A STUDY OF
BIAXIAL VISCOELASTIC BEHAVIOR

by

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I. INTRODUCTION

This investigation is aimed toward the development of simplified methods for evaluating viscoelastic material constants and interpretation of these in terms of a stress-strain law for the material in question.

Herein the analytical solution for the (creep) strain in a lead plate subjected to biaxial tension is presented and compared with experimental results. As part of the solution it was required to postulate a visco-elastic stress-strain law for lead and experimentally determine the co-efficients of each derivative of stress and strain contained therein. Uniaxial test data was used for evaluating these constants.

The immediate objective of this thesis is to see if the above procedure, i.e., using uniaxial test data to determine the constants in a two (or three) dimensional viscoelastic stress-strain law, retains the same accuracy for a two dimensional stress state (as for the uniaxial stress state). Within the limitations imposed herein an affirmative conclusion is made to this question.

In leading to the analytical solution, the procedure will be based upon the viscoelastic theory as presented by E. H. Lee, in numerous papers, such as, "The Proceedings of the First Symposium on Naval Structural Mechanics", 1959, in which he uses the Laplace transform on both the basic equations and the boundary conditions. From the viewpoint of application to stress analysis problems, this method is most convenient when applicable.

Different techniques for solution are required as dictated primarily by the boundary conditions. Problems for which the regions of prescribed surface tractions or displacements do not change with time are directly amenable to the Laplace transform type of solution. On the other hand, certain problems exist where the transform of the boundary conditions can not be obtained in explicit form and other methods are required. J. R. M. Radok has extended the Laplace transform technique to handle such problems in certain cases.

The generalized form of the viscoelastic stress-strain law as given by Lee is derived in detail herein. The approach used is to write the most general relation between all the stress and strain components and subsequently reduce this to the desired form.

Currently the range of applicability of linear viscoelastic theory toward solving practical problems is approaching the extent of the field of elasticity. In spite of these mathematical developments advances are still needed in techniques for experimental measurement of material properties and translation of these into viscoelastic stress-strain laws. A search of the literature showed some work in plastics and other polymers toward this end but none concerning metals. It is hoped that this thesis will aid in this direction.

II. FORMULATION OF THE VISCOELASTIC PROBLEM AND BASIC EQUATIONS

The subject of Viscoelastic Stress Analysis considers a material to have time dependent properties. Hence, the state of stress, strain, etc. in a body is not only dependent upon the body configuration and loading but also depends upon time. The element of time is brought into the mathematical formulation of the theory by means of the viscoelastic stress-strain law which contains derivatives with respect to the time. For the isotropic case such a law will assume the form,

$$(P)S_{ij} = (Q)e_{ij} \quad \text{and} \quad (P')\sigma_{ii} = (Q')\epsilon_{ii}$$

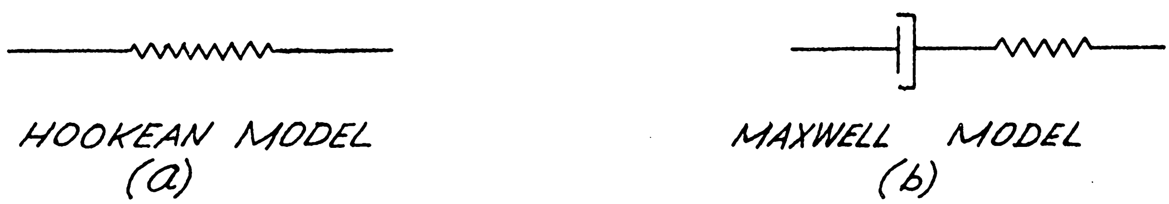
where P, Q, P', Q' are linear operators of the form

$$P = \sum_{n=0}^{N} p_n D^{(n)} \quad \text{and} \quad Q = \sum_{n=0}^{M} q_n D^{(n)} \quad \text{and similarly}$$

for the primed set. The first law covers shear effects where S_{ij} is the stress deviator, e_{ij} the strain deviator, and according to the usual index notation for repeated indices the second law relates the dilatation, ϵ_{ii}, to (three times) the average hydrostatic pressure. The coefficients p_i and q_i are assumed to be constants for the linear theory and their values depend upon the particular viscoelastic behavior that the material exhibits.

The problem chosen for this thesis is an experimental study of the accuracy of such a viscoelastic stress-strain law as applied to biaxial creep of lead. As a supplement to the thesis the general viscoelastic theory is presented including the development of the viscoelastic stress-strain law, which follows immediately.

Considering the one dimensional problem first where only one stress and its corresponding strain are involved, as in a simple tension test, it is helpful to visualize the behavior of the material in terms of a mechanical model. The simplest case would be that in which the material is purely elastic and hence the Hookean Model consisting of the linear spring, shown in Figure (1-a), would apply. Viscous or time dependent properties are characterized by the dashpot. The Maxwell Model, Figure (1-b), is one possible representation using the spring and dashpot as its elements.



- Figure (1) -

The relation between the extension, a, of the Maxwell Model and the force, F, producing the extension is

$$\frac{da}{dt} = \frac{1}{E}\frac{dF}{dt} + \frac{1}{\eta} F \qquad (1)$$

where $\frac{d}{dt}$ is the derivative with respect to time and E and η are the spring and dashpot moduli. E and η correspond to material constants that must be determined experimentally. The extension, a, is analogous to the strain, ε, in the viscoelastic body and the force, F, corresponds to the stress, σ . To generalize for the one dimensional case, the viscoelastic material will obey the following type of force-extension or stress-strain law:

$$P\sigma = Q\varepsilon \qquad (2)$$

Here $P = \sum_{r=0}^{p} p_r \frac{\partial^r}{\partial t^r}$, $Q = \sum_{r=0}^{q} q_r \frac{\partial^r}{\partial t^r}$, and p_r and q_r are material constants. The number of such constants needed to represent a specific material depend upon the particular viscoelastic behavior that it exhibits. Extending these notions to three dimensional stress distribution problems the procedure used to develop Hooke's law will be used. The general linear viscoelastic law must relate linear time operators acting on stress components to such operations on strain components. The most general form would be the following:

$$\overset{(N)}{C}_{ijkl} \frac{\partial^N}{\partial t^N} \sigma_{kl} + \dots + \overset{(1)}{C}_{ijkl} \frac{\partial \sigma_{kl}}{\partial t} + \sigma_{ij} =$$

$$\overset{(0)}{B}_{ijkl} \varepsilon_{kl} + \overset{(1)}{B}_{ijkl} \frac{\partial}{\partial t} \varepsilon_{kl} + \dots + \overset{(M)}{B}_{ijkl} \frac{\partial^M}{\partial t^M} \varepsilon_{kl} \qquad (3)$$

where the summation and range convention is adopted on the subscripts and $\sigma_{ij} = \sigma_{ij}(X_k, t)$ and $\varepsilon_{ij} = \varepsilon_{ij}(X_k, t)$ are the stress and strain tensors depending on the coordinates, X_i, and time, t. The material constants C_{ijkl} and B_{ijkl} take the place of p_r and q_r defined previously. Since only isotropic materials will be considered B_{ijkl} and C_{ijkl}, which describe the material, must be independent of the choice of axes or coordinates, i.e. they must be isotropic. As given by J. L. Synge and A. Schild, Tensor Calculus, page 211, the most general form that these constants can assume is,

$$C_{rsmn} = \lambda \delta_{rs}\delta_{mn} + \mu (\delta_{rm}\delta_{sn} + \delta_{rn}\delta_{sm}) \tag{4}$$

where δ_{ij} is the Kronecker delta: $\delta_{ij} = 1$, $i = j$; $\delta_{ij} = 0$, $i \neq j$ and λ and μ are constants. For brevity, consider only the first two terms on each side of equation (3) and use equation (4) for the coefficients B_{ijkl} and C_{ijkl}. Substitution of equation (4) into (3) gives,

$$\cdots + \left\{\lambda\, \delta_{ij}\delta_{kl} + \mu\, (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right\} \frac{\partial\sigma_{kl}}{\partial t} + \sigma_{ij} =$$
$$\left\{\lambda'\, \delta_{ij}\delta_{kl} + \mu' (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right\} \varepsilon_{kl} +$$
$$\left\{\lambda''\, \delta_{ij}\delta_{kl} + \mu'' (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right\} \frac{\partial\varepsilon_{kl}}{\partial t} + \cdots \tag{5}$$

or,

$$\cdots + \lambda\delta_{ij}\frac{\partial\sigma_{kk}}{\partial t} + \mu \left(\frac{\partial\sigma_{ij}}{\partial t} + \frac{\partial\sigma_{ji}}{\partial t}\right) + \sigma_{ij} = \lambda'\delta_{ij}\varepsilon_{kk} + \mu'\left\{\varepsilon_{ij} + \varepsilon_{ji}\right\} +$$
$$\lambda''\delta_{ij}\frac{\partial\varepsilon_{kk}}{\partial t} + \mu''\left\{\frac{\partial\varepsilon_{ij}}{\partial t} + \frac{\partial\varepsilon_{ji}}{\partial t}\right\} + \cdots \tag{6}$$

Since $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{ij} = \varepsilon_{ji}$, it follows that,

$$\lambda\delta_{ij}\frac{\partial\sigma_{kk}}{\partial t} + 2\mu\frac{\partial\sigma_{ij}}{\partial t} + \sigma_{ij} = \lambda'\delta_{ij}\varepsilon_{kk} + 2\mu'\varepsilon_{ij} + \lambda''\delta_{ij}\frac{\partial\varepsilon_{kk}}{\partial t} +$$
$$2\mu''\frac{\partial\varepsilon_{ij}}{\partial t} + \cdots \tag{7}$$

By introducing the deviator tensor for stress and strain, denoted respectively by S_{ij} and e_{ij}, the effects of shear and hydrostatic tension or compression can be singled out individually. By definition let

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} \quad \text{and} \quad e_{ij} = \varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk} \tag{8}$$

