AN ANALYSIS OF SOME ASPECTS OF POPULATION PROJECTION

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I. INTRODUCTION

1.1 Population problems

In his "Essay on the Principle of Population" Thomas R. Malthus (13) wrote:

> "I think I may fairly make two postulata. First, that food is necessary to the existence of man. Second, that the passion between the sexes is necessary, and will remain nearly in its present state, ... assuming, then, my postulata as granted, I say, that the power of population is infinitely greater than the power of the earth to produce subsistence for man."

The world population was estimated by the United Nations Statistical Office to have reached 2,097 million in 1959 and to be increasing at an average rate which yields an additional 50 million per year. The expected world population for the year 2000, at the present estimated rate of increase, is 6,267 million.(1) These estimates make explicit a situation which the people of the world are only beginning to comprehend. The world population is increasing by such vast amounts that it is impossible to believe present trends can continue for any great length of time. In commenting on some of its estimates the United Nations report reads as follows:

> "Never in the history of mankind have numbers of the human species multiplied as rapidly as in the present century, nor can it easily be conceived that the people of the earth will continue at a similar pace in the century which follows.... Even if it is

> conceded that population growth, after its peak near the end of our century, might diminish gradually and cease within another century, world population would not stop growing until it had reached between 10,000 and 25,000 million. One cannot say that such further growth is utterly impossible, but the vast changes in human organization required to sustain it hardly can be conceived at the present time."(2)

Of course the severity of the problems created by this rapid growth in world population depends upon the area of the world under consideration. The more highly modernized and industrialized countries of the world, including the United States, do not experience the extreme conditions or problems of such countries as China or India. These are experiencing some of the most severe population problems of the present time. The projected growth of the world population is so high, professional demographers can hardly believe their own estimates of what is likely to take place. Since the high fertility populations such as China and India have not yet become accustomed to the idea of family planning, demographers foresee no alternatives to such growth other than a rise in death rates caused by famine, war, malnutrition, and other diseases related to poverty present in such countries. This is the general context within which the future growth of the United States population may be expected to take place, although at a much slower rate and at a time much further in the future than for the high fertility populations.

The preface to a report issued by the Department of Economic and Social Affairs of the United Nations on the future growth of world population describes the situation in the following manner:

> "... while it took 200,000 years for the world's human population to reach 2500 million, it will now take a mere 30 years to add another 2000 million. With the present rate of increase it can be calculated that within 600 years the number of human beings on earth will be such that there will be only one square meter for each to live on. It goes without saying that this can never take place. Something will happen to prevent it ...."(1)

E. M. East, in the earlier part of the century, after conducting a careful study of the statistics available, concluded that it took about 2.5 acres of land to support an individual human being. He stated that there were about 13,000 million acres of land in the world available for agriculture and that of these some 5,000 million acres were then under cultivation. This yielded a maximum population of a little over 5000 million which the earth could support. (13) This is over 1 1/4 million less than the predicted 2000 population.

In 1933 it was estimated that there were approximately 630 million tillable acres in the United States.(23) This acreage would support a maximum of about 250 million persons if 2.5 acres are necessary to support an individual. With our standard of living this acreage would support less than 250 million. Thus if the population continues to increase at the present rate either the acreage tillable must increase, the standard of living decrease, or scientific discoveries produce more per acre.(23)

W. S. Thompson in Population: A Study in Malthusianism stated that, in the United States, either the present standard of living must be simplified or the present rate of increase of population must be lowered, and that probably both must take place in order that we continue having a really progressive civilization.(13)

1.2 The role of population projections

The above statements may be too harsh and overly exaggerated. As the realization of world population growth becomes more widespread and as the economic consequences become more acute, attitudes will change in such a manner as to retard this growth. Although economic, agricultural, scientific, and even political advancements may very well combine to greatly reduce the severity of the population problems, they cannot prevent these problems from occurring at least to some extent. However minor this extent may be, untold millions of persons will be affected by them. Certainly, it must be realized that the more densely populated areas of the world are presently experiencing the most severe population problem. Although the more materially and scientifically advanced nations are doing much to aid the starving, diseased inhabitants of these areas, it takes time and education to build the chain of understanding and acceptance of the methods of population control practiced by the more scientifically advanced nations. Until there is understanding and acceptance, these undernourished, poverty-stricken areas will continue to experience population problems.

The prime question for future civilization is whether the supply of means of subsistence from the earth can prove adequate to meet the basic needs of people and to support the complex requirements of modern culture and economy. The answer to this question will not become evident in days or in years but in a generation. To succeed, we must clearly understand the facts regarding existing and potential resources. Equally, the facts regarding populations, their growths and pressures, must be recognized and dealt with in a deliberate and rational manner.(12)

It is easily seen that a knowledge of population trends and dependable estimates of expected population increase can be of vast importance to anyone who, using such knowledge, will plan today to help insure a more progressive civilization for tomorrow. Thus, population projections can play an important role in preparing a better future for mankind.

It is on the more adequate knowledge of the processes of change in social, economic, political, and demographic fields that the chances for sustained advances in health and material welfare of the human race eventually may well depend.(11)

II. REVIEW OF LITERATURE

The nature of the materials used as principle sources of information should be established. In general, the dates of publication of the majority of the sources cited range from 1880 to 1940. Some recent material was found by Bogue and Benjamin, and the Census Bureau's Current Population Reports. So far as theory of population projection is concerned, there is little material to be found in any recent publications. The last empirical method suggested for population projection which received any favorable attention was the logistic of Pearl and Reed in 1920. Since 1920 the publications on projection concern the analytical methods of Whelpton and the Scripps Foundation, and other similar approaches. Thus, most projections contained in the tables of this work are based on periods up to 1890 or 1910 rather than being based on more recent data.

In many cases, the census counts will differ from source to source, depending on the time of publication. These discrepancies are due to certain revisions of old enumerations which had been questioned since their original release. Estimates of world populations also differ depending on the source. The older reference by the United Nations Population Division (20) and the more recent article by Benjamin (1) give different world population estimates for certain dates. In such instances the more recent data were used. The data given in the *Statistical Abstract of the United States* (1962) were used for the observed population values for the United States.

III. POPULATION GROWTH AND PROJECTION

3.1 Introduction

The tendency of the student of population study has been to explain population growth in terms of naturalistic, mechanistic, or economic factors. Literally hundreds of theories have been developed explaining population growth in terms of such phenomena as human fecundity, natural conditions, economic goods or standard of living, density in space, or various combinations of these and other factors of a comparable nature.

An excess of births over deaths determines growth of world population. This is the simple statistical answer, but this difference varies greatly with various periods of history and in various locations at any given time. The assumption that factors must exist which control the ratio between births and deaths has led to much painstaking research and to the formulation of many so-called laws of population growth. Several estimates of the world's population have been made but this is a task involving so much chance of error that the results are unreliable to a high degree.

Nations in the sphere of industrial civilization, having brought death somewhat under control, first experienced rapid population increase. In the United States there is still some natural increase, but the rate is diminishing. Landis (9) stated that since the net-reproduction rate was too small to replace the generation of his time, population decline might occur.

3.2 Theories on population growth

3.2.1 Naturalistic theories

These studies illustrate attempts to demonstrate "universal laws of population growth" derived from data. Pearl (13) and Gini (9) are fairly recent exponents of naturalistic theories of population growth.

Pearl's universal law of population growth is based on both biological and geographic assumptions and is demonstrated by statistical methods. Any population, he believes, begins its cycle by a slow increase, the rate of increase rising steadily until the midpoint of the cycle is reached, after which the rate decreases until the end of the cycle. The mathematical curve describing this natural and universal tendency of population growth is the logistic curve. Spatial density is supposedly the factor making populations follow this trend of growth. As the size of a population confined within definite spatial limits progresses along the logistic curve, population density automatically increases. Density is one factor in slowing down population growth as it is associated with a lower rate of reproduction. In explaining the curve, Pearl takes culture into account for he believes that stability of environmental and cultural conditions are essential to the normal development of the logistic population curve. A new economic system, such as an industrial revolution, will start a new cycle, or a new logistic curve, and so may interrupt an old cycle before its completion. The possibility of making any kind of population prediction according to the logistic law seems rather hopeless when this exception is considered. It is characteristic of cultures that they are constantly shifting, and change rather than stability is natural with them. One is obliged to begin a new logistic curve whenever the cultural environment brings about radical changes in reproductive behavior.

Willcox (9) stated that he did not believe in any simple law of population growth and criticized the employment of statistical curves and mathematical equations in predicting population growth. Fairchild (9) concluded that the latter portion of a population growth curve does not necessarily bear any consistent relationship to the earlier portion, simply because the behavior of man up to a certain time does not, in any mechanistic sense, predetermine future behavior.

Pearl arrived at his conclusion concerning the effect of density on population growth by studying the reproductive activity of fruit flies in bottles of varied sizes, of laying hens in pens of varied sizes, and of human beings in varying numbers of persons per acre. Upon giving place to the cultural influence in human populations, it seems rather doubtful that one can draw conclusions concerning growth trends as a product of density of humans per acre comparable to those for fruit flies in a jar, or for laying hens in a pen. Human beings in different areas of the land represent so many different cultural conditions that it would be hard to draw any reasonable generalization concerning the effect of spatial density on reproductivity.

Gini (9) approached population growth from a biological standpoint, believing that the parabola describes the reproductive tendencies of a population as well as of an individual family. Populations have their beginnings of rapid expansion, their maturity, and then old-age and decline. He refuted Malthus' theory of a geometrical increase in population and formulated a theory of cyclical rise and fall. He believes that a biological law controls the growth of population rather than a rise to its limits of subsistence. He cites as evidence many populations starting to decline before they reached such limits.

Gini's (9) explanation for the rapid growth of population, its tapering off, and its final decline is in terms of fluctuations in the reproductive energy of the human species. He reasons that during the first phase of the development of a population, the energy increases. Since reproductive energy is somewhat hereditary, Gini assumes that each succeeding generation will consist in a greater degree than its predecessor of individuals possessing this vigor. As the rate of reproductivity thus becomes higher, the population multiplies very rapidly. Later comes a "natural exhaustion" of reproductive powers. Eventually this exhaustion, brought about by "demographic metabolism," gains the upper hand.

Several critics, in direct contrast to Gini, have interpreted declining national birth rates in terms of the development of certain cultural patterns, which place other values above those of offspring.

Both Pearl's and Gini's theories seem to fail to explain adequately the growth of human population. Pearl assumes that density automatically reduces the capacity of the species to reproduce offspring. Gini assumes that competitive struggle of human beings in society automatically reduces their vital energy to the point where they cannot reproduce. Pearl appears to give too little place to motives which seem to determine reproduction. If one accepts the premise that, in most human cultures, there is much human reproductive power that remains unused, then human desire becomes the variable factor. Desire is conditioned by cultural values under which men live and one can, therefore, hardly explain trends in population growth by the effect of density on reproductive capacity when one is dealing with the human species. It also seems more in harmony with observations to explain the lack of offspring, resulting from Gini's theory, by psychological and cultural factors rather than by a lack of vital energy. More successful groups, it would seem, desire no offspring because offspring interfere with the attainment of other values which they consider more immediately desirable.(9)

3.2.2 Environmental-economic theories

Closely allied to the naturalistic theories are the population theories which grow directly out of varying conceptions of geographic influences on food supply and consequently on population growth. These environmental-economic theories represent variations of the Malthusian premise that mankind tends to increase more rapidly than his food supply.

Malthus (9) believed the means of subsistence tended to increase in arithmetical ratio; population to increase in geometrical ratio. Many of his basic premises are acceptable but one can hardly accept his ratio as a statement of a law of nature. Malthus' general premises were that

(1) the capacity of nature to reproduce plant life is

limited;

(2) the tendency of man is to reproduce vigorously;

(3) human population growth normally tends to outrun food supply;

(4) nature tends to cut off surplus population by positive checks -- war, pestilence, and vice;

(5) human beings might, by instituting a system of "preventive" checks reduce the birth rate, thus avoiding in part the serious consequences of the "positive" checks. These "preventive" checks were deferred marriage and celibacy.

In presenting this population dilemma -- that man checks numbers or nature will -- Malthus overlooked two important considerations. First, he underestimated the capacity of man for changing the limits of food supply by his ingenuity at invention. The limits to food supply set by nature for one culture have no meaning in another, and the limits set today may be overcome tomorrow in any culture by inventions. Secondly, he underestimated man's almost unlimited capacity to move both himself and his goods.

Malthus' premise that food supply is limited in nature cannot be denied, but it is just as certain that man has never found its limits. Animals have found such limits and perished, and so have local tribes of people, but man as a species never has found its limits and probably never will.

Possibilities for increasing food supply through invention within the vast limits of nature seem at present to be beyond exhaustion.

One may grant the fundamental geographic-economic assumption that all resources and goods are limited and even that a slow rate of increase in population would, given enough time, ultimately fill the earth with people; but having done so, one has no meaningful explanation for population growth in any particular area. In fact, what the land will produce is a highly variable factor and, in advanced civilizations, more variable than population growth itself.

More recent economic analyses are centered on "population pressure". The concept refers to the supposed tendency in nature for numbers to exceed food supply, thus creating scarcity. In strictly economic analyses, famine with starvation, or malnutrition, becomes the limiting factor in the determination of the maximum numbers a species may attain. In the final analysis, animals feed on plants or on other animals which feed on plants, which in turn depend upon the capacity of the soil, climate, and the whole complex of natural forces.

The fallacy of many economic arguments when applied to population is that they presuppose what would happen in population behavior under minimum conditions of survival, and then assume that man always acts as if he were facing these conditions. Economic analyses which are concerned with the system of economic institutions that affect "food supply" and its distribution, as well as with food as a product of nature, are of a different order.(9)

3.2.3 The sociocultural approach

The explanations of population growth so far considered disregard a number of factors in human experience. These are much more immediately deterministic, as far as birth- and death-rates are concerned, in highly advanced civilizations. Birth and death are controlled in large part not by natural or mechanistic factors but rather by social and cultural factors.

Man's capacity to reproduce and his reproductive behavior are entirely different phenomena and must be analyzed on different levels. Also, due to the fact that customs are as varied as the tribes of mankind, one cannot assume that a given set of social conditions will always result in the same customs. Since sociocultural norms are variable, the scientist cannot safely take a long time perspective, but he must try to understand the phenomenon in the area in which he works at a given time. Thus in the realm of human population trends, prediction is of a rather dubious nature. In a rapidly changing culture the sociologist's predictions can be valid only for a very brief period of time.

A sociocultural approach makes possible a more realistic understanding of a population group, but it removes any easy possibility of formulating universal laws regarding behavior.(9)

3.3 World population growth and projection

3.3.1 World population trends

As stated in Chapter I, the United Nations Statistical Office estimated in 1960 that the world population had reached 2,907 million at the middle of 1959 and was increasing at an average rate of 1.72% per year. This means an annual addition to the world population of approximately 50 million. The nature of the problems created by such fantastic growth was also brought out in Chapter I. It was noted that the problems accompanying world population growth, however, are really the problems of the underdeveloped regions of the world, as is evident from Table 3.3.1.1 showing the annual average birth-, death-, and population growth-rates by regions of the world for the period 1950-1959. The more highly developed regions are only beginning to experience the problems now faced by these lesser developed regions of the world. In a country such as our own, the dilemma of the populations of China and India and other such overcrowded and undernourished nations can hardly be visualized.

Table 3.3.1.1 (1)

Population of the world, 1959; and annual average birth-, death-, and population growth-rates, by regions, 1950-1959[1].

Region	Population mid-year 1959 (millions)	Birth-rates	Death-rates	Population growth-rates
World	2907	36	19	1.7
Africa:				
North Africa	78	45	26	1.9
Tropical and South Africa	159	47	28	1.9
	237	46	27	1.9
America:				
Northern America	196	25[2]	9[2]	1.8[3]
Middle America	65	45[2]	18	2.7
South America	137	42	19	2.3
	398	34	14	2.1[3]
Asia:				
South-west Asia	74	46	21	2.5[3]
South-central Asia	546	44	26	1.8
South-east Asia	208	44	23	2.1
East Asia	794	39[2]	21[2]	1.8
	1622	42	23	1.8[3]
Europe:				
North and West Europe	141	18[2]	11[2]	0.7
Central Europe	137	19[2]	10[2]	0.8[3]
South Europe	145	21[2]	10[2]	0.9[3]
	423	19	10	0.8[3]
Oceania	16	25[2]	9[2]	2.4[3]
U.S.S.R.	211	25[2]	8[2]	1.7

[1] Birth and death-rates are per 1000 population and are annual average for 1950-1959.

[2] Weighted average of recorded rates.

[3] Combined effect of natural increase and migration.

By far the greatest factor accelerating the growth of population in underdeveloped parts of the world has been the decline in mortality as a result of widespread campaigns waged against diseases, and as a consequence of medical and economic intervention on the part of more developed countries, through the United Nations and its special agencies, as well as efforts made by the underdeveloped countries themselves. But those influences tending to reduce fertility which usually accompany advanced economic development have not penetrated these areas and their birth rates have not fallen; in fact they have risen slightly in some instances. In 1958 the United Nations estimated (1) that if these present trends continue to 1975 the birth-rate in Asia will be 43 per thousand while the death-rate will have fallen to 20 per thousand giving a natural increase of 2.3% per year. In Latin America the annual rate of increase will have risen to 2.8%. In Europe, on the other hand, where birth and death rates are already low, the rate of increase will be a mere 0.8%; and in Northern America it will only be 1.2%.

3.3.2 Future world population growth

Knowledge of the world population is limited to a rather brief span of time. Estimates have been projected back to over 300 years ago and although they are valuable in giving a broad approximate picture of the number of persons on the earth during a recent span of history, they are by no means accurate. Even at the present time there is no accurate way of knowing the number of people in the world, for in great areas no population count is ever taken. Thus it seems a lost cause to attempt to project the population of the world into the future since our estimates of past populations and even of the present-day population of the earth are unreliable. Nevertheless, the projections that have been made do give us a broad picture of the number of people in the world and the rate at which they are increasing.

Some of the most significant estimates of world population have been collected by Knibbs.(13) They range from 1,000 million by Riccioli for 1660, to 1,649 million by Knibbs for 1914. Pearl and Reed chose the seemingly most reliable figures from Knibbs' collection and fitted them to their augmented logistic curve of growth. Their relation is

$$P = 445.5 + \frac{1580.5}{1 + 5.342e^{-0.0243t}} . \qquad (3.3.2.1)$$

This gives the population, P, of the world at any future time, where t is time measured in decades. Nothing more will be said here about this equation since an explanation of the nature of the logistic is given in Chapter IV. The resulting equation gives the estimates appearing in Table 3.3.2.1.

