PERTURBATION METHODS FOR SLIGHTLY DAMPED GYROSCOPIC SYSTEMS

by

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Dissertation submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering Mechanics

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ACKNOWLEDGEMENTS

The author must express deep gratitude to his advisor, Dr. Leonard Meirovitch; to his parents, ; and to his department head, Dr. Daniel Frederick, for their patience, encouragement and support during the research for and the writing of this dissertation. He must also thank for her expert preparation of the manuscript.

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Chapter 1

INTRODUCTION

Consider the linear, coupled, ordinary differential equations of motion

$$B\dot{\underset{\sim}{u}}(t) = A\underset{\sim}{u}(t) + \underset{\sim}{U}(t) \tag{1.1}$$

where A and B are $n \times n$ matrices of constant coefficients, $\underset{\sim}{u}(t)$ is an $n \times 1$ vector of generalized coordinates, $\underset{\sim}{U}(t)$ is an $n \times 1$ vector of generalized forces, and t is the independent variable; an overdot representing differentiation with respect to t. We wish to solve for $\underset{\sim}{u}(t)$. To this end, we consider the algebraic eigenvalue problem

$$\lambda_i B\underset{\sim}{u}_i = A\underset{\sim}{u}_i \quad , \quad i = 1,2,\ldots,n \ . \tag{1.2a}$$

Because matrices A and B possess no symmetries, we must also consider the adjoint eigenvalue problem.

$$\lambda_i B^T\underset{\sim}{v}_i = A^T\underset{\sim}{v}_i \quad , \quad i = 1,2,\ldots,n \ . \tag{1.2b}$$

Although Eqs. (1.2) possess the same eigenvalues, λ_i, the right and left eigenvectors, $\underset{\sim}{u}_i$ and $\underset{\sim}{v}_i$, respectively, are generally different.

The right and left eigenvectors possess the biorthogonality property, which can be expressed as

$$\underset{\sim}{v}_i^T B\underset{\sim}{u}_j = \underset{\sim}{u}_i^T B^T\underset{\sim}{v}_j = 0 \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,n \ . \tag{1.3a}$$

From Eqs. (1.2) it follows that

$$\underset{\sim}{v}_i^T A\underset{\sim}{u}_j = \underset{\sim}{u}_i^T A^T\underset{\sim}{v}_j = 0 \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,n \ . \tag{1.3b}$$

The right and left eigenvectors provide the basis for a transformation permitting the diagonalization of A and B, and hence the diagonalization of Eq. (1.1). The resulting decoupled equations of motion can then be solved independently.

Let us suppose matrices A and B can be expressed as

$$A = A_0 + A_1 \quad , \quad B = B_0 + B_1 \tag{1.4}$$

where A_1 and B_1 are small in comparison to A_0 and B_0, respectively. Henceforth, we shall refer to matrices A_1 and B_1 as perturbations. Let us suppose that the unperturbed eigenvalues and eigenvectors, λ_{0i}, $\underset{\sim}{u}_{0i}$ and $\underset{\sim}{v}_{0i}$ $(i = 1,2,\ldots,n)$, corresponding to $A = A_0$ and $B = B_0$ are known. The objective is to produce the perturbed eigenvalues and eigenvectors, λ_i, $\underset{\sim}{u}_i$ and $\underset{\sim}{v}_i$ $(i = 1,2,\ldots,n)$, in terms of known quantities, namely the unperturbed eigenvalues and eigenvectors and the perturbation matrices A_1 and B_1. Furthermore, we wish to avoid solving a new eigenvalue problem. This objective can be achieved, provided no difference between two unperturbed eigenvalues is small, which is to say that the unperturbed eigenvalues are "clearly distinct." Alternatively, if the unperturbed eigenvalue problem possesses multiple, or nearly multiple, eigenvalues, then an eigenvalue problem of the same order as the multiplicity, or near multiplicity, must be solved, and the above stated objective is only partially achieved.

Various authors have investigated this problem. Franklin [1], Wilkinson [2] and Lancaster [3] present equivalent perturbation analyses based upon the statement of the eigenvalue problem itself, which utilize the biorthogonality of the unperturbed eigenvectors, and which are dependent upon the unperturbed eigenvalues being clearly distinct. Franklin and Lancaster begin with the assumption that each eigenvector perturbation must be biorthogonal to the corresponding unperturbed eigenvector. Lancaster presents an analysis for the case when the unperturbed eigenvalue problem possesses multiple eigenvalues, and then generalizes to include nearly multiple unperturbed eigenvalues. His procedure entails solving an eigenvalue problem that is of the same order as the multiplicity. Wilkinson also presents an analysis based on Gerschgorin's discs that is valid irrespective of the distinctness of the unperturbed eigenvalues. Aubrun [4] presents a perturbation analysis that is based on the statement of the eigenvalue problem. His analysis uses right and left eigenvectors for the determination of eigenvalue perturbations. Perturbations to the right eigenvectors are determined using right eigenvectors and a weighting matrix.

Huseyin [5] and Ziegler [6] provide valuable insight regarding the nature of gyroscopic systems with and without damping and/or circulatory effects. Chen and Wada [7] address the problem of a damped, nongyroscopic system and use a perturbation analysis not only in the eigensolution analysis, but also in the determination of the response.

The perturbation techniques presented in this dissertation were developed in response to the following considerations. Space structures have become relatively large in recent years. Thus, the need arises for computationally efficient algorithms whereby the dynamic response of large order systems can be determined. Note that the necessity of performing such computations is encountered in the more general problem of controlling such structures. See, for example, Meirovitch and Baruh [8]. Meirovitch, [9] and [10], has shown how the eigenvalue problem for an undamped, noncirculatory gyroscopic system can be solved in terms of real quantities with relative ease. However, if damping and/or circulatory effects are added, the eigensolution analysis becomes considerably more difficult. Thus, perturbation techniques have been developed so that the assumed small effects of damping and/or circulatory forces can be included in the eigensolution without resolving the eigenvalue problem (see Meirovitch and Ryland [11]).

The perturbation techniques presented in the following chapters are quite general. Indeed, they should be applicable to many seemingly different eigensolution analyses, some of which are as follows. Considering Eqs. (1.2) and (1.4), one could regard matrices A_1 and B_1 as errors associated with matrices A_0 and B_0, respectively. The perturbation procedures then provide the means of calculating the corresponding eigensolution error. Conversely, given approximate eigenvectors, the matrices that take the place of A_1 and B_1 can be regarded as a measure of the accuracy of these eigenvectors. The perturbation techniques should be useful to the analyst who wishes to incorporate small design changes, and yet avoid a complete new solution of the eigenvalue problem. Indeed this is the stated objective of Chen and Wada [7], and Ryland and Meirovitch [12]. The techniques are also applicable to the control problem in which control forces are weak, and are proportional to the generalized coordinates and/or velocities, as is common practice. Indeed, this is the application used by Aubrun [4].

In the following chapters, a perturbation procedure, based upon the biorthonormality of the perturbed eigenvectors, is presented for clearly distinct unperturbed eigenvalues. The eigenvalue problem for slightly damped and/or circulatory gyroscopic systems is discussed and the perturbation theory is applied to such a system. The perturbation theory is generalized to include multiple unperturbed eigenvalues, and is then further generalized to include nearly multiple unperturbed eigenvalues. Rayleigh's quotient is discussed and then used to define a more general basis for application of a perturbation analysis, whereupon an iterative procedure is described. Computation of the dynamic response is described. Examples are given where appropriate.

Chapter 2

GENERAL PERTURBATION THEORY

Consider the algebraic eigenvalue problem

$$\lambda_i B \underset{\sim}{u}_i = A \underset{\sim}{u}_i \quad , \quad i = 1,2,\ldots,n \tag{2.1a}$$

and its adjoint

$$\lambda_i B^T \underset{\sim}{v}_i = A^T \underset{\sim}{v}_i \quad , \quad i = 1,2,\ldots,n \tag{2.1b}$$

where matrices A and B possess no symmetries. The eigenvalues of both eigenvalue problems are the same, which follows from the fact that the determinant of a matrix is equal to the determinant of the transpose of that matrix. Thus, for each eigenvalue, λ_i, there is a corresponding right eigenvector, $\underset{\sim}{u}_i$, and a left eigenvector, $\underset{\sim}{v}_i$. The right and left eigenvectors satisfy the biorthogonality relations

$$\underset{\sim}{v}_i^T B \underset{\sim}{u}_j = \underset{\sim}{u}_i B^T \underset{\sim}{v}_j = 0 \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,n$$

$$\underset{\sim}{v}_i^T A \underset{\sim}{u}_j = \underset{\sim}{u}_i^T A^T \underset{\sim}{v}_j = 0 \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,n$$

If the eigenvectors are normalized such that

$$\underset{\sim}{v}_i^T B \underset{\sim}{u}_i = \underset{\sim}{u}_i^T B^T \underset{\sim}{v}_i = +1 \text{ or } -1 \ ,$$

we can state the biorthonormality relations

$$\underset{\sim}{v}_i^T B \underset{\sim}{u}_j = \underset{\sim}{u}_i^T B^T \underset{\sim}{v}_j = \Delta_{ij} \quad , \quad i,j = 1,2,\ldots,n \tag{2.2a}$$

$$\underset{\sim}{v}_i^T A \underset{\sim}{u}_j = \underset{\sim}{u}_i^T A^T \underset{\sim}{v}_j = \lambda_i \Delta_{ij} \quad , \quad i,j = 1,2,\ldots,n \tag{2.2b}$$

where Δ_{ij} is a modified Kronecker delta; it is zero when $i \neq j$, and when $i = j$ it is either plus or minus one, depending on the value of i.

Allowing the eigenvectors to satisfy either a plus or minus one normalization may seem unnecessary at this stage. However, there are instances where this normalization is convenient.

A word about normalization of right and left eigenvectors is pertinent. For the moment, let us suppose that the right and left eigenvectors are related in some simple fashion, so that in reality we must address only one set of eigenvectors. Then in applying Eq. (2.2a) to normalize the ith eigenvector, we have one equation to determine one normalization constant, and there is no ambiguity. Alternatively, let us suppose the right and left eigenvectors are not related to one another. Then application of Eq. (2.2a) to normalize the ith right and left eigenvectors yields a single equation involving two normalization constants. In the following work, we divide the weighting equally by setting the two normalization constants equal to each other.

Equations (2.1) and (2.2) can be regarded as alternate bases for an eigensolution analysis. Let us compare them. According to Eqs. (2.1), once the eigenvalues are known, the eigenvectors are the corresponding nontrivial solutions. Furthermore, the right-left eigenvector pair corresponding to a distinct eigenvalue is biorthogonal to the remaining eigenvectors. However, if an eigenvalue is repeated, Eqs. (2.1) do not require biorthogonality among the corresponding eigenvectors. In particular, let us suppose that $\lambda_1 = \lambda_2 = \ldots = \lambda_m$, $m \leq n$. Then the condition

$$\underset{\sim}{v}_j^T B \underset{\sim}{u}_i = 0 \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,m$$

must be imposed in addition to Eqs. (2.1). We also observe that eigenvector normalization is not demanded by Eqs. (2.1). Alternatively, Eqs. (2.2) demand that $\underset{\sim}{u}_i$, $\underset{\sim}{v}_i$ and λ_i $(i = 1,2,\ldots,n)$ satisfy not only the eigenvalue problem, but also biorthogonality and normalization. Now since the objective of an eigensolution analysis is to produce a set of biorthogonal vectors, which can be used as a transformation to diagonalize coefficient matrices, regardless of eigenvalue multiplicity, it is more fitting to base an eigensolution analysis on Eqs. (2.2). Thus, the perturbation analyses that follow are based on biorthonormality relations. Franklin [1], Wilkinson [2] and Lancaster [3] present perturbation analyses based upon the statement of the eigenvalue problem itself, i.e., Eqs. (2.1). An adaptation and discussion of their analyses is presented in Appendix A.

At this point let us change to a modal representation, in which we construct matrices U and V from the right and left eigenvectors $\underset{\sim}{u}_i$ and $\underset{\sim}{v}_i$ $(i = 1,2,\ldots,n)$ as follows:

$$U = [\underset{\sim}{u}_1 \mid \underset{\sim}{u}_2 \mid \cdots \mid \underset{\sim}{u}_n]$$

$$V = [\underset{\sim}{v}_1 \mid \underset{\sim}{v}_2 \mid \cdots \mid \underset{\sim}{v}_n] \ .$$

The diagonal matrix of eigenvalues Λ is constructed as

$$\Lambda \equiv \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

and the modified Kronecker deltas can be arranged in the diagonal matrix

$$\Delta \equiv \begin{bmatrix} \Delta_{11} & & & \\ & \Delta_{22} & & \\ & & \ddots & \\ & & & \Delta_{nn} \end{bmatrix}$$

Equations (1.2) and (1.3) can now be expressed as

$$BU\Lambda = AU \tag{2.3a}$$

$$B^T V\Lambda = A^T V \tag{2.3b}$$

and

$$V^T BU = \Delta \tag{2.4a}$$

$$V^T AU = \Lambda\Delta \tag{2.4b}$$

Let us suppose matrices B and A are given as

$$B = B_0 + B_1 \quad , \quad A = A_0 + A_1 \tag{2.5}$$

where the norms of matrices B_1 and A_1 are significantly smaller than the norms of matrices B_0 and A_0 respectively. This difference in magnitude can be used to define an ordering scheme in which matrices B_0 and A_0 are 0(0) quantities while matrices B_1 and A_1 are 0(1) quantities. The scheme can be extended to higher orders, i.e. smaller quantities, which are referenced as 0(2), 0(3), etc.

Substituting Eqs. (2.5) into Eqs. (2.3) and (2.4), we obtain

$$(B_0 + B_1)U\Lambda = (A_0 + A_1)U \tag{2.6a}$$

$$(B_0^T + B_1^T)V\Lambda = (A_0^T + A_1^T)V \tag{2.6b}$$

and

$$V^T(B_0 + B_1)U = \Delta \tag{2.7a}$$

$$V^T(A_0 + A_1)U = \Lambda\Delta \tag{2.7b}$$

Let us suppose, for the moment, that $A_1 = B_1 = 0$. Then Eqs. (2.6) and (2.7) appear as

$$B_0U_0\Lambda_0 = A_0U_0 \tag{2.8a}$$

$$B_0^TV_0\Lambda_0 = A_0^TV_0 \tag{2.8b}$$

and

$$V_0^TB_0U_0 = \Delta \tag{2.9a}$$

$$V_0^TA_0U_0 = \Lambda_0\Delta \tag{2.9b}$$

We refer to Eqs. (2.6) and (2.7) as the perturbed eigenvalue problem. Its solutions, U, V and Λ, are called the perturbed eigensolutions. Similarly, Eqs. (2.8) and (2.9) are referred to as the unperturbed, or O(0), eigenvalue problem, while its solutions are called the unperturbed, or O(0), eigensolutions. Quantities smaller than O(0), i.e. O(1), O(2), ..., etc., are called perturbations, the order of smallness being indicated by the subscript.

Let us assume that the O(0) eigensolutions, satisfying Eqs. (2.9), are available. Our objective, then, is to develop solutions to the perturbed eigenvalue problem, based upon the known O(0) eigensolutions and matrices B_1 and A_1. Furthermore, in generating this solution, we wish to avoid solving another eigenvalue problem. As it turns out, these goals can be met if the O(0) eigenvalues are "clearly distinct", which is to say that the difference $(\lambda_{0i} - \lambda_{0j})$, $i \neq j$

$(i,j = 1,2,\ldots,n)$, is an $O(0)$ quantity. For the remainder of the present chapter, we assume this to be true. In Chapters 6 and 7, we allow some of the $O(0)$ eigenvalues to be less than clearly distinct.

Consider transformation matrices E and Γ, which relate U and V to U_0 and V_0, respectively, as

$$U = U_0\, E \tag{2.10a}$$

$$V = V_0\, \Gamma \tag{2.10b}$$

Note that Eqs. (2.10) can also be written as

$$\underset{\sim}{u}_i = U_0\, \underset{\sim}{\epsilon}_i = \sum_{q=1}^{n} \underset{\sim}{u}_{0q}\, \epsilon_{qi} \quad , \quad i = 1,2,\ldots,n \tag{2.11a}$$

$$\underset{\sim}{v}_i = V_0\, \underset{\sim}{\gamma}_i = \sum_{p=1}^{n} \underset{\sim}{v}_{0p}\, \gamma_{pi} \quad , \quad i = 1,2,\ldots,n \tag{2.11b}$$

where $\underset{\sim}{\epsilon}_i$ and $\underset{\sim}{\gamma}_i$ are the ith columns of E and Γ, respectively, and where ϵ_{qi} and γ_{pi} are the (q,i)th and (p,i)th elements of E and Γ, respectively. Equations (2.11) and hence Eqs. (2.10) <u>can</u> be written because the $\underset{\sim}{u}_{0q}$ $(q = 1,2,\ldots,n)$ and $\underset{\sim}{v}_{0p}$ $(p = 1,2,\ldots,n)$ both span the space L^n in which $\underset{\sim}{u}_i$ and $\underset{\sim}{v}_i$ $(i = 1,2,\ldots,n)$ are given.

Substituting Eqs. (2.10) into Eqs. (2.6) and (2.7), we obtain

$$(B_0 + B_1)U_0E\Lambda = (A_0 + A_1)U_0E \tag{2.12a}$$

$$(B_0^T + B_1^T)V_0\Gamma\Lambda = (A_0^T + A_1^T)V_0\Gamma \tag{2.12b}$$

and

$$\Gamma^T V_0^T(B_0 + B_1)U_0E = \Delta \tag{2.13a}$$

$$\Gamma^T V_0^T(A_0 + A_1)U_0E = \Lambda\Delta \tag{2.13b}$$

If we premultiply Eq. (2.12a) by V_0^T and Eq. (2.12b) by U_0^T, then use of Eqs. (2.9) allows Eqs. (2.12) and (2.13) to be expressed as

$$(\Delta + \hat{B})E\Lambda = (\Lambda_0\Delta + \hat{A})E \qquad (2.14a)$$

$$(\Delta + \hat{B}^T)\Gamma\Lambda = (\Lambda_0\Delta + \hat{A}^T)\Gamma \qquad (2.14b)$$

and

$$\Gamma^T(\Delta + \hat{B})E = \Delta \qquad (2.15a)$$

$$\Gamma^T(\Lambda_0\Delta + \hat{A})E = \Lambda\Delta \qquad (2.15b)$$

where

$$\hat{B} \equiv V_0^T B_1 U_0 \quad , \qquad \hat{A} \equiv V_0^T A_1 U_0 \qquad (2.16)$$

are O(1) quantities. Now according to Eqs. (2.14) and (2.15), E and Γ appear as matrices of right and left eigenvectors, which would indicate that we must solve the algebraic eigenvalue problem for E, Γ and Λ. However, as mentioned previously, we wish to avoid this task.

Since $\hat{B}$ and $\hat{A}$ are small, matrices $(\Delta + \hat{B})$ and $(\Lambda_0\Delta + \hat{A})$ are nearly diagonal. Then from Eqs. (2.15), we can argue that the transformation matrices E and Γ, which diagonalize $(\Delta + \hat{B})$ and $(\Lambda_0\Delta + \hat{A})$, must be near identity transformations. Thus we assume the expansions

$$E = I + E_1 + E_2 + E_3 + \ldots \qquad (2.17a)$$

$$\Gamma = I + \Gamma_1 + \Gamma_2 + \Gamma_3 + \ldots \qquad (2.17b)$$

and

$$\Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda_3 + \ldots \qquad (2.18)$$

where I is the identity matrix, and where the subscripts indicate order of smallness. We assume these expansions to be convergent, which is to say that any one term is one order of magnitude smaller (larger) than the term before (after) it. Indeed, Wilkinson [2] defines a small parameter, which typifies the smallness of A_1 and B_1, or equivalently of $\hat{A}$ and $\hat{B}$, and then argues as to how the perturbed eigenvalues and eigenvectors are expressible as power series in this small parameter. In writing the expansions of Eqs. (2.17) and (2.18) we assume that, as the perturbing matrices A_1 and B_1 tend to zero, we must have

$$E_1, E_2, E_3, \ldots \to 0$$

$$\Gamma_1, \Gamma_2, \Gamma_3, \ldots \to 0$$

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots \to 0$$

which is to say that

$$\Lambda \to \Lambda_0$$

$$E \to I$$

$$\Gamma \to I$$

or, equivalently

$$U \to U_0$$

$$V \to V_0$$

We note that these expansions are infinite. In practice only a finite number of terms are determined, thus the determination of Λ, E and Γ, and hence of U and V, is merely approximate.

Let us embark upon the perturbation analysis. We substitute Eqs. (2.17) and (2.18) into Eqs. (2.15) and separate according to order.

$$O(0): \quad \Delta = \Delta \tag{2.19a}$$

$$\Lambda_0 \Delta = \Lambda_0 \Delta \tag{2.19b}$$

$$O(1): \quad \Delta E_1 + \Gamma_1^T \Delta = - \hat{B} \tag{2.20a}$$

$$\Lambda_0 \Delta E_1 + \Gamma_1^T \Lambda_0 \Delta = - \hat{A} + \Lambda_1 \Delta \tag{2.20b}$$

$$O(2): \quad \Delta E_2 + \Gamma_2^T \Delta = - \hat{B} E_1 - \Gamma_1^T \Delta E_1 - \Gamma_1^T \hat{B} \tag{2.21a}$$

$$\Lambda_0 \Delta E_2 + \Gamma_2^T \Lambda_0 \Delta = - \hat{A} E_1 - \Gamma_1^T \Lambda_0 \Delta E_1 - \Gamma_1^T \hat{A} + \Lambda_2 \Delta \tag{2.21b}$$

$$O(3): \quad \Delta E_3 + \Gamma_3^T \Delta = - \hat{B} E_2 - \Gamma_1^T \Delta E_2 - \Gamma_1^T \hat{B} E_1 - \Gamma_2^T \Delta E_1 - \Gamma_2^T \hat{B} \tag{2.22a}$$

$$\Lambda_0 \Delta E_3 + \Gamma_3^T \Lambda_0 \Delta = - \hat{A} E_2 - \Gamma_1^T \Lambda_0 \Delta E_2 - \Gamma_1^T \hat{A} E_1 - \Gamma_2^T \Lambda_0 \Delta E_1 - \Gamma_2^T \hat{A} + \Lambda_3 \Delta \tag{2.22b}$$

$$\vdots$$

Now from Eqs. (2.20), we have

$$\hat{B} = - (\Delta E_1 + \Gamma_1^T \Delta) \tag{2.23a}$$

$$\hat{A} = - (\Lambda_0 \Delta E_1 + \Gamma_1^T \Lambda_0 \Delta) + \Lambda_1 \Delta \tag{2.23b}$$

Substituting for $\hat{B}$ and $\hat{A}$ in Eqs. (2.21) and (2.22), we obtain

$$O(2): \quad \Delta E_2 + \Gamma_2^T \Delta = \Delta E_1 E_1 + \Gamma_1^T \Delta E_1 + \Gamma_1^T \Gamma_1^T \Delta \tag{2.24a}$$

$$\Lambda_0 \Delta E_2 + \Gamma_2^T \Lambda_0 \Delta = \Lambda_0 \Delta E_1 E_1 + \Gamma_1^T \Lambda_0 \Delta E_1 + \Gamma_1^T \Gamma_1^T \Lambda_0 \Delta - (\Lambda_1 \Delta E_1 + \Gamma_1^T \Lambda_1 \Delta) + \Lambda_2 \Delta \tag{2.24b}$$

and

$$O(3): \quad \Delta E_3 + \Gamma_3^T \Delta = \Delta E_1 E_2 - \Gamma_1^T \hat{B} E_1 + \Gamma_2^T \Gamma_1^T \Delta \tag{2.25a}$$

$$\Lambda_0 \Delta E_3 + \Gamma_3^T \Lambda_0 \Delta = \Lambda_0 \Delta E_1 E_2 - \Gamma_1^T \hat{A} E_1 + \Gamma_2^T \Gamma_1^T \Lambda_0 \Delta - (\Lambda_1 \Delta E_2 + \Gamma_2^T \Lambda_1 \Delta) + \Lambda_3 \Delta \tag{2.25b}$$

The O(0) equations, Eqs. (2.19), are satisfied identically, as anticipated.

Turning attention to the O(1) equations, Eqs. (2.20), let us inspect homologous elements on both sides of these equations. We have

$$\Delta_{ii}\,\epsilon_{1ij} + \gamma_{1ji}\,\Delta_{jj} = -\,\hat{b}_{ij} \quad , \quad i,j = 1,2,\ldots,n \tag{2.26a}$$

$$\lambda_{0i}\,\Delta_{ii}\,\epsilon_{1ij} + \gamma_{1ji}\,\lambda_{0j}\,\Delta_{jj} = -\,\hat{a}_{ij} + \lambda_{1i}\,\Delta_{ij} \quad , \quad i,j = 1,2,\ldots,n \tag{2.26b}$$

where ϵ_{1ij} and γ_{1ij} are the (i,j) elements of E_1 and Γ_1, respectively, and where $\hat{b}_{ij}$ and $\hat{a}_{ij}$ are the (i,j) elements of $\hat{B}$ and $\hat{A}$, respectively. When $i \neq j$, we can solve Eqs. (2.26) independently for ϵ_{1ij} and γ_{1ji}, to obtain

$$\epsilon_{1ij} = -\,\Delta_{ii}\,\frac{\hat{a}_{ij} - \hat{b}_{ij}\,\lambda_{0j}}{\lambda_{0i} - \lambda_{0j}} \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,n$$

$$\gamma_{1ji} = -\,\Delta_{jj}\,\frac{\hat{a}_{ij} - \lambda_{0i}\,\hat{b}_{ij}}{\lambda_{0j} - \lambda_{0i}} \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,n$$

Alternatively, when $i = j$, Eqs. (2.26) appear as

$$\Delta_{ii}(\epsilon_{1ii} + \gamma_{1ii}) = -\,\hat{b}_{ii} \quad , \quad i = 1,2,\ldots,n$$

$$\lambda_{0i}\,\Delta_{ii}(\epsilon_{1ii} + \gamma_{1ii}) = -\,\hat{a}_{ii} + \lambda_{1i}\,\Delta_{ii} \quad , \quad i = 1,2,\ldots,n$$

Clearly, these equations indicate that

$$\lambda_{1i} = \Delta_{ii}(\hat{a}_{ii} - \lambda_{0i}\,\hat{b}_{ii}) \quad , \quad i = 1,2,\ldots,n$$

yet they provide no unique solution for ϵ_{1ii} and γ_{1ii} ($i = 1,2,\ldots,n$). We note that this is the same sort of ambiguity as that encountered in normalization of unrelated right and left eigenvectors. To divide the weighting equally, let us take

$$\epsilon_{1ii} = \gamma_{1ii} = -\frac{1}{2}\Delta_{ii}\,\hat{b}_{ii} \quad , \quad i = 1,2,\ldots,n$$

which is consistent with the condition that E_1 and Γ_1 tend to zero as A_1 and B_1 tend to zero. Thus we have

$$\epsilon_{1ij} = -\Delta_{ii}\frac{\hat{a}_{ij} - \hat{b}_{ij}\lambda_{0j}}{\lambda_{0i} - \lambda_{0j}} \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,n \qquad (2.27a)$$

$$\epsilon_{1ii} = \gamma_{1ii} = -\frac{1}{2}\Delta_{ii}\,\hat{b}_{ii} \quad , \quad i = j = 1,2,\ldots,n \qquad (2.27b)$$

$$\gamma_{1ij} = -\Delta_{ii}\frac{\hat{a}_{ji} - \lambda_{0j}\,\hat{b}_{ji}}{\lambda_{0i} - \lambda_{0j}} \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,n \qquad (2.27c)$$

$$\lambda_{1i} = \Delta_{ii}(\hat{a}_{ii} - \lambda_{0i}\,\hat{b}_{ii}) \quad , \quad i = 1,2,\ldots,n \qquad (2.28)$$

Determination of the perturbed eigensolution is now complete through O(1). The rationale for our assumption of clearly distinct O(0) eigenvalues should be evident from Eqs. (2.27). We also note that the validity of the expression for λ_{1i} ($i = 1,2,\ldots,n$) is not subject to the assumption of clearly distinct O(0) eigenvalues. In fact, Eq. (2.28) can be derived from Rayleigh's quotient, as we shall see in Chapter 8. It is pleasing to note that as A_1, $B_1 \to 0$, ϵ_{1ij}, γ_{1ij} and λ_{1i} tend to zero linearly. Solutions to the O(2) equations can be generated similarly. These are

$$\epsilon_{2ij} = \frac{\Delta_{ii}}{\lambda_{0i} - \lambda_{0j}} \left\{ \begin{aligned} & -(\lambda_{1i}\, \Delta_{ii}\, \epsilon_{1ij} + \gamma_{1ji}\, \lambda_{1j}\, \Delta_{jj}) \\ & + \sum_{p=1}^{n} [(\lambda_{0i} - \lambda_{0j})\Delta_{ii}\, \epsilon_{1ip}\, \epsilon_{1pj} \\ & \qquad + (\lambda_{0p} - \lambda_{0j})\gamma_{1pi}\, \Delta_{pp}\, \epsilon_{1pj}] \end{aligned} \right\} ,$$

$$i \neq j \quad , \quad i,j = 1,2,\ldots,n \quad (2.29a)$$

$$\epsilon_{2ii} = \gamma_{2ii} = \frac{1}{2}\Delta_{ii} \sum_{p=1}^{n} (\Delta_{ii}\, \epsilon_{1ip}\, \epsilon_{1pi} + \gamma_{1pi}\, \Delta_{pp}\, \epsilon_{1pi} + \gamma_{1pi}\, \gamma_{1ip}\, \Delta_{ii}) \quad , \quad i = j \quad , \quad i = 1,2,\ldots,n \quad (2.29b)$$

$$\gamma_{2ij} = \frac{\Delta_{ii}}{\lambda_{0i} - \lambda_{0j}} \left\{ \begin{aligned} & -(\lambda_{1j}\, \Delta_{jj}\, \epsilon_{1ji} + \gamma_{1ij}\, \lambda_{1i}\, \Delta_{ii}) \\ & + \sum_{p=1}^{n} [(\lambda_{0p} - \lambda_{0j})\gamma_{1pj}\, \Delta_{pp}\, \epsilon_{1pi} \\ & \qquad + (\lambda_{0i} - \lambda_{0j})\gamma_{1pj}\, \gamma_{1ip}\, \Delta_{ii}] \end{aligned} \right\} ,$$

$$i \neq j \quad , \quad i,j = 1,2,\ldots,n \quad (2.29c)$$

$$\lambda_{2i} = \lambda_{1i}(\epsilon_{1ii} + \gamma_{1ii}) + \Delta_{ii} \sum_{p=1}^{n} (\lambda_{0i} - \lambda_{0p})\gamma_{1pi}\, \Delta_{pp}\, \epsilon_{1pi} \quad , \quad i = 1,2,\ldots,n \quad (2.30)$$

Recalling that ϵ_{1ij}, γ_{1ij} and λ_{1i} $(i,j = 1,2,\ldots,n)$ tend to zero linearly as A_1 and B_1 tend to zero, we note that ϵ_{2ij}, γ_{2ij} and λ_{2i} $(i,j = 1,2,\ldots,n)$ tend to zero in a quadratic fashion as A_1 and B_1 tend to zero. Third-order eigenvalue perturbations are

$$\lambda_{3i} = \lambda_{1i}(\epsilon_{2ii} + \gamma_{2ii}) + \Delta_{ii} \sum_{p=1}^{n} \gamma_{1pi}(\hat{a}_{pq} - \lambda_{0i}\, \hat{b}_{pq})\, \epsilon_{1qi} \quad , \qquad i = 1,2,\ldots,n \ . \qquad (2.31)$$

It is felt that third-order eigenvector perturbations are unnecessary. We observe that λ_{3i} tends to zero in a cubic fashion as A_1, $B_1 \to 0$.

