SURGERY SPACES OF CRYSTALLOGRAPHIC GROUPS

by

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0. Introduction

Classification of topological spaces is one of the main problems in topology. Although classification of 1- and 2-dimensional manifolds up to homeomorphism is possible, the problem becomes extremely harder for higher dimensional manifolds. The classification problem of manifolds up to homeomorphism can be devided into two problems:

1. classification up to homotopy, and
2. classification up to homeomorphism in a given homotopy type.

The first problem is harder than the classification problem of finitely presented groups, because any such group can be realized as the fundamental group of some compact manifold. In this sense, classification is impossible.

Let us introduce some notation. ([W], [KS]) Let X be a finite Poincaré complex (i.e. a finite CW complex satisfying Poincaré duality) with formal dimension m such that ∂X is a manifold (possibly empty). A <u>homotopy-topological structure</u> on X is represented by a pair (M^m, f) where M^m is a compact manifold and f is a homotopy equivalence: $M^m \longrightarrow X$ which maps ∂M onto ∂X homeomorphically. Two such structures (M, f) (M', f') are equivalent if there exist an h-cobordism (W;M,M') from M to M' rel boundary (i.e. which is a product cobordism from ∂M to $\partial M'$) and a homotopy equivalence $F : W \longrightarrow X \times I$ such that $F(x) = (f(x),0)$ for $x \in M$ and $F(x) = (f'(x),1)$ for $x \in M'$. $\mathcal{S}(X)$ denotes the set of the equivalence classes of structures on X.

Notice that if $m \geqslant 6$, $\partial X = \phi$ and the Whitehead group $Wh(\pi_1 X) = 0$, then the vanishing of $\mathcal{S}(X)$ implies the uniqueness of the homotopy-topological structure on X up to homeomorphism. Thus the computation of $\mathcal{S}(X)$ is important in solving the second problem.

Surgery is a technique devised for studying these problems. Let X be a topological space, M a manifold, and $f: M \longrightarrow X$ be a map. For simplicity let us assume M is smooth and closed. The homotopy group $\pi(f)$ of f measures how near f is to a homotopy equivalence. If X is reasonably nice, then the vanishing of $\pi(f)$ implies that f is a homotopy equivalence. Consider an example. Let M be a 2-dimensional torus T^2 and X be a 2-dimensional sphere S^2. Take a small disk D^2 in T^2, and let P be a point on S^2. Since $S^2 - P$ is an open disk, there is a homeomorphism from $\mathrm{int}(D^2)$ (= interior) onto $S^2 - P$. By sending $T^2 - \mathrm{int}(D^2)$ to P, we get a (degree one) continuous map $f: T^2 \longrightarrow S^2$. Obviously this is not a homotopy equivalence. The loop α(see Fig.0) represents a non-trivial element of the fundamental group $\pi_1 T^2$, but f sends α to a trivial loop in S^2;

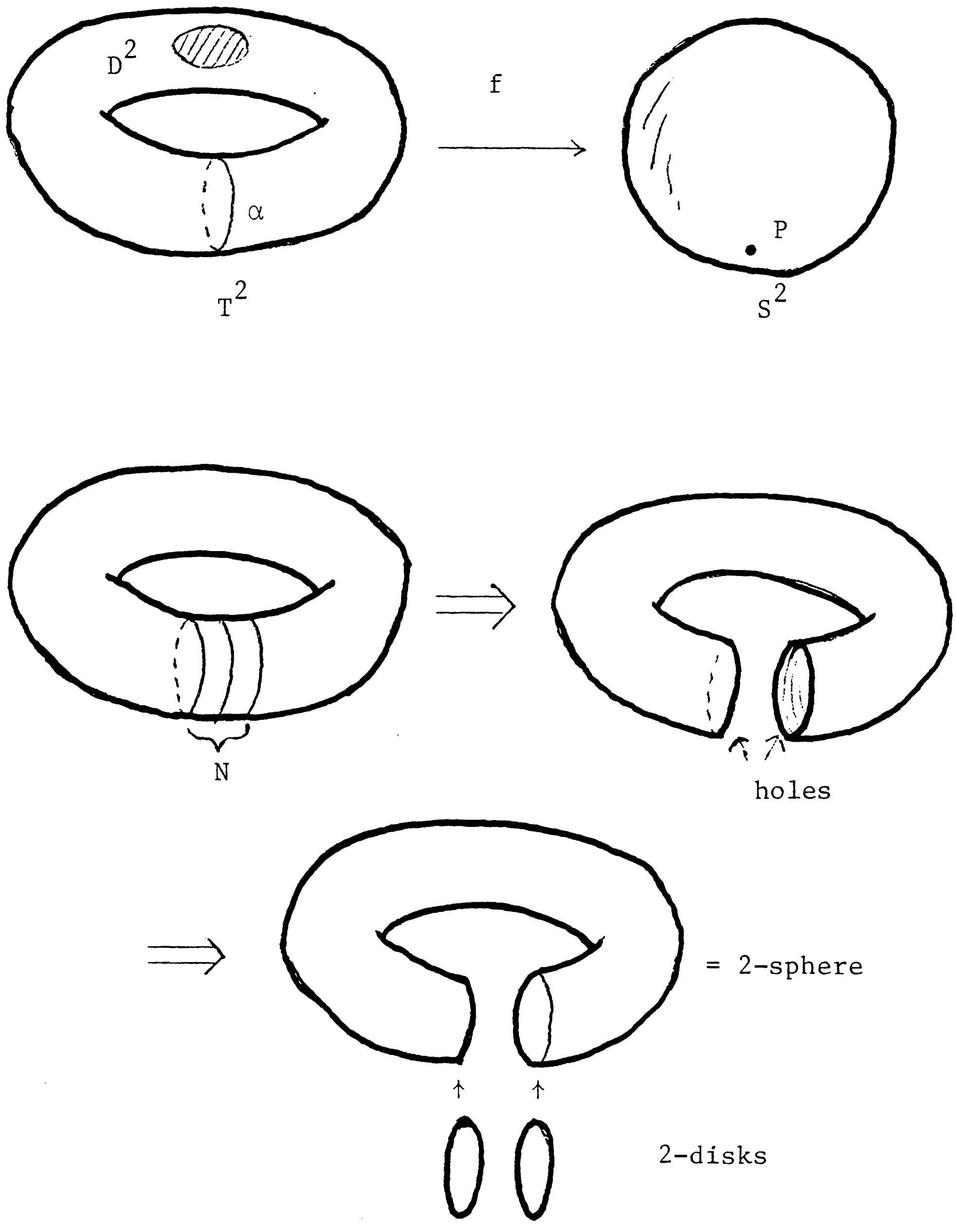


Figure 0. Surgery on T^2.

i.e., α represents a non-trivial element of $\pi_2(f)$. One can "kill" this element by performing surgery on M (and f). It goes as follows. See Fig.0. α has a thin neighborhood homeomorphic to an annulus $S^1 \times D^1$ in T^2; call it N. Delete int(N) from T^2; then the result has two holes, so attach two 2-disks to each hole. The result is a 2-sphere. One also needs to modify the map f by sending the new attached disks to P. The new map is a homotopy equivalence. In general, in order that we can carry out this surgery operation, we have to make some assumptions on $f: M \longrightarrow X$. First of all X has to be a Poincaré complex, and f has to be of degree one. But this is not sufficient. Suppose we want to do surgery on an element $\alpha \in \pi_{r+1}(f)$; then we need to have an embedding $g: S^r \times D^{m-r} \longrightarrow M^m$ such that $g(S^r \times 0)$ represents the element $\partial\alpha$. For this purpose we assume that X has a bundle ν over it, and there exists a stable trivialization F of $\tau_M \oplus f^*(\nu)$, where τ_M denotes the tangent bundle of M.

Wall [W] defined an abelian group $L_m(\pi_1 X,w)$ depending only on the group $\pi_1 X$, the homomorphism $w: \pi_1 X \longrightarrow \{\pm 1\}$, and the value of m modulo 4, and showed that each triple (M,f,F) corresponds to an element of $L_m(\pi_1 X,w)$, called the surgery obstruction, which vanishes if we can perform surgery on (M,f) several times and get a (simple-)homotopy equivalence. If $m \geqslant 5$, then the converse is also true. See [W] for the detail.

Surgery can be applied for uniqueness argument. Suppose we have two homotopy equivalences $f: M \longrightarrow X$ and $f': M' \longrightarrow X$ and they are bordant; i.e., there exist a manifold N whose boundary is the disjoint union of M and M' and a map $g: N \longrightarrow X$ such that $g|M = f$ and $g|M' = f'$. This happens if both f and f' are obtained by doing surgery to the same map. Take a product map $(N;M,M'') \longrightarrow (X \times I; X \times 0, X \times 1)$, and attempt a surgery on this. If we can make this into a homotopy equivalence without touching M and M', then we obtain an h-cobordism.

Now let us go back to the classification problem 2. Recall the Sullivan-Wall surgery exact sequence [W]:

$$[X \times I \text{ rel } \partial, G/TOP] \to L_{m+1}(\pi_1 X) \to \mathcal{S}(X)$$

$$\to [X \text{ rel } \partial, G/TOP] \to L_m(\pi_1 X)$$

where $m = \dim X \geqslant 5$. Farrell and Hsiang proved the following in [FH2].

<u>Theorem</u> Let M^n be a closed aspherical manifold whose fundamental group is virtually nilpotent. Then $\mathcal{S}(M^n) = 0$ for $n \geqslant 4$.

This is proved by showing that the function $[X \text{ rel } \partial, G/TOP] \longrightarrow L_m(\pi_1 X)$ is a bijection. As a corollary they obtained:

<u>Theorem</u> Let N^n be a closed connected flat Riemannian manifold

where $n \neq 3,4$ and let M^n be an aspherical manifold such that $\pi_1(M^n)$ is isomorphic to $\pi_1(N^n)$, then N^n and M^n are homeomorphic.

We would like to prove a similar result for some topological spaces which are almost manifolds but have singularities. Let Γ be a crystallographic group acting on $\mathbb{R}^n$ and consider the orbit space $\mathbb{R}^n/\Gamma$. When Γ is torsion free, $\mathbb{R}^n/\Gamma$ is an n-dimensional aspherical manifold, and Farrell and Hsiang's result will apply to this. If Γ has torsion, $\mathbb{R}^n/\Gamma$ is a stratified space, i.e., a nice union of manifolds. The following is our conjecture:

<u>Conjecture</u> If a stratified space is homotopy equivalent to $\mathbb{R}^n/\Gamma$ in some nice way, then it is homeomorphic to $\mathbb{R}^n/\Gamma$.

This thesis is the first step toward this conjecture. As with the Farrell-Hsiang theorem this conjecture is approached by showing that functions in appropriate "stratified" exact sequences are bijections. Our main result is a partial computation of one of the terms in one of these exact sequences, specifically the surgery group of the group Γ.

Let W_Γ be a free contractible Γ-space. Then the map p: $(\mathbb{R}^n \times W_\Gamma)/\Gamma \longrightarrow \mathbb{R}^n/\Gamma$ has point inverses $p^{-1}(x) = W_\Gamma/\Gamma_x$, which are classifying spaces for "isotropy" subgroups Γ_x. Quinn has defined Ω-spectra $\mathbb{L}(X)$ whose homotopy groups are the surgery obstruction groups $L_i(\pi_1 X)$ [Q1]. This functor can be applied fibrewise to obtain a "sheaf" of spectra $\mathbb{L}(p) \longrightarrow \mathbb{R}^n/\Gamma$, with fibre over x, $\mathbb{L}(p^{-1}(x))$. Next Quinn has defined [Q3] homology groups with spectral sheaf coefficients $H_*(\mathbb{R}^n/\Gamma; \mathbb{L}(p))$. For technical reasons we use a definition of $\mathbb{L}$ using the Poincaré chain complex of Ranicki [R2,3,4]. The homotopy groups are the limits $L_i^{-\infty}$ of Ranicki's lower L-theory $L_i^{(-m)}$ [R1] which may differ from L_i possibly by 2-torsion. The following is our main theorem.

<u>Theorem</u> (4.4.1) If a crystallographic group Γ has no 2-torsion, there is a natural isomorphism:

$$a : H_*(\mathbb{R}^n/\Gamma; \mathbb{L}(p)) \longrightarrow L_*^{-\infty}((\mathbb{R}^n \times W_\Gamma)/\Gamma).$$

The map a is essentially Quinn's "assembly" map. The assembly map is defined in §3.2. The outline of the proof is like that in [FH2], and we use an induction on the size of Γ. If $\Gamma = \Gamma' \rtimes \mathbb{Z}$ for some crystallographic group Γ', then $\mathbb{R}^n/\Gamma$ is a fibre bundle over S^1 with fibre $\mathbb{R}^{n-1}/\Gamma'$, and the theorem is proved by a standard homology property on the left side, the splitting theorem on the right, and the induction hypothesis. Otherwise, the structure theorem for crystallographic groups of [FH2] implies that there exists a surjection $\Gamma \longrightarrow \Gamma_s^\wedge$ onto some finite group and that we can use a hyperelementary induction with respect to the maximal hyperelementary subgroups $\{H\}$ of $\Gamma^{\wedge}_{s}$ (4.1.2). More precisely, we have a commutative diagram:

$$\begin{array}{ccccc} 0 \longrightarrow & H_j(\mathbb{R}^n/\Gamma;\, \mathbb{L}(p)) & \longrightarrow & \bigoplus\limits_H H_j(\mathbb{R}^n/C_H;\, \mathbb{L}(p_H)) & \longrightarrow \\ & \Big\downarrow a & & \Big\downarrow a_H & \\ 0 \longrightarrow & L_j^{-\infty}((\mathbb{R}^n \times W_\Gamma)/\Gamma) & \longrightarrow & \bigoplus\limits_H L_j^{-\infty}((\mathbb{R}^n \times W_\Gamma)/C_H) & \longrightarrow \end{array}$$

where C_H is the preimage of H in Γ. Since the two rows are exact, the map a is proved to be an isomorphism if the next two columns are isomorphisms. But we cannot do this directly, because, for example the size of C_H may not be smaller than that of Γ. We have to use the proof of 5-lemma applied to a single element. For example, pick an element y of $L_j^{-\infty}((\mathbb{R}^n \times W_\Gamma)/\Gamma)$ and represent it by a geometric quadratic Poincaré complex on p, $\bar{y}$, with radius r measured on $\mathbb{R}^n/\Gamma$. We want to show that the restriction image y_H of y in each $L_j^{-\infty}((\mathbb{R}^n \times W_\Gamma)/C_H)$ lies in the image of a_H. If the size of C_H is strictly smaller than the size of Γ, then by induction hypothesis this is the case. If not, then 4.1.2 implies that there is a shrinking map $\alpha : \mathbb{R}^n/C_H \longrightarrow \mathbb{R}^m/\Gamma^{\sim}$ for some crystallographic group $\Gamma^{\sim}$ of rank $m \geq 1$, and that we can make the radius of the restriction image $\bar{y}_H$ of $\bar{y}$ arbitrarily small on $\mathbb{R}^m/\Gamma^{\sim}$, by choosing a very large integer s. Now the method of [Q3] is applied to characterize things of small radius as exactly elements of the sheaf homology groups (3.2.2 and 3.2.3). So y_H comes from an element of $H_j(\mathbb{R}^m/\Gamma^{\sim}; \mathbb{L}(\alpha p_H))$. Lastly this is proved to be isomorphic to $H_j(\mathbb{R}^n/C_H; \mathbb{L}(p_H))$ using the induction hypothesis.

1. Preliminaries

1.1. Chain complexes

In 1.1, we fix the notation concerning chain complexes and review some basic facts. The same notation and sign convention as in [R3] will be used.

Let R denote a ring with involution. An R-module chain complex C is a sequence of R-modules and R-module homomorphisms

$$C : \ldots \longrightarrow C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \xrightarrow{d} \ldots \qquad (r \in \mathbb{Z})$$

such that $d^2 = 0$. See [S] or [R3] for the definitions of chain maps, chain homotopies, chain equivalences, and chain contraction. A finitely generated projective R-module chain complex C is strictly n-dimensional if $C_r = 0$ for $r > n$ and $r < 0$. A finite complex of finitely generated projective R-module is n-dimensional if it is chain equivalent to a strictly n-dimensional chain complex.