Table 3.3.2.1

Estimates of world population by the logistic of Pearl and Reed.

(Population in millions)

Year	Population	Year	Population	Year	Population
Lower Asymptote	445	1870	1246	2020	1987
1650	453	1880	1341	2040	2002
1700	471	1890	1433	2060	2011
1720	487	1900	1520	2080	2017
1740	512	1910	1600	2100	2020
1760	550	1920	1671	Upper Asymptote	2026
1780	609	1930	1734		
1800	695	1940	1787		
1810	750	1950	1832		
1820	814	1960	1870		
1830	887	1970	1901		
1840	969	1980	1926		
1850	1057	1990	1947		
1860	1150	2000	1963		

The fit to the population estimates used by Pearl and Reed are not very good but it is remarkable that it is as good as it is, considering how world population estimates have to be made.

It is, of course, immediately obvious that the use of equation (3.3.2.1) can not be recommended for projection of world population beyond 1920, the time of Pearl and Reed, for the United Nations has made estimates of 2,406 million for 1950, in contrast to 1832 million in Table 3.3.2.1, presenting a difference of 574 million. Thus it is obvious that the estimates made by using equation (3.3.2.1) on a projection basis tend to be far too low since the United Nations estimate for 1950 was already in excess of the upper asymptote proposed by Pearl and Reed by as much as 380 million.

Table 3.3.2.2 gives some more recent estimates of world population from 1650 to the year 2000.(1, 19)

Table 3.3.2.2

Estimates of world population, 1650-2000.

Source	Year	Population (millions)
Willcox	1650	470
Willcox	1750	694
Willcox	1850	1091
Willcox	1900	1571
United Nations	1920	1834
United Nations	1930	2008
United Nations	1940	2216
United Nations	1950	2406
United Nations	1975	3828
United Nations	2000	6267

Table 3.3.2.3 gives the expected population growth by continents to the year 2000.(1)

Table 3.3.2.3

Expected world population growth in different regions, 1975-2000.

Region	Population (millions)		
	1959	1975	2000
World	2907	3828	6267
Africa	237	303	517
North America	196	240	312
Latin America	202	303	592
Asia	1622	2210	3870
Europe	423	476	568
Oceania	16	21	29
U.S.S.R.	211	275	379

The population of the world is thus expected to rise by 1975 by one-third of the existing total to over 3800 million, and by 2000 by more than twice its existing total to over 6200 million. It is the situation in Asia which invites urgent attention. Presently contributing 56%, or more than half of the world population, in 1975 Asia is expected to account for 57% and in 2000 over 61% of the people of the earth. In this monsoon region there are already high concentrations of people and a level of living desperately low by western standards. There is no evidence supporting hope of any immediate relief from the pressure of numbers. Difficulties elsewhere are rather dwarfed by the extremity of the situation in Asia, but they, nevertheless, do exist and are of a different character. In large parts of Latin America and Africa there is ample living space but full development and use of potential resources is retarded by lack of capital and by social and cultural obstacles to the effective use of existing or of much needed immigrant manpower. In Europe and North America population pressure is less acute and the need is for the most efficient use of existing resources and for the preservation of favorable terms of trade to preserve the advanced economic organization now in existence.

We have been speaking of the year 2000 which is only a short distance away. During this relatively brief time the pace of world growth is expected to increase unless major influences, of much greater proportions than yet exerted, emerge to alter this trend. Upon examining the experience of countries undergoing industrialization, it is indicated that the effect of urbanization and education in producing attitudes favoring smaller families may take a generation to yield significant changes in fertility. The rapid growth of the world's population cannot be slowed instantly.(1)

In conclusion, it might be said that there is little point in discussing the population of the world as a whole. The situation is far too heterogeneous to make it useful to do so, and the problems, although of world-wide importance, are the problems of particular peoples and particular areas. The population of the world in the year 2000 will depend very much on events between the present and that date. Even so, the size of the population is much less important than the process by which the population of that date is reached. (11)

3.4 Growth and projection of the United States population

There is information on the nature of population growth in the United States. Also, throughout Chapter IV, the United States is used as the principle model for the several described methods of population projection. Thus there seems to be little necessity in giving any very lengthy description here of either the growth trends or projections of the population of this country.

The people of the United States are standing on the threshold of what may be an era of almost runaway population growth. The generation of parents who, during the 1920's and 1930's, controlled fertility to a point below the replacement level has now completed its reproductive cycle. It has been replaced by younger generations whose members have been bearing many more babies each year and have also been enjoying lower death rates than prevailed when their parents were going through the reproductive cycle. As a consequence, the growth rate of the population has risen from the low of about 0.6% per year in 1932 to over 1.5% at present. In the present period, 1962 to 1965, the infants of this "baby boom" are attaining adulthood, marrying in large numbers, and having babies of their own. If this second generation of the baby boom maintains a fertility rate that is only the same as that of its parents, or even slightly lower, the numerical growth in population will dwarf any previous growth the nation has known. The momentum provided by the baby boom will be simply the starting point for a second spurt of growth.

Table 3.4.1 was proposed by Bogue (2) as a possible set of projections of the population of the United States through the year 2100. It carries the assumption that fertility will continue at the level of the period 1955-1957, that the level of mortality declines moderately, and that net migration averages 300,000 per year.

Table 3.4.1

Projections of United States population (1960-2100).

Year	Population (millions)
1950	150,697
1960	180,126
1970	213,810
1980	259,981
1990	312,338
2000	375,238
2025	593,627
2050	939,117
2075	1,485,684
2100	2,350,352

As of 1960 the United States had been in existence 170 years and had gained over 175 million persons. If it continues to grow at present rates, only about 50 additional years will be required to gain another 175 million. After that, a third set of 175 million will be produced in only 25 years, and a fourth set in about 12 years. This rapid numerical growth results, of course, from the continuous increase in the base population to which the vital rates apply; even a moderate rate of increase applied to a huge base gives a sizable amount of growth.

The interest of the general public in the population changes that lie ahead has become so widespread that the United States Bureau of Census has accepted as one of its special activities the analysis of probable and possible future developments, using the term "illustrative projections". This term is to warn the user that the components of population growth are much too dynamic, and much too likely to change without warning in the economic, political, and sociological aspects of the nation's situation, ever to be predicted with a high degree of assurance.

Whether or not the population grows rapidly will depend largely on the future course of birth rates. Death rates at the ages between birth and the end of the reproductive period have become extremely low, so low that even if they were reduced to zero, they would add only moderately to the population growth. Immigration from abroad is restricted to a small constant annual amount that accounts for only a very small amount of the population beyond 1960, if present trends continue. But the most important factor in determining future population trends is fertility, and this is a most difficult item to predict.

It has been said (2) that more than 90% of the people in the nation know how to limit the size of their families, and that almost 100% of the people would take such action if the necessity to do so arose. Such action could take the form of an almost universal practice of birth control that would restrict family size to the number of children desired. Other possibilities are the delay of marriage, or a practice by larger percentages of the population of not getting married at all. The fact that marriage- and birth-rates are responsive to changing economic conditions seems to indicate that the people of the United States would not knowingly reproduce themselves into a state of semi-starvation, or even endanger their present level of living. Almost at the first signs of such a development they would probably take steps to reduce fertility. Hence, one can easily visualize a situation involving a sharp and sudden drop in growth. Although the members of our population would differ widely as to the speed with which they would reduce their fertility and the extent of reduction, they nevertheless could produce such a sharp decrease of growth almost at will. This is the major difference between the implications of the "population bomb" in the United States and in most other rapidly growing countries.(2)

IV. METHODS OF POPULATION PROJECTION

4.1 Introduction

4.1.1 Classification of methods

Politicians, engineers, sociologists, economists, mathematicians, biologists, and others of greatly varied interests have made long-time forecasts of population. One finds widely different procedures for making such predictions. These divergent methods, however, actually fall into one of two general classes.

In the first method data available for the forecast is studied, usually graphically, then projected into the future in some free-hand fashion according to the best judgement of the projector. The second method uses the same type of data, and some formal functional expression relating population and time is assumed. The parameters for this function are obtained and future estimates of the population determined from the resulting relation.

The population, P, then, may be represented in terms of time, t, by a relation of the form

$$P = f(a, b, c, \ldots, t) \qquad (4.1.1.1)$$

where a, b, c, ..., represent the social, economic, and biological factors of population growth. Whenever known population values are taken at various times and extended into the future, either in terms of a free-hand curve or in terms of a mathematical relation, this functional idea is being used. In the case of the free-hand curve, the number of parameters is unstated and their values undetermined. Whenever a mathematical equation is used the assumed parameters, the part they play in expressing population growth, and the method by which their values are determined, all are brought into play.

There is no fundamental change in the problem if, instead of dealing with the population directly, one turns to the use of the number of births (B), the number of deaths (D), the number of immigrants (I), and the number of emigrants (E) in the area in question over a particular period of time Δt, and considers the expression

$$\frac{\Delta P}{\Delta t} = B - D + I - E \quad . \qquad (4.1.1.2)$$

This equation for the change of population (ΔP) in the time interval (Δt) is obviously exact. Thus if P_0 is the population at the beginning of the time interval Δt, then the population at the end of the interval, or P, will be

$$P = P_0 + \frac{\Delta P}{\Delta t} = P_0 + (B - D + I - E) \quad . \qquad (4.1.1.3)$$

Equation (4.1.1.3) can be used instead of (4.1.1.1) in making population predictions. However, in forecasting by means of this equation, one must project four functional forms into the future instead of one, but the methods of doing so necessarily fall into one of the previously mentioned classes. Once the predictions of B, D, I, and E have been made one may turn the results from the form of equation (4.1.1.3) back to that of equation (4.1.1.1) regardless of the method by which they were obtained.(17)

4.1.2 The nature of a projection

Several early approaches to population projections were, implicitly or explicitly, mathematical in nature. The mathematical expression fitted to the observed counts of a population may be projected into the future. The assumption that the social and economic forces that have molded population growth in the past will continue into the future is also imposed.(19) More will be said of these assumptions later.

For a population of reasonable size, such as a country or large community, it is possible to determine an empirical equation, by ordinary methods of curve fitting, which will describe the normal rate of population growth. Such a determination will not necessarily give any idea as to the underlying organic laws of population growth for a particular population. It will simply give a rather exact empirical statement of the nature of the changes which have occurred in the past. No process of empirically graduating given data with a curve can in and of itself demonstrate the fundamental law which causes the occurring change.(14) This should be kept in mind when examining methods of projection and their applications.

4.2 Early predictions

4.2.1 Straight line and exponential forms

Most of the early workers treated the problem of prediction or projection directly in terms of the formulation of equation (4.1.1.1). By making use of a few known values of the population in question, they made estimates of the population for future times. Undoubtedly many people in the past have carried out this idea in terms of free-hand curves, but the majority of those whose writings remain used formal functional expressions. The favorite mathematical forms used were the straight line,

$$P = a + bt \tag{4.2.1.1}$$

which postulates uniform increase of population with time, and the exponential form,

$$P = e^{a + bt} \tag{4.2.1.2}$$

which assumes a constant percentage increase of population with time. Of these the most common was (4.2.1.2). Since there are only two parameters in each of these forms, it is possible to make predictions in terms of <u>parameters</u> estimated from only two known values. It was a frequent practice to apply one of the above two equations in this way.

Both of these expressions have the unreasonable characteristic that the population will ultimately become infinite. Long-time predictions based on the straight line go astray because of the feature of a constant absolute increase of population for unit increase in time. The exponential form recognizes the necessity for working in terms of relative rates but the feature of constant relative rate is in conflict with observations. These indicate that over long-time periods the relative rate of population growth declines.

Even as early as 1815 Watson (17), using the three census counts available for the United States and the idea of a constant percentage rate of increase, made estimates of the population for each decade up to 1900. For a time the population remained fairly close to his values, but gradually fell below them until in 1900 his estimate exceeded the actual population by 33%.

4.2.2 Other early predictions

At the time of the census of 1850 considerable interest was shown in the matter of the future population of the United States. The census volumes for that year summarize the work of various men on this subject. Darby (17) showed that the rate of increase of 3% per year applied as a compounding rate to the population of 1790 and gave values that agreed very well with the census counts up through 1850. The

Bureau of the Census continued these predictions, using the same rate of increase, for each individual year up to 1901. Their estimate for 1900 was 98.6 million which exceeded the actual count by about 30%.

The census volumes for 1850 also made a variety of predictions based on various modifications of the idea of a constant percentage rate of increase. Some of these estimates were carried through to 1950 and for that year range in their values from 49 million to 497 million. None of these estimates were satisfactory to the Census Bureau, for they recognized that the 60 years of observations available indicated a decline in the percentage rate of increase. They allowed for this decline by making use of a constant decennial rate of 26.95% up through 1890, and then arbitrarily changing to a much lower rate thereafter. They estimated the population of the United States to be 70 million in 1900 and 125 million in 1950. These estimates are low because the percentage rates of increase used were lower than they should have been for the periods covered.

In dealing with the declining percentage rate of increase, Tucker (17) observed that with increasing density the percentage rate of increase would decline by approximately one per-cent per decade. Assuming that migration would continue to increase as it had in the few decades prior to 1850, he estimated for 1900 a population of 80 million. Assuming migration would remain constant at the 1850 level his 1900 estimate was 74 million. The actual population in 1900 was 76 million. Thus his orderly treatment of the principle of declining percentages allowed him to make a fairly accurate prediction for 50 years into the future.

A graphical treatment of this same idea was given in 1909 by Woodruff.(17) He plotted the percentage decennial increases and drew a curve continuing their decrease into the future under the idea that the population would be nearly stationary by the year 2000. His percentage decennial increase for the decade 1990-2000 was 2.6% whereas his last observed percentage for the decade 1890-1900 was 20.6%. His population prediction over a 30 year interval was low. He estimated for 1930 a population of 115 million whereas the observed population was 123 million.

4.3 The Malthusian Law

4.3.1 Development and characteristics

One may express the increase of population, dP, within a small interval of time, dt, as a function of the size of the population, i.e.,

$$\frac{dP}{dt} = \phi (P) \tag{4.3.1.1}$$

Suppose $\phi (P) = rP$ where r is the rate of increase per capita; then

$$\frac{dP}{dt} = r\,P \qquad (4.3.1.2)$$

or $$\frac{1}{P}\frac{dP}{dt} = r$$, so that,

$$\frac{dP}{P} = r\,d\,t$$ and, integrating produces

$$\log_e P = r\,t + \log_e C$$

where $\log_e C$ is a combined constant of integration. From the latter equation we have

$$\log_e \left(\frac{P}{C}\right) = r\,t$$, or,

$$\frac{P}{C} = e^{rt}$$ giving

$$P = C\,e^{rt} \quad . \qquad (4.3.1.3)$$

According to (4.3.1.3), population increases in geometric progression. Population growth by geometric increase is frequently termed the Malthusian Law of growth, after Thomas Malthus (1776-1834) who held that man tends to increase faster than his means of subsistence.(19)

Few empirical expressions used to fit census data have been dignified with the name of "laws" of growth and few have been considered to have any rational basis. This is not true of the Malthusian Law. This law may be rationalized by pointing out that under constant conditions the birth and death rates should remain constant and consequently, if we disregard migration rates, so should the survival rate, r, which is their difference.

If the constant C is eliminated from $P = C\ e^{rt}$ by differentiation, the differential equation $\frac{dP}{dt} = r\ P$ results, and its integration reintroduces the constant C or its equivalent. The constant C is determined for any particular population by using the observed value of P, say P_0, at a given time, t_0, and noting that $P_0 = C\ e^{rt_0}$ must hold, so that $C = P_0\ e^{-rt_0}$ and $P = P_0\ e^{r(t-t_0)}$. It is possible also to eliminate the constant r by a second differentiation to obtain the differential equation

$$\frac{d}{dt}\left(\frac{d\ \log_e P}{dt}\right) = 0 \quad \text{or} \quad \frac{d^2P}{dt^2} - \frac{1}{P}\left(\frac{dP}{dt}\right)^2 = 0 \ . \tag{4.3.1.4}$$

To determine the constants which arise, one needs to know from observation not only the value P_0 of P at the time t_0 but also the value of dP/dt at that time -- or some equivalent conditions, such as the values of P at two specified times. It may be preferred, however, to determine both constants by some method of curve fitting which uses all the observed values of P and t available.

In most problems involving natural laws, one does not seek to eliminate all the constants or parameters which enter into the finite equation. One divides the constants into two categories, a group of disposable constants which will be determined empirically from the observations and another group of natural constants which refer to intrinsic properties of the system studied. In the Malthusian Law, $P = C\,e^{rt}$, a change of the origin of time, which is arbitrary changes the value of C. However, r could well be regarded as a natural constant, namely, the natural survival rate of a species to be determined once and for all and not as a constant to be fitted in each individual case. On this hypothesis we should not proceed to the differential equation (4.3.1.4).(23)

4.3.2 Objections to the Malthusian Law

The law is irrational in two respects:

(1) with indefinite increase of time the population increases indefinitely, which is impossible; and

(2) although two distinct subpopulations, being enumerations, must add to form the total population, $P_1 = C_1\,e^{r_1 t}$ and $P_2 = C_2\,e^{r_2 t}$ do not add to give an expression of the form $P = C\,e^{rt}$ unless r_1 and r_2 are equal.

Malthus himself emphasized the first of these two difficulties. Actually, it is unfitting to call the exponential Malthusian when Malthus' chief thesis was that it could not be followed, but the usage seems established.(23)

There were certain neo-Malthusians such as Bonynge (7) whose estimates made in 1852 did not differ from subsequent census enumerations by more than 3% until 1910, and Gannett (7) who, in 1909, made estimations which were surprisingly close to the next few enumerations. Then estimates were all wide of the mark, as they assumed a much larger increase in population than actually took place. Evidence is abundant to illustrate that the slower than expected increase of population did not result from population increase outrunning means of subsistence as proposed by Malthus.

4.4 Parabolic methods

4.4.1 Pritchett's third degree parabola

4.4.1.1 Development and fitting

In 1891 A. S. Pritchett (15,16) totally dissatisfied with the Malthusian Law, tackled the problem of deriving an equation which would represent the population growth in the United States during the 100 year period from 1790 to 1890 and which might be used to predict growth through future decades. An equation which would fit the growth from 1790 to 1890 would, in his time, form the most probable basis for predicting the population of the future.