The accuracy of the above perturbation procedure is of prime importance. One method of judging accuracy is by simply comparing the magnitudes of successive terms in the expansions of Eqs. (2.17) and (2.18). A less direct, but perhaps more pertinent, measure of eigenvector accuracy is how well the expansions of Eqs. (2.17) diagonalize matrices $(\Delta + \hat{B})$ and $(\Lambda_0\Delta + \hat{A})$. In the examples, such a computation will be referred to as a "biorthonormality check."

In summary, a third-order perturbation analysis has been developed for the algebraic eigenvalue problem

$$(A_0 + A_1)\underset{\sim}{u}_i = \lambda_i(B_0 + B_1)\underset{\sim}{u}_i \quad , \quad i = 1,2,\ldots,n$$

$$(A_0 + A_1)^T\underset{\sim}{v}_i = \lambda_i(B_0 + B_1)^T\underset{\sim}{v}_i \quad , \quad i = 1,2,\ldots,n$$

where matrices A_1 and B_1 are perturbations, where $\underset{\sim}{u}_i$ and $\underset{\sim}{v}_i$ are the respective right and left eigenvectors and where λ_i is an eigenvalue. Eigensolution perturbations, due to the presence of matrices A_1 and B_1, are derived from the biorthonormality relations

$$\underset{\sim}{v}_i^T(B_0 + B_1)\underset{\sim}{u}_j = \Delta_{ij} \quad , \quad i,j = 1,2,\ldots,n$$

$$\underset{\sim}{v}_i^T(A_0 + A_1)\underset{\sim}{u}_j = \lambda_i\Delta_{ij} \quad , \quad i,j = 1,2,\ldots,n$$

The eigenvalue and eigenvector perturbations are expressed in terms of matrices A_1 and B_1, and the unperturbed eigenvalues and eigenvectors, which satisfy biorthonormality relations

$$\underset{\sim}{v}_{0i}^{T} B_0 \underset{\sim}{u}_{0j} = \Delta_{ij} \qquad , \qquad i,j = 1,2,\ldots,n$$

$$\underset{\sim}{v}_{0i}^{T} A_0 \underset{\sim}{u}_{0j} = \lambda_{0i} \Delta_{ij} \qquad , \qquad i,j = 1,2,\ldots,n$$

The validity of the results hinges on the smallness of the perturbing matrices and upon the distinctness of the unperturbed eigenvalues. In Chapters 6 and 7, we present perturbation analyses for which some of the 0(0) eigenvalues may not be clearly distinct.

Chapter 3

GENERAL DAMPED, CIRCULATORY, GYROSCOPIC SYSTEMS

The linear equations of motion describing an n degree of freedom, damped, circulatory gyroscopic system can be written in matrix form as

$$M\,\ddot{\underset{\sim}{q}}(t) + (G + C)\dot{\underset{\sim}{q}}(t) + (K + H)\underset{\sim}{q}(t) = \underset{\sim}{Q}(t) \tag{3.1}$$

where

M = real, symmetric, positive definite $n \times n$ mass matrix

G = real, skew symmetric $n \times n$ gyroscopic matrix

C = real, symmetric, positive semidefinite $n \times n$ damping matrix

K = real, symmetric $n \times n$ generalized stiffness matrix including elastic and centrifugal effects

H = real, skew symmetric $n \times n$ circulatory matrix

$\underset{\sim}{q}(t)$ = $n \times 1$ column vector of generalized coordinates

$\underset{\sim}{Q}(t)$ = $n \times 1$ column vector of generalized forces

Note that the above descriptions of the coefficient matrices can be generalized. Indeed, recalling that any real matrix can be expressed as the sum of two real matrices, one symmetric and the other skew symmetric, one can regard matrices K and H as the respective symmetric and skew symmetric parts of the matrix of coordinate coefficients. Similarly, G and C can be regarded as the skew symmetric and symmetric parts, respectively, of the matrix of velocity coefficients.

It is anticipated that Eqs. (3.1) are the results of the linearization about some equilibrium point, and then the discretization of some partial differential equations whereby the spatial dependence is purged. It must be emphasized that all discretization procedures do not yield coefficient matrices possessing the above specified symmetries and definiteness. The interested reader may wish to consult Meirovitch [14] for a detailed discussion of various discretization procedures.

We recall that matrix K is referred to as a "generalized" stiffness matrix. Indeed, the typical element of this matrix is of the form $k - \Omega^2 m$, where k is a bona fide elastic stiffness coefficient, Ω is an angular velocity, and m is an inertia. We assume that the discretization procedure is such that the matrix of stiffness coefficients is real, symmetric and positive definite or positive semidefinite according to whether or not the system under consideration is constrained. Then if $\Omega = 0$, the generalized stiffness matrix possesses the same definiteness. However, if $\Omega \neq 0$, it should be clear that K may not possess the same definiteness. In fact, if Ω is large enough, K can be negative definite! The eigenvalue problem corresponding to Eq. (3.1) is

$$\lambda_i^2 M \underset{\sim}{q}_i + \lambda_i (G + C)\underset{\sim}{q}_i + (K + H)\underset{\sim}{q}_i = \underset{\sim}{0} \quad , \quad i = 1,2,\ldots,2n \quad (3.2a)$$

where λ_i and q_i $(i = 1,2,\ldots,2n)$ are the ith eigenvalue and eigenvector, respectively. Due to the symmetry and skew symmetry of the various coefficient matrices, we must also address the adjoint eigenvalue problem

$$\lambda_i^2 M^T \underset{\sim}{p}_i + \lambda_i (G^T + C^T)\underset{\sim}{p}_i + (K^T + H^T)\underset{\sim}{p}_i = \underset{\sim}{0}$$

or

$$\lambda_i^2 M \underset{\sim}{p}_i + \lambda_i (-G + C)\underset{\sim}{p}_i + (K - H)\underset{\sim}{p}_i = \underset{\sim}{0} \quad , \quad i = 1,2,\ldots,2n \quad (3.2b)$$

We refer to Eq. (3.2a) as the "right" eigenvalue problem, and its eigenvectors, the $\underset{\sim}{q}_i$ ($i = 1,2,\ldots,2n$), as the "right" eigenvectors. Similarly, we refer to Eq. (3.2b) as the "left" eigenvalue problem, and the $\underset{\sim}{p}_i$ ($i = 1,2,\ldots,2n$) as the "left" eigenvectors. Although the right and left eigenvectors are generally unrelated, the eigenvalues of Eqs. (3.2a) and (3.2b) are the same. To show this, let us write the characteristic polynomial corresponding to Eqs. (3.2). We have

$$F_{R(\lambda)} = |\lambda^2 M + \lambda(G + C) + (K + H)| \tag{3.3a}$$

$$F_{L(\lambda)} = |\lambda^2 M^T + \lambda(G^T + C^T) + (K^T + H^T)| \tag{3.3b}$$

where $F_{R(\lambda)}$ and $F_{L(\lambda)}$ can be regarded as the determinants of $n \times n$ matrices. Now since the determinant of a matrix, and the determinant of its transpose are the same, the polynomials in λ, $F_{R(\lambda)}$ and $F_{L(\lambda)}$, are identical, and hence they possess the same roots. We note that since the coefficient matrices M, G, C, K and H are real, the coefficients in the characteristic polynomial are also real. Thus if an eigenvalue is complex, there must be another eigenvalue that is its complex conjugate. Let us order the complex eigenvalues according to

$$\lambda_{i+n} = \bar{\lambda}_i \quad , \quad \lambda_i = \text{Complex}$$

Introducing primed subscripts by the example $i' \equiv i + n$ ($i = 1,2,\ldots,n$), this relation can be written more compactly as

$$\lambda_{i'} = \bar{\lambda}_i \quad , \quad \lambda_i = \text{Complex} \tag{3.4}$$

According to Eq. (3.4), the corresponding eigenvectors must also occur in complex conjugate pairs as

$$\underset{\sim}{q}_{i'} = \bar{\underset{\sim}{q}}_i \quad , \quad \underset{\sim}{p}_{i'} = \bar{\underset{\sim}{p}}_i \quad , \quad \lambda_i = \text{Complex} \tag{3.5}$$

An eigenvalue, and hence the corresponding eigenvectors, can also be real. We indicate this as

$$\underset{\sim}{q}_i \ , \ \underset{\sim}{p}_i = \text{Real} \quad , \quad \lambda_i = \text{Real} \tag{3.6}$$

However, there appear to be no simple relations, analogous to Eqs. (3.4) and (3.5), for real eigenvalues. Due to the presence of matrices C and H, we do not anticipate pure imaginary eigenvalues.

It is convenient to express the equations of motion in state form. To this end, we write Eqs. (3.1) as

$$\begin{bmatrix} M & 0 \\ 0 & (K+H) \end{bmatrix} \begin{Bmatrix} \ddot{\underset{\sim}{q}}(t) \\ \dot{\underset{\sim}{q}}(t) \end{Bmatrix} + \begin{bmatrix} (G+C) & (K+H) \\ -(K+H) & 0 \end{bmatrix} \begin{Bmatrix} \dot{\underset{\sim}{q}}(t) \\ \underset{\sim}{q}(t) \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{Q}(t) \\ \underset{\sim}{0} \end{Bmatrix} \tag{3.7}$$

We introduce the $2n \times 1$ state vector

$$\underset{\sim}{x}(t) \equiv \{\dot{\underset{\sim}{q}}(t)^T \mid \underset{\sim}{q}(t)^T\}^T \tag{3.8a}$$

and the $2n \times 1$ vector of excitations

$$\underset{\sim}{X}(t) \equiv \{\underset{\sim}{Q}(t)^T \mid \underset{\sim}{0}^T\}^T \tag{3.8b}$$

as well as the $2n \times 2n$ matrices

$$M^* \equiv \begin{bmatrix} M & 0 \\ 0 & (K+H) \end{bmatrix} \quad , \quad K^* \equiv \begin{bmatrix} (G+C) & (K+H) \\ -(K+H) & 0 \end{bmatrix} \tag{3.9a,b}$$

so that Eq. (3.7) can be written compactly as

$$M^*\dot{\underset{\sim}{x}}(t) + K^*\underset{\sim}{x}(t) = \underset{\sim}{X}(t) \tag{3.10}$$

The reader may notice that the inclusion of matrix H in the respective lower right and left corners of matrices M* and K*, respectively, is optional. In fact, Eq. (3.7) remains valid if H is deleted from these locations. However, its inclusion at this stage leads to convenient symmetries which allow net reductions in computation effort. Following the spirit of the above manipulations, we express the right and left eigenvalue problems in state form as

$$\lambda_i M^* \underset{\sim}{x}_i + K^* \underset{\sim}{x}_i = \underset{\sim}{0} \quad , \quad i = 1,2,\ldots,2n \tag{3.11a}$$

$$\lambda_i M^{*T} \underset{\sim}{y}_i + K^{*T} \underset{\sim}{y}_i = \underset{\sim}{0} \quad , \quad i = 1,2,\ldots,2n \tag{3.11b}$$

where M* and K* are given in Eqs. (3.9) and where

$$\underset{\sim}{x}_i \equiv \left\{ \lambda_i \, \underset{\sim}{q}_i^T \;\vdots\; \underset{\sim}{q}_i^T \right\}^T \quad , \quad \underset{\sim}{y}_i \equiv \left\{ -\lambda_i \, \underset{\sim}{p}_i^T \;\vdots\; \underset{\sim}{p}_i^T \right\}^T \quad , \quad i = 1,2,\ldots,2n \tag{3.12a,b}$$

Upon normalization, the right and left eigenvectors, $\underset{\sim}{x}_i$ and $\underset{\sim}{y}_i$ ($i = 1,2,\ldots,2n$) respectively, satisfy biorthonormality relations

$$\underset{\sim}{y}_i^T \, M^* \, \underset{\sim}{x}_j = \underset{\sim}{x}_i^T \, M^{*T} \, \underset{\sim}{y}_j = \Delta_{ij} \quad , \quad i,j = 1,2,\ldots,2n \tag{3.13a}$$

$$\underset{\sim}{y}_i^T \, K^* \, \underset{\sim}{x}_j = \underset{\sim}{x}_i^T \, K^{*T} \, \underset{\sim}{y}_j = -\Delta_{ij} \, \lambda_i \quad , \quad i,j = 1,2,\ldots,2n \tag{3.13b}$$

where Δ_{ij} is the modified Kronecker delta discussed earlier. Note that because the eigenvalues occur in complex conjugate pairs, we can state relations analogous to Eqs. (3.5) and (3.6)

$$\underset{\sim}{x}_{i'} = \bar{\underset{\sim}{x}}_i \quad , \quad \underset{\sim}{y}_{i'} = \bar{\underset{\sim}{y}}_i \quad , \quad \lambda_i = \text{Complex} \tag{3.14}$$

$$\underset{\sim}{x}_i \; , \; \underset{\sim}{y}_i = \text{Real} \quad , \quad \lambda_i = \text{Real} \tag{3.15}$$

Due to the general lack of symmetry of matrices M* and K*, the left and right eigenvectors are not related to each other. Furthermore, the eigenvalues, and hence the eigenvectors must be regarded as generally complex. Computation of such eigensolutions can be tedious, especially for high order systems. Now on physical grounds, the dissipative and circulatory effects described by matrices C and H, respectively, are often small. Then assuming matrices C and H small, we now have the justification for applying a perturbation analysis in which matrices C and H are regarded as perturbations. The motivation for applying a perturbation analysis becomes evident if we observe that when $C = H = 0$, matrices M^* and K^* become symmetric and skew symmetric, respectively, and the eigensolution can be determined with relative ease, as we shall see in the following chapter. Indeed, one could apply a perturbation analysis to the eigenvalue problem, as stated in Eqs. (3.11) with M^* and K^* defined in Eqs. (3.9). However, let us first reduce the equations of motion, and the eigenvalue problem, to a quasi standard form.

Referring to Eq. (3.10), we observe that a bona fide standard form can be obtained if we premultiply by M^{*-1}. Such a maneuver is undesirable because computation of M^{*-1} can be a laborious task, especially for large order systems. Furthermore, even if we neglect matrices C and H, the product $M^{*-1} K^*$ has no symmetries.

Consider the $2n \times 2n$ matrix

$$M_0^* = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}$$

and express it as

$$M_0^* = L\,D\,L^T \tag{3.16}$$

where the $2n \times 2n$ matrices L and D are real, where L is a lower triangular matrix, and where D is a diagonal matrix, with entries equal to either plus or minus one. Note that this decomposition is possible because the matrix M_0^* is real and symmetric. Determination of matrices L and D is accomplished by a procedure similar to the Cholesky decomposition, where the matrix D has been inserted to accommodate minus signs. Note that the end result of such a decomposition is the same as that obtained using Gaussian elimination without row interchanges. Furthermore, we note that the inverse of matrix L is also a lower triangular matrix. See Meirovitch [13].

Due to the partitioned nature of matrix M_0^*, matrices L and D can also be written in partitioned form as follows:

$$L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \quad , \quad D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \qquad (3.17a,b)$$

where L_1, L_2, D_1 and D_2 are $n \times n$ matrices, and where

$$M = L_1 D_1 L_1^T \quad , \quad K = L_2 D_2 L_2^T \qquad (3.18a,b)$$

where the elements of L_1, L_2, D_1 and D_2 are determined as discussed above.

Let us inspect Eq. (3.18a). Because the matrix M is positive definite, D_1 is in fact the identity matrix. Thus the determination of L_1 from M, as in Eq. (3.18a), represents a bona fide Cholesky decomposition. Turning attention to Eq. (3.18b), we recall that no sign definite properties have been atributed to matrix K. This stems from the fact that the stiffness matrix includes not only elastic stiffnesses, but also centrifugal effects. Hence we cannot generally say that D_2 is the identity matrix.

Next, let us rewrite Eq. (3.10) as

$$L^{-1}M^*L^{-T}\cdot L^T\dot{\underset{\sim}{x}}(t) = -L^{-1}K^*L^{-T}\cdot L^T\underset{\sim}{x}(t) + L^{-1}\underset{\sim}{X}(t) \;. \tag{3.19}$$

Then if we define

$$B \equiv L^{-1}M^*L^{-T} \quad , \quad A \equiv -\,L^{-1}K^*L^{-T} \tag{3.20a,b}$$

$$\underset{\sim}{u}(t) \equiv L^T\underset{\sim}{x}(t) \quad , \quad \underset{\sim}{U}(t) \equiv L^{-1}\underset{\sim}{X}(t) \quad , \tag{3.21a,b}$$

Eq. (3.19) can be written compactly as

$$B\dot{\underset{\sim}{u}}(t) = A\underset{\sim}{u}(t) + \underset{\sim}{U}(t) \;. \tag{3.22}$$

Performing the same manipulations with Eqs. (3.11), we obtain

$$\lambda_i B\underset{\sim}{u}_i = A\underset{\sim}{u}_i \quad , \quad \lambda_i B^T\underset{\sim}{v}_i = A^T\underset{\sim}{v}_i \quad , \quad i = 1,2,\ldots,2n \tag{3.23a,b}$$

where matrices A and B are defined in Eqs. (3.20), and where

$$\underset{\sim}{u}_i \equiv L^T\underset{\sim}{x}_i \quad , \quad \underset{\sim}{v}_i \equiv L^T\underset{\sim}{y}_i \quad , \quad i = 1,2,\ldots,2n \;. \tag{3.24a,b}$$

The biorthonormality relations now appear as

$$\underset{\sim}{v}_i^T B\underset{\sim}{u}_j = \Delta_{ij} \quad , \quad \underset{\sim}{v}_i^T A\underset{\sim}{u}_j = \Delta_{ij}\,\lambda_i \quad , \quad i,j = 1,2,\ldots,2n \tag{3.25a,b}$$

Relations analogous to Eqs. (3.5) and (3.6), or Eqs. (3.14) and (3.15), are

$$\underset{\sim}{u}_{i'} = \bar{\underset{\sim}{u}}_i \quad , \quad \underset{\sim}{v}_{i'} = \bar{\underset{\sim}{v}}_i \quad , \quad \lambda_i = \text{Complex} \tag{3.26}$$

$$\underset{\sim}{u}_i,\ \underset{\sim}{v}_i = \text{Real} \quad , \quad \lambda_i = \text{Real} \tag{3.27}$$

The eigensolution must be regarded as generally complex, with left and right eigenvectors that are not related to one another.

It is instructive to express matrices B and A in their partitioned forms. Indeed, substituting into Eqs. (3.20) from Eqs. (3.9) and (3.17), we obtain

$$B = \begin{bmatrix} I & 0 \\ 0 & D_2 + L_2^{-1} H L_2^{-T} \end{bmatrix} \tag{3.28a}$$

$$A = \begin{bmatrix} -L_1^{-1}(G + C)L_1^{-T} & -L_1^{-1}(L_2D_2 + H L_2^T) \\ (D_2L_2^T + L_2^{-1}H)L_1^{-T} & 0 \end{bmatrix} \tag{3.28b}$$

With an eye toward applying a perturbation analysis, let us assume matrices C and H small, and write

$$B = B_0 + B_1 \quad , \quad A = A_0 + A_1 \tag{3.29a,b}$$

where we take

$$B_0 = D = \begin{bmatrix} I & 0 \\ 0 & D_2 \end{bmatrix} \quad , \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & L_2^{-1} H L_2^{-T} \end{bmatrix} \tag{3.30a,b}$$

$$A_0 = \begin{bmatrix} -L_1^{-1} G L_1^{-T} & -L_1^{-1} L_2D_2 \\ D_2 L_2^T L_1^{-T} & 0 \end{bmatrix} , \quad A_1 = \begin{bmatrix} -L_1^{-1} C L_1^{-T} & -L_1^{-1} H L_2^{-T} \\ L_2^{-1} H L_1^{-T} & 0 \end{bmatrix} \tag{3.31a,b}$$

We note that the matrix B_0 is diagonal, matrices B_1 and A_0 are skew symmetric, and that matrix A_1 is symmetric. Furthermore, since the product $L_1^{-1} L_2 D_2$ is lower triangular, matrix A_0 is banded, with half bandwidth n. Thus if matrices C and H, and hence B_1 and A_1 are neglected, we address an eigenvalue problem characterized by a real diagonal matrix, and a real skew symmetric matrix, the eigensolutions of which can be generated with relative ease, as we shall see in the following chapter. At this stage, the equations of motion, Eq. (3.22), the statement of the eigenvalue problem, Eqs. (3.23), and the biorthonormality relations, Eqs. (3.25), with matrices B and A given by Eqs. (3.29), are expressed using notation similar to that used in Chapter 2.

Chapter 4

UNDAMPED, NONCIRCULATORY GYROSCOPIC SYSTEMS

Consider the eigenvalue problem for an undamped, noncirculatory gyroscopic system. In Eqs. (3.28), we set C = H = 0, with the result B = D and A = A_0. Then inserting a zero subscript to indicate this special case, Eqs. (3.23) now appear as

$$\lambda_{0i} D \underset{\sim}{u}_{0i} = A_0 \underset{\sim}{u}_{0i} \quad , \quad \lambda_{0i} D \underset{\sim}{v}_{0i} = A_0^T \underset{\sim}{v}_{0i} \quad , \quad i = 1,2,\ldots,2n \qquad (4.1a,b)$$

where A_0 is a real skew symmetric matrix, and where D is a diagonal matrix, the elements of which are either plus or minus one. From Eqs. (3.25), the biorthonormality relations corresponding to Eqs. (4.1) are

$$\underset{\sim}{v}_{0i}^T D \underset{\sim}{u}_{0j} = \Delta_{ij} \quad , \quad \underset{\sim}{v}_{0i}^T A_0 \underset{\sim}{u}_{0j} = \Delta_{ij}\lambda_{0i} \quad , \quad i,j = 1,2,\ldots,2n \qquad (4.2a,b)$$

We recall that the right and left eigenvalue problems, Eqs. (4.1), possess the same eigenvalues, as can be concluded if we note that the characteristic polynomials,

$$F_{R0(\lambda_0)} \equiv |A_0 - \lambda_0 D| = 0 \qquad (4.3a)$$

$$F_{L0(\lambda_0)} \equiv |A_0^T - \lambda_0 D| = 0 \qquad (4.3b)$$

are the same. Now since A_0 is skew symmetric, Eq. (4.3b) can be written as

$$F_{L0(\lambda_0)} = |-A_0 - \lambda_0 D| = 0$$

or as

$$F_{L0(\lambda_0)} = |A_0 + \lambda_0 D| = 0 \qquad (4.4)$$

Then comparing Eqs. (4.3a) and (4.4), we conclude that if λ_0 is an eigenvalue, then $-\lambda_0$ is also an eigenvalue. Note that this observation is due to Ziegler [6], Sec. 3.1. Recalling the notion of primed subscripts, we have

$$\lambda_{0i'} = -\lambda_{0i} \quad , \quad i = 1,2,\ldots,n \tag{4.5}$$

Let us repeat the statement of the right eigenvalue problem, Eq. (4.1a)

$$A_0 \underset{\sim}{u}_{0i} = \lambda_{0i} D \underset{\sim}{u}_{0i} \quad , \quad i = 1,2,\ldots,n \tag{4.6a}$$

and write the left eigenvalue problem, Eq. (4.1b), as

$$A_0^T \underset{\sim}{v}_{0i'} = \lambda_{0i'} D \underset{\sim}{v}_{0i'} \quad , \quad i = 1,2,\ldots,n \ . \tag{4.6b}$$

The skew symmetry of A_0, and Eq. (4.5), allow Eq. (4.6b) to be written as

$$A_0 \underset{\sim}{v}_{0i'} = \lambda_{0i} D \underset{\sim}{v}_{0i'} \quad , \quad i = 1,2,\ldots,n \ . \tag{4.6c}$$

Then comparing Eqs. (4.6a) and (4.6c), we conclude that

$$\underset{\sim}{v}_{0i'} = \underset{\sim}{u}_{0i} \quad , \quad i = 1,2,\ldots,n \tag{4.7a}$$

from which it follows that

$$\underset{\sim}{v}_{0i} = \underset{\sim}{u}_{0i'} \quad , \quad i = 1,2,\ldots,n \tag{4.7b}$$

Equations (4.5) and (4.7) are quite convenient, and apply to all unperturbed eigenvalues and eigenvectors.

We recall the eigensolution properties, indicated in Eqs. (3.4), (3.26) and (3.27), for the damped, circulatory, gyroscopic system. Since the undamped, noncirculatory system presently under consideration is a special case of that in Chapter 3, these properties are certainly applicable. We have

$$\underset{\sim}{u}_{0i}, \underset{\sim}{v}_{0i} = \text{Real} \quad , \quad \lambda_{0i} = \text{Real} \tag{4.8}$$

$$\lambda_{0i'} = \bar{\lambda}_{0i} \quad , \quad \lambda_{0i} = \text{Complex} \tag{4.9}$$

$$\underset{\sim}{u}_{0i'} = \bar{\underset{\sim}{u}}_{0i} \quad , \quad \underset{\sim}{v}_{0i'} = \bar{\underset{\sim}{v}}_{0i} \quad , \quad \lambda_{0i} = \text{Complex} \tag{4.10}$$

Comparing Eqs. (4.5) and (4.9), we conclude that if an unperturbed eigenvalue is complex, then it is imaginary. Thus we can state that the unperturbed eigenvalues, λ_{0i} $(i = 1,2,\ldots,2n)$ are either real or imaginary. If an unperturbed eigenvalue is real, we can make no additional statements regarding the corresponding eigenvectors, other than Eqs. (4.7) and (4.8). Alternatively, if an unperturbed eigenvalue is imaginary, Eqs. (4.7) and (4.10) yield the convenient property

$$\underset{\sim}{v}_{0i} = \bar{\underset{\sim}{u}}_{0i} \quad , \quad \lambda_{0i} = \text{Imag} \ . \tag{4.11}$$

In summary, all unperturbed eigenvalues and eigenvectors satisfy Eqs. (4.5) and (4.7). If an eigenvalue is real, the corresponding eigenvectors are also, as indicated in Eq. (4.8). If an unperturbed eigenvalue is complex, it is imaginary, and the corresponding eigenvectors satisfy not only Eqs. (4.7), but also Eq. (4.11). We note that when the stiffness matrix K is positive definite, we can guarantee that all the unperturbed eigenvalues are imaginary, and hence that the corresponding right-left eigenvector pairs are complex conjugates of each other. Sadly, the positive definiteness of matrix K cannot be guaranteed.

