GENERAL PROPERTIES OF REAL-VALUED FUNCTIONS

by

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INTRODUCTION

The real-valued functions of a real variable have been studied quite extensively in the past. In particular, the subclass C of real continuous functions defined on a closed interval [0,1] on the real line was found to have certain properties that more general real functions do not have.

The aim of this work is threefold. First, we try to answer the question as to what are some of the elementary properties of C that it inherits from a larger class A of positively continuous functions of which C is a subset. Next, we investigate some topological properties of a few subclasses of F, the collection of all real-valued functions defined on [0,1] on the real line; and finally, we study some properties of the set I of all real-valued functions from [0,1] to [0,1].

Some fundamental results of C will be given followed by a comparison of the properties of A with those of C. Algebraic structures of some subclasses of F will also be studied. The concept of area function, $\overline{G_f}$, defined as the closure of the graph of a function, will be used to characterize the integrals of continuous functions in I. $\overline{G_f}$ will also be applied to bring out the topological implications of monotone functions and several other classes of functions in I. Finally, $\overline{G_f}$ of a function in I will be described by a closed set in U, the closed unit square.

GLOSSARY OF FREQUENTLY USED SYMBOLS AND NOTATIONS

" $\boxtimes$ " - denotes the end of a proof.

" $\in$ " - " an element of" or "elements of".

" $\ni$ " - "such that".

" iff " - "if and only if".

" $\subset$ " - "properly contained in".

" $\subseteq$ " - "contained in".

" $\exists$ " - "there exists" or "there exist".

" $\forall$ " - "for all".

" $\Rightarrow$ " - "implies that".

" $\Leftarrow$ " - "implied by".

" $\cup$ " - "union"

" $\cap$ " - "intersection"

" $S - X$" - the set consists of elements belonging to S but not X.

" $\circ$ " - an operation on the elements of a set.

" $\because$ " - " since "

" $\therefore$ " - "therefore"

" Σ " - "summation"

A slash (/) through a symbol negates the meaning of the symbol.

A bar ($\bar{\ }$) above any symbol designating a set means the closure of the set.

Lower case Greek or English Alphabets denote real numbers or elements of a set unless otherwise specified.

Upper case English Alphabets designate sets in general, unless otherwise stated.

E^n - Euclidean n-dimensional space.

F - the class of real-valued functions defined on $[0,1]$ on the real line.

I - the class of functions $f(x)$ in F such that $f(x) : [0,1] \to [0,1]$.

C - the collection of continuous functions in F.

A - the set of functions $f(x)$ in F such that $|f(x)| \in C$.

J - the interval on the real line.

$P = \{ax : 0 \leq a \leq 1,$ ax in F$\}$.

Ω - an index set for a collection of sets.

G_f - the graph of a function $f(x)$.

U - closed unit square in $E^2 = \{(x,y) \quad 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

i - the element of a set such that $i \circ a = a \circ i = a$ for all a in the set.

a^{-1} - the element of a set such that $a^{-1} \circ a = a \circ a^{-1} = i$ for all a in the set.

SECTION I - ANALYTIC PROPERTIES

Most of the terms that are needed in this section will be defined. A few classical theorems will be used (applied) or stated without proof while others that are more pertinent to our work are established.

Properties of C

Let $f(x)$ be a real-valued function defined on an interval J on the real line.

Definition 1.1

Let x_o be a point in J. $f(x)$ is said to be continuous at x_o iff given any $\varepsilon > 0$, there exists a $\delta(x_o, \varepsilon) > 0$ such that

$|f(x) - f(x_o)| < \varepsilon$ whenever $|x - x_o| < \delta(x_o, \varepsilon)$

or whenever $x \in (x_o - \delta(x_o, \varepsilon), x_o + \delta(x_o, \varepsilon))$

Definition 1.2

If $f(x)$ is continuous at every point of J, we say that $f(x)$ is continuous on J. Otherwise, $f(x)$ is said to be discontinuous on J.

Let F be the class of all real-valued functions defined on the closed interval $[0,1]$ on the real line.

Definition 1.3

$C \subset F$ is the collection of all continuous functions in F, i.e., $f(x) \in C$ iff $f(x)$ is continuous on $[0,1]$.

Definition 1.4

$f(x)$ is bounded on J iff there exists a $\sigma > 0$ such that $|f(x)| \leq \sigma$ for all x in J.

Theorem 1.1

(Heine - Borel Theorem) Let E be a closed bounded point set. Let $\{J_\alpha\}_{\alpha \in \Omega}$ be an infinite or a finite set of intervals such that every point of E is interior to at least one J_j of $\{J_\alpha\}$. Then there exists a finite subset of $\{J_\alpha\}$, say, $J_1', J_2', \ldots J_n'$ such that every point of E is interior to at least one J_j', $1 \leq j \leq n$.

Theorem 1.2

If $f(x) \in C$, then $f(x)$ is bounded on $[0,1]$.

Proof

$$\text{Define } f_1(x) = f(0) \qquad x < 0$$
$$= f(x) \qquad 0 \leq x \leq 1$$
$$= f(1) \qquad x > 1$$

Let x_0 be any point in $[0,1]$ and $\varepsilon = 1$. Since $f(x) \in C$, there exists a $\delta(x_0,1) > 0$ such that

$|f(x) - f(x_0)| < 1$ whenever $x \in (x_0 - \delta(x_0,1),\ x_0 + \delta(x_0,1))$

Also, $|f_1(x) - f_1(x_0)| < 1$ whenever $x \in (x_0 - \delta(x_0,1),\ x_0 + \delta(x_0,1))$

$$\begin{aligned} \text{Then} \quad |f_1(x)| &= |f_1(x) - f_1(x_0) + f_1(x_0)| \\ &\leq |f_1(x) - f_1(x_0)| + |f_1(x_0)| \\ &< 1 + |f_1(x_0)| \quad \text{whenever } x \in (x_0 - \delta(x_0,1),\ x_0 + \delta(x_0,1)) \end{aligned}$$

Let $\sigma_{x_0} = 1 + |f_1(x_0)|$, $J_{x_0} = (x_0 - \delta(x_0,1),\ x_0 + \delta(x_0,1))$.

Then $|f_1(x)| < \sigma_{x_0}$ whenever $x \in J_{x_0}$

Clearly every point of $[0,1]$ is interior to at least one of the intervals $\{J_{x_\alpha}\}_{\alpha \in \Omega}$. By Theorem 1.1, there exists a finite subset of $\{J_{x_\alpha}\}$, say, $J_{x_1}, J_{x_2}, \ldots J_{x_n}$ such that every point of $[0,1]$ is interior to at least one J_{x_j}, $1 \leq j \leq n$.

Let $\sigma = \max(\sigma_{x_1}, \sigma_{x_2}, \ldots \sigma_{x_n})$.

Then $|f_1(x)| < \sigma$ for all $x \in [0,1]$.

Hence $|f(x)| < \sigma$ for all $x \in [0,1]$. ⊠

Definition 1.5

M is the least upper bound (l.u.b.) of $f(x)$ for all x in J iff

(i) $f(x) \leq M$ for all $x \in J$ and

(ii) For any $\varepsilon > 0$, there exists an $x_0 \in J$ such that $f(x_0) > M - \varepsilon$.

Definition 1.6

m is the greatest lower bound (g. l. b.) of $f(x)$ for all x in J iff

(i) $f(x) \geq m$ for all $x \in J$ and

(ii) For any $\varepsilon > 0$, there exists an $x_0 \in J$ such that $f(x) < m + \varepsilon$.

Definition 1.7

M is the maximum of $f(x)$ for all x in J iff

(i) $M = $ l.u.b. $f(x)$ for all x in J.

(ii) there exists an $x_0 \in J$ such that $f(x_0) = M$.

The minimum of $f(x)$, m, is similarly defined.

Theorem 1.3

If $f(x) \in C$, then $f(x)$ attains its maximum in $[0,1]$.

Proof:

We prove the theorem by showing that $f(x)$ attains its l.u.b. in $[0,1]$. The desired conclusion then follows from Definition 1.7.

