INPUT IMPEDANCE OF A

SLOT-CYLINDER ANTENNA

by

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I. INTRODUCTION.

Discussed herein is an antenna consisting of a hollow, thin-walled conducting cylinder with a narrow axial slot connecting the internal and external surfaces. This slot does not extend the full length of the cylinder, as shown in Figure 1. In this thesis an analytical expression is developed for the impedance looking into this antenna when it is fed with a transmission line connected across the center of the slot.

Much of the early work on slotted cylinder antennas was done by Dr. E. C. Jordan and his colleagues at Ohio State University and the University of Illinois.[1,2] The primary application of this antenna was to airplanes, because it is possible to fill the hollow cylinder with a dielectric material and maintain aerodynamic stability. J. R. Wait did considerable work with this antenna in calculating field patterns and in calculating the conductance for a half-wave slot-cylinder.[3] Recently the National Aeronautics and Space Administration has shown much interest in this antenna in connection with space communications.[4] Especially, work has been done on the effect on the radiation field of a plasma coating around the cylinder. The plasma layer adversely affects both the radiation pattern and the input impedance, creating a problem in re-entry communications.

This thesis follows very closely the work done by Dr. C. A. Holt at the University of Illinois in 1949.[5] The principle variation from Dr. Holt's work appears in the calculation of conductance. Dr. Holt assumed a current distribution around the cylinder in order to calculate the radiation losses. In this thesis integration is performed over a spherical surface surrounding the antenna using the radiation field obtained by S. Silver and W. K. Saunders.[6]

An exact analysis of this antenna was not attempted as it is too involved. Instead an approximate analysis was made by considering the slot as a shorted transmission line and the cylinder as a circular waveguide. The justification for this analysis is presented in Section III.

Throughout this thesis all time variations are assumed to be sinusoidal, and complex exponentials are utilized in the manner conventional with electrical engineers. Current, voltage, and the components of the electric and magnetic fields and their potentials become phasors, or complex scalars, which are represented in this thesis by capital letters. Also, capital letters with a bar ($\bar{E}$) are used for complex vectors. These complex quantities have rms (root-mean-square) magnitudes. The rationalized MKS system of units is used throughout.

II. LIST OF ABBREVIATIONS.

α angle at cylinder's axis subtended by the slot width.

α_a attenuation constant.

β phase constant.

γ propagation constant.

ϵ permittivity of free space.

η intrinsic impedance of free space.

λ wavelength in free space.

λ_s wavelength in slot region.

μ permeability of free space.

Φ magnetic flux, or electric scalar potential.

ψ elementary wave function.

ω $2\pi f$.

a cylinder radius (average).

$\bar{A}$ magnetic vector potential.

B slot distributed capacitive susceptance.

$B_n(X)$ Bessel function.

c velocity of light in free space.

C slot distributed capacitance.

C_o, C_n constants in solution of Helmholtz equation.

$\bar{E}$ electric field intensity.

f frequency.

$\bar{F}$ electric vector potential.

G	slot distributed conductance.
$\bar{H}$	magnetic field intensity.
$H_n^{(2)}(X)$	Hankel function of the second kind.
I	axial slot current.
I_c	transverse radiation current.
I_d	displacement current.
I_t	transverse current.
J	current density
$J_n(X)$	Bessel function of the first kind.
k	wave number of medium.
k_ρ, k_z	constants in solution of Helmholtz equation.
L	slot distributed inductance.
L'	slot length.
$N_n(X)$	Bessel function of the second kind.
P_f	radiated power.
R	slot distributed resistance.
$\bar{S}$	Poynting vector.
t	cylinder thickness.
$\bar{U}_z$	unit vector in z direction.
v	volume.
$V(z)$	voltage across slot.
W	slot width; also constant in Fourier transform.
X	slot distributed inductive reactance.
$\hat{y}$	admittivity of medium.

Y	slot distributed admittance.
$\hat{z}$	impedivity of medium.
Z	slot distributed impedance.
Z_{in}	input impedance.
Z_o	slot characteristic impedance.
x, y, z	rectangular coordinate symbols.
ρ, ϕ, z	cylindrical coordinate symbols.
r, θ, ϕ	spherical coordinate symbols.
∇	del operator.

III. TRANSMISSION LINE ANALOGY.

In this section the basis for the analogy between the slot cylinder antenna, shown in Figure 1, and the transmission line is presented. This is done by deriving the transmission line equations for the slot. This follows the method of Holt.[7,8]

The cylinder is assumed to be made of perfectly conducting metal. Since, in most cases, the metal used would be copper and the frequencies are high, this is a valid assumption. The width W of the slot is assumed to be very small ($W \ll \lambda$) compared with the wavelength λ of excitation. Another way of expressing the slot width is

$$W = \alpha a \tag{1}$$

where α is the angle from the z-axis subtended by the slot width and "a" is the average radius of the cylinder. It might be observed that throughout this thesis the cylinder is considered to be thin-walled. This makes possible the use of the average radius rather than necessitating the use of both inner and outer radii. The first transmission line equation will now be derived using the method of Holt.[8]

Consider two small axial strips extending along the sides of

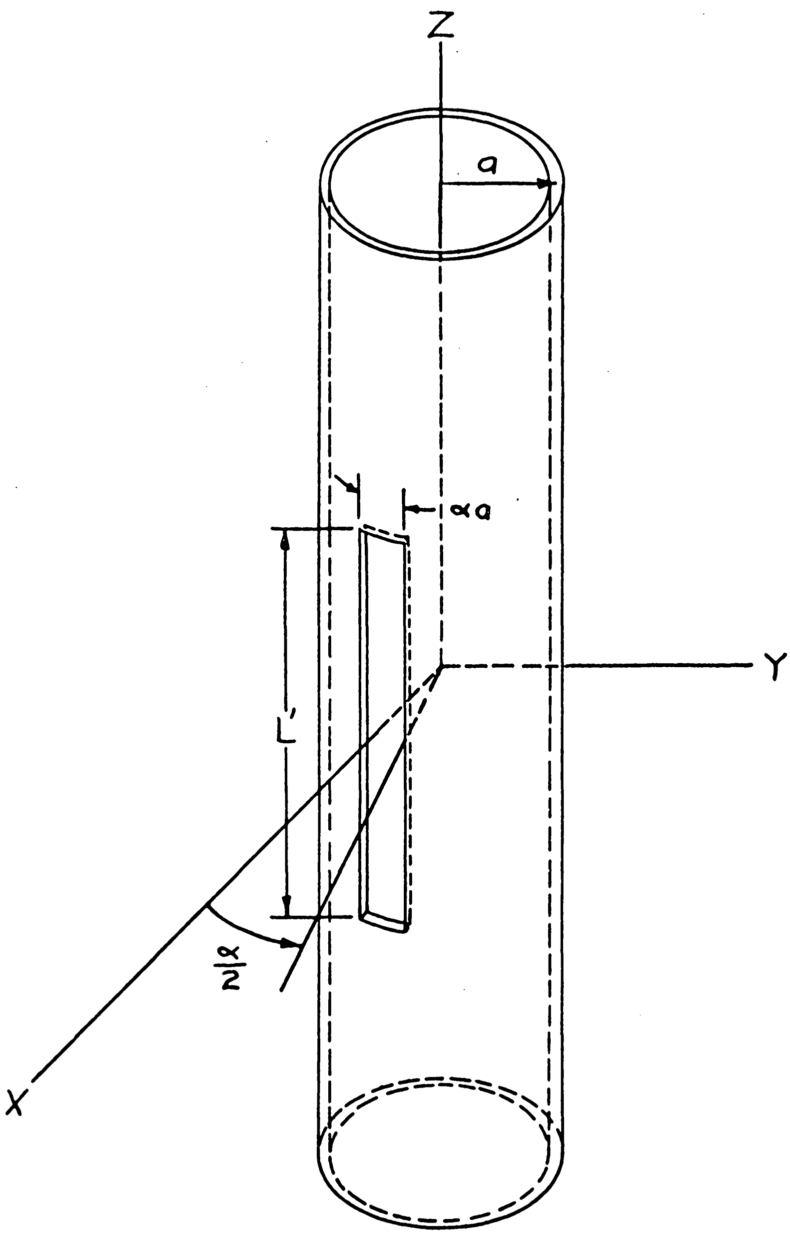

FIGURE 1.
SLOTTED CYLINDER ANTENNA

the slot, as shown in Figure 2(a). These strips are shown in Figure 2(b) when removed from the cylinder. Consider a differential length dz of the upper strip. There are six paths which the current could follow to enter or leave the differential volume dv. If the axial current entering the left side is I and the current leaving the right side is I + dI, then dI denotes the increase in I as z increases by dz. Hence, -dI must be the current entering through the two cross-sections. It is evident from the equation of continuity that -dI must equal the differential transverse current dI_t, leaving the differential volume; i.e.,

$$dI = -dI_t \tag{2}$$

or, making these per-unit-length,

$$\frac{dI}{dz} = - \frac{dI_t}{dz} \tag{3}$$

This transverse current may be broken into two parts; one is in-phase with the voltage (I_c) and the other is ninety degrees out-of-phase with the voltage (I_d). The out-of-phase current in turn has two components; one is the displacement current across the slot (I_{d1}) which leaves the lower side of the differential volume and the other is the out-of-phase or reactive part of the transverse conduction current (I_{d2}) going around the cylinder. This reactive current thus

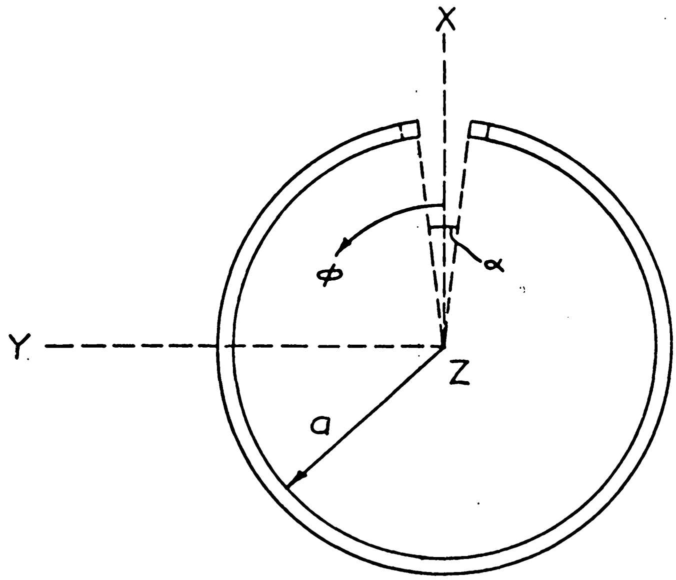

FIGURE 2(a). CROSS-SECTIONAL VIEW OF SLOTTED CYLINDER

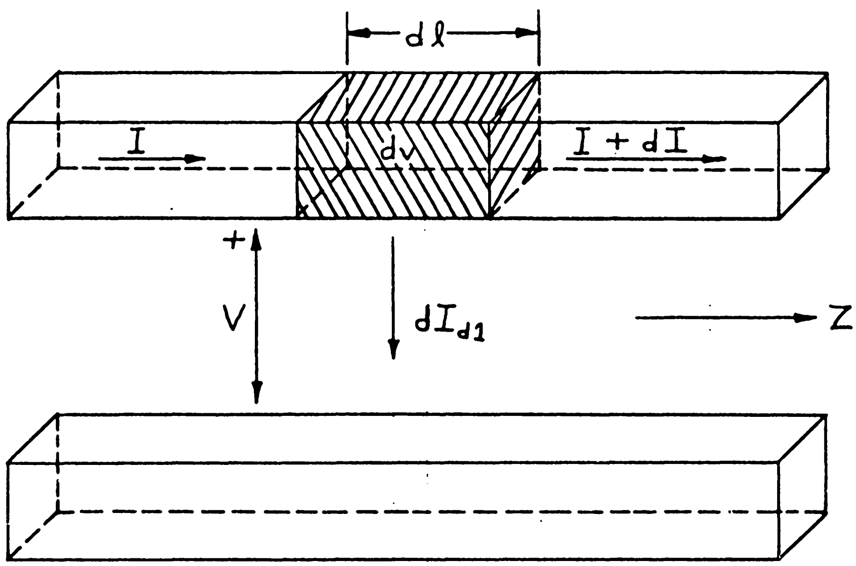

FIGURE 2(b). AXIAL STRIPS ALONG THE EDGES OF THE SLOT leaves the differential volume through the upper side.