The property indicated in Eq. (4.11) is most desirable. However, as will be shown in the following discussion, we cannot always produce right and left eigenvectors, corresponding to an imaginary eigenvalue λ_{0i}, that are complex conjugates of each other, and satisfy a plus one normalization. Let us suppose that eigenvalues and unnormalized right and left eigenvectors, denoted as $\underset{\sim}{a}_{0i}$ and $\underset{\sim}{b}_{0i}$ $(i = 1,2,\ldots,2n)$, respectively, have been determined. To normalize the ith right-left eigenvector pair, we compute the product

$$\underset{\sim}{b}_{0i}^{T} D \underset{\sim}{a}_{0i} \equiv \hat{d}_{ii} . \tag{4.12}$$

If λ_{0i} is real, then $\underset{\sim}{a}_{0i}$ and $\underset{\sim}{b}_{0i}$, and hence $\hat{d}_{ii}$, are real. Alternatively if λ_{0i} is imaginary, then $\underset{\sim}{b}_{0i} = \bar{\underset{\sim}{a}}_{0i}$, and once again $\hat{d}_{ii}$ is real. Thus we can say that $\hat{d}_{ii}$ $(i = 1,2,\ldots,2n)$ is real. Let us consider the possible configurations.

i) If $\hat{d}_{ii}$ is positive, then its square root is real, and we form normalized right and left eigenvectors according to

$$\underset{\sim}{u}_{0i} = \underset{\sim}{a}_{0i}/\sqrt{\hat{d}_{ii}} \quad , \quad \underset{\sim}{v}_{0i} = \underset{\sim}{b}_{0i}/\sqrt{\hat{d}_{ii}} \quad , \quad \hat{d}_{ii} = \text{Pos.} \tag{4.13}$$

Note that a plus one normalization, i.e. $\Delta_{ii} = +1$, is maintained, and if λ_{0i} is imaginary, then Eq. (4.11) remains valid.

ii) If $\hat{d}_{ii}$ is negative, then its square root is imaginary. Following the spirit of Eqs. (4.13), we form normalized eigenvectors according to

$$\underset{\sim}{u}_{0i} = \underset{\sim}{a}_{0i}/i\sqrt{-\hat{d}_{ii}} \quad , \quad \underset{\sim}{v}_{0i} = \underset{\sim}{b}_{0i}/i\sqrt{-\hat{d}_{ii}} \quad , \quad \hat{d}_{ii} = \text{Neg.} \tag{4.14}$$

Now although Eqs. (4.14) indicate normalized eigenvectors that obey a plus one normalization, these eigenvectors possess undesirable properties. If λ_{0i} is imaginary, then Eq. (4.11) ceases to be valid! Alternatively, if λ_{0i} is real, Eq. (4.14) indicates normalized eigenvectors that are imaginary, which is rather unorthodox. One way to obviate this awkwardness is to admit a minus one normalization, i.e. $\Delta_{ii} = -1$, and then form the normalized eigenvectors according to

$$\underset{\sim}{u}_{0i} = \underset{\sim}{a}_{0i}/\sqrt{-\hat{d}_{ii}} \quad , \quad \underset{\sim}{v}_{0i} = \underset{\sim}{b}_{0i}/\sqrt{-\hat{d}_{ii}} \quad ,$$

$$\hat{d}_{ii} = \text{Neg.} \quad , \quad \Delta_{ii} = -1 \tag{4.15}$$

The relationship between the sign definiteness of the stiffness matrix K and the positive versus negative unity normalization can be brought into sharper focus if we return to the "original" variables. Recalling Eqs. (3.16), (3.24), and (3.12), we have

$$M_0^* \equiv L\,D\,L^T \qquad \text{or} \qquad D = L^{-1}\,M_0^*\,L^{-T} \tag{4.16a,b}$$

$$\underset{\sim}{u}_{0i} \equiv L^T\,\underset{\sim}{x}_{0i} \quad , \quad \underset{\sim}{v}_{0i} = L^T\,\underset{\sim}{y}_{0i} \quad , \quad i = 1,2,\ldots,2n \tag{4.16c,d}$$

$$M_0^* \equiv \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} \tag{4.16e}$$

$$\underset{\sim}{x}_{0i} \equiv \begin{Bmatrix} \lambda_{0i}\,\underset{\sim}{q}_{0i} \\ \underset{\sim}{q}_{0i} \end{Bmatrix} \quad , \quad \underset{\sim}{y}_{0i} \equiv \begin{Bmatrix} -\lambda_{0i}\,\underset{\sim}{p}_{0i} \\ \underset{\sim}{p}_{0i} \end{Bmatrix} , \quad i = 1,2,\ldots,2n. \tag{4.16f,g}$$

For i = j, the use of Eqs. (4.16) allow Eq. (4.2a) to be written as

$$\Delta_{ii} = -\lambda_{0i}^2\,\hat{m}_{ii} + \hat{k}_{ii} \quad , \quad i = 1,2,\ldots,2n \tag{4.17}$$

where

$$\hat{m}_{ii} \equiv \underset{\sim}{p}_{0i}^T\,M\,\underset{\sim}{q}_{0i} \quad , \quad \hat{k}_{ii} \equiv \underset{\sim}{p}_{0i}^T\,K\,\underset{\sim}{q}_{0i} \quad , \quad i = 1,2,\ldots,2n \tag{4.18}$$

and where we assume that the vectors $\underset{\sim}{q}_{0i}$ and $\underset{\sim}{p}_{0i}$ $(i = 1,2,\ldots,2n)$ have been normalized so that Δ_{ii} $(i = 1,2,\ldots,2n)$ is either plus or minus one. Once again, we investigate the possible configurations.

CASE 1) If the stiffness matrix K is positive definite, then the eigenvalues are imaginary, $\underset{\sim}{p}_{0i} = \bar{\underset{\sim}{q}}_{0i}$ $(i = 1,2,\ldots,2n)$, and the quantitives $-\lambda_{0i}^2 \hat{m}_{ii}$ and $\hat{k}_{ii}$ are guaranteed real and positive, thus $\Delta_{ii} = +1$ $(i = 1,2,\ldots,2n)$.

CASE 2) If the stiffness matrix K is not positive definite, the eigenvalues may be either real or imaginary.

i) If λ_{0i} is imaginary, then $\underset{\sim}{p}_{0i} = \bar{\underset{\sim}{q}}_{0i}$, and the quantities $-\lambda_{0i}^2 \hat{m}_{ii}$ and $\hat{k}_{ii}$ are real. Furthermore, although the quantity $-\lambda_{0i}^2 \hat{m}_{ii}$ is positive, we must regard $\hat{k}_{ii}$ as sign variable. Thus the possibility exists that the quantity $(-\lambda_{0i}^2 \hat{m}_{ii} + \hat{k}_{ii})$ is negative indicating that $\Delta_{ii} = -1$.

ii) If λ_{0i} is real, then $\underset{\sim}{p}_{0i}$ and $\underset{\sim}{q}_{0i}$ are real yet unrelated, and although the quantities $-\lambda_{0i}^2 \hat{m}_{ii}$ and $\hat{k}_{ii}$ are real, we can make no statements regarding their sign definiteness. Thus, once again, minus one normalizations become a possibility.

We note that the positive definiteness of matrix K is sufficient, but not necessary, to ensure plus one normalizations for all the eigenvectors. As we shall see later, the matter of positive or negative unity normalization plays a role in the determination of stability. We note that according to the ordering indicated by primed subscripts, we have

$$\Delta_{i'i'} = \Delta_{ii} \quad , \quad i = 1,2,\ldots,n \tag{4.19}$$

The eigensolutions of Eqs. (4.1) must be regarded as generally complex, thus they are undesirable from a computational point of view. In the following, we reduce Eqs. (4.1) to a single eigenvalue problem involving a single real matrix, possessing real eigensolutions. We note that the procedure is a slight generalization of that presented by Meirovitch [9], which was developed under the assumption that the stiffness matrix K is positive definite.

Recalling the skew symmetry of matrix A_0, Eqs. (4.1) can be written as

$$A_0 \underset{\sim}{u}_{0i} = \lambda_{0i} D \underset{\sim}{u}_{0i} \quad , \quad i = 1,2,\ldots,2n \tag{4.20a}$$

$$A_0 \underset{\sim}{v}_{0i} = - \lambda_{0i} D \underset{\sim}{v}_{0i} \quad , \quad k = 1,2,\ldots,2n \tag{4.20b}$$

Premultiplication of Eqs. (4.20) by $- A_0 D$ yields

$$- A_0 D A_0 \underset{\sim}{u}_{0i} = - \lambda_{0i} A_0 D^2 \underset{\sim}{u}_{0i} \quad , \quad i = 1,2,\ldots,2n \tag{4.21a}$$

$$- A_0 D A_0 \underset{\sim}{v}_{0i} = \lambda_{0i} A_0 D^2 \underset{\sim}{v}_{0i} \quad , \quad i = 1,2,\ldots,2n \tag{4.21b}$$

Now the matrix D has the convenient property that it is its inverse, i.e.

$$D = D^{-1} \qquad \text{or} \qquad D^2 = I , \tag{4.22}$$

thus Eqs. (4.21) can be written as

$$- A_0 D A_0 \underset{\sim}{u}_{0i} = - \lambda_{0i} A_0 \underset{\sim}{u}_{0i} \quad , \quad i = 1,2,\ldots,2n \tag{4.23a}$$

$$- A_0 D A_0 \underset{\sim}{v}_{0i} = \lambda_{0i} A_0 \underset{\sim}{v}_{0i} \quad , \quad i = 1,2,\ldots,2n \tag{4.23b}$$

Regarding the right sides of Eqs. (4.23), Eqs. (4.20) can be used to express Eqs. (4.23) as

$$- A_0 \, D \, A_0 \, \underset{\sim}{u}_{0i} = - \lambda_{0i}^2 \, D \, \underset{\sim}{u}_{0i} \quad , \quad i = 1,2,\ldots,2n \tag{4.24a}$$

$$- A_0 \, D \, A_0 \, \underset{\sim}{v}_{0i} = - \lambda_{0i}^2 \, D \, \underset{\sim}{v}_{0i} \quad , \quad i = 1,2,\ldots,2n \ , \tag{4.24b}$$

which can also be written as

$$\lambda_{0i}^2 \, \underset{\sim}{u}_{0i} = - C_0 \, \underset{\sim}{u}_{0i} \quad , \quad i = 1,2,\ldots,2n \tag{4.25a}$$

$$\lambda_{0i}^2 \, \underset{\sim}{v}_{0i} = - C_0 \, \underset{\sim}{v}_{0i} \quad , \quad i = 1,2,\ldots,2n \ , \tag{4.25b}$$

where

$$C_0 \equiv - (DA_0)^2 \ . \tag{4.26}$$

Comparing Eqs. (4.25), we observe that $\underset{\sim}{u}_{0i}$ and $\underset{\sim}{v}_{0i}$ $(i = 1,2,\ldots,2n)$ satisfy the <u>same eigenvalue problem</u>, thus one need solve only <u>one</u> eigenvalue problem. As was mentioned earlier, the matrix C_0 is real, and since λ_{0i}^2 is real, the eigenvectors are also real. On balance, the matrix C_0, as expressed in Eq. (4.26), possesses no symmetries. Furthermore, each eigenvalue λ_{0i}^2 $(i = 1,2,\ldots,2n)$ has multiplicity two, thus the determination of orthogonal eigenvectors may be slightly awkward.

Let us suppose that the n eigenvalues λ_{0i}^2, each with multiplicity two, and the corresponding real, orthogonal eigenvectors, of the matrix C_0 are available. The question arises as to how these results should be interpreted. If the eigenvalue λ_{0i}^2 $(1 \le i \le n)$ is positive, then $\lambda_{0i} = +\sqrt{\lambda_{0i}^2}$ and $\lambda_{0i'} = -\sqrt{\lambda_{0i}^2}$ $(1 \le i \le n)$, and we can regard the corresponding eigenvectors of C_0 as $\underset{\sim}{u}_{0i}$ and $\underset{\sim}{v}_{0i}$ $(1 \le i \le n)$. Equations (4.7) provide the eigenvectors $\underset{\sim}{u}_{0i'}$ and $\underset{\sim}{v}_{0i'}$. Alternatively, if the eigenvalue λ_{0i}^2 $(1 \le i \le n)$, is negative, then $\lambda_{0i} = +\, i\sqrt{-\lambda_{0i}^2}$ and $\lambda_{0i'} = -\, i\sqrt{-\lambda_{0i}^2}$ $(1 \le i \le n)$, and, recalling Eq. (4.11), we can regard the corresponding pair of eigenvectors as the real and imaginary parts of $\underset{\sim}{u}_{0i}$ and $\underset{\sim}{v}_{0i}$ $(1 \leq i \leq n)$. Once again, Eq. (4.7) provides the eigenvectors $\underset{\sim}{u}_{0i}$, and $\underset{\sim}{v}_{0i}$. We note that when the stiffness matrix K is positive definite, matrix D is the identity matrix, and C_0 is not only symmetric but also positive definite. The eigenvalues λ_{0i}^2 are all negative leading to eigenvalues λ_{0i} $(i = 1,2,\ldots,2n)$ that are imaginary.

To summarize, the algebraic eigenvalue problem for the undamped, noncirculatory gyroscopic system, described by Eqs. (4.1), has been reduced to a single eigenvalue problem, defined by a single real matrix, and possessing real eigensolutions.

Chapter 5

PERTURBED, GYROSCOPIC SYSTEMS

Let us return to the general damped, circulatory, gyroscopic system discussed in Chapter 3, and regard the damping and circulatory effects as perturbations. We consider the eigenvalue problem as expressed in Eqs. (3.23) with matrices B and A given in Eqs. (3.28) - (3.31). Note that solution of the corresponding unperturbed eigenvalue problem was discussed in Chapter 4.

In the present chapter we discuss some special properties that result from the various symmetries of matrices B_0, B_1, A_0 and A_1, and then apply the perturbation theory to a rotating beam with a lumped mass at its center. Note that these special properties will be derived from the results of Chapter 2, thus excluding the possibility of not clearly distinct unperturbed eigenvalues.

From Chapter 3, we recall that in addition to being real, matrices A_0 and B_1 are skew symmetric, matrix A_1 is symmetric, and matrix $B_0 = D$ is diagonal, its elements being either plus or minus one. These properties indicate that all unperturbed eigenvalues and eigenvectors satisfy

$$\lambda_{0i'} = -\lambda_{0i} \quad , \quad i = 1,2,\ldots,n \tag{5.1a}$$

$$\underset{\sim}{v}_{0i} = \underset{\sim}{u}_{0i'} \quad , \quad i = 1,2,\ldots,n \tag{5.1b}$$

$$\underset{\sim}{u}_{0i} = \underset{\sim}{v}_{0i'} \quad , \quad i = 1,2,\ldots,n \tag{5.1c}$$

Furthermore, we recall that the unperturbed eigenvalues may be either real or imaginary. If one of these eigenvalues is real, then the corresponding left and right eigenvectors are also real. Alternatively, if an unperturbed eigenvalue is imaginary, we have, in addition to Eqs. (5.1b,c)

$$\underset{\sim}{v}_{0i} = \bar{\underset{\sim}{u}}_{0i} \quad , \quad \lambda_{0i} = \text{Imag} \tag{5.2}$$

Equations (5.1b,c) and (5.2b) indicate that

$$\underset{\sim}{u}_{0i'} = \bar{\underset{\sim}{u}}_{0i} \quad , \quad \underset{\sim}{v}_{0i'} = \bar{\underset{\sim}{v}}_{0i} \quad , \quad \lambda_{0i} = \text{Imag} \tag{5.3}$$

Let us state a convenient rule regarding primed subscripts. "A subscript with two primes can be replaced by the same subscript with no primes." To exemplify the use of this rule, let us prime both sides of Eq. (5.1b), to obtain

$$\underset{\sim}{v}_{0i'} = \underset{\sim}{u}_{0i''} \quad , \quad i = 1,2,\ldots,n$$

but $\underset{\sim}{u}_{0i''} = \underset{\sim}{u}_{0i}$, thus

$$\underset{\sim}{v}_{0i'} = \underset{\sim}{u}_{0i}$$

which is the same as Eq. (5.1c). Thus we need have stated only one of Eqs. (5.1b) and (5.1c).

For the following endeavor, it is convenient to reexpress certain quantities as follows. In place of

$$\hat{A} \equiv V_0^T A_1 U_0 \quad , \quad \hat{B} \equiv V_0^T B_1 U_0 \tag{5.4a,b}$$

we use

$$\hat{a}_{ij} = \underset{\sim}{v}_{0i}^T A_1 \underset{\sim}{u}_{0j} \quad , \quad \hat{b}_{ij} = \underset{\sim}{v}_{0i}^T B_1 \underset{\sim}{u}_{0j} \quad , \quad i,j = 1,2,\ldots,2n. \tag{5.5a,b}$$

Furthermore, we recall the eigenvector expansions

$$U = U_0 \,(I + E_1 + E_2 + \ldots) \tag{5.6a}$$

$$V = V_0 \,(I + \Gamma_1 + \Gamma_2 + \ldots) \tag{5.6b}$$

which can also be represented as

$$\underset{\sim}{u}_i = \sum_{k=1}^{2n} \underset{\sim}{u}_{0k}(\delta_{ki} + \epsilon_{1ki} + \epsilon_{2ki} + \ldots) \quad , \quad i = 1,2,\ldots,2n \tag{5.7a}$$

$$\underset{\sim}{v}_i = \sum_{k=1}^{2n} \underset{\sim}{v}_{0k}(\delta_{ki} + \gamma_{1ki} + \gamma_{2ki} + \ldots) \quad , \quad i = 1,2,\ldots,2n \tag{5.7b}$$

Let us write

$$\underset{\sim}{u}_i = \underset{\sim}{u}_{0i} + \underset{\sim}{u}_{1i} + \underset{\sim}{u}_{2i} + \ldots \quad , \quad i = 1,2,\ldots,2n \tag{5.8a}$$

$$\underset{\sim}{v}_i = \underset{\sim}{v}_{0i} + \underset{\sim}{v}_{1i} + \underset{\sim}{v}_{2i} + \ldots \quad , \quad i = 1,2,\ldots,2n\ . \tag{5.8b}$$

Then comparing Eqs. (5.7) and (5.8), we conclude that

$$\underset{\sim}{u}_{1i} = \sum_{k=1}^{2n} \underset{\sim}{u}_{0k}\, \epsilon_{1ki} \quad , \quad i = 1,2,\ldots,2n \tag{5.9a}$$

$$\underset{\sim}{u}_{2i} = \sum_{k=1}^{2n} \underset{\sim}{u}_{0k}\, \epsilon_{2ki} \quad , \quad i = 1,2,\ldots,2n \tag{5.9b}$$

$$\vdots$$

$$\underset{\sim}{v}_{1i} = \sum_{k=1}^{2n} \underset{\sim}{v}_{0k}\, \gamma_{1ki} \quad , \quad i = 1,2,\ldots,2n \tag{5.9c}$$

$$\underset{\sim}{v}_{2i} = \sum_{k=1}^{2n} \underset{\sim}{v}_{0k}\, \gamma_{2ki} \quad , \quad i = 1,2,\ldots,2n \tag{5.9d}$$

$$\vdots$$

Let us look for convenient properties that can be derived from Eqs. (5.1) and the respective symmetry and skew symmetry of matrices

A_1 and B_1. One can show that

$$\hat{a}_{ij'} = \hat{a}_{ji'} \quad , \quad i,j = 1,2,\ldots,n \tag{5.10a}$$

Recalling the rule pertaining to primed subscripts, we note that Eq. (5.10) also indicates that

$$\hat{a}_{ij} = \hat{a}_{j'i'} \quad , \quad i,j = 1,2,\ldots,n$$

$$\hat{a}_{i'j} = \hat{a}_{j'i} \quad , \quad i,j = 1,2,\ldots,n$$

We also have

$$\hat{b}_{ij'} = -\hat{b}_{ji'} \quad , \quad i,j = 1,2,\ldots,n \tag{5.10b}$$

It follows that

$$\lambda_{1i'} = \lambda_{1i} \quad , \quad i = 1,2,\ldots,n \tag{5.11a}$$

$$\epsilon_{1ij'} = -\gamma_{1i'j} \quad , \quad i,j = 1,2,\ldots,n \tag{5.11b}$$

$$\underset{\sim}{v}_{1i'} = -\underset{\sim}{u}_{1i} \quad , \quad i = 1,2,\ldots,n \tag{5.11c}$$

Turning attention to the higher order perturbations, one can show that

$$\lambda_{2i'} = -\lambda_{2i} \quad , \quad i = 1,2,\ldots,n \tag{5.12a}$$

$$\epsilon_{2ij'} = \gamma_{2i'j} \quad , \quad i,j = 1,2,\ldots,n \tag{5.12b}$$

$$\underset{\sim}{v}_{2i'} = \underset{\sim}{u}_{2i} \quad , \quad i = 1,2,\ldots,n \tag{5.12c}$$

$$\lambda_{3i'} = \lambda_{3i} \quad , \quad i = 1,2,\ldots,n \tag{5.12d}$$

Equations (5.11) and (5.12) allow us to write

$$\lambda_{i'} = -\lambda_{0i} + \lambda_{1i} - \lambda_{2i} + \lambda_{3i} + \ldots \quad , \quad i = 1,2,\ldots,n \tag{5.13a}$$

$$\underset{\sim}{v}_{i'} = \underset{\sim}{u}_{0i} - \underset{\sim}{u}_{1i} + \underset{\sim}{u}_{2i} + \ldots \quad , \quad i = 1,2,\ldots,n \tag{5.13b}$$

We note with pleasure the alternating signs in Eqs. (5.13).

The above relations are convenient and labor saving. We note that their existence is dependent upon the inclusion of matrix H in the lower left and right corners of matrices A_1 and B_1, respectively. Indeed, additional labor saving properties can be derived from the fact that the λ_{0i} (i = 1,2,...,2n) are either real or imaginary.

Let us suppose λ_{0i} is real. We can then show that

$$\lambda_{1i}, \lambda_{2i}, \lambda_{3i} = \text{Real} \quad , \quad \lambda_{0i} = \text{Real} \tag{5.14a}$$

$$\underset{\sim}{u}_{1i}, \underset{\sim}{u}_{2i}, \underset{\sim}{v}_{1i}, \underset{\sim}{v}_{2i} = \text{Real} \quad , \quad \lambda_{0i} = \text{Real} . \tag{5.14b}$$

These results are pleasing in that if an unperturbed eigenvalue is real, then the corresponding perturbed eigenvalue, summarized through O(3), is also real. Furthermore, the corresponding perturbed eigenvectors, summarized through O(2), are also real, in agreement with Eq. (3.27). Alternatively, let us suppose that λ_{0i} is imaginary. It follows that

$$\lambda_{1i} = \text{Real} \quad , \quad \lambda_{0i} = \text{Imag} \tag{5.15a}$$

$$\underset{\sim}{v}_{1i} = - \bar{\underset{\sim}{u}}_{1i} \quad , \quad \lambda_{0i} = \text{Imag} \tag{5.15b}$$

$$\lambda_{2i} = \text{Imag} \quad , \quad \lambda_{0i} = \text{Imag} \tag{5.15c}$$

$$\underset{\sim}{v}_{2i} = \bar{\underset{\sim}{u}}_{2i} \quad , \quad \lambda_{0i} = \text{Imag} \tag{5.15d}$$

$$\lambda_{3i} = \text{Real} \quad , \quad \lambda_{0i} = \text{Imag} \tag{5.15e}$$

From Eqs. (5.11a), (5.15a), (5.12a), (5.15c), (5.12d) and (5.15e), we can write

$$\lambda_{i'} = \bar{\lambda}_{0i} + \bar{\lambda}_{1i} + \bar{\lambda}_{2i} + \bar{\lambda}_{3i} + \dots \quad , \quad \lambda_{0i} = \text{Imag} . \tag{5.16}$$

Thus if λ_{0i}, and hence $\lambda_{0i'}$, is imaginary, the corresponding perturbed eigenvalues, summarized through $O_{(3)}$, occur in complex conjugate pairs, in agreement with Eq. (3.4). Furthermore, from Eqs. (5.11c), (5.15b), (5.12c) and (5.15d), we can write

$$\underset{\sim}{u}_{i'} = \bar{\underset{\sim}{u}}_{0i} + \bar{\underset{\sim}{u}}_{1i} + \bar{\underset{\sim}{u}}_{2i} + \dots \quad , \quad \lambda_{0i} = \text{Imag} . \tag{5.17a}$$

Similarly, we have

$$\underset{\sim}{v}_{i'} = \bar{\underset{\sim}{v}}_{0i} + \bar{\underset{\sim}{v}}_{1i} + \bar{\underset{\sim}{v}}_{2i} + \dots \quad , \quad \lambda_{0i} = \text{Imag} . \tag{5.17b}$$

Thus, if λ_{0i}, and hence $\lambda_{0i'}$, is imaginary, then the corresponding perturbed eigenvectors, summarized through $O(2)$, occur in complex conjugate pairs, in agreement with Eq. (3.26).

Equations (5.15a) and (5.15c) suggest an interesting analog. Let us consider the homogeneous equations of motion describing a single-degree-of-freedom spring, mass and dashpot system. We have

$$\ddot{q}(t) + 2\gamma\omega_0 \, q(t) + \omega_0^2 \, q(t) = 0$$

where γ and ω_0^2 are positive. The eigenvalues are

$$\lambda = - \gamma\omega_0 \pm i\omega_0(1 - \gamma^2)^{1/2}.$$

Assuming γ to be small, we can write

$$\lambda = \pm \, i\omega_0 - \gamma\omega_0 \mp i \, \frac{1}{2} \, \gamma^2\omega_0 + \dots$$

The second term, which is $O(1)$, is real, as is λ_{1i}, while the first and third terms, which are $O(0)$ and $O(2)$, respectively, are imaginary, as are λ_{0i} and λ_{2i}, respectively. Sadly, the analog cannot be carried to higher order terms.

The stability of a dynamical system is of great importance, thus we focus attention on the real parts of the eigenvalues. The purpose of the present chapter is to investigate the effects of damping and/or circulatory forces on the undamped, noncirculatory gyroscopic system. Since λ_{1i} $(i = 1,2,\ldots,2n)$ is real, and because it is the largest eigenvalue perturbation, we wish to discuss whether it tends to stabilize or destabilize the unperturbed system. It is instructive to express λ_{1i} in terms of "original" variables as

$$\lambda_{1i} = \Delta_{ii}(\lambda_{0i}^2 \hat{c}_{ii} + \lambda_{0i} \hat{h}_{ii}) \quad , \quad i = 1,2,\ldots,2n \tag{5.18a}$$

where

$$\hat{c}_{ii} \equiv \underset{\sim}{p}_{0i}^T C \underset{\sim}{q}_{0i} \quad , \quad \hat{h}_{ii} \equiv \underset{\sim}{p}_{0i}^T H \underset{\sim}{q}_{0i} \quad , \quad i = 1,2,\ldots,2n \tag{5.18b}$$

Let us suppose λ_{0i} is imaginary. Then $\underset{\sim}{p}_{0i} = \bar{\underset{\sim}{q}}_{0i}$ and the quantity $\lambda_{0i}^2 \hat{c}_{ii}$ is real because of the symmetry of the matrix C, and is negative (nonpositive) due to the positive definiteness (semidefiniteness) of the matrix C. Then, if $\Delta_{ii} = 1$, we observe that the dissipative effects of the matrix C do indeed attenuate the free response. The quantity $\lambda_{0i} \hat{h}_{ii}$ is also real because of the skew symmetry of the matrix H. However, we can make no statement regarding the sign of this quantity. Then, even if $\Delta_{ii} = 1$, the possibility exists that the inclusion of circulatory effects can lead to unstable solutions. This variety of instability is discussed by Ziegler [6] and Huseyin [5], and is known as flutter.

Alternatively, let us assume that λ_{0i} is real. Then, $\underset{\sim}{p}_{0i}$ and $\underset{\sim}{q}_{0i}$ are real, but we cannot make any statements regarding the sign of $\hat{c}_{ii}$. In general, there are two ways that λ_{1i} can be negative, and two ways that it can be positive. We list them as the following four cases.

Case i.

$$\Delta_{ii} = 1 \quad , \quad \lambda_{0i}^2 \hat{c}_{ii} + \lambda_{0i} \hat{h}_{ii} < 0 \Rightarrow \lambda_{1i} < 0 \tag{5.19a}$$

Case ii.

$$\Delta_{ii} = 1 \quad , \quad \lambda_{0i}^2 \hat{c}_{ii} + \lambda_{0i} \hat{h}_{ii} > 0 \Rightarrow \lambda_{1i} > 0 \tag{5.19b}$$

Case iii.

$$\Delta_{ii} = -1 \quad , \quad \lambda_{0i}^2 \hat{c}_{ii} + \lambda_{0i} \hat{h}_{ii} < 0 \Rightarrow \lambda_{1i} > 0 \tag{5.19c}$$

Case iv.

$$\Delta_{ii} = -1 \quad , \quad \lambda_{0i}^2 \hat{c}_{ii} + \lambda_{0i} \hat{h}_{ii} > 0 \Rightarrow \lambda_{1i} < 0 \tag{5.19d}$$

Consider the small bending motions of a slender, rotating beam with a rigid disk afixed at its center, as depicted in Fig. 5.1. Coordinates X, Y and Z are inertial while x, y and z are fixed in the shaft. Due to the assumption of small deflections, the coordinates Z and z are approximately coincident along the shaft, while x and y rotate with respect to X and Y at the constant angular speed $\dot{\theta} = \Omega$. The disk has mass M, while the shaft has a constant mass per unit length m_0. The shaft has constant, distributed stiffnesses EI_y and EI_x in the local directions $\vec{e}_x$ and $\vec{e}_y$, respectively. At the ends we specify zero deflections and restoring moments that are proportional to the angular deflections, via the concentrated stiffnesses K_1 at $z = 0$ and K_2 at $z = L$, which are independent of the coordinates X and Y or x and y. Furthermore, we include constant distributed internal dissipation that is proportional to the local speed via the constant c. Similarly, we assume constant distributed external dissipation, that is proportional to the absolute velocity via the constant h.