$\because f(x) \in C$. $\therefore$ By Theorem 1.2, $f(x)$ is bounded. Hence l.u.b. of $f(x)$ exists. Let $M = $ l.u.b. $f(x)$.

Suppose that the theorem is false, i.e., $\exists$ no $x_0 \in [0,1]$ such that $f(x_0) = M$. Then $M - f(x) > 0$ for all $x \in [0,1]$.

Also, $M - f(x)$ is continuous on $[0,1]$.

Hence, $\frac{1}{M-f(x)} > 0$ for all $x \in [0,1]$, and $\frac{1}{M-f(x)}$ is continuous on $[0,1]$.

By Theorem 1.2 $\exists\ \sigma > 0$ such that $|1/M-f(x)| < \sigma$, i.e., $1/M-f(x) < \sigma \quad \forall\ x \in [0,1]$.

It follows that $M - f(x) > 1/\sigma$

$$f(x) < M - 1/\sigma \text{ for all } x \in [0,1].$$

This is a contradiction to the definition of M.

Thus, $\exists$ an $x_o \in [0,1]$ such that $f(x_o) = M$. ⊠

Theorem 1.4

If $f(x) \in C$, then $f(x)$ attains its minimum in $[0,1]$.

Proof:

The proof is analogous to that of Theorem 1.3. ⊠

Definition 1.8

Let x be a real number, and let $\varepsilon > 0$ be any positive number. Then the totality of points in $[x - \varepsilon, x + \varepsilon]$ is called a closed neighborhood of x and the totality of points in $(x - \varepsilon, x + \varepsilon)$ is called an open neighborhood of x.

Theorem 1.5

If $f(x) \in C$, then $f(x)$ attains all values between its maximum and its minimum in $[0,1]$.

Proof:

$\because f(x) \in C$. $\therefore \exists$ an x' and an x'' in $[0,1]$ such that

$f(x') = \text{g.l.b. } f(x) = m$, $f(x'') = \text{l.u.b. } f(x) = M$.

Consider the closed interval $[x', x'']$, where for definiteness we have assumed $x' < x''$. (If $x' = x''$, the theorem is trivial.)

Let μ be such that $m < \mu < M$.

Then since $f(x'') = M$, $\exists$ a neighborhood of x'' such that $f(x) > \mu$ for all x in this neighborhood.

For suppose for every neighborhood of x'', $\exists$ a point x_o in the neighborhood such that $f(x_o) \leq \mu$, then $f(x)$ will not be continuous at x''.

Consider the totality of neighborhoods $y \leq x \leq x''$ for which $f(x) > \mu$. The y's are bounded below by x'. Hence g.l.b. of the y's exists. Let γ = g.l.b. y's. Then $f(\gamma) = \mu$.

For suppose $f(\gamma) > \mu$. Then $\because$ $f(x)$ is continuous at γ,

for $\varepsilon = 1/2 \ (f(\gamma) - \mu)$, $\exists$ a $\delta > 0$ such that

$|f(x) - f(\gamma)| < 1/2 \ (f(\gamma) - \mu)$ whenever $x \in (\gamma-\delta, \gamma+\delta)$

i.e., $f(x) > f(\gamma) - 1/2 \ (f(\gamma) - \mu)$

$f(x) > 1/2 \ (f(\gamma) + \mu)$

$f(x) > \mu$ whenever $x \in (\gamma-\delta, \gamma+\delta)$

This is a contradiction to the assumption that γ = g.l.b. y's. On the other hand, suppose $f(\gamma) < \mu$. By the same reasoning as above there would exist an x^* on the right of γ such that $f(x^*) < \mu$. But x^* is in one of the y-neighborhoods and hence $f(x^*) > \mu$. We again have a contradiction. $\boxtimes$

The converse of this theorem is not true. That is, there exists $f(x) \in F$ such that $f(x)$ takes all values between its maximum and its minimum in $[0,1]$, and yet $f(x)$ is discontinuous on $[0,1]$. We give the following example.

Let $f(x) = x$ x rational in $[0,1]$

$= 1 - x$ x irrational in $[0,1]$.

Clearly, $f(x)$ takes all values between 0 and 1 but it is discontinuous at every point in $[0,1]$.

Definition 1.9

Let $\{a_n\}$ be a sequence of real numbers. We say that $\{a_n\}$ converges to the limit a iff given any $\varepsilon > 0$, there exists an integer $N > 0$ such that $|a_n - a| < \varepsilon$ for all $n > N$.

Theorem 1.6

(Cauchy Convergence Theorem) A necessary and sufficient condition that $\{a_n\}$ converges is: Given any $\varepsilon > 0$, there exists an integer $N > 0$ such that $|a_n - a_{n+p}| < \varepsilon$ for all $n > N$, p any positive integer.

Definition 1.10

Let $f(x)$ be a function defined in $[a,b]$. If, for some $x_0 \in [a,b]$, $f(x_0) = x_0$, x_0 is called a fixed point.

Theorem 1.7

Let $f(x)$ be continuous on $[a,b]$, and $a \leq f(x) \leq b \quad \forall \quad x \in [a,b]$. Then $f(x)$ has a fixed point.

Proof:

Let c be the midpoint of $[a,b]$.

Consider the closed interval (either $[a,c]$ or $[c,b]$) such that the values of $f(x)$ at the end points are on both sides of the line $y = x$. Call it J_1.

Consider the midpoint of J_1. Let J_2 be the half interval such that the values of $f(x)$ at the end points are on opposite sides of $y = x$, and so on.

Note that if J_j cannot be defined for some j, then the theorem would have been proved. Therefore assume J_j is defined for $j = 1,2,3, \ldots$.

In this way we construct an infinite set of closed intervals

$J_1, J_2, \ldots$ such that

(i) $J_{j+1} \subset J_j$ for $j = 1,2,3, \ldots$.

(ii) The lengths of the J_j's are decreasing to zero, and

(iii) For each J_j, the values of $f(x)$ at the end points lie on opposite sides of $y = x$.

Let α_j = the left hand end point of the J_j interval.

β_j = the right hand end point of the J_j interval.

Since given any $\varepsilon > 0$, $\exists$ an integer $N > 0 \ni |\alpha_n - \alpha_{n+p}| < \varepsilon$ $n > N$, and every positive integer p (take N such that $J_N = 1/2^N (b-a) < \varepsilon$), by Theorem 1.6 $\{\alpha_n\}$ converges to some number α $\cdot$ α is contained in each J_j.

For suppose it were not. Then there would exist a J_Q, Q some positive integer, such that $\alpha_n - \alpha > 0$ for all $n \geq Q$. This is impossible, for if we let $\varepsilon = 1/2\,(\alpha_Q - \alpha) > 0$, then there would exist an integer $N' > 0$ such that $|\alpha_n - \alpha| < \varepsilon$ for all $n > N'$.

Let $n > \max(Q, N')$. Then $\alpha_Q - \alpha \leq \alpha_n - \alpha = |\alpha_n - \alpha| < \alpha_Q - \alpha/2$ which is absurd.

Similarly, $\{\beta_n\}$ converges to β, and β is contained in each J_j.

We now assert that $\alpha = \beta$.

Suppose $\alpha \neq \beta$. Then $\beta - \alpha = \mu > 0$. Since the J_j's are decreasing in length, we can find an integer $N > 0$ such that for all $n > N$, the lengths of J_n is less than μ. This is a contradiction since both α and β are in J_n.

Let $\alpha = \beta = \gamma$. Then $f(\gamma) = \gamma$.

Suppose $f(\gamma) \neq \gamma$. With no loss of generality assume $f(\gamma) - \gamma = \mu_1 > 0$. Let $\varepsilon = (1/2)\mu_1 > 0$. Then since $f(x)$ is continuous at γ, there exists an $\mu_2 > 0$ such that

$|f(\gamma) - f(x)| < \mu_1/2$. whenever $x \in (\gamma - \mu_2, \gamma + \mu_2)$

Let $\mu = \min((1/2)\mu_1, \mu_2)$. Then

$|f(\gamma) - f(x)| < (1/2)\mu_1$ whenever $x \in (\gamma - \mu, \gamma + \mu)$.