Since the total displacement current ($I_d = I_{d1} + I_{d2}$) is proportional to the electric field, it must also be proportional to the slot voltage V; that is,

$$\frac{dI_d}{dz} = j\omega CV \tag{4}$$

where C is the capacitance per unit length. The in-phase component of the transverse conduction current, the transverse radiation current (I_c), is also proportional to the voltage across the slot:

$$\frac{dI_c}{dz} = GV \tag{5}$$

The significance of G will be discussed later. Thus,

$$-\frac{dI}{dz} = \frac{dI_t}{dz} = \frac{dI_d}{dz} + \frac{dI_c}{dz} = GV + j\omega CV \tag{6}$$

Or,

$$\frac{dI}{dz} = -(G + j\omega C)V = -YV \tag{7}$$

where Y is the slot distributed admittance. Equation (7) is the first transmission-line equation.

Next, the second transmission-line equation will be derived using the method of Holt.[7] The slot distributed inductance is defined by

$$L = \frac{\Phi}{I} \tag{8}$$

where Φ is the magnetic flux passing through the slot per unit length. (See Figure 2(a).) Thus,

$$\Phi = \int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} (\mu H_\rho)(a d\phi) \tag{9}$$

From the Maxwell-Faraday equation,

$$\nabla \times \bar{E} = -\hat{z}\bar{H} \tag{10}$$

where

$$\hat{z} = j\omega\mu$$

which gives

$$\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} = -\hat{z} H_\rho \tag{11}$$

Since the conducting material of the cylinder was assumed to be perfect, E_z is zero. Then,

$$\frac{\partial E}{\partial z} = \hat{z} H_\rho \tag{12}$$

Substituting into Eqs. (8) and (9),

$$L = \frac{1}{I}\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} \left(\mu \frac{1}{j\omega\mu} \frac{\partial E_\phi}{\partial z}\right) (a d\phi)$$

$$L = \frac{1}{j\omega I} \frac{\partial}{\partial z} \int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} E_\phi \; a d\phi \tag{13}$$

Let the voltage across the slot at any point be

$$V = -\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} E_\phi \; a d\phi \tag{14}$$

Then

$$L = \frac{1}{j\omega I} \frac{\partial}{\partial z} (-V)$$

Or,

$$-\frac{\partial V}{\partial z} = j\omega L I = ZI \tag{15}$$

where Z is the slot distributed impedance. This equation is the second transmission-line equation. Thus, the analogy between the slot and a transmission line is established.

It might be observed that the conductance G used above is not the same conductance that is encountered ordinarily in transmission line theory. In common transmission lines, the conductance arises from the lossy dielectric between the two conductors which results in a leakage current. For the slot cylinder antenna the dielectric and conductor are assumed to be lossless. However, there is energy radiated which does not appear in transmission-line theory. This radiation is accounted for by a conductance. Hence, the slot distributed conductance used in the line equation (Eq. (7)) is that which accounts for the radiation losses of the transverse currents on the cylinder. It can be seen from the second line equation (Eq. (15)) that the slot distributed resistance has been neglected. For conventional transmission lines the assumption of a perfect conductor leads to R = 0. The radiation losses from the two equal and opposite axial currents are negligible.

Equations will now be obtained which relate the input impedance Z_{in} of a shorted transmission line to the parameters L, C, and G. In accordance with the transmission-line analogy, it becomes necessary to assume that the line is uniform; i.e., Y and Z are independent of the axial coordinate z. From transmission-line theory the propagation constant and characteristic impedance are given by

$$\gamma = \alpha_a + j\beta = \sqrt{ZY} \tag{16}$$

$$Z_o = \sqrt{\frac{Z}{Y}} \tag{17}$$

Also, the input impedance to a shorted transmission line is given by

$$Z_{in} = Z_o \tanh \gamma L' \tag{18}$$

For the case of the center-fed slot, the input impedance becomes that of two identical lines in parallel. Or,

$$Z_{in} = \frac{Z_o}{2} \tanh \frac{\gamma L'}{2} \tag{19}$$

where L' is the slot length.

Some more straight forward relations between the attenuation constant α_a, the phase constant β , and the line parameters can be derived. Let,

$$\begin{aligned} Y &= G + j\omega C = G + jB \\ Z &= j\omega L = jX \\ \gamma &= \alpha_a + j\beta = \sqrt{YZ} \\ Z_o &= \sqrt{\frac{Z}{Y}} \end{aligned} \qquad (20$$

Then, according to Holt[9],

$$\alpha_a = \frac{GX}{2\beta} = \sqrt{\frac{X}{2}}\sqrt{\left(B^2+G^2\right)^{\frac{1}{2}} - B} \tag{21}$$

$$\beta = \frac{GX}{2\alpha_a} = \sqrt{\frac{X}{2}}\sqrt{\left(B^2+G^2\right)^{\frac{1}{2}}+B} \tag{22}$$

$$\beta^2 + \alpha_a^2 = BX \tag{23}$$

$$Z_o = \frac{X}{\beta - j\alpha_a} \tag{24}$$

Hence, this completes the equations necessary for the line theory.

All that remains now is to determine expressions for the line parameters B, G, and C in terms of the wavelength of excitation and the antenna dimensions. It might be wise at this time to consider one of the problems that will arise from the equations for the line parameters. These expressions will contain an unknown quantity, the phase constant β . Thus, the equations for G and C which contain β must be solved simultaneously with Eq. (22). Also, since the equations for G and C contain infinite series, the method of trial-and-error must be used.

In the following section an analysis is presented of an infinitely long cylinder containing an infinite slot. This provides a basis for the determination of the slot distributed inductance and capacitance.

IV. ANALYSIS OF INFINITELY LONG CYLINDER CONTAINING INFINITE SLOT.

Consider the problem of an infinitely-long cylinder containing a slot running its entire length. Assume that the slot is center fed. As stated previously, for $W << \lambda$ the field across the slot is assumed uniform. The slot then acts as an infinite transmission line, with traveling waves of the form,

$$E_\phi = E_o \epsilon^{-j\beta|z|} \tag{25}$$

and

$$E_z = 0$$

for

$$-\frac{\alpha}{2} < \phi < \frac{\alpha}{2} \quad \text{and} \quad \rho = a$$

Since E_z was taken as zero in the slot, it must be zero both internal and external to the cylinder. This suggests a field which is transverse electric (TE) to z. As will be shown later, the boundary conditions are satisfied by a field TE to z. A general solution can now be determined which will satisfy the boundary conditions.

The solution of the infinite slot problem will be used to determine the line parameters X and B. For the determination of G it is feasible to restrict the slot to a finite length. Thus, instead of having traveling waves along the slot, there will be reflections resulting in standing waves. This is discussed in detail when the equation for conductance is derived.

A solution to the fields of a slotted cylinder may be obtained in two parts. First, the field is determined for the homogeneous, source-free region interior to cylinder and then other expressions are determined for a similar region exterior to the cylinder. For any homogeneous, source-free region, the fields must satisfy Maxwell's equations,

$$\nabla \times \bar{E} = -\hat{z}\,\bar{H} \tag{26}$$

$$\nabla \times \bar{H} = \hat{y}\,\bar{E} \tag{27}$$

$$\nabla \cdot \bar{H} = 0 \tag{28}$$

$$\nabla \cdot \bar{E} = 0 \tag{29}$$

where $\hat{z} = j\omega\mu$ and $\hat{y} = j\omega\epsilon$. Since the divergence of $\bar{E}$ is zero, $\bar{E}$ can be expressed as the curl of some other vector $\bar{F}$; thus

$$\bar{E} = -\nabla \times \bar{F} \tag{30}$$

where $\bar{F}$ is the electric vector potential. Substituting into Eq. (27),

$$\nabla \times \bar{H} = \hat{y}\left(\nabla \times \bar{F}\right)$$

$$\nabla \times \left(\bar{H} - \hat{y}\,\bar{F}\right) = 0 \tag{31}$$

Since any vector whose curl is zero must be the gradient of some scalar,

$$\bar{H} - \hat{y}\,\bar{F} = \nabla \Phi \tag{32}$$

where Φ is any scalar. Substituting Eqs. (30) and (32) into Eq. (26),

$$\nabla \times \left(\nabla \times \bar{F}\right) = -\hat{z}\left(\hat{y}\bar{F} + \nabla\Phi\right)$$

$$\nabla \times \nabla \times \bar{F} + \hat{z}\,\hat{y}\bar{F} = -\hat{z}\nabla\Phi \qquad (33)$$

If k is the wave number of the medium,

$$k = \sqrt{-\hat{z}\hat{y}} \qquad (34)$$

then

$$\nabla \times \nabla \times \bar{F} - k^2\bar{F} = -\hat{z}\nabla\Phi \qquad (35)$$

Consider the vector identity,

$$\nabla \times \nabla \times \bar{F} = \nabla\left(\nabla \cdot \bar{F}\right) - \nabla^2\bar{F}$$

Substituting this into Eq. (35) gives

$$\nabla\left(\nabla \cdot \bar{F}\right) - \nabla^2\bar{F} - k^2\bar{F} = -\hat{z}\nabla\Phi \qquad (36)$$

Because the electric vector potential is merely an arbitrary vector that is chosen to facilitate the solution of the problem, let

$$\nabla \cdot \bar{F} = -\hat{z}\Phi \qquad (37)$$

Thus, from Eq. (36),

$$\nabla^2 \bar{F} + k^2 \bar{F} = 0 \tag{38}$$

This is the Helmholtz equation. Note that the rectangular components of $\bar{F}$ satisfy the scalar Helmholtz equation,

$$\nabla^2 \psi + k^2 \psi = 0 \tag{39}$$

The total field is given by Eqs. (3), (32), and (37), or,

$$\bar{E} = - \nabla \times \bar{F} \tag{40}$$

$$\bar{H} = - y \bar{F} + \frac{1}{\hat{z}} \nabla(\nabla \cdot \bar{F}) \tag{41}$$

For a given problem, any electric vector potential which is chosen must satisfy the Helmholtz equation. Thus, the scalar Helmholtz equation can be solved to obtain the required potential, and consequently the field.

Consider the case

$$\bar{F} = \bar{U}_z \psi \tag{42}$$

where $\bar{U}_z$ is the unit vector in the +z direction. Expanding by Eqs. (40) and (41) in cylindrical coordinates gives

$$E_\rho = -\frac{1}{\rho} \frac{\partial \psi}{\partial \phi}$$

$$E_\phi = \frac{\partial \psi}{\partial \rho}$$

$$E_z = 0$$

$$H_\rho = \frac{1}{\hat{z}} \frac{\partial^2 \psi}{\partial \rho \partial z} \tag{43}$$

$$H_\phi = \frac{1}{\hat{z}\rho} \frac{\partial^2 \psi}{\partial \phi \partial z}$$

$$H_z = \frac{1}{\hat{z}}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$$

Since E_z is zero, this is a field TE to z. This is the required field for the slotted cylinder which satisfies the condition that $E_z = 0$. Solutions to the scalar Helmholtz equation can now be substituted into Eq. (43) to obtain the field.