The results of the enumerations of the census for the years 1790 to 1890 appear in Table 4.4.1.1. Preliminary examination showed these values could be approximately.

Table 4.4.1.1.1

Census counts of U. S. population for 1790 - 1890

Year	Population	Year	Population
1790	3,929,214	1850	23,191,876
1800	5,308,483	1860	31,443,321
1810	7,239,881	1870	38,558,371
1820	9,633,822	1880	50,155,783
1830	12,866,020	1890	62,622,280
1840	17,069,453		

represented by a "parabola," and would be closely represented by an equation of the form

$$P = A + Bt + Ct^2 + Dt^3 \qquad (4.4.1.1.1)$$

where P represents population, t represents time in decades from some assumed year, and A, B, C, and D are constants to be determined.

It should be noted here that the term "parabola" does not have the same connotation today as it did during Pritchett's time. The portion of the graph of the cubic equation (4.4.1.1.1) which appears when the equation is applied to population growth is, however, parabolic in form. I will henceforth apply the term "cubic" instead of "parabola" as used by Pritchett.

If we express the population in millions and fractions of a million, and t in decades counting from 1840, the observations furnish the following 11 equations of condition for determining the constants A, B, C, and D:

Equations of Condition	v.
$A - 5B + 25C - 125D - 3.9292 = 0$	+0.083
$A - 4B + 16C - 64D - 5.3085 = 0$	-0.041
$A - 3B + 9C - 27D - 7.2399 = 0$	-0.181
$A - 2B + 4C - 8D - 9.6338 = 0$	-0.065
$A - B + C - D - 12.8660 = 0$	+0.119
$A - 17.0695 = 0$	+0.415
$A + B + C + D - 23.1919 = 0$	+0.058
$A + 2B + 4C + 8D - 31.4433 = 0$	-0.975
$A + 3B + 9C + 27D - 38.5584 = 0$	+0.754
$A + 4B + 16C + 64D - 50.1588 = 0$	-0.181
$A + 5B + 25C + 125D - 62.6222 = 0$	+0.012

(4.4.1.1.2)

The column headed "v." in equations (4.4.1.1.2) consists of the values obtained by substituting the solutions for A, B, C, and D into each equation of condition. These values determine how closely the resulting equation fits the observed population values.

Solving equations (4.4.1.1.2) by least squares yields the following normal equations:

$$\begin{aligned}
11A + 0B + 110C + 0D - 262.017 &= 0 \\
0A + 110B + 0C + 1958D - 620.753 &= 0 \\
110A + 0B + 1958C + 0D - 3163.765 &= 0 \\
0A + 1958B + 0C + 41030D - 11237.254 &= 0
\end{aligned}$$

From the normal equations one finds:

$A = 17.4841$; $B = 5.1019$; $C = 0.6336$; $D = 0.0304$.

The population, P, at any decade t after 1840 will then be given by the equation

$$P = 17.4841 + 5.1019t + 0.6336t^2 + 0.0304t^3 \qquad (4.4.1.1.3)$$

This equation is evidently not what might be called a normal or natural population curve. It has no asymptotes and P becomes zero for a value of t equal to about -8.8, corresponding to the year 1752. For larger negative values of t, P becomes negative. This, however, can be expected from the data used, since the populations there given are not the result of a slow natural growth from an original small beginning, but are largely the result of influences from outside.

How accurately formula (4.4.1.1.3) represents the observed values of the population will be seen from the graphical representation in Figure 4.4.1.1.1. The observed values of the population for each decade are represented by the black dots, and the black-line curve is furnished by formula (4.4.1.1.3). With the exception of the values for 1860 and 1870, the curve fits the observations with considerable exactness.

The population as given by the Census Bureau and as determined by formula (4.4.1.1.3) appear in Table 4.4.1.1.2.

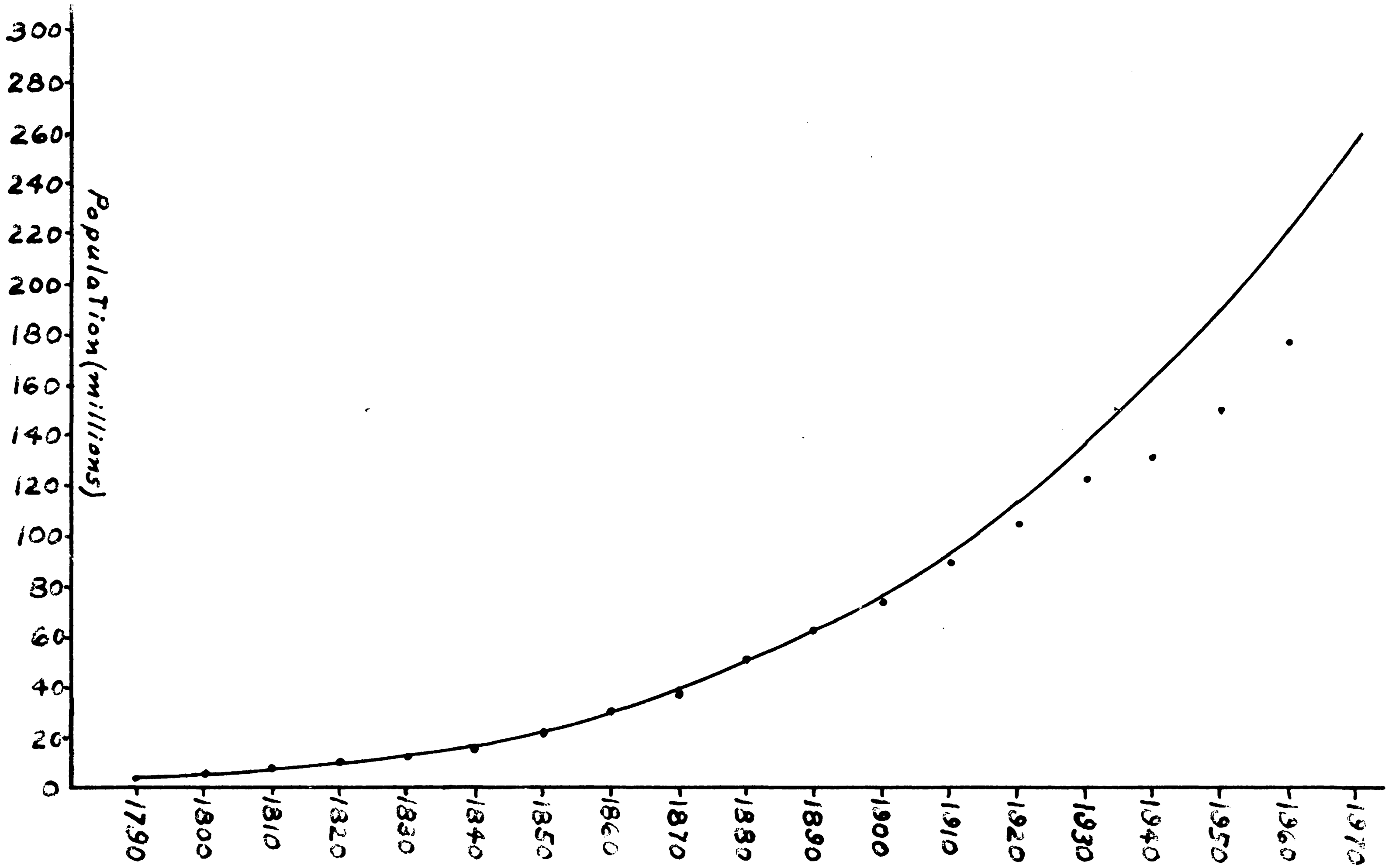

Fig. 4.4.1.1.1

Growth curve for United States according to Pritchett's cubic, fitted 1790 - 1890. Black dots represent observed population.

Table 4.4.1.1.2

Populations of the United States (1790 - 1890) by census counts and by formula (4.4.1.1.3). All values are in thousands.

Year	Observed Population	Computed Population	Discrepancy
1790	3,929	4,012	+ 83
1800	5,308	5,267	- 41
1810	7,240	7,059	-181
1820	9,634	9,569	- 65
1830	12,866	12,985	+119
1840	17,069	17,484	+415
1850	23,192	23,250	+ 58
1860	31,443	30,468	-975
1870	38,558	39,312	+754
1880	50,156	49,975	-181
1890	62,622	62,634	+ 12

The explanation of the larger discrepancies is easily found. The effect of the Civil War upon population growth would show itself in the census of 1870 and succeeding years. This effect would be to give a value of the population in 1870 much below that which would be expected. This is precisely the case, since the census count for that year falls 754,000 below the computed value. The abnormally small value for 1870, of course, had its effect upon the population of succeeding decades, and would give an apparent difference of opposite sign to the observed population in 1860. There is, however, good reason to believe that the value of the population as determined by the census in 1870 is much smaller than the actual population, and there can be little question that the computed value is much nearer the truth than the census count of that date. The Superintendent of the Census at that time, Mr. R. P. Porter, stated: "There is but little question that the population of the United States in 1870 was at least 40,000,000, instead of 38,558,371, as stated."(15)

The computed value just given is 39,312,000 which is, of course, affected to some extent by the error in the census of 1870 which entered into the computation of formula (4.4.1.1.3). To compute a value for 1870 which would be derived from data unaffected by the deficit due to the war, it would be necessary to discuss the observations from 1790 to 1860 alone. The solution of the eight equations of condition furnished by the data yields the following formula:

$$P = 17.189 + 5.2103t + 0.8202t^2 + 0.0623t^3 \quad . \tag{4.4.1.1.4}$$

How closely equation (4.4.1.1.4) fits the observed values can be determined from the residuals for the eight equations of condition for 1790 to 1860. These residuals show that during the 70 years from 1790 to 1860 the growth of the population followed the law expressed by equation (4.4.1.1.4) very accurately, and also that this rate of growth was more rapid than that of later decades. Had this rate of growth continued to 1870, the population would have amounted at that time to a point just short of 42 million. The decline during the decade due to those actually killed, to decreased immigration, and decreased birth-rate cannot be stated with exactness, but probably is very near 1,700,000. After considering this loss it does not seem possible that the population in 1870 could have been less than 40 million.

Had the population continued to grow after 1860 at the same rate as before, we would have had a population in 1890 of over 71 million, or nine million more than were counted. The difference is hardly due to the war alone, but is probably due in part to a continued diminishing birth rate.

4.4.1.2 Value of the formula for prediction

How closely formula (4.4.1.1.3) would continue to represent the population growth after 1890 depended, of course, upon the continuance of the same conditions of growth. A decided change in the birth rate, or immigration rate, or a destructive war, would bring out a large discrepancy between the observed and computed values. The general law governing the increase of population, as stated by Pritchett, is that, when not disturbed by extraneous causes, such as wars, pestilences, immigration, emigration, and the like, the increase of population goes on at a constantly diminishing rate. This means that the percentage of increase from decade to decade diminishes. The law of growth expressed by equation (4.4.1.1.3) involves such a decrease in the percentage of growth.

Differentiating (4.4.1.1.3) and dividing by P gives

$$\frac{\frac{dP}{dt}}{P} = \frac{B + 2Ct + 3Dt^2}{A + Bt + Ct^2 + Dt^3}$$

which diminishes as t increases, and approaches zero as t becomes infinite. In 1790 the percentage increase per decade was 32%; in 1880, 24%; in 1990 it would be 13%, and in 2100 it would be below 3%.(15)

Pritchett expected the projected values, from this equation, to depart more and more from the observed values, but until the year 2000 he anticipated a close representation of the growth of population. Table 4.4.1.2.1 contains these values.

Table 4.4.1.2.1

Population of the United States as projected by Pritchett (16), in 1890.

Year	Computed Population (1,000's)
1900	77,472
1910	94,673
1920	114,416
1930	136,887
1940	162,268
1950	190,740
1960	222,067
1970	257,688
1980	296,814
1990	339,193
2000	385,860
2100	1,112,867
2500	11,856,302
2900	40,852,273

Since Pritchett obtained his projection equation, seven census counts have been made in the United States. Using this added data and the methods of Pritchett, starting with the year 1800 and using 1880 as an origin, one obtains the following values of the coefficients of equation 4.4.1.1.3.

$$A = 50.9737$$
$$B = 10.9560$$
$$C = 0.6107$$
$$D = -0.0071$$

Hence the equation

$$P = 50.9737 + 10.9560t + .6107t^2 - .0071t^3 \quad (4.4.1.2.1)$$

Figure 4.4.1.2.1 gives a graphical representation of the census counts against the populations given by formula (4.4.1.2.1) in a similar manner as did Figure 4.4.1.1.1 for formula (4.4.1.1.3).

Table 4.4.1.2.2 compares formulae (4.4.1.1.3) and (4.4.1.2.1) and gives their respective forecasts through the year 2500. It is noted that the discrepancies of formula (4.4.1.1.3) are very large after the year 1890. These discrepancies represent projections for the years 1900 - 1960. The discrepancies of formula (4.4.1.2.1) after 1890 are much smaller but this is to be expected because this formula was based on the years 1800 - 1960. Considering only the respective periods of time on which the two formulae are based, formula (4.4.1.2.1) results in considerably larger discrepancies than does formula (4.4.1.1.3) in almost every instance. It will be noted that since (4.4.1.2.1) is based on all census counts to the present, it gives projections which are much more likely than those of (4.4.1.1.3) since the projections of this relation for 1900 - 1960 are higher than the corresponding census counts in increasing amounts with each succeeding decade.

Table 4.4.1.2.2

Population in 1,000's of the United States (1800 - 2500) as given by formulae (4.4.1.1.3) and (4.4.1.2.1).

Year	Observed Population[1]	By Formula (4.4.1.1.3)[2]	By Formula (4.4.1.2.1)	Discrepancy of (4.4.1.1.3)	Discrepancy of (4.4.1.2.2)
1800	5,308	5,267	6,040	-41	+732
1810	7,240	7,059	6,638	-181	-602
1820	9,638	9,569	8,753	-69	-885
1830	12,866	12,985	12,347	+119	-519
1840	17,069	17,484	17,374	+415	+305
1850	23,192	23,250	23,793	+58	+601
1860	31,443	30,468	31,561	-975	+118
1870	38,558	39,312	40,635	+754	+2,077
1880	50,156	49,975	50,974	-181	+818
1890	62,948	62,634	62,219	-314	-415
1900	75,995	77,472	75,272	+1,477	-723
1910	91,972	94,673	89,146	+2,701	-2,826
1920	105,711	114,416	104,114	+8,705	-1,597
1930	122,775	136,887	120,139	+14,112	-2,636
1940	131,669	162,268	137,161	+30,599	+5,492
1950	150,697	190,740	155,154	+40,043	+4,457
1960	178,464	222,067	174,071	+43,603	-4,393
1970	---	257,688	193,869		
1980	---	296,814	214,505		
1990	---	339,193	235,937		
2000	---	385,860	258,121		
2100	---	1,112,867	512,023		
2500	---	11,856,302	1,386,826		

[1]According to Statistical Abstract of the United States (1962)
[2]Pritchett (16)

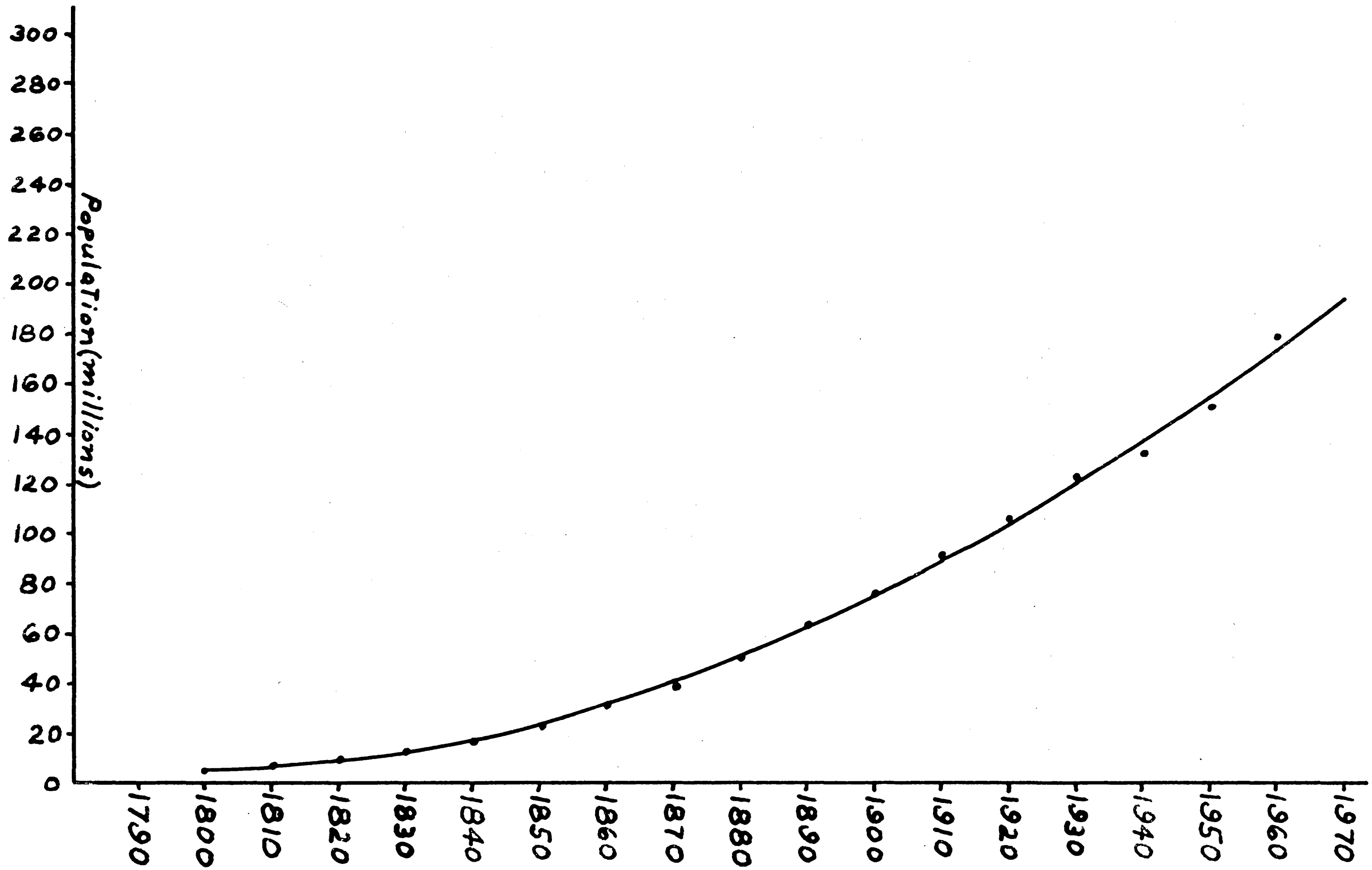

Fig. 4.4.1.2.1

Growth curve for United States according to Pritchett's cubic, fitted 1800 - 1960. Black dots represent observed populations.