Since γ is the common limit of the sequences of left and right hand end points of the J_j's, therefore there exists an integer $n > 0$ such that $J_n = [x_n, x_n'] \subset (\gamma - \mu, \gamma + \mu)$.

But then

$f(x_n) > f(\gamma) - \mu_1/2 > 1/2\ (f(\gamma) + \gamma) > 1/2\ (\gamma + \mu_1 + \gamma) > \gamma + (1/2)\ \mu_1 > \gamma + \mu$. Also, $f(x_n') > \gamma + \mu$. Thus both $f(x_n)$ and $f(x_n')$ are on the same side of $y = x$. This is a contradiction. $\boxtimes$

Properties of A

Definition 1.11

Let $A \subset F$ be the set of all $f(x) \in F$ such that $|f(x)| \in C$. A is called the class of positively continuous functions.

Theorem 1.8

$C \subset A$.

Proof:

(i) $f(x) \in C \Rightarrow$ for every $x_1 \in [0,1]$, and any $\varepsilon > 0$, there exists a $\delta(x_1, \varepsilon) > 0 \ni$:

$$\Rightarrow |f(x) - f(x_1)| < \varepsilon \quad \text{whenever } |x - x_1| < \delta(x_1, \varepsilon)$$

$$\Rightarrow \big|\, |f(x)| - |f(x_1)| \,\big| < \varepsilon \text{ whenever } |x - x_1| < \delta(x_1, \varepsilon)$$

$$\Rightarrow |f(x)| \in C$$

$\therefore C \subseteq A.$

(ii) There exists $f(x) \in A$ such that $f(x) \notin C$, i.e., $A \nsubseteq C$.

Let $f(x) = x$ $\quad$ x irrational in $[0,1]$

$\qquad\quad = -x$ $\quad$ x rational in $[0,1]$

Then $|f(x)| \in C$, i.e., $f(x) \in A$. But clearly $f(x) \notin C$.

By (i) and (ii), we have $C \subset A$. $\boxtimes$

<u>Theorem 1.9</u>

If $f(x) \in C$, then $g(x) = \dfrac{|f(x)|}{1 + |f(x)|}$ has a fixed point in $[0,1]$.

<u>Proof:</u>

Since $f(x) \in C$. Therfore $f(x) \in A$. Hence $|f(x)| \in C$, $1 + |f(x)| \in C$.

Since $1 + |f(x)| \neq 0$ for all $x \in [0,1]$. $\therefore g(x) = \dfrac{|f(x)|}{1 + |f(x)|} \in C$.

Also, $0 \leq g(x) < 1$ for all $x \in [0,1]$ and for all $f(x) \in C$.

$\therefore$ by Theorem 1.7, $g(x)$ has a fixed point in $[0,1]$. $\boxtimes$

<u>Theorem 1.10</u>

If $f(x) \in A$, then $f(x)$ is bounded on $[0,1]$.

<u>Proof:</u>

$$
\begin{aligned}
f(x) \in A &\Longrightarrow |f(x)| \in C \\
&\Longrightarrow \exists\ \sigma > 0 \text{ such that } \big||f(x)|\big| \leq \sigma \quad \text{for all } x \in [0,1] \\
&\Longrightarrow |f(x)| \leq \sigma \text{ for all } x \in [0,1]
\end{aligned}
$$

Hence $f(x)$ is bounded on $[0,1]$. $\boxtimes$

Although $f(x) \in C$ implies that $f(x)$ attains its maximum and its minimum in $[0,1]$, there exist functions in A such that this is not true. The following example illustrates this fact for a maximum:

Let $f(x) = x^2 - 1$ $\qquad$ x rational in $[0,1]$

$\qquad\quad = 1 - x^2$ $\qquad$ x irrational in $[0,1]$

Clearly $|f(x)| \in C$. Hence $f(x) \in A$, but $f(x)$ has no maximum in $[0,1]$.

Theorem 1.11

If $f(x) \in A$, then $g(x) = \dfrac{|f(x)|}{1 + |f(x)|}$ has a fixed point in $[0,1]$.

Proof:

$g(x)$ is continuous on $[0,1]$ and $0 \leq g(x) < 1$ for all $x \in [0,1]$. Therefore, by Theorem 1.7, $g(x)$ has a fixed point in $[0,1]$. $\boxtimes$

Theorem 1.12

If $f(x) \in A$ and $0 \leq f(x) \leq 1$ for all $x \in [0,1]$, then $f(x)$ has a fixed point.

Proof:

Since $f(x) \in A$. Therefore $|f(x)| \in C$.

Also, since $0 \leq f(x) \leq 1$, for all $x \in [0,1]$, thus $0 \leq |f(x)| \leq 1$ for all $x \in [0,1]$.

By Theorem 1.8, $|f(x)|$ has a fixed point. Hence $f(x)$ has a fixed point. $\boxtimes$

SECTION II - SOME ALGEBRAIC PROPERTIES

In this section, some elementary algebraic properties of C are given. A comparison between the properties of C and A will be made. Algebraic structures of F and some of its subclasses will also be investigated, together with an introduction of the concept of a right (left) ideal in F.

Properties of C

Theorem 2.1

If $f(x)$, $g(x) \in C$, then $f(x) \pm g(x) \in C$.

Theorem 2.2

If $f(x)$, $g(x) \in C$, then $[f(x)][g(x)]$ is in C.

Theorem 2.3

If $f(x) \in C$, and $f(x) \neq 0$ for all x in $[0,1]$, then $1/f(x) \in C$.

Theorem 2.4

If $f(x)$, $g(x) \in C$, and $0 \leq g(x) \leq 1$ for all x in $[0,1]$, then $f(g(x))$ is in C.

Theorem 2.5

If $f(x)$, $g(x) \in C$ and $g(x) \neq 0$ for all x in $[0,1]$, then $f(x)/g(x)$ is in C.

Proof:

The result follows from Theorem 2.2 and 2.3. $\boxtimes$

Theorem 2.6

Let $f(x) \in C$ and $g(x) \notin C$. Then (a) $\pm f(x) \pm g(x) \notin C$. If in addition, $f(x) \neq 0$ for all x in $[0,1]$, then (b) $[f(x)][g(x)] \notin C$ and (c) $g(x)/f(x) \notin C$.

Proof:

(a) Suppose $f(x) + g(x) \in C$. Since $f(x) \in C$, by Theorem 2.1, $f(x) + g(x) - f(x) = g(x) \in C$. This is a contradiction.

The other cases can be proved in a similar manner.

(b) Suppose $[f(x)][g(x)] \in C$. By hypothesis $f(x) \in C$, and $f(x) \neq 0$ for all x in $[0,1]$. Therefore, by Theorem 2.2 and 2.3, $[f(x)\ g(x)]/f(x) = g(x) \in C$. This is a contradiction.

(c) Suppose $g(x)/f(x) \in C$. We have $f(x) \in C$. Therefore, by Theorem 2.2, $[g(x)/f(x)]\,[f(x)] = g(x) \in C$. This is a contradiction. $\boxtimes$

Properties of A

The class of positively continuous functions $A \subset F$ shares some of the algebraic properties of C but there are some characteristics of C that A does not possess.

The sum, difference of two positively continuous functions in general is not a positively continuous function, as may be seen by the following example for the sum:

$$\text{Let } f(x) = \begin{cases} 1 & 0 \leq x < 1/2 \\ -1 & 1/2 \leq x \leq 1 \end{cases} \qquad g(x) = 1 \text{ for all } x \text{ in } [0,1]$$

$$\text{Then, } f(x)+g(x) = \begin{cases} 2 & 0 \leq x < 1/2 \\ 0 & 1/2 \leq x \leq 1. \end{cases}$$

Clearly, $f(x), g(x) \in A$, but $f(x) + g(x) \notin A$.

However, by imposing some restrictive properties on the functions, a result equivalent to Theorem 2.1 can be derived for A.

Theorem 2.7

If $f(x), g(x) \in A$, and $f(x), g(x)$ do not change sign in $[0,1]$, then $f(x) \pm g(x) \in A$.

Proof:

Assume $f(x) \geq 0$, $g(x) \geq 0$ for all x in $[0,1]$.