For cylindrical coordinates, the scalar Holmholtz equation becomes

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0 \tag{44}$$

Solutions to this equation are of the form

$$\psi_{k_\rho, n, \beta} = B_n(k_\rho \rho) h(n\phi) h(\beta z) \tag{45}$$

where $B_n(k_\rho \rho)$ is a Bessel function and $h(n\phi)$ and $h(\beta z)$ are sine and/or cosine functions. Also,

$$k_\rho^2 = k^2 - \beta^2 \tag{46}$$

gives the relation between the constants of Eq. (45). In order to achieve a form similar to that of Eq. (25), the z and ϕ functions are chosen as

$$\begin{aligned} h(n\phi) &= \cos n\phi \\ h(\beta z) &= \epsilon^{-j\beta|z|} \end{aligned} \tag{47}$$

Since the field inside the cylinder must be finite, let

$$B_n(k_\rho \rho) = J_n(k_\rho \rho) \tag{48}$$

for $\rho \leq a$. Because the external field must be finite as ρ approaches infinity, let

$$B_n(k_\rho \rho) = H_n^{(2)}(k_\rho \rho) \tag{49}$$

for $\rho \geq a$. Thus the wave functions ψ become

$$\psi^-_{k_\rho,n,\beta} = J_n(k_\rho \rho) \cos n\phi \; \epsilon^{-j\beta|z|} \qquad \rho \leq a \tag{50}$$

$$\psi^+_{k_\rho,n,\beta} = H_n^{(2)}(k_\rho \rho) \cos n\phi \; \epsilon^{-j\beta|z|} \qquad \rho \geq a \tag{51}$$

Since it is anticipated that the boundary conditions will be given as a Fourier series in ϕ , the wave functions ψ may be rewritten as

$$\psi^- = \sum_{n=0}^{\infty} C_n J_n(k_\rho \rho) \cos n\phi \, \epsilon^{-j\beta|z|} \qquad \rho \leq a \tag{52}$$

$$\psi^+ = \sum_{n=0}^{\infty} D_n H_n^{(2)}(k_\rho \rho) \cos n\phi \, \epsilon^{-j\beta|z|} \qquad \rho \geq a \tag{53}$$

For convenience, consider only the fields in the +z direction, replacing $|z|$ by z. The only difference between the fields in the +z and -z directions is that H_ρ and H_ϕ will have different signs. This difference in signs will not affect the final result.

For the interior fields, substitute Eq. (52) into Eq. (43) giving

$$E_\phi = k_\rho \sum_{n=0}^{\infty} C_n J_n'(k_\rho \rho) \cos n\phi \, \epsilon^{-j\beta z} \tag{54}$$

$$E_\rho = \frac{1}{\rho} \sum_{n=0}^{\infty} n C_n J_n(k_\rho \rho) \sin n\phi \, \epsilon^{-j\beta z} \tag{55}$$

$$H_\rho = -\frac{j\beta k_\rho}{\hat{z}} \sum_{n=0}^{\infty} C_n J_n'(k_\rho \rho) \cos n\phi \, \epsilon^{-j\beta z} \tag{56}$$

$$H_\phi = \frac{j\beta}{\hat{z}\rho} \sum_{n=0}^{\infty} n C_n J_n(k_\rho \rho) \sin n\phi \, \epsilon^{-j\beta z} \tag{57}$$

$$H_z = \frac{k^2 - \beta^2}{\hat{z}} \sum_{n=0}^{\infty} C_n J_n(k_\rho \rho) \cos n\phi \, \epsilon^{-j\beta z} \tag{58}$$

where the primes on the Bessel functions denote derivatives with respect to the argument. Determination of the constant C_n can now be achieved by applying the boundary conditions. At $\rho = a$:

$$\left.\begin{aligned} E_\phi &= E_o \epsilon^{-j\beta z} \\ E_z &= 0 \end{aligned}\right\} \quad -\frac{\alpha}{2} < \phi < \frac{\alpha}{2} \tag{59}$$

Expressing E_ϕ as a Fourier series in ϕ,

$$E_\phi = \left[\frac{E_o \alpha}{2\pi} + \frac{2E_o}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\alpha}{2}\cos n\phi\right]\epsilon^{-jkz} \tag{60}$$

Equating Eqs. (60) and (54) at $\rho = a$ gives

$$E_\phi = \left[\frac{E_o \alpha}{2\pi} + \frac{2E_o}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\alpha}{2}\cos n\phi\right]\epsilon^{-j\beta z}$$

$$= \left[k_\rho C_o J_o'(k_\rho a) + \sum_{n=1}^{\infty} k_\rho C_n J_n'(k_\rho a)\cos n\phi\right]\epsilon^{-j\beta z}$$

Thus,

$$C_o = \frac{E_o \alpha}{2\pi k_\rho J_o'(k_\rho a)} \tag{61}$$

$$C_n = \frac{2E_o \sin\frac{n\alpha}{2}}{\pi n k_\rho J_n'(k_\rho a)} \tag{62}$$

Therefore, for $\rho \leq a$,

$$E_\rho = \frac{2E_o}{\pi k_\rho \rho} \sum_{n=1}^{\infty} \frac{J_n(k_\rho \rho)}{J_n'(k_\rho a)} \sin\frac{n\alpha}{2} \sin n\phi \, \epsilon^{-j\beta z} \tag{63}$$

$$E_\phi = \left[\frac{E_o \alpha}{2\pi} \frac{J_o'(k_\rho \rho)}{J_o'(k_\rho a)} + \frac{2E_o}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{J_n'(k_\rho \rho)}{J_n'(k_\rho a)} \sin\frac{n\alpha}{2} \cos n\phi \right] \epsilon^{-j\beta z} \tag{64}$$

$$E_z = 0 \tag{65}$$

$$H_\rho = \left[-\frac{j\beta E_o \alpha}{2\pi \hat{z}} \frac{J_o'(k_\rho \rho)}{J_o'(k_\rho a)} - \frac{j2E_o\beta}{\pi \hat{z}} \sum_{n=1}^{\infty} \frac{1}{n} \frac{J_n'(k_\rho \rho)}{J_n'(k_\rho a)} \sin\frac{n\alpha}{2} \cos n\phi \right] \epsilon^{-j\beta z} \tag{66}$$

$$H_\phi = \frac{j2E_o\beta}{\pi k_\rho \hat{z}} \sum_{n=1}^{\infty} \frac{J_n(k_\rho \rho)}{J_n'(k_\rho a)} \sin\frac{n\alpha}{2} \sin n\phi \, \epsilon^{-j\beta z} \tag{67}$$

$$H_z = \left[\frac{E_o \alpha k_\rho}{2\pi \hat{z}} \frac{J_o(k_\rho \rho)}{J_o'(k_\rho a)} + \frac{2E_o k_\rho}{\pi \hat{z}} \sum_{n=1}^{\infty} \frac{1}{n} \frac{J_n(k_\rho \rho)}{J_n'(k_\rho a)} \sin\frac{n\alpha}{2} \cos n\phi \right] \epsilon^{-j\beta z} \tag{68}$$

Similar expressions may be written for the external fields by merely replacing the Bessel functions of the first kind with the respective Hankel functions of the second kind.

It can now be shown that for cylinders of small radii the field pattern inside the cylinder is very similar to that of the dominant mode of a circular waveguide. This is the TE_{11} mode. The fields of this waveguide mode are

$$E_\rho = \frac{1}{\rho} J_1\left(\frac{X_{11}'\rho}{a}\right) \sin\phi \, \epsilon^{-j\beta z}$$

$$E_\phi = \frac{X_{11}'}{a} J_1'\left(\frac{X_{11}'\rho}{a}\right) \cos\phi \, \epsilon^{-j\beta z}$$

$$E_z = 0$$

$$H_\rho = -\frac{j\beta X'_{11}}{a\hat{z}} J'_1\left(\frac{X'_{11}\rho}{a}\right) \cos\phi\, \epsilon^{-j\beta z}$$

$$H_\phi = \frac{j\beta}{\hat{z}\rho} J_1\left(\frac{X'_{11}\rho}{a}\right) \sin\phi\, \epsilon^{-j\beta z} \tag{69}$$

$$H_z = \frac{k_\rho^2}{\hat{z}} J_1\left(\frac{X'_{11}\rho}{a}\right) \cos\phi\, \epsilon^{-j\beta z}$$

where X'_{11} is the first zero of $J'_1(X)$. For small radius, the fields of the slotted cylinder can be approximated by taking only the zero and first order terms of the infinite series

$$E_\rho = \left[\frac{2E_o}{\pi k_\rho} \frac{\sin\frac{\alpha}{2}}{J'_1(k_\rho a)}\right] \frac{1}{\rho} J_1(k_\rho \rho) \sin\phi\, \epsilon^{-j\beta z}$$

$$E_\phi = \left[\frac{E_o \alpha}{2\pi J'_o(k_\rho a)}\right] J'_o(k_\rho \rho) \epsilon^{-j\beta z} + \left[\frac{2E_o}{\pi} \frac{\sin\frac{\alpha}{2}}{J'_1(k_\rho a)}\right] J'_1(k_\rho \rho) \cos\phi\, \epsilon^{-j\beta z}$$

$$E_z = 0 \tag{70}$$

$$H_\rho = \left[-\frac{j\beta E_o \alpha}{2\pi \hat{z} J'_o(k_\rho a)}\right] J'_o(k_\rho \rho) \epsilon^{-j\beta z} + \left[-\frac{j2E_o\beta}{\pi \hat{z}} \frac{\sin\frac{\alpha}{2}}{J'_1(k_\rho a)}\right] J'_1(k_\rho \rho) \cos\phi\, \epsilon^{-j\beta z}$$

$$H_\phi = \left[\frac{j2E_o\beta}{\pi k_\rho \hat{z}} \frac{\sin\frac{\alpha}{2}}{J_1(k_\rho a)}\right] \frac{1}{\rho} J_1(k_\rho \rho) \sin\phi\, \epsilon^{-j\beta z}$$

$$H_z = \left[\frac{E_o \alpha k_\rho}{2\pi \hat{z} J'_o(k_\rho a)}\right] J_o(k_\rho \rho) \epsilon^{-j\beta z} + \left[\frac{2E_o k_\rho}{\pi \hat{z}} \frac{\sin\frac{\alpha}{2}}{J'_1(k_\rho a)}\right] J_1(k_\rho \rho) \cos\phi\, \epsilon^{-j\beta z}$$

In comparing the two sets of equations it is evident that the H_ϕ and E_ρ terms have the same form. The H_z, H_ρ, and E_ϕ terms for the antenna all contain an extra term which accounts for the field in the slot region. Thus, for cylinders with $a << \lambda$, the field approximates that of the TE_{11} circular waveguide mode if modifications are made to account for the field around the slot. The field patterns are shown in Figure 3.

Using the fields given by Eqs. (63) to (68) and the similar ones for the external region, it is possible to derive equations for the slot distributed inductance L and the slot distributed capacitance C. Expressions for L and C are obtained in the next two sections.