The algebraic mapping cone C(f) of a chain map $f: C \longrightarrow D$ is the R-module chain complex defined by

$$d_{C(f)} = \begin{pmatrix} d_D & (-)^{r-1}f \\ 0 & d_C \end{pmatrix} : C(f)_r = D_r \oplus C_{r-1} \longrightarrow C(f)_{r-1} = D_{r-1} \oplus C_{r-2}.$$

The algebraic mapping cylinder M(f) of f is the R-module chain complex defined by

$$d_{M(f)} = \begin{pmatrix} d_D & (-)^{r-1}f & 0 \\ 0 & d_C & 0 \\ 0 & (-)^r & d_C \end{pmatrix}$$

$$: M(f)_r = D_r \oplus C_{r-1} \oplus C_r \longrightarrow M(f)_{r-1} = D_{r-1} \oplus C_{r-2} \oplus C_{r-1}.$$

A triad of R-module chain complexes

$$\begin{tikzcd} & C \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "k"] & D \arrow[d, "h"] \\ \mathcal{T} : & C' \arrow[r, "f'"'] & D' \end{tikzcd}$$

consists of R-module chain maps

$$f: C \longrightarrow D,\ f': C' \longrightarrow D',\ g: C \longrightarrow C',\ h: D \longrightarrow D'$$

and an R-module chain homotopy

$$k: f'g \simeq hf: C \longrightarrow D' \quad (\text{i.e., } dk + kd = f'g - hf).$$

1.1.1. Lemma A triad $\mathcal{T}$ (given above) of finite dimensional R-module chain complexes induces an R-module chain map from the algebraic mapping cone of g to that of h, which is an R-module chain equivalence if both f and f' are R-module chain equivalences.

Proof. The map is given by a matrix

$$\begin{pmatrix} f' & (-)^{r-1}k \\ 0 & f \end{pmatrix} : C(g)_r = C'_r \oplus C_{r-1} \longrightarrow C(h)_r = D'_r \oplus D_{r-1}.$$

The second part is obtained by applying 5-lemma to the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_r(C) \arrow[r] \arrow[d, "\cong"] & H_r(C') \arrow[r] \arrow[d, "\cong"] & H_r(C(g)) \arrow[r] \arrow[d] & H_{r-1}(C) \arrow[r] \arrow[d, "\cong"] & H_{r-1}(C') \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H_r(D) \arrow[r] & H_r(D') \arrow[r] & H_r(C(h)) \arrow[r] & H_{r-1}(D) \arrow[r] & H_{r-1}(D') \arrow[r] & \cdots
\end{tikzcd}$$

where the two rows are exact.

The suspension SC (resp. desuspension $S^{-1}C$) of an R-module chain complex C is the R-module chain complex defined by

$$d_{SC} = d_C : (SC)_r = C_{r-1} \longrightarrow (SC)_{r-1} = C_{r-2}$$

$$(\text{ resp. } \; d_{S^{-1}C} = d_C : (S^{-1}C)_r = C_{r+1} \longrightarrow (S^{-1}C)_{r-1} = C_r).$$

A diagram of R-module chain maps

$$A \xleftarrow{f} B \xrightarrow{g} C$$

induces a triad

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, "h"] & C \arrow[d, "j"] \\
A \arrow[r, "i"'] & A \cup_B C
\end{tikzcd}$$

defined by

$$A \cup_B C = C\left(\begin{pmatrix} f \\ g \end{pmatrix} : B \longrightarrow A \oplus C\right)$$

$$d_{A \cup_B C} = \begin{pmatrix} d_A & (-)^{r-1}f & 0 \\ 0 & d_B & 0 \\ 0 & (-)^{r-1}g & d_C \end{pmatrix}$$

$$: (A \cup_B C)_r = A_r \oplus B_{r-1} \oplus C_r$$

$$\longrightarrow (A \cup_B C)_{r-1} = A_{r-1} \oplus B_{r-2} \oplus C_{r-1}$$

$$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : A_r \longrightarrow (A \cup_B C)_r$$

$$j = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} : C_r \longrightarrow (A \cup_B C)_r$$

$$h = \begin{pmatrix} 0 \\ (-)^r \\ 0 \end{pmatrix} : B_r \longrightarrow (A \cup_B C)_{r+1}.$$

This push-out triad is natural in the following sense. Suppose we have a diagram which commutes up to chain homotopy:

$$\begin{tikzcd}
A \arrow[d, "r_A"'] & B \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "r_B"] \arrow[dl, bend right, "h_1"'] \arrow[dr, bend left, "h_2"] & C \arrow[d, "r_C"] \\
A' & B' \arrow[l, "f'"] \arrow[r, "g'"'] & C'
\end{tikzcd}$$
.

Then it induces a diagram which commutes strictly:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "r_A"'] & A \cup_B C \arrow[d, "r"] & C \arrow[l, "j"'] \arrow[d, "r_C"] \\
A' \arrow[r, "i'"'] & A' \cup_{B'} C' & C' \arrow[l, "j'"]
\end{tikzcd}$$
,

where

$$r = \begin{pmatrix} r_A & (-)^{r-1}h_1 & 0 \\ 0 & r_B & 0 \\ 0 & (-)^r h_2 & r_C \end{pmatrix}$$

$$: (A \cup_B C)_r = A_r \oplus B_{r-1} \oplus C_r \longrightarrow (A' \cup_{B'} C')_r = A'_r \oplus B'_{r-1} \oplus C'_r .$$

Next we consider the dual situation. A diagram of R-module chain maps:

$$A \xrightarrow{\ i\ } D \xleftarrow{\ j\ } C$$

induces a triad:

$$\begin{tikzcd} A \cap^D C \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, bend left, "h"] & C \arrow[d, "j"] \\ A \arrow[r, "i"'] & D \end{tikzcd}$$

defined by

$$A \cap^D C = S^{-1}C((i \quad j) : A \oplus C \longrightarrow D)$$

$$d_{A \cap^D C} = \begin{pmatrix} d_A & 0 & 0 \\ (-)^r i & d_D & (-)^r j \\ 0 & 0 & d_C \end{pmatrix}$$

$$: (A \cap^D C)_r = A_r \oplus D_{r+1} \oplus C_r \longrightarrow (A \cap^D C)_{r-1} = A_{r-1} \oplus D_r \oplus C_{r-1}$$

$$f = (-1 \quad 0 \quad 0) : (A \cap^D C)_r \longrightarrow A_r$$

$$g = (0 \quad 0 \quad 1) : (A \cap^D C)_r \longrightarrow C_r$$

$$h = (0 \quad (-)^{r+1} \quad 0) : (A \cap^D C)_r \longrightarrow D_{r+1}.$$

This "pullback" triad is also natural; i.e., a chain-homotopy commutative diagram:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "r_A"'] \arrow[dr, bend left, "h_1"] & D \arrow[d, "r_D"] & C \arrow[l, "j"'] \arrow[d, "r_C"] \arrow[dl, bend left, "h_2"'] \\ A' \arrow[r, "i'"'] & D' & C' \arrow[l, "j'"] \end{tikzcd}$$

induces a strictly commutative diagram:

$$\begin{tikzcd}
A \arrow[d, "r_A"'] & A \cap^D C \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "r"] & C \arrow[d, "r_C"] \\
A' & A' \cap^{D'} C' \arrow[l, "f'"] \arrow[r, "g'"'] & C'
\end{tikzcd}$$

where

$$r = \begin{pmatrix} r_A & 0 & 0 \\ (-)^{r+1}h_1 & r_D & (-)^r h_2 \\ 0 & 0 & r_C \end{pmatrix}$$

$$:(A \cap^D C)_r = A_r \oplus D_{r+1} \oplus C_r \longrightarrow (A' \cap^{D'} C')_r = A'_r \oplus D'_{r+1} \oplus C'_r .$$

"Push-outs" and "pullbacks" are universal:

1.1.2. Lemma A triad

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, bend left, "h"] & C \arrow[d, "j"] \\
A \arrow[r, "i"'] & D
\end{tikzcd}$$

induces chain maps

$$(i \quad (-)^{r-1}h \quad -j): (A \cup_B C)_r = A_r \oplus B_{r-1} \oplus C_r \longrightarrow D_r$$

$${}^t(-f \quad (-)^r h \quad g): B_r \longrightarrow (A \cap^D C)_r = A_r \oplus D_{r+1} \oplus C_r ,$$

which make the following diagrams commutative:

$$\begin{tikzcd}
A \arrow[r, "{}^t(1 \quad 0 \quad 0)"] \arrow[d, equal] & A \cup_B C \arrow[d] & C \arrow[l, "{}^t(0 \quad 0 \quad -1)"'] \arrow[d, equal] \\
A \arrow[r, "i"'] & D & C \arrow[l, "j"]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[d, equal] & B \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & C \arrow[d, equal] \\
A & A \cap^D C \arrow[l, "(-1 \quad 0 \quad 0)"] \arrow[r, "(0 \quad 0 \quad 1)"'] & C
\end{tikzcd}$$

"Push-out" and "pullback" are essentially inverses of each other. First begin with a diagram

$$A \xrightarrow{i} D \xleftarrow{j} C \ ;$$

then the pullback triad

$$\begin{tikzcd}
A \cap^D C \arrow[r] \arrow[d] \arrow[dr, "h"] & C \arrow[d, "j"] \\
A \arrow[r, "i"'] & D
\end{tikzcd}$$

$$(\quad h = (0 \quad (-)^{r+1} \quad 0) \quad)$$

induces a chain equivalence

$$(i \quad 0 \quad 1 \quad 0 \quad -j) : A \cup_{A \cap^{D} C} C \longrightarrow D.$$

The inclusion map

$${}^{t}(0 \quad 0 \quad 1 \quad 0 \quad 0) : D \longrightarrow A \cup_{A \cap^{D} C} C$$

gives a chain homotopy inverse. Next begin with a diagram

$$A \xleftarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C;$$

then the push-out triad

$$\begin{tikzcd} B \arrow[r] \arrow[d, "f"'] \arrow[dr, bend left, "h"] & C \arrow[d] \\ A \arrow[r] & A \cup_B C \end{tikzcd} \qquad h = \begin{pmatrix} 0 \\ (-)^{r} \\ 0 \end{pmatrix}$$

induces a chain equivalence

$${}^{t}(-f \quad 0 \quad 1 \quad 0 \quad g) : B \longrightarrow A \cap_{A \cup_B C} C,$$

and the projection map

$$(0 \quad 0 \quad 1 \quad 0 \quad 0) : A \cap_{A \cup_B C} C \longrightarrow B$$

gives a chain homotopy inverse.

Let C be an R-module chain complex. C^* denotes the R-module chain complex defined by

$$d_{C^*} = (d_C)^*: (C^*)_r = C^{-r} \longrightarrow (C^*)_{r-1} = C^{-r+1},$$

where C^{-r} is the dual of C_{-r}. And C^{n-*} denotes the R-module chain complex defined by

$$d_{C^{n-*}} = (-)^r (d_C)^*: (C^{n-*})_r = C^{n-r} \longrightarrow (C^{n-*})_{r-1} = C^{n-r+1}.$$

The generator $T \in \mathbb{Z}_2$ acts on $\mathrm{Hom}_R(C^*, C)$, the R-module chain complex of R-module chain maps from C^* to C, as follows:

$$T: \mathrm{Hom}_R(C^p, C_q) \longrightarrow \mathrm{Hom}_R(C^q, C_p);\ f \longmapsto (-)^{pq} f^*.$$

Now let us recall the definition of quadratic complexes and pairs by A. Ranicki.

1.1.3. <u>Definition</u> An <u>n-dimensional</u> <u>quadratic</u> <u>complex</u> <u>over</u> <u>R</u> (C, ψ) is an n-dimensional R-module chain complex C together with an element $\psi \in Q_n(C)$. Such a complex is <u>Poincaré</u> if $(1+T)\psi_0$: $C^{n-*} \longrightarrow C$ is a chain equivalence. A <u>map</u> (resp. <u>homotopy</u> <u>equivalence</u>) of n-dimensional quadratic complexes over R

$$f: (C, \psi) \longrightarrow (C', \psi')$$

is an R-module chain map (resp. chain equivalence) $f: C \longrightarrow C'$ such that $f_\%(\psi) = \psi' \in Q_n(C')$.

An <u>(n+1)-dimensional</u> <u>quadratic</u> <u>pair</u> over R $(f: C \longrightarrow D, (\delta\psi, \psi))$ is a chain map f from an n-dimensional chain complex C to an (n+1)-dimensional chain complex D together with an element $(\delta\psi, \psi) \in Q_{n+1}(f)$. Such a pair is <u>Poincaré</u> if the R-module chain map D^{n+1-*}

$\longrightarrow C(f)$ defined (up to chain homotopy) by

$$\begin{pmatrix} (1+T)\delta\psi_0 \\ \\ (1+T)\psi_0 f^* \end{pmatrix} : D^{n+1-r} \longrightarrow C(f)_r = D_r \oplus C_{r-1}$$

is a chain equivalence, in which case the <u>boundary</u> n-dimensional quadratic complex (C, ψ) is Poincaré. A <u>map</u> (resp. <u>homotopy equivalence</u>) of $(n+1)$-dimensional quadratic pairs over R $(g,h;k)$: $(f: C \longrightarrow D, (\delta\psi,\psi) \in Q_{n+1}(f)) \rightarrow (f': C' \longrightarrow D', (\delta\psi', \psi') \in Q_{n+1}(f'))$ is a chain complex triad of the type

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "k"] & D \arrow[d, "h"] \\ C' \arrow[r, "f'"'] & D' \end{tikzcd}$$

such that $g: C \longrightarrow C'$ and $h: D \longrightarrow D'$ are R-module chain maps (resp. chain equivalences) and

$$(g,h;k)_\%(\delta\psi, \psi) = (\delta\psi', \psi') \in Q_{n+1}(f').$$

See Ranicki's [R2,3] for the precise definition of $Q_n(C)$, $Q_{n+1}(f)$, $f_\%$, and $(g,h;k)_\%$. An element $\psi \in Q_n(C)$ is represented by a collection

$$\{\psi_s \in \mathrm{Hom}_R(C^{n-r-s}, C_r) \mid r \in \mathbb{Z}, \quad s \geq 0\}$$

such that

$$d\psi_s + (-)^r\psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1}T\psi_{s+1}) = 0$$

$$: C^{n-r-s-1} \longrightarrow C_r \qquad (s \geqslant 0)$$

and if $f: C \longrightarrow D$ is a chain map, $f_\%\psi$ is represented by $\{f\psi_s f^*\}$. An element $(\delta\psi,\psi) \in Q_{n+1}(f)$ is represented by a collection

$$\{(\delta\psi,\psi)_s = (\delta\psi_s,\psi_s) \in \mathrm{Hom}_R(D^*,D)_{n-s+1} \oplus \mathrm{Hom}_R(C^*,C)_{n-s} \mid s \geqslant 0\}$$

such that

$$(d(\delta\psi_s) + (-)^r(\delta\psi_s)d^* + (-)^{n-s}(\delta\psi_{s+1} + (-)^{s+1}T\delta\psi_{s+1}) + (-)^n f\psi_s f^*,$$

$$d\psi_s + (-)^r\psi_s d^* + (-)^{n-s-1}(\psi_{s+1} + (-)^{s+1}T\psi_{s+1}))$$

$$= 0 \quad \sum_{r=-\infty}^{\infty} \mathrm{Hom}_R(D^{n-r-s},D_r) \oplus \mathrm{Hom}_R(C^{n-r-s-1},C_r) \qquad (s \geqslant 0),$$

and $(g,h;k)_\%(\delta\psi,\psi)$ is represented by

$$\{(h\delta\psi_s h^* + (-)^{n-1}k\psi_s f^*h^* + (-)^p f'g\psi_s k^*$$

$$+ (-)^{n+p}kT\psi_{s+1}k^*,\ g\psi_s g^*) \mid s \geqslant 0\}.$$

1.1.4. <u>Definition</u> A <u>cobordism</u> of n-dimensional quadratic Poincaré complexes over R (C,ψ), (C',ψ') is an $(n+1)$-dimensional quadratic Poincaré pair over R

$$((f \quad f'): C \oplus C' \longrightarrow D,\ (\delta\psi,\psi \oplus -\psi') \in Q_{n+1}((f \quad f')))$$

with boundary $(C \oplus C', \psi \oplus -\psi')$.

1.1.5. Proposition (Ranicki) Cobordism is an equivalence relation on the set of n-dimensional quadratic Poincaré complexes over R, such that homotopy equivalent Poincaré complexes are cobordant. The cobordism classes define an abelian group, the n-dimensional quadratic L-group of R, $L_n(R)$ $(n \geqslant 0)$, with addition and inverses by

$$(C,\psi) + (C',\psi') = (C + C', \psi + \psi'), \quad -(C,\psi) = (C,-\psi).$$

1.2. Geometric modules

We define geometric modules and their homomorphisms, following Connell and Hollingsworth [CH] and Quinn [Q2,3]. We start with the constant coefficient case.

1.2.1. Definition Suppose X is a metric space, and R is a ring.
1) If $\{x_\alpha\}$, $\alpha \in A$, is a collection of points of X, then the geometric R-module with basis $\{x_\alpha\}$ is the free module $R(\{x_\alpha\})$.
2) If $h: M \longrightarrow M'$ is an R-module homomorphism of geometric modules with bases $\{x_\alpha\}$, $\{y_\beta\}$ respectively, then the underlying set function is obtained by $\underline{h}(x_\alpha) = \{y_\beta \mid y_\beta$ has non-zero coefficient in $h(x_\alpha)\}$.
3) h has radius r if $\underline{h}(x_\alpha) \subseteq x_\alpha{}^r$ (= the r-neighborhood of x_α).
4) A homomorphism with radius r, $h: M \longrightarrow M'$, is an r-isomorphism (with support $C \subseteq X$) if there exists a homomorphism $g: M' \longrightarrow M$ with radius r such that $hg = 1$ and $gh = 1$ ($hg = 1$ on C and $gh = 1$ on C).