Then $|f(x)| = f(x)$, $|g(x)| = g(x)$.

By Theorem 2.1 and 1.8, $f(x) \pm g(x) \in A$.

The other cases can be proved similarly. $\boxtimes$

Theorem 2.8

If $f(x)$, $g(x) \in A$, then $[f(x)][g(x)] \in A$.

Proof:

Since $f(x)$, $g(x) \in A$, therefore $|f(x)|$, $|g(x)| \in C$. By Theorem 2.2, $[|f(x)|][|g(x)|] \in C$. Hence $|[f(x)][g(x)]| \in C$ and $[f(x)][g(x)] \in A$. $\boxtimes$

Theorem 2.9

If $f(x) \in A$ and $f(x) \neq 0$ for all x in $[0,1]$, then $1/f(x) \in A$.

Proof:

Since $f(x) \in A$, therefore $|f(x)| \in C$.

Also, $f(x) \neq 0$ for all x in $[0,1]$, Thus $|f(x)| \neq 0$ for all x in $[0,1]$.

By Theorem 2.3, $1/|f(x)| = |1/f(x)| \in C$. Hence $1/f(x) \in A$. $\boxtimes$

In general, A is not closed with respect to composition. Even if the hypothesis of Theorem 2.4 is satisfied, its conclusion still does not hold for A. The following example illustrates this fact.

Let
$$f(x) = \begin{cases} 1/2 & 0 \leq x < 1/2 \\ -x & 1/2 \leq x \leq 1 \end{cases} \qquad g(x) = 1 \text{ for all } x \text{ in } [0,1]$$

$$|f(x)| = \begin{cases} 1/2 & 0 \leq x < 1/2 \\ x & 1/2 \leq x \leq 1 \end{cases}$$

Therefore, $f(x)$, $g(x) \in A$ and $0 \leq g(x) \leq 1$ for all x in $[0,1]$.

Since $f(g(x)) = 1/2 \qquad 0 \leq x < 1/2$

$\qquad\qquad = -1 \qquad 1/2 \leq x \leq 1$

Thus, $|f(g(x))| = 1/2 \qquad 0 \leq x < 1/2$

$\qquad\qquad = 1 \qquad 1/2 \leq x \leq 1$

Clearly, $|f(g(x))| \notin C$. Hence $f(g(x)) \notin A$.

The next theorem, however, is true.

Theorem 2.10

If $f(x)$, $g(x) \in A$ and $g(x)$ is $\ni$; $0 \leq g(x) \leq 1$ for all x in $[0,1]$, then $f(g(x))$ is in A.

Proof:

By hypothesis, $|f(x)| = f(x)$, $|g(x)| = g(x)$.

Also, $0 \leq g(x) \leq 1$ for all x in $[0,1]$.

Therefore, by Theorem 2.4 and 1.8, $f(g(x))$ is in A. $\boxtimes$

Theorem 2.11

If $f(x)$, $g(x) \in A$ and $g(x) \neq 0$ for all x in $[0,1]$, then $f(x)/g(x)$ is in A.

Proof:

Since $g(x) \in A$ and $g(x) \neq 0$ for all x in $[0,1]$, therefore by Theorem 2.9, $[1/g(x)]$ is in A.

By Theorem 2.8, $[f(x)][1/g(x)] = f(x)/g(x) \in A$. $\boxtimes$

The following statements are also true for A:

(i) There exist functions $f(x)$, $g(x)$ in A such that $f(x)^{g(x)}$ is not in A. Example: Let $f(x) = x \quad 0 \leq x \leq 1 \quad g(x) = x \quad 0 \leq x < 1/2$

$\qquad = -x \quad 1/2 \leq x < 1$

Then $|f(x)^{g(x)}| = f(x)^{g(x)} = x^x \qquad 0 < x < 1/2$

$= 0 \qquad x = 0$

$= x^{-x} \qquad 1/2 \leq x \leq 1$

Clearly, $x^x < 1$ for all x in $[0,1/2)$ and $f(1/2)^{g(1/2)} = (1/2)^{-1/2} = 2^{1/2} > 1$. Hence $f(x)$ is not continuous at $x = 1/2$.

Therefore, $|f(x)^{g(x)}| \notin C$, and $f(x)^{g(x)} \notin A$.

(ii) There exist functions $f(x)$, $g(x)$ in F such that $f(x)^{g(x)}$ is in A, but neither $f(x)$ nor $g(x)$ is in A.

Example: Let $f(x) = 2 \quad 0 \leq x < 1/2 \qquad g(x) = 2 \quad 0 \leq x < 1/2$

$= 4 \quad 1/2 \leq x \leq 1 \qquad\qquad = 1 \quad 1/2 \leq x \leq 1$

Then $\qquad f(x)^{g(x)} = 4 \quad 0 \leq x \leq 1$

Therefore, $f(x)^{g(x)}$ is in A, but neither $f(x)$ nor $g(x)$ is in A.

Algebraic Structures of F and Some of Its Subclasses

F together with some subclasses of F have certain algebraic structures that are worth noting and shall be given to conclude this section. Only definitions that are more pertinent to the results are listed.

Definition 2.1

A semi-group is a system consisting of a set E and an operation (o) satisfying the following axioms:

1. closure: If $a,b \in E$, then $a \circ b \in E$.

2. associative: If $a,b,c, \in E$, then $a \circ (b \circ c) = (a \circ b) \circ c$.

If the operation (o) is ordinary multiplication, $a \circ b$ is written ab.

Definition 2.2

A group is a semi-group $(E, \circ)$ such that

1. There exists a unique element i in E such that

   $i \circ a = a \circ i = a$ for all a in E.

2. For every element a in E, there exists a unique a^{-1} in E such

   that $a \circ a^{-1} = a^{-1} \circ a = i$.

Definition 2.3

A commutative group is a group $(E, \circ)$ such that $a \circ b = b \circ a$ for all a, b in E.

Definition 2.4

A ring is a system consisting of a set E and two operations $(+)$ and $(\circ)$ such that

1. $(E, +)$ is a commutative group.

2. $(E, \circ)$ is a semi-group.

3a. $a \circ (b + c) = a \circ b + a \circ c$ and

3b. $(b + c) \circ a = b \circ a + c \circ a$ for all a, b, c in E.

Theorem 2.12

$(C, +, \circ)$ is a ring, where $(+)$, $(\circ)$ are the ordinary addition and multiplication.

Proof:

1. $(C, +)$ is a commutative group.

a. closure: By Theorem 2.1, $f(x) + g(x) \in C$ for all $f(x)$, $g(x)$ in C.

b. associative: It is clear that

$$(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x)) \text{ for all}$$

$f(x)$, $g(x)$, $h(x)$ in C.

c. $i = 0$, since $f(x) + 0 = 0 + f(x) = f(x)$ for all $f(x)$ in C.

d. $f(x)^{-1} = -f(x)$, since $f(x) + (-f(x)) = 0$ for all $f(x)$ in C.

e. $f(x) + g(x) = g(x) + f(x)$ for all $f(x)$, $g(x)$ in C.

2. (C, o) is a semi-group.

a. closure: By Theorem 2.2, $f(x)g(x) \in C$ for all $f(x)$, $g(x)$ in C.

b. associative: It is clear that $f(x)[g(x)h(x)] = [f(x)g(x)]\, h(x)$ for all $f(x)$, $g(x)$, $h(x)$ in C.

3a, 3b of Definition 2.4 can be easily seen to be satisfied.

Hence by Definition 2.4, C is a ring with respect to ordinary addition and multiplication. ⊠

However, C in general is not a ring with respect to ordinary addition and composition. A subclass of C that can easily be shown to satisfy the postulates for a ring is given below:

<u>Definition 2.5</u>

Let $P = \{ax: 0 \leq a \leq 1, ax \text{ in } F\}$. Clearly, $P \subset C$.

<u>Theorem 2.13</u>

P is a ring with respect to ordinary addition and composition.

<u>Proof:</u>

1. (P, +) is a commutative group, by Theorem 2.12.

2. (P, o) is a semi-group.

a. closure: By Theorem 2.4, a_1x, a_2x in P implies that $a_1x \circ a_2x = a_1a_2x$ is in P.

b. associative: Let a_1x, a_2x , a_3x be in P.