Therefore, upon substitution of equation (8) into (7),

$$\cdots + \lambda\delta_{ij}\frac{\partial\sigma_{kk}}{\partial t} + 2\mu\frac{\partial}{\partial t}(S_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk}) + (S_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk}) = \lambda'\delta_{ij}\varepsilon_{kk} +$$

$$2\mu'(e_{ij} + \frac{1}{3}\delta_{ij}\varepsilon_{kk}) + \lambda''\delta_{ij}\frac{\partial\varepsilon_{kk}}{\partial t} + 2\mu''(\frac{\partial e_{ij}}{\partial t} + \frac{1}{3}\delta_{ij}\frac{\partial\varepsilon_{kk}}{\partial t}) + \cdots \tag{9}$$

or

$$\cdots + (\lambda + \frac{2}{3}\mu)\delta_{ij}\frac{\partial\sigma_{kk}}{\partial t} + 2\mu\frac{\partial}{\partial t}(S_{ij}) + S_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk} =$$

$$(\lambda' + \frac{2}{3}\mu')\delta_{ij}\varepsilon_{kk} + 2\mu' e_{ij} + (\lambda'' + \frac{2}{3}\mu'')\delta_{ij}\frac{\partial\varepsilon_{kk}}{\partial t} + 2\mu''\frac{\partial e_{ij}}{\partial t} \tag{10}$$

which leads immediately to,

$$\cdots + (p_1'\frac{\partial\sigma_{kk}}{\partial t} + p_0'\sigma_{kk})\delta_{ij} + p_1\frac{\partial S_{ij}}{\partial t} + p_0 S_{ij} = (q_1'\frac{\partial\varepsilon_{kk}}{\partial t} + q_0'\varepsilon_{kk})\delta_{ij}$$

$$+ q_1\frac{\partial e_{ij}}{\partial t} + q_0 e_{ij} + \cdots \tag{11}$$

where the p's and q's are constants.

This can be written in compact form as,

$$\delta_{ij}(\Sigma p_r'\frac{\partial^r}{\partial t^r})\sigma_{kk} + (\Sigma p_r\frac{\partial^r}{\partial t^r})S_{ij} = \delta_{ij}(\Sigma q_r'\frac{\partial^r}{\partial t^r})\varepsilon_{kk} + (\Sigma q_r\frac{\partial^r}{\partial t^r})e_{ij} \tag{12}$$

Therefore equation (3) becomes,

$$\delta_{ij}\left(\sum^{N} p'_r \frac{\partial^r}{\partial t^r}\right)\sigma_{kk} + \left(\sum^{N} p_r \frac{\partial^r}{\partial t^r}\right) S_{ij} = \delta_{ij}\left(\sum^{M} q'_r \frac{\partial^r}{\partial t^r}\right)\varepsilon_{kk} + \left(\sum^{M} q_r \frac{\partial^r}{\partial t^r}\right) e_{ij} \tag{13}$$

Contracting the subscripts, i.e. letting i = j, and noting by virtue of its definition $S_{ii} = e_{ii} = 0$, yields the following relation:

$$\sum^{N} p'_r \frac{\partial^r}{\partial t^r} \sigma_{kk} = \sum^{M} q'_r \frac{\partial^r}{\partial t^r} \varepsilon_{kk} \tag{14}$$

Hence, it follows that,

$$\sum^{N} p_r \frac{\partial^r}{\partial t^r} S_{ij} = \sum^{M} q_r \frac{\partial^r}{\partial t^r} e_{ij} \tag{15}$$

Therefore, structurally, the same form of relation exists between the deviator stress and deviator strain components, and σ_{kk} and ε_{kk}, as between the single stress and strain components in the one dimensional case. Abbreviating the above expressions, the general isotropic linear viscoelastic law becomes:

$$\begin{aligned} (P)\, S_{ij} &= (Q)\, e_{ij} \\ (P')\, \sigma_{kk} &= (Q')\, \varepsilon_{kk} \end{aligned} \tag{16}$$

where,

$$P = \sum^{N} p_r \frac{\partial^r}{\partial t^r}, \quad Q = \sum^{M} q_r \frac{\partial^r}{\partial t^r}, \quad P' = \sum^{N} p'_r \frac{\partial^r}{\partial t^r}, \text{ and } Q' = \sum^{M} q'_r \frac{\partial^r}{\partial t^r} \tag{17}$$

To complete the formulation of the viscoelastic problem the procedure presented by E. H. Lee (1) * will be followed. Consideration will be given to bodies subjected to prescribed body forces $f_i(X_j, t)$ per unit volume, surface tractions $T_i(X_j, t)$ and/or surface displacements $U_i(X_j, t)$. In order to determine an exact solution the stresses $\sigma_{ij}(X_k, t)$ and the displacements $U_i(X_j, t)$ must satisfy the stress-strain relations (16), and the equilibrium equations:

$$\sigma_{ij,j} = f_i(X_k, t) \tag{18}$$

where the subscript after the comma indicates differentiation with respect to the corresponding space coordinate. The strain components are connected with the displacements through the relation valid for small strains:

$$\varepsilon_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}) \tag{19}$$

The displacement must be compatible with the prescribed surface displacement, and over the portion of the surface where tractions are prescribed the following relation must hold:

$$T_i = \sigma_{ij} n_j \tag{20}$$

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* See reference 1.

where the n_j are the direction cosines of the outward normal to the surface.

Application of the Laplace Transform to the above system of equations 16, 18, 19 and 20 removes the time dependence, and the viscoelastic problem becomes an elastic problem in the transformed variables. The Laplace transform will be denoted by a bar above the corresponding function. With zero initial conditions an operator in $\frac{\partial}{\partial t}$ merely becomes the same function of the transform parameter, p, so that the governing equations become

$$P(p)\,\overline{S}_{ij}(X_k, p) = Q(p)\overline{e}_{ij}(X_k, p)$$

$$P'(p)\overline{\sigma}_{ii}(X_j, p) = Q'(p)\overline{\varepsilon}_{ii}(X_j, p) \qquad (21)$$

$$\overline{\sigma}_{ij,j} = \overline{f}_i(X_k, p) \qquad (22)$$

$$\overline{\varepsilon}_{ij} = \frac{1}{2}(\overline{U}_{i,j} + \overline{U}_{j,i}) \qquad (23)$$

$$\overline{T}_i(X_j, p) = \overline{\sigma}_{ij}n_j \qquad (24)$$

Equations 21 through 24 represent a stress analysis problem, termed the associated elastic problem, for an elastic body of the same shape as the viscoelastic body with elastic constants a function of the parameter p by virtue of equations (21). The correspondence between equations (21) and the analogous equations for an isotropic elastic body, namely,

$$\begin{aligned} s_{ij} &= 2Ge_{ij} \\ \sigma_{ii} &= 3K\epsilon_{ii} \end{aligned} \qquad (21')$$

where G is the shear modulus and K the bulk modulus, should be noted. Comparison of equations (21) and (21') indicates that in the associated elastic problem the shear modulus, G_v, should be replaced by $\frac{1}{2}\frac{Q(p)}{P(p)}$ and the bulk modulus, K_v, by $\frac{1}{3}\frac{Q'(p)}{P'(p)}$. Here the subscript v indicates we are working with the transformed variables. Since Young's Modulus , E, is given in terms of the shear modulus and the compressibility by

$$E = \frac{9KG}{3K + G}$$

the corresponding viscoelastic modulus, E_v, becomes

$$E_v = \frac{\frac{3}{2}\frac{Q'}{P'}\frac{Q}{P}}{\frac{Q'}{P'} + \frac{1}{2}\frac{Q}{P}} \qquad (25)$$

In a similar manner the viscoelastic operator, ν_v, corresponding to Poisson's ratio becomes

$$\nu_v = \frac{\frac{Q'}{P'} - \frac{Q}{P}}{2(\frac{Q'}{P'} + \frac{1}{2}\frac{Q}{P})} \qquad (26)$$

Hence these substitutions can be made directly in the associated elastic problem.

If the associated elastic problem can be solved, $\bar{\sigma}_{ij}(X_k, p)$ is the transform of the stress variation of the original viscoelastic problem, and inversion to give $\sigma_{ij}(X_k, t)$ provides the desired viscoelastic stress distribution. Thus the extensive literature of the theory of elasticity can be utilized to solve viscoelastic stress analysis problems.

The above concludes the general theory for the Laplace transform technique. Whether or not a problem is amenable to this method of solution depends on the prescribed surface conditions for T_i and U_i . In the case of a mixed boundary value problem for which the regions of prescribed T_i and U_i, denoted respectively by S_1 and S_2, change with time, as in the Hertz contact problem *, a variation of technique is required. Here, there exist fixed surface points which are traversed by the boundary separating the two regions S_1 and S_2 and consequently neither T_i or U_i are prescribed throughout the loading process, so that their transforms cannot be obtained. Thus the associated elastic problem is not defined. However, for many practical problems this type of situation does not arise and the theory as outlined applies directly.

To complete any viscoelastic problem a choice must be made for the differential operators in equation (16) for a given material. As this is based strictly on experimental data let us consider the following hypothetical situation to illustrate the procedure.

For example, one might suspect that a certain material obeys a

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* See reference 1.

stress-strain law in shear given by

$$\frac{\partial \varepsilon_{ij}}{\partial t} = A \frac{\partial}{\partial t}(S_{ij}) + B S_{ij} \tag{27}$$

but is perfectly elastic under hydrostatic tension or compression, i.e.

$$\sigma_{kk} = C \varepsilon_{kk} \tag{28}$$

A, B, and C are constants of the material that must be evaluated experimentally. Recourse will be made to uniaxial data to avoid the added complexitites of two or three dimensional tests. For such a case it is first necessary to derive by means of equations (27) and (28) the proper stress-strain relation for the single stress component and its corresponding strain. Considering simple tension, where σ_{11} is the principal stress, the equation resulting from (27) and (28) is

$$\left(\frac{2}{3} A + \frac{1}{3C} \right) \frac{\partial \sigma_{11}}{\partial t} + \frac{2}{3} B \sigma_{11} = \frac{\partial \varepsilon_{11}}{\partial t} \tag{29}$$

The constants A, B, and C may now be evaluated by the following procedure:

> The constant B is determined by measuring the slope of the linear (secondary region) portion of a uniaxial creep test. For this test $\frac{\partial \sigma_{11}}{\partial t} = 0$. A stress relaxation test, in

which $\frac{\partial \epsilon_{11}}{\partial t} = 0$, provides means of writing two simultaneous equations for A and C. With the constants evaluated, comparison of theory and experiment should be made for additional tests of various types. If it is found that the results are in good agreement then the proper choice was made for the model.