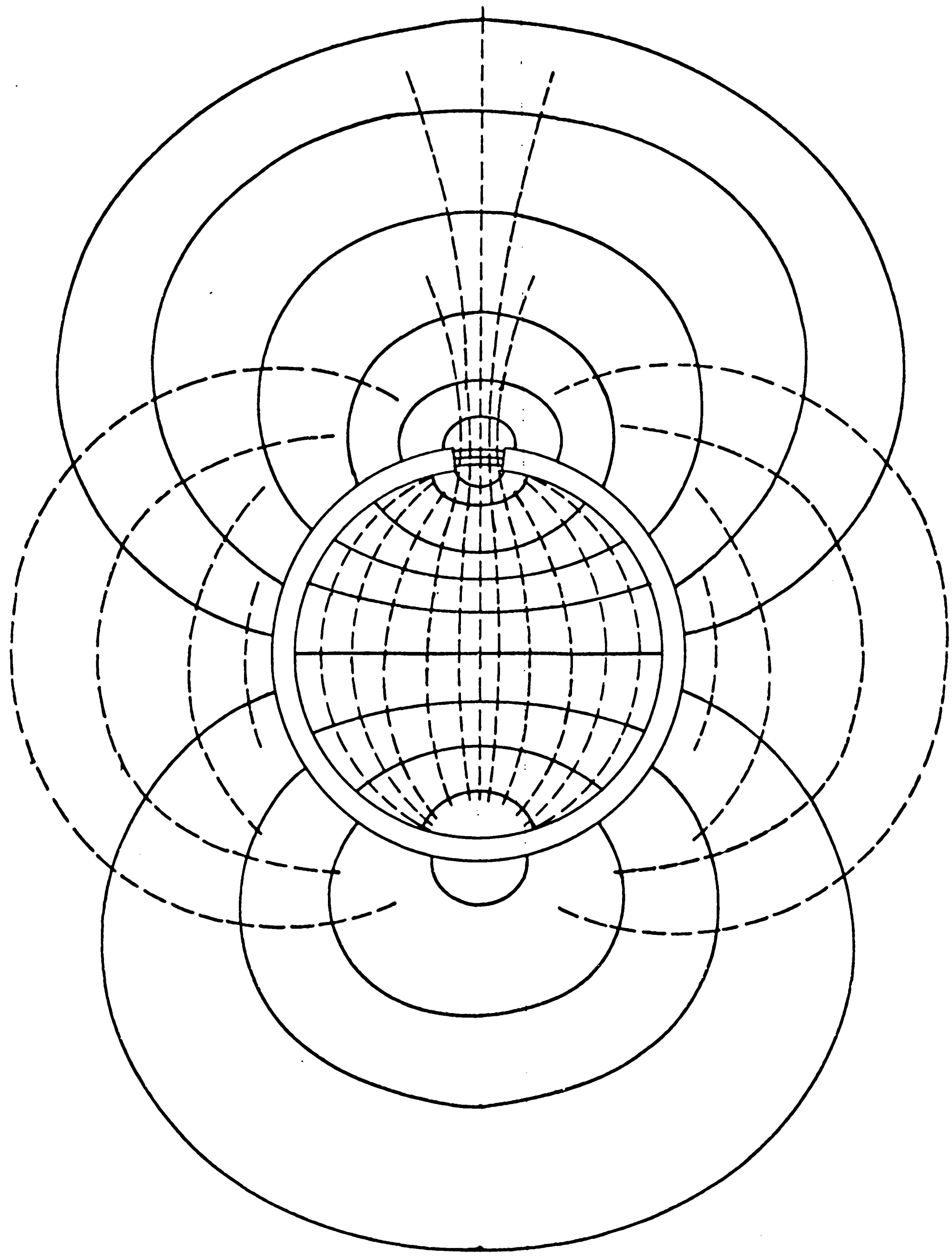

FIGURE 3. END VIEW OF INFINITE SLOT CYLINDER SHOWING H (DASHED LINES) AND E(SOLID) FIELDS

V. DETERMINATION OF SLOT DISTRIBUTED CAPACITANCE.

In this section, an expression for the slot distributed susceptance B will be obtained, with $B = \omega C$.

When considering the capacitance of the slotted cylinders, it must be noted that there are displacement currents in the entire volume around the cylinders; that is, there is a transverse current across the slot, and there are also displacement currents from the exterior of the cylinder and currents across the interior region. Thus, the problem is to find an expression for the net transverse reactive current. This is done by first determining the total transverse current at the slot. The reactive current is then just the imaginary part of the total current.

At the slot ($\phi = \alpha/2$ and $\rho = a$), the total transverse current is the sum of the displacement current across the slot and the transverse conduction current due to the cylinder. In determining the transverse current across the slot, it is assumed that there is no fringing in the slot region; that is, the field lines are uniform across the slot. This is justified by observing that the interior and exterior sides of the cylinder just adjacent to the slot will act as guard plates. Thus, the E field emanating from the slot face will be confined primarily to the slot region. The displacement current per unit length in the slot is thus given by

$$J_\phi^s = j\omega\epsilon t E_o \epsilon^{-j\beta z} \tag{71}$$

where t is the thickness of the cylinder wall.

Since the cylinder is assumed a perfect conductor, the transverse conduction current at $\phi = \alpha/2$ due to the cylinder may be divided into two parts - the current on the outer surface of the cylinder and the current on the inner surface. This transverse current per unit length is given by[10]

$$J_s = \bar{n} \times \bar{H} \tag{72}$$

where $\bar{n}$ is the unit vector normal to the cylinder and pointing into the field. Thus, these transverse conduction currents at $\phi = \alpha/2$ are $J_\phi^- = H_z^-$ for the inner surface and $J_\phi^+ = -H_z^+$ for the outer surface where H_z is evaluated at $\phi = \alpha/2$ and $\rho = a$. Therefore, from Eq. (68) and the similar equation for the external field,

$$J_\phi^- = \left[\frac{E_o \alpha k_\rho}{2\pi \hat{z}} \frac{J_o(k_\rho a)}{J_o'(k_\rho a)} + \frac{E_o k_\rho}{\pi \hat{z}} \sum_{n=1}^{\infty} \frac{1}{n} \frac{J_n(k_\rho a)}{J_n'(k_\rho a)} \sin n\alpha\right] \epsilon^{-j\beta z} \tag{73}$$

and

$$J_\phi^+ = \left[-\frac{E_o \alpha k_\rho}{2\pi \hat{z}} \frac{H_o^{(2)}(k_\rho a)}{H_o^{(2)\prime}(k_\rho a)} - \frac{E_o k_\rho}{\pi \hat{z}} \sum_{n=1}^{\infty} \frac{1}{n} \frac{H_n^{(2)}(k_\rho a)}{H_n^{(2)\prime}(k_\rho a)} \sin n\alpha\right] \epsilon^{-j\beta z} \tag{74}$$

The total transverse current J_ϕ at the slot is then the sum of

Eqs. (71), (73) and (74),

$$J_\phi = J_\phi^+ + J_\phi^- + J_\phi^s \tag{75}$$

As stated above, the displacement current is the imaginary part of the total current of Eq. (75).

The slot distributed susceptance is

$$B = \frac{I_m(J_\phi)}{V\epsilon^{-j\beta z}} \tag{76}$$

where

$$V = E_o W \tag{77}$$

Thus,

$$B = \frac{I_m(J_\phi)}{E_o W \epsilon^{-j\beta z}} \tag{78}$$

An expression for the numerator $\left[I_m(J_\phi)\right]$ of the preceding equation may be obtained by substituting Bessel functions of the first and second kind for the Hankel functions of Eq. (74) and simplifying. (See Appendix A.) Equation (76) then becomes

$$B = \frac{\omega\epsilon t}{W} - \frac{\alpha}{\pi^2 W \mu a \omega}\left\{\frac{N_o'(k_\rho a)}{J_o'(k_\rho a)\left[J_o'^2(k_\rho a) + N_o'^2(k_\rho a)\right]}\right\} + \tag{79}$$

$$- \frac{2}{\pi^2 \omega \mu a W}\sum_{n=1}^{\infty}\frac{N_n'(k_\rho a)}{J_n'(k_\rho a)\left[J_n'^2(k_\rho a) + N_n'^2(k_\rho a)\right]}\frac{\sin n\alpha}{n}$$

Equation (79) is thus the required equation for the slot distributed susceptance in terms of the slot-cylinder dimensions and the wavelength. It might be noted that the real part of the total transverse current could be used to determine a value of conductance. This method of determining conductance is not used because a more accurate equation will be derived in a later section based on the field of a slot of finite length.

In the next section the solution of the fields to the problem with the infinite slot is used to determine the slot distributed inductance.

VI. DETERMINATION OF SLOT DISTRIBUTED INDUCTANCE.

This section is devoted to obtaining an expression for the distributed inductance L of the slot cylinder antenna.

In an earlier section where the transmission-line equations for the slot were determined, the slot distributed inductance was given by (Eq. (8)),

$$L = \frac{\Phi}{I} \tag{80}$$

where Φ is the magnetic flux passing through the slot per unit length. As with the transmission line analogy, it is assumed that all the axial current flows in thin strips along the faces of the slot, the axial current is given by Eq. (72) with $\bar{n} = -\bar{U}_\phi$. Or,

$$J_z = H_\rho \tag{81}$$

and

$$I_z = H_\rho \; t \quad \text{at} \quad \rho = a \quad \text{and} \quad \phi = \alpha/2 \tag{82}$$

From Eq. (9), the magnetic flux is

$$\Phi = \int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} (\mu H_\rho)(a\, d\phi) \tag{83}$$

Substituting Eq. (66) into Eq. (82) and evaluating at $\rho = a$ and $\phi = \alpha/2$ gives

$$I_z=\left[-\frac{j\beta E_o \alpha t}{2\pi\hat{z}}-\frac{j2E_o\beta t}{\pi\hat{z}}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\alpha}{2}\cos\frac{n\alpha}{2}\right]\epsilon^{-j\beta z}$$

$$=\frac{-jE_o\beta t}{\pi\hat{z}}\left[\frac{\alpha}{2}+\sum_{n=1}^{\infty}\frac{1}{n}\sin n\alpha\right]\epsilon^{-j\beta z} \qquad (84)$$

Summing the series[11],

$$I_z=-\frac{jE_o\beta t}{\pi\hat{z}}\left[\frac{\alpha}{2}+\frac{1}{2}(\pi-\alpha)\right]\epsilon^{-j\beta z}$$

$$=-\frac{jE_o\beta t}{2\hat{z}}\epsilon^{-j\beta z} \qquad (85)$$

Substituting Eq. (66) into Eq. (83), evaluating at $\rho = a$ and $\phi = \alpha/2$, and integrating give

$$\Phi=\left[-\frac{j\beta E_o\alpha^2\mu a}{2\pi\hat{z}}-\frac{j4\beta E_o\mu a}{\pi\hat{z}}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\frac{n\alpha}{2}\right]\epsilon^{-j\beta z}$$

$$=-\frac{j\beta E_o\mu a}{\pi\hat{z}}\left[\frac{\alpha^2}{2}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\frac{n\alpha}{2}\right]\epsilon^{-j\beta z} \qquad (86)$$

Making a trigonometric substitution,

$$\Phi=-\frac{j\beta E_o\mu a}{\pi\hat{z}}\left[\frac{\alpha^2}{2}+2\sum_{n=1}^{\infty}\frac{1}{n^2}-2\sum_{n=1}^{\infty}\frac{\cos n\alpha}{n^2}\right]\epsilon^{-j\beta z} \qquad (87)$$

Summing the two series[11],

$$\Phi = -\frac{j\beta E_o \mu a}{\pi \hat{Z}}\left[\frac{\alpha^2}{2} + 2\left(\frac{\pi^2}{6}\right) - 2\left(\frac{\pi^2}{6} - \frac{\pi\alpha}{2} + \frac{\alpha^2}{4}\right)\right] \epsilon^{-j\beta z} \tag{88}$$

$$= -\frac{j\beta E_o \mu a \alpha}{\hat{Z}} \epsilon^{-j\beta z}$$

Substituting Eqs. (88) and (85) into Eq. (80) gives

$$L = \frac{2\alpha\mu a}{t} \tag{89}$$

This is the required equation for the slot distributed inductance. It should be noted that this is not a function of frequency as was the susceptance, and so may be evaluated directly from the dimensions of the antenna.

The only remaining line parameter is the slot distributed conductance G. In the following section a basis is provided for the derivation of G by considering the radiation field from an infinite cylinder with a finite slot. From this field the conductance is obtained.

VII. ANALYSIS OF INFINITELY LONG CYLINDER CONTAINING FINITE SLOT.

In the last two sections the inductance and susceptance parameters were determined from the field of an infinite cylinder containing an infinite slot. For that analysis the total field was considered; that is, both the near and far fields were used. As was pointed out in the derivation of the transmission-line equations, the conductance G for the slotted cylinder antenna is not the conductance present in a conventional transmission line. It is a conductance used to account for the radiation losses from the cylinder. Thus, it is necessary to use only the radiation field in the determination of conductance. Also, a different model will be used in finding the field of the cylinder; that is, the radiation field for an infinite cylinder with a finite slot will be found. The purpose of this section is to determine the radiation field for an infinite cylinder containing a finite axial slot. From this field, the conductance will be determined in the next section.