Then $(a_1x \circ a_2x) \circ a_3x = a_1a_2x \circ a_3x$

$$= a_1a_2a_3x$$

$$= a_1x \circ a_2a_3x = a_1x \circ (a_2x \circ a_3x).$$

3a. $a_1x \circ (a_2x+a_3x) = a_1(a_2x+a_3x) = a_1a_2x+a_1a_3x$

$= a_1x \circ a_2x+a_1x \circ a_3x$

3b. $(a_2x+a_3x) \circ a_1x=(a_2+a_3)x \circ a_1x = (a_2+a_3)a_1x$

$= a_2a_1x+a_3a_1x$

$= a_2x \circ a_1x+a_3x \circ a_1x$

Hence by definition 2.4, F is a ring with respect to ordinary addition and composition. ⊠

Definition 2.6

Let E, E' be sets. Then $E \times E' = \{(e,e'): e \in E, e' \in E'\}$.

Definition 2.7

A mapping from E to E' is a function which associates to each element e of E precisely one element e' of E', designated $E \rightarrow E'$.

Definition 2.8

A partial (half) groupoid is a set E together with a mapping of a non-empty subset of $E \times E$ into E.

Theorem 2.14

F is a half groupoid under composition.

Proof:

Let f(x), g(x) be in F. Then f[g(x)] is in F iff $0 \leq g(x) \leq 1$ for all x in [0,1].

Let $E_1 = \{g(x) \in F: 0 \leq g(x) \leq 1$ for all x in $[0,1]\}$. Then under composition, $F \times E_1$ is mapped into F.

Hence F is a half groupoid under composition. ⊠

Definition 2.9

Let $I = \{f(x) \mid f(x): [0,1] \rightarrow [0,1]\}$. Clearly, $I \subset F$.

Theorem 2.15

I is a semi-group with respect to composition.

Proof:

Let $f_1(x), f_2(x), f_3(x)$ be in I. Then $0 \leq f_1(x) \leq 1$, $0 \leq f_2(x) \leq 1$, $0 \leq f_3(x) \leq 1$ for all x in $[0,1]$.

1. closure: Clearly, $f_1(x) \circ f_2(x) = f_1[f_2(x)]$ is in I.
2. associative: $[f_1(x) \circ f_2(x)] \circ f_3(x) = f_1[f_2(x)] \circ f_3(x)$

$$= f_1\{f_2[f_3(x)]\}$$

$$= f_1(x) \circ f_2[f_3(x)] =$$

$f_1(x) \circ [f_2(x) \circ f_3(x)]$.

Thus by Definition 2.1, I is a semi-group with respect to composition. ⊠

Definition 2.10

A left (right) ideal in F is a subset E of F such that

a. E is not empty and

b. FE (EF) $\subseteq$ E. Where $FE = \{f \circ e: f \in F, e \in E, f \circ e \text{ is defined}\}$.

Theorem 2.16

I is a right ideal in F with respect to composition.

Proof:

Since identity map is in I, hence I is not empty.

Let $IF = \{i[f(x)]: f(x) \in F, i(x) \in I, i[f(x)] \text{ is defined}\}$.

Then since $i[f(x)]$ is defined, therefore $0 \leq i[f(x)] \leq 1$ for all x in $[0,1]$. Thus $i[f(x)]$ is in I and $IF \subseteq I$.

By Definition 2.10, I is a right ideal in F with respect to composition.

⊠

SECTION III - CLOSURE OF THE GRAPH OF A FUNCTION - AREA FUNCTION

In this section, we introduce the concept of "area function", which is an extension of the usual definition of area in the Riemann sense. Only functions in I are investigated. Some preliminary topological concepts that are necessary for the development will be given preceding the study.

Topological Preliminaries

Definition 3.1

A collection of subsets $\{B_\alpha\}_{\alpha \in \Omega}$ of a given set S is called a <u>basis</u> for a topology in S iff

a. $\bigcup_{\alpha \in \Omega} B_\alpha = S$ and

b. if p is a point of $B_\alpha \cap B_\beta$ then there exists an element B_γ of $\{B_\alpha\}$ which contains p and which itself is contained in $B_\alpha \cap B_\beta$.

Definition 3.2

A set S is a <u>metric space</u> with metric d iff there exists a real-valued function $d(x,y)$ on pairs of elements of S such that:

(i) $d(x,y) \geq 0$. $d(x,y) = 0$ iff $x = y$.

(ii) $d(x,y) = d(y,x)$.

(iii) $d(x,y) + d(y,z) \geq d(x,z)$ (the triangle inequality).

Definition 3.3

Let S be a metric space with metric d, and let r be a positive number.

<u>Spherical neighborhood</u> $S(x,r)$ of the point x is the set of all points y in S such that $d(x,y) < r$. r is called the <u>radius</u> of the spherical neighborhood.

Note that with the usual metric for E^n, the Euclidean n-dimensional space, i.e., $d(x,y)$ = distance between two points in E^n, $S(x,r)$ in E^1 becomes an open interval $(x - r, x + r)$, and $S(x,r)$ in E^2 is the interior of a circle with center at x and radius r.

Lemma 3.1

The set of all spherical neighborhoods in S satisfies the conditions for a basis.

Proof:

a. $\bigcup_{r \in \Omega} S(x,r) = S$. Trivial.

b. Let p be a point in $S(x_1,r_1) \cap S(x_2,r_2)$.

Let $r = \min [r_1 - d(p,x_1), r_2 - d(p,x_2)]$.

Since p is in $S(x_1,r_1) \cap S(x_2,r_2)$, therefore r is positive.

Suppose q is a point in $S(p,r)$. Then for $j = 1$ or 2,

$d(q,x_j) \leq d(q,p) + d(p,x_j) < r + d(p,x_j) < r_j - d(p,x_j) + d(p,x_j) < r_j$.

Thus q lies in $S(x_j,r_j)$, $j = 1, 2$.

Also, $S(p, r) \subseteq S(x_1,r_1) \cap S(x_2, r_2)$. $\boxtimes$

Definition 3.4

A subset X of S is open iff it is a union of elements of a basis in S.

X is closed iff $S - X$ is open, i.e., iff its complement is open.

Definition 3.5

A point p is a limit point of a subset X of S iff every open set containing p also contains a point of X distinct from p.

Definition 3.6

$\bar{X}$ is the closure of X iff $\bar{X} = X \cup \{$all limit points of X$\}$.

Attention Patron:

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Theorem 3.2

If $X \subseteq S$, then X is closed iff $X = \bar{X}$.

Proof:

$(\Rightarrow)$: If X is closed, then $S - X$, is open.

If p is any point of $S - X$, then $S - X$ itself is an open set containing p but no other points of X. Hence no point of $S - X$ can be a limit point of X. Therefore, $X = \bar{X}$.

$(\Leftarrow)$: Suppose $X = \bar{X}$. Then no point of $S - X$ is a point or a limit point of X. For every point p in $S - X$, there exists an open set O_p containing no point of X. The union of all the sets O_p, p in $S - X$, is open. Clearly this union is $S - X$, and X is closed. ⊠

Definition 3.7

A subset X of S is said to be dense in S iff every point of S is a point or a limit point of X, i.e., $S = \bar{X}$.

Definition 3.8

A space is separable if it has a countable dense subset.

Lemma 3.3

E^n is separable.

Proof:

The set of all points whose coordinates are all rationals is countable and dense in E^n. ⊠

Theorem 3.4

Every separable metric space has a countable basis.

Proof:

Let S be a metric space with metric $d(x,y)$ and having a countable dense subset $X = \{x_j\}$, $j = 1, 2, 3, \ldots$.

For each rational number $r > 0$ and each j, there is a $S(x_j, r)$ and the set B of all these is countable.

We shall prove that B is a basis in S.