An illustration has been cited where the stress-strain law was postulated for the general three dimensional stress state and subsequently reduced to the uniaxial case for tension. Furthermore, it is assumed that such a procedure provides a valid means for determining the constants. The aim of this thesis is to investigate this point, i.e. is it possible to determine the material constants for a general three dimensional viscoelastic stress-strain law by means of one dimensional tests? It was decided that a criterion for answering this question would be a comparison of biaxial creep data with theory in which the material constants are evaluated by means of uniaxial tests. The procedure of the investigation follows.

III. THE INVESTIGATION

3.1 Object of Investigation

The purpose of this thesis is to determine whether or not it is possible to evaluate the material constants of a general three dimensional viscoelastic stress-strain law by means of one dimensional tests. To answer this question a lead plate was tested in creep under biaxial tension. The experimental data was then compared with the viscoelastic theory in which the material constants were evaluated by means of uniaxial tests.

3.2 Method of Procedure

The first step in the analysis is to postulate a viscoelastic model for lead. Here it was decided to limit the analysis in the respect that the model need only give an accurate representation of creep, since only biaxial creep data would be available. Therefore, the criterion for selecting the model is whether or not its one dimensional stress-strain relation yields an accurate representation of one dimensional creep.

Consider first the following form of viscoelastic stress-strain law:

Let

$$\sum_{0}^{M} \left(A_r \frac{\partial^r}{\partial t^r}\right) S_{ij} = B \frac{\partial^2}{\partial t^2} e_{ij} + C \frac{\partial}{\partial t} e_{ij} \tag{30}$$

cover shear effects, and assume

$$\sigma_{ii} = k \varepsilon_{ii} \tag{31}$$

applies for hydrostatic tension or compression. The form of the left hand side of equation (30) is yet to be determined. The one dimensional law implied by equations (30) and (31) for the case of simple tension, where σ_{11} denotes the principal stress, is obtained in the following manner. By definition

$$\sigma_{11} = S_{11} + \frac{1}{3}\sigma_{kk} \tag{32}$$

Equation (30), when written in operator form, becomes

$$\left(\sum_{0}^{M} A_r D^r\right) S_{ij} = (BD^2 + CD)\, e_{ij} \tag{33}$$

where D indicates differentiation with respect to time. Therefore,

$$S_{11} = \frac{(BD^2 + CD)}{\left(\sum_{0}^{M} A_r D^r\right)} e_{11} \tag{34}$$

Substituting equation (34) into (32) yields,

$$\sigma_{11} = \frac{(BD^2 + CD)}{\left(\sum_{0}^{M} A_r D^r\right)} e_{11} + \frac{\sigma_{kk}}{3} \tag{35}$$

By using the definition of e_{11} and equation (31), the above expression becomes,

$$\sigma_{11} = \frac{(BD^2 + CD)}{\left(\sum_{0}^{M} A_r D^r\right)} \left(\varepsilon_{11} - \frac{\sigma_{kk}}{3k}\right) + \frac{\sigma_{kk}}{3} \tag{36}$$

But for simple tension, $\sigma_{kk} = \sigma_{11}$, and therefore,

$$\frac{2}{3}\sigma_{11} = \frac{(BD^2 + CD)}{(\sum_0^M A_r D^r)}\left(\varepsilon_{11} - \frac{\sigma_{11}}{3k}\right) \tag{37}$$

or

$$\frac{2}{3}\left(\sum_0^M A_r D^r\right)\sigma_{11} = (BD^2 + CD)\left(\varepsilon_{11} - \frac{\sigma_{11}}{3k}\right) \tag{38}$$

which reduces to,

$$\frac{2}{3}\left(\sum_1^M A_r' D^r\right)\sigma_{11} + \frac{2}{3}A_0\sigma_{11} = BD^2\varepsilon_{11} + CD\varepsilon_{11} \tag{39}$$

From equation (39), the governing equation for a one dimensional creep test, where $\sigma_{11} = \sigma_0$ = a constant, becomes

$$\frac{2}{3}A_0\sigma_0 = B\frac{\partial^2\varepsilon_{11}}{\partial t^2} + C\frac{\partial\varepsilon_{11}}{\partial t} \tag{40}$$

Thus, one dimensional behavior in creep will be governed by equation (40). The solution of equation (40) is,

$$\varepsilon_{11} = C_1 + C_2 e^{-C/B\,t} + \frac{2}{3}\frac{A_0}{C}\sigma_0 t \tag{41}$$

where C_1 and C_2 are constants of integration.

The process of evaluating the material constants will now be considered. First observe from equation (41) that these constants appear only in ratios, i.e. in the form (C/B) and (A_0/C). Hence, as far as one dimensional creep is concerned, numerical values are required only for the two ratios and the individual values of A_0, B, and C are unnecessary. With this thought in mind let us rewrite equation (40), in the form shown below and use it to write two simultaneous equations for (B/C) and (A_0/C) once the derivatives of the strain have been determined experimentally.

$$\frac{2}{3}\left(\frac{A_0}{C}\right) \sigma_0 - \left(\frac{B}{C}\right) \frac{\partial^2 \varepsilon_{11}}{\partial t^2} = \frac{\partial \varepsilon_{11}}{\partial t} \qquad (42)$$

One device for obtaining the derivatives is to measure $\frac{\partial \varepsilon_{11}}{\partial t}$ and the corresponding value of $\frac{\partial^2 \varepsilon_{11}}{\partial t^2}$, as explained on page (21), at two points of a uniaxial creep curve. Another approach is to measure the aforementioned derivatives from two different creep curves i.e. at one point of each of two curves. In this case of course the stress level, σ_0, would be different for each curve. Both methods were tried to see how closely the results would compare. Unfortunately, differences of the order of 30% could arise depending on what point of the curve the measurements were taken and the stress levels of the tests. This large variation might be attributed to errors in the graphical measurements, which are outlined below, but it is felt that the main inadequacy is due to one or both of the following reasons.

Number one, there is a definite need for a nonlinear theory, as explained on page 24, in which the material constants are functions of the stresses. Without this extension of the theory a set of "constants" can only be expected to apply to a limited range of stress levels. Consequently the experimental creep curves herein are presented only for a restricted range of values.

A second shortcoming could be due to the selection of the terms in the right hand side of the stress-strain law, equation (30). All possibilities for the right hand side of the equation were considered, with derivatives up to the second order, before concluding that equation (30) as it stands gives the best results. Therefore it can only be concluded that a more accurate analysis would require higher order derivatives of the strain. Such a case would not only complicate the mathematics but would also require more sophisticated experimental procedures to evaluate the constants. Nevertheless, much insight to the problem at hand can still be gained from the analysis employed.

Returning to the mechanics used herein for measuring $\frac{\partial \varepsilon_{11}}{\partial t}$ and $\frac{\partial^2 \varepsilon_{11}}{\partial t^2}$, it was decided to determine these quantities at two points of a creep curve in which the stress level was approximately the same as that for the biaxial creep data. This would be the 1200 p.s.i. curve shown in Figure (2). An enlarged plot of this curve was made and the slope was measured graphically at time $t = 17.5$, 20.0, and 22.5 minutes and at $t = 65, 75, 85$ minutes. Hence the value of $\frac{\partial \varepsilon_{11}}{\partial t}$ is known at, for example, $t = 20$ minutes and its corresponding value of $\frac{\partial^2 \varepsilon_{11}}{\partial t^2}$ was approximated by taking the average rate of change of $\frac{\partial \varepsilon_{11}}{\partial t}$ over the five minute interval from 17.5 to 22.5 minutes according to the formula,

$$\frac{\partial^2 \varepsilon_{11}}{\partial t^2} = \frac{\left.\frac{\partial \varepsilon_{11}}{\partial t}\right|_{t = 22.5 \text{ min.}} - \left.\frac{\partial \varepsilon_{11}}{\partial t}\right|_{t = 17.5 \text{ min.}}}{(22.5 - 17.5)} \tag{43}$$

The accuracy of the first order derivatives is felt to be reasonable, whereas, admittedly, the approximation from equation (43) for the second order derivatives is quite crude. The magnitude of error that this approximation introduces will be considered after conclusion of the immediate analysis.

Table (1) lists the data for the above measurements and calculations. A description of the test apparatus is given in the appendix.