Consider a cylinder of infinite length containing an axial slot of length L' and width $W = \alpha a$, as shown in Figure 1. When the slot is fed at the center it can be considered to be two similar shorted transmission lines in parallel. From this analogy, it is evident that the excitation will result in standing waves along the length of the slot. For a narrow slot ($W << \lambda$), the field in the slot can be assumed to have the form

$$E_\phi(a,\phi,z) = \frac{V}{\alpha a}\sin\left[\beta\left(\frac{L'}{2}-|z|\right)\right] \qquad \begin{cases} -\frac{L'}{2} < z < \frac{L'}{2} \\ -\frac{\alpha}{2} < \phi < \frac{\alpha}{2} \end{cases} \tag{90}$$

where V is the maximum value of the rms voltage along the slot. Also, as with the previous analyses,

$$E_z = 0 \tag{91}$$

From Eq. (90) it is seen that the attenuation along the slot is assumed negligible. Using the specified slot field (Eqs. (90) and (91)), it is possible to determine the radiation field by the method of Harrington[12].

The method used by Harrington is to transform a three-dimensional problem, such as this one, into a two-dimensional problem where it can be solved easier. This elimination of one dimension is accomplished by taking the Fourier transform of the field at $\rho = a$ with respect to the z-coordinate and thus eliminating the z-dimension. After solving the problem in two dimensions, the inverse Fourier transformation can be used to change it back into three dimensions. The Fourier transform of E_ϕ with respect to z and its corresponding inverse transformation are given, respectively, by

$$\tilde{E}_\phi(n,w) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, E_\phi(a,\phi,z)\,\epsilon^{-jn\phi}\,\epsilon^{-jwz} \tag{92}$$

$$E_\phi(a,\phi,z) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \epsilon^{-jn\phi}\int_{-\infty}^{\infty} \tilde{E}_\phi(n,w)\,\epsilon^{jwz}\,dw \tag{93}$$

where the wavy line ($\sim$) denotes a Fourier transform.

Since $E_z = 0$, the field is again TE to z as in the infinite slot problem. A solution can be constructed which satisfies the Helmholtz equation in cylindrical coordinates. As will be shown, this solution is constructed to conform with the anticipated inverse Fourier transformation. As with the infinite slot, a field TE to z is obtained by letting

$$\bar{F} = \bar{U}_z \psi \tag{94}$$

where the E field is given by

$$\bar{E} = - \nabla \times \bar{F} \tag{95}$$

A possible solution to the scalar Helmholtz equation is

$$\psi = \sum_n \int_{k_z} f_n(k_z) B_n(k_\rho \rho) h(n\phi) h(k_z z)\, dk_z \tag{96}$$

where f_n is a function to be determined from the boundary conditions. If

$$k_z = W \tag{97}$$

to conform with the Fourier transform symbols, then

$$k_\rho = \sqrt{k^2 - W^2} \tag{98}$$

and Eq. (96) becomes

$$\psi = \sum_n \int_w f_n(w) B_n\left(\rho\sqrt{k^2-w^2}\right) h(n\phi) h(wz)\, dw \tag{99}$$

Since the radiation field is desired, Hankel functions of the second kind are chosen for B_n. Also, the functions $h(n\phi)$, $h(wz)$, and the constant $1/2\pi$ are chosen to conform with the inverse Fourier transform. Equation (99) is then

$$\psi = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \epsilon^{jn\phi} \int_{-\infty}^{\infty} f_n(w) H_n^{(2)}\left(\rho\sqrt{k^2-w^2}\right) \epsilon^{jwz}\, dw \tag{100}$$

From Eq. (95) the field is

$$\bar{E} = -\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\bar{u}_\rho + \frac{\partial\psi}{\partial\rho}\bar{u}_\phi \tag{101}$$

Since the tangential field is the only component of interest,

$$E_\phi = \frac{\partial\psi}{\partial\rho}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \epsilon^{jn\phi} \int_{-\infty}^{\infty} f_n(w) \sqrt{k^2-w^2}\, H_n^{(2)\prime}\left(\rho\sqrt{k^2-w^2}\right) \epsilon^{jwz}\, dw \tag{102}$$

f_n can now be determined from the boundary condition. This boundary condition is given by Eq. (93); that is, at $\rho = a$ Eq. (93) must equal Eq. (102). Equating these two, f_n is obtained as

$$f_n = \frac{\tilde{E}_\phi(n,w)}{\sqrt{k^2-w^2}\, H_n^{(2)\prime}\left(a\sqrt{k^2-w^2}\right)} \tag{103}$$

where $\tilde{E}_\phi$ is the Fourier transform of Eq. (90). The required field is then obtained by performing the integration indicated in Eq. (102). This is a difficult integration to perform except in the radiation zone. For this case the field is given by Harrington as

$$E_{\phi}\xrightarrow[r\to\infty]{} -jk\frac{\epsilon^{-jkr}}{\pi r}\sin\theta\sum_{n=-\infty}^{\infty}\epsilon^{jn\phi}j^{n+1}f_n(-k\cos\theta) \tag{104}$$

which applies to all cases except when θ is 0 or π.

The only remaining task is to determine f_n from Eq. (103). Taking the Fourier transform of Eq. (90) gives (See Appendix B.)

$$\tilde{E}_{\phi}(n,w)=\frac{2V\beta}{n\pi\alpha a}\,\frac{\cos\frac{wL'}{2}-\cos\frac{\beta L'}{2}}{\beta^2-w^2}\sin\frac{n\alpha}{2} \tag{105}$$

Thus,

$$f_n(w)=\frac{2V\beta\left(\cos\frac{wL'}{2}-\cos\frac{\beta L'}{2}\right)\sin\frac{n\alpha}{2}}{n\pi\alpha a(\beta^2-w^2)\sqrt{k^2-w^2}\,H_n^{(2)\prime}\left(a\sqrt{k^2-w^2}\right)}$$

which results in a radiation field,

$$E_{\phi}=\frac{V\epsilon^{-jkr}}{\pi^2\beta a r}\left[\frac{\cos\left(\frac{kL'}{2}\cos\theta\right)-\cos\frac{\beta L'}{2}}{1-\left(\frac{k}{\beta}\cos\theta\right)^2}\right]\sum_{n=-\infty}^{\infty}\frac{j^n\epsilon^{jn\phi}}{H_n^{(2)\prime}(ak\sin\theta)}\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2} \tag{106}$$

Note that for a very narrow slot the two terms involving α at the end of Eq. (106) become unity.

Equation (106) is thus the required radiation field for an axial slot of length L' in an infinite cylinder. In the following section this field will be used to find the final transmission-line parameter, the slot distributed conductance G.

VIII. DETERMINATION OF SLOT DISTRIBUTED CONDUCTANCE.

The slot distributed conductance G is determined by first using the radiation field found in the last section to get the radiated power. This expression for the radiated power is then equated to another expression which involves the conductance G. The resulting formula can then be solved for G.

Integrating the radial component of the Poynting vector over the surface of a sphere surrounding the slotted cylinder antenna gives the radiated power as

$$P_f = \iint_{sphere} S_r \, ds \tag{107}$$

where P_f is the radiated power and S_r is the radial component of the Poynting vector. For spherical coordinates,

$$P_f = \int_0^{2\pi} \int_0^{\pi} S_r r^2 \sin\theta \, d\theta \, d\phi \tag{108}$$

Since the power radiated will be the same in each of four sectors, the total power can be determined by integrating over one sector and multiplying by four:

$$P_f = 4r^2 \int_0^{\pi} \int_0^{\frac{\pi}{2}} S_r \sin\theta \, d\theta \, d\phi \tag{109}$$

Since the Poynting vector is given by

$$\bar{S} = \bar{E} \times \bar{H}^{*} \tag{110}$$

where * denotes the complex conjugate, the radial component is

$$S_r = E_\phi \, H_\theta^{*} \tag{111}$$

For the radiation field,

$$H_\theta = -\frac{E_\phi}{\eta} \tag{112}$$

where η is the intrinsic impedance of the medium. Thus,

$$S_r = \frac{|E_\phi|^2}{\eta} \tag{113}$$

Substituting the radiation field of the last section into Eq. (109) gives

$$P_f = \frac{4|V|^2}{\pi^4 \beta^2 a^2 \eta} \int_0^{\frac{\pi}{2}} \left| \frac{\cos\left(\frac{kL'}{2}\cos\theta\right) - \cos\frac{\beta L'}{2}}{1-\left(\frac{k}{\beta}\cos\theta\right)^2} \right| \sin\theta \int_0^{\pi} \left| \sum_{n=-\infty}^{\infty} \frac{j^n \epsilon^{jn\phi}}{H_n^{(2)\prime}(ka\sin\theta)} \left(\frac{2}{n\alpha}\right) \sin\frac{n\alpha}{2} \right|^2 d\phi \, d\theta \tag{114}$$

Integration can be performed with respect to ϕ as shown in Appendix C. This gives

$$P_f = \frac{4|V|^2}{\pi^4 \beta^2 a^2 \eta} \int_0^{\frac{\pi}{2}} \left| \frac{\cos\left(\frac{kL'}{2}\cos\theta\right) - \cos\frac{\beta L'}{2}}{1-\left(\frac{k}{\beta}\cos\theta\right)^2} \right|^2 \sin\theta \; \times$$

$$\times \, \pi \left\{ \frac{1}{J_1^2(x) + N_1^2(x)} + 8 \sum_{n=1}^{\infty} \frac{\left(\frac{2}{n\alpha}\right)^2 \sin^2\frac{n\alpha}{2}}{\left[J_{n-1}(x) - J_{n+1}(x)\right]^2 + \left[N_{n-1}(x) - N_{n+1}(x)\right]^2} \right\} d\theta \tag{115}$$

where $x = ka \sin\theta$. In order to integrate this expression further it is necessary to employ numerical methods.

An expression for the total power supplied to the slot will now be derived in terms of the slot distributed conductance G, following Holt[13]. From the transmission line analogy the conductance of an elemental section dz is Gdz. The voltage across the slot as a function of z is a standing wave of the same form as the E field.

$$V(z) = V \sin \beta\left(\frac{L'}{2} - |z|\right) \tag{116}$$

Thus the power supplied to this elemental section is

$$dP_f = (Gdz)\left[V(z)\right]^2 \tag{117}$$

Integrating over the slot length,

$$\begin{aligned} P_f &= \int_{-\frac{L'}{2}}^{\frac{L'}{2}} G\left[V(z)\right]^2 dz \\ &= GV^2 \int_{-\frac{L'}{2}}^{\frac{L'}{2}} \sin^2\beta\left(\frac{L'}{2} - |z|\right) dz \\ &= \frac{GV^2}{2\beta}\left(\beta L' - \sin\beta L'\right) \end{aligned} \tag{118}$$

Equating this to Eq. (115) and solving for G give

$$G = \frac{8}{\pi^3 \beta a^2 \eta\left(\beta L' - \sin\beta L'\right)} \int_0^{\frac{\pi}{2}} \left| \frac{\cos\left(\frac{kL'}{2}\cos\theta\right) - \cos\frac{\beta L'}{2}}{1 - \left(\frac{k}{\beta}\cos\theta\right)^2} \right|^2 \sin\theta \times$$

$$\times \left\{ \frac{1}{J_1^2(x) + N_1^2(x)} + 8\sum_{n=1}^{\infty} \frac{\left(\frac{2}{n\alpha}\right)^2 \sin^2\frac{n\alpha}{2}}{\left[J_{n-1}(x) - J_{n+1}(x)\right]^2 + \left[N_{n-1}(x) - N_{n+1}(x)\right]^2} \right\} d\theta \tag{119}$$

Thus, Eq. (119) is the required expression for the slot distributed conductance.