Let p be any point of S and let O be an open set containing p. Then there is a positive number ε such that $S(p, \varepsilon)$ is contained in O, by the definition of O. There is a point x_j of X such that $d(x_j, p) < \varepsilon/3$, since X is dense. Let r be a rational number satisfying $\varepsilon/3 < r < 2\varepsilon/3$, and consider $S(x_j, r)$. Certainly $S(x_j, r)$ contains p, and if y is any point of $S(x_j, r)$, then $d(y, p) \leq d(y, x_j) + d(x_j, p) < 2\varepsilon/3 + \varepsilon/3 < \varepsilon$. Thus y is in $S(p, \varepsilon)$. Therefore $S(x_j, r)$ is an element of B that contains p and lies in O. It follows that O is a union of elements of B and that B is a basis in S. $\boxtimes$

Definition 3.9

A space is completely separable iff it has a countable basis.

Theorem 3.5

Every separable metric space is completely separable.

Proof:

By Theorem 3.4 and definition above. $\boxtimes$

Theorem 3.6

Every completely separable space is separable.

Proof:

Let $\{B_j\}$ be a countable basis for a space S that is completely separable.

We claim that if $S_1 = \{b_j\}$, where b_j lies in B_j, then $\bar{S}_1 = S$.

It is clear that the set S_1 is countable.

Suppose a point p is in S but not in S_1. Then for every open set O that contains p, there exists a B_n of $\{B_j\}$ such that p is in B_n and B_n is contained in O. Since p is not in S_1, there exists at least one b_k of S_1 such that $b_k \neq p$ and b_k is in O. Hence p is a limit point of S_1.

It follows that $S = \bar{S}_1$. ⊠

Theorem 3.7

If S is a completely separable space, then every subspace of S is completely separable, and hence every subspace is separable.

Proof:

If $\{B_n\}$ is a countable basis for S and X is any subset of S, then $\{B_n \cap X\}$ is a countable basis for the subspace topology of X.

Assertion:

(1) $\bigcup_{n=1}^{\infty} (B_n \cap X) = X \cap \bigcup_{n=1}^{\infty} B_n$. But $\bigcup_{n=1}^{\infty} B_n = S$, therefore

$\bigcup_{n=1}^{\infty} (B_n \cap X) = X \cap S = X$.

(2) p in $(B_j \cap X) \cap (B_t \cap X)$

$\Rightarrow$ p is in both $B_j \cap X$ and $B_t \cap X$.

$\Rightarrow p \in$ both $B_j \cap B_t$ and X.

$\Rightarrow$ there exists a B_s of $\{B_n\}$ such that p is in B_s and B_s is contained in $B_j \cap B_t$

$\Rightarrow$ p is in $B_s \cap X$

x in $B_s \cap X \Rightarrow$ x is in both B_s and X

$\Rightarrow$ x is in both $B_j \cap B_t$ and X

$\Rightarrow$ x is in both $B_j \cap X$ and $B_t \cap X$

$\Rightarrow$ x is in $(B_j \cap X) \cap (B_t \cap X)$.

Therefore, $B_s \cap X \subseteq (B_j \cap X) \cap (B_t \cap X)$.

Thus X is completely separable and by Theorem 3.6, X is separable.

⊠

Henceforth, the usual metric is assumed to have been defined for E^n. This, together with Lemma 3.3 and Theorem 3.5, implies that E^n is completely separable.

Definition 3.10

A point x in a metric space S is an isolated point iff there exists a (spherical) neighborhood of x such that it contains no y in S other than x.

Theorem 3.8

A set L E^2 has only countably many isolated points.

Proof:

Since E^2 is completely separable, thereore, by Theorem 3.7, every subspace of E^2 is separable. Thus L is separable, i.e., L contains a countable dense subset. Hence, L has only countably many isolated points.

Topological Properties of I and Some of Its Subclasses

Definition 3.11

Let f(x) be a function in I. The graph of f(x), G_f, consists of the pairs (x, f(x)) for all x in [0,1].

Note that G_f is a set of points in E^2.

Definition 3.12

Let $\bar{G}_f$ be the closure of G_f.

Let f(x), g(x) be in I.

Then f(x) and g(x) are said to have the same area function iff $\bar{G}_f = \bar{G}_g$.

Theorem 3.9

Let f(x), g(x) be any two continuous functions in I such that $\bar{G}_f = \bar{G}_g$. Then f(x) and g(x) have the same integrals over [0,1].

Proof:

We first show that G_f is closed, i.e., $\overline{G}_f = G_f$.

Let (x_1, y_1) be a limit point of G_f. Then x_1 is in $[0,1]$.

For suppose x_1 is not in $[0,1]$, there would exist an $S(x, \varepsilon)$, $\varepsilon > 0$, such that $S(x, \varepsilon)$ contains no point of G_f. This is a contradiction.

Consider $(x_1, f(x_1))$. Suppose $f(x_1) \neq y_1$. Without loss of generality assume $y_1 - f(x_1) = k > 0$. Since $f(x)$ is continuous, thus for $\varepsilon = k/3$, there exists a $\delta_1 > 0$ such that $|f(x) - f(x_1)| < k/3$ whenever x is in $(x_1 - \delta_1, x_1 + \delta_1)$.

Let $\delta = \min. (\delta_1, k/3)$.

Then $|f(x) - f(x_1)| < k/3$ whenever x is in $(x_1 - \delta, x_1 + \delta)$.

But there exists an x_n in $(x_1 - \delta, x_1 + \delta)$ such that $(x_n, f(x_n))$ is in $S[(x_1, y_1), \delta]$, i.e., $|f(x_n) - y_1| < \delta \leq k/3$.

$f(x_n) > y_1 - k/3 = f(x_1) + k - k/3 = f(x_1) + 2k/3$.

Therefore $f(x_n) - f(x_1) > k/3$, a contradiction.

Similarly, $\overline{G}_g = G_g$.

By hypothesis, $\overline{G}_g = \overline{G}_f$. Hence $G_f = G_g$. Thus $f(x)$ and $g(x)$ have the same integrals over $[0,1]$. $\boxtimes$

The converse of this theorem is not true, as may be seen by the following example.

Let $f(x) = x$, $g(x) = 1 - x$. Clearly $g(x)$ and $f(x)$ have equal integrals over $[0,1]$ but $\overline{G}_g \neq \overline{G}_f$.

Theorem 3.10

Let U be the closed unit square in E^2, i.e., $U = \{(x,y): 0 \leq x \leq 1,$

$0 \leq y \leq 1$. Then there exists an $f(x)$ in I such that $\bar{G}_f = U$.

Proof:

The theorem will be demonstrated by explicitly constructing a function $f(x)$ satisfying the required conditions.

(a) Divide U into four squares. In $[0, 1/2]$, choose any two distinct rational numbers x_1, x_2 and define $f(x_1)$, $f(x_2)$ such that each of $(x_1, f(x_1))$, $(x_2, f(x_2))$ lies in the interior of each of the two quarters directly above $[0, 1/2]$. Similarly for the other interval $[1/2, 1]$.

(b) Subdivide each of the four squares into four quarters. In $[0, 1/4]$, for every square directly above $[0, 1/4]$ that contains no previously determined $(x, f(x))$, select a rational number x' and define $f(x')$ such that $(x', f(x'))$ lies within this square. Each of the remaining intervals $[1/4, 1/2]$, $[1/2, 3/4]$, $[3/4, 1]$ is then considered similarly.

(c) Further subdivide each of the 16 squares into four quarters, and so on.

(d) Complete the construction by defining

$$
\begin{aligned}
f(x) &= 0 \qquad x \text{ irrational in } [0,1] \\
&= x \qquad x \text{ rational in } [0,1] \text{ but such that} \\
&\qquad\quad f(x) \text{ is not defined by the above scheme.}
\end{aligned}
$$

(e) By construction, the set G_f is dense in U. Hence $\bar{G}_f = U$. ⊠

Definition 3.13

If $f(x)$, defined in a set S, is such that $f(x_1) \geq f(x_2)$ when x_1 and x_2 are points of S with $x_1 > x_2$, then $f(x)$ is non-decreasing in S. There is a similar definition for non-increasing functions. Functions which are non-increasing or non-decreasing are called monotone.

Theorem 3.11

If $f(x)$ is a monotone function in I, then $\bar{G}_f$ contains no non-degenerate square u in U.