From Table (1) and equation (42) the following equations may be written,

$$\frac{2}{3}\left(\frac{A_0}{C}\right)(1200) - (-.26)(10^{-6})\left(\frac{B}{C}\right) = 10.12(10^{-6}) \tag{44}$$

$$\frac{2}{3}\left(\frac{A_0}{C}\right)(1200) - (-.053)(10^{-6})\left(\frac{B}{C}\right) = (5.47)(10^{-6}) \tag{45}$$

Upon solution, these give,

$$\frac{B}{C} = 22.322 \text{ min.}, \quad \text{or} \quad \frac{C}{B} = .0448 \text{ min.}^{-1},$$

and

$$\frac{A_0}{C} = (.539)(10^{-8}) \ \frac{\text{in.}^2}{\text{Lb. Min.}}$$

Table (1)

Numerical data for uniaxial creep test (σ_0 = 1200 p.s.i.)

time (minutes)	$\frac{\partial \epsilon}{\partial t}$(in/in/min.) x 10^{-6}	$\frac{\partial^2 \epsilon}{\partial t^2}$(in/in/min.^{2}) x 10^{-6}
17.5	10.87	
20.0	10.12	-.26
22.5	9.57	
65	6.08	
75	5.47	-.053
85	5.03	
Remarks	Measured	Calculated

Table (2)

time(minutes)	stress (p.s.i.)	ϵ(in/in) x 10^{-6}	$\frac{\partial \epsilon}{\partial t}$(in/in/min.) x 10^{-6}
0	1100	630	-
40	1100	-	5.25
0	1150	720	-
40	1150	-	6.02
0	1200	780	-
40	1200	-	7.32

With the above ratios of constants equation (41) may be used to predict one dimensional creep at various stress levels, i.e.

$$\epsilon_{11} = C_1 + C_2 e^{-.0448t} + \frac{2}{3}(.539)(10^{-8})\,\sigma_0 t \qquad (46)$$

A plot of equation (46) is compared to experimental creep curves in Figure 2, 3 and 4 for stress levels of 1200, 1150, and 1100 p.s.i. The constants C_1 and C_2 were evaluated using the information listed in Table (2) which was read directly from the experimental creep curves at time $t = 0$ and $t = 40$ minutes.

The agreement between theory and experiment is quite good for the 1200 p.s.i. test, which, it is noted, was used to evaluate the material constants. Inspection of the other two curves, i.e. the 1150 and 1100 p.s.i. tests, shows increasingly less accuracy as σ_0, the stress level, varies away from $\sigma_0 = 1200$ p.s.i. Hence, this fact justifies the previous concern for the need of a non-linear theory (or perhaps a more complicated stress-strain law). Therefore any attempt for increased experimental accuracy in the present analysis would be of a secondary nature and consequently unwarrented.

As a sidelight to illustrate that more accurate means of measurement might lead to slightly better agreement between equation (41) and experiment, let us consider the following hypothetical set of data which closely approximates that given in Table (1) except for the (questionable) values of $\frac{\partial^2\varepsilon}{\partial t^2}$.

time (minutes)	$\frac{\partial\varepsilon}{\partial t}$ (in/in/min.) x 10^{-6}	$\frac{\partial^2\varepsilon}{\partial t^2}$ (in/in/min.^{2}) x 10^{-6}
20.0	10.1	-.217
75.0	5.5	-.0585

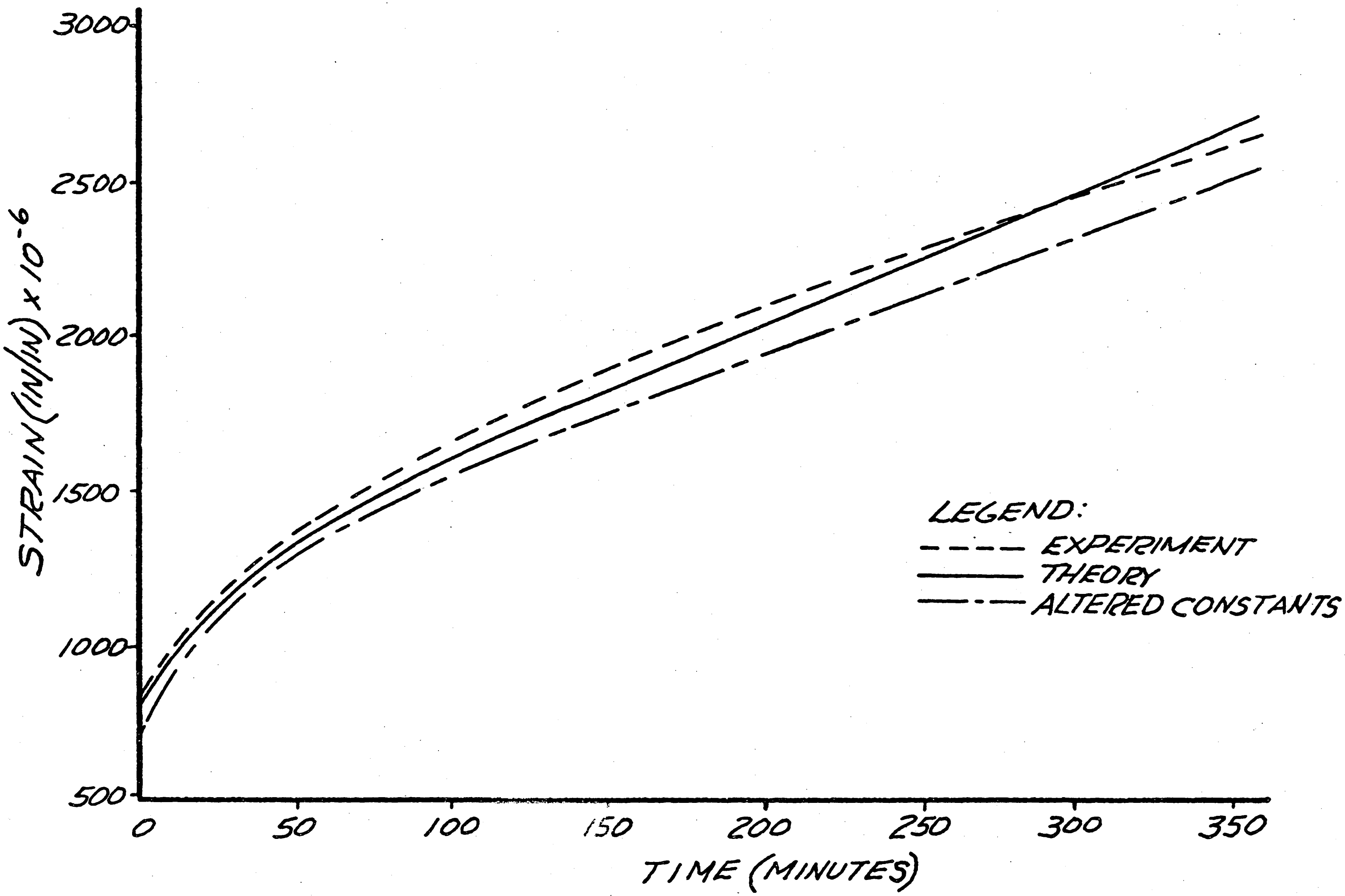
Table (3)

These numbers lead to the following values: $\frac{A_0}{C} = .475(10^{-8})\ \frac{\text{In.}^2}{\text{Lb. Min.}}$ and $\frac{C}{B} = .0345\ \text{min.}^{-1}$. Therefore, using the hypothetical data, equation (41) becomes,

$$\varepsilon_{11} = C_1 + C_2 e^{-.0345t} + \tfrac{2}{3}(.475)(10^{-8})\,\sigma_0 t \qquad (46')$$

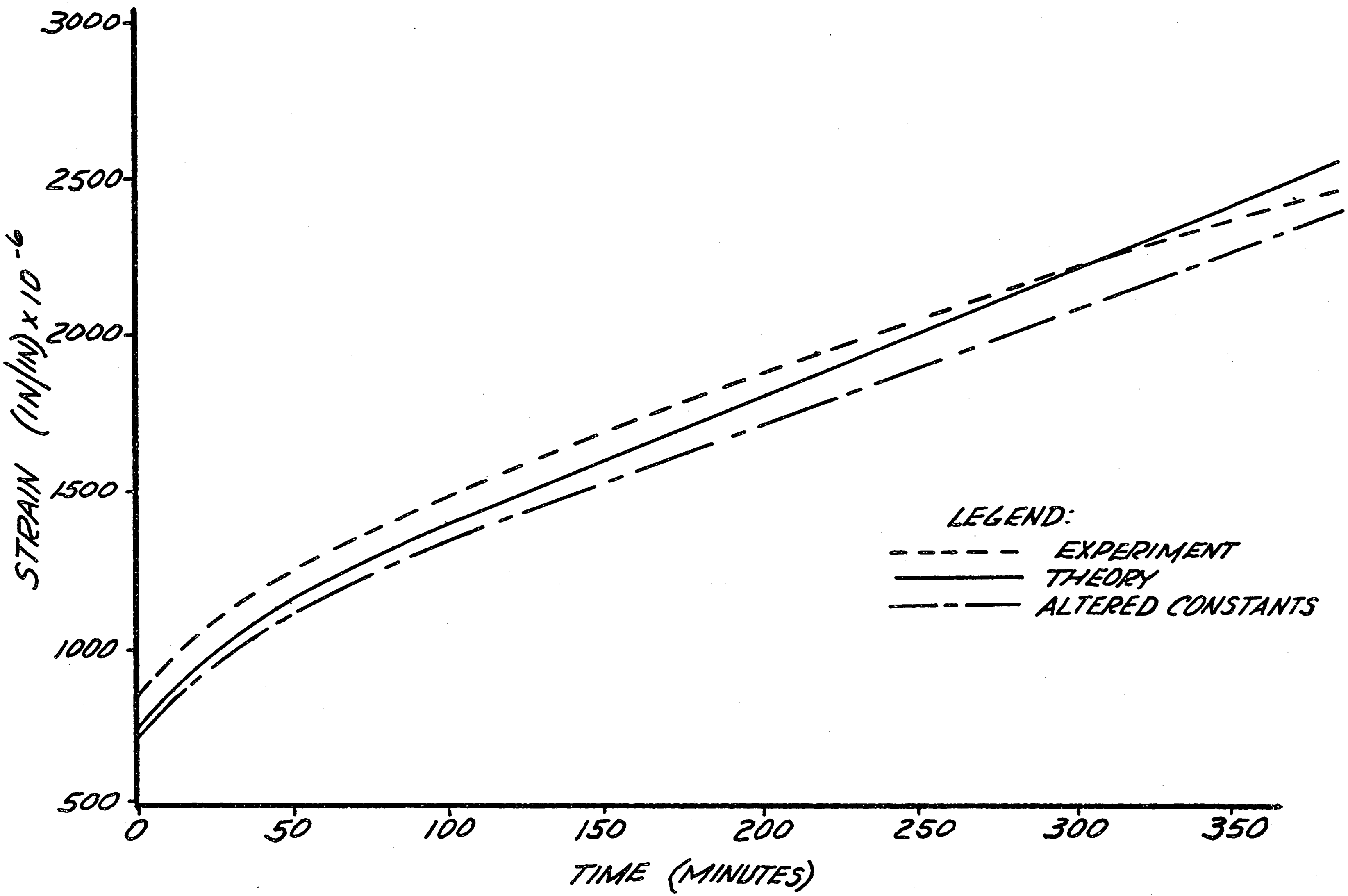
Equation (46') is also plotted in Figures 2, 3, and 4 and shows an improved agreement with experiment. (The constants C_1 and C_2 were evaluated as in the previous case.)