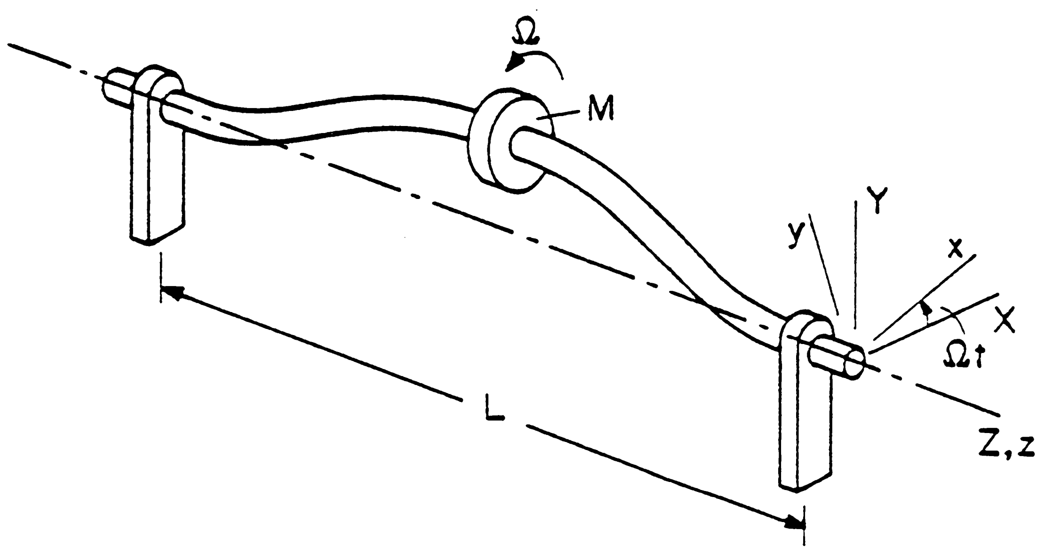

Figure 5.1

A Rotating Beam

Note that the deflections $X(z,t)$ and $Y(z,t)$ are related to the deflections $x(z,t)$ and $y(z,t)$ according to

$$X(z,t) = x(z,t)\cos\theta - y(z,t)\sin\theta \tag{5.20a}$$

$$Y(z,t) = x(z,t)\sin\theta + y(z,t)\cos\theta \tag{5.20b}$$

where $\theta = \Omega t$. We observe that

$$\dot{X}^2(z,t) + \dot{Y}^2(z,t) = (\dot{x}(z,t) - y(z,t)\Omega)^2 + (\dot{y}(z,t) + x(z,t)\Omega)^2 \tag{5.21}$$

where the overdots represent partial derivatives with respect to t. The kinetic energy is

$$\begin{aligned} T(t) &= \frac{1}{2}\int_{z=0}^{z=L} [m_0 + M\delta(z - \tfrac{1}{2}L)](\dot{X}^2(z,t) + \dot{Y}^2(z,t))dz \\ &= \frac{1}{2}\int_{z=0}^{z=L} [m_0 + M\delta(z - \tfrac{1}{2}L)] \\ &\qquad [(\dot{x}(z,t) - y(z,t)\Omega)^2 + (\dot{y}(z,t) + x(z,t)\Omega)^2]dz \end{aligned} \tag{5.22}$$

while the potential energy is

$$\begin{aligned} V(t) = {} & \frac{1}{2}\int_{z=0}^{z=L} [EI_y(x''(z,t))^2 + EI_x(y''(z,t))^2]dz \\ & + \frac{1}{2}K_1[(x'(0,t))^2 + (y'(0,t))^2] \\ & + \frac{1}{2}K_2[(x'(L,t))^2 + (y'(L,t))^2] \end{aligned} \tag{5.23}$$

where the primes represent partial derivatives with respect to z. The dissipative function is

$$F(t) = \frac{1}{2} c \, (\dot{x}^2(z,t) + \dot{y}^2(z,t)) + \frac{1}{2} h \, (\dot{X}^2(z,t) + \dot{Y}^2(z,t))$$

$$= \frac{1}{2} c \, (\dot{x}^2(z,t) + \dot{y}^2(z,t))$$

$$+ \frac{1}{2} h \, [(\dot{x}(z,t) - y(z,t)\Omega)^2 + (\dot{y}(z,t) + x(z,t)\Omega)^2] \quad (5.24)$$

Application of Hamilton's principle, see Meirovitch [14], yields the equations of motion and boundary conditions. These are

$$\vec{e}_x: \quad [m_0 + M\delta(z - \frac{1}{2} L)](\ddot{x}(z,t) - 2\Omega\dot{y}(z,t) - \Omega^2 x(z,t))$$

$$+ (c+h)\dot{x}(z,t) + EI_y \, x''''(z,t) - h\Omega y(z,t) = F_x(z,t) \quad (5.25a)$$

$$x(0,t) = 0 \quad , \quad EI_y \, x''(0,t) - K_1 \, x'(0,t) = 0 \quad (5.25b,c)$$

$$x(L,t) = 0 \quad , \quad EI_y \, x''(L,t) + K_2 \, x'(L,t) = 0 \quad (5.25d,e)$$

$$\vec{e}_y: \quad [m_0 + M\delta(z - \frac{1}{2} L)](\ddot{y}(z,t) + 2\Omega\dot{x}(z,t) - \Omega^2 y(z,t))$$

$$+ (c+h)\dot{y}(z,t) + EI_x \, y''''(z,t) + h\Omega x(z,t) = F_y(z,t) \quad (5.26a)$$

$$y(0,t) = 0 \quad , \quad EI_x \, y''(0,t) - K_1 \, y'(0,t) = 0 \quad (5.26b,c)$$

$$y(L,t) = 0 \quad , \quad EI_x \, y''(L,t) + K_2 \, y'(L,t) = 0 \quad (5.26d,e)$$

where F_x and F_y are the components of the nonconservative forces, resolved along the local directions $\vec{e}_x$ and $\vec{e}_y$, respectively.

We wish to discretize Eqs. (5.25a) and (5.26a), which is to say that we wish to eliminate the spatial dependence. The interested reader may wish to consult Meirovitch [13,14] for a discussion of various discretization procedures. In the following, we shall use Galerkin's method with admissible functions, and integrate by parts to include the natural boundary conditions. Admissible functions, suitable for both the $\vec{e}_x$ and $\vec{e}_y$ motions, can be generated from the simpler problem of a uniform, simply supported, nonrotating beam. These functions are

$$\phi_j(z) = \sqrt{2}\,\sin \frac{j\pi}{L} z \quad , \qquad j = 1,2,\ldots \tag{5.27}$$

Let us take

$$x(z,t) = \sum_{j=1}^{p} \phi_j(z)\, q_j(t) \tag{5.28a}$$

$$y(z,t) = \sum_{j=p+1}^{2p} \phi_j(z)\, q_j(t) \tag{5.28b}$$

where $\phi_j(z) = \phi_{j+p}(z)$ $(j = 1,2,\ldots,p)$. We substitute these expansions into Eqs. (5.25a) and (5.26a), then multiply Eq. (5.25a) by $\phi_i(z)$, $(i = 1,2,\ldots,p)$ and multiply Eq. (5.26a) by $\phi_i(z)$, $(i = p+1,\ldots,2p)$. Upon integration with respect to z from z = 0 to z = L, the resulting ordinary differential equations of motion can be written in the matrix form

$$M\, \ddot{\underset{\sim}{q}}(t) + (G + C)\, \dot{\underset{\sim}{q}}(t) + (K + H)\, \dot{\underset{\sim}{q}}(t) = \underset{\sim}{Q}(t) \tag{5.29}$$

which is identical to Eq. (3.1) if we take 2p = n. The $n \times n$ coefficient matrices in Eq. (5.29) are partitioned into $p \times p$ submatrices

$$M = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix} \tag{5.30a}$$

$$G = \begin{bmatrix} 0 & G_{12} \\ -G_{12} & 0 \end{bmatrix} \quad , \qquad C = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix} \tag{5.30b,c}$$

$$K = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix} \quad , \quad H = \begin{bmatrix} 0 & H_{12} \\ -H_{12} & 0 \end{bmatrix} \tag{5.30d,e}$$

the elements of which are given as

$$[M_{11}]_{ij} = [M_{22}]_{ij} \equiv \int_{z=0}^{z=L} \phi_i(z)[m_0 + M\delta(z - \tfrac{1}{2} L)]\phi_j(z)dz \quad ,$$

$$i,j = 1,2,\ldots,p$$

$$= m_0 \, L \, \delta_{ij} + 2M \sin \frac{i\pi}{2} \sin \frac{j\pi}{2} \quad , \tag{5.31a}$$

$$i,j = 1,2,\ldots,p$$

$$[G_{12}]_{ij} \equiv \int_{z=0}^{z=L} \phi_i(z)[m_0 + M\delta(z - \tfrac{1}{2} L)](-2\Omega)\phi_j(z)dz \quad ,$$

$$i,j = 1,2,\ldots,p$$

$$= - 2\Omega \, [M_{11}]_{ij} \quad , \qquad i,j = 1,2,\ldots,p \tag{5.31b}$$

$$[C_{11}]_{ij} = [C_{22}]_{ij} \equiv \int_{z=0}^{z=L} \phi_i(z)(c+h)\phi_j(z)dz \quad ,$$

$$i,j = 1,2,\ldots,p$$

$$= (c+h) \, L \, \delta_{ij} \quad , \qquad i,j = 1,2,\ldots,p \tag{5.31c}$$

$$[K_{11}]_{ij} \equiv \int_{z=0}^{z=L} \phi_i(z)\{EI_y \, \phi_j''''(z) - \Omega^2[m_0 + M\delta(z - \tfrac{1}{2} L)]\phi_j(z)\}dz$$

$$= \left(\phi_i(z) \, EI_y \, \phi_j'''(z)\right|_{z=0}^{z=L} - \left(\phi_i'(z) \, EI_x \, \phi_j''(z)\right|_{z=0}^{z=L}$$

$$+ \int_{z=0}^{z=L} \{\phi_i''(z) \, EI_y\phi_j''(z)$$

$$- \phi_i(z) \, \Omega^2[m_0 + M\delta(z - \tfrac{1}{2} L)]\phi_j(z)\}dz$$

$$[K_{11}]_{ij} = \phi_i'(L)\ K_2\phi_j'(L) + \phi_i'(0)\ K_1\phi_j'(0)$$

$$+ \int_{z=0}^{z=L} \{\phi_i''(z)\ EI_y\phi_j''(z)$$

$$- \phi_i(z)\ \Omega^2[m_0 + M\delta(z - \tfrac{1}{2}L)]\phi_j(z)\}dz$$

$$= 2(K_2 \cos i\pi \cos j\pi + K_1)\ \frac{i\pi}{L}\frac{j\pi}{L}$$

$$+ EI_y\left(\frac{i\pi}{L}\right)^2\left(\frac{j\pi}{L}\right)^2 L\ \delta_{ij} - \Omega^2[M_{11}]_{ij} \quad , \qquad (5.31d)$$

$$i,j = 1,2,\ldots,p$$

Similarly

$$[K_{22}]_{ij} = 2(K_2 \cos i\pi \cos j\pi + K_1)\ \frac{i\pi}{L}\frac{j\pi}{L}$$

$$+ EI_x\left(\frac{i\pi}{L}\right)^2\left(\frac{j\pi}{L}\right)^2 L\ \delta_{ij} - \Omega^2[M_{11}]_{ij} \quad , \qquad (5.31e)$$

$$i,j = 1,2,\ldots,p$$

$$[H_{12}]_{ij} \equiv \int_{z=0}^{z=L} \phi_i(z)\ (-h\Omega)\ \phi_j(z)dz \quad , \quad i,j = 1,2,\ldots,p$$

$$= -\,h\Omega\, L\ \delta_{ij} \quad , \quad i,j = 1,2,\ldots,p \qquad (5.31f)$$

The vector $\underset{\sim}{Q}(t)$ of nonconservative forces is also partitioned. Its elements are given by

$$Q_j(t) = \begin{cases} \displaystyle\int_{z=0}^{z=L} \phi_j(z)\ F_x(z,t)\ dz \quad , \quad j = 1,2,\ldots,p \\ \displaystyle\int_{z=0}^{z=L} \phi_j(z)\ F_y(z,t)\ dz \quad , \quad j = p+1,\ldots,n \end{cases} \qquad (5.32)$$

The eigenvalue problem associated with Eq. (5.29) is

$$\lambda_i^2\ M\ \underset{\sim}{q}_i + \lambda_i(G + C)\ \underset{\sim}{q}_i + (K + H)\ \underset{\sim}{q}_i = \underset{\sim}{0} \quad , \quad i = 1,2,\ldots,2n \qquad (5.33)$$

In the sums of Eqs. (5.28), let us take p = 3, so that the submatrices M_{11}, G_{12}, C_{11}, K_{11}, K_{22} and H_{12} have dimensions 3 × 3. Matrices M, G, C, K and H have dimensions 6 × 6, so that the equations of motion and the eigenvalue problem have dimensions 12 × 12 when expressed in state form. Let us choose the numerical values

$$m_0 = 1 \text{ kg/m} \tag{5.34a}$$

$$M = 1 \text{ kg} \tag{5.34b}$$

$$L = 1 \text{ m} \tag{5.34c}$$

$$EI_y = 4L^3/5\pi^2 = 0.081,057 \text{ n m}^2 \tag{5.34d}$$

$$EI_x = 9L^3/5\pi^2 = 0.182,378 \text{ n m}^2 \tag{5.34e}$$

$$K_1 = K_2 = L^2/20 = 0.05 \text{ n m} \tag{5.34f}$$

$$c = h = 0.25 \text{ n sec/m} \tag{5.34g}$$

For the first example, let us take

$$\Omega = \pi/2 = 1.570,796 \text{ rad/sec} \tag{5.35}$$

The coefficient matrices are

$$M_{11} = M_{22} = \begin{bmatrix} 3.0 & 0 & -2.0 \\ 0 & 1.0 & 0 \\ -2.0 & 0 & 3.0 \end{bmatrix} \tag{5.36a}$$

$$G_{12} = \begin{bmatrix} -9.425 & 0 & 6.283 \\ 0 & -3.142 & 0 \\ 6.283 & 0 & -9.425 \end{bmatrix} \tag{5.36b}$$

$$C_{11} = C_{22} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \tag{5.36c}$$

$$K_{11} = \begin{bmatrix} 2.467 & 0 & 10.857 \\ 0 & 131.759 & 0 \\ 10.857 & 0 & 649.913 \end{bmatrix} \tag{5.36d}$$

$$K_{22} = \begin{bmatrix} 12.337 & 0 & 10.857 \\ 0 & 289.673 & 0 \\ 10.857 & 0 & 1449.351 \end{bmatrix} \tag{5.36e}$$

$$H_{12} = \begin{bmatrix} -0.393 & 0 & 0 \\ 0 & -0.393 & 0 \\ 0 & 0 & -0.393 \end{bmatrix} \tag{5.36f}$$

Following the developments of Chapters 3 and 4, let us regard matrices C and H as perturbations. In reducing the eigenvalue problem to quasi-standard form, we find that the 6 × 6 matrix D_2 is the identity matrix, which is an indication that K is positive definite. Upon solving the unperturbed eigenvalue problem, we find that all of the unperturbed eigenvalues are imaginary and clearly distinct from each other. All of the corresponding unperturbed eigenvectors satisfy plus one normalizations.

The perturbation analysis described in Chapter 2 is applicable to the problem presently under consideration. In the following tables and figures, we list a few pertinent results from that analysis. In Table 5.1 we present the first six eigenvalues summarized through various orders, along with exact eigenvalues for comparison. Because the eigenvalues indexed 7 through 12 are the complex conjugates of the eigenvalues indexed 1 through 6, we do not list them. Recalling the properties listed in Eqs. (5.15), we observe that the eigenvalue perturbations λ_{1i} and λ_{3i} $(i = 1,2,\ldots,12)$ are indeed real, while the eigenvalue perturbations λ_{2i} $(i = 1,2,\ldots,12)$ are imaginary. Because the real parts of all the eigenvalues are negative, we conclude that the system under consideration is stable. In particular, because λ_{1i} $(i = 1,2,\ldots,12)$ is not only real, but negative, and because $\Delta_{ii} = +1$ $(i = 1,2,\ldots,12)$, we observe that all the λ_{1i} are in the Case i category, described in Eq. (5.19a). In Table 5.2 we present the eigenvector $\underset{\sim}{u}_1$ summarized through various orders, along with the exact eigenvector for comparison. A more meaningful measure of eigenvector accuracy is available in the form of a biorthonormality check in which we inspect the products $Z^T(A_0 + A_1)W$ and $Z^T(B_0 + B_1)W$. In place of the trial vectors W and Z we use the respective right and left eigenvectors, summarized through various orders. Note that if exact eigenvectors are used in place of the trial vectors W and Z, the product $Z^T(B_0 + B_1)W$ is the diagonal matrix of modified Kronecker delta's. In Table 5.3 we present the upper left 3×3 portion of the matrix $Z^T(B_0 + B_1)W$ as representative of the entire 12×12 matrix. We observe that by considering only the upper left 3×3 portion, we restrict our attention to the eigenvectors indexed 1, 2 and 3. Note that three generally complex values are presented at each location in this table. The uppermost is the result of using the unperturbed eigenvectors $\underset{\sim}{u}_{0i}$ and $\underset{\sim}{v}_{0i}$ $(i = 1,2,3)$ as trial vectors, while the second and third values were computed using the perturbed eigenvectors summarized through $O(1)$

TABLE 5.1

Eigenvalue Summary, $c = h = 1/4$, $\Omega = \pi/2$

	0(0)	0(0) + 0(1)	0(0) + ... + 0(2)	0(0) + ... + 0(3)	Exact
λ_1	i0.455,727	-0.053,213 +i0.455,727	-0.053,213 +i0.453,851	-0.053,198 +i0.453,851	-0.053,198 +i0.453,849
λ_2	i11.150,651	-0.243,245 +i11.150,651	-0.243,245 +i11.148,030	-0.243,244 +i11.148,030	-0.243,244 +i11.148,030
λ_3	i19.786,046	-0.216,605 +i19.786,046	-0.216,605 +i19.784,345	-0.216,601 +i19.784,345	-0.216,601 +i19.784,345
λ_4	i3.799,358	-0.109,946 +i3.799,358	-0.109,946 +i3.798,118	-0.109,967 +i3.798,118	-0.109,967 +i3.798,118
λ_5	i17.520,408	-0.256,755 +i17.520,408	-0.256,755 +i17.518,511	-0.256,756 +i17.518,511	-0.256,756 +i17.518,511
λ_6	i29.990,346	-0.220,236 +i29.990,346	-0.220,236 +i29.989,158	-0.220,234 +i29.989,158	-0.220,234 +i29.989,158

TABLE 5.2

Eigenvector Summary for $\underset{\sim}{u}_1$,

$c = h = 1/4, \ \Omega = \pi/2$

O(0)	O(0) + O(1)	O(0) + ... + O(2)	Exact
0.0 +i0.366,442	-0.021,980 +i0.366,442	-0.021,980 +i0.366,909	-0.022,019 +i0.336,911
0.0 +i0.0	0.0 +i0.0	0.0 +i0.0	0.0 +i0.0
0.0 -i0.005,138	0.000,409 -i0.005,138	0.000,409 -i0.005,131	0.000,409 -i0.005,131
0.137,286 +i0.0	0.137,286 +i0.011,038	0.136,892 +i0.011,038	0.136,891 +i0.011,036
0.0 +i0.0	0.0 +i0.0	0.0 +i0.0	0.0 +i0.0
-0.001,338 +i0.0	-0.001,338 -i0.000,166	-0.001,329 -i0.000,166	-0.001,329 -i0.000,165
0.659,713 +i0.0	0.659,713 -i0.038,819	0.658,925 -i0.038,819	0.658,933 -i0.038,788
0.0 +i0.0	0.0 +i0.0	0.0 +i0.0	0.0 +i0.0
-0.214,311 +i0.0	-0.214,311 +i0.007,979	-0.213,957 +i0.007,979	-0.213,959 +i0.007,971
0.0 -i0.598,542	-0.022,303 -i0.598,542	-0.022,303 -i0.596,732	-0.022,171 -i0.596,737
0.0 +i0.0	0.0 +i0.0	0.0 +i0.0	0.0 +i0.0
0.0 +i0.086,263	-0.000,605 +i0.086,263	-0.000,605 +i0.086,126	-0.000,618 +i0.086,127

TABLE 5.3

Biorthonormality Check of First Three Eigenvectors,

$c = h = 1/4, \ \Omega = \pi/2$

1.0 + i0.062,089	0.0 + i0.0	0.0 + i0.001,627
1.003,148 - i0.000,130	0.0 + i0.0	-0.000,055 + i0.0
0.999,991 - i0.000,234	0.0 + i0.0	0.0 - i0.000,003
0.0 + i0.0	1.0 + i0.000,606	0.0 + i0.0
0.0 + i0.0	0.999,888 + i0.0	0.0 + i0.0
0.0 + i0.0	1.0 + i0.0	0.0 + i0.0
0.0 + i0.001,627	0.0 + i0.0	1.0 + i0.000,107
-0.000,055 + i0.0	0.0 + i0.0	0.999,944 + i0.0
0.0 - i0.000,003	0.0 + i0.0	1.0 + i0.0

and O(2), respectively. In summary, it appears that the perturbation theory of Chapter 2 has produced reasonably accurate representations of the eigenvalues and eigenvectors.

For a second example, let us retain the values listed in Eqs. (5.34), and set

$$\Omega = \sqrt{21.6}\ \pi = 14.600{,}803 \text{ rad/sec.} \tag{5.37}$$

The coefficient matrices $M_{11} = M_{22}$ and $C_{11} = C_{22}$ remain as given in Eqs. (5.36a) and (5.36c), respectively, while the other coefficient matrices become

$$G_{12} = \begin{bmatrix} -87.605 & 0.0 & 58.403 \\ 0.0 & -29.202 & 0.0 \\ 58.403 & 0.0 & -87.605 \end{bmatrix} \tag{5.38a}$$

$$K_{11} = \begin{bmatrix} -629.681 & 0.0 & 432.289 \\ 0.0 & -78.957 & 0.0 \\ 432.289 & 0.0 & 17.765 \end{bmatrix}$$

$$K_{22} = \begin{bmatrix} -619.811 & 0.0 & 432.289 \\ 0.0 & 78.957 & 0.0 \\ 432.289 & 0.0 & 817.203 \end{bmatrix} \tag{5.38c}$$

$$H_{12} = \begin{bmatrix} -3.650 & 0.0 & 0.0 \\ 0.0 & -3.650 & 0.0 \\ 0.0 & 0.0 & -3.650 \end{bmatrix} \tag{5.38d}$$

In reducing the eigenvalue problem to quasi standard form, we find that

$$D_2 = \begin{bmatrix} -1 & & & & & \\ & -1 & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & +1 & \\ & & & & & +1 \end{bmatrix} \tag{5.39}$$

The unperturbed eigenvalues λ_{02} and λ_{08} are real, while the remaining unperturbed eigenvalues are imaginary. The unperturbed eigenvectors indexed 1, 2, 7 and 8 satisfy -1 normalizations, while the remaining unperturbed eigenvectors satisfy +1 normalizations. Because the unperturbed eigenvalues are clearly distinct, the perturbation analysis of Chapter 2 is once again applicable. The results of this analysis are presented in Tables 5.4 - 5.6.

Let us inspect Table 5.4. Note that the eigenvalues indexed 7, 9, 10, 11 and 12 are the complex conjugates of the eigenvalues indexed 1, 3, 4, 5 and 6, respectively, thus, they are not listed. We also note that the eigenvalues indexed 2 and 8 are real, and since λ_2 is positive, we conclude that the system is unstable. We observe that the O(1) eigenvalue perturbations λ_{12} and λ_{18} are negative. Then since $\Delta_{22} = \Delta_{88} = -1$, these perturbations are examples of Case iv, as in Eq. (5.19d). Alternatively, the O(1) eigenvalue perturbations λ_{11} and λ_{17} are positive. Then, because $\Delta_{11} = \Delta_{77} = -1$, these perturbations are examples of Case iii, as in Eq. (5.19c). The remaining O(1) eigenvalue perturbations are negative, and since the corresponding unperturbed eigenvectors satisfy plus one normalizations, they are examples of Case i, as in Eq. (5.19a). Thus, the system is unstable due to one eigenvalue that is real and positive, and due to two eigenvalues with positive real parts. From Table 5.5 we observe that the eigenvector $\underset{\sim}{u}_2$ is real, as was predicted earlier. Table 5.6 presents a biorthonormality check, following the same format as Table 5.3. Once again, it appears that the perturbation theory has produced reasonably accurate representations of the system's eigenvalues and eigenvectors, although not as accurate as in the first example.

TABLE 5.4

Eigenvalue Summary, $c = h = 1/4$, $\Omega = \sqrt{21.6}\,\pi$

	0(0)	0(0) + 0(1)	0(0) + ... + 0(2)	0(0) + ... + 0(3)	Exact
λ_1	i12.394,400	0.189,662 +i12.394,400	0.189,662 +i12.379,192	0.187,768 +i12.379,192	0.187,815 +i12.379,469
λ_2	2.692,432	2.565,343	2.565,395	2.565,409	2.565,408
λ_3	i8.799,942	-0.158,855 +i8.799,942	-0.158,855 +i8.798,580	-0.158,846 +i8.798,580	-0.158,846 +i8.798,580
λ_4	i16.805,929	-0.352,790 +i16.805,929	-0.352,790 +i16.821,616	-0.350,904 +i16.821,616	-0.350,951 +i16.821,338
λ_5	i29.325,467	-0.372,910 +i29.325,467	-0.372,910 +i29.323,856	-0.372,924 +i29.323,856	-0.372,924 +i29.323,856
λ_6	i40.434,096	-0.278,016 +i40.434,096	-0.278,016 +i40.432,571	-0.278,018 +i40.432,571	-0.278,018 +i40.432,571
λ_8	-2.692,432	-2.819,522	-2.819,575	-2.819,561	-2.819,561

TABLE 5.5

Eigenvector Summary for $\underset{\sim}{u}_2$,

$c = h = 1/4,\ \Omega = \sqrt{21.6}\ \pi$

O(0)	O(0) + O(1)	O(0) + ... + O(2)	Exact
0.0	0.0	0.0	0.0
0.222,939	0.218,363	0.218,226	0.218,227
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
-0.203,329	-0.197,437	-0.197,476	-0.197,474
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.735,760	0.755,388	0.755,848	0.755,869
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
-0.671,043	-0.683,271	-0.683,963	-0.683,986
0.0	0.0	0.0	0.0

TABLE 5.6

Biorthonormality Check of

First Three Eigenvectors,

$c = h = 1/4$, $\Omega = \sqrt{21.6}\,\pi$

-1.0 - i0.021,920 -1.004,805 + i0.000,064 -1.000,059 + i0.001,044	0.0 + i0.0 0.0 + i0.0 + i0.0	0.0 + i0.003,096 0.000,297 + i0.0 0.000,002 - i0.000,036
0.0 + i0.0 0.0 + i0.0 0.0 + i0.0	-0.954,350 + i0.0 -0.998,449 + i0.0 -0.999,938 + i0.0	0.0 + i0.0 0.0 + i0.0 0.0 + i0.0
0.0 + i0.003,096 0.000,297 + i0.0 0.000,002 - i0.000,036	0.0 + i0.0 0.0 + i0.0 0.0 + i0.0	1.0 + i0.006,705 1.000,030 + i0.0 1.0 + i0.000,003

Chapter 6

PERTURBATION THEORY FOR MULTIPLE UNPERTURBED EIGENVALUES

The assumption in Chapter 2 regarding clearly distinct O(0) eigenvalues is indeed necessary for the validity of the expressions for ϵ_{1ij}, γ_{1ij}, ϵ_{2ij} and γ_{2ij}, $i \neq j$. We also note that the expressions for ϵ_{1ii}, γ_{1ii}, ϵ_{2ii} and γ_{2ii} and for λ_{1i}, λ_{2i} and λ_{3i} do not involve division by $(\lambda_{0i} - \lambda_{0j})$, $i \neq j$, and hence are not directly affected if some of the O(0) eigenvalues are not clearly distinct.

As an example of the difficulties caused by the O(0) eigenvalues not being clearly distinct and to motivate the following manipulations, let us inspect the expression for ϵ_{1ij} given in Eq. (2.27a). We suppose that m of the O(0) eigenvalues are close to each other and that the remaining n-m O(0) eigenvalues are clearly distinct. Without loss of generality, let us say that the O(0) eigenvalues that are not clearly distinct are indexed 1, 2,...,m. Thus

$$(\lambda_{0i} - \lambda_{0j}) \leqslant O(1) \quad , \quad i,j = 1,2,\ldots,m$$

and

$$(\lambda_{0i} - \lambda_{0j}) = O(0)$$

when i and j are not simultaneously in the range (1,2,...,m). We have

$$\epsilon_{1ij} \leqslant O(0) \quad , \quad i,j = 1,2,\ldots,m$$

which is contrary to our assumption that ϵ_{1ij} $(i,j = 1,2,\ldots,n)$ be an O(1) quantity. Note that when i and j are not simultaneously in the range (1,2,...,m), the expression for ϵ_{1ij} remains valid, as do those for γ_{1ij}, ϵ_{2ij} and γ_{2ij}. Thus, perturbations to the remaining n-m eigenvectors can be constructed as in Chapter 2.

The message here is that eigenvector perturbations at one order of magnitude larger than O(1), namely O(0), must be allowed. However, these perturbations need affect only the first m eigenvectors.

In the present chapter we address the extreme of not clearly distinct O(0) eigenvalues. We assume the O(0) eigensolution to have multiplicity m, which is to say that m of the O(0) eigenvalues are equal, and that the remaining n-m O(0) eigenvalues are clearly distinct. We have

$$(\lambda_{0i} - \lambda_{0j}) = 0 \quad , \quad i,j = 1,2,\ldots,m \tag{6.1}$$

and

$$(\lambda_{0i} - \lambda_{0j}) = O(0)$$

when i and j are not simultaneously in the range 1,2,...,m. In the following chapter we generalize to include nearly multiple O(0) eigenvectors.

Let us return to Eqs. (2.10) and modify them to read as

$$U = U_0 \, \mathcal{E} \, \mathrm{E} \tag{6.2a}$$

$$V = V_0 \, \mathcal{G} \, \Gamma \tag{6.2b}$$

where matrices $\mathcal{E}$ and $\mathcal{G}$ are the respective O(0) right and left eigenvector perturbations. The n × n matrix $\mathcal{E}$ has the partitioned form

$$E = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1m} & 0 & \cdots & 0 \\ e_{21} & e_{22} & \cdots & e_{2m} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ e_{m1} & e_{m2} & \cdots & e_{mm} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}$$

which is to say that, excepting the upper left $m \times m$ portion, matrix E is the identity matrix. Matrix G has the same form. Note that we have specifically constructed matrices E and G so that only the first m of the 0(0) eigenvectors are affected.