Next we consider the non-constant coefficient case. Although a more general definition is possible, we restrict ourselves to the following special case.

1.2.2. Definition Let $p: E \longrightarrow X$ be a continuous map, where X is a metric space, $q: \tilde{E} \longrightarrow E$ a covering space of E ($\tilde{E}$ may not be connected), and Γ the group of covering transformations of q. Let

$\{x_\alpha\}$ be a subset of E. A <u>geometric</u> <u>$\mathbb{Z}\Gamma$-module</u> M on p generated by $\{x_\alpha\}$ is a free abelian group generated by the points of $q^{-1}(\{x_\alpha\})$ together with the $\mathbb{Z}\Gamma$-module structure induced by the action of Γ. Although $\tilde{E}$ is not mentioned explicitly, it is part of the information, and we always fix a lifting $\bar{x}_\alpha$ of x_α in $\tilde{E}$.

Let M and M' be geometric modules on p generated by $\{x_\alpha\}$, $\{y_\beta\}$ respectively. A $\mathbb{Z}\Gamma$-module homomorphism h: $M \longrightarrow M'$ has <u>radius</u> r if, for each pair v and v' of points in $q^{-1}(\{x_\alpha\})$ and $q^{-1}(\{y_\beta\})$ such that v' has a non-zero coefficient in h(v) written as a linear combination of points in $q^{-1}(\{y_\beta\})$ with integral coefficients, there exists an arc connecting v and v' inside $(pq)^{-1}(pq(v)^r)$.

Geometric chain complexes, chain maps, chain homotopies, chain equivalences, etc. on p (with support $C \subseteq X$) are defined in the obvious way. For example, a <u>geometric chain complex on</u> p (with support C) is a sequence of homomorphisms of geometric modules on p:

$$\cdots \longrightarrow M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} M_{i-2} \longrightarrow \cdots$$

such that $d_{i-1}d_i = 0$ ($d_{i-1}d_i|_C = 0$) for each i. The radius of a chain complex is $\sup_i$(radius of d_i). An <u>r-chain equivalence</u> is not just a chain equivalence of radius r. We also require that there is a chain homotopy inverse with radius r and that the two chain homotopies involved have radius r.

For the dual M^* of a geometric module M on p generated by $\{x_\alpha\}$, we use the standard dual basis, denoted by $\{x_\alpha{}^*\}$. x^*_α (resp. $\bar{x}^*_\alpha$) corresponds to the same point in E (resp. $\tilde{E}$) as x_α (resp. $\bar{x}_\alpha$).

In general it is hard to work with geometric modules on p, but if p behaves like a product (at least locally) then we can understand geometric modules on p with small radii pretty well. The following definition describes such a nice p.

1.2.3. <u>Definition</u> 1) If $p: E \longrightarrow X$ and $X \supseteq Y$, then Y is a <u>p-NDR</u> (<u>neighborhood deformation retract</u>) subset of X if there is a neighborhood U of Y, and homotopies $H: U \times I \longrightarrow X$, $\hat{H}: p^{-1}(U) \times I \longrightarrow E$ such that H is identity on $U \times \{0\}$ and $Y \times I$, $H(U \times \{1\}) \subseteq Y$, $\hat{H}$ is the identity on $p^{-1}(U) \times \{0\}$and $p^{-1}(Y) \times I$, and the diagram

$$\begin{array}{ccc} p^{-1}(U) \times I & \xrightarrow{\hat{H}} & E \\ {\scriptstyle p \times 1}\downarrow & & \downarrow{\scriptstyle p} \\ U \times I & \xrightarrow[H]{} & X \end{array}$$

commutes.

2) A <u>stratified system of fibrations</u> on a space X consists of a map $p: E \longrightarrow X$ and a closed finite filtration of X, $X = X_N \supseteq \ldots \supseteq X_0$, such that each X_j is a p-NDR subset of X, and each $p: p^{-1}(X_j - X_{j-1}) \longrightarrow X_j - X_{j-1}$ is a fibration.

1.2.4. Remark If $R = \mathbb{Z}\Gamma$, then the involution of R is given by $x \longmapsto x^{-1}$ for $x \in \Gamma$. Thus we consider only the "orientable" case.

2. Glueing and Splitting

2.1. Glueing of quadratic pairs and triads

In this section we review from Ranicki [R2,3] to fix notation. Consider two n-dimensional quadratic pairs over R whose boundaries are the inverses of each other:

$$c' = (g' : B \longrightarrow C', (\delta\psi', -\psi))$$

$$c'' = (g'' : B \longrightarrow C'', (\delta\psi'', \psi)).$$

The <u>union</u> of c' and c'' along the (n-1)-dimensional quadratic complex $b = (B, \psi)$ is an n-dimensional quadratic complex $(C' \cup_B C'', \delta\psi' \cup_\psi \delta\psi'')$ defined by

$$(\delta\psi' \cup_\psi \delta\psi'')_s = \begin{pmatrix} \delta\psi'_s & 0 & 0 \\ (-)^{n-r-1}\psi_s g'^* & (-)^{n-r-s} T\psi_{s+1} & 0 \\ 0 & (-)^s g''\psi_s & \delta\psi''_s \end{pmatrix}$$

$$: (C' \cup_B C'')^{n-r-s} = C'^{n-r-s} \oplus B^{n-r-s-1} \oplus C''^{n-r-s}$$

$$\longrightarrow (C' \cup_B C'')_r = C'_r \oplus B_{r-1} \oplus C''_r,$$

and will be denoted by $c' \cup_b c''$. If both c' and c'' are Poincaré, then so is $c' \cup_b c''$.

To state the splitting lemma for quadratic Poincaré pairs, we need to introduce the notion of "quadratic triad," which is a quadratic pair with a splitting of its boundary into two pieces.

More precisely, an <u>(n+1)-dimensional quadratic triad over R</u> $(\mathcal{J},\Psi)$ is a triad of R-module chain complexes:

$$\mathcal{J}:\begin{tikzcd} B \arrow[r, "g''"] \arrow[d, "g'"'] \arrow[dr, bend left, "h"] & C'' \arrow[d, "f''"] \\ C' \arrow[r, "f'"'] & D \end{tikzcd}$$

such that B is (n-1)-dimensional, C' and C" are n-dimensional, and D is (n+1)-dimensional, together with a representative

$$\Psi = (\psi,\delta\psi',\delta\psi'',\ \delta\overline{\psi})$$

of an element of the triad Q-group $Q_{n+1}(\mathcal{J})$. Such a quadratic triad is <u>Poincaré</u> if

i) the n-dimensional quadratic pairs over R

$$c' = (g'\colon B \longrightarrow C',\ (\delta\psi',-\psi))$$

$$c'' = (g''\colon B \longrightarrow C'',\ (\delta\psi'',\psi))$$

are Poincare, and

ii) the (n+1)-dimensional quadratic pair over R

$$d = ((f' \quad (-)^{r-1}h \quad -f'')\colon C' \cup_B C'' \longrightarrow D,\ (\delta\overline{\psi},\ \delta\psi' \cup_{\psi} \delta\psi''))$$

is Poincaré.

Suppose we have two (n+1)-dimensional quadratic triads over R:

$$\mathcal{J}_1:\begin{tikzcd} B \arrow[r, "g^!"] \arrow[d, "g'"'] \arrow[dr, bend left, "h'"] & C^! \arrow[d, "f_1^!"] \\ C' \arrow[r, "f'"'] & D' \end{tikzcd} \qquad \Psi_1 = (\psi,\delta\psi',-\delta\psi^!,\delta\overline{\psi}')$$

$$\mathcal{J}_2: \quad \begin{tikzcd} B \arrow[r, dash, "g''"] \arrow[d, "g^!"'] \arrow[dr, "h''"] & C'' \arrow[d, "f''"] \\ C^! \arrow[r, dash, "f_2^!"'] & D'' \end{tikzcd} \qquad \Psi_2 = (\psi, \delta\psi^!, \delta\psi'', \delta\bar{\psi}'').$$

Then their union $(\mathcal{J}_1 \cup \mathcal{J}_2,\ \Psi_1 \cup \Psi_2)$ is an (n+1)-dimensional quadratic triad over R:

$$\mathcal{J}_1 \cup \mathcal{J}_2: \quad \begin{tikzcd} B \arrow[r, dash, "g''"] \arrow[d, "g'"'] \arrow[dr, "h"] & C'' \arrow[d, "j''f''"] \\ C' \arrow[r, dash, "j'f'"'] & D' \cup_{C^!} D'' \end{tikzcd} \qquad \Psi_1 \cup \Psi_2 = (\psi, \delta\psi', \delta\psi'', \delta\bar{\psi})$$

where

$$j' = {}^t(1 \quad 0 \quad 0): D'_r \longrightarrow (D' \cup_{C^!} D'')_r = D'_r \oplus C^!_{r-1} \oplus D''_r$$

$$j'' = {}^t(0 \quad 0 \quad -1): D''_r \longrightarrow (D' \cup_{C^!} D'')_r$$

$$h = {}^t(h' \quad (-)^r g^! \quad -h''): B_r \longrightarrow (D' \cup_{C^!} D'')_r$$

$$\delta\bar{\psi}_s = (\delta\bar{\psi}' \cup_{\delta\psi^!} \delta\bar{\psi}'')_s = \begin{pmatrix} \delta\bar{\psi}'_s & 0 & 0 \\ (-)^{n-r}\delta\psi^!_s f_1^{!*} & (-)^{n+1-r-s} T\delta\psi^!_{s+1} & 0 \\ 0 & (-)^s f_2^! \delta\psi^!_s & \delta\bar{\psi}''_s \end{pmatrix}$$

$$: (D' \cup_{C^!} D'')^{n+1-r-s} = D'^{n+1-r-s} \oplus C^{!n-r-s} \oplus D''^{n+1-r-s}$$

$$\longrightarrow (D' \cup_{C^!} D'')_r = D'_r \oplus C^!_{r-1} \oplus D''_r.$$

If both of $(\mathcal{J}_1, \Psi_1)$ and $(\mathcal{J}_2, \Psi_2)$ are Poincaré, then so is

$(\mathcal{J}_1 \cup \mathcal{J}_2, \Psi_1 \cup \Psi_2)$.

2.2. Splitting lemma for quadratic Poincaré complexes and pairs

The object is to give a sort of inverse to the union operation of 2.1: given a Poincaré pair we find the smallest amount of information required to show that it is equivalent to a union. Let $c = (f\colon C \longrightarrow D, (\delta\psi, \psi))$ be an $(n+1)$-dimensional quadratic Poincaré pair over R and C' (resp. D') be a subcomplex of C(resp. D). We assume that $(C/C')_r$ and $(D/D')_r$ are projective for each r, and that the image of C' by f lies in D'. Let i_C denote the inclusion map of C' into C, and p_C denote the projection map of C onto C/C'. We fix the splitting maps $j_C\colon (C/C')_r \longrightarrow C_r$ and $q_C\colon C_r \longrightarrow C'_r$ for the short exact sequence:

$$C'_r \xrightarrow{\quad i_C \quad} C_r \xrightarrow{\quad p_C \quad} (C/C')_r .$$

This gives an identification of C_r with $C'_r \oplus (C/C')_r$, and if we define a chain map $\rho_C\colon (C/C')_r \longrightarrow (SC')_r = C'_{r-1}$ by $(-)^{r-1} q_C d j_C$, then the boundary map of C is given by a matrix

$$\begin{pmatrix} d & (-)^{r-1}\rho_C \\ 0 & d \end{pmatrix} : C'_r \oplus (C/C')_r \longrightarrow C'_{r-1} \oplus (C/C')_{r-1}$$

under this identification. We define maps i_D, p_D, j_D, q_D, and ρ_D for D in the same way. We further assume

2.2.1. $H_i(p_C(1+T)\psi_0 p_C^*) = 0$ for $i \leqslant 0$, and

2.2.2. $H_i(p_D(1+T)\delta\psi_0 p_D^*) = 0$ for $i \leqslant 0$.

These conditions will be used to prove that B in Lemma 2.2.3 below is $(n-1)$-dimensional, etc. The auther is not sure whether these are actually necessary or not. But anyway, in geometric situation, these are automatically satisfied. See 2.2.6.

Let C'' denote the chain complex $(C/C')^{n-*}$.

By the assumption on D', f induces a chain map $f' = q_D f i_C$: $C' \longrightarrow D'$ such that $i_D f' = f i_C$. The algebraic mapping cone $C(f')$ of f' is a subcomplex of the algebraic mapping cone $C(f)$ of f. Define D'' by $(C(f)/C(f'))^{n+1-*}$, which is same as $C(p_D f j_C)^{n+1-*}$. There is a chain map (inclusion map)

$$f'' = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : C''_r = (C/C')^{n-r} \longrightarrow D''_r = (D/D')^{n+1-r} \oplus (C/C')^{n-r}$$

from C'' to D''.

2.2.3. <u>Lemma</u> (Splitting lemma for quadratic Poincaré pairs) Let c, C', D', C'', D'' be as above. Then there are $(n+1)$-dimensional quadratic Poincaré triads over R:

$$\left(\mathcal{J}_1 : \begin{tikzcd} B \arrow[r, dash] \arrow[d] \arrow[dr, bend left] & C^! \arrow[d] \\ C' \arrow[r, dash] & D' \end{tikzcd}, \ \Psi_1\right) \qquad \left(\mathcal{J}_2 : \begin{tikzcd} B \arrow[r, dash] \arrow[d] \arrow[dr, bend left] & C'' \arrow[d] \\ C^! \arrow[r, dash] & D \end{tikzcd}, \ \Psi_2\right)$$

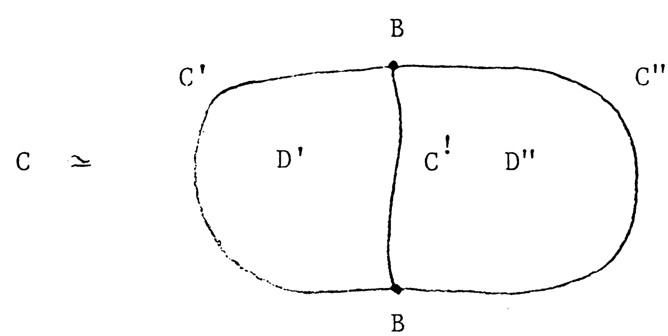
such that the (n+1)-dimensional quadratic Poincaré pair induced by their union is homotopy equivalent to c.

2.2.4. Corollary (Splitting lemma for quadratic Poincaré complexes) If $c = (C,\psi)$ is an n-dimensional quadratic Poincaré complex over R and C' is a subcomplex of C such that C/C' is projective and 2.2.1 holds, then there exist two Poincaré pairs:

$$(B \longrightarrow C',\ (\delta\overline{\psi}',-\overline{\psi})) \text{ and } (B \longrightarrow C'',\ (\delta\overline{\psi}'',\overline{\psi}))$$

such that c is homotopy equivalent to their union.

The following illustrates the splitting lemma for pairs.



Proof of Splitting Lemma: By the naturality of pull-backs, we have the following (partly homotopy-)commutative diagram:

$$\begin{tikzcd}
B \arrow[rr, "g''_C"] \arrow[dr, "g^!"] \arrow[dd, "g'_C"'] & & C'' \arrow[dr, "f''"] \arrow[dd] & \\
& C^! \arrow[rr, "g''_D"'] \arrow[dd, "g'_D"] & & D'' \arrow[dd] \\
C' \arrow[rr, "i_C"] \arrow[dr, "f'"'] & & C \arrow[dr, "f"] & \\
& D' \arrow[rr, "i_D"'] & & D
\end{tikzcd}$$

where the unlabeled vertical maps are

$$(1+T)\psi_0 p_C^* \; : \; C'' \longrightarrow C$$

$$((1+T)\delta\psi_0 p_D^* \quad f(1+T)\psi_0 p_C^*) : D'' \longrightarrow D$$

and the front and the back squares are pull-back triads; thus $B = C' \cap^C C''$, $g'_C = (-1 \quad 0 \quad 0)$, $g''_C = (0 \quad 0 \quad 1)$, and $C^! = D' \cap^D D''$, $g'_D = (-1 \quad 0 \quad 0)$, $g''_D = (0 \quad 0 \quad 1)$. The chain map $g^! : B \longrightarrow C^!$ is given by a matrix

$$\begin{pmatrix} f' & & \\ & f & \\ & & f'' \end{pmatrix} : C'_r \oplus C_{r+1} \oplus C''_r \longrightarrow D'_r \oplus D_{r+1} \oplus D''_r .$$

We define an n-dimensional quadratic Poincaré structure $(\delta\overline{\psi}, -\overline{\psi})$ on the pair $g^! : B \longrightarrow C^!$ as follows:

$$\delta\overline{\psi}_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-)^{n-r-s}T\delta\psi_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (s \geqslant 1)$$

$$: C^{!n-r-s} = D'^{n-r-s} \oplus D^{n+1-r-s} \oplus D''^{n-r-s}$$

$$\longrightarrow C^!_r = D'_r \oplus D_{r+1} \oplus D''_r$$

$$\delta\overline{\psi}_0 = \begin{pmatrix} 0 & q_D(1+T)\delta\psi_0 & (-)^{nr+r}\rho' \\ 0 & 0 & (-)^{nr+r+1}j' \\ 0 & 0 & 0 \end{pmatrix}$$

$$: C^{!n-r} = D'^{n-r} \oplus D^{n+1-r} \oplus D''^{n-r}$$

$$\longrightarrow C^!_r = D'_r \oplus D_{r+1} \oplus D''_r$$

$$\overline{\psi}_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-)^{n-r-s-1}T\psi_{s-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (s \geqslant 1)$$

$$: B^{n-1-r-s} = C'^{n-1-r-s} \oplus C^{n-r-s} \oplus C''^{n-1-r-s}$$

$$\longrightarrow B_r = C'_r \oplus C_{r+1} \oplus C''_r$$

$$\overline{\psi}_0 = \begin{pmatrix} 0 & q_C(1+T)\psi_0 & (-)^{nr}\rho_C \\ 0 & 0 & (-)^{nr+1}j_C \\ 0 & 0 & 0 \end{pmatrix}$$

$$: B^{n-1-r} = C'^{n-1-r} \oplus C^{n-r} \oplus C''^{n-1-r}$$

$$\longrightarrow B_r = C'_r \oplus C_{r+1} \oplus C''_r,$$

where $\rho' = (\rho_D \quad q_D f j_C) : D''^{n-r} = (D/D')_{r+1} \oplus (C/C')_r \longrightarrow D'_r$
and $j' = (j_D \quad 0) : D'^{n-r} \longrightarrow D_{r+1}$.