This completes the derivation of all the expressions for the line parameters L, G, and B. In the next section a summary is given of the general method.

IX. SUMMARY AND CONCLUSIONS.

In this thesis analytical expressions are derived which make possible the computation of the input impedance to a cylindrical antenna containing an axial slot. The cylinder is made of thin-walled conducting material. Therefore, the average radius is found and used in all calculations. Also, the cylinder is assumed to be perfectly conducting. An input signal is applied to the slot by connecting a transmission line across the slot at its midpoint. In order to insure that the field is uniform across the slot, the slot width is assumed to be very much less than the wavelength of the excitation. This condition is nearly always met in practice.

For the general method of analysis an analogy is drawn between the slotted cylinder antenna and a transmission line. This is done by taking two thin strips along the faces of the slot and deriving the transmission-line equations for them. Since the slot is center-fed, it is analyzed as two similar shorted transmission lines in parallel. The equation for input impedance is thus taken from transmission-line theory. This equation involves the propagation constant and the characteristic impedance of the line which in turn are functions of the line parameters. It then becomes necessary to determine the line parameters R, L, G, and C. These parameters are assumed independent of the axial coordinate. The slot distributed resistance is taken as zero since the cylinder is a perfect conductor, and the radiation from oppositely directed axial currents along the slot is negligible.

The slot distributed inductance and capacitance are assumed to be the same as that which would be present if a cylinder of infinite length containing an infinite slot were used. By assuming a traveling wave along the slot, the fields of this cylinder are determined. From these fields it is possible to calculate the capacitance and the inductance. The capacitance arises from displacement currents flowing across the slot, across the interior of the cylinder, and around the outside. For the calculation of inductance, the total axial current is assumed to be flowing along the sides of the slot. The inductance is then taken as the flux-per-unit-current passing through the slot.

Since the slot distributed conductance is used to account for the radiation losses due to the circumferential currents, a different field analysis was used. The radiation field for an infinite cylinder containing a finite slot was determined by first postulating the existence of standing waves in the slot. From this radiation field, an expression for the total power radiated was found. By equating this expression for the radiated power with another equation containing G for the total power supplied to the slot, it is possible to obtain an equation for the slot distributed conductance G.

Thus, expressions have been derived for the line parameters L, G, and C. It must be remembered that the expressions for G and C contain the phase constant β . This means that they cannot be evaluated explicitly. Thus the problem becomes one of solving the three simultaneous equations shown below,

$$
\begin{aligned}
G &= f_1(\beta) \\
C &= f_2(\beta) \\
\beta &= f_3(G,C)
\end{aligned}
\tag{120}
$$

Since neither of the first two equations can be solved for β, this becomes a trial-and-error solution.

As is readily seen when looking at the equations derived in this thesis, the only reasonable way to obtain numerical results is to employ a high-speed digital computer. Most of these equations could be adapted to a computer language such as FORTRAN. A numerical example is presented in the next section.

It is believed that the equations derived in this thesis approximately describe the behavior of the input impedance over a frequency range of 400 to 1100 mc. Experimental results would be necessary to verify this belief.

X. CALCULATION PROCEDURE.

Numerical calculations were made on the IBM 7040 digital computer using the equations derived in this thesis. These calculations were made for an antenna of the size used by Holt[14]:

$a = 0.0262$ m

$t = 0.003175$ m

$W = 0.00635$ m

$\alpha = 14^{\circ}$

$L' = 0.54$ m

Thus, a plot of frequency versus input resistance and reactance was obtained as shown in Figure 4.

The method of calculation is as follows. Due to a scarcity of Bessel function tables, computer programs were written which would generate tables of Bessel functions of the first and second kinds, using the method of Abramowitz and Stegun[15]. These tables were for arguments between 0.01 and 1.00 in increments of 0.01 with integer orders between zero and ten. Subprograms were then written which would interpolate the tables and extend them below the 0.01 argument. A subprogram was written to calculate the slot distributed conductance from Eq. (119). Simpson's rule was used to perform the numerical integration. Only about five terms are required in the infinite summation to obtain an answer accurate to the nearest 0.001. Another subprogram was written for the slot distributed susceptance using Eq. (79). By employing the approximations derived by Holt[16], the infinite series requires about ten terms to converge to 0.001 accuracy.

By using these subprograms in conjunction with Eq. (22), it is possible, by a trial-and-error method, to obtain sufficiently accurate values for the phase constant and the line parameters. These are then used in the transmission-line equations to calculate the input impedance. By varying the frequency over the range of 400 to 1100 mc it was possible to obtain values for curves of input resistance and reactance versus frequency.

The FORTRAN program is shown in Appendix D.

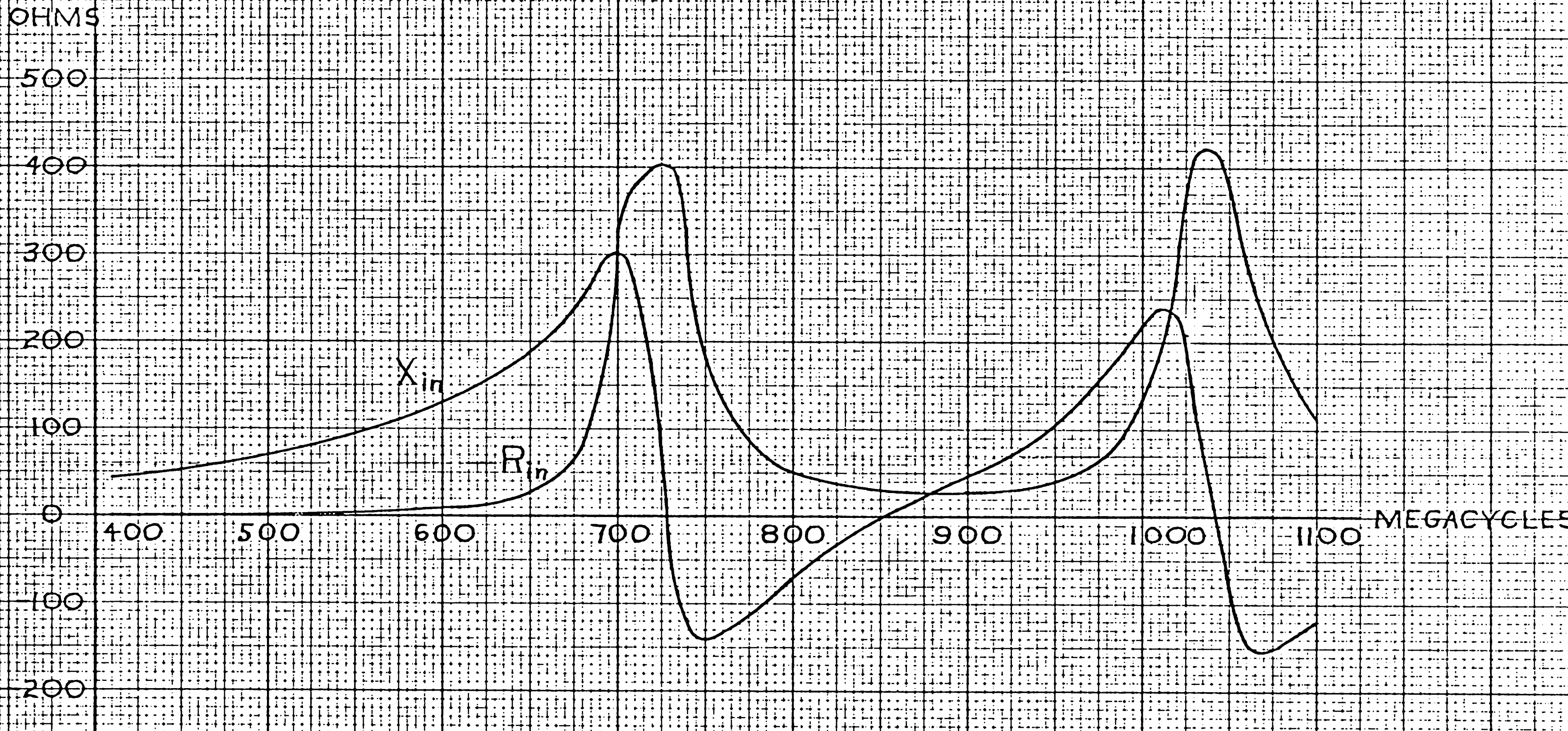

FIGURE 4.
ANTENNA INPUT RESISTANCE AND REACTANCE AS A FUNCTION OF FREQUENCY

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XIV. APPENDICES.

Appendix A.

Simplification of Susceptance Expression.

From Eqs. (73) to (77),

$$B=\frac{Im(J_\phi)}{V\epsilon^{-j\beta z}}$$

$$=Im\left[\frac{\alpha k_\rho}{2\pi\hat{z}W}\frac{J_0(k_\rho a)}{J_0'(k_\rho a)}+\frac{k_\rho}{\pi\hat{z}W}\sum_{n=1}^{\infty}\frac{J_n(k_\rho a)}{J_n'(k_\rho a)}\sin n\alpha+\frac{j\omega\epsilon t}{W}+\right.$$

$$\left.-\frac{\alpha k_\rho}{2\pi\hat{z}W}\frac{H_0^{(2)}(k_\rho a)}{H_0^{(2)\prime}(k_\rho a)}-\frac{k_\rho}{\pi\hat{z}W}\sum_{n=1}^{\infty}\frac{1}{n}\frac{H_n^{(2)}(k_\rho a)}{H_n^{(2)\prime}(k_\rho a)}\sin n\alpha\right]$$

For convenience, let the argument of the Bessel functions $k_\rho a$ be understood. Then, substituting $\hat{z}=j\omega\mu$,

$$B=Im\left(-\frac{j\alpha k_\rho}{2\pi\omega\mu W}\frac{J_0}{J_0'}-\frac{jk_\rho}{\pi\omega\mu W}\sum_{n=1}^{\infty}\frac{1}{n}\frac{J_n}{J_n'}\sin n\alpha+\frac{j\omega\epsilon t}{W}+\right.$$

$$\left.+\frac{j\alpha k_\rho}{2\pi\omega\mu W}\frac{H_0^{(2)}}{H_0^{(2)\prime}}+\frac{jk_\rho}{\pi\omega\mu W}\sum_{n=1}^{\infty}\frac{1}{n}\frac{H_n^{(2)}}{H_n^{(2)\prime}}\sin n\alpha\right)$$

$$=Im\left[-\frac{j\alpha k_\rho}{2\pi\omega\mu W}\left(\frac{J_0}{J_0'}-\frac{H_0^{(2)}}{H_0^{(2)\prime}}\right)-\frac{jk_\rho}{\pi\omega\mu W}\sum_{n=1}^{\infty}\left(\frac{J_n}{J_n'}-\frac{H_n^{(2)}}{H_n^{(2)\prime}}\right)\sin n\alpha\right]+\frac{\omega\epsilon t}{W}$$