In the analysis which follows, the original material constants will be used when drawing conclusions, etc. and in some instances the hypothetical set will be considered for comparison.



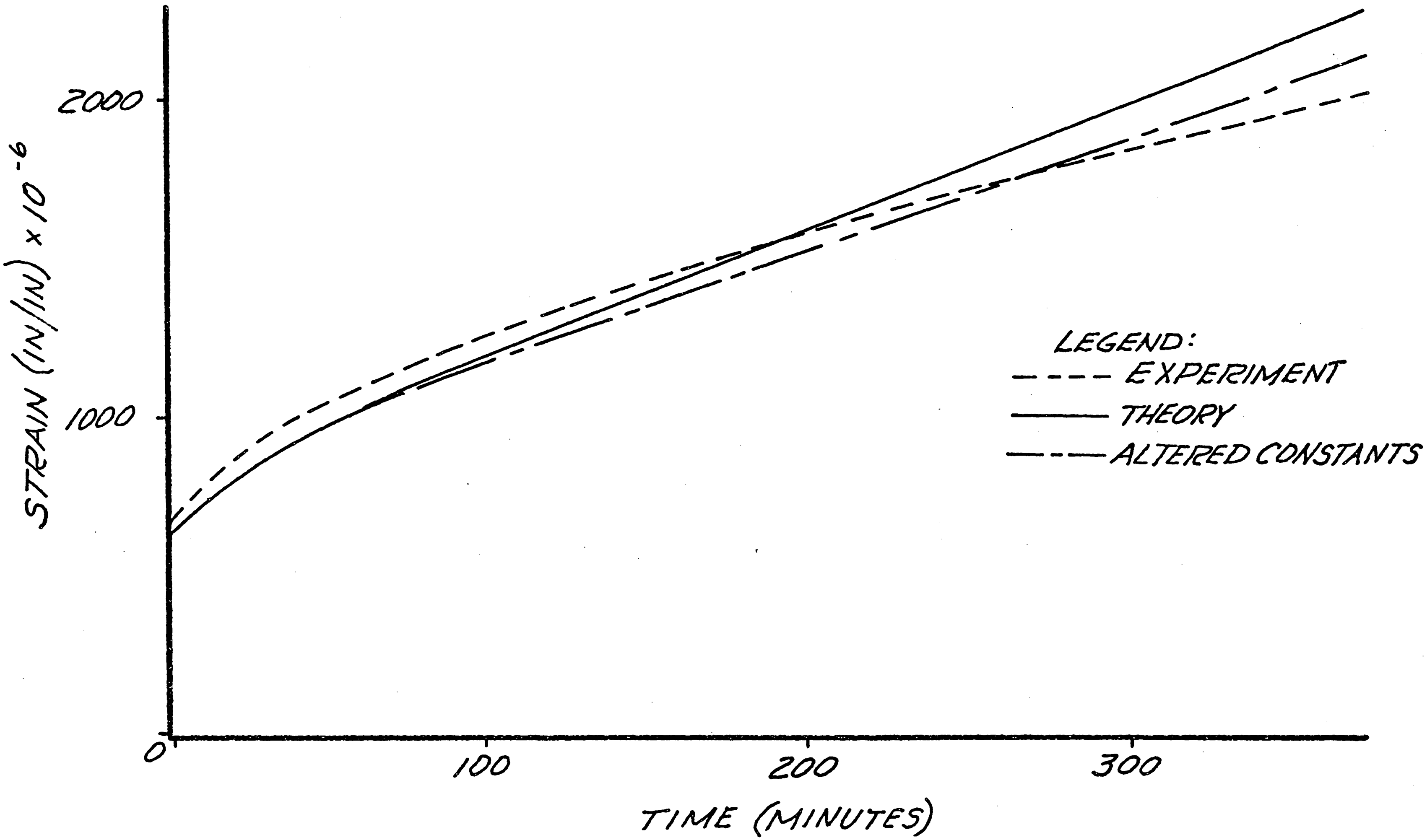
UNIAXIAL CREEP TEST, σ_0 = 1200 P.S.I.

FIGURE (2)



UNIAXIAL CREEP TEST, $\sigma_o = 1150$ P.S.I.

FIGURE (3)



UNIAXIAL CREEP TEST, $\sigma_0 = 1100$ P.S.I.

FIGURE 4

Summarizing, it is seen that, based on one dimensional tests, the selection for the right hand side of equation (30) is sufficiently accurate. Other types of tests, such as stress relaxation, would be necessary to determine the complete form of the model and material constants for lead. Fortunately, the exact form of the left hand side of equation (30) does not have to be known to complete the analysis herein. This point will become evident later, but first it is necessary to derive the viscoelastic solution for the plate.

As described in the derivation of equations 21 through 24, the associated elastic problem will be of the same shape as the viscoelastic body, but with transformed boundary conditions. Figure (5-a) shows the viscoelastic plate of constant thickness subjected to the uniform tensile stresses $T_x(\pm \frac{L}{2}, t) = T_y(\pm \frac{L}{2}, t) = T$ which are constant with respect to time. Application of the Laplace transform to the boundary conditions indicates that the associated elastic problem will be that shown in Figure (5-b) where the boundary conditions become $\overline{T}_x(\pm \frac{L}{2}, p) = \overline{T}_y(\pm \frac{L}{2}, p) = T/p$, p being the transform parameter.

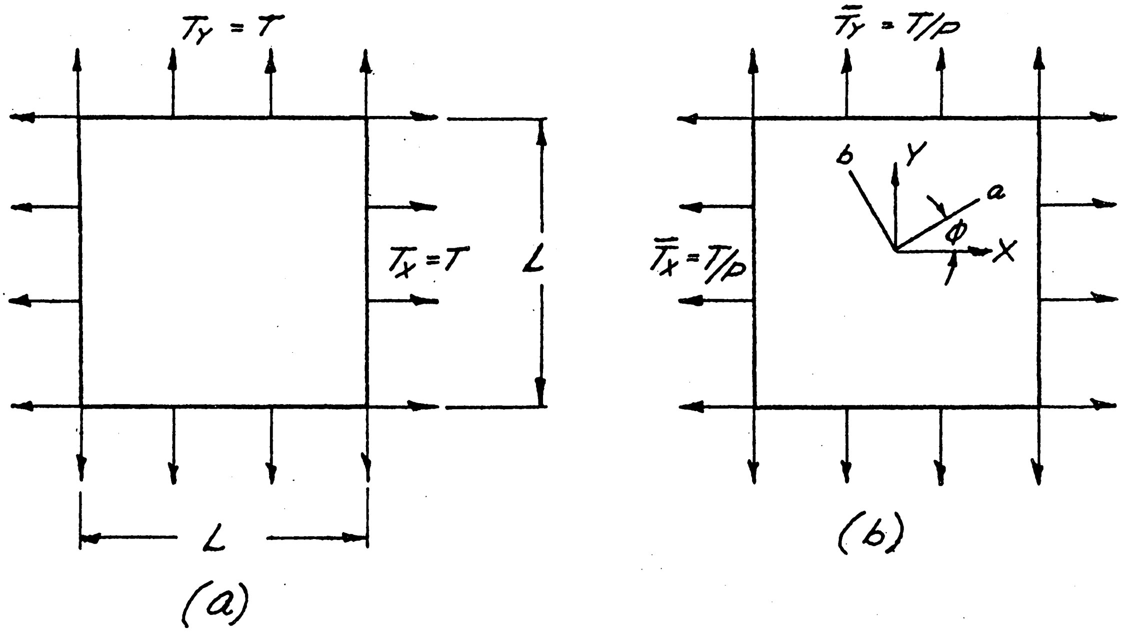


Figure (5)

Once the solution is found for the associated elastic problem of Figure 5-b inversion will yield the viscoelastic solution. From elementary strength of materials it follows immediately that

$$\bar{e}_a(p) = \left(\frac{1-\nu_v}{2E_v}\right)(\bar{T}_x + \bar{T}_y) + \left(\frac{1+\nu_v}{2E_v}\right)(\bar{T}_x - \bar{T}_y)\cos 2\phi \qquad (47)$$

and

$$\bar{\gamma}_{ab}(p) = \left(\frac{1+\nu_v}{E_v}\right)(\bar{T}_x - \bar{T}_y)\sin 2\phi \qquad (48)$$

where $\bar{e}_a$ and $\bar{\gamma}_{ab}$ are respectively the normal and shearing strains on an element oriented at an angle ϕ with respect to the applied load $\bar{T}_x$ (see Figure 5-b). Substitution into equations (47) and (48) is made for the transformed moduli E_v and ν_v by equations (25) and

(26), which after simplifying give,

$$\overline{e}_a = \frac{1}{3}\left(\frac{P}{2Q} + \frac{P'}{Q'}\right)(\overline{T}_x + \overline{T}_y) + \frac{1}{2}\,\frac{P}{Q}\,(\overline{T}_x - \overline{T}_y)\cos 2\phi \tag{49}$$

and

$$\overline{\gamma}_{ab} = \frac{P}{Q}\,(\overline{T}_x - \overline{T}_y)\sin 2\phi \tag{50}$$

For our problem $\overline{T}_x = \overline{T}_y = \frac{T}{p}$, and therefore

$$\overline{e}_a = \frac{1}{3}\left(\frac{P}{2Q} + \frac{P'}{Q'}\right)\left(\frac{T}{p} + \frac{T}{p}\right) \tag{51}$$

and

$$\overline{\gamma}_{ab} = 0 \tag{52}$$

From equations (30) and (31)

$$\frac{P(p)}{Q(p)} = \frac{\sum_0^M A_r p^r}{Bp^2 + Cp} \tag{53}$$

and

$$\frac{P'(p)}{Q'(p)} = \frac{1}{k} \tag{54}$$

Therefore,

$$\overline{e}_a = \frac{2}{3}\left\{\frac{1}{2}\left(\frac{\sum_0^M A_r p^r}{Bp^2 + Cp}\right) + \frac{1}{k}\right\}\frac{T}{p} \tag{55}$$

Equation (55), when inverted, will give the desired expression for the strain, $e_a(X_i, t)$, as a function of time. But first, attention must be given to the term,

$$\left(\frac{\sum_{0}^{M} A_r p^r}{Bp^2 + Cp} \right) \frac{1}{p}$$

contained therein.