2.2.5. Proposition $(g^! : B \longrightarrow C^!, (\delta\overline{\psi}, -\overline{\psi}))$ is an n-dimensional quadratic Poincaré complex.

Proof: First of all the chain map

$$((1+T)\delta\overline{\psi}_0 \quad g^!(1+T)(-\overline{\psi}_0)) : C(g^!)^{n-*} \longrightarrow C^!$$

is given by a matrix

$$\begin{pmatrix} 0 & q_D(1+T)\delta\psi_0 & (-)^{nr+r}\rho_D & (-)^{nr+r}q_D f j_C & 0 & -f'q_C(1+T)\psi_0 & (-)^{nr+1}f'\rho_C \\ (-)^r(1+T)\delta\psi_0 g_D^* & 0 & (-)^{nr+r+1}j_D & 0 & (-)^{r+1}f(1+T)\psi_0 q_C^* & 0 & (-)^{nr}f j_C \\ \rho_D^* & -j_D^* & 0 & 0 & 0 & 0 & 0 \\ j_C^*\, f^*\, q_D^* & 0 & 0 & 0 & -\rho_C^* & j_C^* & 0 \end{pmatrix}$$

$$: (D'^{n-r} \oplus D^{n+1-r}) \oplus (D/D')_{r+1} \oplus (C/C')_r \oplus (C'^{n-1-r} \oplus C^{n-r}) \oplus (C/C')_{r+1}$$

$$\longrightarrow (D'_r \oplus D_{r+1}) \oplus (D/D')^{n+1-r} \oplus (C/C')^{n-r},$$

which is a chain equivalence made up of three chain equivalences combined together in the way described in Lemma 1.1.1. Next notice that B is chain equivalent to $S^{-1}C(p_C(1+T)\psi_0 p_C^*)$; therefore by 2.2.1, $H_i(B) = 0$ for $i < 0$, and $H^{n-1+j}(B) = 0$ for $j > 0$ owing to the duality of B. The standard "folding" argument [Co] shows that B is (n-1)-dimensional. Similarly $C^!$ is n-dimensional. A direct calculation shows that $(\delta\overline{\psi},-\overline{\psi})$ is an n-dimensional quadratic structure. Thus the proposition is proved.

Now we have two (n+1)-dimensional quadratic triads over R:

$$(\mathcal{J}_1: \quad \begin{tikzcd} B \arrow[r, dash, "g^!"] \arrow[d, "g'_C"'] \arrow[dr, bend left, "0"] & C^! \arrow[d, "g'_D"] \\ C' \arrow[r, dash, "f'"'] & D' \end{tikzcd} \quad , \quad \Psi_1 = (\overline{\psi},\ 0,\ -\delta\overline{\psi},\ 0))$$

$$(\mathcal{J}_2: \quad \begin{tikzcd} B \arrow[r, dash, "g''_C"] \arrow[d, "g^!"'] \arrow[dr, bend left, "0"] & C \arrow[d, "f''"] \\ C^! \arrow[r, dash, "g''_D"'] & D'' \end{tikzcd} \quad , \quad \Psi_2 = (\overline{\psi},\ \delta\overline{\psi},\ 0,\ 0)),$$

which turn out to be Poincaré; in fact the duality maps of the (n+1)-dimensional quadratic pairs induced from these can be written as follows:

$$\text{(I)} \quad (0 \qquad (f' \quad 0 \quad -g'_D)(1+T)(0 \ \cup_{\overline{\psi}}(-\delta\overline{\psi}))_0)$$

$$: C((f' \quad 0 \quad -g'_D))^{n+1-*} \longrightarrow D'$$

$$= \left(0 \quad 0 \quad 0 \quad \boxed{-q_D f(1+T)\psi_0} \quad (-)^{nr+1} f'\rho_C \quad 0 \right.$$

$$\left. \boxed{-q_D(1+T)\delta\psi_0} \quad \boxed{(-)^{nr+r+1}\rho_D} \quad (-)^{nr+r+1} q_D f j_C \right)$$

$$: D'^{n+1-r} \oplus C'^{n-r} \oplus C'^{n-1-r} \oplus C^{n-r} \oplus (C/C')_{r+1} \oplus D'^{n-r}$$

$$\oplus (D'^{n+1-r} \oplus (D/D')_{r+1}) \oplus (C/C')_r \longrightarrow D'_r,$$

which is essentially same as the following composition of chain equivalences:

$$C^{n-r} \oplus D^{n+1-r} \oplus (D/D')_{r+1}$$

$$\xrightarrow{(f(1+T)\psi_0 \quad (1+T)\delta\psi_0 \quad 1)} D_r \oplus (D/D')_{r+1}$$

$$\xrightarrow{(-q_D \quad (-)^{nr+r+1}\rho_D)} D'_r,$$

(II) $\quad (0 \quad (g''_D \quad 0 \quad -f'')(1+T)(\delta\overline{\psi} \cup_{\overline{\psi}} 0)_0)$

$$: C((g''_D \quad 0 \quad -f''))^{n+1-*} \longrightarrow D''$$

$$= \begin{pmatrix} 0 & \boxed{\rho^*_D \quad -j^*_D} & 0 & 0 & 0 & 0 & 0 \\ 0 & j^*_C f^* q^*_D \quad 0 & 0 & \boxed{(-)^{nr}\rho^*_C \quad (-)^{nr+1} j^*_C} & & 0 & 0 \end{pmatrix}$$

$$: D''^{n-1+r} \oplus (D'^{n-r} \oplus D^{n+1-r}) \oplus D''^{n-r} \oplus (C'^{n-1-r} \oplus C^{n-r}) \oplus C''^{n-1-r}$$

$$\oplus C''^{n-r} \longrightarrow (D/D')^{n+1-r} \oplus (C/C')^{n-r}.$$

We now claim that the (n+1)-dimensional quadratic Poincaré pair

$$(F = f' \oplus g^! \oplus (-f'') : (C' \cup_B C'')_r = C'_r \oplus B_{r-1} \oplus C''_r$$

$$\longrightarrow (D' \cup_{C^!} D'')_r = D'_r \oplus C^!_{r-1} \oplus D''_r, \quad (0 \cup_{\delta\bar{\psi}} 0,\ 0 \cup_{\bar{\psi}} 0))$$

which is induced by the union of $(\mathcal{J}_1, \Psi_1)$ and $(\mathcal{J}_2, \Psi_2)$ is homotopy equivalent to the original Poincaré pair $(f\colon C \longrightarrow D,\ (\delta\psi, \psi))$. Here,

$$(0 \cup_{\delta\bar{\psi}} 0)_s = 0 \oplus 0 \oplus \delta\psi_s \oplus 0 \oplus 0$$

$$: D'^{n+1-r} \oplus D'^{n-r} \oplus D^{n+1-r} \oplus D''^{n-r} \oplus D''^{n+1-r}$$

$$\longrightarrow D'_r \oplus D'_{r-1} \oplus D_r \oplus D''_{r-1} \oplus D''_r$$

$$(0 \cup_{\bar{\psi}} 0)_s = 0 \oplus 0 \oplus \psi_s \oplus 0 \oplus 0$$

$$: C'^{n-r} \oplus C'^{n-1-r} \oplus C^{n-r} \oplus C''^{n-1-r} \oplus C''^{n-r}$$

$$\longrightarrow C'_r \oplus C'_{r-1} \oplus C_r \oplus C''_{r-1} \oplus C''_r.$$

It is already shown that the inclusion maps

$${}^t(0\ \ 0\ \ 1\ \ 0\ \ 0)\colon C \longrightarrow C' \cup_B C''$$

$${}^t(0\ \ 0\ \ 1\ \ 0\ \ 0)\colon D \longrightarrow D' \cup_{C^!} D''$$

are chain equivalences. The following diagram:

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "{}^t(0\ 0\ 1\ 0\ 0)"'] & D \arrow[d, "{}^t(0\ 0\ 1\ 0\ 0)"] \\
C' \cup_B C'' \arrow[r] & D' \cup_{C^!} D''
\end{tikzcd}$$

commutes, and

$$^t(0\quad 0\quad 1\quad 0\quad 0)\psi_s(0\quad 0\quad 1\quad 0\quad 0) = (0 \cup_{\bar{\psi}} 0)_s$$

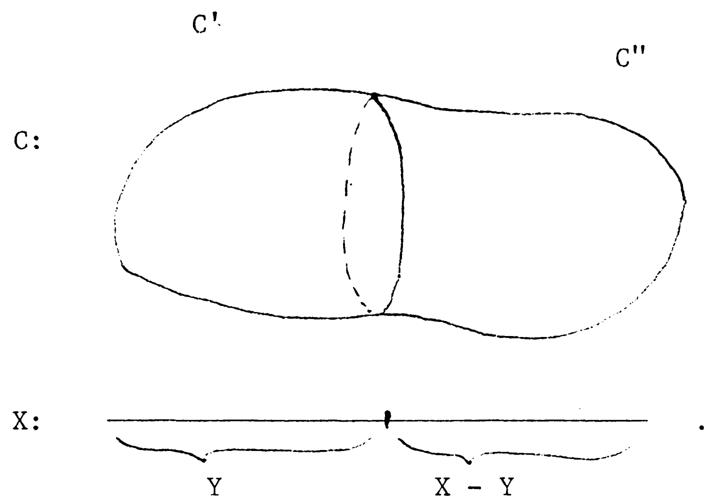
$$^t(0\quad 0\quad 1\quad 0\quad 0)\delta\psi_s(0\quad 0\quad 1\quad 0\quad 0) = (0 \cup'_{\delta\bar{\psi}} 0)_s$$

for each $s \geqslant 0$. Therefore, these two Poincaré pairs are homotopy equivalent, and the lemma is proved.

2.2.6. Remarks i) If c is strictly $(n+1)$-dimensional, then the conditions 2.2.1 and 2.2.2 are satisfied when the images of C'^n and D'^{n+1} by $(1+T)\psi_0$ and $(1+T)\delta\psi_0$ lie in C'_0 and D'_0 respectively. The vanishing of $H_0((1+T)\psi_0)$ will imply that of $H_0(p_C(1+T)\psi_0 p_C^*)$, etc.
ii) If c is free, the above argument can be carried through in the category of free complexes.
iii) (Relative Splitting) If the splitting of the boundary is already given, then we can modify the construction so that the result has the given splitting of the boundary.

2.3. Splitting lemma for geometric quadratic Poincaré complexes and pairs

Let (C,ψ) be a strictly n-dimensional geometric quadratic Poincaré complex on X with radius less than δ, and let Y be a subset of X. We would like to use 2.2.4 to split (C,ψ) into two parts, one lying over Y and the other mostly over $X - Y$, whose common boundary lies along the border between Y and $X - Y$:



As C', we can use the following subcomplex of C:

$$C'_i = C_i | Y^{-i\delta}.$$

Then $C'' = (C/C')^{n-*}$ lies over $X - Y^{-n\delta}$; 2.2.4 gives an equivalence with radius 0 from (C,ψ) to a union such that everything has radius less than 2δ. The only defect is that the common boundary B of 2.2.4 may lie all over X. It is easy to cut off the portion of B lying over $Y^{-n\delta}$; B is chain equivalent to $S^{-1}C(p_C(1+T)\psi_0 p_C^*)$. Next notice that $p_C(1+T)\psi_0 p_C^*$ is a chain equivalence over $X - Y^{\delta}$. This produces a "chain contraction" over $X - Y^{\delta'}$ for some $\delta' > 0$. We will use this, trying to eliminate the portion lying over $X - Y^{\delta'}$.

We begin with defining "chain contraction" mentioned above. Let (C,d) be a geometric R-module chain complex on X, and Y be a subset of X.

2.3.1. <u>Definition</u> A collection $\{s_i : C_i \longrightarrow C_{i+1}\}$ of R-module homomorphisms is called a <u>chain contraction of</u> C <u>over</u> Y if

$$d_{i+1}s_i + s_{i-1}d_i = 1 \qquad \text{on } Y.$$

A chain contraction $\{s_i\}$ over Y <u>has radius less than</u> ε if each s_i has radius less than ε.

Roughly speaking, if there is a chain contraction of C over Y, we can do the "localized folding argument" over Y to eliminate most of the portion of C over Y. See [Co §14] for the standard folding argument.

2.3.2. <u>Lemma</u> If a geometric R-module chain complex (C,d) on X with radius less than ε has a chain contraction over Y with radius less than ε and satisfies the following:

$$C_\ell | Y = 0 \qquad \text{for } \ell < k,$$

then C is 2ε chain equivalent to a geometric R-module chain complex (C',d') on X with radius less than ε such that

i) $C'_\ell | Y^{-\varepsilon} = 0 \qquad$ for $\ell < k$,

ii) $C' = C \qquad$ on $X - Y$, and

iii) there exists a chain contraction of C' over $Y^{-2\varepsilon}$ with radius less than 3ε.

<u>Proof:</u> Let $\{s_i\}$ denote the chain contraction of C over Y, and let

i, j, p, q denote the following canonical inclusion maps and projection maps:

$$C_k|X-Y^{-\varepsilon} \underset{p}{\overset{i}{\rightleftarrows}} C_k \underset{j}{\overset{q}{\rightleftarrows}} C_k|Y^{-\varepsilon}.$$

Notice the identities $d_{k+1}s_kj = j$ and $d_kj = 0$. Define (C',d') as in Diagram 1. T is the trivial complex with $T_{k+2} = T_{k+1} = C_k|Y^{-\varepsilon}$, $T_\ell = 0$ otherwise, and $d_T = 1\colon T_{k+2} \longrightarrow T_{k+1}$; and T' is the de-suspension of T. There are obvious chain equivalences $f\colon C \longrightarrow C \oplus T$ and $h : C' \oplus T' \longrightarrow C'$. Define a chain map $g : C \oplus T \longrightarrow C' \oplus T'$ by

$$g_\ell = 1 \qquad \text{if } \ell \neq k+1$$

$$g_{k+1} = \begin{pmatrix} 1 & s_kj \\ qd_{k+1} & 0 \end{pmatrix}$$

$$: C_{k+1} \oplus C_k|Y^{-\varepsilon} \longrightarrow C_{k+1} \oplus C_k|Y^{-\varepsilon}.$$

Since g_{k+1} can be decomposed as

$$g_{k+1} = \begin{pmatrix} 1 & 0 \\ qd_{k+1} & -1 \end{pmatrix}\begin{pmatrix} 1 & s_kj \\ 0 & 1 \end{pmatrix},$$

g is an ε-isomorphism. Composing f, g, and h, we obtain a 2ε chain equivalence between C and C'. A desired chain contraction $\{s'_\ell\}$ of C' over $Y^{-\varepsilon}$ is defined by $s'_\ell = 0$ for $\ell \leqslant k$, $s'_{k+1} = {}^t(s_{k+1} \quad qd_{k+1})$, $s'_{k+2} = (s_{k+2} \quad -s_{k+2}s_{k+1}s_kj)$, and $s'_\ell = s_\ell$ for $\ell \geqslant k+3$.

$$
\begin{tikzcd}[column sep=large, row sep=large]
C: \cdots \arrow[r] & C_{k+3} \arrow[r, "d_{k+3}"] & C_{k+2} \arrow[r, "d_{k+2}"] \arrow[d, "f"'] \arrow[r, phantom, shift right=8, "\cong"] & C_{k+1} \arrow[r, "d_{k+1}"] & C_k \arrow[r, "d_k"] & C_{k-1} \arrow[r, "d_{k-1}"] & {} \\
C \oplus T: \cdots \arrow[r] & C_{k+3} \arrow[r, "d_{k+3}"] \arrow[dr, "0"'] & C_{k+2} \arrow[r, "d_{k+2}"] \arrow[d, phantom, "\oplus"] & C_{k+1} \arrow[r, "d_{k+1}"] \arrow[u, "f^{-1}"'] \arrow[d, phantom, "\oplus"] & C_k \arrow[r, "d_k"] & C_{k-1} \arrow[r, "d_{k-1}"] & {} \\
 & & C_k|Y^{-\varepsilon} \arrow[r, "1"'] \arrow[d, "g"'] \arrow[r, phantom, shift right=10, "\cong"] & C_k|Y^{-\varepsilon} \arrow[ur, "0"'] & & & \\
 & & {} & {} \arrow[u, "g^{-1}"'] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
C' \oplus T' : & \cdots \arrow[r] & C_{k+3} \arrow[r, "d_{k+3}"] \arrow[ddr, "0"'] & C_{k+2} \arrow[r, "d_{k+2}"] & C_{k+1} \arrow[r, "pd_{k+1}"] & C_k|X-Y^{-\varepsilon} \arrow[r, "d_k i"] & C_{k-1} \\
 & & & \oplus & \oplus & \oplus & \\
 & & & C_k|Y^{-\varepsilon} \arrow[uur, "s_k j"] \arrow[d, "h"] & C_k|Y^{-\varepsilon} \arrow[r, "1"'] & C_k|Y^{-\varepsilon} \arrow[uur, "0"'] & \\
C' : & \cdots \arrow[r] & C_{k+3} \arrow[r, "d_{k+3}"] \arrow[ddr, "0"'] & C_{k+2} \arrow[r, "d_{k+2}"] & C_{k+1} \arrow[r, "pd_{k+1}"] \arrow[u, "h^{-1}"'] & C_k|X-Y^{-\varepsilon} \arrow[r, "d_k i"] & C_{k-1} \\
 & & & \oplus & & & \\
 & & & C_k|Y^{-\varepsilon} \arrow[uur, "s_k j"'] & & &
\end{tikzcd}
$$

Diagram 1.