Proof:

Suppose on the contrary, $\bar{G}_f$ contains u, u any non-degenerate square in U. Without loss of generality assume that the edges of the square u are parallel to the axes. Let s be the edge length of u.

Let (x_o, y_o) be the lower left hand corner of u. Then there exist a $(x_1, f(x_1))$ such that $(x_1, f(x_1))$ is in $S[(x_o, y_o + s), s/8]$, a $(x_2, f(x_2))$ such that $(x_2, f(x_2))$ is in $S[(x_o + s/2, y_o), s/8]$, and a $(x_3, f(x_3))$ such that $(x_3, f(x_3))$ is in $S[(x_o + s, y_o + s), s/8]$.

Therefore, $x_1 < x_o + s/8$; $x_o + 3s/8 < x_2 < x_o + 5s/8$; $x_o + 7s/8 < x_3$; i.e., $x_1 < x_2 < x_3$.

But, $f(x_1) > y_o + 7s/8$; $f(x_2) < y_o + s/8$; $f(x_3) > y_o + 7s/8$; i.e., $f(x_2) < f(x_1)$ and $f(x_3) > f(x_2)$.

This is a contradiction. Hence $\bar{G}_f \not\supseteq u$. $\boxtimes$

Theorem 3.12

Let $f(x)$ be a monotone function in I. Then $\bar{G}_f$ contains no O, O any open set in U.

Proof:

Suppose $\bar{G}_f \supseteq O$, O an open set in U. Then there would exist a square u such that u is contained in O. This is impossible by Theorem 3.11. $\boxtimes$

Corollary to Theorem 3.12

Let $f(x)$ be a function in I. If in any non-degenerate closed interval in $[0,1]$, $\bar{G}_f \supseteq O$, O an open set, then $f(x)$ is not monotonic.

Proof:

By Theorem 3.12, the hypothesis implies that $f(x)$ is not monotonic in any non-degenerate closed interval in $[0,1]$, in particular, $[0,1]$ itself. $\boxtimes$

Definition 3.14

Let $f(x)$ be a real-valued function defined on an interval $[a,b]$.

Let $[a,b]$ be subdivided into a finite number of non-overlapping intervals, i.e.,

$$a = x_o < x_1 < x_2 \ldots < x_n = b.$$

If there exists a non-negative number K such that K = l.u.b. $\sum_{j=0}^{n-1} |f(x_{j+1}) - f(x_j)|$ for all such subdivisions of $[a,b]$, then $f(x)$ is said to be of bounded variation on $[a,b]$.

Theorem 3.13

Let $f(x)$ be defined, finite valued, and monotone on $[a,b]$. Then $f(x)$ is of bounded variation.

Proof:

To be specific suppose $f(x)$ is non-decreasing on $[a,b]$. Then for every subdivision

$$a = x_o < x_1 < x_2 \ldots < x_n = b,$$

$$\sum_{j=0}^{n-1} |f(x_{j+1}) - f(x_j)| = \sum_{j=0}^{n-1} [f(x_{j+1}) - f(x_j)] = f(b) - f(a) = K,\ K \geq 0.$$

K is the common value of all the sums, and is therefore their least upper bound.

Hence $f(x)$ is of bounded variation. $\boxtimes$

Definition 3.15

If there is a finite number b which is such that for every $\varepsilon > 0$, there exists a $\delta > 0$ for which $|f(x) - b| < \varepsilon$ whenever $0 < |x - a| < \delta$, then b is the limit of f(x) as x tends to a. The notation for this is

$\lim_{x \to a} f(x) = b.$

Note that if $\lim_{x \to a} f(x) = f(a)$, then f(x) is continuous at x = a.

Definition 3.16

If there exists a number $\bar{b}$ such that for every $\varepsilon > 0$,

(a) every interval with a as an interior point contains a point $x \neq a$ for which $f(x) > \bar{b} - \varepsilon$ and

(b) there exists a $\delta > 0$ such that $f(x) < \bar{b} + \varepsilon$ whenever $0 < |x-a| < \delta$,

then $\bar{b}$ is the limit superior or upper limit of f(x) as x tends to a. There is a corresponding definition for the limit inferior or lower limit $\underline{b}$. In symbols $\limsup_{x \to a} f(x) = \bar{b}$, $\liminf_{x \to a} f(x) = \underline{b}$.

It follows that $\lim_{x \to a} f(x)$ exists iff $\bar{b} = \underline{b}$.

Definition 3.17

Let f(x) be defined on a set S and let x_o be a limit point on the right of points of S. If f(x) tends to a limit as $x \to x_o$, $x > x_o$, x in S, this limit is denoted by $\lim_{x \to x_o+} f(x) = f(x_o+)$. The limit $f(x_o-)$ is similarly defined.

As a natural consequence, $\lim_{x \to x_o} f(x)$ exists iff $f(x_o+) = f(x_o-)$.

Theorem 3.14

If f(x) is of bounded variation on [a,b], then for every point x_o in [a,b], the limits $f(x_o+)$, $f(x_o-)$ exist.

Proof:

Let x_o be a point of [a,b]. Then x_o is a limit point of [a,b].

Consider points of [a,b] that are to the right of x_o. Then $f(x_o+)$ exists.

For, suppose that $\limsup_{x \to x_o+} f(x) - \liminf_{x \to x_o+} f(x) > K > 0$.

It is then possible to get a sequence of points

$x_1 > x_1' > x_2 > x_2' > \ldots > x_n > x_n'$

that tends to x_o with

$|f(x_j') - f(x_j)| > K$.

$$\sum_{j=0}^{n-1} |f(x_{j+1}') - f(x_j)| > nK,$$

where n may be taken arbitrarily large.

Hence f(x) is not of bounded variation on [a,b].

This is a contradiction.

Similarly, $f(x_o-)$ exists. ⊠

Corollary to Theorem 3.14

Let f(x) be defined, finite valued, and monotone on [a,b]. Then for every point x_o in [a,b], the limits $f(x_o+)$, $f(x_o-)$ exist.

Proof:

By Theorem 3.13, our hypothesis implies that f(x) is of bounded variation on [a,b]. Hence, by Theorem 3.14 above, the limits $f(x_o+)$, $f(x_o-)$ exist. ⊠

Definition 3.18

A set S is said to be dense in itself if every point of the set is a limit point of S.

Definition 3.19

A set S is perfect iff every limit point of S belongs to S, and every point of S is a limit point of S.

An alternate definition for a perfect set can easily be seen to be "a set which is closed and dense in itself."

Since $\overline{G}_f = G_f$ if f(x) is a continuous function in I, it is clear that a necessary condition that $\overline{G}_f$ contains no graph of a continuous function in I is that f(x) be discontinuous. That this condition is not sufficient may be seen from the following example:

Let $f(x) = 1 \quad x$ rational in $[0,1]$

$\qquad = 0 \quad x$ irrational in $[0,1]$

Then $\overline{G}_f = \{(x,1): 0 \leq x \leq 1\} \cup \{(x,0): 0 \leq x \leq 1\}$.

Therefore, $\overline{G}_f \supset G_g$. $g(x) = 1$ is a continuous function in I.

However we have the following theorem.

Theorem 3.15

Let f(x) be a discontinuous function in I such that f(x) is of bounded variation and $\overline{G}_f$ is perfect. Then $\overline{G}_f \not\supset G_g$, where g(x) is a continuous function.

Proof:

Since f(x) is of bounded variation, by Theorem 3.14, f(x+), f(x−) exist for all x in [0,1].

Also, f(x) is discontinuous. Hence there exists at least one point x_o in [0,1] such that either (1) $f(x_o+) \neq f(x_o-)$ or (2) $f(x_o+) =$

$f(x_o-) \neq f(x_o)$. (2) can not happen since $\overline{G}_f$ is perfect. Therefore $f(x_o+) \neq f(x_o-)$. This implies that $\overline{G}_f$ cannot contain the graph of a continuous function. ⊠

Theorem 3.16

Let $f(x)$ be a monotonic discontinuous function in I. Then $\overline{G}_f$ contains no graph of a continuous function.

Proof:

Since $f(x)$ is monotonic, therefore, by Theorem 3.14 and Theorem 3.15 $f(x+)$, $f(x-)$ exist for all x in $[0,1]$.