Substituting $H_n^{(2)}=J_n-jN_n$,

$$B=-\frac{k_\rho}{\pi\omega\mu W}Im\left[j\frac{\alpha}{2}\left(\frac{J_0}{J_0'}-\frac{J_0-jN_0}{J_0'-jN_0'}\right)+j\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{J_n}{J_n'}-\frac{J_n-jN_n}{J_n'-jN_n'}\right)\sin n\alpha\right]+\frac{\omega\epsilon t}{W}$$

$$
\begin{aligned}
B = &-\frac{k_\rho}{\pi\omega\mu w}\,\mathrm{Im}\left[j\frac{\alpha}{2}\left(\frac{J_0}{J_0'}-\frac{J_0J_0'-jN_0J_0'+jJ_0N_0'+N_0N_0'}{J_0'^2+N_0'^2}\right)+\right.\\
&\left.+j\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{J_n}{J_n'}-\frac{J_nJ_n'-jN_nJ_n'+jJ_nN_n'+N_nN_n'}{J_n'^2+N_n'^2}\right)\sin n\alpha\right]+\frac{\omega\epsilon t}{w}\\
= &-\frac{k_\rho}{\pi\omega\mu w}\,\mathrm{Im}\left[\frac{\alpha}{2}\left(j\frac{J_0}{J_0'}-j\frac{J_0J_0'+N_0N_0'}{J_0'^2+N_0'^2}-j\frac{jJ_0N_0'-jN_0J_0'}{J_0'^2+N_0'^2}\right)+\right.\\
&\left.+\sum_{n=1}^{\infty}\frac{1}{n}\left(j\frac{J_n}{J_n'}-j\frac{J_nJ_n'+N_nN_n'}{J_n'^2+N_n'^2}-j\frac{jJ_nN_n'-jN_nJ_n'}{J_n'^2+N_n'^2}\right)\sin n\alpha\right]+\frac{\omega\epsilon t}{w}\\
= &-\frac{k_\rho}{\pi\omega\mu w}\left[\frac{\alpha}{2}\left(\frac{J_0}{J_0'}-\frac{J_0J_0'+N_0N_0'}{J_0'^2+N_0'^2}\right)+\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{J_n}{J_n'}-\frac{J_nJ_n'+N_nN_n'}{J_n'^2+N_n'^2}\right)\sin n\alpha\right]+\frac{\omega\epsilon t}{w}\\
= &-\frac{k_\rho}{\pi\omega\mu w}\left\{\frac{\alpha}{2}\left[\frac{J_0J_0'^2+J_0N_0'^2-J_0J_0'^2-N_0N_0'J_0'}{J_0'\left(J_0'^2+N_0'^2\right)}\right]+\right.\\
&\left.+\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{J_nJ_n'^2+J_nN_n'^2-J_nJ_n'^2-J_n'N_nN_n'}{J_n'\left(J_n'^2+N_n'^2\right)}\right]\sin n\alpha\right\}+\frac{\omega\epsilon t}{w}\\
= &-\frac{k_\rho}{\pi\omega\mu w}\left\{\frac{\alpha}{2}\left[\frac{N_0'\left(J_0N_0'-N_0J_0'\right)}{J_0'\left(J_0'^2+N_0'^2\right)}\right]+\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{N_n'\left(J_nN_n'-N_nJ_n'\right)}{J_n'\left(J_n'^2+N_n'^2\right)}\right]\sin n\alpha\right\}+\frac{\omega\epsilon t}{w}
\end{aligned}
$$

Substituting the Wronskian,

$$J_nN_n'-N_nJ_n'=\frac{2}{\pi k_\rho a}$$

gives

$$B=\frac{\omega\epsilon t}{w}-\frac{\alpha}{\pi^2\omega\mu a w}\left[\frac{N_0'}{J_0'\left(J_0'^2+N_0'^2\right)}\right]-\frac{2}{\pi^2\omega\mu a w}\sum_{n=1}^{\infty}\frac{N_n'}{J_n'\left(J_n'^2+N_n'^2\right)}\,\frac{\sin n\alpha}{n}$$

Appendix B.

Fourier Transform of Field in Finite Slot.

From Eqs. (90) and (92),

$$\tilde{E}_\phi(n,w)=\frac{1}{2\pi}\int_0^{2\pi}d\phi\int_{-\infty}^{\infty}dz\,E_\phi(a,\phi,z)\epsilon^{-jn\phi}\epsilon^{-jwz}$$

$$=\frac{1}{2\pi}\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}}\epsilon^{-jn\phi}d\phi\int_{-\frac{L'}{2}}^{\frac{L'}{2}}\frac{V}{\alpha a}\sin\left[\beta\left(\frac{L'}{2}-|z|\right)\right]\epsilon^{-jwz}dz$$

Integrating with respect to ϕ,

$$\tilde{E}_\phi(n,w)=\frac{V}{2\pi\alpha a}\left[\frac{\epsilon^{-jn\phi}}{-jn}\right]_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}}\int_{-\frac{L'}{2}}^{\frac{L'}{2}}\sin\beta\left(\frac{L'}{2}-|z|\right)\epsilon^{-jwz}dz$$

$$=\frac{V}{\pi\alpha an}\sin\frac{n\alpha}{2}\left[\int_{-\frac{L'}{2}}^{0}\epsilon^{-jwz}\sin\beta\left(\frac{L'}{2}+z\right)dz+\right.$$

$$\left.+\int_0^{\frac{L'}{2}}\epsilon^{-jwz}\sin\beta\left(\frac{L'}{2}-z\right)dz\right]$$

Integrating with respect to z,

$$\tilde{E}_\phi(n,w)=\frac{V}{\pi\alpha an}\sin\frac{n\alpha}{2}\left\{\frac{\epsilon^{-jwz}}{\beta^2-w^2}\left[-jw\sin\beta\left(\frac{L'}{2}+z\right)-\beta\cos\beta\left(\frac{L'}{2}+z\right)\right]_{-\frac{L'}{2}}^{0}+\right.$$

$$\left.+\frac{\epsilon^{-jwz}}{\beta^2-w^2}\left[-jw\sin\beta\left(\frac{L'}{2}-z\right)+\beta\cos\beta\left(\frac{L'}{2}-z\right)\right]_0^{\frac{L'}{2}}\right\}$$

$$\tilde{E}_\phi(n,w) = \frac{V \sin\frac{n\alpha}{2}}{\pi\alpha a n(\beta^2 - w^2)} \left\{ \epsilon^0 \left[-jw \sin\frac{\beta L'}{2} - \beta \cos\frac{\beta L'}{2} \right] + \right.$$

$$- \epsilon^{\frac{jwL'}{2}} \left[-jw \sin 0 - \beta \cos 0 \right] + \epsilon^{-\frac{jwL'}{2}} \left[-jw \sin 0 + \right.$$

$$\left. \left. + \beta \cos 0 \right] - \epsilon^0 \left[-jw \sin\frac{\beta L'}{2} + \beta \cos\frac{\beta L'}{2} \right] \right\}$$

$$= \frac{2V\beta \sin\frac{n\alpha}{2}}{\pi\alpha a n(\beta^2 - w^2)} \left[-\cos\frac{\beta L'}{2} + \frac{1}{2}\left(\epsilon^{-\frac{jwL'}{2}} + \epsilon^{\frac{jwL'}{2}} \right) \right]$$

Substituting for the exponential functions,

$$\tilde{E}_\phi(n,w) = \frac{2V\beta \sin\frac{n\alpha}{2}}{\pi\alpha a n(\beta^2 - w^2)} \left(\cos\frac{wL'}{2} - \cos\frac{\beta L'}{2} \right)$$

Appendix C.

Integration of the Expression for Radiated Power.

Consider only the part of Eq. (114) containing ϕ :

$$\int_0^{\pi}\left|\sum_{n=-\infty}^{\infty}\frac{j^n\epsilon^{jn\phi}}{H_n^{(2)\prime}(ka\sin\theta)}\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\right|^2 d\phi$$

Let the argument of the Bessel function k a sin θ be understood.

Changing the limits of the summation,

$$\int_0^{\pi}\left|\sum_{n=-\infty}^{\infty}\frac{j^n\epsilon^{jn\phi}}{H_n^{(2)\prime}}\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\right|^2 d\phi =$$

$$=\int_0^{\pi}\left|\frac{1}{H_1^{(2)}}+2\sum_{n=1}^{\infty}\frac{j^n\cos n\phi}{H_n^{(2)\prime}}\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\right|^2 d\phi$$

Using the identities,

$$j^n=\epsilon^{j\frac{n\pi}{2}}=\cos\frac{n\pi}{2}+j\sin\frac{n\pi}{2}$$

and

$$H_n^{(2)\prime}=\frac{1}{2}\left(J_{n-1}-jN_{n-1}-J_{n+1}+jN_{n+1}\right)$$

Substituting,

$$\int_0^{\pi}\left|\sum_{n=-\infty}^{\infty}\frac{j^n\epsilon^{jn\phi}}{H_n^{(2)\prime}}\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\right|^2 d\phi =$$

$$=\int_0^{\pi}\left|\frac{-1}{J_1-jN_1}+4\sum_{n=1}^{\infty}\frac{\left(\cos\frac{n\pi}{2}+j\sin\frac{n\pi}{2}\right)\cos n\phi}{\left(J_{n-1}-J_{n+1}\right)-j\left(N_{n-1}-N_{n+1}\right)}\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\right|^2 d\phi$$

$$\int_0^{\pi}\left|\sum_{n=-\infty}^{\infty}\frac{j^n\epsilon^{jn\phi}}{H_n^{(2)\prime}}\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\right|^2 d\phi=$$

$$=\int_0^{\pi}\left|\frac{-J_1-jN_1}{J_1^2+N_1^2}+4\sum_{n=1}^{\infty}\frac{\left(\cos\frac{n\pi}{2}+j\sin\frac{n\pi}{2}\right)\cos n\phi\left[\left(J_{n-1}-J_{n+1}\right)+j\left(N_{n-1}-N_{n+1}\right)\right]}{\left(J_{n-1}-J_{n+1}\right)^2+\left(N_{n-1}-N_{n+1}\right)^2}\times\right.$$

$$\left.\times\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\right|^2 d\phi$$

$$=\int_0^{\pi}\left|\frac{-J_1-jN_1}{J_1^2+N_1^2}+4\sum_{n=1}^{\infty}\frac{\cos n\phi}{\left(J_{n-1}-J_{n+1}\right)^2+\left(N_{n-1}-N_{n+1}\right)^2}\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\times\right.$$

$$\times\left\{\left[\left(\cos\frac{n\pi}{2}\right)\left(J_{n-1}-J_{n+1}\right)-\left(\sin\frac{n\pi}{2}\right)\left(N_{n-1}-N_{n+1}\right)\right]+j\left[\left(\cos\frac{n\pi}{2}\right)\left(N_{n-1}-N_{n+1}\right)+\right.\right.$$

$$\left.\left.\left.+\left(\sin\frac{n\pi}{2}\right)\left(J_{n-1}-J_{n+1}\right)\right]\right\}\right|^2 d\phi$$

Let,

$$A_0=\frac{-J_1}{J_1^2+N_1^2}\qquad;\qquad B_0=\frac{-N_1}{J_1^2+N_1^2}$$

$$A_n=\frac{\left[\left(\cos\frac{n\pi}{2}\right)\left(J_{n-1}-J_{n+1}\right)-\left(\sin\frac{n\pi}{2}\right)\left(N_{n-1}-N_{n+1}\right)\right]}{\left(J_{n-1}-J_{n+1}\right)^2+\left(N_{n-1}-N_{n+1}\right)^2}4\left(\frac{n\alpha}{2}\right)\sin\frac{n\alpha}{2}$$

$$B_n=\frac{\left[\left(\cos\frac{n\pi}{2}\right)\left(N_{n-1}-N_{n+1}\right)+\left(\sin\frac{n\pi}{2}\right)\left(J_{n-1}-J_{n+1}\right)\right]}{\left(J_{n-1}-J_{n+1}\right)^2+\left(N_{n-1}-N_{n+1}\right)^2}4\left(\frac{n\alpha}{2}\right)\sin\frac{n\alpha}{2}$$

Then,

$$\int_0^\pi \left|\sum_{n=-\infty}^{\infty} \frac{j^n \epsilon^{jn\phi}}{H_n^{(2)\prime}}\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\right|^2 d\phi = \int_0^\pi \left|A_0+B_0+\sum_{n=1}^{\infty}\left(A_n+jB_n\right)\cos n\phi\right|^2 d\phi$$

$$=\int_0^\pi \left|\left(A_0+A_1\cos\phi+A_2\cos 2\phi+\cdots\right)+j\left(B_0+B_1\cos\phi+B_2\cos 2\phi+\cdots\right)\right|^2 d\phi$$

$$=\int_0^\pi \left[\left(A_0+A_1\cos\phi+A_2\cos 2\phi+\cdots\right)^2+\left(B_0+B_1\cos\phi+B_2\cos 2\phi+\cdots\right)^2\right] d\phi$$