4.4.1.3 Evaluation of the method of Pritchett

Pritchett treated the decline in percentage increase analytically when he attempted to allow for this by the use of his third order parabola. Although his curve allowed for decreasing decennial percentages, however, it did permit the population to become infinite and so was, in that sense, no better than the earlier use of the straight line or the exponential curve. The advance that Pritchett made over previous workers was that he used an analytical function allowing for declining percentage increases and that he determined his parameters by an explicit method so that his resulting predictions were free of the arbitrary element present in the free-hand curve, such as that of Woodruff previously discussed.

4.4.2 Pearl's logarithmic parabola

4.4.2.1 Development and application

In 1907, Pearl (14) demonstrated the use of a logarithmic curve of the form

$$P = A + Bt + Ct^2 + D \log_{10} t \qquad (4.4.2.1.1)$$

to represent growth changes, using data from an aquatic plant. Following the application of this curve to the growth of this plant it was found equally useful in representing a wide range of other growth and related changes.

While the increase in size of a population cannot on "a priori" grounds be regarded, except by rather loose analogy, as the same thing as the growth of an organism in size, nevertheless it is essentially a growth phenomenon. Thus it seems entirely reasonable that this type of curve should give as adequate a representation of population increase as did Pritchett's method.

Equation (4.4.2.1.1) was fitted by least squares to census data for 1790 - 1910, taking 1780 as an origin. As in previous equations, P denotes population, t denotes time, and A, B, C, and D are constants to be found in fitting. The equation thus deduced by Pearl was

$$P = 9{,}064{,}900 - 6{,}281{,}430t + 842{,}377t^2 + 19{,}829{,}500 \log_{10} t \quad (4.4.2.1.2)$$

4.4.2.2 Comparison of the two "parabolic" methods

The populations according to equation (4.4.2.1.2) for 1790 - 2000 are compared with Pritchett's results and actual census counts in Table 4.4.2.2.1. The data clearly shows that the logarithmic parabola gives a distinctly better graduation than does Pritchett's cubic equation, when fitted for the periods indicated in the table. A graph showing the curve of equation (4.4.2.1.2) and the observed counts is shown in Figure 4.4.2.2.1.

It is evident that as a purely empirical representation of population growth within the United States, the methods of both Pritchett and Pearl give results with fair accuracy. Estimates projected through the year 2000, as shown in Table 4.4.2.2.1 for both methods, indicate that Pearl's method, or equation (4.4.2.1.2), gives projections which are much less erroneous through 1960. It must be remembered, however, that Pearl had two more census counts than did Pritchett, which would increase the accuracy of the former's predictions.

In section 4.4.1.2, Pritchett's cubic was fitted for the period 1790 - 1960 and the resulting relationship, equation (4.4.1.2.1), gave a fairly good fit and projections that were much more reasonable than those resulting from the fit for 1790 - 1890. In fitting Pearl's logarithmic relationship for the period 1790 - 1960, the equation resulting was

$$P = 5{,}247{,}910 - 1{,}087{,}000t + 595{,}230t^2 - 2{,}803{,}040 \log_{10} t \qquad (4.4.2.2.1)$$

This equation, however, is a poor fit to the census data. Seemingly the logarithmic equation of Pearl is not as readily applicable as is the cubic of Pritchett fitted over extended periods of time.

Table 4.4.2.2.1

Showing (a) the actual population by census counts,

(b) estimated population according to Pritchett's method, fitted 1790 - 1890,

(c) estimated population according to logarithmic parabola, fitted 1790 - 1910,

and (d), (e) discrepancies of (b), (c).

(All numbers are in 1,000's.)

Year	(a) Observed Population[1]	(b) Pritchett's Estimates[1]	(c) Log. Parabola Estimates[2]	(d) Discrepancy of (b)	(e) Discrepancy of (c)
1790	3,929	4,012	3,693	+83	-236
1800	5,308	5,267	5,865	-41	+577
1810	7,240	7,059	7,239	-181	+53
1820	9,638	9,569	9,404	-69	-234
1830	12,866	12,985	12,577	+199	-289
1840	17,069	17,484	17,132	+415	+63
1850	23,192	23,250	23,129	+58	-63
1860	31,443	30,468	30,633	-975	-810
1870	38,558	39,312	39,687	+754	+1,129
1880	50,156	49,975	50,318	-181	+162
1890	62,948	62,634	62,547	-314	-401
1900	75,995	77,472	76,389	+1,477	+394
1910	91,972	94,673	91,647	+2,701	-325
1920	105,711	114,416	108,958	+8,705	+3,247
1930	122,775	136,887	127,700	+14,112	+4,925
1940	131,669	162,268	148,088	+30,599	+16,419
1950	150,697	190,740	170,127	+40,043	+19,430
1960	178,464	222,067	193,821	+43,603	+15,357
1970		257,688	219,173		
1980		296,814	246,186		
1990		339,193	274,862		
2000		385,860	305,203		

[1]From Table 4.4.1.2.2.

[2]Pearl (14)

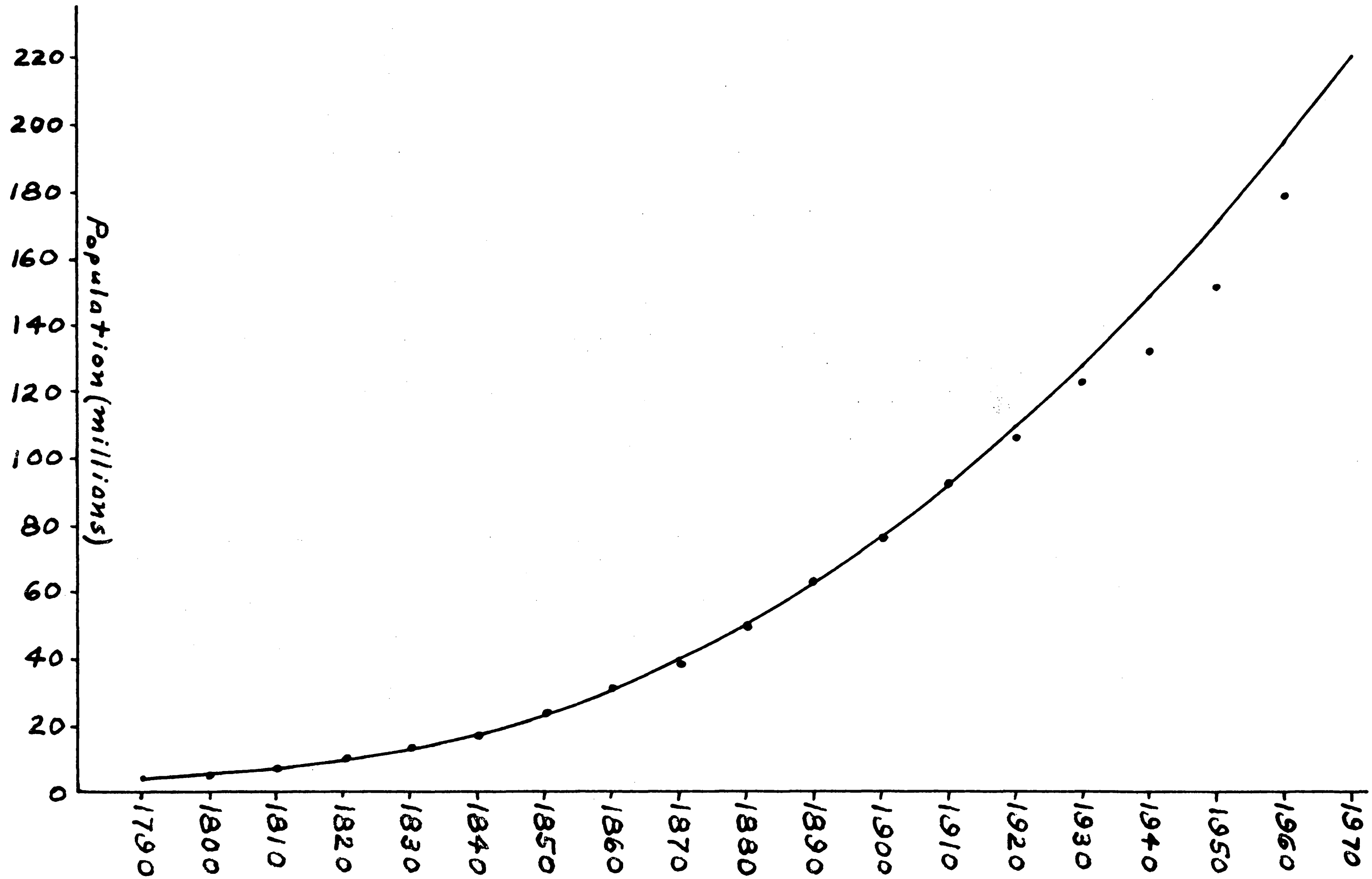

Fig. 4.4.2.2.1
Growth curve for United States according to Pearl's logarithmic parabola, fitted 1790 - 1910. Black dots represent observed populations.

ERRATA

Due to an error in numbering, there is no page 63.

4.5 The logistic

4.5.1 Early development by Verhulst

Verhulst (13, 25), a Belgian mathematician, in 1838, proposed that if a population is expanding freely over unoccupied territory, the percentage rate of increase is constant. If the population is growing in a limited area, the percentage rate of increase must tend to decrease as the population increases. Hence the percentage rate of increase is a function of the population itself, which decreases continuously as the population increases. The following modification of (4.3.1.2) was proposed by Verhulst.

$$\frac{dP}{dt} = r\,P - \psi(P) \quad . \tag{4.5.1.1}$$

The function of $\psi(P)$ is to be chosen so as to increase with rising values of P. This function is introduced to account for the retarding effect by the increased number of inhabitants on the rate of growth of the population. The simplest form of $\psi(P)$ is to suppose $\psi(P) = r\,a\,P^2$, so we have

$$\frac{1}{P}\frac{dP}{dt} = r - r\,a\,P \quad . \tag{4.5.1.2}$$

Integrating yields

$$P = \frac{1}{a + b\,e^{-rt}} = \frac{1/a}{1 + \frac{b}{a}\,e^{-rt}} \tag{4.5.1.3}$$

where P represents population size, t is time, r is the survival rate, b is a constant of integration, and 1/a is the limiting population size. Verhulst proposed this solution for P to be the required probable form for the law of growth of a population confined to a given area.

Instead of taking $\psi(P) = r\,a\,P^2$, we may assume other relations such as $\psi(P) = r\,a\,P^{\alpha}$, or $\psi(P) = r\,a \log P$. All these assumptions agree equally well with the empirical data, but they give very different values for the upper limit to the population. Verhulst applied his formula to the annual populations of France from 1817 - 1831; of Belgium, 1815 - 1833; and the county of Essex, England, 1811 - 1831. In these three cases the agreement of the formulae with the data is excellent, but at that time he had not stated how he determined the values to be assigned to the constants.

In 1843, Verhulst (25) presented a slightly different argument. A freely-expanding population, he proposed, must increase in geometric progression. He tried data for the United States, 1790 - 1840, to illustrate the point. But suppose the population has expanded to the point when the difficulty of finding good land has begun to make itself felt. Let the population at such time, which will be taken as zero time, be m, which was called by Verhulst the "normal population." The "retarding function," $\psi(P)$, now comes into play, and the differential equation may be written

$$\frac{1}{P}\frac{dP}{dt} = \ell - f(P - m) = \ell - \psi(P) \quad .$$

Two conditions are necessary for the retarding function in its new form:

(1) it must increase indefinitely with the population if $P > m$, and

(2) it must vanish when $P = m$.

The simplest form to assume is $n(P - m)$, which gives

$$\frac{1}{P}\frac{dP}{dt} = \ell - n(P - m) \quad ,$$

or, writing $r = \ell + n\,m$ for brevity,

$$\frac{1}{P}\frac{dP}{dt} = r - n\,P \quad .$$

This is precisely the same form as (4.5.1.2) where $n = r\,a$. Verhulst called the curve a "logistic." He developed the principle properties, pointing out that the curve is symmetric with respect to the point of inflection, $1/2\ a$. In this paper Verhulst presented equations for determining the constants in terms of three equidistant ordinates, a method described in section 4.5.3.2 of this thesis.

Probably owing to the fact that Verhulst's thinking was greatly in advance of his time and that the then existing data were quite inadequate to form any effective test of his views, his work went unnoticed for many years.

4.5.2 Independent development by Pearl and Reed (1920)

4.5.2.1 Introduction

Pearl and Reed (14), in 1920, derived an empirical curve which agreed with certain reasonable postulates concerning population growth. This was an independent discovery of the formula formerly suggested by Verhulst, which as previously mentioned, had virtually remained unnoticed.

It was first pointed out by Malthus (14), although in non-mathematical terms, that in any restricted area, such as the United States, a time must eventually occur when population will press so closely upon subsistence that its rate of increase per unit time must be reduced to the vanishing point. A population curve may start with a convex face to the base, but presently it must develop a point of inflection, and from that point on present a concave face to the x-axis, and finally approach an asymptote which represents the maximum number of people which can be supported on the given fixed area.

It is here assumed that the average standard of living, methods of agricultural production, and other such phenomena, either do not change during the time period of interest or if such changes do occur they will be negligible when compared with such factors as reproduction and immigration.

Pearl's logarithmic curve, (4.4.2.1.1), in due time, will develop a point of inflection and become concave to the base, but it never becomes asymptotic. It, therefore, cannot be regarded as a hopeful approach to a true law of population growth.

One, of course, cannot expect to accurately predict the population many years ahead. But any proposed law of population growth ought to give approximate indications of the number of people living in an area at that time, provided no alterations of major circumstance have occurred.

4.5.2.2 Mathematical theory

There are certain factors which, according to Pearl and Reed (13, 14), should be taken into account in any mathematical theory of population growth. For instance, the area of the earth capable of supporting human life must necessarily be a finite upper limit to population itself, or to the number of persons capable of living in such an area. The existence of a finite upper limit is as much a physical as a biological matter. This is obviously true whatever the future may provide in the way of agricultural discoveries or improvements. Future discoveries do not influence the limit based on all known or unknown resources of the earth; these discoveries only alter the approach to the limit and our estimates of that limit. But a limit still exists.

History reveals that each advancement in cultural level has been accompanied by the possibility of additional growth within any defined area. Each geographical unit which has been inhabited for any great length of time has, so far as available evidence indicates, had a succession of waves or cycles of population growth. Within each cycle of growth the rate of population increase has not been constant in time. Instead, at first the population grows slowly and the rate constantly increases to a maximum point. This point may be taken to present the optimum relation between the numbers in the population and the means of subsistence of the defined area. This point of maximum rate of growth is the point of inflection of the population growth curve. After that point is passed the rate becomes progressively slower until the curve finally closely approaches the upper asymptote which belongs to the particular cultural cycle and area involved (13).

Thus the following are characteristics of the population growth curve:

(1) asymptotic to a line $P = L$, when $t = +\infty$,

(2) asymptotic to a line $P = 0$, when $t = -\infty$,

(3) existence of a point of inflection,

(4) concave upwards to the left, and concave downwards to the right of the point of inflection,

(5) no horizontal slope except at $t = \pm\infty$,

(6) values of P varying continuously from 0 to L as t

varies from $-\infty$ to $+\infty$,

(7) cultural or cyclical character of growth, successive cycles being additive.

Pearl and Reed's first approximation to such a law expressing population growth was

$$P = \frac{b}{c + e^{-at}} \quad , \qquad (4.5.2.2.1)$$

where $L = b/c$ is the limiting population and a is a function of time.

This satisfied all of the above points except the last. The above equation is a modified form of the equation of Verhulst and represents a first approximation to a true law of population growth. There are several characteristics of this curve which are too rigid and inelastic to meet the requirements of such a law. In this relation the point of inflection must necessarily be exactly half-way between the asymptotes. The curve is also symmetrical about the point of inflection which implies that the forces which act to decrease the rate of growth during the latter half of the population history of the area are equal in magnitude, and exactly similarly distributed in time, to the forces which act to increase the rate of growth during the first half of the history. It does not seem feasible that such postulates as these could be characteristic of human population growth.

What needs to be done is to generalize equation

(4.5.2.2.1) in some way to free it from the restrictive features (location of point of inflection and symmetry) and will retain its other essential features.