Substituting for U and V from Eqs. (6.2) into Eqs. (2.7), we obtain

$$\Gamma^T G^T V_0^T (B_0 + B_1) U_0 E E = \Delta \tag{6.3a}$$

$$\Gamma^T G^T V_0^T (A_0 + A_1) U_0 E E = \Lambda\Delta \tag{6.3b}$$

Use of Eqs. (2.9) and (2.16) allows Eqs. (6.3) to be written as

$$\Gamma^T(G^T\Delta E + G^T\hat{B}E)E = \Delta \tag{6.4a}$$

$$\Gamma^T(G^T\Lambda_0\Delta E + G^T\hat{A}E) E = \Lambda\Delta \tag{6.4b}$$

We substitute the expansions of Eqs. (2.17) and (2.18) into Eqs. (6.4) and separate according to order. We obtain

$$0(0): \quad G^T\Delta E = \Delta \tag{6.5a}$$

$$G^T\Lambda_0\Delta E = \Lambda_0\Delta \tag{6.5b}$$

$$0(1): \quad G^T\Delta E E_1 + G^T\hat{B}E + \Gamma_1^T G^T\Delta E = 0 \tag{6.6a}$$

$$G^T\Lambda_0\Delta E E_1 + G^T\hat{A}E + \Gamma_1^T G^T\Lambda_0\Delta E = \Lambda_1\Delta \tag{6.6b}$$

The use of Eqs. (6.5) allows Eqs. (6.6) to be written as

$$\Delta E_1 + \Gamma_1^T \Delta = - G^T \hat{B} E \qquad (6.7a)$$

$$\Lambda_0 \Delta E_1 + \Gamma_1^T \Lambda_0 \Delta = - G^T \hat{A} E + \Lambda_1 \Delta \qquad (6.7b)$$

Let us restrict attention to the upper left $m \times m$ portion of these matrix equations. Subject to this restriction, it is necessary to consider only the upper left $m \times m$ portion of Λ_0. Then since the first m $O(0)$ eigenvalues are the same, we can write Eqs. (6.5) and (6.7) as

$$O(0): \quad G^T \Delta E = \Delta \qquad (6.8a)$$

$$\Lambda_0 G^T \Delta E = \Lambda_0 \Delta \qquad (6.8b)$$

$$O(1): \quad \Delta E_1 + \Gamma_1^T \Delta = - G^T \hat{B} E \qquad (6.9a)$$

$$\Lambda_0 (\Delta E_1 + \Gamma_1^T \Delta) = - G^T \hat{A} E + \Lambda_1 \Delta \qquad (6.9b)$$

Note that Eqs. (6.8a) and (6.8b) are the same. Regarding Eqs. (6.9) we observe that due to the multiplicity of the $O(0)$ eigenvalues, we cannot solve for the elements of matrices E_1 and Γ_1 as in Chapter 2. However, let us combine Eqs. (6.9) so as to eliminate the common quantity $(\Delta E_1 + \Gamma_1^T \Delta)$. We obtain

$$G^T (\hat{A} - \hat{B} \Lambda_0) E = \Lambda_1 \Delta \qquad (6.10)$$

Inspection reveals that Eqs. (6.8a) and (6.10) are in fact the biorthonormality relations for the $m \times m$ right and left eigenvalue problems

$$(\hat{A} - \hat{B} \Lambda_0) \underset{\sim}{e}_i = \lambda_{1i} \, \Delta \, \underset{\sim}{e}_i \quad , \qquad i = 1,2,\ldots,m \qquad (6.11a)$$

$$(\hat{A}^T - \Lambda_0 \hat{B}^T) \underset{\sim}{g}_i = \lambda_{1i} \, \Delta \, \underset{\sim}{g}_i \quad , \qquad i = 1,2,\ldots,m \qquad (6.11b)$$

where the m × 1 vectors $\underset{\sim}{e}_i$ and $\underset{\sim}{g}_i$ (i = 1,2,...,m) are the ith columns of matrices E and G, respectively. Solution of this eigenvalue problem yields the respective 0(0) eigenvector perturbations, E and G. As a bonus, we obtain the 0(1) eigenvalue perturbations λ_{1i} (i = 1,2,...,m). Let us assume that λ_{1i} (i = 1,2,...,m) to be clearly distinct, that is

$$(\lambda_{1i} - \lambda_{1j}) = 0(1) \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,m$$

As we shall see, this assumption is made for the same reasons that the 0(0) eigenvalues are assumed clearly distinct in Chapter 2.

At this stage we are faced with a choice. We can either continue the analysis as begun above, or absorb the 0(0) eigenvector perturbations E and G, and begin anew. Having investigated both alternatives, we choose the latter. Let us redefine

$$U_0 E \rightarrow U_0 \quad \text{and} \quad V_0 G \rightarrow V_0$$

where the redefined U_0 and V_0 are referred to as "0(0) perturbed" eigenvectors. Conversely, eigenvectors U_0 and V_0 that do not include 0(0) perturbations are referred to as "0(0) unperturbed" eigenvectors. Note that the above redefinition is reflected in the perturbing matrices $\hat{A}$ and $\hat{B}$. In particular, Eq. (6.10) now appears as

$$\hat{A} - \Lambda_0 \hat{B} = \Lambda_1 \Delta \tag{6.12}$$

which is to say that the off-diagonal elements in the upper left m × m portion of the matrix $(\hat{A} - \Lambda_0 \hat{B})$ are zero. The diagonal elements of Eq. (6.12) yield

$$\lambda_{1i} = \Delta_{ii}(\hat{a}_{ii} - \lambda_{0i}\, \hat{b}_{ii}) \quad , \quad i = 1,2,\ldots,n \tag{6.13}$$

which are identical to those given in Eq. (2.28).

We wish to determine $O(1)$ eigenvector perturbations. The expressions given in Eqs. (2.27) for ϵ_{1ij} and γ_{1ij} are valid when i and j are not simultaneously in the range $(1,2,\ldots,m)$. However, when $i,j = 1,2,\ldots,m$, Eq. (6.12) indicates that the expressions given for ϵ_{1ij} and γ_{1ij}, $i \neq j$, in Eqs. (2.27a) and (2.27b) are indeterminate. This indeterminacy is of course a direct consequence of using $O(0)$ perturbed eigenvectors. To determine ϵ_{1ij} and γ_{1ij} $(i,j = 1,2,\ldots,m)$ we address Eqs. (2.24) and combine them so as to eliminate the common quantity $(\Delta E_2 + \Gamma_2^T \Delta)$. The result is

$$\Lambda_1 \Delta E_1 + \Gamma_1^T \Lambda_1 \Delta = - \hat{\hat{A}} + \Lambda_2 \Delta \tag{6.14}$$

where

$$\begin{aligned} \hat{\hat{A}} &= - \Gamma_1^T \Lambda_0 \Delta E_1 + \Lambda_0 \Gamma_1^T \Delta E_1 \\ &= - \Gamma_1^T \Lambda_0 \Delta E_1 + \Gamma_1^T \Delta E_1 \Lambda_0 \end{aligned} \tag{6.15}$$

For clarity, the (i,j) element of $\hat{\hat{A}}$ can be written as

$$\hat{\hat{a}}_{ij} = - \sum_{p=m+1}^{n} \gamma_{1pi}(\lambda_{0p} - \lambda_{0i})\Delta_{pp}\, \epsilon_{1pj} \quad , \quad i,j = 1,2,\ldots,n \tag{6.16}$$

Note that we need consider only the upper left $m \times m$ portion of matrix $\hat{\hat{A}}$. We now solve Eqs. (2.20a) and (6.14) for ϵ_{1ij} and γ_{1ij} $(i,j = 1,2,\ldots,m)$ as in Chapter 2. As a bonus, we obtain an expression for λ_{2i} $(i = 1,2,\ldots,m)$. The results are

$$\epsilon_{1ij} = - \frac{\Delta_{ii}(\hat{\hat{a}}_{ij} - \hat{b}_{ij}\,\lambda_{1j})}{\lambda_{1i} - \lambda_{1j}} \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,m \tag{6.17a}$$

$$\epsilon_{1ii} = \gamma_{1ii} = -\frac{1}{2}\Delta_{ii}\,\hat{b}_{ii} \quad , \quad i = j \quad , \quad i = 1,2,\ldots,m \qquad (6.17b)$$

$$\gamma_{1ij} = -\frac{\Delta_{ii}(\hat{\hat{a}}_{ji} - \lambda_{1j}\,\hat{b}_{ji})}{\lambda_{1i} - \lambda_{1j}} \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,m \qquad (6.17c)$$

$$\lambda_{2i} = \Delta_{ii}(\hat{\hat{a}}_{ii} - \lambda_{1i}\,\hat{b}_{ii}) \quad , \quad i = 1,2,\ldots,m \qquad (6.18)$$

Thus the 0(1) eigenvector perturbations are fully determined. The reason for our assumption of clearly distinct 0(1) eigenvalue perturbations, λ_{1i} $(i = 1,2,\ldots,m)$, should be evident from Eqs. (6.17a) and (6.17c). We note that the expressions for ϵ_{1ii} and γ_{1ii} given in Eqs. (2.27b) and (6.17b) are indeed the same. Furthermore, if we substitute for $\hat{\hat{a}}_{ii}$ and $\hat{b}_{ii}$ into Eq. (6.18), we obtain the same expression for λ_{2i} as that given in Eq. (2.30).

Let us pursue 0(2) eigenvector perturbations. We note that the expressions for ϵ_{2ij} and γ_{2ij}, given in Eqs. (2.29) are valid whenever i and j are not simultaneously in the range 1,2,...,m. To determine ϵ_{2ij} and γ_{2ij} $(i,j = 1,2,\ldots,m)$ we address the 0(3) biorthonormality relations, Eqs. (2.25), and combine them so as to eliminate the common quantity $(\Delta E_3 + \Gamma_3^T\Delta)$. We obtain

$$\Lambda_1\Delta E_2 + \Gamma_2^T\Lambda_1\Delta = -\Gamma_1^T(\hat{A} - \Lambda_0\hat{B})E_1 + \Lambda_3\Delta \qquad (6.19)$$

For clarity, it is best to work with the (i,j), $(i,j = 1,2,\ldots,m)$ element of the first matrix on the right side of Eq. (6.19). Substituting for $\hat{A}$ and $\hat{B}$ from Eqs. (2.23), we obtain

$$- \sum_{p,q=1}^{n} \gamma_{1pi}(\hat{a}_{pq} - \lambda_{0i}\,\hat{b}_{pq})\epsilon_{1qj}$$

$$= \sum_{q=1}^{n} \left[\sum_{p=m+1}^{n} \gamma_{1pi}(\lambda_{0p} - \lambda_{0i})\Delta_{pp}\,\epsilon_{1pq} \right] \epsilon_{1qj}$$

$$- \sum_{p=1}^{n} \gamma_{1pi}\,\lambda_{1p}\Delta_{pp}\,\epsilon_{1pj}$$

$$+ \sum_{p=1}^{n} \gamma_{1pi} \left[\sum_{q=m+1}^{n} \gamma_{1qp}(\lambda_{0q} - \lambda_{0i})\Delta_{qq}\,\epsilon_{1qj} \right] ,$$

$$i,j = 1,2,\ldots,m \qquad (6.20)$$

From Eq. (6.16), we recognize the defined quantities $\hat{\hat{a}}_{ip}$ and $\hat{\hat{a}}_{pj}$. Thus,

$$- \sum_{p,q=1}^{n} \gamma_{1pi}(\hat{a}_{pq} - \lambda_{0i}\,\hat{b}_{pq})\epsilon_{1qj}$$

$$= - \sum_{p=1}^{n} (\hat{\hat{a}}_{ip}\,\epsilon_{1pj} + \gamma_{1pi}\,\lambda_{1p}\,\Delta_{pp}\,\epsilon_{1pj} + \gamma_{1pi}\,\hat{\hat{a}}_{pj}) \quad ,$$

$$i,j = 1,2,\ldots,m\ . \qquad (6.21)$$

and Eq. (6.19) can be expressed as

$$\Lambda_1\Delta E_2 + \Gamma_2^T\Lambda_1\Delta = - (\hat{\hat{A}}E_1 + \Gamma_1^T\Lambda_1\Delta E_1 + \Gamma_1^T\hat{\hat{A}}) + \Lambda_3\Delta \qquad (6.22)$$

Finally, we substitute for $\hat{\hat{A}}$ from Eq. (6.14) to obtain

$$\Lambda_1\Delta E_2 + \Gamma_2^T\Lambda_1\Delta = \Delta_1\Delta E_1 E_1 + \Gamma_1^T\Lambda_1\Delta E_1 + \Gamma_1^T\Gamma_1^T\Lambda_1\Delta$$

$$- (\Lambda_2\Delta E_1 + \Gamma_1^T\Lambda_2\Delta) + \Lambda_3\Delta \qquad (6.23)$$

Equations (2.24a) and (6.23) can now be solved for ϵ_{2ij}, γ_{2ij} and λ_{3i} $(i,j = 1,2,\ldots,m)$. The results are

$$\epsilon_{2ij} = \frac{\Delta_{ii}}{\lambda_{1i}-\lambda_{1j}} \left\{ \begin{array}{l} - (\lambda_{2i}\Delta_{ii}\epsilon_{1ij} + \gamma_{1ji}\lambda_{2j}\Delta_{jj}) \\ + \sum_{p=1}^{n} [(\lambda_{1i}-\lambda_{1j})\Delta_{ii}\epsilon_{1ip}\epsilon_{1pj} + (\lambda_{1p}-\lambda_{1j})\Delta_{pp}\gamma_{1pi}\epsilon_{1pj}] \end{array} \right\} ,$$

$$i \neq j \quad , \quad i,j = 1,2,\ldots,m \quad (6.24a)$$

$$\epsilon_{2ii} = \gamma_{2ii} = \frac{1}{2} \Delta_{ii} \sum_{p=1}^{n} (\Delta_{ii}\epsilon_{1ip}\epsilon_{1pi} + \Delta_{pp}\gamma_{1pi}\epsilon_{1pi} + \Delta_{ii}\gamma_{1pi}\gamma_{1ip}) \quad ,$$

$$i = j \quad , \quad i = 1,2,\ldots,m \quad (6.24b)$$

$$\gamma_{2ij} = \frac{\Delta_{ii}}{\lambda_{1i}-\lambda_{1j}} \left\{ \begin{array}{l} - (\lambda_{2j}\Delta_{jj}\epsilon_{1ji} + \gamma_{1ij}\lambda_{2i}\Delta_{ii}) \\ + \sum_{p=1}^{n} [(\lambda_{1p}-\lambda_{1j})\Delta_{pp}\gamma_{1pj}\epsilon_{1pi} + (\lambda_{1i}-\lambda_{1j})\Delta_{ii}\gamma_{1pj}\gamma_{1ip}] \end{array} \right\} ,$$

$$i \neq j \quad , \quad i,j = 1,2,\ldots,m \quad (6.24c)$$

$$\lambda_{3i} = \lambda_{2i}(\epsilon_{1ii} + \gamma_{1ii}) + \Delta_{ii} \sum_{p=1}^{n} (\lambda_{1i}-\lambda_{1p})\Delta_{pp}\gamma_{1pi}\epsilon_{1pi} \quad ,$$

$$i = 1,2,\ldots,m \quad (6.25)$$

Thus the 0(2) eigenvector perturbations have been fully determined. Once again we note that the expressions for ϵ_{2ii} and γ_{2ii} given in Eqs. (2.29b) and (6.24b) are the same. Manipulations to show that the expressions for λ_{3i} given in Eqs. (2.31) and (6.25) are identical, follow those between Eqs. (6.19) and (6.23).

In summary, the manipulations of Chapter 2 have been modified to include multiplicity in the 0(0) eigensolution. As in Chapter 2, eigensolution perturbations are derived from the biorthonormality relations, and hence normalization is preserved through the order to which eigenvector perturbations are determined. Note that the definition of the quantity $\hat{\hat{a}}_{ij}$ allows the results of the present section to be presented in forms that are analogous to the corresponding results of Chapter 2.

As an example of the developments in this chapter, we consider the eigenvalue problem in standard form

$$A\underset{\sim}{u}_i = \lambda_i \underset{\sim}{u}_i \quad , \quad A^T \underset{\sim}{v}_i = \lambda_i \underset{\sim}{v}_i \quad , \quad i = 1,2,\ldots,n \tag{6.26a,b}$$

and the associated biorthonormality relations

$$\underset{\sim}{v}_i^T \underset{\sim}{u}_j = \delta_{ij} \quad , \quad \underset{\sim}{v}_i^T A \underset{\sim}{u}_j = \lambda_i \delta_{ij} \quad , \quad i,j = 1,2,\ldots,n \tag{6.27a,b}$$

where

$$A = A_0 + A_1 \tag{6.28}$$

Let us take

$$A_0 = \begin{bmatrix} 3/2 & \sqrt{3/4} & 0 \\ \sqrt{3/4} & 5/2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad ; \quad A_1 = \begin{bmatrix} 0 & 1/10 & 2/10 \\ -1/10 & 0 & 1/10 \\ -2/10 & -1/10 & 0 \end{bmatrix} \tag{6.29}$$

A few comments are in order. Because the real matrices A_0 and A_1 are symmetric and skew symmetric, respectively, the sum $(A_0 + A_1)$ possesses no symmetries, so that the perturbed eigensolution is generally complex, with unrelated right and left eigenvectors. However, since the sum $A_0 + A_1$ is real, any complex eigenvalues, and the corresponding eigenvectors, must occur in complex conjugate pairs. Because the eigenvalue problem under consideration here is of dimension 3×3, and because complex quantities must occur in conjugate pairs, we are guaranteed that at least one eigenvalue, and the corresponding eigenvectors, is real.

We now turn our attention to the O(0) eigenvalue problem

$$A_0 \underset{\sim}{u}_{0i} = \lambda_{0i} \underset{\sim}{u}_{0i} \quad , \quad A_0^T \underset{\sim}{v}_{0i} = \lambda_{0i} \underset{\sim}{v}_{0i} \quad , \quad i = 1,2,\ldots,n \qquad (6.30a,b)$$

and the associated biorthonormality relations

$$\underset{\sim}{v}_{0i}^T \underset{\sim}{u}_{0j} = \delta_{ij} \quad , \quad \underset{\sim}{v}_{0i}^T A_0 \underset{\sim}{u}_{0j} = \lambda_{0i} \delta_{ij} \quad , \quad i,j = 1,2,\ldots,n \qquad (6.31a,b)$$

Because the matrix A_0 is real and symmetric, the O(0) eigenvalues must be real. As is the usual practice, we shall express the corresponding eigenvectors in terms of real quantities. However, we observe that neither the statement of the eigenvalue problem, nor the associated biorthonormality relations, require that the eigenvectors, corresponding to the real eigenvalue of a real matrix, be real. Indeed, in the latter part of the next chapter we shall encounter complex eigenvectors corresponding to the real eigenvalue of a real matrix.

The O(0) eigenvalue problem possesses the eigenvalues

$$\lambda_{01} = \lambda_{02} = 3 \quad , \quad \lambda_{03} = 1 \ , \qquad (6.33)$$

thus we have multiplicity two. The corresponding real, unnormalized eigenvectors are

$$\underset{\sim}{u}_{01} = \underset{\sim}{v}_{01} = \begin{Bmatrix} 1 \\ \sqrt{3} \\ a \end{Bmatrix} \quad , \quad \underset{\sim}{u}_{02} = \underset{\sim}{v}_{02} = \begin{Bmatrix} 1 \\ \sqrt{3} \\ b \end{Bmatrix} \quad , \quad \underset{\sim}{u}_{03} = \underset{\sim}{v}_{03} = \begin{Bmatrix} \sqrt{3} \\ -1 \\ 0 \end{Bmatrix} \qquad (6.34)$$

where the presence of the real, arbitrary parameters a and b is due to the multiplicity. We observe that $\underset{\sim}{u}_{03}$ is orthogonal to $\underset{\sim}{u}_{01}$ and $\underset{\sim}{u}_{02}$. Demands that $\underset{\sim}{u}_{01}$ be orthogonal to $\underset{\sim}{u}_{02}$ yields the single relation

$$ab + 4 = 0 \tag{6.35}$$

Equation (6.35) is certainly satisfied if we pick the convenient values

$$a = -2 \quad , \quad b = +2 \quad , \tag{6.36}$$

however, we must emphasize that this choice is by no means unique. Using these values for a and b, we obtain the normalized 0(0) eigenvectors

$$U_0 = V_0 = \begin{bmatrix} 0.353,553 & 0.353,553 & 0.866,025 \\ 0.612,372 & 0.612,372 & -0.5 \\ -0.707,107 & 0.707,107 & 0.0 \end{bmatrix} \tag{6.37}$$

Note that the unperturbed eigenvectors satisfy the biorthonormality relations

$$V_0^T U_0 = I \quad , \quad V_0^T A_0 U_0 = \Lambda_0 \tag{6.38}$$

where the elements of the diagonal matrix Λ_0 are given in Eq. (6.33).

Let us form the product

$$V_0^T A_1 U_0 = \hat{A} = \begin{bmatrix} 0.0 & 0.186,603 & \cdots \\ -0.186,603 & 0.0 & \\ \vdots & & \end{bmatrix} \tag{6.39}$$

From Eqs. (6.11) and (6.39), we have the 2 × 2 right and left eigenvalue problems

$$\hat{A}\, \underset{\sim}{e}_i = \lambda_{1i}\, \underset{\sim}{e}_i \quad , \quad i = 1,2 \tag{6.40a}$$

$$\hat{A}^T \underset{\sim}{g}_i = \lambda_{1i}\, \underset{\sim}{g}_i \quad , \quad i = 1,2 \tag{6.40b}$$

The eigensolution is

$$\Lambda_1 = \begin{bmatrix} -i0.186,603 & & \\ & +i0.186,603 & \\ & & \ddots \end{bmatrix} \tag{6.41}$$

$$E = \begin{bmatrix} 0.707,107 & 0.707,107 & 0.0 \\ -i0.707,107 & i0.707,107 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \tag{6.42a}$$

$$G = \begin{bmatrix} 0.707,107 & 0.707,107 & 0.0 \\ i0.707,107 & -i0.707,107 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \tag{6.42b}$$

We now construct O(0) perturbed eigenvectors from the products U_0E and V_0G.

$$U_0 = \begin{bmatrix} \begin{matrix} 0.25 \\ -i0.25 \end{matrix} & \begin{matrix} 0.25 \\ +i0.25 \end{matrix} & 0.866,025 \\ \begin{matrix} 0.433,013 \\ -i0.433,013 \end{matrix} & \begin{matrix} 0.433,013 \\ +i0.433,013 \end{matrix} & -0.5 \\ \begin{matrix} -0.5 \\ -i0.5 \end{matrix} & \begin{matrix} -0.5 \\ +i0.5 \end{matrix} & 0 \end{bmatrix} \tag{6.43a}$$

$$V_0 = \begin{bmatrix} \begin{matrix} 0.25 \\ +i0.25 \end{matrix} & \begin{matrix} 0.25 \\ -i0.25 \end{matrix} & 0.866,025 \\ \begin{matrix} 0.433,013 \\ +i0.433,013 \end{matrix} & \begin{matrix} 0.433,013 \\ -i0.433,013 \end{matrix} & -0.5 \\ \begin{matrix} -0.5 \\ +i0.5 \end{matrix} & \begin{matrix} -0.5 \\ -i0.5 \end{matrix} & 0 \end{bmatrix} \tag{6.43b}$$

Comparing Eqs. (6.37) and (6.43), we note that $\underset{\sim}{u}_{03}$ and v_{03} are unchanged since these vectors correspond to the distinct eigenvalue $\lambda_{03} = 1$. We also note that the above eigenvectors satisfy biorthonormality relations

$$V_0^T U_0 = I \qquad \text{and} \qquad V_0^T A_0 U_0 = \Lambda_0 \tag{6.44}$$

which are the same as those given by Eqs. (6.31). Using Eqs. (6.43), we compute the redefined perturbation matrix

$$\hat{A} = \begin{bmatrix} -i0.186,603 & 0 & \begin{matrix} 0.011,603 \\ -i0.111,603 \end{matrix} \\ 0 & i0.186,603 & \begin{matrix} 0.011,603 \\ +i0.111,603 \end{matrix} \\ \begin{matrix} -0.011,603 \\ -i0.111,603 \end{matrix} & \begin{matrix} -0.011,603 \\ +i0.111,603 \end{matrix} & 0 \end{bmatrix} \tag{6.45}$$

where the zeros in the (1,2) and (2,1) locations are a direct consequence of solving the 0(1) eigenvalue problem, Eqs. (6.40).

From the diagonal elements of Eq. (6.45), we have the 0(1) eigenvalue perturbations

$$\Lambda_1 = \begin{bmatrix} -i0.186,603 & & \\ & +i0.186,603 & \\ & & 0.0 \end{bmatrix} \tag{6.46}$$

Furthermore, from Eq. (6.45) we can determine the elements of E_1 and Γ_1 for which i and j are not simultaneously in the range (1,2). From these elements, we can construct

$$\hat{\hat{A}} = \begin{bmatrix} -0.006,295 & \begin{matrix} 0.006,160 \\ +i0.001,295 \end{matrix} & \\ \begin{matrix} 0.006,160 \\ -i0.001,295 \end{matrix} & -0.006,295 & \\ & & \ddots \end{bmatrix} \tag{6.47}$$

from which the remaining elements of E_1 and Γ_1 can be determined. The result is

$$E_1 = \begin{bmatrix} 0 & \begin{matrix}0.003{,}470\\ -i0.016{,}506\end{matrix} & \begin{matrix}-0.005.801\\ +i0.055{,}801\end{matrix} \\[2ex] \begin{matrix}0.003{,}470\\ +i0.016{,}506\end{matrix} & 0 & \begin{matrix}-0.005{,}801\\ -i0.055{,}801\end{matrix} \\[2ex] \begin{matrix}-0.005{,}801\\ -i0.055{,}801\end{matrix} & \begin{matrix}-0.005{,}801\\ +i0.055{,}801\end{matrix} & 0 \end{bmatrix} \qquad (6.48a)$$

$$\Gamma_1 = - E_1^T \qquad (6.48b)$$

Second-order perturbations are as follows

$$\Lambda_2 = \begin{bmatrix} -0.006{,}295 & & \\ & -0.006{,}295 & \\ & & 0.012{,}590 \end{bmatrix} \qquad (6.49)$$

$$E_2 = \begin{bmatrix} 0.001{,}716 & \begin{matrix}-0.001{,}540\\ -i0.000{,}324\end{matrix} & \begin{matrix}-0.005{,}206\\ -i0.000{,}541\end{matrix} \\[2ex] \begin{matrix}-0.001{,}540\\ +i0.000{,}324\end{matrix} & 0.001{,}716 & \begin{matrix}-0.005{,}206\\ +i0.000{,}541\end{matrix} \\[2ex] \begin{matrix}0.004{,}265\\ -i0.000{,}443\end{matrix} & \begin{matrix}0.004{,}265\\ +i0.000{,}443\end{matrix} & 0.003{,}147 \end{bmatrix} \qquad (6.50a)$$

$$\Gamma_2 = \begin{bmatrix} 0.001{,}716 & \begin{matrix}0.001{,}540\\ -i0.000{,}324\end{matrix} & \begin{matrix}-0.005{,}206\\ +i0.000{,}541\end{matrix} \\[2ex] \begin{matrix}0.001{,}540\\ +i0.000{,}324\end{matrix} & 0.001{,}716 & \begin{matrix}-0.005{,}206\\ -i0.000{,}541\end{matrix} \\[2ex] \begin{matrix}0.004{,}265\\ +i0.000{,}443\end{matrix} & \begin{matrix}0.004{,}265\\ -i0.000{,}443\end{matrix} & 0.003{,}147 \end{bmatrix} \qquad (6.50b)$$

Third-order eigenvalue perturbations are

$$\Lambda_3 = \begin{bmatrix} -i0.000{,}481 & & \\ & +i0.000{,}481 & \\ & & 0.0 \end{bmatrix} \qquad (6.51)$$

A summary is in order. In Table 6.1 we present a compilation of eigenvalue estimates produced by the perturbation analysis, along with exact eigenvalues for comparison. Note that estimates for λ_2 are not listed as they are the complex conjugates of λ_1. In Table 6.2 we present the right eigenvectors $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_3$ summarized through various orders. Note that $\underset{\sim}{u}_2$ is not listed because it is the complex conjugate of $\underset{\sim}{u}_1$. Inspection of Tables 6.1 and 6.2 reveals that, after the O(0) eigenvector perturbations are included, convergence appears to be uniform. Because a tabulation of the left eigenvectors reveals the same convergence qualities, they are not displayed.

A biorthonormality check can also be used to judge eigenvector accuracy. In Table 6.2 we present the product $Z^T(A_0 + A_1)W$, where in place of W and Z we take the respective right and left eigenvectors, summarized through various orders. At each location in this table, we present four generally complex numbers, the first of which was computed using $W = U_0$ and $Z = V_0$, where U_0 and V_0 are <u>unperturbed</u> O(0) eigenvectors. The second number was computed using the O(0) perturbed eigenvectors U_0 and V_0, while the third and fourth were computed using $U_0\mathcal{E}(I + E_1)$, $V_0\mathcal{G}(I + \Gamma_1)$ and using $U_0\mathcal{E}(I + E_1 + E_2)$, $V_0\mathcal{G}(I + \Gamma_1 + \Gamma_2)$, respectively. We recall that if exact eigenvectors are used, then the product $Z^T(A_0 + A_1)W$ is the diagonal matrix of eigenvalues.

TABLE 6.1

Eigenvalue Summary, Multiple O(0) Eigenvalues

	O(0)	O(0) + O(1)	O(0) + O(1) + O(2)	O(0) + ... + O(3)	Exact
λ_1	3.0	3.0 -i0.186,603	2.993,705 -i0.186,603	2.993,705 -i0.187,084	2.993,721 -i0.187,086
λ_3	1.0	1.0	1.012,590	1.012,590	1.012,559

TABLE 6.2

Right Eigenvector Summary, Multiple O(0) Eigenvalues

	O(0)	O(0) Corrected	O(0) + O(1)	O(0) + O(1) + O(2)	Exact
$\underset{\sim}{u}_1$	$\begin{Bmatrix} 0.353,553 \\ 0.612,372 \\ -0.707,107 \end{Bmatrix}$	$\begin{Bmatrix} 0.25 - i0.25 \\ 0.433,013 - i0.433,013 \\ -0.5 - i0.5 \end{Bmatrix}$	$\begin{Bmatrix} 0.241,717 - i0.293,331 \\ 0.430,268 - i0.396,462 \\ -0.509,988 - i0.506,518 \end{Bmatrix}$	$\begin{Bmatrix} 0.245,374 - i0.294,448 \\ 0.428,072 - i0.397,510 \\ -0.510,238 - i0.508,308 \end{Bmatrix}$	$\begin{Bmatrix} 0.245,486 - i0.294,362 \\ 0.428,175 - i0.397,442 \\ -0.510,056 - i0.508,416 \end{Bmatrix}$
$\underset{\sim}{u}_3$	$\begin{Bmatrix} 0.866,025 \\ -0.5 \\ 0.0 \end{Bmatrix}$	$\begin{Bmatrix} 0.866,025 \\ -0.5 \\ 0.0 \end{Bmatrix}$	$\begin{Bmatrix} 0,891,025 \\ -0.456,699 \\ 0.061,603 \end{Bmatrix}$	$\begin{Bmatrix} 0.890,877 \\ -0.463,250 \\ 0.066,268 \end{Bmatrix}$	$\begin{Bmatrix} 0.890,848 \\ -0.463,243 \\ 0.066,339 \end{Bmatrix}$

TABLE 6.3

Biorthonormality Check, Multiple 0(0) Eigenvalues

3.0 + i0.0 3.0 - i0.186,603 2.983,409 - i0.186,443 2.993,687 - i0.188,530	0.186,603 + i0.0 0.0 + i0.0 0.009,240 + i0.001,942 0.009,132 + i0.002,522	0.016,408 + i0.0 0.011,603 - i0.111,603 0.011,409 + i0.000,653 0.000,096 - i0.000,331
-0.186,603 + i0.0 0.0 + i0.0 0.009,240 - i0.001,942 0.009,132 - i0.002,522	3.0 + i0.0 3.0 + i0.186,603 2.983,409 + i0.186,443 2.993,687 + i0.188,530	-0.157,830 + i0.0 0.011,603 + i0.111,603 0.011,408 - i0.000,653 0.000,096 + i0.000,331
-0.016,408 + i0.0 -0.011,603 - i0.111,603 0.011,299 - i0.001,707 0.000,060 - i0.000,168	0.157,830 + i0.0 -0.011,603 + i0.111,603 0.011,299 + i0.001,707 0.000,060 + i0.000,168	1.0 + i0.0 1.0 + i0.0 1.006,295 + i0.0 1.012,624 + i0.0

Chapter 7

PERTURBATION THEORY FOR NEARLY MULTIPLE UNPERTURBED EIGENVALUES

In the present chapter we extend the derivation of Chapter 6 to include nearly multiple O(0) eigenvalues. Let us suppose that the difference between any two of the O(0) eigenvalues indexed (1,2,...,m) is an O(1) quantity, and that the remaining n-m O(0) eigenvalues are clearly distinct. This is to say that

$$(\lambda_{0i} - \lambda_{0j}) = O(1) \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,m \tag{7.1}$$

and

$$(\lambda_{0i} - \lambda_{0j}) = O(0)$$

when i and j are not simultaneously in the range 1,2,...,m.