Using the integrals,

$$\int_0^\pi \cos n\phi \, d\phi = 0 \quad , \quad m \neq 0$$

$$\int_0^\pi \cos m\phi \cos n\phi \, d\phi = 0 \, , \; m \neq n$$

$$\int_0^\pi A_0^2 \, d\phi = \pi A_0^2$$

$$\int_0^\pi \cos^2 m\phi \, d\phi = \frac{\pi}{2} \, , \; m \neq 0$$

Then,

$$\int_0^\pi \left|\sum_{n=-\infty}^{\infty} \frac{j^n \epsilon^{jn\phi}}{H_n^{(2)\prime}}\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\right|^2 d\phi =$$

$$=\int_0^\pi \left[\left(A_0^2+A_1^2\cos^2\phi+A_2^2\cos^2 2\phi+\cdots\right)+\left(B_0^2+B_1^2\cos^2\phi+B_2^2\cos^2 2\phi+\cdots\right)\right] d\phi$$

$$=\left(\pi A_0^2+A_1^2\frac{\pi}{2}+A_2^2\frac{\pi}{2}+\cdots\right)+\left(\pi B_0^2+B_1^2\frac{\pi}{2}+B_2^2\frac{\pi}{2}+\cdots\right)$$

$$\int_0^{\pi}\left|\sum_{n=-\infty}^{\infty}\frac{j^n\epsilon^{jn\phi}}{H_n^{(2)\prime}}\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\right|^2 d\phi=$$

$$=\frac{\pi}{2}\left(2A_0^2+2B_0^2+A_1^2+B_1^2+A_2^2+B_2^2+\cdots\right)$$

$$=\frac{\pi}{2}\left[2\left(A_0^2+B_0^2\right)+\sum_{n=1}^{\infty}\left(A_n^2+B_n^2\right)\right]$$

where

$$2\left(A_0^2+B_0^2\right)=2\frac{J_1^2+N_1^2}{\left(J_1^2+N_1^2\right)^2}=\frac{2}{J_1^2+N_1^2}$$

and

$$A_n^2+B_n^2=\frac{\left[\left(\cos\frac{n\pi}{2}\right)\left(J_{n-1}-J_{n+1}\right)-\left(\sin\frac{n\pi}{2}\right)\left(N_{n-1}-N_{n+1}\right)\right]^2+\left[\left(\cos\frac{n\pi}{2}\right)\left(N_{n-1}-N_{n+1}\right)+\left(\sin\frac{n\pi}{2}\right)\left(J_{n-1}-J_{n+1}\right)\right]^2}{\left[\left(J_{n-1}-J_{n+1}\right)^2+\left(N_{n-1}-N_{n+1}\right)^2\right]^2}\times$$

$$\times\left[4\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\right]^2$$

$$=\frac{\left(J_{n-1}-J_{n+1}\right)^2+\left(N_{n-1}-N_{n+1}\right)^2}{\left[\left(J_{n-1}-J_{n+1}\right)^2+\left(N_{n-1}-N_{n+1}\right)^2\right]^2}\left[4\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\right]^2$$

$$=\frac{16\left(\frac{2}{n\alpha}\right)^2\sin^2\frac{n\alpha}{2}}{\left(J_{n-1}-J_{n+1}\right)^2+\left(N_{n-1}-N_{n+1}\right)^2}$$

Thus,

$$\int_0^{\pi}\left|\sum_{n=-\infty}^{\infty}\frac{j^n\epsilon^{jn\phi}}{H_n^{(2)\prime}}\left(\frac{2}{n\alpha}\right)\sin\frac{n\alpha}{2}\right|^2 d\phi=\pi\left[\frac{1}{J_1^2+N_1^2}+8\sum_{n=1}^{\infty}\frac{\left(\frac{2}{n\alpha}\right)^2\sin^2\frac{n\alpha}{2}}{\left(J_{n-1}-J_{n+1}\right)^2+\left(N_{n-1}-N_{n+1}\right)^2}\right]$$

Appendix D

```
D1      DIMENSION BESJ(13,100),BESN(11,100),BE(70)
        COMMON BESJ,BESN
        READ(5,21) (BE(I),I=1,70)
21      FORMAT(14F5.2)
        DO 3 M=1,100
        BESJ(13,M)=0.
        BESJ(12,M)=1.
        DO 2 N=1,11
        A=FLOAT(N)
        B=.01*FLOAT(M)
        I=12-N
        J=13-N
        K=14-N
 2      BESJ(I,M)=(2.*(12.-A)/B)*BESJ(J,M)-BESJ(K,M)
        DELTA=BESJ(1,M)+2.*(BESJ(3,M)+BESJ(5,M)+BESJ(7,M)+BESJ(9,M)+BESJ(1
       11,M))
        DO 3 N=1,12
 3      BESJ(N,M)=BESJ(N,M)/DELTA
        DO 5 M=1,100
        B=.01*FLOAT(M)
        BESN(1,M)=.63662*(BESJ(1,M)*(ALOG(B/2.)+.577216)+2.*(BESJ(3,M)-
       1BESJ(5,M)/2.+BESJ(7,M)/3.-BESJ(9,M)/4.+BESJ(11,M)/5.))
        BESN(2,M)=(BESJ(2,M)*BESN(1,M)-.63662/B)/BESJ(1,M)
        DO 5 N=2,10
        A=FLOAT(N)
5       BESN(N+1,M)=(2.*(A-1.)/B)*BESN(N,M)-BESN(N-1,M)
        AL=.54
        DO 20 IA=1,70
        IB=111-IA
        FREQ=1E7*FLOAT(IB)
        OMEGA=6.2832*FREQ
        AK=OMEGA/3E8
        X=5.0687E-6*OMEGA
        BETA1=0.
        BETA=BE(IA)*2.
4       BETA=(BETA+BETA1)/2.
        AKRHO=SQRT(AK**2-BETA***2)
        B=-KARHO/(2.0687E-7*OMEGA)*AP(AK,BETA)+4.3901E-12*OMEGA
        BETA1=SQRT(X/2.)*SQRT(SQRT(B**2+G(AK,BETA,AL)**2)+B)
        AB2=ABS(BETA1-BETA)
        IF(AB2-.001)7,7,4
7       CONTINUE
        ALPHA=SQRT(X/2.)*SQRT(SQRT(B**2+G(AK,BETA,AL)**2)-B)
        COMPLEX ZO,HTAN,ZIN
        ZO=COMPLX(X,O.)/COMPLX(BETA,-ALPHA)
```

```
      R=TANH(ALPHA*AL/2.)
      S=SIN(BETA*AL/2.)/COS(BETA*AL/2.)
      T=R*S
      HTAN=COMPLX(R,S)/COMPLX(1.,T)
      ZIN=ZO/2.*HTAN
      Y=REAL(ZIN)
      Z=AIMA(ZIN)
      WRITE(7,40)IB,Y,Z
20    CONTINUE
40    FORMAT(I4,2F8.3)
      END

SUB1  FUNCTION ABESJ(N,S,AESJ)
      DIMENSION AESJ(13,100)
      IF(S-.01)7,4,4
4     I=N+1
      Z=S*100.
      DO 2 M=1,100
      Y=FLOAT(M)
      IF(Z-Y)5,6,2
2     CONTINUE
5     ABESJ=AESJ(I,M-1)+(Z-(Y-1,))*(AESJ(I,M)-AESJ(I,M-1))
      RETURN
6     ABESJ=AESJ(I,M)
      RETURN
7     J=N
      C=S
      AN=1.
      DO 8L=1,J
      K=J-L+1
      A=FLOAT(K)
8     AN=AN*A
      ABESJ=((C/2.)**J)*(1./AN)
      RETURN
      END

SUB2  FUNCTION ABESN(N,X,AESN)
      DIMENSION AESN(11,100)
      IF(X-.01)7,4,4
4     I=N+1
      R=X*100.
      DO 2M=1,100
      Y=FLOAT(M)
      IF(R-Y)5,6,2
2     CONTINUE
5     ABESN=AESN(I,M-1)+(R-(Y-1.))*(AESN(I,M)-AESN(I,M-1))
      RETURN
```

```
6       ABESN=AESN(I,M)
        RETURN
7       J=N
        W=X
        IF(J)9,8,9
8       ABESN=.63662*ALOG(W)
        RETURN
9       J1=N
        W=X
        AN=1.
        J=J1-1
        DO 10 L=1,J
        K=J-L+1
        A=FLOAT(K)
10      AN=AN*A
        ABESN=-.31831*AN*(2./W)**J1
        RETURN
        END

SUB3    FUNCTION G(AK,BETA,AL)
        DIMENSION GCN(21),BESJ(13,100),BESN(11,100)
        COMMON BESJ,BESN
        GCN(1)=0.
        DO 5 I=2,21
        IA=I-1
        TH=.078540*FLOAT(IA)
        ARG=.0262*AKSIN(TH)
        HA=ABESJ(1,ARG,BESJ)**2+ABESN(1,ARG,BESN)**2
        DA=COS(AK*AL/2.*COS(TH))-COS(BETA*AL/2.)
        DB=1.-(AK/BETA*COS(TH))**2
        DAB=(DA/DB)**2
        BA=BETA*(BETA*AL-SIN(BETA*AL))
        HNA=0.
        DO 6 N=1,4
        GA=1./(FLOAT(N)*.1222)
        GB=1./GA
        GN=(GA*SIN(GB))**2
        HN=(ABESJ(N-1,ARG,BESJ)-ABESJ(N+1,ARG,BESJ))**2+(ABESN(N-1,
       1ARG,BESN)-ABESN(N+1,ARG,BESN))**2
        X=GN/HN
6       HNA=HNA+X
5       GCN(I)=.99702/BA*DAB*SIN(TH)*(1./HA+8.*HNA)
        SUM=0.
        DO 7 I=1,9
7       SUM=SUM+4.*GCN(2*I)+2.*GCN(2*I+1)
        G=.078540/3.*(GCN(1)+SUM+4.*GCN(20)+GCN(21)
        RETURN
        END
```

```
SUB4    FUNCTION AP(AK,BETA)
        COMMON BESJ(13,100),BESN(11,100)
        AKRHO=SQRT(AK**2-BETA**2)
        BNA=0.
        ARG=AKRHO*.0262
        DO 2 N=1,10
        X=ABESJ(N-1,ARG,BESJ)-ABESJ(N+1,ARG,BESJ)
        Y=ABESN(N-1,ARG,BESN)-ABESN(N+1,ARG,BESN)
        AN=(97.2052/AKRHO)*Y/(X*(X**2+Y**2))
        GA=1./(FLOAT(N)*.2444)
        GB=1./GA
        G2N=GA*SIN(GB)
        BN=G2N*(AN-(.1048*AKRHO)/FLOAT(N))
2       BNA=BNA+BN
        AO=(24.3013/AKRHO)*(ABESN(1,ARG,BESN)/ABESJ(1,ARG,BESJ))/
       1(ABESJ(1,ARG,BESJ)**2+ABESN(1,ARG,BESN)**2)
        AP=AO+AKRHO*.16893+BNA
        RETURN
        END
```

ABSTRACT

In this thesis a mathematical analysis is made of the input impedance to a cylinder antenna with an axial slot. It is excited by a parallel-wire line connected across the center of the slot. This causes standing waves along the slot. The analysis is then based on an analogy between the slot and a transmission line. Following this analogy equations for the transmission-line parameters are developed for the slotted cylinder. The slot distributed inductance and capacitance are determined by assuming an infinite slot length, while the conductance is obtained for a finite slot. The phase constant, which is contained in the expressions for the line parameters, is a function of the line parameters. Thus, a final answer requires the solution of simultaneous equations. This is done on a digital computer.

This analysis applies to the case where the wavelength of excitation is of the same order of magnitude as the diameter of the cylinder. Possible frequencies for which this antenna might be used are in the microwave range.