Also, $f(x)$ is discontinuous. Hence there exists at least one point x_o in $[0,1]$ such that either (1) $f(x_o+) = f(x_o-) \neq f(x_o)$ or (2) $f(x_o+) \neq f(x_o-)$. (1) cannot happen since $f(x)$ is monotonic. Therefore $f(x_o+) \neq f(x_o-)$, and thus, the conclusion. ⊠

Theorem 3.17

Let $f(x)$ be a function in I. Then $\lim\limits_{n \to \infty} [f(x)]^n$ exists.

Proof:

Since $0 \leq f(x) \leq 1$ for all x in $[0,1]$, therefore $\lim\limits_{n \to \infty} [f(x_1)]^n = 0$ whenever $f(x_1) \neq 1$, x_1 in $[0,1]$ and

$\lim\limits_{n \to \infty} [f(x_2)]^n = 1$ whenever $f(x_2) = 1$, x_2 in $[0,1]$.

Hence $\lim\limits_{n \to \infty} [f(x)]^n = g(x)$. Where

$$g(x) = \begin{cases} 1 & \text{for all } x \text{ in } [0,1] \text{ such that } f(x) = 1 \\ 0 & \text{for all } x \text{ in } [0,1] \text{ such that } f(x) \neq 1. \end{cases}$$ ⊠

Theorem 3.18

Let $f(x)$ be a function in I, and let $g(x) = \lim\limits_{n \to \infty} [f(x)]^n$. Then $\overline{G}_g$

contains the interval [0,1] iff $f(x)$ is not identically one on any non-degenerate interval in [0,1].

Proof:

$\Rightarrow$: Suppose $\overline{G}_g$ contains the interval [0,1]. Let x be any point in [0,1]. Then every open neighborhood N_x of x contains at least one point $x_o \neq x$ such that $(x_o, g(x_o) = 0)$ is in G_g. Hence $f(x_o) \neq 1$. Thus $f(x)$ is not identically one on any non-degenerate interval in [0,1].

$\Leftarrow$: Suppose $f(x)$ is not identically one on any non-degenerate interval in [0,1]. Let x be any point in [0,1]. Then for every open neighborhood N_x of x, there exists at least one point x_o in N_x and $x_o \neq x$ such that $f(x_o) \neq 1$.

Since $f(x)$ is in I, therefore, $g(x_o) = 0$, and $(x_o, 0)$ is in G_g. This implies that x is a limit point of G_g. Thus x is in $\overline{G}_g$. $\boxtimes$

Theorem 3.19

Let $f(x)$ be in I and let $g(x) = \lim_{n\to\infty} [f(x)]^n$. Then $g(x)$ is continuous iff either $f(x)$ is identically one or never one.

Proof:

$\Rightarrow$: Suppose $g(x)$ is continuous. Clearly $g(x)$ is in I. If there exist x_o, x_1 in [0,1] such that $g(x_o) = 1$, $g(x_1) = 0$, then since $g(x)$ is continuous, by Theorem 1.5, there would exist an x_2 in [0,1] such that $g(x_2) = 1/2$. This is impossible. Therefore, $g(x)$, and hence $f(x)$, is either identically one or never one.

$\Leftarrow$: Suppose $f(x) \equiv 1$. Then $g(x) \equiv 1$. Hence $g(x)$ is continuous. The same conclusion holds when $f(x) \not\equiv 1$. $\boxtimes$

Theorem 3.20

Let T be a closed set in U. Then there exists a function f(x) in I such that $\tilde{G}_f = T$ iff

(1) T meets every vertical segment of length one in U in at least one point.

(2) No two isolated points of T (if they exist) have the same x-coordinates.

(3) Every spherical neighborhood (interior of a circle) of a point of T that is not an isolated point contains points of T of different x-coordinates.

Proof:

I. Sufficiency:

Let D = set of isolated points of T. Then $T = D \cup cD$, where cD is the complement of D with respect to T.

Let $D_x = \{x: (x,y) \in D\}$, $cD_x = \{x: (x,y) \in cD,\ x \notin D_x\}$.

(i) Define f(x) = y for every x in D_x so that $(x,y) \in D$.

(ii) Define f(x) for all x in cD_x by the following scheme, considering only cD. A point T lying on the boundary of a square at any stage of subdivision will be considered as contained in either one of the two squares (in case it lies at the corner, four squares) sharing the boundary.

(a) Divide U into four squares. Select one point (x_{1j}, y_{1j}) in cD from each of the four squares (if possible) such that x_{1j} is in cD_x and define $f(x_{1j}) = y_{1j}$, $1 \leq j \leq 4$.

(b) Subdivide each of the four squares into four quarters. Choose one point (x_{2n}, y_{2n}) in cD from each of the squares (if possible) that contains no (x_{1j}, y_{1j}) for some j such that x_{2n} is in cD_x and $x_{2n} \neq x_{1j}$ for all j, n and define $f(x_{2n}) = y_{2n}$, $1 \leq n \leq 16$.

(c) Further subdivide each of the 16 squares into four squares and so on.

(iii) Complete the construction of $f(x)$ by defining $f(x_1) = y_1$ where x_1 is in $T_1 = \{x : f(x)$ have not been defined$\}$ and y_1 is in $T_2 = \{y : (x_1, y)$ in $T\}$.

(iv) Proof:

Since $G_f \subseteq T$, therefore $\bar{G}_f \subseteq \bar{T} = T$.

We show that $T \subseteq \bar{G}_f$. It is clear that $D \subseteq \bar{G}_f$.

Let (x,y) be a point in cD. Then (x,y) is a limit point of G_f.

Suppose that (x,y) is not a limit point of G_f. Then there exists a $S[(x,y), r]$, r some positive constant, such that $S[(x,y), r]$ contains no point of G_f. Since (x,y) is in cD, $S[(x,y), r]$ contains points of T with infinitely many distinct x-coordinates.

Let $T_{r_x} = \{x : (x,y) \in T,\ (x,y) \in S[(x,y), r]\}$.

$D_{x_r} = \{x : x \in D_x,\ x \in (x-r, x+r)\}$.

x_n = set of x-coordinates that have been defined by the grid-method after n^{th} stage of subdivision.

Therefore, the assumption implies that for some j, $T_{r_x} = D_{x_r} \cup x_j$. This is impossible since $D_{x_r} \cup x_j$ is in fact a proper subset of T_{r_x}.

Hence $cD \subseteq \bar{G}_f$.

Thus $T = D \cup cD \subseteq \bar{G}_f$, and $T = \bar{G}_f$.

II. Necessity:

That (1) is necessary is trivial. (2) Suppose there exist two isolated points d_1, d_2 of T such that they have the same x-coordinates. Then, since d_1, d_2 cannot be limit points of G_f, d_1, d_2 are in G_f. This is impossible since f(x) is single-valued. (3) Suppose that for some (x_o, y_o) in cD, there exists a $S[(x_o, y_o), r]$, r some positive constant, such that the points of T that are in $S[(x_o, y_o), r]$ have the same x-coordinates. This will force a contradiction since f(x) is single-valued. $\boxtimes$

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ABSTRACT

Many general properties of real continuous functions defined on the closed interval [0,1] on the real line have been studied in the past. The present work started with a review of some known analytical results of this class C. An extended class A of positively continuous functions was then defined and its properties compared with that of C. While many of the characteristics of C were inherited from A, some properties of C are not shared by A.

The first part of the second section (chapter) is devoted to a study of some elementary algebraic properties of A and C. Results obtained for the two classes showed differences. The rest of the section (chapter) deals with the algebraic structures of some subclasses of F, the set of all real-valued functions in [0,1] on the real line. The concept of an ideal in F was introduced for the class of real functions from [0,1] to the real line.

In the last section (chapter), the concept of area function, $\overline{G_f}$, defined as the closure of the graph of a function, is used to study the properties of elements of I. Integrals of continuous functions in I are completely determined by their $\overline{G_f}$'s. Some topological implications of a few analytical subclasses of I were also revealed. This section concluded with an important theorem that fully characterizes the $\overline{G_f}$ of a real function in I by a closed set in the closed square $U = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.