From the theory of Laplace transforms, for example, Operational Mathematics by R. V. Churchill, page 14, the above mentioned term must vanish as p tends to infinity. Therefore, the most general form that $\sum_{0}^{M} A_r p^r$ can assume is,

$$\sum_{0}^{M} A_r p^r = \sum_{0}^{2} A_r p^r = A_2 p^2 + A_1 p + A_0$$

With this substitution for $\sum_{0}^{M} A_r p^r$, $\bar{e}_a$ becomes,

$$\bar{e}_a = \frac{2T}{3} \left\{ \frac{1}{2} \frac{(A_2 p^2 + A_1 p + A_0)}{(Bp^2 + Cp)\, p} + \frac{1}{kp} \right\} \tag{56}$$

This can be inverted by partial fractions to give,

$$e_a(X_i, t) = \frac{2T}{3} \left\{ \frac{1}{2} \left(\frac{A_2}{B} e^{-\frac{C}{B}t} + \frac{A_1}{C} \left(1 - e^{-\frac{C}{B}t}\right) + \frac{A_0 B}{C^2} e^{-\frac{C}{B}t} - \frac{A_0 B}{C^2} + \frac{A_0 t}{C} \right) + \frac{1}{k} \right\} \tag{57}$$

or

$$e_a(X_i, t) = \frac{T}{3} \left\{ \left(\frac{A_2}{B} - \frac{A_1}{C} + \frac{A_0 B}{C^2} \right) e^{-\frac{C}{B}t} + \frac{A_0 t}{C} + \left(\frac{A_1}{C} - \frac{A_0 B}{C^2} + \frac{2}{k} \right) \right\} \tag{58}$$

Noting that numerical values for the ratios C/B and A_0/C have already been determined, the other constants A_0, A_1, etc., displayed in equation (58) must be evaluated to completely solve for $e_a(X_i, t)$. An alternative approach to the problem would be to evaluate the two groups of constants, i.e. $\left(\frac{A_2}{B} - \frac{A_1}{C} + \frac{A_0 B}{C^2}\right)$ and $\left(\frac{A_1}{C} - \frac{A_0 B}{C^2} + \frac{2}{k}\right)$, from the experimental biaxial creep data. (See appendix for description of the test and data.) For such a case, equation (58) may be written as

$$e_a(X_i, t) = T\left\{ C_1' e^{-\frac{C}{B}t} + \frac{1}{3}\frac{A_0}{C}t + C_2' \right\} \tag{59}$$

The constants C_1' and C_2' will be evaluated by using the initial strain of the plate and the slope of the creep curve at, say, $t = 40$ minutes. The uniform edge stress, T, was 1190 p.s.i. From the experimental biaxial creep curve of Figure (6) it is found that

$$e_a\Big|_{t=0} = 439(10^{-6})(\text{in/in}) \quad \text{and} \quad \frac{\partial e_a}{\partial t}\Big|_{t = 40\text{ min.}} =$$

$$5.25\,(10^{-6})(\text{in/in/min.})$$

which give

$$C_1' = -.35(10^{-6})(\text{in.}^2/\text{Lb.}) \quad \text{and} \quad C_2' = .719\,(10^{-6})(\text{in.}^2/\text{Lb.})$$

and therefore,

$$e_a(X_i, t) = T\left\{ -.35e^{-.0448t} + .0018t + .719 \right\}(10^{-6}) \quad (\text{in/in}) \tag{60}$$

This equation (60) gives the theoretical expression for the strain (in any direction) as a function of time for the lead plate with uniform edge stresses, T.

The final step of the analysis is to compare equation (60) with the experimental data. If there is close agreement then it can be concluded affirmatively that uniaxial tests may be used in evaluating the material constants of a general viscoelastic stress-strain law.

As mentioned earlier on page (25) a parallel comparison to experiment would be carried out using the altered or hypothetical values for $\frac{A_0}{C}$ and $\frac{C}{B}$. For such a case equation (59) would lead to equation (61) below where the constants C_1' and C_2' are evaluated as in the previous case.

$$e_a(X_i, t) = T\left\{-.328e^{-.0345t} + .00158t + .697\right\}(10^{-6}) \quad (\text{in/in}) \tag{61}$$

In Figure (6) the biaxial experimental creep curve is compared with equations (60) and (61).

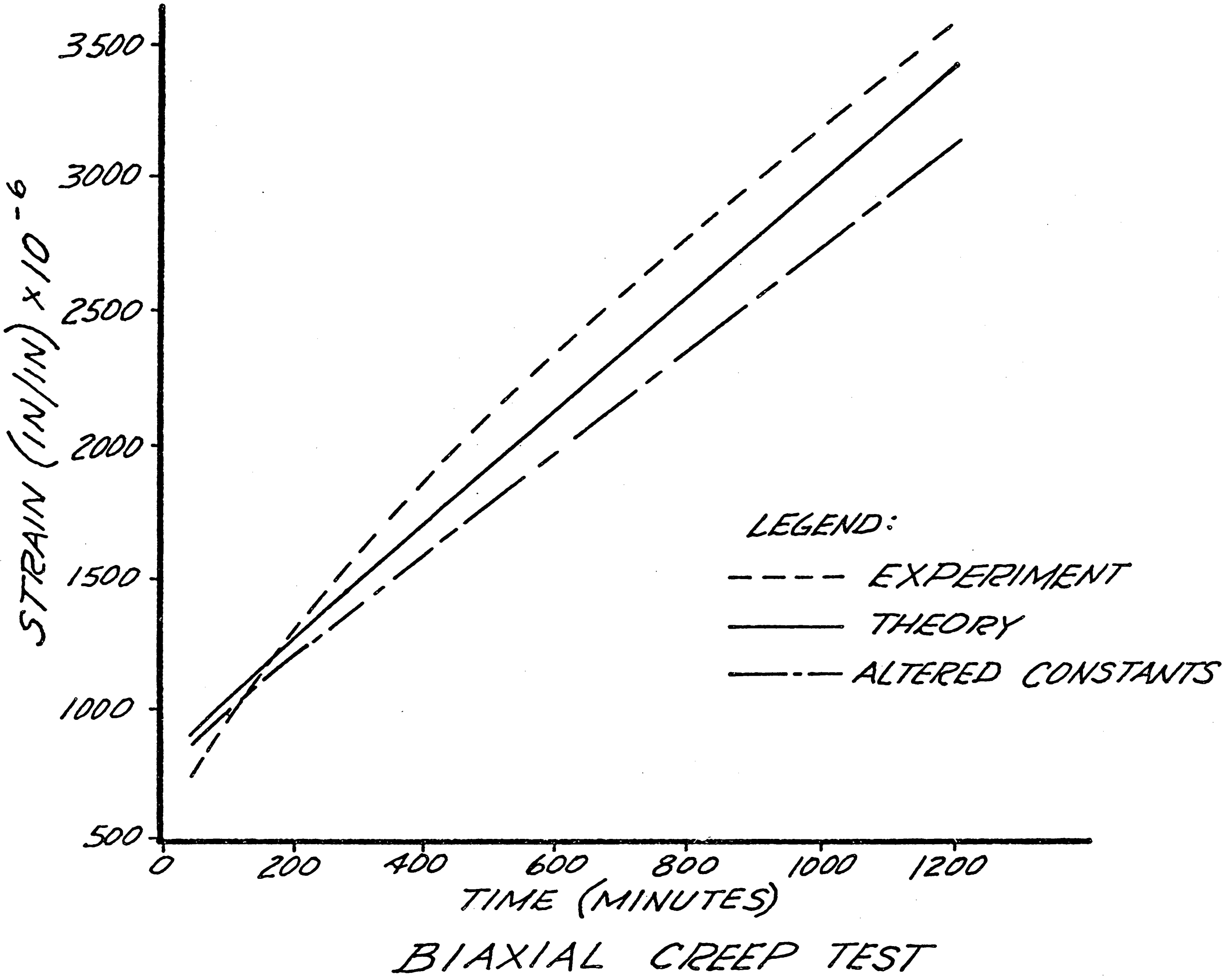

3500
3000
2500
2000
1500
1000
500
STRAIN (IN/IN) x 10^-6
0
200
400
600
800
1000
1200
TIME (MINUTES)
LEGEND:
EXPERIMENT
THEORY
ALTERED CONSTANTS
BIAXIAL CREEP TEST

FIGURE (6)

IV. DISCUSSION OF RESULTS AND CONCLUSIONS

Examination of figure 6 shows reasonably close agreement between experiment and theory as given by equation 60. Beyond a time of 400 minutes there exists a difference of approximately 250 micro inches/inch between the curves, however, this is of minor importance as for extended periods of time this difference would only be a small fraction of the total strain.