Using this lemma repeatedly, we get the following:

2.3.3. <u>Corollary</u> Fix a positive integer n. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any strictly n-dimensional geometric R-module chain complex C on X with radius less than δ which has a chain contraction over Y with radius less than δ, there exists a strictly n-dimensional geometric R-module chain complex C' on X with radius less than ε satisfying

i) $C'_0|Y^{-\varepsilon} = \ldots = C'_{n-2}|Y^{-\varepsilon} = 0$,

ii) $C = C'$ on $X - Y$,

iii) there exists an R-module homomorphism $s : C'_{n-1} \longrightarrow C'_n$ with radius less than ε such that

$$d_n s = 1 \text{ and } sd_n = 1 \quad \text{on } Y^{-\varepsilon},$$

iv) C is ε chain equivalent to C'.

It is in general impossible to finish this elimination, but if we "stabilize" everything, we can avoid the difficulty as follows. (We will use a special case of this which corresponds to taking the product with $\mathbb{R}$.) Let (E, d_E) be a strictly 1-dimensional geometric $\mathbb{Z}$-module chain complex on a metric space Z such that $E_0 = E_1 = \mathbb{Z}[P]$, where P is a set of points of Z. Then $C' \otimes E$ is a strictly (n+1)-dimensional geometric R-module chain complex with radius less than ε on $X \times Z$ (we measure the radius after projecting points to X):

$$0 \longrightarrow C'_n \otimes E_1 \xrightarrow{\begin{pmatrix} (-)^n 1 \otimes d_E \\ d_n \otimes 1 \end{pmatrix}} \begin{matrix} C'_n \otimes E_0 \\ \oplus \\ C'_{n-1} \otimes E_1 \end{matrix} \xrightarrow{\begin{pmatrix} d_n \otimes 1 & (-)^{n-1} 1 \otimes d_E \\ 0 & d_{n-1} \otimes 1 \end{pmatrix}} \begin{matrix} C'_{n-1} \otimes E_0 \\ \oplus \\ C'_{n-2} \otimes E_1 \end{matrix} \longrightarrow \cdots$$

and $(C' \otimes E)_i | Y^{-\varepsilon} \times Z = 0$ if $i \leqslant n-2$. The maps

$$(s'_i \otimes 1) \oplus (s'_{i-1} \otimes 1) : (C'_i \otimes E_0) \oplus (C'_{i-1} \otimes E_1) \longrightarrow (C'_{i+1} \otimes E_0) \oplus (C'_i \otimes E_1),$$

where $s'_{n-1} = s$, $s'_i = 0$ if $i \neq n-1$, define a chain contraction of $C' \otimes E$ over $Y^{-\varepsilon} \times Z$ with radius less than ε. Use 2.3.2 to get a strictly $(n+1)$-dimensional geometric R-module chain complex $\overline{C}$ on $X \times Z$ with radius less than ε such that

$$\overline{C}_i | Y^{-2\varepsilon} \times Z = 0 \text{ if } i \leqslant n-1 ;$$

the boundary map $d : \overline{C}_{n+1} \longrightarrow \overline{C}_n$ is given by the following matrix:

$$d = \begin{pmatrix} (-)^n 1 \otimes d_E & sj \otimes 1 \\ d_n \otimes 1 & 0 \end{pmatrix}$$

$$: (C'_n \otimes \mathbb{Z}[P]) \oplus (C'_{n-1} | Y^{-2\varepsilon} \otimes \mathbb{Z}[P]) \to (C'_n \otimes \mathbb{Z}[P]) \oplus (C'_{n-1} \otimes \mathbb{Z}[P]),$$

where j is the inclusion map of $C'_{n-1} | Y^{-2\varepsilon}$ into C'_{n-1}. Now consider the following ε-isomorphism of $\bar{C}_{n+1}$ to itself:

$$h = \begin{pmatrix} 1 & (-)^n sj \otimes d_E \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -sj \otimes 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ qd_n \otimes 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & sj \otimes 1 \\ 0 & -1 \end{pmatrix}$$

where q is the projection map of C'_{n-1} onto $C'_{n-1}|Y^{-2\varepsilon}$. Then on $Y^{-3\varepsilon} \times Z$, $h = d$. If we replace the boundary map $d\colon \bar{C}_{n+1} \to \bar{C}_n$ by dh^{-1}, we get a new geometric R-module chain complex $\bar{C}'$, which is ε-isomorphic to $\bar{C}$ and the boundary map dh^{-1} is the identity on $Y^{-3\varepsilon} \times Z$. Now we can delete $\bar{C}'_{n+1}|Y^{-4\varepsilon} \times Z$ and $\bar{C}'_n|Y^{-4\varepsilon} \times Z$ from $\bar{C}'$. Thus we obtain the following:

2.3.4. <u>Lemma</u> Fix n and E. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any strictly n-dimensional geometric R-module chain complex C on X with radius less than δ which has a chain contraction over Y with radius less than δ, there exists a strictly $(n+1)$-dimensional geometric R-module chain complex $\tilde{C}$ on $X \times Z$ with radius less than ε (measured in X) satisfying

i) $\tilde{C}$ lies over $(X - Y^{-\varepsilon}) \times Z$,

ii) $C \otimes E = \tilde{C}$ on $(X - Y) \times Z$,

iii) $C \otimes E$ is ε chain equivalent to $\tilde{C}$.

In our application of this lemma for splitting, we will use the following complex as E. Consider a geometric $\mathbb{Z}$-module $\mathbb{Z}[\mathbb{Z}]$

on $\mathbb{R}^1$ which is generated by all the integers, and a $\mathbb{Z}$-module isomorphism $t\colon \mathbb{Z}[\mathbb{Z}] \longrightarrow \mathbb{Z}[\mathbb{Z}]$ defined by

$$t([n]) = [n+1] .$$

Define a strictly 1-dimensional geometric $\mathbb{Z}$-module chain complex E by

$$E_0 = E_1 = \mathbb{Z}[\mathbb{Z}]$$

$$d = t-1 : E_1 \longrightarrow E_0 .$$

Although E is not finitely generated, we define a symmetric structure ϕ on E as shown below.

$$\begin{tikzcd}
E^{1-*}: & 0 \arrow[r] \arrow[d, "0"'] & E^0 \arrow[r, "-d^*"] \arrow[d, "\phi_0=t^{-1}"] \arrow[dl, "0"'] & E^1 \arrow[r] \arrow[d, "\phi_0=1"] \arrow[dl, "\phi_1=-1"'] & 0 \arrow[d, "0"] \arrow[dl] \\
E: & 0 \arrow[r] & E_1 \arrow[r, "d"'] & E_0 \arrow[r] & 0 .
\end{tikzcd}$$

For $s \geqslant 2$, $\phi_s = 0$; $d^* = t^{-1} - 1$. Obviously ϕ_0 is an isomorphism, thus (E,ϕ) is a 1-dimensional "symmetric" Poincaré complex.

Now we state the stable splitting lemma for geometric quadratic Poincaré complexes and pairs.

2.3.5. <u>Lemma</u> Fix n. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any strictly n-dimensional geometric quadratic Poincaré complex (C,ψ) on X with radius less than δ, and a subset Y of X, there exist two strictly n-dimensional quadratic Poincaré pairs on X with radii less than ε:

$$(B \longrightarrow C', (\delta\overline{\psi}', -\overline{\psi}))$$

$$(B \longrightarrow C'', (\delta\overline{\psi}'', \overline{\psi}))$$

such that

i) C' (resp. C'') lies over Y (resp. $X - Y^{-\varepsilon}$),

ii) C is homotopy equivalent to their union along B,

iii) $(B, \overline{\psi}) \otimes (E, \phi)$ is ε homotopy equivalent to a strictly n-dimensional geometric quadratic Poincaré complex on $X \times \mathbb{R}$ which lies over $(Y^{\varepsilon} - Y^{-\varepsilon}) \times \mathbb{R}$.

A strictly (n+1)-dimensional geometric quadratic Poincaré pair on X with radius less than δ has a similar splitting into two geometric triads.

Proof: If $(f: C \longrightarrow D, (\delta\psi, \psi))$ is the given pair, define subcomplexes C', D' of C, D as follows:

$$C'_i = C_i | Y^{-(i+1)\delta}$$

$$D'_i = D_i | Y^{-i\delta}.$$

Then $f(C'_i) \subseteq D'_i$. The conditions 2.2.1 and 2.2.2 are satisfied because $(1+T)\psi_0(C'^n) \subseteq C'_0$ and $(1+T)\delta\psi_0(D'^{n+1}) \subseteq D'_0$. It is easy to construct chain contractions of B and $C^!$ over $Y^{-\delta'} \cup (X - Y^{\delta'})$ for some δ', and we can apply Lemma 2.3.4 to B and $C^!$. The proof for complexes is similar.

We will call $(B,\bar{\psi}) \otimes (E,\psi)$ the <u>external suspension</u> of $(B, \bar{\psi})$ and denote it by $\Sigma(B, \bar{\psi})$.

2.4. <u>Glueing and splitting over a triangulation of a manifold</u>

The notion of pairs(= 2-ads) and triads(= 3-ads) naturally extends to "(k+2)-ads." ([R4]) These are defined inductively as follows:

(1) a <u>(k+2)-ad</u> x is a collection of

i) k+1 (k+1)-ads $\partial_0 x, \ldots, \partial_k x$ satisfying

$$(*) \qquad \partial_j \partial_i x = \partial_i \partial_{j+1} x \quad \text{if } 0 \leqslant i \leqslant j \leqslant k,$$

ii) a pair $\partial_i x \longrightarrow \| x \|$ ($\| x \|$ is the <u>underlying chain complex</u> of x), and

(2) suppose there are k+2 (k+2)-ads $\partial_0 y, \ldots, \partial_{k+1} y$ which satisfy (*), then $\partial_0 y \cup \ldots \cup \partial_i y$ are defined inductively (on i) using the glueing method of §2.1.

Let x be a (k+2)-ad. When $i_1 < \ldots < i_\ell$, we define $\partial_{\{i_1,\ldots,i_\ell\}} x$ by $\partial_{i_1} \partial_{i_2} \ldots \partial_{i_\ell} x$.

An <u>n-dimensional quadratic Poincaré (k+2)-ad</u> x is a (k+2)-ad x together with structure maps

$$(\psi_\alpha)_s : \| \partial_\alpha x \|^{n-|\alpha|-r-s} \longrightarrow \| \partial_\alpha x \|, \qquad \alpha \subset \{0,1,\ldots,k\}$$

such that

(1) $\partial_i x$ is an $(n-1)$-dimensional quadratic Poincaré $(k+1)$-ad for each i, and

(2) $(\cup \partial_i x \longrightarrow \|x\|, (\psi_\phi, \cup \psi_i))$ is an n-dimensional quadratic Poincaré pair.

Here $|\alpha|$ denotes the size of the subset α, and $\cup \psi_i$ is defined using the glueing operation of §2.1 repeatedly.

Let X be a metric space. If we use geometric chain complexes on X, we can define geometric quadratic Poincaré n-ads on X. Such a thing has radius ε (resp. support $C \subseteq X$) if all the involved chain complexes, chain maps, chain homotopies, and the possible compositions of maps have radii ε (resp. support C). An n-ad x is called special if $\partial_{\{0,1,\ldots,n-2\}} x = 0$. Recall that an "n-ad" in the usual sense is a topological space together with $n-1$ subsets. We call this a topological n-ad to distinguish it from n-ads of chain complexes. Let $(X, \partial_* X)$ be a topological n-ad of metric space. A geometric quadratic Poincaré n-ad x on $(X, \partial_* X)$ is a geometric quadratic Poincaré n-ad on X such that $\partial_i x$ lies over $\partial_i X$. Let $(\Delta, \partial_* \Delta)$ be a topological $(n+2)$-ad induced by an n-simplex. An $(n+2)$-ad on $(\Delta, \partial_* \Delta)$ is automatically special.

As an application of previous sections, let us consider the following problem. Let M be an n-dimensional compact manifold with a triangulation K, and fix a metric on M. Suppose each n-simplex Δ of K is given a geometric m-dimensional quadratic Poincaré $(n+2)$-ad x on $(\Delta, \partial_* \Delta)$ such that

2.4.1. (compatibility) if, for an (n-1)-simplex Δ^{n-1} of K, $\partial_i\Delta = \Delta^{n-1} = \partial_j\Delta'$ and $\Delta \neq \Delta'$, then $\partial_i x_\Delta = -\partial_j x_{\Delta'}$.

We would like to glue all these (n+2)-ads to get a geometric quadratic pair on (M, ∂M). We can also consider a problem of the inverse direction. Notice that there is a small difficulty. For example, consider an annulus (Fig. 1 (i)) with 2n 2-simplices which are given geometric quadratic Poincaré special 4-ad structures. Let us try to glue these together inductively in the order as shown in Fig.1(ii). This turns out to be a wrong order. If we try to glue the (2n-1)-st 2-simplex, then we have to do so along the union of a 1-simplex and a 0-simplex, which is not codimension one. But if we glue (1) and (2), (3) and (4), ..., (2n-1) and (2n) respectively, then we have n 2-cells such that any two 2-cells are disjoint or meet along a codimension one cell. Now we can glue these in any order. This is the basic idea. First we glue locally so that the local blocks behave nicely, and then we glue the blocks (in any order). When we split something, we first split it into several blocks so that each block is a union of simplices in some nice way, and we split each block into pieces.

