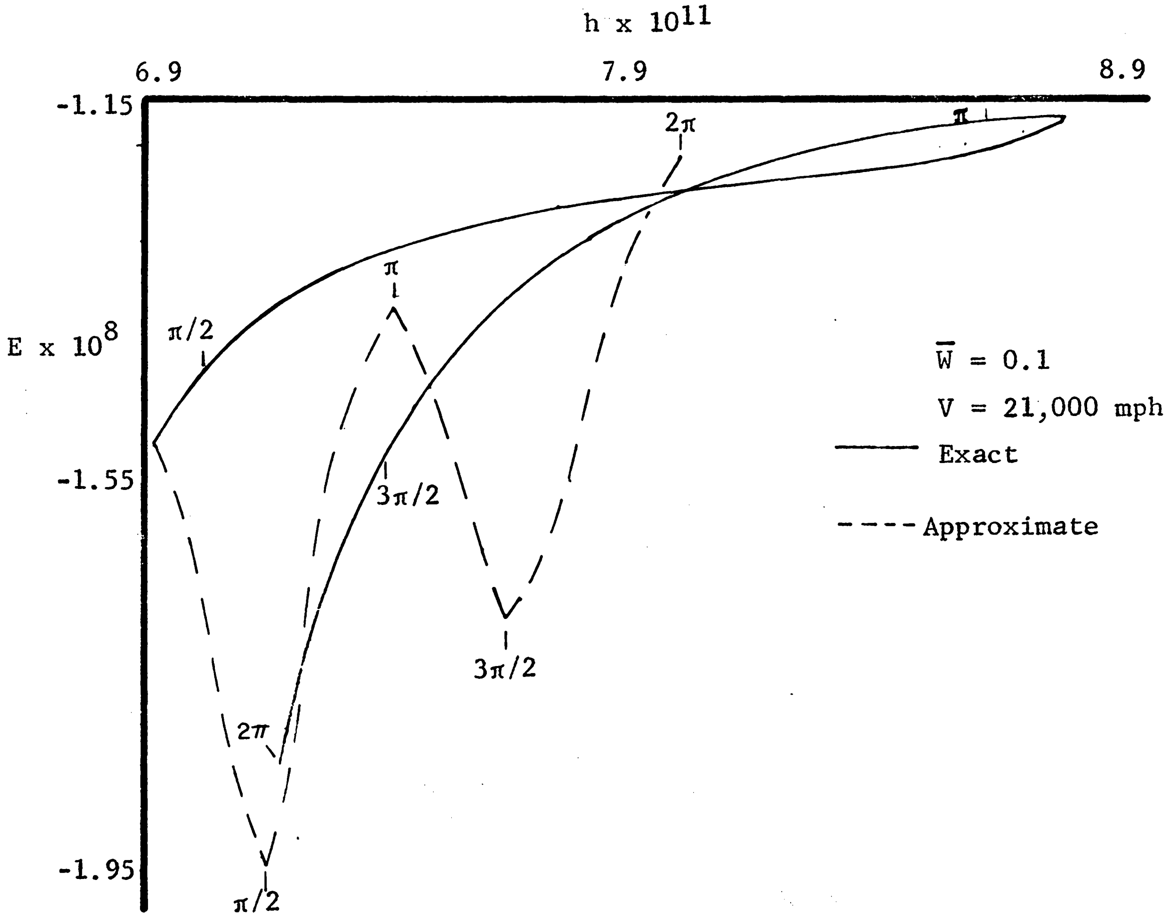
Tangential Thrust

FIGURE 22



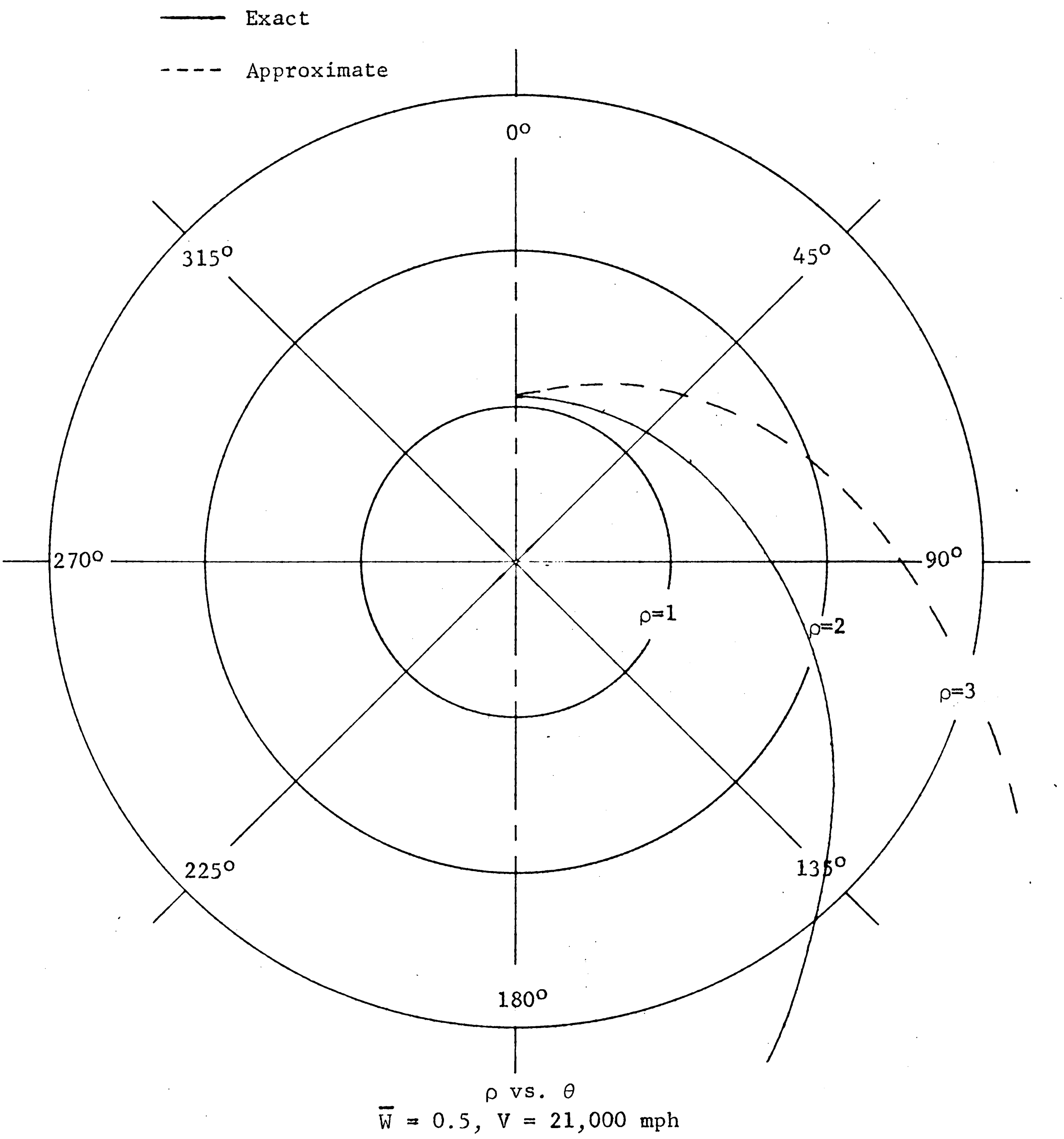
Eccentricity vs. p

FIGURE 23



Energy vs. Angular Momentum, Tangential Thrust

FIGURE 24

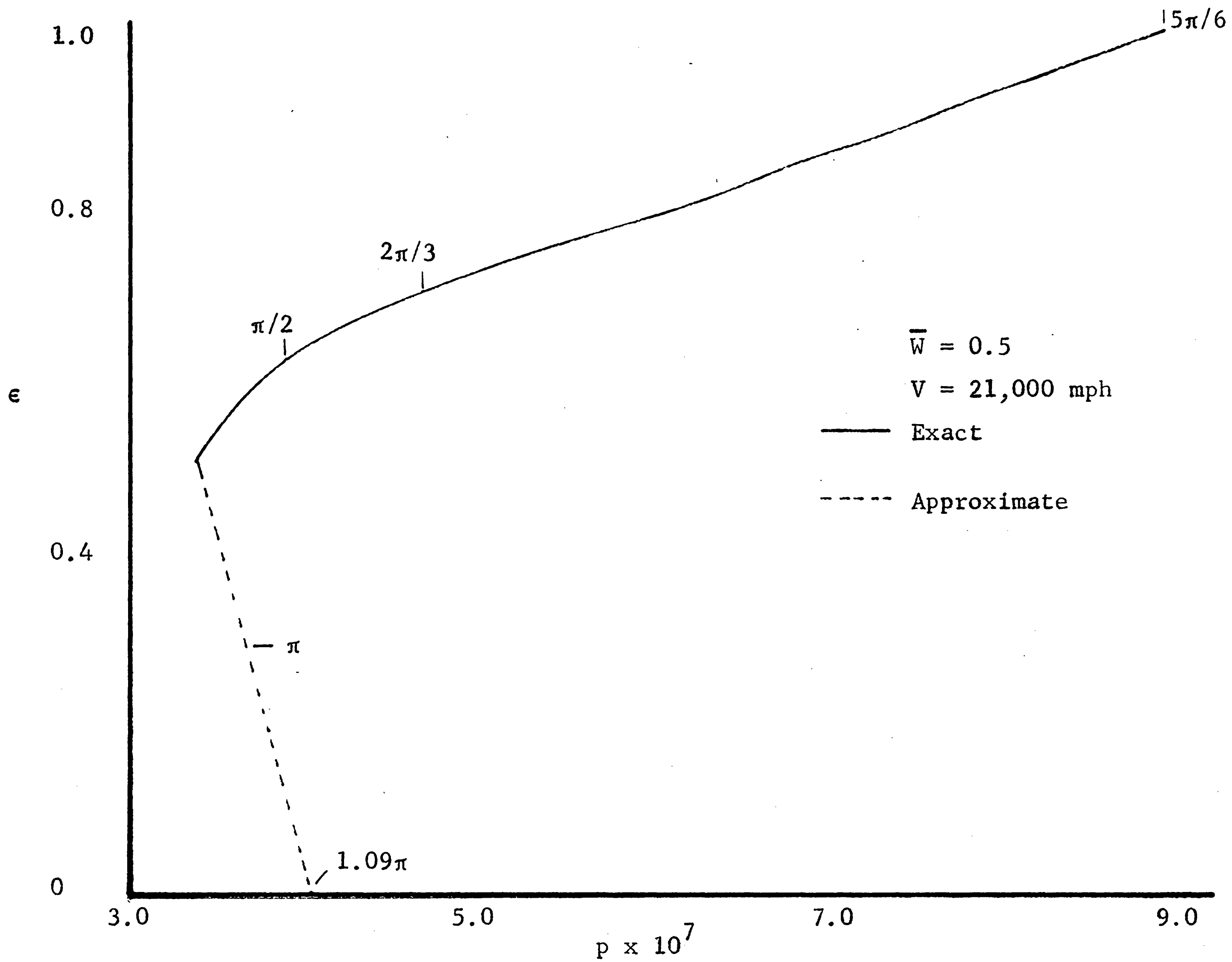


ρ vs. θ

$\overline{W}$ = 0.5, V = 21,000 mph

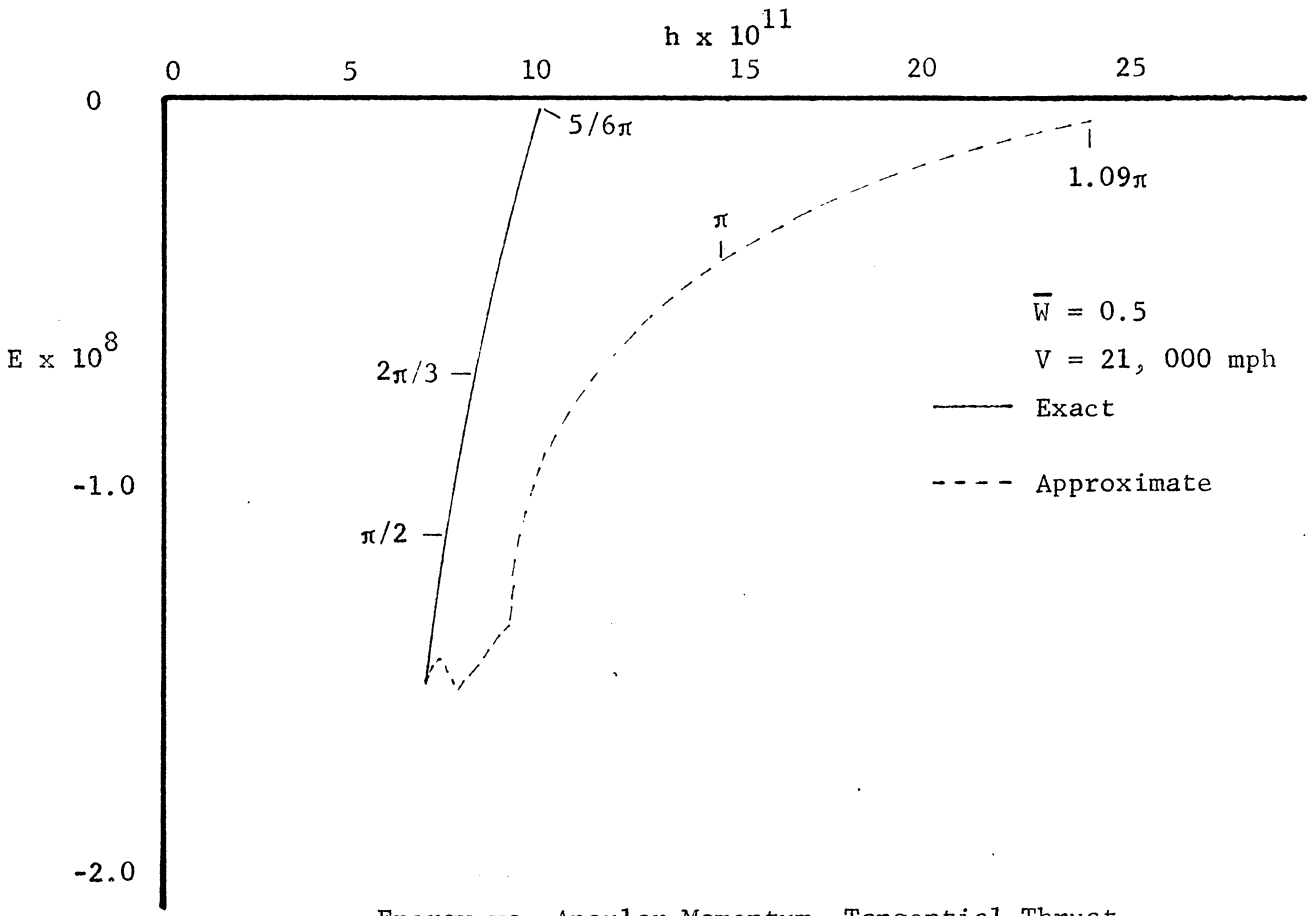
Tangential Thrust

FIGURE 25



Eccentricity vs. p

FIGURE 26



Energy vs. Angular Momentum, Tangential Thrust

FIGURE 27

A STUDY OF THE RANGE OF VALIDITY FOR THE METHOD OF KRYLOFF AND BOGOLIUBOFF AS APPLIED TO A SATELLITE IN MOTION WITH A SPECIFIED CONSTANT THRUST

by

Richard D. Johnson

ABSTRACT

The solution to the problem of a satellite with a small constant thrust and under the influence of a central force field presents difficulties due to equations non-linearity. An investigation was made to determine what range of values of the thrust parameter could be utilized to obtain a valid approximate solution for the case of tangential thrusting.

The investigation was accomplished by developing programs for the 1620 High Speed Digital Computer, since the solutions to the exact and approximate equations would otherwise be exceedingly laborious.

Under the assumption of no atmosphere, and neglecting the earth's oblateness, the study showed that the validity of the Kryloff and Bogoliuboff method was dependent on both the specific thrust and the vehicle speed. For a speed corresponding to an orbit of smaller eccentricity it was determined that the method of Kryloff and

Bogoliuboff remained valid for larger values of $\overline{W}$ than when V was a velocity corresponding to an orbit with a greater eccentricity.

The method of Kryloff and Bogoliuboff represents a practical approach to the solution of satellite motion both from the aspect of ease of application and reasonable calculation times.