More encouraging conclusions may be drawn from observation of the curves representing experiment and theory using the altered constants, i. e. equation 61. Recalling that the most accurate representation of uniaxial strain was given by the altered constants, here again the same is true, as the slopes of the curves in question are almost identical. In fact, had the biaxial test been conducted for a longer period of time, it appears as if the slope of the experimental curve and equation 61 would coincide. Unfortunately, difficulties arose, as explained in the description of the test program in the appendix, that necessitated stopping the biaxial test at 1200 minutes, thus preventing a more definite conclusion. But based on the available information, it appears that the material constants in the general viscoelastic stress-strain law can be evaluated by means of uniaxial tests.

V. ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to for his assistance and guidance throughout this study, and also to for his numerous suggestions.

VI. BIBLIOGRAPHY

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VIII. APPENDIX

DESCRIPTION OF EXPERIMENTAL TESTS

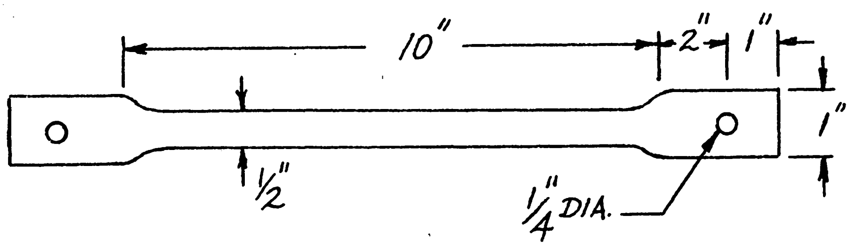
8.A UNIAXIAL TESTS

1.) Creep Test

The uniaxial creep tests, four in number, were run simultaneously to decrease laboratory time and temperature variation. The results of three tests, all at different stress levels, are given in figures (2), (3), and (4). A fourth test run at a stress level of 1250 p. s. i., showed a highly increased creep rate due to closely approaching the ultimate stress and is not included. Figure (8) shows a typical stress-strain curve for lead.

The strain was measured by SR-4, type A7, strain gages with one mounted longitudinally on each side of the specimen to give an average reading in case of bending. Lateral measurements were not taken. Strain readings were picked up by a Baldwin SR-4 strain scanner fifty channel unit and recorded directly in micro inches per inch by a Brown unit. The range sensitivity was 5000 micro inches per inch strain for a chart width of ten inches. Included in the system was a temperature compensating gage as well as three unloaded gages to check for zero drift, which did not occur.

The average room temperature of $77\frac{1}{2}\ ^{\circ}F$ was maintained within $\pm\ 1^{\circ}$ by two heating elements regulated through a thermostat.



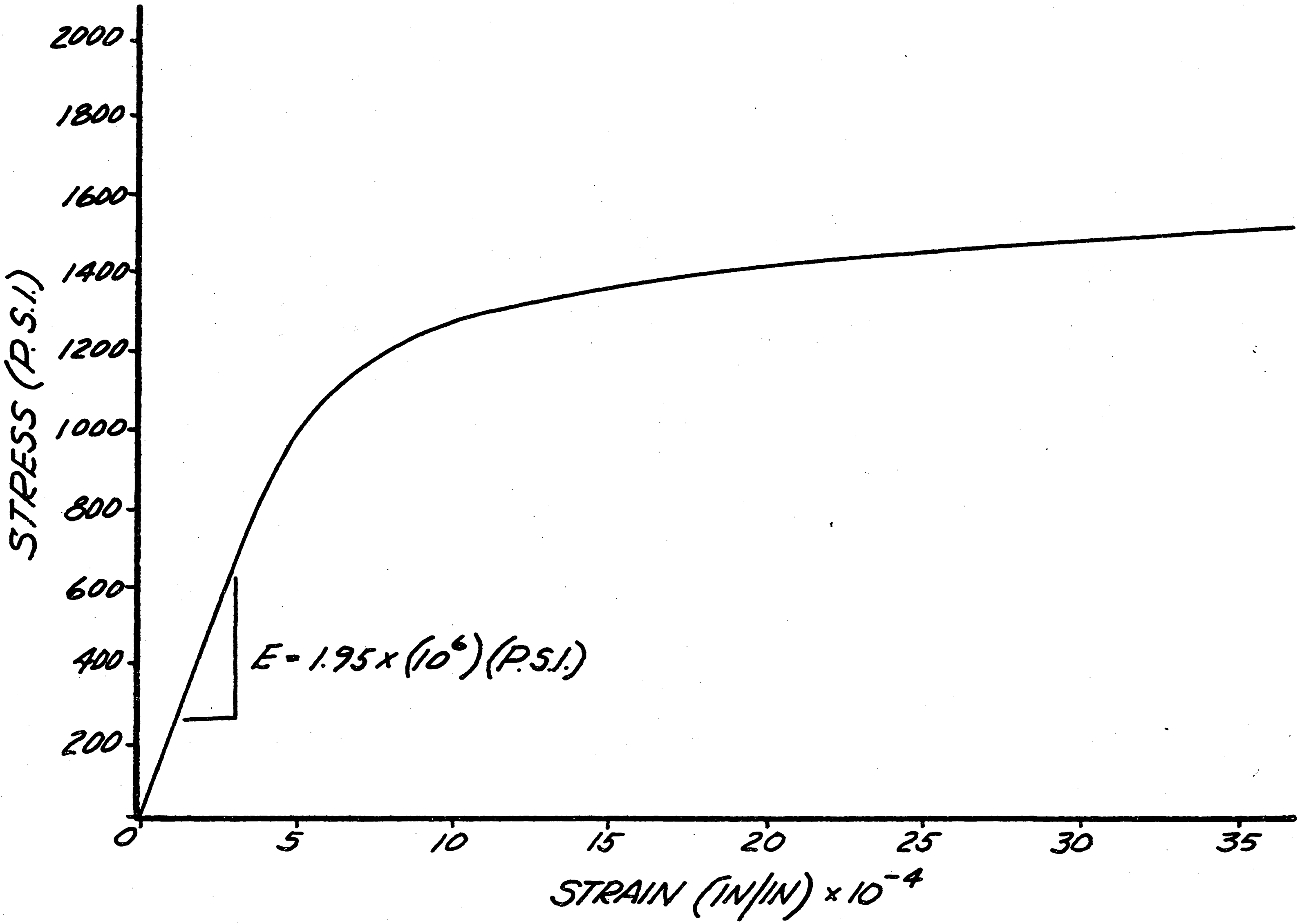
- Figure (7) -

A typical specimen is shown in figure (7). These were cut from one of two lead plates made specifically for this test program, the remaining plate being used in the biaxial creep test. The specimens were first rough cut and then machine milled to their final dimensions as shown. A 1/4 inch bolt was inserted through a hole at each end of the specimen, and left untightened to insure a two force member.

The dead weight loads were applied by hand at "zero" time with the first reading taken twenty seconds later. The time interval required for a sequence of readings for the four specimens was less than 10 seconds and therefore was not of any consequence. At the beginning of the tests a sequence of readings was taken at least twice a minute. The tests were continued until fracture due to "necking down" of the cross section.

2.) Stress-Strain Test

A typical stress-strain curve for the type of lead used in the experiment, designated as pure lead, is shown in figure (8). The specimen configuration was identical with that used in the creep tests as shown in figure (7). A mechanical extensometer with a ten inch gage length was used to measure the deflection in .001" increments. The loads were measured by means of a proving ring coupled with a Ames dial that afforded readings in increments of .023 pounds. The loads were applied by a Tinus Olsen testing machine that loaded by a constant speed head movement which in this case was .05 inches per minute.