Let us define an average of the nearly multiple O(0) eigenvalues and distinguish each one via a deviation from that average. To simplify the notation, we redefine λ_{0i} $(i = 1,2,\ldots,m)$ as the average and refer to the deviation as $\delta\lambda_{0i}$ $(i = 1,2,\ldots,m)$. We have

$$\lambda_{0i} \rightarrow \lambda_{0i} + \delta\lambda_{0i} \; (i = 1,2,\ldots,m)$$

where the λ_{0i} $(i = 1,2,\ldots,m)$ now satisfy Eq. (6.1), and where $\delta\lambda_{0i}$ $(i = 1,2,\ldots,m)$ is an O(1) quantity.

For convenience let us construct the $n \times n$ diagonal matrix

$$\delta\Lambda_0 = \begin{bmatrix} \delta\lambda_{01} & & & & & \\ & \delta\lambda_{02} & & & & \\ & & \ddots & & & \\ & & & \delta\lambda_{0m} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

Equations (2.9) now appear as

$$V_0^T B_0 U_0 = \Delta \tag{7.2a}$$

$$V_0^T A_0 U_0 = (\Lambda_0 + \delta\Lambda_0)\Delta \tag{7.2b}$$

and the eigenvalue expansion of Eq. (2.18) must be modified to

$$\Lambda = \Lambda_0 + (\delta\Lambda_0 + \Lambda_1) + \Lambda_2 + \Lambda_3 + \ldots \tag{7.3}$$

The use of Eqs. (7.2) allows Eqs. (6.4) to be written as

$$\Gamma^T(G^T\Delta E + G^T\hat{B}E)E = \Delta \tag{7.4a}$$

$$\Gamma^T(G^T\Lambda_0\Delta E + G^T\delta\Lambda_0\Delta E + G^T\hat{A}E)E = \Lambda\Delta \tag{7.4b}$$

Substituting the expansions of Eqs. (2.17) and (7.3) into Eqs. (7.4) and separating according to order, we obtain

$$0(0): \quad G^T\Delta E = \Delta \tag{7.5a}$$

$$G^T\Lambda_0\Delta E = \Lambda_0\Delta \tag{7.5b}$$

$$0(1): \quad G^T\Delta EE_1 + G^T\hat{B}E + \Gamma_1^T G^T\Delta E = 0 \tag{7.6a}$$

$$G^T\Lambda_0\Delta EE_1 + G^T(\delta\Lambda_0\Delta + \hat{A})E + \Gamma_1^T G^T\Lambda_0\Delta E = (\delta\Lambda_0 + \Lambda_1)\Delta \tag{7.6b}$$

Use of Equations (7.5) allows Eqs. (7.6) to be expressed as

$$O(1): \quad \begin{aligned} \Delta E_1 + \Gamma_1^T \Delta &= - G^T \hat{B} E && (7.7a) \\ \Lambda_0 \Delta E_1 + \Gamma_1^T \Lambda_0 \Delta &= - G^T(\hat{A} + \delta\Lambda_0 \Delta) E + (\delta\Lambda_0 + \Lambda_1)\Delta && (7.7b) \end{aligned}$$

Restricting attention to the upper left m × m corner of these matrix equations, we can write Eqs. (7.5) and (7.7) as

$$O(0): \quad \begin{aligned} G^T \Delta E &= \Delta && (7.8a) \\ \Lambda_0 G^T \Delta E &= \Lambda_0 \Delta && (7.8b) \end{aligned}$$

$$O(1): \quad \begin{aligned} \Delta E_1 + \Gamma_1^T \Delta &= - G^T \hat{B} E && (7.9a) \\ \Lambda_0(\Delta E_1 + \Gamma_1^T \Delta) &= - G^T(\hat{A} + \delta\Lambda_0 \Delta) E + (\delta\Lambda_0 + \Lambda_1)\Delta && (7.9b) \end{aligned}$$

As in Chapter 6, Eqs. (7.8a) and (7.8b) are identical. We combine Eqs. (7.9) so as to eliminate the common quantity $(\Delta E_1 + \Gamma_1^T \Delta)$. The result is

$$G^T(\hat{A} + \delta\Lambda_0 \Delta - \hat{B}\Lambda_0) E = (\delta\Lambda_0 + \Lambda_1)\Delta \qquad (7.10)$$

Equations (7.8a) and (7.10) represent biorthonormality relations for the m × m right and left eigenvalue problems

$$(\hat{A} + \delta\Lambda_0 \Delta - \hat{B}\Lambda_0)\, \underset{\sim}{e}_i = (\delta\lambda_{0i} + \lambda_{1i})\, \underset{\sim}{e}_i \quad , \quad i = 1,2,\ldots,m \qquad (7.11a)$$

$$(\hat{A}^T + \delta\Lambda_0 \Delta - \Lambda_0 \hat{B}^T)\, \underset{\sim}{g}_i = (\delta\lambda_{0i} + \lambda_{1i})\, \underset{\sim}{g}_i \quad , \quad i = 1,2,\ldots,m \qquad (7.11b)$$

Solution of the O(1) eigenvalue problem, Eqs. (7.11), yields the O(1) eigenvalue perturbations λ_{1i} (i = 1,2,...,m) and the O(0) right and left eigenvector perturbations E and G, respectively.

Following the manipulations of Chapter 6, we redefine

$$U_0 E \rightarrow U_0 \quad \text{and} \quad V_0 G \rightarrow V_0 \quad ,$$

and redefine $\hat{B}$ accordingly. Due to the near multiplicity of the O(0) eigenvalues, the redefinition of $\hat{A}$ is not so straight forward. It is simplest to express the redefined matrix $\hat{A}$ as

$$\hat{A} \equiv V_0^T(A_0 + A_1)U_0 - \Lambda_0\Delta \tag{7.12}$$

where U_0 and V_0 are the respective $O(0)$ perturbed right and left eigenvectors, and where Λ_0 is the diagonal matrix of redefined $O(0)$ eigenvalues, the first m of which are the same. If we perform the substitution

$$\Lambda_1 \rightarrow \delta\Lambda_0 + \Lambda_1$$

the remaining analysis can be taken from Chapter 6.

As an example of the developments presented earlier in the present chapter, let us consider the eigenvalue problem in standard form

$$A\underset{\sim}{u}_i = \lambda_i \underset{\sim}{u}_i \quad , \quad A^T\underset{\sim}{v}_i = \lambda_i \underset{\sim}{v}_i \quad , \quad i = 1,2,\ldots,n \tag{7.13a,b}$$

and the associated biorthonormality relations

$$\underset{\sim}{v}_i^T\underset{\sim}{u}_j = \delta_{ij} \quad , \quad \underset{\sim}{v}_i^T A\underset{\sim}{u}_j = \lambda_i\delta_{ij} \quad , \quad i,j = 1,2,\ldots,n \tag{7.14a,b}$$

where

$$A = A_0 + A_0 \; . \tag{7.15}$$

Let us take

$$A_0 = \begin{bmatrix} 3/2 & (\sqrt{3/4} + 1/10) & 2/10 \\ (\sqrt{3/4} - 1/10) & 5/2 & 1/10 \\ -2/10 & -1/10 & 3 \end{bmatrix} \tag{7.16a}$$

$$A_1 = \begin{bmatrix} 0 & -1/10 & -2/10 \\ 1/10 & 0 & -1/10 \\ 2/10 & 1/10 & 0 \end{bmatrix} \tag{7.16b}$$

Thus the exact eigensolution of the example in Chapter 6 becomes the O(0) eigensolution for the present example. Furthermore, since the perturbing matrix of the present example is the negative of that in Chapter 6, the exact eigensolution of the present example should be the O(0) eigensolution of the example in Chapter 6. Indeed the objective here is to investigate how well this challenge is met.

The O(0) eigensolution is

$$\underset{\sim}{u}_{01} = \begin{Bmatrix} 0.245,486 \\ -i0.294,362 \\ \\ 0.428,175 \\ -i0.397,442 \\ \\ -0.510,056 \\ -i0.508,416 \end{Bmatrix} \quad , \quad \underset{\sim}{v}_{01} = \begin{Bmatrix} 0.261,886 \\ +i0.207,875 \\ \\ 0.433,394 \\ +i0.470,668 \\ \\ -0.490,422 \\ +i0.495,158 \end{Bmatrix} \tag{7.17a}$$

$$\underset{\sim}{u}_{02} = \bar{\underset{\sim}{u}}_{01} \quad , \quad \underset{\sim}{v}_{02} = \bar{\underset{\sim}{v}}_{01} \tag{7.17b}$$

$$\underset{\sim}{u}_{03} = \begin{Bmatrix} 0.890,848 \\ \\ -0.463,243 \\ \\ 0.066,339 \end{Bmatrix} \quad , \quad \underset{\sim}{v}_{03} = \begin{Bmatrix} 0.840,817 \\ \\ -0.549,900 \\ \\ -0.056,944 \end{Bmatrix} \tag{7.17c}$$

$$\begin{aligned} \lambda_{01} &= 2.993,721 - i0.187,086 \\ \lambda_{02} &= \bar{\lambda}_{01} \\ \lambda_{03} &= 1.012,559 \end{aligned} \tag{7.18}$$

Note that Eqs. (7.17) and (7.18) satisfy the biorthonormality relations

$$V_0^T U_0 = I \quad , \quad V_0^T A_0 U_0 = \Lambda_0 \tag{7.19}$$

Let us say that λ_{01} and λ_{02} are nearly multiple, and hence redefine

$$\lambda_{01} = \lambda_{02} = 2.993,720 \tag{7.20a}$$

$$\delta\lambda_{01} = -i0.186,603 \quad , \quad \delta\lambda_{02} = \overline{\Delta\lambda}_{01} \tag{7.20b}$$

We compute the product

$$V_0^T A_1 U_0 = \hat{A} = \begin{bmatrix} \begin{matrix} 0.012,527 \\ +i0.188,057 \end{matrix} & \begin{matrix} -0.006,187 \\ -i0.001,301 \end{matrix} & --- \\ & & \\ \begin{matrix} -0.006,187 \\ +i0.001,301 \end{matrix} & \begin{matrix} 0.012,527 \\ -i0.188,057 \end{matrix} & \\ \vdots & & \end{bmatrix} \tag{7.21}$$

and address the O(1) eigenvalue problem

$$(\hat{A} + \delta\Lambda_0)\underset{\sim}{e}_i = (\delta\lambda_{0i} + \lambda_{1i})\underset{\sim}{e}_i \quad , \quad i = 1,2 \tag{7.22a}$$

$$(\hat{A}^T + \delta\Lambda_0)\underset{\sim}{g}_i = (\delta\lambda_{0i} + \lambda_{1i})\underset{\sim}{g}_i \quad , \quad i = 1,2 \tag{7.22b}$$

where

$$(\hat{A} + \delta\Lambda_0) = \begin{bmatrix} \begin{matrix} 0.012,527 \\ +i0.000,971 \end{matrix} & \begin{matrix} -0.006,187 \\ -i0.001,301 \end{matrix} \\ & \\ \begin{matrix} -0.006,187 \\ +i0.001,301 \end{matrix} & \begin{matrix} 0.012,527 \\ -i0.000,971 \end{matrix} \end{bmatrix} \tag{7.22c}$$

Solutions of Eqs. (7.22) are

$$\delta\Lambda_0 + \Lambda_1 = \begin{bmatrix} 0.006,277 & \\ & 0.018,774 \end{bmatrix} \tag{7.23a}$$

$$E = \begin{bmatrix} \begin{matrix} 0.705,326 \\ +i0.092,283 \end{matrix} & \begin{matrix} 0.682,790 \\ +i0.199,496 \end{matrix} & \begin{matrix} 0.0 \\ +i0.0 \end{matrix} \\ & & \\ \begin{matrix} 0.709,225 \\ +i0.054,778 \end{matrix} & \begin{matrix} -0.709,225 \\ +i0.054,778 \end{matrix} & \begin{matrix} 0.0 \\ +i0.0 \end{matrix} \\ & & \\ \begin{matrix} 0.0 \\ +i0.0 \end{matrix} & \begin{matrix} 0.0 \\ +i0.0 \end{matrix} & \begin{matrix} 1.0 \\ +i0.0 \end{matrix} \end{bmatrix} \tag{7.23b}$$

$$G = \begin{bmatrix} 0.682{,}790 & 0.705{,}326 & 0.0 \\ -i0.199{,}496 & -i0.092{,}283 & +i0.0 \\ & & \\ 0.709{,}225 & -0.709{,}225 & 0.0 \\ +i0.054{,}778 & +i0.054{,}778 & +i0.0 \\ & & \\ 0.0 & 0.0 & 1.0 \\ +i0.0 & +i0.0 & +i0.0 \end{bmatrix} \tag{7.23c}$$

We now construct the O(0) perturbed eigenvectors from the products $U_0 E$ and $V_0 G$

$$U_0 = \begin{bmatrix} 0.358{,}293 & 0.036{,}110 & \\ +i0.037{,}249 & -i0.347{,}337 & 0.890{,}848 \\ & & \\ 0.620{,}581 & 0.046{,}198 & \\ +i0.064{,}518 & -i0.444{,}372 & -0.463{,}243 \\ & & \\ -0.702{,}431 & 0.087{,}060 & \\ -i0.073{,}027 & -i0.837{,}417 & 0.066{,}339 \end{bmatrix} \tag{7.24a}$$

$$V_0 = \begin{bmatrix} 0.417{,}407 & 0.029{,}549 & \\ -i0.043{,}395 & +i0.284{,}227 & 0.840{,}817 \\ & & \\ 0.722{,}969 & 0.067{,}527 & \\ -i0.075{,}162 & +i0.649{,}529 & -0.549{,}900 \\ & & \\ -0.556{,}769 & 0.074{,}731 & \\ +i0.057{,}883 & +i0.718{,}821 & -0.056{,}944 \end{bmatrix} \tag{7.24b}$$

Comparing Eqs. (7.17c) and (7.24), we note that $\underset{\sim}{u}_{03}$ and $\underset{\sim}{v}_{03}$ are unchanged since they correspond to the distinct eigenvalue $\lambda_{03} = 1.012{,}559$.

Using the vectors expressed in Eqs. (7.24), we construct the redefined matrix $\hat{A}$ according to Eq. (7.12).

$$\hat{A} = \begin{bmatrix} 0.006{,}279 & 0 & 0 \\ & & \\ 0 & 0.018{,}774 & \begin{matrix} 0.016{,}397 \\ +i0.157{,}723 \end{matrix} \\ & & \\ 0 & \begin{matrix} -0.016{,}397 \\ +i0.157{,}723 \end{matrix} & -0.025{,}053 \end{bmatrix} \tag{7.25}$$

Note that while the zeros in the (1,2) and (2,1) locations are a direct consequence of solving the O(1) eigenvalue problem, the zeros in the (1,3) and (3,1) locations are unexpected boni!

From the diagonal elements of Eq. (7.25), we have the O(1) eigenvalue perturbations

$$\delta\Lambda_0 + \Lambda_1 = \begin{bmatrix} 0.006,279 & & \\ & 0.018,774 & \\ & & -0.025,053 \end{bmatrix} \tag{7.26}$$

From Eq. (7.25) we determine the elements of E_1 and Γ_1 for which i and j are not simultaneously in the range (1,2). From these elements, we construct

$$\hat{\hat{A}} = \begin{bmatrix} 0 & 0 & \\ 0 & -0.012,692 & \\ & & \ddots \end{bmatrix} \tag{7.27}$$

from which we can determine the remaining elements of E_1 and Γ_1. The result is

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \begin{matrix}-0.008,277 \\ -i0.079,611\end{matrix} \\ 0 & \begin{matrix}-0.008,277 \\ +i0.079,611\end{matrix} & 0 \end{bmatrix} \tag{7.28a}$$

$$\Gamma_1 = - E_1^T \tag{7.28b}$$

Second order perturbations are

$$\Lambda_2 = \begin{bmatrix} 0 & & \\ & -0.012{,}692 & \\ & & +0.012{,}692 \end{bmatrix} \tag{7.29}$$

and

$$E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.003{,}203 & \begin{matrix} 0.000{,}183 \\ +i0.001{,}761 \end{matrix} \\ 0 & \begin{matrix} 0.000{,}183 \\ -i0.001{,}761 \end{matrix} & 0.003{,}203 \end{bmatrix} \tag{7.30a}$$

$$\Gamma_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.003{,}203 & \begin{matrix} -0.000{,}183 \\ +i0.001{,}761 \end{matrix} \\ 0 & \begin{matrix} -0.000{,}183 \\ -i0.001{,}761 \end{matrix} & 0.003{,}203 \end{bmatrix} \tag{7.30b}$$

Third-order eigenvalue perturbations are

$$\Lambda_3 = \begin{bmatrix} 0 & & \\ & 0.000{,}281 & \\ & & -0.000{,}281 \end{bmatrix} \tag{7.31}$$

In Table 7.1 we present a summary of eigenvalue estimates produced by the above perturbation analysis, along with exact eigenvalues for comparison. Estimates for λ_2 are not included as they are the complex conjugates of λ_1. Table 7.2 presents the right eigenvectors summarized through various orders. Once again, we observe that after the O(0) eigenvector perturbations are included, convergence appears to be uniform. Because a tabulation of estimates for the left eigenvectors reveals the same convergence qualities, they are not listed.

The results of a biorthonormality check, with respect to the matrix $A_0 + A_1$, are presented in Table 7.3 using the same format as Table 6.3.

We recall the brief discussion following Eqs. (6.31) regarding the existence of complex eigenvectors corresponding to the real eigenvalue of a real matrix. Furthermore, we recall that the exact eigenvectors of Table 7.2 should coincide with the O(0) unperturbed eigenvectors of Table 6.2. Comparing these tables, we find the same values for $\underset{\sim}{u}_3$, yet we do not find the same values for $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$. This apparent discrepancy can be removed if we divide the exact representations of $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$, as given in Table 7.2, by their first elements, yielding the real, unnormalized eigenvectors

$$\underset{\sim}{u}_1 = \begin{Bmatrix} 1.0 \\ 1.732{,}051 \\ -1.960{,}495 \end{Bmatrix} , \quad \underset{\sim}{u}_2 = \begin{Bmatrix} 1.0 \\ 1.732{,}051 \\ 2.998{,}777 \end{Bmatrix} . \tag{7.32a}$$

A similar manipulation with the left eigenvectors yields

$$\underset{\sim}{v}_1 = \begin{Bmatrix} 1.0 \\ 1.732{,}051 \\ -1.333{,}877 \end{Bmatrix} , \quad \underset{\sim}{v}_2 = \begin{Bmatrix} 1.0 \\ 1.732{,}051 \\ 2.040{,}301 \end{Bmatrix} \tag{7.32b}$$

The vectors of Eqs. (7.32) remain biorthogonal, which is to say that $\underset{\sim}{v}_1^T \underset{\sim}{u}_2 = \underset{\sim}{v}_2^T \underset{\sim}{u}_1 = 0$. Then recalling the arbitrariness of the parameters a and b, used in Eqs. (6.34) and (6.35), we conclude that the exact eigenvectors of Table 7.2 are indeed equivalent to the O(0) unperturbed eigenvectors of Table 6.2

TABLE 7.1

Eigenvalue Summary, Nearly Multiple O(0) Eigenvalues

	O(0)	O(0) + O(1)	O(0) + O(1) + O(2)	O(0) + ... + O(3)	Exact
λ_1	2.993,721 -i0.187,086	3.0	3.0	3.0	3.0
λ_2	2.993,721 +i0.187,086	3.012,495	2.999,802	3.000,083	3.0
λ_3	1.012,559	0.987,505	1.000,198	0.999,917	1.0

TABLE 7.2

Right Eigenvector Summary, Nearly Multiple 0(0) Eigenvalues

	0(0)	0(0) Perturbed	0(0) + 0(1)	0(0) + 0(1) + 0(2)	Exact
$\underset{\sim}{u}_1$	0.245,486 -i0.294,363 0.428,175 -i0.397,442 -0.510,055 -i0.508,416	0.358,293 +i0.037,249 0.620,581 +i0.064,518 -0.702,431 -i0.073,027	0.358,293 +i0.037,249 0.620,581 +i0.064,518 -0.702,431 -i0.073,027	0.358,293 +i0.037,249 0.620,581 +i0.064,518 -0.702,431 -i0.073,027	0.358,293 +i0.037,249 0.620,581 +i0.064,518 -0.702,431 -i0.073,027
$\underset{\sim}{u}_2$	0.245,486 +i0.294,363 0.428,175 +i0.397,442 -0.510,055 +i0.508,416	0.036,110 -i0.347,337 0.046,198 -i0.444,372 0.087,060 -i0.837,417	0.028,737 -i0.276,415 0.050,032 -i0.481,251 0.086,511 -i0.832,135	0.029,016 -i0.279,096 0.050,096 -i0.481,859 0.086,802 -i0.834,935	0.028,942 -i0.278,385 0.050,129 -i0.482,176 0.086,790 -i0.834,813
$\underset{\sim}{u}_3$	0.890,848 -0.463,243 0.066,339	0.890,848 -0.463,243 0.066,339	0.862,898 -0.499,002 -0.001,049	0.866,370 -0.499,695 0.000,654	0.866,025 -0.5 0.0

TABLE 7.3

Biorthonormality Check, Nearly Multiple O(0) Eigenvalues

3.006,247 + i0.000,971 3.0 3.0 3.0	-0.006,187 - i0.001,301 0.0 0.0 0.0	-0.020,269 + i0.110,963 0.0 0.0 0.0
-0.006,187 + i0.001,301 0.0 0.0 0.0	3.006,247 - i0.000,971 3.012,495 2.980,078 3.000,873	-0.020,269 - i0.110,963 0.016,397 + i0.157,723 -0.000,258 - i0.002,479 0.000,161 + i0.001,551
-0.002,990 + i0.112,759 0.0 0.0 0.0	0.002,990 - i0.112,759 -0.016,397 + i0.157,723 0.000,258 - i0.002,479 -0.000,161 + i0.001,551	0.987,505 0.987,505 0.993,590 1.000,289

Chapter 8

RAYLEIGH'S QUOTIENT

Rayleigh's quotient can be used to improve eigenvalue estimates. For the eigenvalue problem given by Eqs. (2.1), we can express Rayleigh's quotient as

$$\mathcal{R}_i = \frac{\underset{\sim}{z}_i^T A \underset{\sim}{w}_i}{\underset{\sim}{z}_i^T B \underset{\sim}{w}_i} \quad , \quad i = 1,2,\ldots,n \tag{8.1}$$

where the trial vectors $\underset{\sim}{w}_i$ and $\underset{\sim}{z}_i$ are approximations to $\underset{\sim}{u}_i$ and $\underset{\sim}{v}_i$ $(i = 1,2,\ldots,n)$, respectively.

For the moment, let us suppose $\underset{\sim}{z} = \underset{\sim}{v}_i$ and $\underset{\sim}{w} = \underset{\sim}{u}_i$, there $\underset{\sim}{v}_i$ and $\underset{\sim}{u}_i$ are the ith left and right eigenvectors of Eqs. (2.1, respectively. Then

$$\mathcal{R}_i = \frac{\underset{\sim}{v}_i^T A \underset{\sim}{u}_i}{\underset{\sim}{v}_i^T B \underset{\sim}{u}_i} \tag{8.2}$$

Recalling Eqs. (2.2), we observe that

$$\mathcal{R}_i = \lambda_i \tag{8.3}$$

We wish to investigate the sensitivity of Rayleigh's quotient to small variations in the trial vectors when z and w are near a left-right eigenvector pair.

For self-adjoint systems, Meirovitch [13] has shown that Rayleigh's quotient is stationary when the trial vectors are in the neighborhood of an eigenvector. Lancaster [3] proves stationarity for non-self-adjoint systems, however his proof is based upon a more general non-self-adjoint system than that under consideration here. The following proof parallels that of Meirovitch closely.

Let us express the trial vectors $\underset{\sim}{z}$ and $\underset{\sim}{w}$ as linear sums of the left and right eigenvectors, respectively.

$$\underset{\sim}{z} = \sum_{p=1}^{n} \beta_p \underset{\sim}{v}_p \quad , \qquad \underset{\sim}{w} = \sum_{q=1}^{n} \alpha_q \underset{\sim}{u}_q \tag{8.4}$$

Let us suppose that

$$\underset{\sim}{z} \simeq \underset{\sim}{v}_i \quad \text{and} \quad \underset{\sim}{w} \simeq \underset{\sim}{u}_i \ . \tag{8.5}$$

Without loss of generality, we can set

$$\alpha_i = \beta_i = 1 \ . \tag{8.6}$$

Then the near equalities (8.5) indicate that

$$\beta_p << 1 \quad , \quad p \neq i \quad , \quad p = 1,2,\ldots,n \tag{8.7a}$$

$$\alpha_q << 1 \quad , \quad q \neq i \quad , \quad q = 1,2,\ldots,n \tag{8.7b}$$

We substitute Eqs. (8.4) into Eq. (8.1), and use the biorthonormality relations (2.2) to obtain

$$R = \frac{\sum_{p=1}^{n} \alpha_p \beta_p \Delta_{pp} \lambda_p}{\sum_{p=1}^{n} \alpha_p \beta_p \Delta_{pp}} \tag{8.8}$$

Equations (8.6) and (8.7) allow us to write

$$R = \frac{\lambda_i + \Delta_{ii} \sum_{\substack{p=1 \\ p\neq i}}^{n} \alpha_p \beta_p \Delta_{pp} \lambda_p}{1 + \Delta_{ii} \sum_{\substack{p=1 \\ p\neq i}}^{n} \alpha_p \beta_p \Delta_{pp}} \ . \tag{8.9}$$

We carry out a binomial series expansion of the denominator to obtain

$$R = \lambda_i + \Delta_{ii} \sum_{\substack{p=1 \\ p \neq i}}^{n} \alpha_p \beta_p \Delta_{pp} (\lambda_p - \lambda_i) + \ldots \tag{8.10}$$

Due to the absence of terms linear in the α's and β's, it is clear that Rayleigh's quotient is stationary when the trial vectors are in the vicinity of a left-right eigenvector pair. This is to say that if we define a small parameter μ to typify trial vector error, then Rayleigh's quotient provides an eigenvalue estimate that errs by μ^2.

The use of trial vectors $\underset{\sim}{w}_i$ and $\underset{\sim}{z}_i$, in place of the unperturbed eigenvectors $\underset{\sim}{u}_{0i}$ and $\underset{\sim}{v}_{0i}$ $(i = 1,2,\ldots,n)$, respectively, leads to a more general definition of the perturbing matrices $\hat{A}$ and $\hat{B}$ than that given in Eq. (2.16). In the process of determining these generalized perturbing matrices we use Rayleigh's quotient to compute improved eigenvalue estimates. Then, let us assume the availability of matrices W and Z, the columns of which are the trial vectors $\underset{\sim}{w}_i$ and $\underset{\sim}{z}_i$. The latter are reasonably accurate approximations to the exact eigenvectors $\underset{\sim}{u}_i$ and $\underset{\sim}{v}_i$ $(i = 1,2,\ldots,n)$, respectively. In analogy with Eqs. (2.10), let us take

$$U = WE \tag{8.11a}$$

$$V = Z\Gamma \tag{8.11b}$$

and substitute into Eqs. (2.4) to obtain

$$\Gamma^T Z^T B W E = \Delta \tag{8.12a}$$

$$\Gamma^T Z^T A W E = \Lambda\Delta \tag{8.12b}$$

Let us express the products Z^TBW and Z^TAW as

$$Z^TBW = \Delta + \hat{B}_R \tag{8.13a}$$

$$Z^TAW = \mathcal{R}\Delta + \hat{A}_R \ . \tag{8.13b}$$

Now the elements of the diagonal matrix $\mathcal{R}$ can be determined according to Rayleigh's quotient as

$$\mathcal{R}_i = \frac{\tilde{z}_i^T A \tilde{w}_i}{\tilde{z}_i^T B \tilde{w}_i} \quad , \qquad i = 1,2,\ldots,n \tag{8.14}$$

Once these eigenvalue estimates are known, matrices $\hat{A}_R$ and $\hat{B}_R$ are given by

$$\hat{B}_R = Z^TBW - \Delta \tag{8.15a}$$

$$\hat{A}_R = Z^TAW - \mathcal{R}\Delta \tag{8.15b}$$

Since we have assumed $\tilde{w}_i \approx \tilde{u}_i$ and $\tilde{z}_i \approx \tilde{v}_i$ $(i = 1,2,\ldots,n)$, it follows that matrices $\hat{A}_R$ and $\hat{B}_R$ are small. Substitution of Eqs. (8.13) into Eqs. (8.12) yields

$$\Gamma^T(\Delta + \hat{B}_R)E = \Delta \tag{8.16a}$$

$$\Gamma^T(\mathcal{R}\Delta + \hat{A}_R)E = \Lambda\Delta \tag{8.16b}$$

which are identical in form to Eqs. (2.15). Thus the perturbation analyses of the previous chapters can be used in their entirety.