We have from equation (4.5.2.2.1)

$$P = \frac{b}{c + e^{-at}}$$

which may be written as

$$P = \frac{L}{1 + m\, e^{La't}}$$

where $L = b/c$, $m = 1/c$, $La' = -a$. Then

$$L = P + P\, m\, e^{La't} \quad .$$

Differentiating with respect to t and equating to zero gives

$$\frac{dP}{dt} + m\, e^{La't}\, \frac{dP}{dt} + P\, m\, L\, a'\, e^{La't} = 0$$

or,
$$\frac{dP}{dt} = \frac{-P\, m\, L\, a'\, e^{La't}}{1 + m\, e^{La't}} \quad .$$

Substituting for L in the numerator coefficient gives

$$\frac{dP}{dt} = -P\, m\, a'\, e^{La't} \cdot P$$

$$\frac{dP}{dt} = -(P\, m\, e^{La't})(Pa') \quad .$$

Again using the relation for L we have

$$\frac{dP}{dt} = -a'\, P(L - P)$$

or,
$$\frac{\frac{dP}{dt}}{P(L-P)} = -a' \quad . \qquad (4.5.2.2.2)$$

If population, P, is the variable changing with respect to time, equation (4.5.2.2.2) implies that the rate of change of population with respect to time varies directly as P and (L - P). The constant L is the upper limit of growth, or $\lim_{t \to \infty} P = L$. Now since the rate of growth of P is dependent upon factors that vary with time one may generalize (4.5.2.2.2) by using f(t) instead of -a', where f(t) is some yet undefined function of t. Then,

$$\frac{dP}{P(L-P)} = f(t)\, dt \quad .$$

Integration gives

$$\frac{1}{L} \int \frac{L dP}{PL-P^2} = \int f(t)\, dt \quad ,$$

$$\frac{1}{L} \int \left[\frac{1}{P} + \frac{1}{(L-P)}\right] dP = \int f(t)\, dt \quad ,$$

$$\frac{1}{L} \left[\ln P - \ln(L - P)\right] + \ln m = \int f(t)\, dt \quad ,$$

$$\ln \frac{mP}{L-P} = L \int f(t)\, dt \quad ,$$

$$\frac{mP}{L-P} = e^{L \int f(t)\, dt} \quad ,$$

$$\frac{L-P}{mP} = e^{-L \int f(t)\, dt} \quad ,$$

Then $$P = \frac{L}{1 + m\, e^{-L \int f(t)\, dt}} = \frac{L}{1 + m\, e^{F(t)}} \qquad (4.5.2.2.3)$$

where $$F(t) = -L \int f(t)\, dt \quad .$$

Thus the assumption that the rate of growth of the dependent variable P varies as P, L - P, and an arbitrary function of time, t, leads to (4.5.2.2.3), which is of the same form as (4.5.2.2.1) where F(t) = -at . If one now assumes that f(t) may be represented by a Taylor's series, then

$$P = \frac{L}{1 + m\,e^{a_1 t + a_2 t^2 + \dots + a_n t^n}} \quad . \qquad (4.5.2.2.4)$$

If $a_2 = a_3 = a_4 = \dots = a_n = 0$, then (4.5.2.2.4) becomes (4.5.2.2.1) with $a_1 = -a$.

If m becomes negative the curve becomes discontinuous within finite time. Since this cannot occur in the case of population growth, further consideration of the equation shall be restricted to $m \geq 0$ only. Also since negative L would give negative P, which is also impossible in the case of population growth, L shall be limited to positive values. With m, $L \geq 0$, then $P \leq L$. Thus the curve in complete form always falls between the x-axis and a line parallel to it at a distance L above it. Also, if

$$\begin{aligned}
F(t) &\to +\infty \quad , \quad P \to 0 \\
F(t) &\to -\infty \quad , \quad P \to L \\
F(t) &\to -0 \quad , \quad P \to L/(1 + m) \text{ from above} \\
F(t) &\to +0 \quad , \quad P \to L/(1 + m) \text{ from below} \quad .
\end{aligned}$$

The maximum and minimum points of (4.5.2.2.3) occur when $\frac{dP}{dt} = 0$. But

$$\frac{dP}{dt} = \frac{-L\ m\ e^{F(t)}\ F'(t)}{[1 + m\ e^{F(t)}]^2} = -P\ \frac{m\ e^{F(t)}}{1 + m\ e^{F(t)}}\ F'(t)$$

and since $L - P = \dfrac{L\ m\ e^{F(t)}}{1 + m\ e^{F(t)}}$ from (4.5.2.2.3)

$$\frac{dP}{dt} = -\frac{P(L - P)}{L}\ F'(t)$$

and, $\quad P(L - P)\ F'(t) = 0$

for a maximum or a minimum. Hence maximum and minimum points appear wherever $F'(t) = 0$. The result, $\frac{d^2P}{dt^2}$, is needed to determine points of inflection.

$$\frac{d^2P}{dt^2} = \frac{2P - L}{L}\frac{dP}{dt}\ F'(t) + \frac{P(P - L)}{L}\ F''(t) = 0$$

$$= \frac{P(P - L)(2P - L)}{L^2}\ [F'(t)]^2 + \frac{P(P - L)}{L}\ F''(t) = 0$$

$$= \frac{P(P - L)}{L}\ [\frac{2P - L}{L}\ [F'(t)]^2 + F''(t)] = 0$$

$$= F''(t) + \frac{2P - L}{L}\ [F'(t)]^2 = 0$$

$$= L + (2P - L)\ \frac{[F'(t)]^2}{F''(t)} = 0\ .$$

Therefore,

$$P\ \frac{[F'(t)]^2}{F''(t)} = -\frac{L}{2} + \frac{L}{2}\ \frac{[F'(t)]^2}{F''(t)}$$

$$P = \frac{L}{2} - \frac{L}{2}\ \frac{F''(t)}{[F'(t)]^2}\ . \qquad (4.5.2.2.5)$$

The points of inflection are thus determined by the intersections of (4.5.2.2.3) with (4.5.2.2.5).

Dropping all powers of t above the n'th we have two cases to consider, one when n is even and one when n is odd. When n is even and $a_n > 0$ the curve will be of the type shown in Figure 4.5.2.2.1. If $a_n < 0$ the curve will be of the same form except it will be asymptotic to the line AB at both $t = +\infty$ and $t = -\infty$ and will be between the line AB and the x-axis. If n is odd, $a_n < 0$, the curve is of the form shown in Figure 4.5.2.2.2. If $a_n > 0$, the curve of (4.5.2.2.4) is reversed and becomes asymptotic to AB at $t = -\infty$ and to the x-axis at $t = +\infty$. Thus $a_n < 0$ is a case of growth, and $a_n > 0$ is a case of decay. We may limit equation (4.5.2.2.4) still further by stopping at the third power of t, since we are seldom justified in using over five arbitrary constants in any practical problem of this type. Now we have

$$P = \frac{L}{1 + m\,e^{a_1 t + a_2 t^2 + a_3 t^3}} . \qquad (4.5.2.2.6)$$

Having determined that growth within any one cycle may be approximately represented by (4.5.2.2.1), or more accurately by (4.5.2.2.6), the next question is that of treating several cycles of growth in the same population. Theoretically some form of (4.5.2.2.4) may be found with proper adjustments such that one equation with several constants would describe

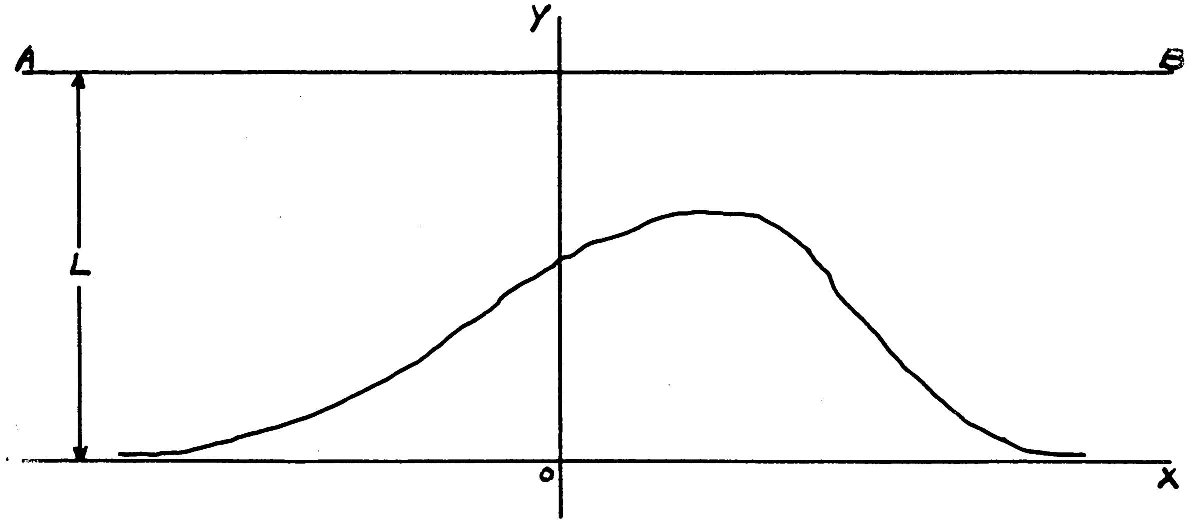

Fig. 4.5.2.2.1

Graphical form of equation (4.5.2.2.4) for n even, $a_n > 0$.

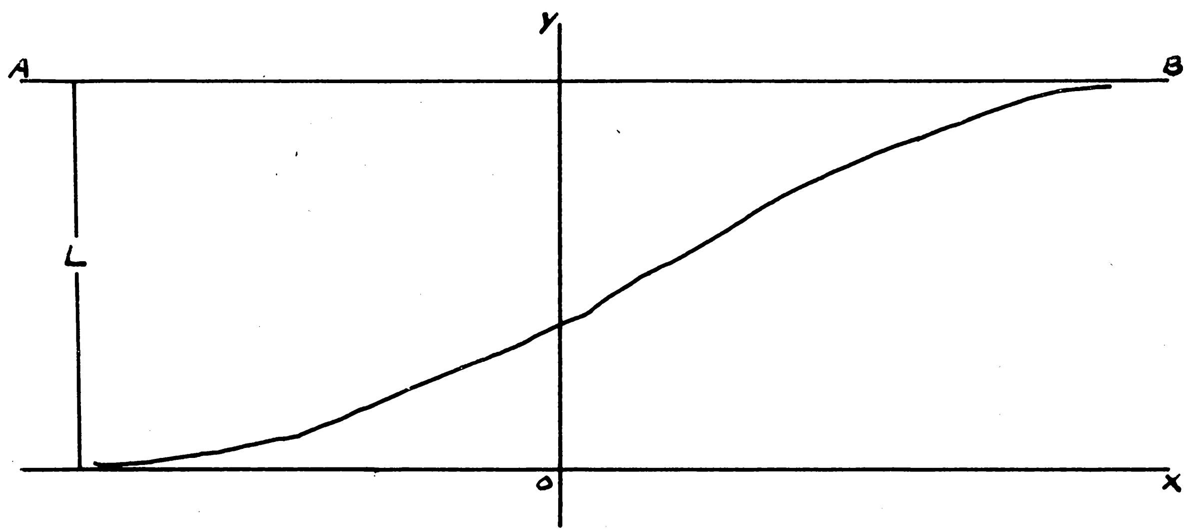

Fig. 4.5.2.2.2

Graphical form of equation (4.5.2.2.4) for n odd, $a_n < 0$.

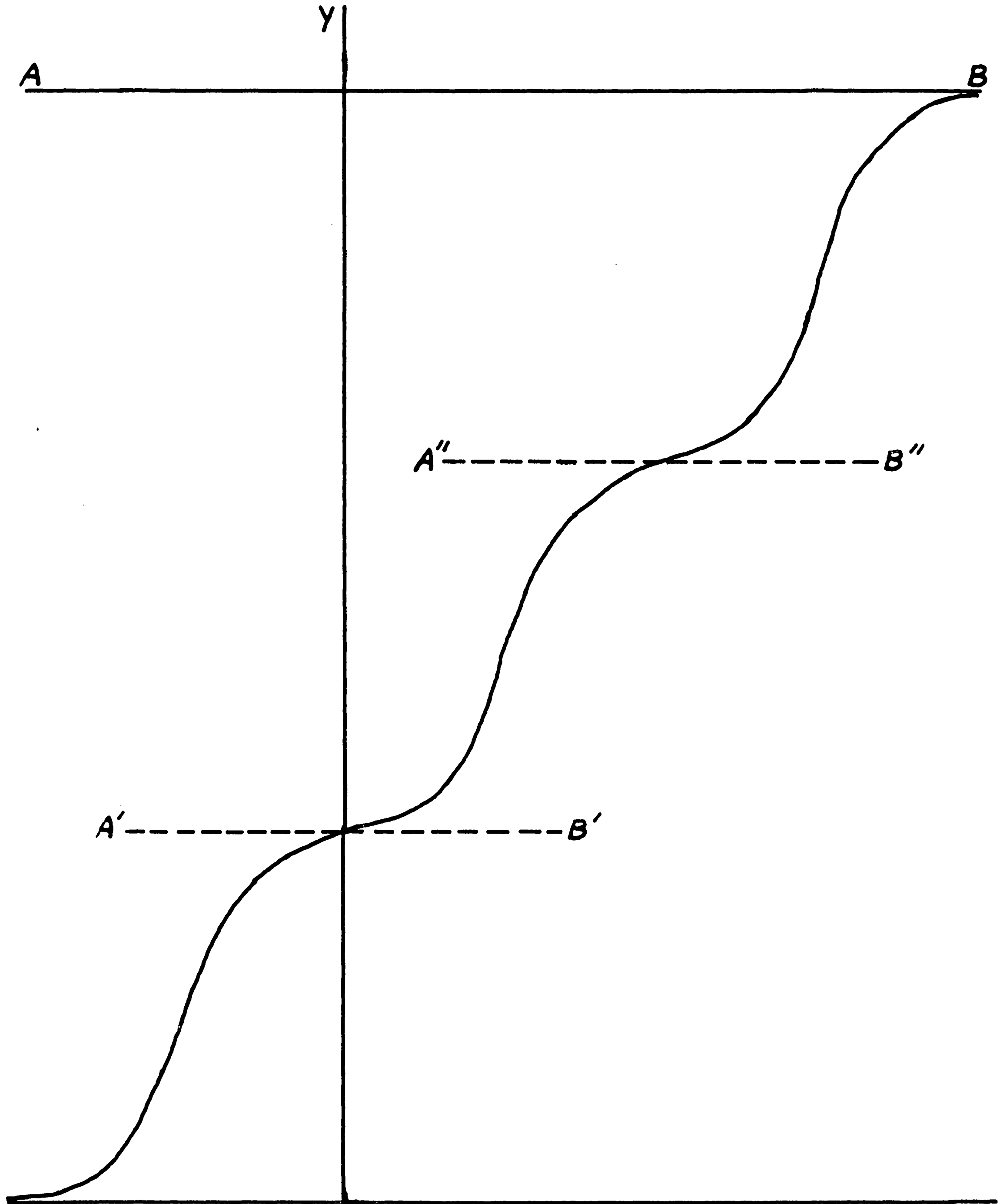

Fig. 4.5.2.2.3

A growth curve of several cycles.

a long history of growth involving several cycles. Practically, however, it is much easier to treat each cycle separately. Since the cycles of any case of growth are additive we may use for any single cycle the equation

$$P = d + \frac{L}{1 + m\,e^{La't}} \quad , \qquad (4.5.2.2.7)$$

or more generally,

$$P = d + \frac{L}{1 + m\,e^{a_1t+a_2t^2+a_3t^3}} \qquad (4.5.2.2.8)$$

where d represents the total growth attained in all previous cycles. This new form of the logistic is termed the "augmented" logistic. The term d is the lower asymptote of the cycle of growth under consideration and d + L is its upper asumptote. The general picture of such a growth curve of several cycles is shown in Figure 4.5.2.2.3. In treating any two adjacent cycles it should be noted that the lower asymptote of the second cycle is frequently below the upper asymptote of the first cycle due to the fact that the second cycle is often started before the first one has had time to reach its natural level. This could happen, for instance, when a population entered upon an industrial era before the country had reached the limiting population possible under purely agricultural conditions.

Whenever the growth within different cycles is symmetrical or nearly so there is considerable advantage in using equation (4.5.2.2.7) instead of (4.5.2.2.8). In this case the labor of fitting the curve will be less and the constant a' will give rates of growth for different cycles.

Recall the differential equation (4.5.2.2.2)

$$\frac{dP}{dt} = - a' \, P(L - P) = \frac{a \, P(L - P)}{L} \quad .$$

If one lets $a = \frac{1}{\alpha}$, then

$$\frac{1}{P} \frac{dP}{dt} = \frac{1}{\alpha} \left(\frac{L - P}{L}\right) \quad . \tag{4.5.2.2.9}$$

The constant α is termed the "standard interval," by analogy with the "standard deviation," and is of the dimensions of time. The solution of (4.5.2.2.9) is

$$P = \frac{L}{1 + e^{\frac{\beta - t}{\alpha}}} \tag{4.5.2.2.10}$$

where β is a constant of integration, and is also in dimensions of time. This is Verhulst's form of the logistic. If $t \to \infty$, $P \to L$ and if $t = \beta$, $P = \frac{L}{2}$. Differentiating (4.5.2.2.9) a second time gives

$$\frac{d^2P}{dt^2} = \left[\frac{1}{\alpha} \frac{L - P}{L} - \frac{1}{\alpha} \frac{P}{L}\right] \frac{dP}{dt}$$

$$= \frac{1}{L\alpha} (L - 2P) \cdot \frac{P}{\alpha} \frac{L - P}{L}$$

$$= \frac{P(L - P)(L - 2P)}{\alpha^2 \, L^2} \quad .$$

Setting $\frac{d^2P}{dt^2} = 0$ gives $P = \frac{L}{2}$, $t = \beta$ as the point of inflection. Also,

$$P_{\beta+h} = \frac{L}{1 + e^{-h/\alpha}} = L - \frac{L}{1 + e^{h/\alpha}}$$

$$= L - P_{\beta-h} \quad ,$$

so that the curve, form (4.5.2.2.10), is symmetrical about the point of inflection.

The smaller P is, compared with L, the more nearly the differential equation (4.5.2.2.9) approach the simple form

$$\frac{1}{P}\frac{dP}{dt} = \frac{1}{\alpha} \quad .$$

But the solution of this form is

$$P = A\ e^{t/\alpha} \quad . \qquad (4.5.2.2.11)$$

This shows that the early stages of the logistic are the same as a logarithmic curve, or the curve of a geometric progression. We would then expect the early stages of growth of a population to be essentially geometric.

A fundamental property of the logistic is that the instantaneous percentage rate of increase is a linear function of the population, P. A similar property may be stated for the percentage increases over finite intervals of time.

$$P_t = \frac{L}{1 + e^{\frac{\beta-t}{\alpha}}}$$

$$P_{t+h} = \frac{L}{1 + e^{\frac{\beta-t-h}{\alpha}}}$$

Hence $$\frac{P_{t+h} - P_t}{P_t} = \frac{e^{\frac{\beta-t}{\alpha}} - e^{\frac{\beta-t-h}{\alpha}}}{1 + e^{\frac{\beta-t-h}{\alpha}}}$$

$$= \frac{e^{\frac{\beta-t-h}{\alpha}}}{1 + e^{\frac{\beta-t-h}{\alpha}}} \left(e^{\frac{h}{\alpha}} - 1\right)$$

$$= \left(1 - \frac{P_{t+h}}{L}\right)\left(e^{\frac{h}{\alpha}} - 1\right) \quad . \qquad (4.5.2.2.12)$$

If, therefore, the percentage or proportional increases $\frac{P_{t+h} - P_t}{P_t}$ are plotted to the values P_{t+h} (the populations at the end of the corresponding intervals) on the x-axis, the resulting points lie on a line with slope $\frac{e^{\frac{h}{\alpha}} - 1}{L}$ which passes through the point $P_{t+h} = L$ on the x-axis.(25)

4.5.3 Methods of fitting the logistic

4.5.3.1 Introduction

There are several methods of fitting the logistic. If the logistic formula is written in the form

$$P = \frac{L}{1 + e^{\frac{\beta-t}{\alpha}}} \qquad (4.5.3.1.1)$$

the three constants, L (the limiting population), α, and β must be determined, in the process of fitting. Following are three proposed methods of fitting the logistic as written above.