Let M be an n-dimensional compact manifold with a triangulation K. Assume that K is the first barycentric subdivision of another triangulation L. For each vertex v of L, its star S(v) in K, or a dual cone, is the local block mentioned above. Two such dual

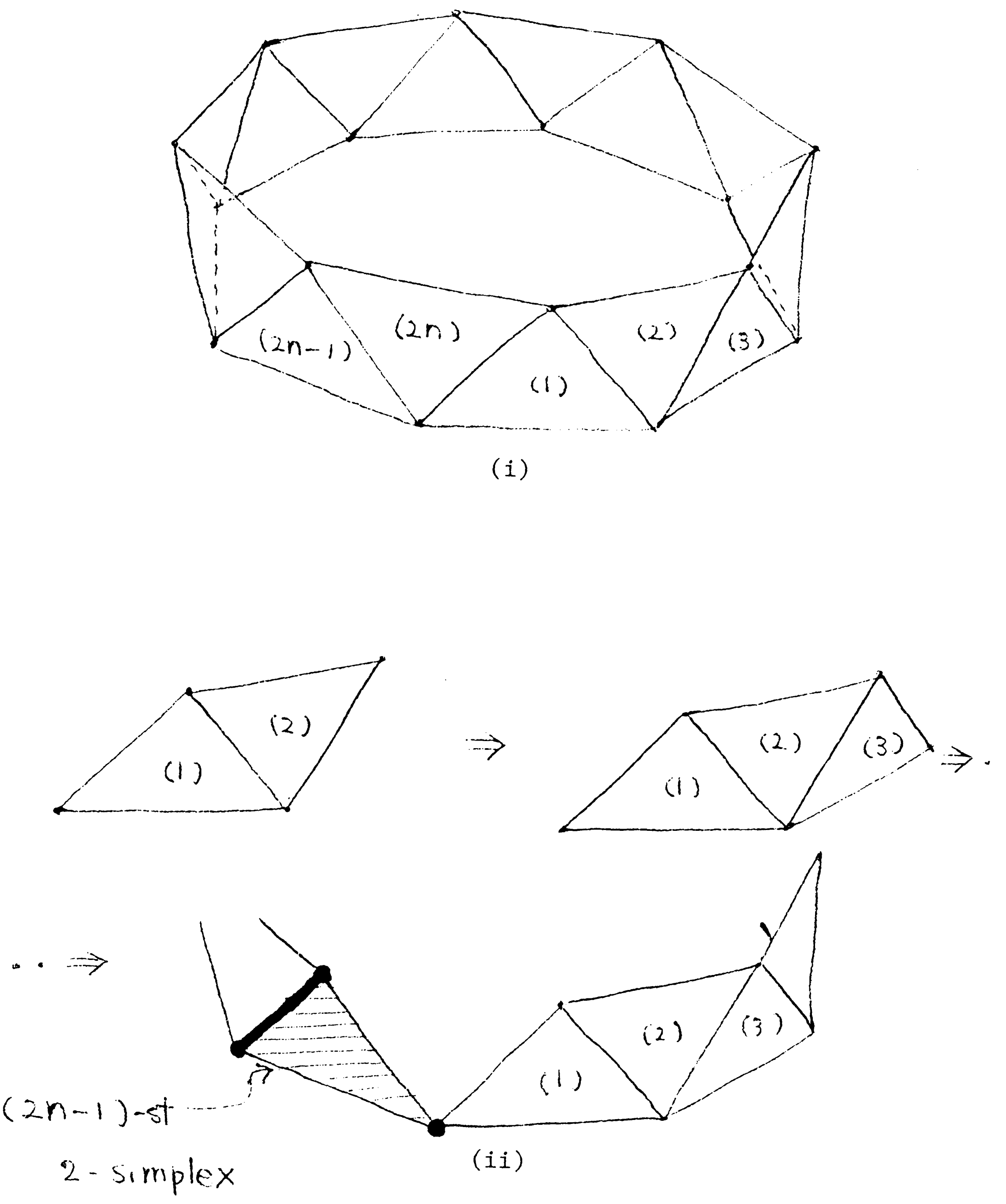
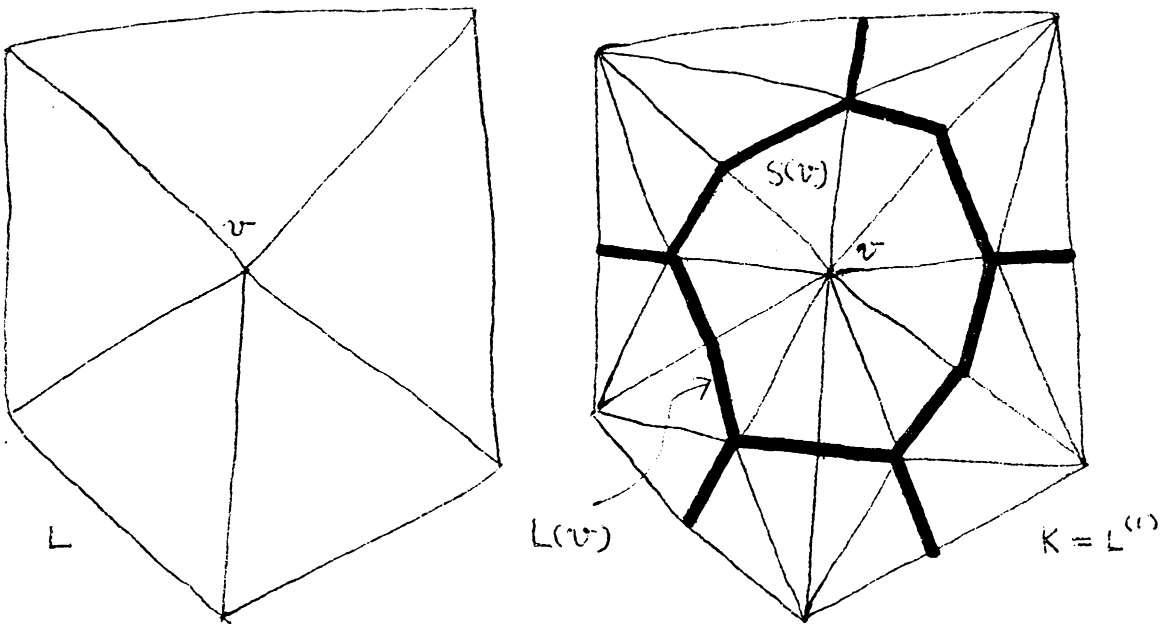


Figure 1. Glueing on an annulus.



cones are either disjoint or meet along codimension 1 cell(s). The glueing and splitting problem over S(v) can be solved by looking at the link L(v) of v in K. Note that L(v) is an (n-1)-dimensional sphere and the triangulation is the first barycentric subdivision of another. Thus we can keep on reducing the dimension until the link becomes a circle as above, and in this case there is a natural order of 2-simplices and glueing and splitting can be done. Thus we have

2.4.2. Theorem (Glueing over a manifold) Let K be the first barycentric subdivision of a triangulation of a compact n-dimensional manifold with a metric, and suppose each n-simplex Δ is given an m-dimensional geometric Poincaré special (n+2)-ad on $(\Delta, \partial_* \Delta)$ which are compatible on common faces (in the sense of 2.4.1). Then one can glue them together to get a geometric quadratic Poincaré pair on K such that its boundary lies over ∂K.

2.4.3. Theorem (Stable splitting over a manifold) Let us fix an integer m, and let K be the first barycentric subdivision of a triangulation of a compact manifold M with a metric. Let ε be any positive number. Then there exists $\delta > 0$ such that any geometric m-dimensional quadratic Poincaré pair with radius $< \delta$ on $(M, \partial M)$ can be stably split into pieces each of which lies in an ε-neighborhood of the corresponding simplex of K.

2.4.4. Remarks 1) Until now we have considered chain complexes with constant coefficients. The result can be extended to the case arising from stratified systems of fibrations (1.2.2 and 1.2.3), if the filtration is compatible with the triangulation and each simplex is sufficiently small.

2) If a splitting of the boundary is already given, then the result has the given splitting of the boundary.

3. Surgery space and assembly

3.1. Surgery spaces

In 3.1 and 3.2, we will immitate what Quinn did in [Q3, §5]. Fix a stratified system of fibrations $p: E \longrightarrow X$, an integer n, and a covering space $\tilde{E}$ of E. A <u>primitive k-simplex</u> x (<u>of degree</u> n) is a strictly $(n+k+\ell)$-dimensional geometric quadratic Poincaré special (k+2)-ad on $pr: E \times \mathbb{R}^{\ell} \longrightarrow X$, where r is the obvious projection $E \times \mathbb{R}^{\ell} \longrightarrow E$. Here we are using $\tilde{E} \times \mathbb{R}^{\ell}$ as the fixed covering space of $E \times \mathbb{R}^{\ell}$. We always assume that x has bounded radius when projected to $\mathbb{R}^{\ell}$ and that it is locally finitely generated on $X \times \mathbb{R}^{\ell}$. If x has radius ε and support C, then its faces $\partial_0 x, \ldots, \partial_k x$ are primitive (k-1)-simplices with radius ε and support C.

For a primitive k-simplex of degree n with radius C and radius ε, we have the following operations.

(1) <u>Reduction</u> Suppose $C' \subseteq C$ is compact, and $\varepsilon' \geq \varepsilon$. Then x can be regarded as a primitive k-simplex of degree n with support C' and radius ε'. This is called a reduction of x.

(2) <u>Suspension</u> The external suspension operation $\Sigma^{\ell} = \Sigma \times \ldots \times \Sigma$ defined in §2.3 can be naturally extended to an operation on primitive simplices. The result $\Sigma^{\ell} x$ has the same support and radius as x.

We define the space of quadratic Poincaré ads.

3.1.1. <u>Definition</u> $\tilde{\mathbb{P}}_n(X;p;E)$ is the Δ-set with simplices (which will be called <u>elaborate simplices</u>) defined inductively: an elaborate 0-simplex is a primitive 0-simplex of degree n, i.e., a strictly $(n+\ell)$-dimensional geometric quadratic Poincaré special 2-ad (= complex) on pr: $E \times \mathbb{R}^\ell \longrightarrow X$ for some ℓ (with unrestricted compact support and radius). An elaborate k-simplex σ consists of an underlying primitive k-simplex $|\sigma|$ of degree n, together with k+1 elaborate (k-1)-simplices $\partial_0\sigma, \ldots, \partial_k\sigma$. We require these to satisfy the usual $\partial_i\partial_j$ identities, and in addition require that the external suspension of a reduction of the underlying primitive (k-1)-simplex $|\partial_i\sigma|$ of $\partial_i\sigma$ be different from the i-th face $\partial_i|\sigma|$ of the underlying primitive k-simplex $|\sigma|$ only in the structure maps, and that the structure maps be homologous by a chain whose radius is less than that of $|\sigma|$. We define the <u>support</u> and <u>radius</u> of an elaborate simplex σ to be those of $|\sigma|$.

Suppose $C \subseteq X$ is compact and $\varepsilon > 0$, then $\tilde{\mathbb{P}}_n(X,C,p,\varepsilon;E)$ is the subset of $\tilde{\mathbb{P}}_n(X;p;E)$ made up of all the simplices with support containing C and radius not exceeding ε. If $\tilde{E}$ is the universal cover of E, then we omit $\tilde{E}$ in these notations and write $\mathbb{P}_n(X;p)$ and $\mathbb{P}_n(X,C,p,\varepsilon)$ respectively. We introduce a control of radii and supports in the following definition.

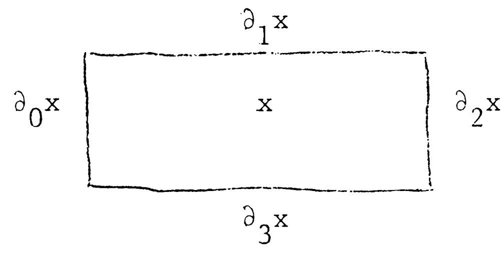
3.1.2. <u>Definition</u> Suppose X is locally compact metric, p: E $\longrightarrow$

X is a stratified system of fibrations, and $\tilde{E}$ is a covering space of E. Then a k-simplex of $\mathbb{L}_n(X;p;\tilde{E})$ is defined to be a simplicial map $\Delta^k \times [0,\infty) \longrightarrow \mathbb{P}_n(X;p;\tilde{E})$ which satisfies the following condition: there are a sequence of compact sets $C_i \subseteq X$ with C_i $(C_{i+1})^o$, $\cup C_i = X$, and a sequence ε_i of numbers monotone decreasing to 0, such that the image of $\Delta^k \times [i,\infty)$ lies in $\mathbb{P}_n(X,C_i,p,\varepsilon_i;\tilde{E})$. If $\tilde{E}$ is the universal cover of E, we write $\mathbb{L}_n(X;p)$ instead of $\mathbb{L}_n(X;p;\tilde{E})$.

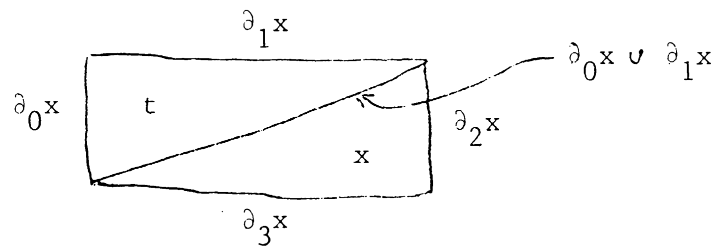
Since we are interested in $\mathbb{P}_n(X;p)$ and $\mathbb{L}_n(X;p)$, we only deal with these in the rest of this section. The general $\tilde{E}$ will appear in the definition of a homology spectrum in §3.2.

First of all, $\mathbb{P}(X;p)$ (we omit the subscript n) satisfies the Kan condition. Consider for example two elaborate 1-simplices σ_0, σ_1 of $\mathbb{P}(X;p)$ with $\partial_0\sigma_0 = \partial_0\sigma_1$. The Kan condition asserts that there is an elaborate 2-simplex τ with $\partial_0\tau = \sigma_0$, $\partial_1\tau = \sigma_1$. We construct such τ as follows. Suspend $|\sigma_0|$, $|\sigma_1|$, and $|\partial_0\sigma_0|$ if necessary so that we can make the union $|\sigma_0| \cup |\sigma_1|$ along $|\partial_0\sigma_0|$. Consider the trivial cobordism x between $|\sigma_0| \cup |\sigma_1|$ and itself. Then τ is given by $(x;\sigma_0,\sigma_1,\sigma_0 \cup \sigma_1)$, where $\sigma_0 \cup \sigma_1$ denotes the elaborate 1-simplex $(|\sigma_0| \cup |\sigma_1|;\ \partial_1\sigma_0,\ \partial_1\sigma_1)$. The same argument works for any simplicial map $\Lambda \longrightarrow \mathbb{P}_n(X;p)$. Thus $\mathbb{P}(X;p)$ satisfies the Kan condition.

3.1.3. Remark The same technique will be used quite often in this chapter. Suppose we have a 5-ad $(x, \partial_0 x, \ldots, \partial_3 x)$:



Then considering the trivial cobordism t of $\partial_0 x \cup \partial_1 x$ to itself, we can triangulate this as follows:



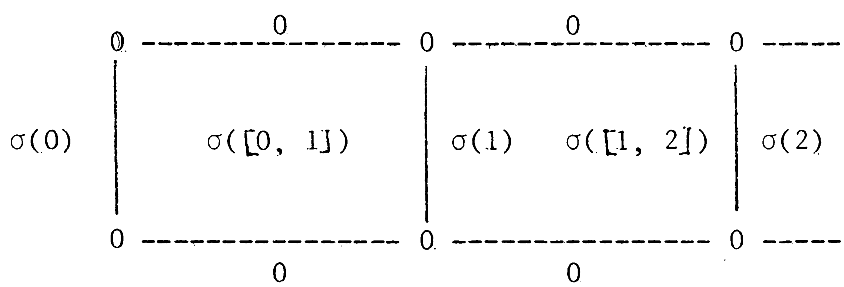
This will be called the "triangulation argument."

The next result describes the spectrum structure.

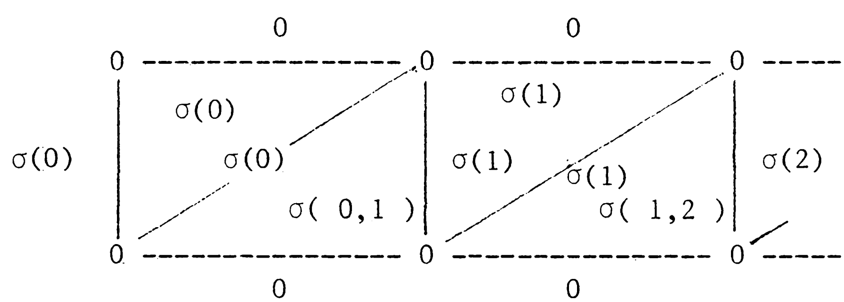
3.2.4. Theorem Let p be as in 3.1.2. Then there is a natural homotopy equivalence $T\colon \Omega\, \mathbb{L}_n(X;p) \longrightarrow \mathbb{L}_{n+1}(X;p)$.

Proof: A k-simplex of $\Omega\, \mathbb{L}_n(X;p)$ is a Δ-map $\sigma\colon \Delta^k \times [0,\infty) \times I \to \mathbb{P}_n(X;p)$. We define $T\sigma\colon \Delta^k \times [0,\infty) \longrightarrow \mathbb{P}_{n+1}(X;p)$ as follows. Let τ be an m-simplex of $\Delta^k \times [0,\infty)$. $\sigma(\tau \times I)$ consists of several simplices of $\mathbb{P}_n(X;p)$. Since $\sigma(\tau \times 0) = \sigma(\tau \times 1) = 0$, if we glue these together after some necessary suspensions, we can regard $\sigma(\tau \times I)$ to be an m-simplex of $\mathbb{P}_{n+1}(X;p)$, and this defines a k-simplex $T\sigma$ of $\mathbb{L}_{n+1}(X;p)$.

We will show that T is a homotopy equivalence. First of all, each 0-simplex of $\mathbb{P}_{n+1}(X;p)$ can be naturally regarded as a 1-simplex of $\mathbb{P}_n(X;p)$ with two 0-faces. A 0-simplex $\sigma\colon [0,\infty) \longrightarrow \mathbb{P}_{n+1}$ of $\mathbb{L}_{n+1}(X;p)$ can be expressed as in the following picture.