We now have two bases upon which a perturbation analysis can be applied, namely that given in Eqs. (2.9), (2.15) and (2.16) and that given in Eqs. (8.14) - (8.16). The latter basis is more general, and hence preferable, in that its use is independent of trial vectors that satisfy an eigenvalue problem. Furthermore, the latter basis provides more accurate eigenvalue estimates since they are determined from Rayleigh's quotient.

Let us suppose Eqs. (8.14) - (8.16) are used as the basis of a perturbation analysis. Then, from Eq. (2.28),

$$\lambda_{1i} = \Delta_{ii}(\hat{a}_{Rii} - \mathcal{R}_i\, \hat{b}_{Rii}) \quad , \quad i = 1,2,\ldots,n \tag{8.17}$$

From Eqs. (8.15b) and (8.14), we have

$$\hat{a}_{Rii} = \underset{\sim}{z}_i^T A \underset{\sim}{w}_i - \frac{\underset{\sim}{z}_i^T A \underset{\sim}{w}_i}{\underset{\sim}{z}_i^T B \underset{\sim}{w}_i} \Delta_{ii} \quad , \quad i = 1,2,\ldots,n \tag{8.18}$$

which can be expressed as

$$\hat{a}_{Rii} = \frac{\underset{\sim}{z}_i^T A \underset{\sim}{w}_i}{\underset{\sim}{z}_i^T B \underset{\sim}{w}_i} (\underset{\sim}{z}_i^T B \underset{\sim}{w}_i - \Delta_{ii}) \quad , \quad i = 1,2,\ldots,n \tag{8.19}$$

Then, bearing in mind Eqs. (8.14) and (8.15a), we have

$$\hat{a}_{Rii} = \mathcal{R}_i\, \hat{b}_{Rii} \quad , \quad i = 1,2,\ldots,n\ . \tag{8.20}$$

It follows from Eqs. (8.17) and (8.20) that

$$\lambda_{1i} = 0 \quad , \quad i = 1,2,\ldots,n. \tag{8.21}$$

We note that if some of the $\mathcal{R}_i$ $(i = 1,2,\ldots,n)$ are not clearly distinct, Eq. (8.21) remains valid if the appropriate $O(0)$ eigenvector perturbations, matrices E and G from Chapters 6 or 7, are included in the trial vectors. When the $\mathcal{R}_i$ $(i = 1,2,\ldots,n)$ are clearly distinct, the developments of Chapter 2 are applicable, and Eq. (8.21) indicates convenient simplifications. Alternatively, when some of the $\mathcal{R}_i$ $(i = 1,2,\ldots,n)$ are not clearly distinct, Eq. (8.21) is a liability because multiple $O(1)$ eigenvalue perturbations are specifically excluded from the developments in Chapters 6 and 7. Thus if some of the R_i ($i = 1,2,\ldots,n$) are not clearly distinct, a perturbation analysis based on Eqs. (8.14) - (8.16) must use R_i ($i = 1,2,\ldots,n$) generated from $O(0)$ unperturbed trial vectors, thereby invalidating Eq. (8.21).

Bearing in mind that the R_i ($i = 1,2,\ldots,n$) are derived from Rayleigh's quotient, Eq. (8.21) implies that the information usually contained in λ_{1i} ($i = 1,2,\ldots,n$) is included in R_i ($i = 1,2,\ldots,n$). Evidence of this speculation is provided by taking

$$\underset{\sim}{w}_i = \underset{\sim}{u}_{0i} \quad , \quad \underset{\sim}{y}_i = \underset{\sim}{v}_{0i} \quad , \quad i = 1,2,\ldots,n \tag{8.22}$$

Then substituting Eqs. (8.22) and (2.5) into Eq. (8.14), we obtain

$$R_i = \frac{\underset{\sim}{v}_{0i}^T\,(A_0 + A_1)\underset{\sim}{u}_{0i}}{\underset{\sim}{v}_{0i}^T\,(B_0 + B_1)\underset{\sim}{u}_{0i}} \quad , \quad i = 1,2,\ldots,n \tag{8.23}$$

Use of Eqs. (2.9) and (2.16) allow Eq. (8.23) to be expressed as

$$R_i = \frac{\lambda_{0i}\Delta_{ii} + \hat{a}_{ii}}{\Delta_{ii} + \hat{b}_{ii}} \quad , \quad i = 1,2,\ldots,n \tag{8.24}$$

We expand the denominator to obtain

$$R_i = \lambda_{0i} + \Delta_{ii}(\hat{a}_{ii} - \lambda_{0i}\,\hat{b}_{ii}) + \ldots \quad , \quad i = 1,2,\ldots,n \tag{8.25}$$

Comparison with Eq. (2.28) indicates that Eq. (8.25) can be written as

$$R_i = \lambda_{0i} + \lambda_{1i} + \ldots \quad , \quad i = 1,2,\ldots,n \tag{8.26}$$

Equation (8.26) is pleasing in that it vividly exemplifies the stationarity, and hence the improvement of accuracy, afforded by the use of Rayleigh's quotient. Furthermore, we observe that the perturbation analysis and Rayleigh's quotient do indeed provide the same $O(1)$

eigenvalue perturbations. As a footnote, if we substitute for $\underset{\sim}{w}_i$ and $\underset{\sim}{z}_i$ the ith columns of $U_0(I + E_1)$ and $V_0(I + \Gamma_1)$, respectively, it can be shown that

$$R_i = \lambda_{0i} + \lambda_{1i} + \lambda_{2i} + \lambda_{3i} + \dots \quad , \quad i = 1,2,\dots,n \tag{8.27}$$

Equation (8.27) once again exemplifies the stationarity of Rayleigh's quotient.

From the above, computation of Rayleigh's quotient emerges as a means whereby the numerical accuracy of reasonably accurate eigenvalue estimates can be enhanced. Alternatively, if accuracy is not of great importance, or if the perturbing matrices A_1 and B_1 are extremely small, computation of Rayleigh's quotient is a means whereby eigenvalue estimates can be upgraded to include the effects of B_1 and A_1, or equivalently of $\hat{B}$ and $\hat{A}$. During the course of actual computations, Rayleigh's quotient is relatively simple to compute, especially if biorthonormality computations have already been performed as a check on eigenvector accuracy.

To exemplify the use of Rayleigh's quotient, let us return to the examples of Chapters 5-7. In Tables 8.1 and 8.2 we present a summary of eigenvalue estimates, computed via Rayleigh's quotient, using eigenvectors from the two examples of Chapter 5 summarized through various orders. Comparison of Tables 8.1 and 8.2 with Tables 5.1 and 5.4, respectively, reveals the increased accuracy afforded by the use of Rayleigh's quotient. The results presented in Tables 8.3 and 8.4 were computed using eigenvectors from the examples of Chapters 6 and 7, respectively. Once again, comparing Tables 8.3 and 8.4 with Tables 6.1 and 7.1, respectively, the increase in accuracy is evident.

TABLE 8.1

Summary of Rayleigh's Quotients,

Using First Example in Chapter 5,

$c = h = 1/4$, $\Omega = \pi/2$

	O(0)	O(0) + O(1)	O(0) + ... + O(2)	Exact
λ_1	-0.053,009 +i0.459,019	-0.053,198 +i0.453,850	-0.053,198 +i0.453,849	-0.053,198 +i0.453,849
λ_2	-0.243,245 +i11.150,798	-0.243,244 +i11.148,030	-0.243,244 +i11.148,030	-0.243,244 +i11.148,030
λ_3	-0.216,605 +i19.786,069	-0.216,601 +i19.784,345	-0.216,601 +i19.784,345	-0.216,601 +i19.784,345
λ_4	-0.109,940 +i3.798,539	-0.109,967 +i3.798,118	-0.109,967 +i3.798,118	-0.109,967 +i3.798,118
λ_5	-0.256,755 +i17.520,309	-0.256,756 +i17.518,511	-0.256,756 +i17.518,511	-0.256,756 +i17.518,511
λ_6	-0.220,236 +i29.990,330	-0.220,234 +i29,989,158	-0.220,234 +i29.989,158	-0.220,234 +i29.989,158

TABLE 8.2

Summary of Rayleigh's Quotients,

Using Second Example in Chapter 5,

$c = h = 1/4$, $\Omega = \sqrt{21.6}\,\pi$

	0(0)	0(0) + 0(1)	0(0) + ... + 0(2)	Exact
λ_1	0.189,571 +i12.390,245	0.187,778 +i12.379,277	0.187,816 +i12.379,475	0.187,815 +i12.379,469
λ_2	2.559,263	2.565,406	2.565,408	2.565,408
λ_3	-0.158,848 +i8.801,007	-0.158,846 +i8.798,580	-0.158,846 +i8.798,580	-0.158,846 +i8.798,580
λ_4	-0.352,698 +i16.800,243	-0.350,907 +i16.821,612	-0.350,952 +i16.821,331	-0.350,951 +i16.821,338
λ_5	-0.372,904 +i29.323,904	-0.372,924 +i29.323,856	-0.372,924 +i29.323,856	-0.372,924 +i29.323,856
λ_6	-0.278,016 +i40.433.685	-0.278,018 +i40.432,571	-0.278,018 +i40.432,571	-0.278,018 +i40.432,571
λ_8	-2.813,974	-2.819,559	-2.819,561	-2.819,561

TABLE 8.3

Summary of Rayleigh's Quotients,

Using Example in Chapter 6,

Multiple 0(0) Eigenvalues

	0(0)	0(0) Perturbed	0(0) + 0(1)	0(0) + 0(1) + 0(2)	Exact
λ_1	3.0	3.0 -i0.186,603	2.993,683 -i0.187,085	2.993,721 -i0.187,086	2.993,721 -i0.187,086
λ_3	1.0	1.0	1.012,669	1.012,559	1.012,559

TABLE 8.4

Summary of Rayleigh's Quotients,

Using Example in Chapter 7,

Nearly Multiple 0(0) Eigenvalues

	0(0)	0(0) Perturbed	0(0) + 0(1)	0(0) + 0(1) + 0(2)	Exact
λ_1	3.006,247 +i0.000,971	3.0	3.0	3.0	3.0
λ_2	3.006,247 -i0.000,971	3.012,495	3.000,003	3.000,001	3.0
λ_3	0.987,505	0.987,505	0.999,997	0.999,999	1.0

Chapter 9

AN ITERATIVE PROCEDURE

In the present chapter, we utilize the generalized basis for perturbation analyses, expressed by Eqs. (8.14) - (8.16), as the basis for an iterative eigensolution procedure. We use a superscript in parentheses to indicate the iteration number.

Following the developments between Eqs. (8.11) and (8.16) we presume the availability of the matrices $W^{(k)} \simeq U$ and $Z^{(k)} \simeq V$, that are approximately biorthonormal with respect to matrices A and B as in Eqs. (2.4), along with the corresponding diagonal matrix of eigenvalue estimates $R^{(k)}$ and the error matrices $\hat{A}_R^{(k)}$ and $\hat{B}_R^{(k)}$, where

$$\hat{B}_R^{(k)} = Z^{(k)T} B W^{(k)} - \Delta \tag{9.1a}$$

$$\hat{A}_R^{(k)} = Z^{(k)T} A W^{(k)} - R^{(k)} \Delta \tag{9.1b}$$

Let us take

$$W^{(k+1)} = W^{(k)} E^{(k)} (I + E_1^{(k)} + \ldots + E_p^{(k)}) \tag{9.2a}$$

$$Z^{(k+1)} = Z^{(k)} G^{(k)} (I + \Gamma_1^{(k)} + \ldots + \Gamma_p^{(k)}) \ . \tag{9.2b}$$

Using $R^{(k)}$, $\hat{A}_R^{(k)}$ and $\hat{B}_R^{(k)}$ in place of Λ_0, $\hat{A}$ and $\hat{B}$, respectively, we determine the elements of matrices $E^{(k)}$, $E_1^{(k)}$, ..., $E_p^{(k)}$, $G^{(k)}$, $\Gamma_1^{(k)}$, ..., $\Gamma_p^{(k)}$ according to the results of Chapters 2, 6 and 7. We recall that if all of the $R_i^{(k)}$ $(i = 1,2,\ldots,n)$ are clearly distinct, matrices $E^{(k)}$ and $G^{(k)}$ become the identity matrices and we use the results of Chapter 2. Otherwise, we use the results of Chapters 6 or 7, as appropriate. Because the expansions indicated in Eqs. (9.2) are finite, the matrices $W^{(k+1)}$ and $Z^{(k+1)}$, although more accurate than $W^{(k)}$ and $Z^{(k)}$, respectively, remain approximations to U and V, respectively. Thus we compute the products $Z^{(k+1)T}BW^{(k+1)}$ and $Z^{(k+1)T}AW^{(k+1)}$. From the diagonal elements of these products, we compute eigenvalue estimates according to

$$R_i^{(k+1)} = \frac{\underset{\sim}{z}_i^{(k+1)T}A\underset{\sim}{w}_i^{(k+1)}}{\underset{\sim}{z}_i^{(k+1)T}B\underset{\sim}{w}_i^{(k+1)}} \quad , \quad i = 1,2,\ldots,n \quad , \tag{9.3}$$

whereupon the error matrices $\hat{A}_R^{(k+1)}$ and $\hat{B}_R^{(k+1)}$ can be constructed according to

$$\hat{B}_R^{(k+1)} = Z^{(k+1)T}BW^{(k+1)} - \Delta \tag{9.4a}$$

$$\hat{A}^{(k+1)} = Z^{(k+1)T}AW^{(k+1)} - R^{(k+1)}\Delta \ . \tag{9.4b}$$

This completes one iteration. We consider convergence to have been attained when the elements of matrices $\hat{A}_R$ and $\hat{B}_R$ are smaller than some previously specified threshold value.

A few comments are in order. We note that the utility of the iteration procedure hinges upon the availability of initial trial vectors $W^{(1)}$ and $Z^{(1)}$ for which the matrices $\hat{A}_R^{(1)}$ and $\hat{B}_R^{(1)}$ are sufficiently small that the perturbation analyses of Chapters 2, 6 and 7 are applicable. The procedure is indeed iterative since the expansions of Eqs. (9.2) are finite. Of course, the user is at liberty to retain as many terms as are deemed necessary - more terms retained should indicate fewer iterations to convergence and vice versa. Let us define the small parameter $\mu^{(k)}$, which typifies the smallness of the matrices $\hat{A}^{(k)}$ and $\hat{B}^{(k)}$. This is to say that

$$\hat{A}_R^{(k)}\ ,\ \hat{B}_R^{(k)} \sim \mu^{(k)}$$

Then, if we retain terms through O(p), as indicated in Eqs. (9.2), subsequent iterations yield

$$\hat{A}_R^{(k+1)}\ ,\ \hat{B}_R^{(k+1)} \sim (\mu^{(k)})^{p+1}$$
$$\hat{A}_R^{(k+2)}\ ,\ \hat{B}_R^{(k+2)} \sim (\mu^{(k)})^{(p+1)^2}$$
$$\hat{A}_R^{(k+3)}\ ,\ \hat{B}_R^{(k+3)} \sim (\mu^{(k)})^{(p+1)^3}$$
$$\vdots$$

Thus the convergence for eigenvector estimates is fairly rapid. Furthermore, due to the Rayleigh's quotient method in which eigenvalues are estimated, the eigenvalue estimates will be as accurate as the eigenvectors computed in the next iteration.

As a final point we mention that the procedure set forth here does not allow the accumulation of computational errors from one iteration to the next. This is due to the fact that in each iteration we use trial vectors $W^{(k)}$ and $Z^{(k)}$ to determine nearly diagonal matrices $(\Delta + \hat{B}^{(k)})$ and $(\Lambda^{(k)}\Delta + \hat{A}^{(k)})$ based upon the original matrices A and B.

Toward presenting an example, let us use the iterative procedure in its simplest form, thus, in the eigenvector expansions of Eqs. (9.2) we retain terms only through O(1), and assume clearly distinct eigenvalues. Equations (9.2) appear as

$$W^{(k+1)} = W^{(k)}(I + E_1^{(k)}) \tag{9.5a}$$

$$Z^{(k+1)} = Z^{(k)}(I + \Gamma_1^{(k)}) \tag{9.5b}$$

Let us use the iterative procedure, based upon Eqs. (9.5), to generate the exact eigenvalues and eigenvectors of the example in Chapter 6.

Nominally, this example is a poor one in that this example possesses nearly multiple eigenvalues while the expansions of Eq. (9.5) presume clearly distinct eigenvalues. This difficulty is neatly obviated if we use the O(0) perturbed eigenvectors of Eqs. (6.43) as starting trial vectors.

From Eqs. (6.26) - (6.29) we have the eigenvalue problem

$$A\underset{\sim}{u}_i = \lambda_i \underset{\sim}{u}_i \quad , \quad A^T \underset{\sim}{v}_i = \lambda_i \underset{\sim}{v}_i \quad , \quad i = 1,2,3 \tag{9.6a,b}$$

where

$$A = \begin{bmatrix} 3/2 & \sqrt{3/4} + 1/10 & 2/10 \\ \sqrt{3/4} - 1/10 & 5/2 & 1/10 \\ -2/10 & -1/10 & 3 \end{bmatrix} . \tag{9.7}$$

From Eqs. (6.43) we have the starting trial vectors

$$W^{(0)} = \begin{bmatrix} \begin{array}{l} 0.25 \\ -i0.25 \end{array} & \begin{array}{l} 0.25 \\ +i0.25 \end{array} & 0.866{,}025 \\ \begin{array}{l} 0.433{,}013 \\ -i0.433{,}013 \end{array} & \begin{array}{l} 0.433{,}013 \\ +i0.433{,}013 \end{array} & -0.5 \\ \begin{array}{l} -0.5 \\ -i0.5 \end{array} & \begin{array}{l} -0.5 \\ +i0.5 \end{array} & 0.0 \end{bmatrix} \tag{9.8a}$$

$$Z^{(0)} = \bar{W}^{(0)} \tag{9.8b}$$

As a means of monitoring the computations, we present the matrices $\hat{A}_R^{(k)}$ and the eigenvalue estimates $R^{(k)}$ $(k = 1,2,3,\ldots)$. Note that the procedure does converge to the exact eigenvectors presented in Table 6.2.

k = 1 iteration:

$$\hat{A}_R^{(1)} = \begin{bmatrix} 0.0 & 0.0 & 0.011{,}603 \\ +i0.0 & +i0.0 & -i0.111{,}603 \\ & & \\ 0.0 & 0.0 & 0.011{,}603 \\ +i0.0 & +i0.0 & +i0.111{,}603 \\ & & \\ -0.011{,}603 & -0.011{,}603 & 0.0 \\ +i0.111{,}603 & +i0.111{,}603 & +i0.0 \end{bmatrix}$$

$$R^{(1)} = \begin{bmatrix} 3.0 & 3.0 & 1.0 \\ -i0.186{,}603 & +i0.186{,}603 & \end{bmatrix}$$

k = 2 iteration

$$\hat{A}_R^{(2)} = \begin{bmatrix} -0.009{,}288 & -0.015{,}268 & 0.000{,}136 \\ -i0.001{,}154 & +i0.003{,}209 & -i0.000{,}684 \\ & & \\ 0.015{,}268 & -0.009{,}288 & 0.000{,}136 \\ -i0.003{,}209 & +i0.001{,}154 & +i0.000{,}684 \\ & & \\ -0.000{,}007 & -0.000{,}007 & -0.006{,}210 \\ -i0.000{,}697 & +i0.000{,}697 & \end{bmatrix}$$

$$R^{(2)} = \begin{bmatrix} 2.993{,}741 & 2.993{,}741 & 1{,}012{,}558 \\ -i0.187{,}190 & +i0.187{,}190 & \end{bmatrix}$$

k = 3 iteration

$$\hat{A}_R^{(3)} = \begin{bmatrix} -0.000{,}868 & 0.000{,}047 & 0.000{,}007 \\ +i0.000{,}203 & +i0.000{,}010 & +i0.000{,}009 \\ & & \\ 0.000{,}047 & -0.000{,}868 & 0.000{,}007 \\ -i0.000{,}010 & -i0.000{,}203 & +i0.000{,}009 \\ & & \\ 0.000{,}005 & 0.000{,}005 & -0.000{,}029 \\ -i0.000{,}010 & +i0.000{,}010 & \end{bmatrix}$$

$$R^{(3)} = \begin{bmatrix} 2.993{,}721 & 2.993{,}721 & 1{,}012{,}559 \\ -i0.187{,}086 & +i0.187{,}086 & \end{bmatrix}$$

The elements of $\hat{A}_R^{(4)}$ are less than 10^{-6}, while the eigenvalue estimates remain unchanged. The rate of convergence is more easily grasped if we inspect the norm of $\hat{A}_R^{(k)}$ (k = 1,2,...), presented in Table 9.1.

TABLE 9.1

Norm of $\hat{A}_R^{(k)}$ $(k = 1,2,\ldots)$

k	$\lVert \hat{A}_R^{(k)} \rVert$
1	0.224×10^{0}
2	0.265×10^{-1}
3	0.126×10^{-2}
4	0.313×10^{-6}
5	0.249×10^{-13}
$\vdots$	$\vdots$

Chapter 10

DYNAMIC RESPONSE

Consider the set of linear, coupled ordinary differential equations

$$B\dot{\underset{\sim}{u}}(t) = A\underset{\sim}{u}(t) + \underset{\sim}{U}(t) \tag{10.1}$$

where B and A are constant n × n coefficient matrices, $\underset{\sim}{u}(t)$ is the n × 1 column vector of coordinates, $\underset{\sim}{U}(t)$ is an n × 1 column vector of excitations, which are generally functions of the time t.

A direct approach to the solution of Eq. (10.1) is to write it as

$$\dot{\underset{\sim}{u}}(t) = B^{-1}A\underset{\sim}{u}(t) + B^{-1}\underset{\sim}{U}(t) \tag{10.2}$$

from which the solution can be written as

$$\underset{\sim}{u}(t) = \Phi(t)\ \underset{\sim}{u}(0) + \int_{\tau=0}^{\tau=t} d\tau\ \Phi(t-\tau)\ B^{-1}\underset{\sim}{U}(\tau) \tag{10.3}$$

where

$$\Phi(t) = e^{B^{-1}At}$$

is the so-called state transition matrix. The incremental form is often more useful. This is

$$\underset{\sim}{u}(t+\delta t) = \Phi(\delta t) \left\{\underset{\sim}{u}(t) + \int_{\tau=t}^{\tau=t+\delta t} \Phi(t-\tau)\ B^{-1}U(\tau)\right\} \tag{10.4a}$$

If we allow the approximations

$$\Phi(t-\tau) \simeq \Phi(0) = I \quad , \quad t \leq \tau \leq t + \delta t$$

$$\underset{\sim}{U}(\tau) \simeq \underset{\sim}{U}(t) \quad , \quad t \leq \tau \leq t + \delta t \quad ,$$

then Eq. (10.4a) can be written in discrete time form as

$$\underset{\sim}{u}_{k+1} = \Phi(T)(\underset{\sim}{u}_k + T\, B^{-1}\underset{\sim}{U}_k) \quad , \quad k = 0,1,2,\ldots \tag{10.4b}$$

where T is the increment. Implementation of this result can be difficult. In particular, computational difficulties may be encountered in the determination of $\Phi(T)$ from the matrix series

$$\Phi(t) \equiv e^{B^{-1}AT} = I + (B^{-1}A)T + \frac{1}{2!}(B^{-1}A)^2T^2 + \ldots \quad , \tag{10.5}$$

especially if matrices B and A are large.

One way to avoid this difficulty is to perform a modal analysis. The first stage of such an analysis entails solving the eigenvalue problem for the biorthogonal matrices U and V, while in the second stage, matrices U and V are used as the basis of a transformation which diagonalizes the coefficient matrices A and B. The resulting decoupled equations can then be solved independently. The advantage of a modal analysis is that the state transition matrix is rendered diagonal, thus alleviating the computational difficulties mentioned above. In balance, we must solve the eigenvalue problem, which can be a tedious task. However, many problems in mechanics can be discretized such that the coefficient matrices A and B possess properties that ease the tedium of eigensolution analysis. In the present endeavor, the modal analysis is considered prefereable.

We consider the algebraic eigenvalue problem

$$BU\Lambda = AU \tag{10.6a}$$

and its adjoint

$$B^TV\Lambda = A^TV \tag{10.6b}$$

where the columns of the $n \times n$ modal matrices U and V are the right and left eigenvectors, respectively, and where Λ is the diagonal matrix of eigenvalues. Let us assume that the solutions of Eqs. (10.6) are known, and that the eigenvectors have been normalized such that

$$V^T BU = \Delta \tag{10.7a}$$

$$V^T AU = \Lambda\Delta \tag{10.7b}$$

where Δ is a diagonal matrix, the elements of which are either +1 or -1.

Returning to Eqs. (10.1), let us take

$$\underset{\sim}{u}(t) = U\underset{\sim}{\eta}(t) \tag{10.8}$$

where $\eta(t)$ is another $n \times 1$ column vector of coordinates. Note that Eq. (10.8) can also be written as

$$\underset{\sim}{u}(t) = \sum_{p=1}^{n} \underset{\sim}{u}_p \eta_p(t) \tag{10.9}$$

thus illustrating that Eq. (10.8) amounts to taking $\underset{\sim}{u}(t)$ as a linear combination of the right eigenvectors $\underset{\sim}{u}_p$ $(p = 1,2,\ldots,n)$. We substitute Eq. (10.8) into Eqs. (10.1) and premultiply by V^T to obtain

$$V^T BU\dot{\underset{\sim}{\eta}}(t) = V^T AU\underset{\sim}{\eta}(t) + V^T\underset{\sim}{U}(t) \tag{10.10}$$

Recalling Eqs. (10.7), Eqs. (10.10) can now be written in the <u>de-coupled</u> form

$$\dot{\underset{\sim}{\eta}}(t) = \Lambda\underset{\sim}{\eta}(t) + \underset{\sim}{N}(t) \tag{10.11}$$

where

$$\underset{\sim}{N}(t) \equiv \Delta V^T\underset{\sim}{U}(t) \tag{10.12}$$

Thus we observe that solution of the eigenvalue problem provides the transformation matrices, U and V, through which Eqs. (10.1) can be uncoupled.

The solution of Eq. (10.11) can be written in terms of the convolution integral as

$$\underset{\sim}{\eta}(t) = e^{\Lambda t}\underset{\sim}{\eta}(0) + \int_{\tau=0}^{\tau=t} d\tau \; e^{\Lambda(t-\tau)}\underset{\sim}{N}(\tau) \tag{10.13}$$

where the diagonal matrix $e^{\Lambda t}$ is constructed as

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \tag{10.14}$$

and the diagonal matrix $e^{\Lambda(t-\tau)}$ is constructed similarly. The discrete time form of Eqs. (10.13), analogous to Eq. (10.4b), is

$$\underset{\sim}{\eta}_{k+1} = e^{\Lambda T}\{\underset{\sim}{\eta}_k + T\ \underset{\sim}{N}_k\} \quad , \quad k = 0,1,2,... \tag{10.15}$$

Once $\underset{\sim}{\eta}(t)$ is known, we substitute into Eq. (10.8) to obtain $\underset{\sim}{u}(t)$. Note that $\underset{\sim}{u}(t)$ and $\underset{\sim}{U}(t)$ are related to $\underset{\sim}{\eta}(t)$ and $\underset{\sim}{N}(t)$, respectively, according to

$$\underset{\sim}{u}(t) \equiv U\underset{\sim}{\eta}(t) \Longleftrightarrow \underset{\sim}{\eta}(t) = \Delta V^T B\underset{\sim}{u}(t) \tag{10.16a}$$

$$\underset{\sim}{U}(t) = BU\underset{\sim}{N}(t) \Longleftrightarrow \underset{\sim}{N}(t) \equiv \Delta V^T \underset{\sim}{U}(t) \tag{10.16b}$$

Now in practice, exact eigensolutions are rarely available. However, let us assume the availability of approximate eigenvectors $W \simeq U$ and $Z \simeq V$. Following Eq. (10.8), let us take

$$\underset{\sim}{u}(t) = W\underset{\sim}{\eta}(t) \tag{10.17}$$

and substitute into Eq. (10.1). Upon premultiplication by Z^T we obtain

$$Z^T BW\dot{\underset{\sim}{\eta}}(t) = Z^T AW + Z^T \underset{\sim}{U}(t) \tag{10.18}$$

Let us approximate the products $Z^T BW$ and $Z^T AW$ as

$$Z^T BW \simeq \lceil Z^T BW \rfloor \tag{10.19a}$$

$$Z^T AW \simeq \lceil Z^T AW \rfloor \tag{10.19b}$$

where the notation $\lceil\ \rfloor$ indicates that all but the diagonal elements are zero. Equation (10.18) can now be expressed as

$$\dot{\underset{\sim}{\eta}}(t) = \mathcal{R}\underset{\sim}{\eta}(t) + \underset{\sim}{N}(t) \tag{10.20}$$

where

$$\mathcal{R} \equiv \lceil Z^T BW \rfloor^{-1} \lceil Z^T AW \rfloor \tag{10.21}$$

is a diagonal matrix, and where

$$\underset{\sim}{N}(t) \equiv \lceil Z^T BW \rfloor^{-1} Z^T \underset{\sim}{U}(t) \tag{10.22}$$

We note from Eq. (10.21) that the elements of $\mathcal{R}$ are given by

$$\mathcal{R}_i = \frac{\underset{\sim}{z}_i^T A \underset{\sim}{w}_i}{\underset{\sim}{z}_i^T B \underset{\sim}{w}_i} \quad , \quad i = 1,2,\ldots,n \ . \tag{10.23}$$

The solution of Eq. (10.20) for $\underset{\sim}{\eta}(t)$ can be constructed as previously.

It is pleasing to observe how the modal analysis, based upon approximations to the exact matrices, readily allows a Rayleigh's quotient interpretation of the eigenvalues, as in Eq. (10.23), and hence the accompanying increase in accuracy. Note that when approximate eigenvectors are used, Eqs. (10.16) must be modified to read as

$$\underset{\sim}{u}(t) = W\underset{\sim}{\eta}(t) \Rightarrow \underset{\sim}{\eta}(t) = [Z^T BW]^{-1} Z^T B\underset{\sim}{u}(t) \qquad (10.24a)$$

$$\underset{\sim}{U}(t) = BW\underset{\sim}{N}(t) \Rightarrow \underset{\sim}{N}(t) = [Z^T BW]^{-1} Z^T \underset{\sim}{U}(t) \qquad (10.24b)$$

The preceding developments are applicable to any modal analysis, irrespective of how the modal matrices are determined.