4.5.3.2 Three equidistant ordinates

This is the method used by Verhulst (25) and by Pearl and Reed (13). It is quite adequate for populations that increase in a fairly regular pattern. The work is simplified if one regards the time of the first census of the three chosen as zero time, and the common interval between the first and second, and second and third, as the time unit. Then the equations for the three given populations, or ordinates, may be written

$$\frac{1}{P_0} = \frac{1}{L}\,(1 + e^{\beta/\alpha})$$

$$\frac{1}{P_1} = \frac{1}{L}\,(1 + e^{(\beta-1)/\alpha}) \qquad (4.5.3.2.1)$$

$$\frac{1}{P_2} = \frac{1}{L}\,(1 + e^{(\beta-2)/\alpha}) \quad .$$

Let
$$d_1 = \frac{1}{P_0} - \frac{1}{P_1} = \frac{1}{L}\,e^{\beta/\alpha}\,(1 - e^{-1/\alpha})$$
$$(4.5.3.2.2)$$
$$d_2 = \frac{1}{P_1} - \frac{1}{P_2} = \frac{1}{L}\,e^{(\beta-1)/\alpha}\,(1 - e^{-1/\alpha})$$

Then
$$e^{1/\alpha} = d_1/d_2 \qquad (4.5.3.2.3)$$

from which we have

$$\frac{1}{\alpha}\log_{10} e = \log_{10}\frac{d_1}{d_2} \quad . \qquad (4.5.3.2.4)$$

The constant α can be determined from the latter equation. Also,

$$\frac{d_1^2}{d_1-d_2} = \frac{1}{L}\, e^{\beta/\alpha} \qquad (4.5.3.2.5)$$

and hence, from the first relation in (4.5.3.2.1),

$$\frac{1}{L} = \frac{1}{P_0} - \frac{d_1^2}{d_1-d_2} \qquad (4.5.3.2.6)$$

from which L can be obtained. Thus, given α and L, equation (4.5.3.2.5) gives β, the distance of the point of inflection from zero time. Then

$$\frac{1}{\alpha} \log_{10} e \cdot \beta = \log_{10} \left(\frac{d_1^2}{d_1-d_2} \cdot L\right) \qquad (4.5.3.2.7)$$

where $\frac{1}{\alpha} \log_{10} e$ can be obtained from equation (4.5.3.2.4).

EXAMPLE: Take as an example the population of the United States in 1790, 1850, and 1910 so the censuses are 60 years apart. We can now form the following table of values:

Year	P_i (millions)	$1/P_i$	First Difference	Second Difference
1790	3.929	0.254518	---	---
1850	23.192	0.043118	0.211400	---
1910	91.972	0.010873	0.032245	0.179155

From (4.5.3.2.3) we have

$$e^{1/\alpha} = \frac{d_1}{d_2} = 6.556055$$

$$\frac{1}{\alpha} \log_{10} e = 0.8166425$$

$$\alpha = 0.53180 \text{ of } 60 \text{ years}$$

$$= 31.908 \text{ years}$$

$$\frac{1}{L} = \frac{1}{P_0} - \frac{d_1^2}{d_1-d_2} = 0.254518 - 0.249449 = 0.005069$$

So $\quad L = 197.28$ million.

From (4.6.2.7),

$$\frac{1}{\alpha} \log_{10} e \cdot \beta = \log_{10} \left(\frac{d_1^2}{d_1-d_2} \cdot L\right)$$

and $\quad 0.8166425\beta = 1.6920595$.

Thus,

$$\beta = 2.071971 \text{ sixty-year units}$$

or $\quad \beta = 124.32$ years

Hence since the origin was taken at 1790 the point of inflection is β years later or in 1914. We now have the equation

$$P = \frac{197.28}{1 + e^{\frac{12.432-t}{3.1908}}} \qquad (4.5.3.2.8)$$

as our growth curve. In order to have integral values 0, 1, 2, ... for t, the values found for α and β are divided by 10 before substitution into the logistic formula.

4.5.3.3 The method of sums of reciprocals

This method, proposed by Yule (25) uses all the data available, provided the number of censuses available is a multiple of three. One considers the reciprocals of the populations as given by the logistic in the form of equations (4.5.3.2.1), treating the common interval between the censuses as the unit of time and the time of the first census as zero time. Now the reciprocals are summed in three successive groups of r each as follows:

$$S_1 = \frac{r}{L} + \frac{C}{L}\, e^{\beta/\alpha}$$

$$S_2 = \frac{r}{L} + \frac{C}{L}\, e^{(\beta-r)/\alpha} \qquad (4.5.3.3.1)$$

$$S_3 = \frac{r}{L} + \frac{C}{L}\, e^{(\beta-2r)/\alpha}$$

where

$$C = \frac{1 - e^{-r/\alpha}}{1 - e^{-1/\alpha}} \; . \qquad (4.5.3.3.2)$$

Hence

$$D_1 = S_1 - S_2 = \frac{C}{L}\,(1 - e^{-r/\alpha})\, e^{\beta/\alpha}$$

$$(4.5.3.3.3)$$

$$D_2 = S_2 - S_3 = \frac{C}{L}\,(1 - e^{-r/\alpha})\, e^{(\beta-r)/\alpha}$$

and $$e^{r/\alpha} = D_1/D_2 \qquad (4.5.3.3.4)$$

which is the equation for determining α, similar to (4.5.3.2.3). Also,

$$\frac{D_1^2}{D_1-D_2} = \frac{C}{L} e^{\beta/\alpha} \qquad (4.5.3.3.5)$$

and hence

$$\frac{r}{L} = S_1 - \frac{D_1^2}{D_1-D_2} \qquad (4.5.3.3.6)$$

from which one obtains L. Having obtained α and L, equation (4.5.3.3.5) gives β.

EXAMPLE: Take data from the censuses for the United States from 1800 to 1910 inclusive as shown in Table 4.4.2.2.1 and form the reciprocals of the populations in millions, as follows:

Year	$1/P_i$	Year	$1/P_i$	Year	$1/P_i$
1800	0.188395	1840	0.058586	1880	0.019938
1810	0.138121	1850	0.043118	1890	0.015886
1820	0.103756	1860	0.031804	1900	0.013159
1830	0.077724	1870	0.025935	1910	0.010873
S_1	0.507996	S_2	0.159443	S_3	0.059856

$$D_1 = 0.348553 \quad , \quad D_2 = 0.099587$$

$$D_1 - D_2 = 0.248966 \quad , \quad r = 4$$

$$e^{4/\alpha} = \frac{D_1}{D_2} \qquad = 3.499984$$

$$\frac{4}{\alpha}\log_{10} e = 0.5440661$$

$$\alpha = 3.19295 \text{ decades}$$

$$= 31.93 \text{ years}$$

$$S_1 - \frac{D_1^2}{D_1 - D_2} = 0.507996 - 0.487975 = 0.020020 \quad .$$

So $\qquad L = 199.790$ million .

From the value of α and equation (4.5.3.3.2) we have $C = 2.656434$. Hence from (4.5.3.3.5),

$$2.656434\, e^{\beta/\alpha} = 0.487975\ (199.790)$$

which gives

$$\beta = 11.5035 \text{ decades}$$

or $\qquad \beta = 115.035$ years .

Hence since the origin was taken at 1800, the point of inflection is β years later, or in 1915. This method then gives a growth curve

$$P = \frac{199.790}{1 + e^{\frac{11.5035 - t}{3.193}}} \qquad (4.5.3.3.7)$$

where t is measured in decades from the origin, 1800. The values of α and β are also divided by 10, as before.

4.5.3.4 The method of percentage increases

This third method is based on equation (4.5.2.2.12). One calculates the percentage increases (preferably expressed as simple proportionate increases without multiplying by 100) over successive intervals h; plots these to the populations at the end of the intervals, P_{t+h}; and then, fits a line to these points to get α and L. To determine β, one forms the sum of the reciprocals of the populations for the whole series available, and the first equation of (4.5.3.3.1) gives the answer required.

EXAMPLE: If one works out the proportional increases of the population of the United States over successive 50 year periods (overlapping periods for 1800 - 1910), plots them to the terminal populations, and fits a line by least squares, the result is

$$z = 3.8204877 - 0.0197631\, P_{t+h}$$

where z denotes the proportional increase $(P_{t+h} - P_t)/P_t$, and P_{t+h} the terminal population in millions. Equating z to zero, the limiting population is L = 193.314 million. Further,

$$e^{5/\alpha} - 1 = 3.8204877$$

$$\alpha = 3.17889 \text{ decades}$$

$$= 31.7889 \text{ years} \quad .$$

Finally, adding together the three S's of the last example, we have, for the 12 censuses 1800 - 1910, S = 0.727295. Here our equation for β might be written

$$0.727295 = \frac{12}{193.314} + \frac{3.620073}{193.314}\, e^{\frac{\beta}{3.17889}} \quad ,$$

which gives $\beta = 11.349$ decades of 113.49 years, so that the date of the point of inflection is now at 1913.49. The curve resulting is

$$P = \frac{193.314}{1 + e^{\frac{11.349-t}{3.17889}}} \qquad (4.6.4.1)$$

where t is measured in decades from 1800.

4.5.3.5 Comparison of the three methods of fit

Upon examining the growth curves for the United States as given by the three examined methods of fitting, we have

$$P = \frac{197.28}{1 + e^{\frac{12.432-t}{3.1908}}}$$, from the method of three equidistant ordinates,

$$P = \frac{199.790}{1 + e^{\frac{11.5035-t}{3.193}}}$$, from the method of sums of reciprocals, and

$$P = \frac{193.314}{1 + e^{\frac{11.349-t}{3.17889}}}$$, from the method of percentage increases.

Although using different censuses for data, and involving methods of calculation which are also different, the three methods of fit yield growth curves which are indeed very similar. The corresponding values of α, β, and L found in each of the three cases show little variation.

Some illustrations of these three methods of fit appear in the following section.

4.5.4 Illustrations of the logistic

4.5.4.1 Introduction

Following are some applications of the logistic curve to actual data. The first two illustrations employ all three of the methods of fitting described in the previous section. The third illustration employs only the method of sums of reciprocals. This method appears to be the best method of fitting presented, and is applied to the United States Census counts through 1960. These illustrations demonstrate the applicability of the logistic curve to population growth as well as the methods of fitting.

4.5.4.2 England and Wales

Table 4.5.4.2.1 shows the census populations of England and Wales for the 12 censuses from 1801 - 1911. The census enumerations were used in fitting the logistic by each of the three methods of fit described in the previous section. Logistic (I) represents the fit by three equidistant censuses, logistic (II), the fit by the method of sums of reciprocals, and logistic (III), the fit by percentage increases. The growth curves are

$$\text{logistic (I):}\quad P = 91.221/\left[1+\exp\left(\frac{13.506-t}{6.113}\right)\right]$$

$$\text{logistic (II):}\quad P = 97.300/\left[1+\exp\left(\frac{14.319-t}{6.248}\right)\right] \qquad (4.5.4.2.1)$$

$$\text{logistic (III):}\quad P = 100.054/\left[1+\exp\left(\frac{14.696-t}{6.344}\right)\right]$$

Logistic (I) gives the census values for the years 1801, 1851, and 1901, chosen because the three censuses were equidistant in time. The values given by this logistic are very close to the observed values. The errors are, of course, zero in 1801, 1851, and 1901. The largest errors occur for 1861 and 1871, but these represent less than 2% of the actual population.

The values given by logistic (II) are not greatly different from those given by logistic (I), though its constants were determined by the method of sums of reciprocals from the three groups 1801 - 1831, 1841 - 1871, and 1881 - 1911. The fit is slightly improved as judged by the sum of the errors without regard to sign being reduced from 1.73 to 1.68 million, but it is more notably improved in another way. Logistic (I) makes the sum of the positive errors much larger than the sum of the negative errors with the result that the curve runs somewhat too high. Logistic (II) makes the two sums much more nearly equal with the result that the fit is very good through 1911. The value given by the logistic for 1921 is naturally very high, due to the losses in the war and the influenza epidemic of 1918 - 1919.

The fit given by logistic (III) is distinctly worse than that of logistics (I) and (II), indicating, perhaps, that it is not as good a method of fitting. However, the values given by all three logistics are fairly consistent and

Table 4.5.4.2.1

Census populations of England and Wales, 1801 - 1911, and the values given by the logistic curves of equations (4.5.4.2.1).[1] All values are in millions.

Year	Census Counts	Value by logistic (I)	Value by logistic (II)	Value by logistic (III)	Error (I)	Error (II)	Error (III)
1801	8.89	8.89	8.93	8.98	0	+0.04	+0.09
1811	10.16	10.29	10.35	10.35	+0.13	+0.16	+0.19
1821	12.00	11.88	11.89	11.91	-0.12	-0.11	-0.09
1831	13.90	13.68	13.66	13.67	-0.22	-0.24	-0.23
1841	15.91	15.69	15.65	15.64	-0.22	-0.26	-0.27
1851	17.93	17.93	17.87	17.83	0	-0.06	-0.10
1861	20.07	20.40	20.33	20.26	+0.33	+0.26	+0.19
1871	22.71	23.11	23.02	22.93	+0.40	+0.31	+0.22
1881	25.97	26.05	25.95	25.83	+0.08	-0.02	-0.14
1891	29.00	29.19	29.11	28.96	+0.19	+0.11	-0.04
1901	32.53	32.53	32.47	32.31	0	-0.06	-0.22
1911	36.07	36.03	36.02	35.85	-0.04	-0.05	-0.22
Sums of positive errors					+1.13	+0.88	+0.69
Sums of negative errors					-0.60	-0.80	-1.31
Sums of errors ignoring signs					1.73	1.68	2.00

[1] Yule (25)

indicate that the logistic formula is a valid approximate description of growth.

4.5.4.3 United States data, fitted for 1790 - 1910

Table 4.5.4.3.1 shows the population of the United States by census counts from 1790 - 1910 inclusive, and the values given by the three logistics fitted by the methods described earlier. The growth curves were

$$\begin{aligned} &\text{logistic (I):} && P = 197.274/[1+\exp(\tfrac{12.432-t}{3.191})] \\ &\text{logistic (II):} && P = 199.790/[1+\exp(\tfrac{11.504-t}{3.193})] \qquad (4.5.4.3.1) \\ &\text{logistic (III):} && P = 193.314/[1+\exp(\tfrac{11.349-t}{3.179})] \end{aligned}$$

The fits are not quite as close as in the case of England and Wales, but are still quite good overall. Logistic (II) actually increases slightly the sum of the errors without regard to sign, but, as before, notably improves the fit in so far as it makes the sums of the negative and positive errors much more nearly equal. Also, as before, logistic (III) gives the poorest fit of the three logistics.

In Table 4.5.4.3.2 extrapolations of logistic (I) give some approximations to the populations of the United States before 1790 and some predictions for years after 1910. The results are represented graphically in Figure 4.5.4.3.1.

Table 4.5.4.3.1

Populations of the United States (1790 - 1910) and the values given by logistic curves from equations (4.5.4.3.1).[1] All values are in millions.

		Value by the logistic			Error		
Year	Census	(I)	(II)	(III)	(I)	(II)	(III)
1790	3.929	3.929	3.902	3.983	0	-0.027	-0.036
1800	5.308	5.336	5.299	5.293	+0.028	-0.007	-0.015
1810	7.240	7.228	7.178	7.177	-0.012	-0.062	-0.063
1820	9.638	9.757	9.690	9.698	+0.119	+0.052	+0.060
1830	12.866	13.109	13.022	13.041	+0.243	+0.156	+0.175
1840	17.069	17.506	17.395	17.427	+0.437	+0.326	+0.358
1850	23.192	23.192	23.054	23.100	0	-0.138	-0.092
1860	31.443	30.412	30.294	30.300	-1.031	-1.194	-1.143
1870	38.558	39.372	39.191	39.229	+0.814	+0.633	+0.671
1880	50.156	50.177	49.998	49.982	+0.021	-0.158	-0.174
1890	62.948	62.769	62.623	62.486	-0.179	-0.325	-0.462
1900	75.995	76.870	76.801	76.451	+0.875	+0.806	+0.456
1910	91.972	91.972	92.035	91.358	0	+0.063	-0.614
	Sum of positive errors				+2.537	+2.036	+1.720
	Sum of negative errors				-1.222	-1.911	-2.599
	Sum of errors ignoring sign				3.759	3.947	4.319

[1] Yule (25)

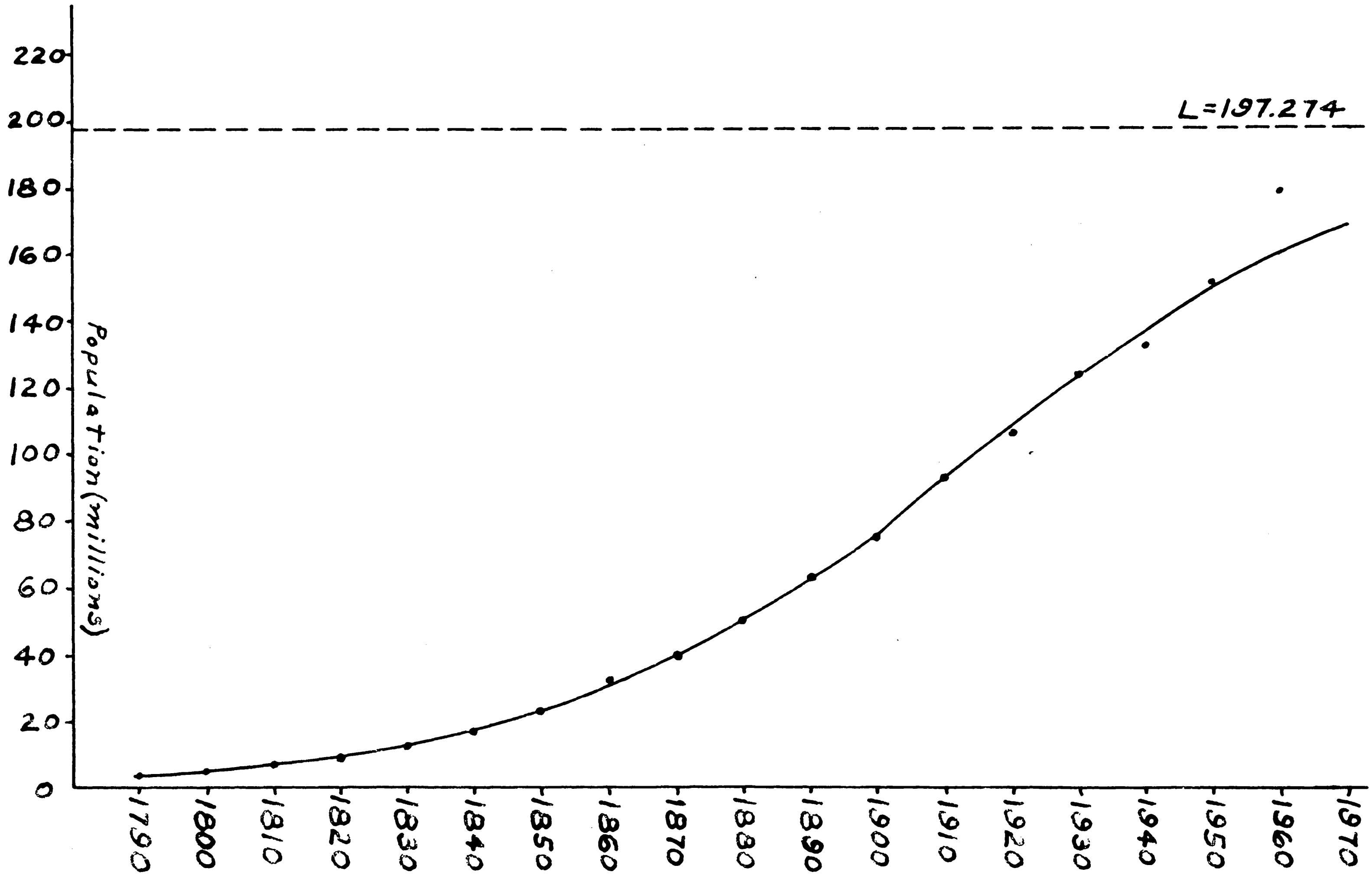

Fig. 4.5.4.3.1

Growth curve for United States according to logistic (I), fitted 1790 - 1910.
Black dots represent observed populations.