By inserting trivial cobordisms we can "triangulate" this (3.1.3):



This defines a 0-simplex $[0,\infty) \times I \longrightarrow \mathbb{P}_n(X;p)$ of $\Omega\,\mathbb{L}_n(X;p)$. If we apply T to this, then the result is different from the original by trivial cobordisms; therefore these two can be connected by a 1-simplex of $\mathbb{L}_{n+1}(X;p)$; i.e., T maps into every component. Next consider an element of the relative homotopy $\pi_j(T)$. It represented by a map $\sigma\colon \Delta^j \times [0,\infty) \longrightarrow \mathbb{P}_{n+1}(X;p)$ such that $\sigma|\partial_i\Delta^j = 0$ for $i < j$, and $\sigma|\partial_j\Delta^j = T\rho$ for some $\rho\colon \Delta^{j-1} \times [0,\infty) \times I \longrightarrow \mathbb{P}_n(X;p)$. We need a deformation of σ rel $\partial_j\sigma$ to a map in the image of T. An extension $\rho'\colon \Delta^j \times [0,\infty) \times I — \mathbb{P}_n(X;p)$ of ρ can be defined by first letting $\rho'(\tau) = 0$ for τ in $(\bigcup_{i<j} \partial_i\Delta^j) \times [0,\infty) \times I \cup \Delta^j \times [0,\infty) \times \{0,1\}$ and then using the triangulation argument. The natural cobordism will give a simplex connecting σ and $T\rho'$.

3.2. Homology theory and assembly map

Let $(F, \partial_*F) = (F, \partial_0F,\ldots, \partial_jF)$ be a topological (j+2)-ad. We consider only special ones; i.e., we assume $\cap\, \partial_iF = \phi$. Fix a covering space $\tilde{F}$ of F. $\widetilde{\partial_\beta F}$ denotes the restriction of $\tilde{F}$ over $\partial_\beta F$, for $\beta \subseteq \{0,1,\ldots j\}$. We construct a Δ-set $\mathbb{L}_n(F,\partial_*F;\tilde{F})$ modifying the construction of $\mathbb{P}_n$. If $j = 0$, $\mathbb{L}_n(F;\tilde{F}) = \mathbb{L}_n(F,\phi;\tilde{F})$ is defined to be $\mathbb{P}_n(*,*,F\longrightarrow *,0;\tilde{F})$, where $*$ denotes a single point. If $j > 0$, a primitive k-simplex x for $\mathbb{L}_n(F,\partial_*F;\tilde{F})$ is a strictly $(n+k+j+\ell)$-dimensional geometric quadratic Poincaré $(k+j+3)$-ad on $F \times \mathbb{R}^\ell \longrightarrow *$ such that

(1) it has the same structure as $\Delta^k \times \Delta^j$; i.e., the faces have two indices: $\partial_{\alpha,\beta}x$, $\alpha \subseteq \{0,1,\ldots,k\}$, $\beta \subseteq \{0,1,\ldots,j\}$, and $\partial_{\alpha,\beta}x = 0$ if $\alpha = \{0,1,\ldots,k\}$ or $\beta = \{0,1,\ldots,j\}$,

(2) $\partial_{\alpha,\beta}x$ is actually a $(n+k+j+\ell-|\alpha|-|\beta|)$-dimensional geometric quadratic Poincaré $(k+j-|\alpha|-|\beta|+3)$-ad on $\partial_\beta F \times \mathbb{R}^\ell \longrightarrow *$, and

(3) everything has radius 0 and support $*$.

The i-th face of a primitive k-simplex x is defined by using the first index, for i= 0, 1, ..., k. The elaborate simplices of $\mathbb{L}_n(F,\partial_* F;\tilde{F})$ are defined by allowing stabilization and reduction in the identification of faces as before.

For each i = 0,1,...,j, we have a natural map $\partial_i: \mathbb{L}_n(F,\partial_* F;\tilde{F}) \longrightarrow \mathbb{L}_n(\partial_i F,\partial_*\partial_i F;\widetilde{\partial_i F})$ by taking the i-th face with respect to the second index.

The next step is to do the above construction to each simplex of a simplicial complex and fit them together using the second index to form a "bundle of spectra." Let us consider a stratified system of fibrations $p: E \longrightarrow X = |K|$, where K is a simplicial complex. For each j-simplex Δ of K, we have a topological (j+2)-ad $(p^{-1}(\Delta), p^{-1}(\partial_*\Delta))$. Let $\tilde{E}$ be the universal cover of E and let $p^{-1}(\Delta)^{\sim}$ be the restriction of $\tilde{E}$ to $p^{-1}(\Delta)$. Now apply $\mathbb{L}_n(\ ,\ ;\ )$ to this for each Δ, then it defines a Δ^2-set (= Δ-(Δ-set)) $\mathbb{L}_n(p)$:

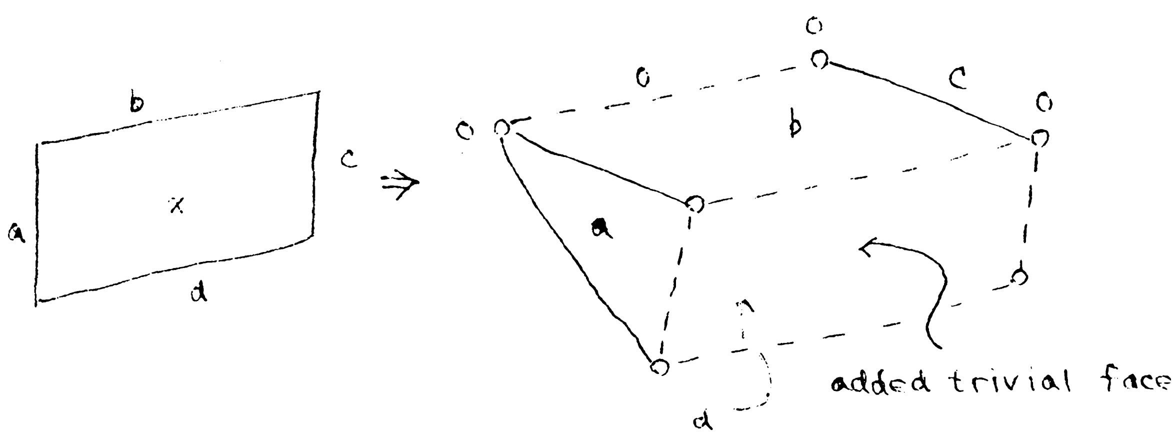
$$\begin{aligned}&\mathbb{L}_n(p)(j,k)\\ &= \{\, \mathbb{L}_n(p^{-1}(\Delta), p^{-1}(\partial_*\Delta); p^{-1}(\Delta)^{\sim})(k) \mid \Delta \text{ is a j-simplex of } K\}.\end{aligned}$$

Let $|\mathbb{L}_n(p^{-1}(\Delta), p^{-1}(\partial_*\Delta); p^{-1}(\Delta)^{\sim})|$ and $|\partial_i|$ denote the geometric realizations. We define the geometric realization $|\mathbb{L}_n(p)|$ of $\mathbb{L}_n(p)$ by:

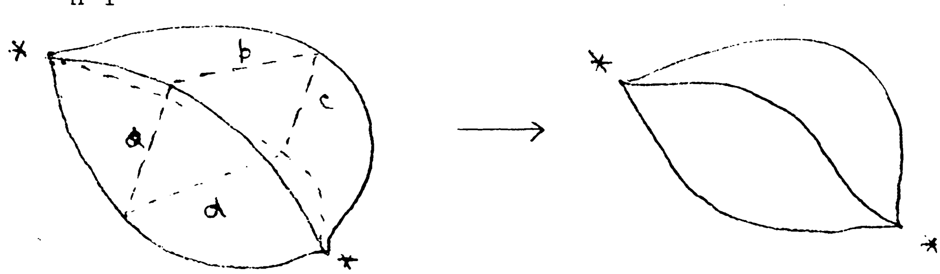
$$|\mathbb{L}_n(p)| = \coprod_{\Delta \in K} |\mathbb{L}_n(p^{-1}(\Delta), p^{-1}(\partial_*\Delta); p^{-1}(\Delta)^{\sim})| \times \Delta/\sim,$$

where the equivalence relation is generated by: $(a,t) \in |\mathbb{L}_n(p^{-1}(\Delta), p^{-1}(\partial_*\Delta); p^{-1}(\Delta)^\sim)| \times \Delta$ is equivalent to $(|\partial_i|a,t) \in |\mathbb{L}_n(p^{-1}(\partial_i\Delta), p^{-1}(\partial_*\partial_i\Delta); p^{-1}(\partial_i\Delta)^\sim)| \times \partial_i\Delta$, provided $t \in \partial_i\Delta < \Delta$. We have a projection map $p_*: |\mathbb{L}_n(p)| \longrightarrow |K|$ defined by $p_*(a,t) = t$. p_* has a natural zero-section $i: |K| \longrightarrow |\mathbb{L}_n(p)|$. $|\mathbb{L}_n(p)|$ has a triangulation obtained by assembling the standard product triangulation of $\Delta^k \times \Delta^j$.

Notice that we can regard a k-simplex of $\mathbb{L}_{-n}(p^{-1}(\Delta), p^{-1}(\partial_*\Delta); p^{-1}(\Delta)^\sim)$ as a (k+1)-simplex of $\mathbb{L}_{-n-1}(p^{-1}(\Delta), p^{-1}(\partial_*\Delta); p^{-1}(\Delta)^\sim)$, by adding a trivial face as follows.



This correspondence can be realized as a map from the reduced suspension of $|\mathbb{L}_{-n}(p^{-1}(\Delta), p^{-1}(\partial_*\Delta); p^{-1}(\Delta)^\sim)| \times \Delta/i(\Delta)$ to $|\mathbb{L}_{-n-1}(p^{-1}(\Delta), p^{-1}(\partial_*\Delta); p^{-1}(\Delta)^\sim)| \times \Delta/i(\Delta)$:



(Two *'s are identified in each picture.)

This assembles to a map from the reduced suspension of $|\mathbb{L}_{-n}(p)|/i(X)$ to $|\mathbb{L}_{-n-1}(p)|/i(X)$, and it gives a well-defined map $|\mathbb{L}_{-n}(p)|/i(X) \longrightarrow \Omega(|\mathbb{L}_{-n-1}(p)|/i(X))$. Taking Ω^{n-j} of this we have a map

$$\Omega^{n-j}(|\mathbb{L}_{-n}(p)|/i(X)) \longrightarrow \Omega^{n-j+1}(|\mathbb{L}_{-n-1}(p)|/i(X)).$$

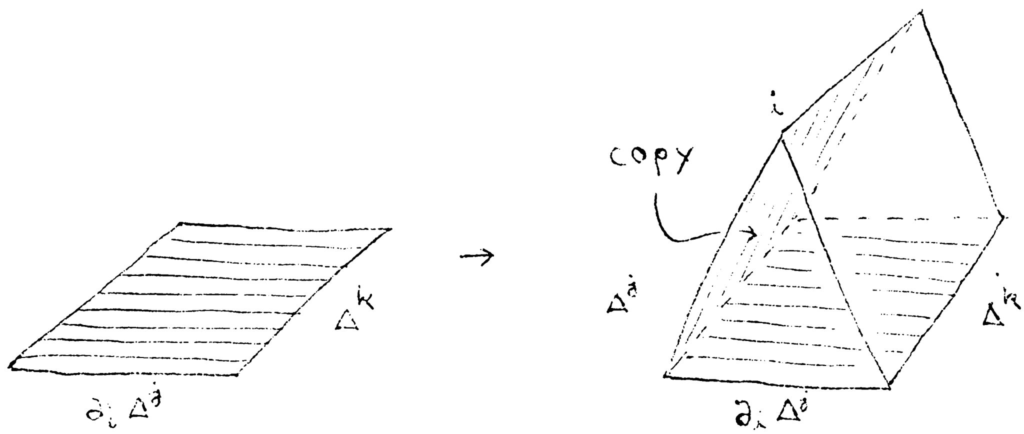
$\mathbb{L}_{-*}(p)$ is Quinn's "ex-spectrum" [Q3, §8].

3.2.1. <u>Definition</u> The homology spectrum $\mathbb{H}(X; \mathbb{L}(p))$ is the Ω-spectrum defined by $\mathbb{H}(X; \mathbb{L}(p)) = \lim\limits_{n\to\infty} \Omega^n(|\mathbb{L}_{-n}(p)|/i(X))$.

3.2.2. <u>Proposition</u> The functor $\mathbb{L}(\ ,\ ;\ )$ which was used to construct $\mathbb{H}$ is <u>homotopy invariant</u>; i.e., 1) a homotopy equivalence of $(j+2)$-ads $(E, \partial_0 E, \ldots, \partial_j E) \longrightarrow (F, \partial_0 F, \ldots, \partial_j F)$ covered by a homotopy equivalence $\tilde{E} \longrightarrow \tilde{F}$ induces a homotopy equivalence $\mathbb{L}(E, \partial_* E; \tilde{E}) \longrightarrow \mathbb{L}(F, \partial_* F; \tilde{F})$, and 2) ∂_i: $\mathbb{L}(F \times \Delta^n, F \times \partial_* \Delta^n; \tilde{F} \times \Delta^n) \longrightarrow \mathbb{L}(F \times \partial_i \Delta^n, F \times \partial_* \partial_i \Delta^n; \tilde{F} \times \partial_i \Delta^n)$ is a homotopy equivalence.

<u>Proof</u>: (1) By assumption, they use the same coefficients; therefore the result is obvious.

(2) The coefficients are constant. The homotopy inverse of ∂_i is defined as shown below:



Use the trivial cobordism to fill in $\Delta^j \times \Delta^k$. The same trick will also give the necessary homotopy.

According to Quinn [Q3, §8], this implies that $\mathbb{H}(\ ;\mathbb{L}(p))$ is a homology theory on the category of polyhedra with stratified systems of fibrations with polyhedral fibrations.

Let us think about triangulation of $|\mathbb{L}_{-n}(p)|$ for a moment. Each of its building blocks $\Delta^k \times \Delta^j$ is given a structure of quadratic Poincaré (k+j+3)-ad of dimension $(n+k+j+\ell)$ on $p^{-1}(\Delta) \times \mathbb{R}^\ell \longrightarrow *$. By the triangulation argument 3.1.3, we may assume that each m-simplex σ in a triangulation of $\Delta^k \times \Delta^j$ is given a structure of quadratic Poincaré special (m+2)-ad of dimension $(n+m+\ell)$ on $p^{-1}(p_*\sigma) \times \mathbb{R}^\ell \longrightarrow *$. (This is possible because p is a fibration over the interior of each simplex of K.) Replacing this map by $p^{-1}(p_*\sigma) \times \mathbb{R}^\ell \longrightarrow p_*\sigma$, we can regard it to be a structure on $p \times \mathbb{R}^\ell = pr : E \times \mathbb{R}^\ell \longrightarrow X$ with radius $\leqslant$ diameter of $p_*\sigma$.

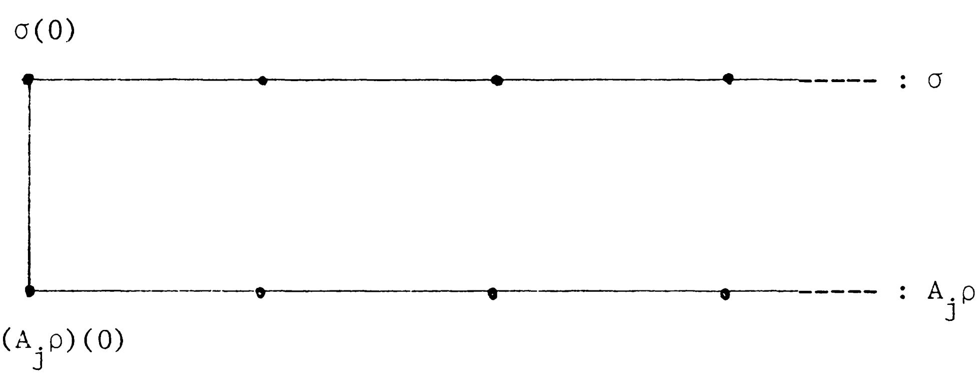
Using this, let us define $A_j : \mathbb{H}_j(X; \mathbb{L}(p)) \longrightarrow \mathbb{L}_{-j}(X;p)$. A k-simplex of $\Omega^{n-j}(|\mathbb{L}_{-n}(p)|/i(X))$ is a map $\rho: S^{n-j} \times \Delta^k \longrightarrow |\mathbb{L}_{-n}(p)|/i(X)$. By modifying ρ a little if necessary, we may assume that there exist a compact codimension 0 submanifold V of $S^{n-j} \times \Delta^k$ and a map $\rho': V \longrightarrow |\mathbb{L}_{-n}(p)|$ such that ρ sends the complement of int V to the base point $[i(X)]$ and $\rho|V$ factors through ρ'. For each (n-j+k)-simplex Δ of V, $\rho'(\Delta)$ is given a structure of $(-j+k+\ell)$-dimensional quadratic Poincaré special (n-j+k+2)-ad on $p \times \mathbb{R}^\ell$ with radius = the diameter of $p_*\rho'(\Delta)$. Glueing all these , after a barycentric subdivision if necessary, we obtain a $(-j+k+\ell)$-dimensional quadratic Poincaré special (k+2)-ad on $p \times \mathbb{R}^\ell$ with radius $\max_{\Delta \in V}$ (diameter of $p_*\rho'(\Delta)$), which is a simplex of $\mathbb{P}_{-j}(X;p)$. If we use finer triangulations (e.g. barycentric subdivisions) of V and $|\mathbb{L}_{-n}(p)|$, the radius of the result becomes smaller, and it differs from the original result by several inserted trivial cobordisms and a small change of the positions of some of the generators of modules. Therefore there is a homotopy from the simplex in $\mathbb{P}$ corresponding to the original triangulation to the one corresponding to the subdivision. Consider this as $\Delta^k \times [t,t+1] \longrightarrow \mathbb{P}_{-j}(X;p)$. Repeated application of this generates a map $\Delta^k \times [0,\infty) \longrightarrow \mathbb{P}_{-j}(X;p)$. Since the radius goes to 0 as $t \to \infty$, this defines a simplex of $\mathbb{L}_{-j}(X;p)$. This is a very elaborate version of the "assembly map" defined by Quinn in his thesis [Q1].