To this point in the present chapter, perturbation analyses have not been mentioned. Let us suppose the approximate eigenvectors W and Z are the result of a perturbation analysis, and thus include the effects of the perturbation matrices A_1 and B_1. Then, according to the earlier developments in this chapter, although an unperturbed response may have been determined, we must repeat the response computations for the perturbed problem. In spite of the fact that response computations are relatively simple to perform, one may wish to express the perturbed response in terms of the unperturbed response and the perturbations A_1 and B_1. Taking special forms for the excitation, Chen and Wada [7] have done just that. However, to maintain the generality of Eq. (10.13), it appears that one cannot represent the perturbed response conveniently in terms of the unperturbed response and the perturbations A_1 and B_1.

Let us return to the examples of Chapter 5 and compute the response of the system depicted in Fig. 5.1. Starting from zero initial conditions, let us suppose the system is acted upon by the force

$$\vec{F}(z,t) = \vec{e}_x F_0 \, \delta(z - \tfrac{1}{2}L) \, \delta(t) \; ,$$

$$F_0 = 1 \text{ n/m sec} \qquad (10.25)$$

Using eigenvectors, from the first example of Chapter 5, summarized through various orders, we compute the response according to Eqs. (10.17) - (10.23), along with the exact response for comparison. In Fig. 10.1 we plot the deflection of the center of the beam in inertial coordinates, $Y(\frac{1}{2}L,t)$ versus $X(\frac{1}{2}L,t)$, with t as a parameter, for $0 \leqslant t \leqslant 10$ sec. The curve labelled O(0) was generated using the unperturbed eigenvectors. We note that within the accuracy of this plot, the curves generated using the perturbed eigenvectors summarized through O(1) and O(2) are identical to the exact response. Once again, it should be evident that the system is stable.

We repeat the response computations for the second example in Chapter 5 and present the results in Fig. 10.2 as a plot of $Y(\frac{1}{2}L,t)$ versus $X(\frac{1}{2}L,t)$, for $0 \leqslant t \leqslant 7$ sec. The instability should be evident.

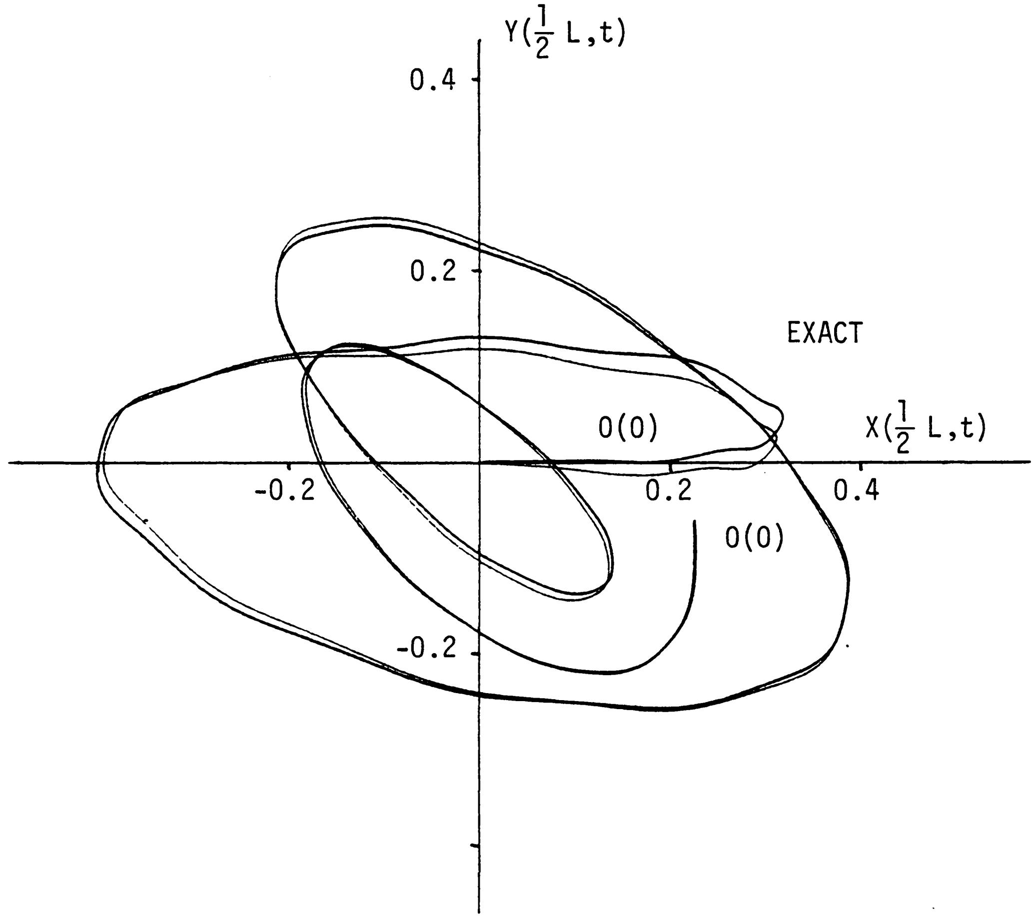

Figure 10.1

Dynamic Response $Y(\frac{1}{2}L,t)$ Versus $X(\frac{1}{2}L,t)$,

$0 \leqslant t \leqslant 10$ sec, $c = h = 1/4$, $\Omega = \pi/2$

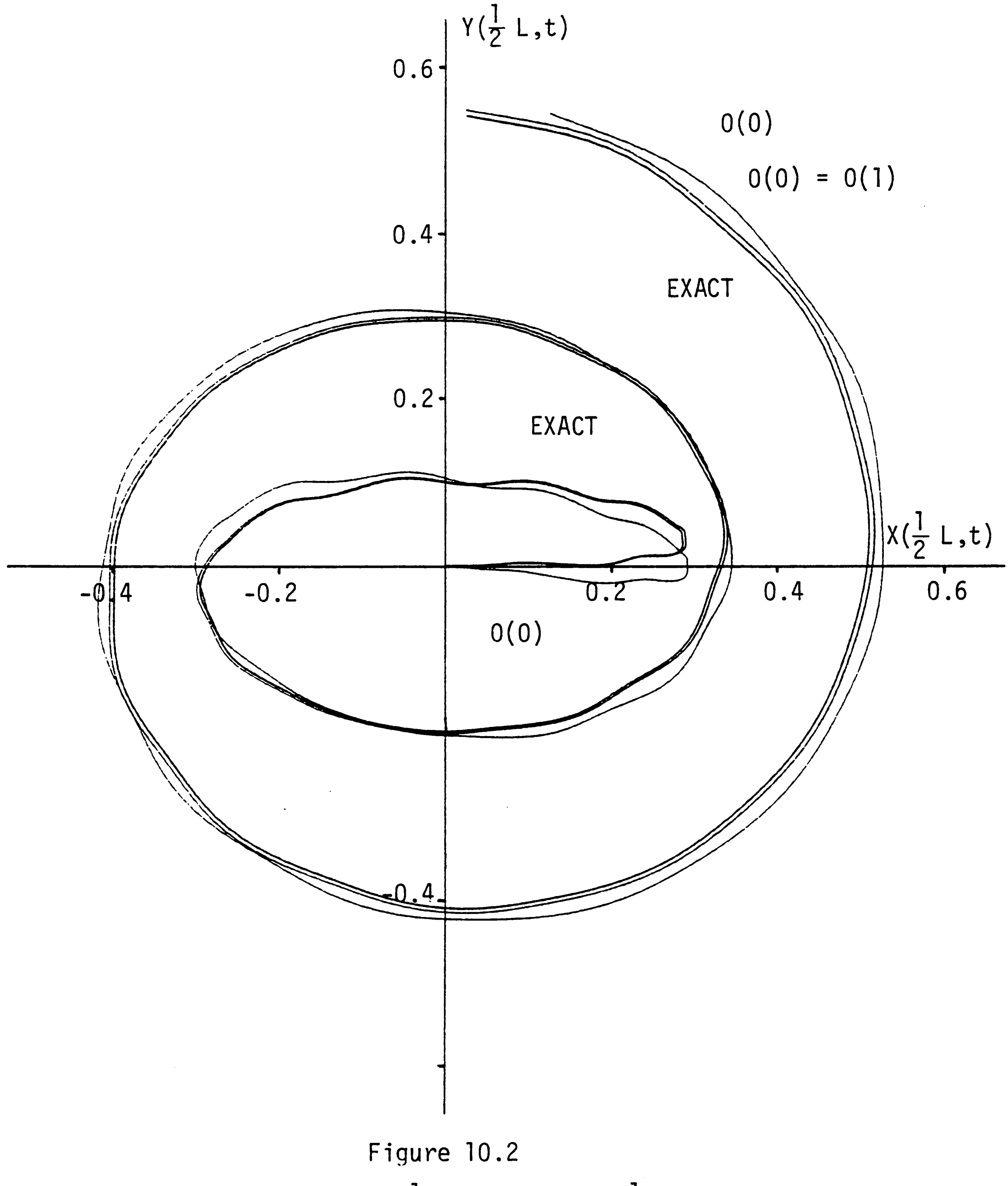

Figure 10.2

Dynamic Response $Y(\frac{1}{2}L,t)$ Versus $X(\frac{1}{2}L,t)$,

$0 \leqslant t \leqslant 7$ sec, $c = h = 1/4$, $\Omega = \sqrt{21.6}\,\pi$

Chapter 11

CONCLUSIONS

This dissertation concerns a perturbation theory for the algebraic eigenvalue problem. The perturbation scheme is based on biorthonormality relations. When the unperturbed eigenvalues are clearly distinct, the perturbation theory provides expressions for the perturbed eigenvalues and eigenvectors in terms of the unperturbed eigenvalues and eigenvectors and the perturbing matrices, without solving a new eigenvalue problem. Alternatively, when some of the unperturbed eigenvalues are not clearly distinct, the perturbation theory requires the solution of an eigenvalue problem that is of the same dimensions as the number of not clearly distinct unperturbed eigenvalues.

The perturbation theory should prove useful in many seemingly different analyses. The topic addressed in Chapter 5 is that of gyroscopic systems subjected to small internal and external damping forces. For this application of the perturbation theory, the perturbing matrices describe physical effects, namely damping, not included in the unperturbed eigenvalue problem. A slightly different application is that in which the perturbing matrices are corrections to the same physical effects described by the unperturbed eigenvalue problem. If we regard the perturbing matrices as errors, then the perturbation theory provides expressions for the corresponding eigenvalue and eigenvector errors. Conversely, given approximate eigenvectors, the matrices computed from these eigenvectors and the perturbing matrices, and taking the place of the perturbing matrices, can be regarded as measures of the errors associated with the approximate eigenvectors.

In Chapter 7 it is shown that Rayleigh's quotient provides eigenvalue estimates that include the first eigenvalue perturbation indicated by the perturbation theory. Furthermore, the use of Rayleigh's quotient allows the definition of a generalized vector basis for application of the perturbation theory, which does not depend on the use of eigenvectors satisfying an eigenvalue problem. This generalized basis allows the definition of an iterative eigensolution algorithm in which one iteration entails an application of the perturbation theory.

The perturbation theory has been applied to the rotating beam, depicted in Fig. 5.1, where the beam is subject to small internal and external damping forces. This particular system is interesting in that one configuration of it is unstable on two accounts: one eigenvalue is real and positive, while one complex conjugate pair of eigenvalues possess positive real parts, which is to say that the system exhibits divergence and flutter simultaneously.

For future endeavors, an investigation into the convergence of the eigenvalue and eigenvector expansions, as in Eqs. (2.17) and (2.18) may be fruitful. As of this writing, it appears that the perturbation theory for not clearly distinct unperturbed eigenvalues can be rendered more usable. Additional investigation of the stability and/or instability of slightly damped gyroscopic systems should be interesting.

REFERENCES

1. Franklin, Joel N., Matrix Theory, Prentice-Hall, Englewood Cliffs, N. J., 1968, Sec. 6.12.

2. Wilkinson, James H., The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1965, Secs. 2.5 - 2.7.

3. Lancaster, Peter, Lambda Matrices and Vibrating Systems, Pergamon Press, Oxford, 1966.

4. Aubrun, J. N., "Theory of the Control of Structures by Low-Authority Controllers", Journal of Guidance and Control, Vol. 3, No. 5, Sept. - Oct. 1980, pp. 444-451.

5. Huseyin, K., Vibrations and Stability of Multiple Parameter Systems, Sijthoff-Nordhoff International Publishers, Alphen aan den Rijn, The Netherlands and Rockville, Maryland, 1978.

6. Ziegler, Hans, Principles of Structural Stability, Blaisdell Publishing Co., Waltham, Mass., 1968.

7. Chen, J. C. and Wada, B. K., "Matrix Perturbation for Structural Dynamic Analysis", AIAA Journal, Vol. 15, No. 8, Aug. 1977, pp. 1095-1100.

8. Meirovitch, L. and Baruh, H., "Optimal Control of Damped Flexible Gyroscopic Systems", Journal of Guidance and Control, Vol. 4, No. 2, Mar. - Apr. 1981, pp. 157-163.

9. Meirovitch, Leonard, "A New System of Solution of the Eigenvalue Problem for Gyroscopic Systems", AIAA Journal, Vol. 12, No. 10, Oct. 1974, pp. 1337-1342.

10. Meirovitch, Leonard, "A Modal Analysis for the Response of Linear Gyroscopic Systems", Journal of Applied Mechanics, Vol. 42, No. 2, June 1975, pp. 446-450.

11. Meirovitch, L. and Ryland, G., "Response of Slightly Damped Gyroscopic Systems", Journal of Sound and Vibration, Vol. 67, Nov. - Dec. 1979, pp. 1-19.

12. Ryland, G. and Meirovitch, L., "Response of Vibrating Systems with Perturbed Parameters", Journal of Guidance and Control, Vol. 3, No. 4, Jul. - Aug. 1980, pp. 298-303.

13. Meirovitch, Leonard, Computational Methods in Structural Dynamics, Sijthoff-Nordhoff International Publishers, The Netherlands, 1980.

14. Meirovitch, Leonard, Analytical Methods in Vibrations, The MacMillan Co., New York, 1967.

APPENDIX

The perturbation analyses presented in the main text are based upon the biorthonormality relations for the eigenvalue problem. Of course, these analyses could have been based upon some other condition. Indeed, the derivations presented by Franklin [1], Wilkinson [2] and Lancaster [3] are based on the statement of the eigenvalue problem itself. In the following, we present a derivation that has been adapted from these three authors, and then compare it with the developments of Chapter 2.

Consider the algebraic eigenvalue problem in nonstandard form

$$\lambda_j B\underset{\sim}{u}_j = A\underset{\sim}{u}_j \quad , \quad j = 1,2,\ldots,n \tag{A.1a}$$

and its adjoint

$$\lambda_j B^T\underset{\sim}{v}_j = A^T\underset{\sim}{v}_j \quad , \quad j = 1,2,\ldots,n \tag{A.1b}$$

where

$$A = A_0 + A_1 \tag{A.2a}$$

$$B = B_0 + B_1 \tag{A.2b}$$

and where the elements of matrices A_1 and B_1 are small in comparison with the elements of matrices A_0 and B_0, respectively. We assume the availability of $O(0)$ eigensolutions $\underset{\sim}{u}_{0j}$, $\underset{\sim}{v}_{0j}$ and λ_{0j} $(j = 1,2,\ldots,n)$, which satisfy the biorthonormality relations

$$\underset{\sim}{v}_{0i}^T B_0\underset{\sim}{u}_{0j} = \underset{\sim}{u}_{0i}^T B_0^T\underset{\sim}{v}_{0j} = \Delta_{ij} \quad , \quad i,j = 1,2,\ldots,n \tag{A.3a}$$

$$\underset{\sim}{v}_{0i}^T A_0\underset{\sim}{u}_{0j} = \underset{\sim}{u}_{0i}^T A_0^T\underset{\sim}{v}_{0j} = \lambda_{0i}\Delta_{ij} \quad , \quad i,j = 1,2,\ldots,n \tag{A.3b}$$

Let us assume the expansions

$$\lambda_j = \lambda_{0j} + \lambda_{1j} + \cdots \quad , \quad j = 1,2,\ldots,n \tag{A.4a}$$

$$\underset{\sim}{u}_j = \underset{\sim}{u}_{0j} + \underset{\sim}{u}_{1j} + \cdots \quad , \quad j = 1,2,\ldots,n \tag{A.4b}$$

$$\underset{\sim}{v}_j = \underset{\sim}{v}_{0j} + \underset{\sim}{v}_{1j} + \cdots \quad , \quad j = 1,2,\ldots,n \tag{A.4c}$$

where λ_{1j}, $\underset{\sim}{u}_{1j}$ and $\underset{\sim}{v}_{1j}$ are small in comparison with λ_{0j}, $\underset{\sim}{u}_{0j}$ and $\underset{\sim}{v}_{0j}$ $(j = 1,2,\ldots,n)$, respectively. Furthermore, we observe that λ_{1j}, $\underset{\sim}{u}_{1j}$ and $\underset{\sim}{v}_{1j}$ $(j = 1,2,\ldots,n)$ must tend to zero as the elements of matrices A_1 and B_1 tend to zero. We state this as

$$\lambda_{1j},\ \underset{\sim}{u}_{1j},\ \underset{\sim}{v}_{1j} \to 0 \text{ as } A_1,\ B_1 \to 0 \quad , \quad j = 1,2,\ldots,n \tag{A.5}$$

Now, because the $\underset{\sim}{u}_{0j}$ $(j = 1,2,\ldots,n)$ span the space L^n in which the $\underset{\sim}{u}_{1j}$ are defined, we can express the $\underset{\sim}{u}_{1j}$ $(j = 1,2,\ldots,n)$ as a linear combination of the $\underset{\sim}{u}_{0j}$. Thus, we write

$$\underset{\sim}{u}_{1j} = \sum_{p=1}^{n} \underset{\sim}{u}_{0p}\, \epsilon_{1pj} \quad , \quad j = 1,2,\ldots,n \tag{A.6a}$$

Similar arguments allow us to write

$$\underset{\sim}{v}_{1j} = \sum_{p=1}^{n} \underset{\sim}{v}_{0p}\, \gamma_{1pj} \quad , \quad j = 1,2,\ldots,n \tag{A.6b}$$

We note that the smallness of $\underset{\sim}{u}_{1j}$ and $\underset{\sim}{v}_{1j}$ $(j = 1,2,\ldots,n)$ has been transferred to the coefficients ϵ_{1pj} and γ_{1pj} $(p,j = 1,2,\ldots,n)$, respectively. At this stage, Franklin [1] and Wilkinson [2] state that in order to maintain Condition (A.5), we must have

$$\epsilon_{1jj} = \gamma_{1jj} = 0 \quad , \quad j = 1,2,\ldots,n \tag{A.7}$$

Lancaster [3] invokes Eq. (A.7) for no stated reason.

Let us substitute Eqs. (A.2) and (A.4) into the statement of the right eigenvalue problem, Eq. (A.1a). We then substitute for $\underset{\sim}{u}_{1j}$ $(j = 1,2,\ldots,n)$ from Eq. (A.6a) and premultiply the result by $\underset{\sim}{v}_{0i}^T$ $(i = 1,2,\ldots,n)$ to obtain

$$\underset{\sim}{v}_{0i}^T(\lambda_{0j} + \lambda_{1j} + \ldots)(B_0 + B_1)(\underset{\sim}{u}_{0j} + \sum_{p=1}^{n} \underset{\sim}{u}_{0p}\, \epsilon_{1pj} + \ldots)$$

$$= \underset{\sim}{v}_{0i}^T(A_0 + A_1)(\underset{\sim}{u}_{0j} + \sum_{p=1}^{n} \underset{\sim}{u}_{0p}\, \epsilon_{1pj} + \ldots) \quad , \qquad i,j = 1,2,\ldots,n \qquad \text{(A.8)}$$

Separation according to order yields the equations

$$O(0): \quad \lambda_{0j}(\underset{\sim}{v}_{0i}^T B_0 \underset{\sim}{u}_{0j}) = (\underset{\sim}{v}_{0i}^T A_0 \underset{\sim}{u}_{0j}) \quad , \quad i,j = 1,2,\ldots,n \qquad \text{(A.9a)}$$

$$O(1): \quad \lambda_{0j} \sum_{p=1}^{n} (\underset{\sim}{v}_{0i}^T B_0 \underset{\sim}{u}_{0p})\, \epsilon_{1pj} + \lambda_{0j}(\underset{\sim}{v}_{0i}^T B_1 \underset{\sim}{u}_{0j}) + \lambda_{1j}(\underset{\sim}{v}_{0i}^T B_0 \underset{\sim}{u}_{0j})$$

$$= \sum_{p=1}^{n} (\underset{\sim}{v}_{0i}^T A_0 \underset{\sim}{u}_{0p})\, \epsilon_{1pj} + (\underset{\sim}{v}_{0i}^T A_1 \underset{\sim}{u}_{0j}) \quad , \qquad i,j = 1,2,\ldots,n \qquad \text{(A.9b)}$$

$$\vdots$$

If we utilize the biorthonormality expressed in Eqs. (A.3), and define

$$\hat{a}_{ij} \equiv v_{0i}^T A_1 u_{0j} \quad , \qquad i,j = 1,2,\ldots,n \qquad \text{(A.10a)}$$

$$\hat{b}_{ij} \equiv v_{0i}^T B_1 u_{0j} \quad , \qquad i,j = 1,2,\ldots,n \quad , \qquad \text{(A.10b)}$$

then Eqs. (A.9) can be written more compactly as

$$O(0): \quad \lambda_{0j}\Delta_{ij} = \lambda_{0i}\Delta_{ij} \quad , \qquad i,j = 1,2,\ldots, \qquad \text{(A.11a)}$$

$$O(1): \quad \lambda_{0j}\Delta_{ii}\epsilon_{1ij} + \lambda_{0j}\hat{b}_{ij} + \lambda_{1j}\Delta_{ij} = \lambda_{0i}\Delta_{ii}\epsilon_{1ij} + \hat{a}_{ij} \quad , \quad i,j = 1,2,\ldots,n \qquad \text{(A.11b)}$$

$$\vdots$$

Equation (A.11a) is an identity, as anticipated. Turning our attention to Eq. (A.11b), let us express it as

$$\Delta_{ii}(\lambda_{0i} - \lambda_{0j})\epsilon_{1ij} - \lambda_{1j}\Delta_{ij} = -(\hat{a}_{ij} - \lambda_{0j}\,\hat{b}_{ij}) \quad , \quad i,j = 1,2,\ldots,n \qquad \text{(A.12)}$$

When $i \neq j$, $\Delta_{ij} = 0$, and we can solve for ϵ_{1ij}.

$$\epsilon_{1ij} = -\Delta_{ii}\frac{\hat{a}_{ij} - \lambda_{0j}\,\hat{b}_{ij}}{\lambda_{0i} - \lambda_{0j}} \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,n \qquad \text{(A.13)}$$

Alternatively, when $i = j$, $\Delta_{ij} \neq 0$, yet ϵ_{1ij} is no longer present due to the coefficient $(\lambda_{0i} - \lambda_{0j})$. Thus we can solve for λ_{1j} $(j = 1,2,\ldots,n)$.

$$\lambda_{1j} = \Delta_{jj}(\hat{a}_{jj} - \lambda_{0j}\,\hat{b}_{jj}) \quad , \quad j = 1,2,\ldots,n \qquad \text{(A.14)}$$

If we repeat the above derivation with the left eigenvalue problem, Eq. (A.1b), the results are

$$\gamma_{1ij} = -\Delta_{ii}\frac{\hat{a}_{ji} - \lambda_{0j}\,\hat{b}_{ji}}{\lambda_{0i} - \lambda_{0j}} \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,n \qquad \text{(A.15)}$$

$$\lambda_{1j} = \Delta_{jj}(\hat{a}_{jj} - \lambda_{0j}\,\hat{b}_{jj}) \quad , \quad j = 1,2,\ldots,n \qquad \text{(A.16)}$$

At this point the analyses of Franklin, Wilkinson and Lancaster are complete through $O(1)$. One may note that the above derivation can be readily extended through higher orders. It is reassuring to note that the expressions given for λ_{1j} $(j = 1,2,\ldots,n)$ in Eqs. (A.14) and (A.16) are identical. Furthermore, a comparison with the results of Chapter 2 reveals that the same expressons for λ_{1j} $(j = 1,2,\ldots,n)$ and for ϵ_{1ij} and γ_{1ij} $(i \neq j,\ i,j = 1,2,\ldots,n)$ have been obtained.

In the preceding derivation, we recall that the opportunities to determine ϵ_{1jj} and γ_{1jj} $(j = 1,2,\ldots,n)$ never presented themselves. Then we can, in fact, relax Eq. (A.7) and accept <u>any</u> expressions for ϵ_{1jj} and γ_{1jj} $(j = 1,2,\ldots,n)$ that are consistent with Condition (A.5). Thus it appears that we have the opportunity to impose some additional conditions on the eigensolution. Because eigenvector biorthogonality and normalization are desirable, let us investigate these properties via the product $\underset{\sim}{v}_i^T B \underset{\sim}{u}_j$ $(i,j = 1,2,\ldots,n)$. To this end, we substitute from Eqs. (A.2b), (A.4b) and (A.4c) and then from Eqs. (A.6). The result is

$$\underset{\sim}{v}_i^T B \underset{\sim}{u}_j = (\underset{\sim}{v}_{0i}^T + \sum_{p=1}^{n} \underset{\sim}{v}_{0p}^T \gamma_{1pi} + \ldots)(B_0 + B_1)(\underset{\sim}{u}_{0j} + \sum_{p=1}^{n} \underset{\sim}{u}_{0p} \epsilon_{1pj} + \ldots) \quad , \quad i,j = 1,2,\ldots,n \qquad (A.17)$$

Recalling Eqs. (A.3a) and Eqs. (A.10b), Eq. (A.17) can be expressed as

$$\underset{\sim}{v}_i^T B \underset{\sim}{u}_j = \Delta_{ij} + \Delta_{ii}\, \epsilon_{1ij} + \hat{b}_{ij} + \Delta_{jj}\, \gamma_{1ji} + O(2) \quad , \quad i,j = 1,2,\ldots,n \qquad (A.18)$$

The first term on the right side of Eq. (A.18) is $O(0)$, while the terms of interest, i.e. the second through fourth, are $O(1)$. When $i \neq j$, we can substitute from Eqs. (A.13) and (A.15) for ϵ_{1ij} and γ_{1ij} $(i \neq j,\ i,j = 1,2,\ldots,n)$ to obtain

$$\underset{\sim}{v}_i^T B \underset{\sim}{u}_j = \Delta_{ij} + 0 + O(2) \quad , \quad i \neq j \quad , \quad i,j = 1,2,\ldots,n \qquad (A.19)$$

This result is pleasing in that the above first order analysis has produced eigenvectors that are biorthogonal through O(1).

Alternatively, let us write Eq. (A.18) for $i = j$ ($j = 1,2,\ldots,n$).

$$\underset{\sim}{v}_j^T B \underset{\sim}{u}_j = \Delta_{jj} + \Delta_{jj}(\epsilon_{1jj} + \gamma_{1jj}) + \hat{b}_{jj} + O(2) \quad , \quad j = 1,2,\ldots,n \qquad (A.20)$$

Clearly, if we utilize Eq. (A.7), the O(1) terms are nonzero, thus normalization is not preserved through O(1). Recalling that ϵ_{1jj} and γ_{1jj} ($j = 1,2,\ldots,n$) were not determined in the perturbation analysis, it appears that the relaxation of Eq. (A.7) will allow us the additional luxury of normalized eigenvectors. Indeed, the eigenvectors will satisfy normalization through O(1) if, in Eq. (A.20), we equate to zero the O(1) terms. That is

$$\Delta_{jj}(\epsilon_{1jj} + \gamma_{1jj}) = - \hat{b}_{jj} \quad , \quad j = 1,2,\ldots,n \qquad (A.21)$$

Any expressions for ϵ_{1jj} and γ_{1jj} ($j = 1,2,\ldots,n$) that satisfy Eq. (A.21) and Condition (A.5) are acceptable. It is pleasing to recall that Eq. (A.21) was addressed in Chapter 2 and that Eq. (2.27b) is an acceptable solution.

Thus, it appears that if the perturbation analysis is based upon the statement of the eigenvalue problem, the normalization must be imposed as a separate condition. An alternative is to base the perturbation analysis upon the biorthonormality relations. Such an analysis is guaranteed to produce eigensolutions that satisfy the eigenvalue problem and normalization through any order.

Hopefully, the chore of reading this is at least one order of magnitude less than the chore of writing it!

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PERTURBATION METHODS FOR

SLIGHTLY DAMPED GYROSCOPIC SYSTEMS

by

Garnett Ryland

(ABSTRACT)

This dissertation concerns the development of a perturbation theory applicable to the algebraic eigenvalue problem. The objective is to express the perturbed eigenvalues and eigenvectors in terms of the unperturbed eigenvalues and eigenvectors and the perturbing matrices, without solving a new eigenvalue problem. If all of the unperturbed eigenvalues are clearly distinct, which is to say that the difference between no two of them is small, this objective can be accomplished. Alternatively, if some of the unperturbed eigenvalues are not clearly distinct, an eigenvalue problem, of the same dimension as the number of not clearly distinct unperturbed eigenvalues, must be solved.

It is shown that Rayleigh's quotient is related to the perturbation theory. The use of Rayleigh's quotient is a key ingredient in defining a generalized basis for application of the perturbation theory, which does not depend upon the use of eigenvectors satisfying an eigenvalue problem. This generalized basis is then used to define an iterative eigensolution algorithm.

The perturbation theory was initially developed as an efficient means of solving the eigenvalue problem associated with linear gyroscopic systems for which the damping and/or circulatory effects are

sufficiently small that they can be regarded as perturbations. The perturbation theory is applied to an example of such a system. One interesting aspect of this example is that for one of the combinations of parameters, it exhibits both divergence and flutter instability, simultaneously. Other numerical examples are given.