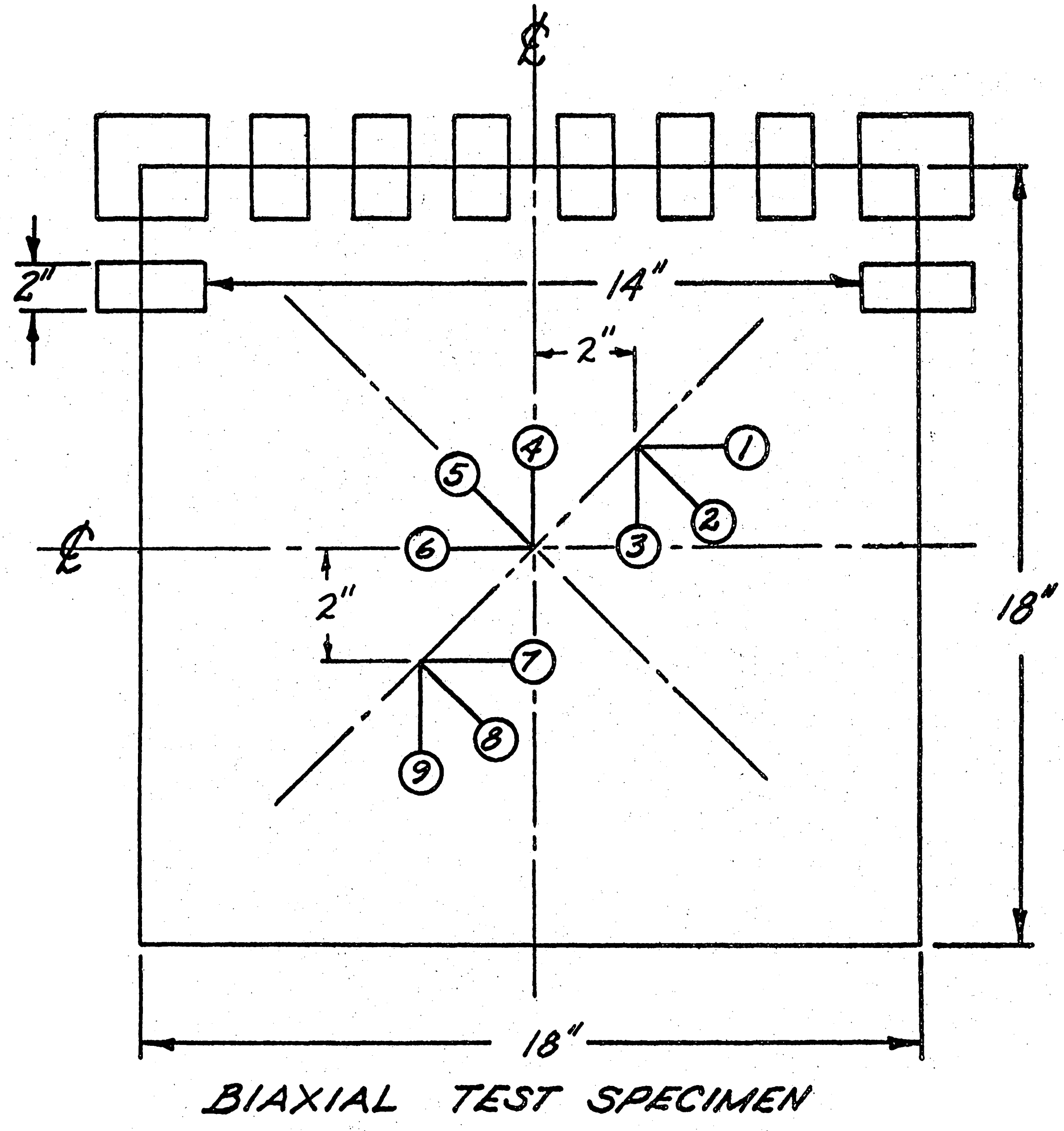


STRESS Vs STRAIN, PURE LEAD 77°F

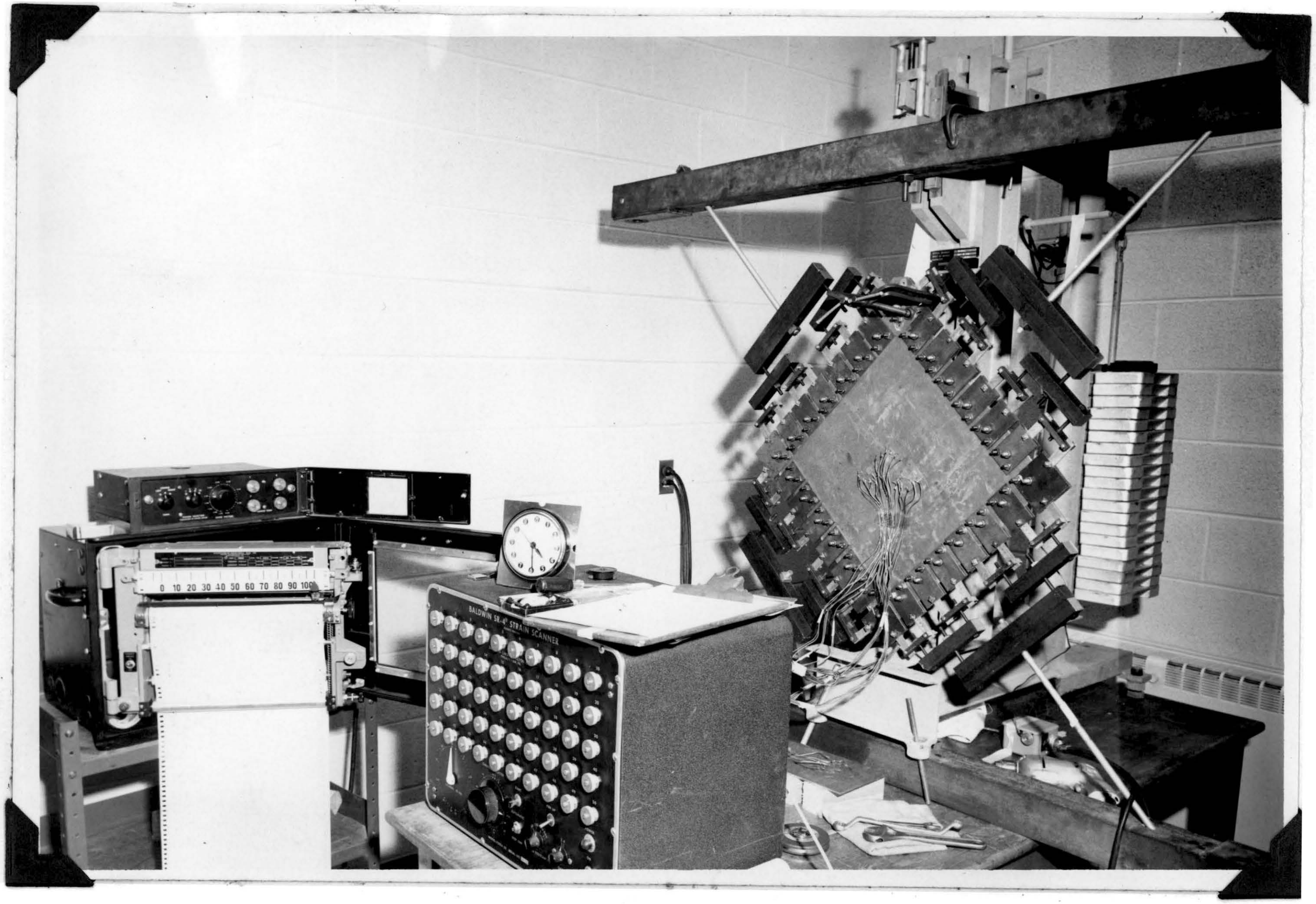
FIGURE (8)

8.B Biaxial Tests

The photograph on the following gage shows the general test set up for the biaxial creep test. Nominal dimensions of the lead plate and strain gage numbering and orientation are shown below in figure (9).



- Figure (9) -



BIAXIAL TEST

The plate was procured through Noland Company, Roanoke, Virginia and was found to have a uniform thickness of .135 inches within $\pm$ .002 inch.

The device used to load the plate is evident from the photograph. A lever system with a ten to one ratio was used to apply an upward vertical force to the top horizontal box beam. This in turn transmitted the force via the 1/2 inch diameter bolts to the first leg of the "Whiffletree" which equally divided the load among the eight grips along each edge of the plate. The grips consisted of two inch wide flat bar, each fastened to the plate by one 5/16 inch and two 1/4 inch diameter bolts. These were torques by hand as much as possible to further distribute the stress. The exposed area of the plate was approximately 14" x 14" and it is assumed that St. Venant's Principle held in the center region.

Baldwin Post Yield rosette strain gages, designated as PAR-7, were attached opposite one another to both sides of the plate at the center. This provided a check for bending. Two other rosettes, attached to only one surface as shown in the photograph, provided check on the stress distribution across the plate in the directions of the load. As expected, the readings of gages 7 and 1 were 10 - 15% higher than gage 6 and similarly in the other direction.

The recording equipment etc. shown in the photograph is as described for uniaxial tests. Temperature compensation was accomplished again by using a dummy gage and zero drift (which did not occur) was checked by reading three unloaded gages. Both the uniaxial and biaxial creep tests were conducted at a temperature of $77\frac{1}{2}^{\circ} \pm 1^{\circ}$.

The testing procedure began by first eliminating the slack in the loading system through adjustment of the 1/2 inch diameter bolts and then "zeroing" the strain gages. Subsequently the weights were applied to the lever, after which the first strain readings were taken. The twelve loaded and three unloaded gages required approximately 15 seconds to read automatically, hence no problem of time lag between individual gage readings existed.

The measuring and loading system worked smoothly with one exception. After the test had proceeded approximately $2\frac{1}{2}$ hours a corner grip began pulling loose from the plate. This is shown occurring in the photograph at the upper corner of the plate. In an attempt to more widely distribute the stress over the area of the grip, the C-clamps were installed. This temporarily alleviated the problem but during the 20th hour of testing the grip pulled loose.

The stress distribution was altered, of course, once the grip began slipping but it is felt that it would be of no serious consequence. This is born out by a plot of the data which did not indicate any "jumps" or other irratic behavior.

The plate was tested at a stress level of 1190 p.s.i. The data presented in figure (6) is the average of gages 5, 6, 11, and 12 which, as seen from figure (9), are located at the center of the plate. Gages 4 and 10 were not included as they gave readings that were approximately 20% higher than the group consisting of 5, 6, 11, 12 where the greatest deviation among the group was less than 10%.

LIST OF SYMBOLS

σ_{ij}	Stress tensor
ϵ_{ij}	Strain tensor
S_{ij}	Stress deviator
e_{ij}	Strain deviator
δ_{ij}	Kronecker delta
P_r, q_r, P_r', q_r'	Material constants
$D^{(n)}$	Derivative with respect to time.
$\frac{\partial}{\partial t}$	Derivative with respect to time.
μ, λ	Material Constants
p	Laplace transform parameter.
T_i	Surface tractions.
U_i	Surface displacements.
n_i	Unit normal vector.
E_v, ν_v	Viscoelastic Moduli.
e_a, γ_a	Normal and shearing strains respectively.
A_r, B, C	Viscoelastic constants.
T_x, T_y	Surface tractions on plate.
$\overline{T}_x, \overline{T}_y$	Transformed surface tractions.

ABSTRACT

An experimental study is made to investigate the feasiability of determining a viscoelastic stress-strain law for two (or three) dimensional stress conditions by means of one dimensional tests, The conclusions are based upon comparison of theory and experiment for a creep test of a lead plate, subjected to biaxial tension.

The stress-strain law that was selected is given by

$$S_{ij} = B \frac{\partial^2}{\partial t^2} e_{ij} + c \frac{\partial}{\partial t} e_{ij}$$

to cover shear effects and by

$$\sigma_{ii} = k \varepsilon_{ii}$$

for hydrostatic tension or compression. S_{ij} and e_{ij} are respectively the stress and strain deviator and σ_{ii} and ε_{ii} are the stress and strain tensors. Uniaxial test data was used to evaluate the constants in the above laws.

The Laplace transform technique was used to obtain the analytical solution for the strain in the plate as a function of time. Agreement between theory and experiment for the duration of the test, 1200 minutes, was quite good.