Table 4.5.4.3.2

Estimations of United States populations for 1700 - 1790, and predictions for 1920 - 2100 by logistic (I).[1] All values are in millions.

Year	Logistic (I)	Census
Lower Asymptote	0.000	---
1700	0.239	---
1720	0.446	---
1740	0.833	---
1760	1.533	---
1780	2.887	---
1920	107.394	105.711
1930	122.397	122.775
1940	136.318	131.669
1950	148.678	150.697
1960	159.230	178.464
1970	167.945	---
1980	174.941	---
1990	180.437	---
2000	184.678	---
2020	190.341	---
2040	193.509	---
2060	195.249	---
2080	196.337	---
2100	196.681	---
Upper Asymptote	197.274	---

[1]Pearl (13)

The point of inflection occurred about April 1, 1914, according to fitting by the method of three equidistant ordinates, on the assumption that equation (4.5.4.3.1) and the constants determined reliably represent the law of population growth in the United States. That is, from that date, the population curve of the United States exhibited a progressively decreasing instead of an increasing rate of growth. If one could rely upon these numerical values, the United States would be judged to have already passed the period of most rapid population growth, unless some unknown factor should arise to make the growth more rapid. The calculated population at the point of inflection was 98.637 million, which was in fact very near to the actual population of the country in 1914.

As noted previously, all of the methods of projection presented here originated well before World War II and the recent "baby boom," both of which have greatly affected the population growth in the country for the last few decades. Neither of these phenomena were foreseen in 1920 or earlier. The result is that, in most every case, the census counts exceed the corresponding projected populations as estimated by Pearl and others. As seen in Table 4.5.4.3.2, for instance, the upper asymptote, or value of L, is only 197.274 million. But the Census Bureau's <u>Current Population Reports, Population Estimates</u> (April 12, 1963) gives an estimate in excess of 186 million for the United States (Alaska and Hawaii excluded) on March 1, 1963. It is clear that Pearl's limiting population will easily be exceeded by the census count of 1970. After the year 2100, when the limiting population as given by logistic (I) was to be reached, the census count likely will be far in excess of the proposed limiting value. The fact that Pearl, and those before him, could not foresee the present-day increase in birth-rate should definitely be taken into account when evaluating their projections beyond 1930.

4.5.4.4 United States data, fitted for 1790 - 1960

Table 4.5.4.4.1 gives the census counts through 1960 along with the estimates according to the logistic formula when fitted by the method of sums of reciprocals for the data 1790 - 1960. The growth curve derived was

$$P = 198.564/[1 + \exp\left(\frac{12.425-t}{3.177}\right)] \qquad (4.5.4.4.1)$$

which will give an estimated population of the United States t decades from the origin of 1790. Although the logistic was fitted to all census counts to the present, the projections are much too low beyond 1950. As was indicated earlier, the Census Bureau's estimated population for March 1, 1963, was in excess of 186 million (Alaska and Hawaii excluded). This census estimate for 1963 is thus in excess of the projection by equation (4.5.4.4.1) for the year 2000. Thus the

Table 4.5.4.4.1

United States population estimates using logistic formula with results of sum of reciprocals method of fit (1790 - 2000). All values are given to the nearest 1,000.

Year	Census Count	By (4.5.4.4.1)	Discrepancy
1790	3,929	3,896	-33
1800	5,308	5,299	-9
1810	7,240	7,189	-51
1820	9,638	9,719	+81
1830	12,866	13,072	+206
1840	17,069	17,495	+426
1850	23,192	23,202	+10
1860	31,443	30,469	-974
1870	38,558	39,500	+942
1880	50,156	50,410	+254
1890	62,948	63,116	+168
1900	75,995	77,383	+1,388
1910	91,972	92,657	+685
1920	105,711	108,239	+2,528
1930	122,775	123,393	+618
1940	131,669	137,453	+5,748
1950	150,697	149,916	-781
1960	178,464	160,534	-17,930
1970	---	169,293	---
1980	---	176,313	---
1990	---	181,812	---
2000	---	186,050	---

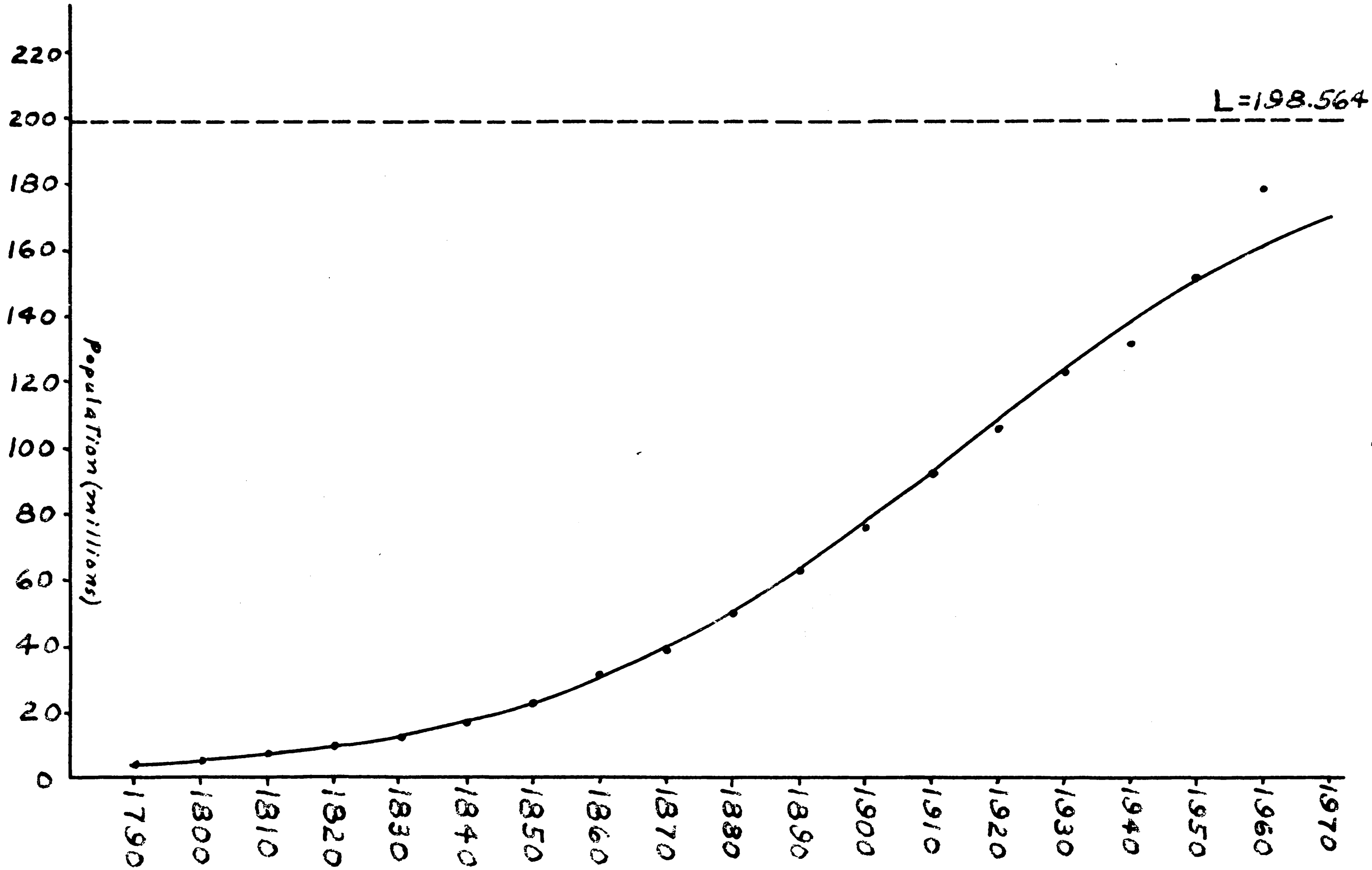

Fig. 4.5.4.4.1

Growth curve for United States according to logistic (II), fitted 1790 - 1960. Black dots represent observed populations.

consideration of the additional census counts for 1920 - 1960 has done little in this particular case to give a better equation of projection. Figure 4.5.4.4.1 gives a graphical representation of the results in Table 4.5.4.4.1.

4.5.5 Criticisms of the logistic

Although much support by Yule (25), Lotka (10), and others has been given the logistic for application to population growth there has also been much criticism and general skepticism in regard to its use.(3) The logistic equation is a concise mathematical expression which reveals what may be regarded as a fundamental law of population growth, that is, that population cannot increase indefinitely in constant geometric progression. There is, however, no reason to suppose, on an "a priori" basis, that the decreasing rate of increase in population growth is of so regular or uniform a nature that a mathematical function of the same form represents the growth in all times and places. Any other curve which gives as good an agreement has similar claims for representing the series of records. The closeness of the agreement may be unduly accented by Pearl and Yule. For instance, Table 4.5.4.3.1 gives, for the Unites States, a recorded increase in population from 1850 to 1860 of 8.251 million, while the change in the curve for logistic (I) is only 7.220 million; and from 1890 to 1900 the recorded and calculated increases are 13.047 million and 14.101 million respectively. It is unlikely that any one would guess this from Figure 4.5.4.3.1. If one has three constants available and ignores differences of indefinite magnitude in 13 or more entries, a satisfactory curve can be found if the phenomenon is continuous.

Bowley (3) attempted to define other formulas which represent the growth of certain populations as well, or almost as well, as does the logistic. In Table 4.5.5.1 the results of logistic (II) from Table 4.5.4.2.1 and the results of two additional formulas defined here are compared for England and Wales (1801 - 1911).

Formula 1 -

$$P = 19.09 + 24.56t + 0.112t^2$$

$$t_0 = 1856 \quad ; \quad \alpha = 1 \text{ year}$$

Formula 2 -

$$P = 107.85 \left(1 - 2 \int_0^t \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt\right)$$

$$t_0 = 2046 \quad ; \quad \alpha = 140 \text{ years}$$

Logistic (II) -

$$P = 97.300/[1 + \exp\left(\frac{14.319-t}{6.248}\right)]$$

$$t_0 = 1944 \quad ; \quad \alpha = 62.475 \text{ years}$$

In all three formulas, of course, t_0 is the date of the proposed point of inflection and α is the standard interval

Table 4.5.5.1

Population of England and Wales, 1801 - 1911, and the values given by formulas 1 and 2 and logistic (II), together with errors.[1] All values are in millions.

Year	Census Counts	Formula 1	Formula 2	Logistic (II)
1801	8.89	8.97	8.80	8.93
1811	10.16	10.31	10.22	10.35
1821	12.00	11.86	11.82	11.89
1831	13.90	13.65	13.58	13.66
1841	15.91	15.66	15.62	15.65
1851	17.93	17.89	17.84	17.87
1861	20.07	20.35	20.30	20.33
1871	22.71	23.02	22.97	23.02
1881	25.97	25.93	25.92	25.95
1891	29.00	29.06	29.12	29.11
1901	32.53	32.41	32.59	32.47
1911	36.07	35.99	36.32	36.02
Maximum Error	---	31.00	32.00	31.00
Mean Error	---	15.00	17.00	14.00
Mean Square Error	---	18.00	19.00	17.00

[1]Bowley (3)

or unit for the particular formula at hand.

In a similar manner, Table 4.5.5.2 compares the results of logistic (II) from Table 4.5.4.3.1 with those of the formula below, for the United States (1790 - 1910).

Formula 1 -

$$P = 23.48 + 65.95t + 0.687t^2 + 0.00215t^3$$

$$t_0 = 1850 \quad ; \quad \alpha = 1 \text{ year}$$

Logistic (II) -

$$P = 199.790/[1 + \exp\left(\frac{12.504-t}{3.193}\right)]$$

$$t_0 = 1915 \quad ; \quad \alpha = 31.93 \text{ years}$$

For England and Wales an ordinary parabola, fitted by least squares, gives values almost identical with those of logistic (II). For the United States a cubic equation was used and the resulting errors differ very little from those of the logistic. Thus, on the test of goodness of fit alone, the logistic equation has no special claims, so far as representation of past records is concerned.(3)

A parabolic or cubic relation cannot be used as a permanent law of population growth. The equation used for England and Wales would suggest a declining population before 1746 and an indefinitely increasing population in the future.

The logistic equation is only one considered that does not yield impossible results when used for predicting populations for the far-distant future. A formula chosen nearly at random

Table 4.5.5.2

Population of the United States, 1790 - 1910, and the values given by formula 1 and logistic (II), together with errors.[1] All values are in millions.

Year	Census Counts	Formula 1	Logistic (II)
1790	3.93	4.00	3.90
1800	5.31	4.99	5.30
1810	7.24	6.72	7.18
1820	9.64	9.30	9.69
1830	12.87	12.87	13.02
1840	17.07	17.55	17.40
1850	23.19	23.48	23.05
1860	31.44	30.78	30.29
1870	38.56	39.59	39.19
1880	50.16	50.03	50.00
1890	62.95	62.23	62.62
1900	76.00	76.32	76.80
1910	91.97	92.43	92.04
Maximum Error	---	103.00	119.00
Mean Error	---	41.00	30.00
Mean Square Error	---	49.00	47.00

[1] Bowley (3)

by Bowley, the integral of the normal error function, which lies between the limits $\pm \frac{1}{2}$ and is skew-symmetrical, represents the England-Wales population from 1801 - 1911 very well and is barely distinguishable from the logistic at any date of this period.

Bowley (3) concluded that, although the logistic is well adapted to represent rather roughly the population growth in selected countries, no further use of it has been justified and, after the considerable advertisement it received, many erroneous conclusions may result from its use.

The logistic is in agreement with our rational ideas about population growth, and it is a theoretically valid description of many different types of growth phenomena. The question is, does the logistic represent past growth of the United States population with such reliability that it can, with confidence, be used to predict the future? When the logistic is fitted to the population of the United States from 1790 - 1910, it points to a maximum population of about 197 million which will practically be attained by 2100. The standard error of this population, as computed by Pearl and Reed (13) is $\pm$ 0.8 million. The standard error of this population, when computed by the methods advocated by Schultz (18), is 10.5 million, a much more reasonable result. Schultz' method also takes into account the standard error of the function itself, as well as of the parameters.

If, by the statement that the logistic is the law of population growth, one means only that the formula is well suited to fitting the census enumerations for the period of a century of so, we can take no exception to it. But if the statement is to be considered as signifying that the formula affords a rational law permitting the extrapolation of the curve for forecasting purposes, we are forced to take exception. One finds that there are too many instances in which the curve becomes infinite in finite time or has a negative lower asymptote, or both. Also the constants are too often so poorly determined as to be of no practical value. In all of these cases we must at least withhold judgment until the populations have developed close enough to a limiting point that the fitting of the curve will give more reasonably well determined indications of its usefulness.(23)

4.6 Summary of the empirical methods of projection

Several empirical equations have been examined and their usefulness for purposes of projections of future populations has been considered and evaluated in each case. For each of these methods, the assumption is necessary that the factors influencing the rate of population growth are essentially constant; that is, that these factors will affect population growth in the future to essentially the same degree as they did in the past. That the total effect of such phenomena as war, famine, pestilence, and even immigration on the rate of growth of any particular population should be relatively constant over any two consecutive periods of time, as is implied by this assumption, seems very unlikely in most instances.

Geometrical progression, the basis of such empirical equations as the Malthusian Law, should be used only for short extrapolations in any attempt to predict populations of the future on account of its tendency to give values which may be much too large. The long-range predictions of the parabolic methods of Pritchett and Pearl are also much too large. Pritchett, however, did make a major contribution in recognizing that there is a tendency for population to grow at a diminishing rate with increase in time.

Seemingly, neither Pearl nor Reed maintained that the logistic was anything more than an empirical curve which described exceptionally well the past growth of certain populations. The use of the logistic for forecasting purposes and the interpretation of the constants as constants of nature is to be questioned. The fact that the population must be past its point of inflection, given by the curve, before future growth can be described with even rough accuracy is a serious handicap of the logistic.

One may doubt whether there is any general rational law of population growth. To give expectation that there is such a law one should have to assume that conditions governing growth remain constant during at least that length of time which is interesting in a practical way for the problem at hand. A study of the history of populations in the past shows that populations have been known not only to rise but also to fall. A noteworthy case is found in Ireland which reached a maximum population of just over eight million about 1841 and then declined to about four million. It is far from certain that a part of a rational population law for humans would be a steady approach to an asymptote as indicated by the logistic.

The question is whether any law is satisfactory over a reasonable range of time and for a considerable number of cases. Knibbs (25) has maintained that the census facts show rather clearly that populations do not grow at a continuously diminishing rate as assumed by Verhulst (25), but at a constant rate for a period of time and then at another constant rate for a subsequent period, and so on.

If the possibilities for variation which arise from graphical methods are added to the indetermination resulting from specified arithmetical methods of fitting, it becomes clear that the logistic which any one person may obtain must often be largely arbitrary and personal. The fitting of the curve becomes a process by which one gets what he wants, and the values forecast from the curve become personal estimates with no great degree of validity. For scientific purposes we should have a method of fitting which is agreed upon and the results of which can be duplicated by anyone. One, of course, naturally thinks of least squares procedures but there are certain mathematical problems which arise in the application of this method to logistics. The method also indicates, through the standard errors of the unknowns, a considerable degree of instability in the values of the constants. An erroneous extrapolation may very well be due to errors in the determination of the parameters of an empirical equation, on the assumption that the curve fitted is the curve which ought to be fitted and that there are no errors in the independent variables.(23)

Schultz (18) said concerning all empirical methods, "... there is no necessary relation between the goodness of fit of a curve to past observations and its reliability for forecasting purposes; a curve may fit the data for the past hundred years with a high degree of accuracy, and yet fail to predict the situation for the next year or so."

It seems obvious that if we wish merely to give a summary of the past 18 censuses, or so, the empirical methods presented may be very useful; but if we wish to predict the population of a century hence we must take into account underlying causes. These underlying causes will be given more consideration in the following section.

4.7 Methods other than empirical

4.7.1 Introduction

Up to this point consideration of population forecasting has been based on the idea that a population count is a statistical quantity having a certain orderliness with time, the forecasts being made by projecting the population into the future on the basis of its past orderliness. The problem may also be approached by projecting into the future birth-rates, death-rates, and migration-rates, and from these rates obtaining a predicted rate of change of the population at any subsequent time. Estimates of the future population can then be made by applying these rates successively, starting with the last known population. This general procedure may be varied by making the rates specific for age, racial group, and other such population characteristics.

4.7.2 Early developments by Dublin and others

In the early 1930's a different method of analysis of population growth had gained widespread acceptance. In 1911, Sharpe and Lotka (7) had shown that a population continuously subject to a fixed set of fertility rates for women of each age and a fixed set of mortality rates for each age, in the absence of migration, would ultimately assume a stable age distribution. The ultimate birth-rate, death-rate, and rate of natural increase, therefore, could be computed and compared with those prevailing at any time prior to stabilization. These ultimate rates were called true birth and death rates and their difference, the true rate of natural increase.

By 1920 some, if not all demographers, realized that the change in total population size was not a reliable guide to the future growth of the population of the United States. As annual statistics of births and deaths became more generally available, population change was analyzed in terms of the crude rate of natural increase and migration.

Dublin (7, 17) in 1925, applied these concepts to the analysis of population growth in the United States. He pointed out that as a result of high fertility in the past, the current age distribution of the population was unusually favorable to a high crude rate of natural increase. However, if the then existing fertility and mortality rates should continue unaltered, in the absence of migration the age distributions would change in such a manner that the annual rate of increase would be reduced by 50%. A method of computing the ratio of total births in two generations was shown by Lotka (7) and this ratio, since known as the net-reproduction rate, was given wide publicity.

Net reproduction rates were computed for all possible segments of the population, and when the required birth and death statistics were lacking, demographers invented ingenious substitutions for the net reproduction rate. The results showed not only that the true or potential rate of population growth was rapidly declining to a level which would eventually result in a decrease in the size of the total population but also that the actual excess of births over deaths was becoming smaller and smaller.

The population forecasts based on the true rate of natural increase reflected the pessimism of the time. In 1931, Dublin (7) presented in London two projections of the United States population before the General Assembly of the International Union for the Scientific Investigation of Population Problems. The most optimistic forecast assumed that the birth rate would continue to fall until it reached a level that would support a stationary population under the best mortality conditions foreseeable. It seemed that the maximum expected life at birth by 1970 would be 70 years. Accordingly, Dublin assumed that the true birth and death rates would be equal and stabilized at 14.3 by 1970 and they would so continue until the year 2100. With these assumptions the population of the United States, in absence of migration would reach a maximum of 154 million between 1980 and 1990 and decline to 140 million by 2100.

His second estimate assumed that the true birth rate would fall to a level of 13 per 1000 by 1970 and then gradually decline until it reached a level of 10 per 1000 by 2100. With the same mortality assumption as before and in the absence of migration, a maximum population of 148 million would be reached by 1970 followed by a decline to 140 million by 2000 and to 76 million by 2100.

Obviously, the true rate of natural increase and the net reproduction rate, in spite of their theoretical importance, yielded values which fall far short of the actual population counts. These rates did not hold the key to the future and, in fact, they seemed to obscure the future.

4.7.3 Whelpton and the Scripps Foundation

The most authoritative and widely accepted series of population estimates to this time have been those of Whelpton (7, 17, 21) of the Scripps Foundation. His procedure which he terms the "analytical method," "consists of

(1) analyzing mortality, fertility, and migration trends in different population segments;

(2) framing hypotheses regarding the future trends of these factors by observing their previous behavior and appraising the influence of changing industrial, social, and legal conditions; and

(3) building up successive hypothetical populations, beginning with the last actual population, by applying hypothetical factors to different population segments. The results thus show the composition as well as the size of the indicated future population." (7)

By 1930 the population growth in the United States had been slowing up more rapidly than most people seemed to realize. Whelpton (21) attempted to show why this rapid decrease was taking place and what its future course might be. The birth-rate, which apparently started downward over a century before this time, fell abruptly during World War I, took an upward trend in the first post-war years, and then again began a rapid decline. Whelpton proposed this decreasing birth-rate would continue into the future. This would not mean that child bearing would cease, but probably that a stabilization of one- to three-child families throughout a large proportion of the population.

Mankind has found it easier to prevent births than to prolong life. The death rates at the younger ages have been lowered greatly in the United States, particularly by the infant mortality-rate. The trend of death-rates at ages over 40 had been downward, but in the 1930's was rising due to increasing deaths caused by cancer, diseases of the heart, and other degenerative diseases.

Compared with birth- and death-rates, the future trend of immigration was naturally hard to forecast for the United States since the laws might be changed in regard to immigration at any time. Whelpton used the estimate of 200,000 persons per year for the net admission of aliens which seemed fair for the next few decades after the 1930's.

On the basis of the projected trends in specific birth- and death-rates and in net immigration, the Scripps Foundation computed the population for the period 1930 - 1980, starting with 122,536,000 persons on January 1, 1930. The results are shown in Table 4.7.3.1. A maximum population of 144,600,000 was indicated.

Table 4.7.3.1

Possible future increase of United States population (1920 - 1980)[1], (in thousands).

Year	Population (Jan. 1)	Decennial Increase	
		Thousands	Per-cent
1920	105,711	---	---
1930	122,536	16,825	15.9
1940	132,500	9,964	8.1
1950	139,800	7,300	5.5
1960	143,900	4,100	2.9
1970	144,600	700	0.5
1980	142,900	-1,700	-1.2

[1]Whelpton (21)

A set of estimates prepared in 1927 by the Scripps Foundation (7) indicated that a population of 175 million might be reached by 1975. President Hoover's Research Committee on Social Trends accepted the projections of the Scripps Foundation. The report of the committee in 1933 contained two alternative forecasts prepared by the Scripps Foundation. The low estimates differed little from Whelpton's 1931 estimates. But the high estimates predicted 190 million by 1980.

It should be noted that Whelpton maintained at all times that, "No claim is made that the Scripps Foundation estimates represent a law of population growth. .... These estimates represent simply what will happen under certain conditions of immigration, birth-rates, and death-rates, conditions that are believed to be reasonable, based on the experience of recent years."(7) However, his computations have been widely regarded and used as forecasts of future populations not only by fellow demographers but also by other social scientists and by governmental officials and businessmen.

The onset of World War I, a somewhat more rapid decline in mortality than had been anticipated and the continuation in the upward turn in the birth rate which had started in the middle 1930's led to revised forecasts. Although the Scripps Foundation saw no reason to materially alter its long range projections of fertility and mortality trends, it was apparent that the "baby boom" had discredited the base population on which future populations rested. Accordingly, a set of revised estimates were prepared in 1947 under the sponsorship of the Bureau of the Census. These indicated a maximum population of 165 million would be reached by 1990 after which a decline would occur.

But population increase outstripped the speed of the government's constant revision of forecasts. In 1949 the Bureau of the Census issued revised forecasts of population for 1948 - 1955. The forecast for July 1, 1950, was nearly 2 million greater than the highest and 5 million greater than the lowest 1947 forecast for that date. As of December, 1949, it was likely that this revised "revised" estimate would be nearly two million too low.

Table 4.7.3.2 shows estimates through 1980 prepared by the Bureau of the Census in 1958.(6) The values are not offered as predictions of the future size of the population but indicate rather the approximate future level of our population under given alternative assumptions as to future fertility, mortality, and net immigration. The set of projections is consistent with current estimates of the population for July 1, 1957. All of the four series presented employ assumptions which are tied in initially with the recent level of fertility (1955 - 1957 average), and the recent level of mortality (1955).

Table 4.7.3.2

Revised population projections by the Census Bureau, 1958, for 1960 to 1980[1], (in millions).

Year (July 1)	Series I	Series II	Series III	Series IV
1960	181.2	180.1	179.8	179.4
1965	199.0	195.7	193.6	191.5
1970	219.5	213.8	208.2	202.5
1975	243.9	235.2	225.6	215.8
1980	272.6	260.0	245.4	230.8

[1]Bureau of Census (6)

A "component" method was used to develop the population projections in Table 4.3.7.2. This method involves the preparation of separate projections of births, deaths, and net immigration on the basis of certain assumptions and the combination of the projections with estimates of the current population. This method is the same as the one used by the Census Bureau in making its earlier national population projections. The assumptions used for the individual series are:

<u>Series I</u> - For the whole projection period 1958 to 1980, fertility will average 10% above the 1955 -

1957 level (gross reproduction rate of 1.97).

Series II - Fertility will remain constant at the 1955 - 1957 level throughout the projection period (gross reproduction rate of 1.79).

Series III - Fertility will decline from the 1955 - 1957 level to the 1949 - 1951 level by 1965 - 1970, then remain at this level to 1957 - 1980 (ultimate gross reproduction rate of 1.54).

Series IV - Fertility will decline from the 1955 - 1957 level to the 1942 - 1944 level by 1965 - 1970, then remain at this level to 1975 - 1980 (ultimate gross reproduction rate of 1.28).

A detailed description of the method used is given in reference (6) and will not be given here since the primary interest of this paper centers on "empirical" rather than "analytical" methods.

4.8 The empirical versus other methods

Within recent years there have been attempts at forecasting the population which differ from those given here. They generally fall into two classes that were omitted from the present discussion. Either the population is estimated over a short-time interval through the use of the correlation between population counts and some function of whose value reasonably short-time estimates can be made, or long-time estimates are made by the use of freehand curves the interpretation of which is very difficult.

This leaves those estimates based on the empirical methods, such as the logistic, and those based on predictions in terms of birth-, death-, and immigration rates for our consideration. Any estimate must naturally have within it the element of personal judgment, the distinction between two methods being in the point at which such judgment is applied. In using the logistic or any similar mathematical function for the purposes of population prediction, judgment is exercised as to whether or not the functional form contains the main rational elements of the problem. Once this judgment is made, predictions are determined by fitting this functional form to the observations. One does not, however, need to leave this judgment entirely unchecked, for by testing a wide variety of populations, one sees whether or not the functional form has the ability to take the variety of shapes seen in the curves of population growth. The logistic appears to be fairly well adaptable for large elements of human population. The few cases where the curve fails badly are usually the result of some single outstanding event in the history of the population. In using the logistic one is following the line of thought applied in many fields of science when any empirical equation is found to fit an observed set of data and then is used for purposes of extrapolation beyond the range of observation.

When one considers the procedures used by Dublin or Whelpton, one sees that they exercise their judgment to state directly what the future birth- and death-rates will be and their population forecasts, being the direct arithmetic application of these rates, have, therefore, the values to be ascribed to the judgment of these workers. There is obviously no check on how good this judgment may be until future observations are available. They place a considerable weight on the events of the few years immediately preceding their predictions. Such a policy would seem to be dangerous when making predictions for as long a time period as 50 years or more.

Another striking characteristic of the two methods of forecasting is the ease with which they may be used for further demographic analysis. The summarizing of any set of population predictions in terms of a simple mathematical equation greatly increases its usefulness when turning to other population problems. It is inconceivable that many workers, who have added greatly to our knowledge of the interaction of demographic factors, could have proceded without the use of a formal expression for the change of population with time.(17) But it should also be realized that the factors affecting population growth may not remain essentially constant over long periods of time. Thus it may be well to

Table 4.8.1

Various estimates of the population of the United States (in millions)

Year	Census	Pritchett (1891)	Pritchett (1963)	Pearl's Parabola (1907)	Logistic (1920)	Logistic (1963)	Dublin (1931) High	Dublin (1931) Low	Scripps (1931)	Scripps (1947) High	Scripps (1947) Med.	Scripps (1947) Low	Bureau of Census (1949)	Bureau of Census (1958) Series I	Series II	Series III	Series IV
1900	76.0	77.5															
1910	92.0	94.7		91.6													
1920	105.7	114.4		109.0	107.4												
1930	122.8	136.9		127.7	122.4												
1940	131.7	162.3		148.1	136.3		131.0	131.0	132.5								
1950	150.7	190.7		170.1	148.7		139.0	139.0	139.8	148.0	146.0	144.9	149.9				
1960	178.5	222.1		193.8	159.2		147.0	146.0	143.9	162.0	155.1	149.8	160.0	181.2	180.1	179.8	179.4
1970		257.7	193.9	219.2	167.9	169.3	151.0	138.0	144.6	177.1	162.0	151.6		219.5	213.8	208.2	202.5
1980		296.8	214.5	246.2	174.9	176.3	154.0	148.0	142.9	increase thereafter	increase until about 2000, then decline	decline thereafter		272.6	260.0	245.4	230.8
1990		339.2	235.9	274.9	180.4	181.8	154.0	145.0	decline thereafter								
2000		385.9	258.1	305.2	184.7	186.1	152.0	140.0									
					ultimate 197.3	ultimate 198.6											

Table 4.8.1 presents a summary of the predictions made by various empirical equations and the methods of Dublin, the Scripps Foundation, and the Bureau of the Census.

consider such factors separately in their effect on population growth as do the methods of Dublin, Whelpton, and others.

4.9 Conclusions

The two principle methods of approach to the study of population growth which yield population projections into the future have been examined. Although chief emphasis was given to the empirical approach, enough attention was also given to the general approach of Dublin and Whelpton to allow certain criticisms of demographers in general.

Many demographers have given the impression that their projected populations were relatively inevitable and certain. They did not seem to realize that their projections could be as inaccurate as they were eventually shown to be. Thus, too little consideration was given to probable error.

Most early forecasters underestimated the effects of scientific developments on lowering mortality rates. Even forecasters of today are very likely to underestimate the effects of scientific achievement on the mortality rates of the future.

Those proposing empirical methods of projection, almost without exception, believed that demographic development of a theoretical stable population must inherently characterize current demographic developments in an actually instable population. The theory developed from a hypothetical population is not necessarily applicable to a particular existing population.

Too much reliance has been placed on the observed fertility and mortality of a single year or short period of years as a guide to future trends. Undue weight has also been given to certain cyclical fluctuations in fertility.

The popular assumption that birth-rates must inevitably continue to decline because they decline for several generations is not necessarily valid. In fact, the voluntary control of fertility can cause the birth rate to rise as well as to fall, contrary to the opinion of many demographers.

Forecasters in general have consistently overestimated the decrease in rate of growth of the population in the United States. This, of course, leads to predictions of future populations which tend to be too low.

In general demographers are too uncritical of the work of their fellow demographers. Certainly any new demographic theory which stems from the workd of other demographers can be of little value unless the work has been proven valid; and unless there is no doubt as to the reliability of the basis of the new theory, it should not be accepted or used in further developments until validated.(7)

There are certain problems which only recently have been attacked and their full consequences are yet to be seen in many cases. Likewise many challenging problems remain whose solutions may very well be turning points for population forecasting in the future.

Formal population curves should be fitted to the populations of different areas and the parameters thus determined should be correlated with certain social and economic forces which influence population changes. This would clarify the meaning of the parameters of the curve. The curves should also be fitted to the population counts for different racial groups living in the same area and the constants of these curves studied for their meaning in terms of social, economic, and biological factors. Finally, formal functional expressions should be considered for the time changes of the birth- and death-rates, specific for such elements as age and race, and the results of such an analysis should be related to the growing body of knowledge in this field.(17)

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ABSTRACT

An Analysis of Some Aspects of Population Projection

The rising awareness of existing problems created by rapid population expansion has resulted in systematic investigations of the characteristics of population growth. These investigations have produced methods for projection of future populations.

Attempts have been made to project world population, but the situation is too heterogeneous to provide useful results. Population problems, although of world-wide importance, are problems of particular peoples and particular areas.

Some of the earliest methods of projection used in the United States were based on the Malthusian Law and geometric progression. Pritchett and Pearl, in the late 1800's and early 1900's, devised parabolic methods of projection. These early projections were good for short term projection but generally unrealistic for long range use.

In 1920 Pearl and Reed devised an empirical curve, later known as the logistic curve of population growth. This method received considerable attention. The logistic was supported by many later demographers and the resulting projections satisfied all but a few critics.

Whelpton's "analytical method," and other similar methods, have been widely accepted. They give emphasis to birth-, death-, and net-reproduction-rates and not to mathematical growth curves.

Many of the above methods are used to make projections based on census counts to date. These projections are compared and tables used to show the results.