The following is the main theorem of this chapter.

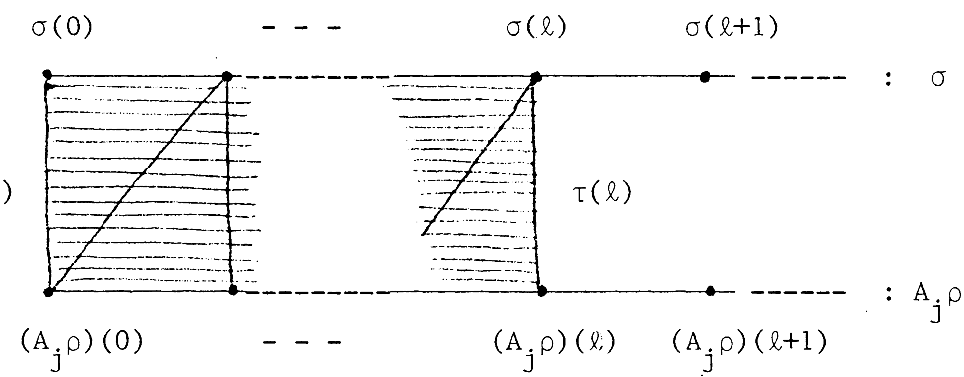
3.2.2. Theorem (Characterization theorem) Let $p\colon E \longrightarrow X$ be a stratified system of fibrations, where X is a finite polyhedron. Then the assembly map $A_j\colon \mathbb{H}_j(X;\mathbb{L}(p)) \longrightarrow \mathbb{L}_{-j}(X;p)$ is a homotopy equivalence.

Proof: First fix some notations. Since X is compact, we will assume that everything has support X in the following proof. Embed X in S^{n-j} for a sufficiently large n, and let W and r denote a regular neighborhood of X in S^{n-j} and the retraction: $W \to X$. Let $p^{\sim}\colon E \longrightarrow W$ be the stratified system of fibrations obtained as the pull-back of p by r. $p^{\sim}$ has the advantage that the base space is a manifold.

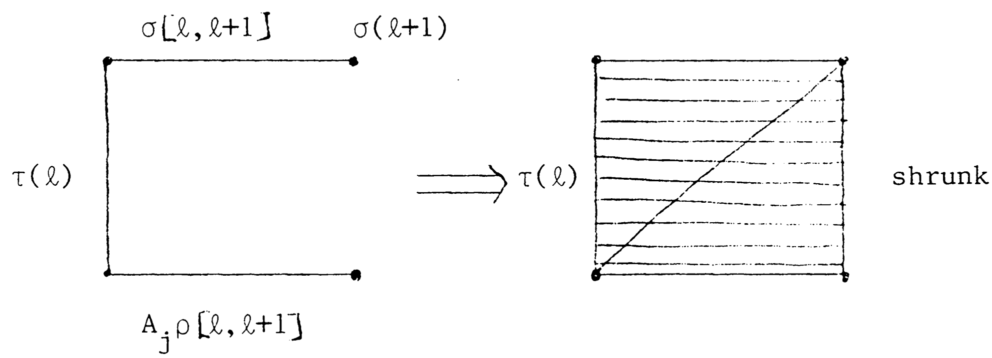
We will show that A_j maps into every component. Let σ be a 0-simplex of $\mathbb{L}_{-j}(X;p)$; σ is a map from $[0,\infty)$ to $\mathbb{P}_{-j}(X;p)$ with a sequence ε_i monotone decreasing to 0 such that $\sigma([i,\infty)) \subseteq \mathbb{P}_{-j}(X,p,\varepsilon_i) = \mathbb{P}_{-j}(X,X,p,\varepsilon_i)$. $\sigma(0)$ can be regarded as an $(\ell-j)$-dimensional quadratic complex on $p^{\sim} \times \mathbb{R}^{\ell}$ with radius ε_0. Try to split $\sigma(0)$ into pieces lying over the simplices of $W^{(1)}$, the first barycentric subdivision of W. Choose ε which is sufficiently small compared with the size of any simplex of $W^{(1)}$. The stable splitting theorem (§2.4) gives $\delta > 0$ such that if $\sigma(0)$ has radius less than δ, then each piece lies over an ε-neighborhood of the corresponding simplex of $W^{(1)}$. We may assume that this is the case. If $\varepsilon_0 \geq \delta$, then choose i such that $\varepsilon_i < \delta$ and use another 0-simplex σ' defined by $\sigma'(t) = \sigma(t+i)$, which is in the same component as σ. Construct a map of $p^{\sim}$ to itself with radius ε which sends $p^{\sim-1}$(ε neighborhood of Δ) to $p^{\sim-1}(\Delta)$ for each simplex $\Delta \in W^{(1)}$. This map induces an ε isomorphism which makes each piece lie exactly over the corresponding simplex of W. There are only trivial ads (i.e., ads made up of zero complexes) over the simplices in ∂W; therefore this gives a 0-simplex ρ of $\Omega^{n-j}(|\mathbb{L}_{-n}(p)|/i(X)$ after a suitable "subdivision." We will show that A_j sends ρ into the same component as σ. First of all, if we glue all the ads on simplices of (a subdivision of) $W^{(1)}$ together, then we get a quadratic Poincaré complex $(A_j\rho)(0)$ which is cobordant to $\sigma(0)$.



This is the first step of the inductive construction of a 1-simplex τ connecting σ and $A_j\rho$. Assume we have filled in 2-simplices of $\mathbb{P}_{-j}(X;p)$ up to ℓ as follows:



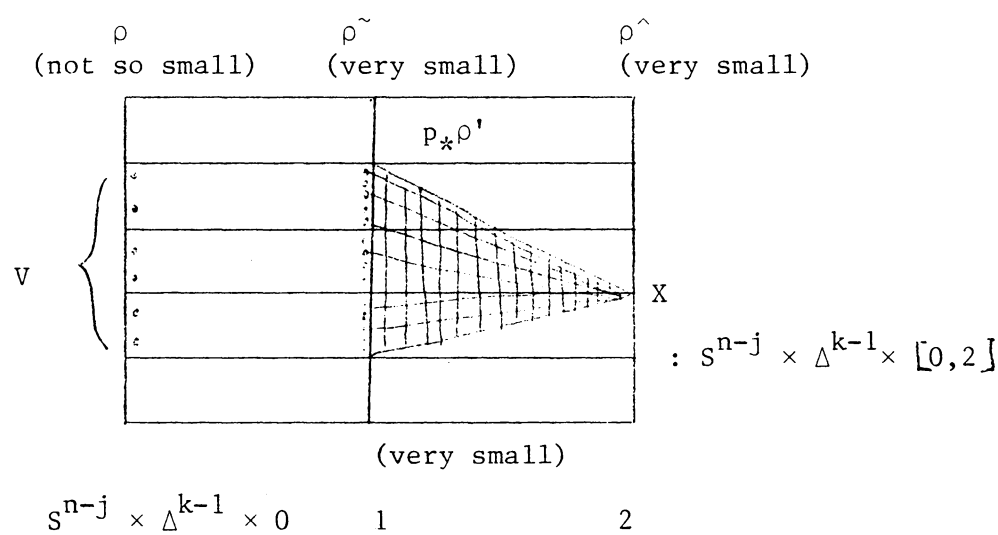
The next step is obtained by applying the "barycentric subdivision" shrinking argument to the union $\sigma([\ell,\ell+1]) \cup \tau(\ell) \cup A_j\rho([\ell,\ell+1])$ and then applying the triangulation argument to the resulted cobordism.



Thus A_j maps into every component.

Next we will show that the relative homotopy group $\pi_k(A_j)$ vanishes for $k > 1$. The idea is basically same as the first part. Its element is represented by a map $\sigma\colon \Delta^k \times 0,\infty) \longrightarrow \mathbb{P}_{-j}(X;p)$ such that $\sigma|\partial_i\Delta^k = 0$ for $i < k$ and $\sigma|\partial_k\Delta^k = A_j\rho$ for some ρ: $S^{n-j} \times \Delta^{k-1} \longrightarrow |\mathbb{L}_{-n}(p)|/i(X)$ such that $\rho(S^{n-j} \times \partial\Delta^{k-1}) = [i(X)]$.

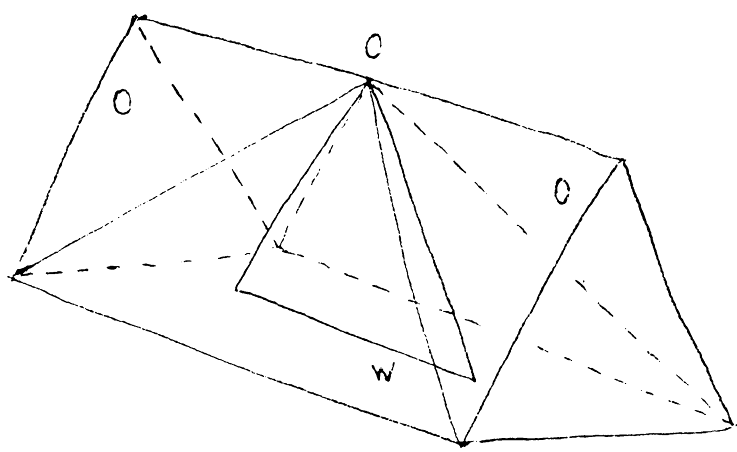
Let $\rho': V \longrightarrow |\mathbb{L}_{-n}(p)|$ be the map defined on a codimension 0 submanifold of $S^{n-j} \times \Delta^{k-1}$ associated with ρ. (See the definition of A_j.) We may assume X and $W(\subset S^{n-j})$ are subcomplexes of V $(\subset S^{n-j} \times \Delta^{k-1})$ via $S^{n-j} = S^{n-j} \times (\text{center of } \Delta^{k-1}) \subset S^{n-j} \times \Delta^{k-1}$, by using subdivisions if necessary. When we defined A_j, we associated a (n-j+k-1)-ad on $p \times \mathbb{R}^{\ell}$ to each (n-j+k-1)-simplex Δ of V. But notice that it can be also regarded to be over the simplex Δ, choosing the centers of simplices as the basis elements. Similarly "subdivided" small thing, $\tilde{\rho}$, can be also realized on V. The map $p_*\rho': V \longrightarrow X$ puts them back into the original things over X. Now we replace ρ by $\hat{\rho}$ for which everything lies on X (and hence on W), via homotopy. The following diagram shows the construction.



On $S^{n-j} \times \Delta^{k-1} \times [0,1]$ we apply the triangulation argument to the cobordism between the original and the subdivided things.

On $S^{n-j} \times \Delta^{k-1} \times [1,2]$, we immerse the mapping cylinder of $p_*\rho'$ whose cylindrical part is a composition of very thin cobordisms. $\rho^{\sim}$ and this cylinder should be chosen very small so that we can do splitting here.

Thus, from the beginning, we may assume that ρ is lying over X. Now apply the relative version of splitting lemma to the pair induced by σ over W, then we can obtain a k-simplex of $\Omega^{n-j}(|\mathbb{L}_{-n}(p)|/i(X))$, ρ_σ, as follows:



Do the same argument as in the first part to obtain a homotopy which pushes σ down to $A_j\rho_\sigma$. This completes the proof.

The proof actually gives the following:

3.2.3. <u>Corollary</u> (Shrinking lemma) Let p be as in 3.2.2. Then, for any positive integer n, there exists a positive number ε such that for any $0 < \delta < \varepsilon$, there is a function("shrinking function") $S\colon \mathbb{P}_j(X,p,\delta)^{(n)} \to \mathbb{L}_j(X;p)$ such that the following composition is homotopic to the inclusion map:

$$\mathbb{P}_j(X,p,\delta)^{(n)} \xrightarrow{S} \mathbb{L}_j(X;p) \xrightarrow{\text{restriction}} \mathbb{P}_j(X,p,\delta),$$

where $\mathbb{P}_j(X,p,\delta)^{(n)}$ denotes the subset of $\mathbb{P}_j(X,p,\delta)$ which is made up of ads of dimension less than or equal to n.

Now let us consider a special case when X is a single point *. In this case, $\mathbb{L}_j(*;E \to *)$ is homotopy equivalent to $\mathbb{P}_j(*,E \to *,0)$ $= \mathbb{L}_j(E)$.

3.2.4. Proposition There is a natural isomorphism

$$\theta : L_n^{-\infty}(\pi) \longrightarrow \pi_n \ \mathbb{L}_0(B_\pi),$$

where B_π is the classifying space of the group π.

$L_n^{-\infty}$ is the limit $\lim\limits_{j\to\infty} L_n^{(-j)}$ of Ranicki's lower L-groups [R1]. $L_n^{(-j)}(\pi)$ is defined to be the kernel of the product of projection maps

$$L_{n+j+1}^{(1)}(\pi \times T^{j+1}) \longrightarrow \prod^{j+1} L_{n+j+1}^{(1)}(\pi \times T^j),$$

where T denotes the infinite cyclic group. The map $L_n^{(-j)} \longrightarrow$

$L_n^{(-j-1)}$ is induced by the map

$$L_{n+j+1}^{(1)}(\pi \times T^{j+1}) \longrightarrow L_{n+j+2}^{(1)}(\pi \times T^{j+2})$$

which corresponds to taking product with S^1.

Notice that the external suspension operation introduced in §2.3 actually changes a geometric $\mathbb{Z}\pi$-module chain complex C on X into a geometric $\mathbb{Z}[\pi \times T]$-module chain complex $C \otimes E$. Here we are regarding E to be corresponding to S^1. So if C represents an element of $L_n^{(1)}(\pi)$, then $C \otimes E$ represents an element of $L_{n+1}^{(1)}(\pi \times T)$. Let us project this to $L_{n+1}^{(1)}(\pi)$. Then the result is $C \otimes \varepsilon(E)$, where $\varepsilon(E)$ is the $\mathbb{Z}$-module symmetric Poincaré complex of S^1. Obviously $\varepsilon(E)$ is cobordant to 0, and hence so is $C \otimes \varepsilon(E)$. This implies that $C \otimes E$ actually represents an element of $L_n^{(0)}(\pi) \subset L_{n+1}^{(1)}(\pi \times T)$.

<u>Proof of 3.2.4</u>: First let us define θ. An element of $L_n^{(-j)}(\pi)$ can be represented by a free (n+j+1)-dimensional quadratic Poincaré complex over $\mathbb{Z}[\pi \times T^{j+1}]$. Consider this as a geometric quadratic Poincaré complex on $B_\pi \times \mathbb{R}^{j+1} \longrightarrow *$. Obviously it has a bounded radius in the $\mathbb{R}^{j+1}$ coordinate. So we can regard this as an n-simplex of $\mathbb{L}_0(B_\pi)$ with empty boundary, and hence as an element of $\pi_n \mathbb{L}_0(B_\pi)$. $(\times S^1)$ on the left side corresponds to $(\times \mathbb{R}^1)$ on the right, so the diagram

$$\begin{tikzcd}
L_n^{(-j)}(\pi) \arrow[d] \arrow[dr, dashed] & \\
& \pi_n \mathbb{L}_0(B_\pi) \\
L_n^{(-j-1)}(\pi) \arrow[ur, dashed] &
\end{tikzcd}$$

commutes, and it defines a homomorphism θ of the direct limit.

Next we show that θ is onto. Take an element of $\pi_n \mathbb{L}_0(B_\pi)$ and represent it by a ($\mathbb{Z}\pi$-module) quadratic Poincaré complex on $B_\pi \times \mathbb{R}^j \longrightarrow *$ for some j. Call the complex C. Split C along $B_\pi \times \mathbb{R}^{j-1} \times 0$; we obtain a decomposition $C \simeq C_+ \cup_B C_-$, but B may lie all over $B_\pi \times \mathbb{R}^j$. Recall that $B \otimes E$ is homotopy equivalent to another complex, say D, which lies in a small neighborhood of $B_\pi \times \mathbb{R}^{j-1} \times 0$. We should note that not only $B \otimes E$ but also D can be regarded to be a $\mathbb{Z}[\pi \times T]$-module quadratic Poincaré complex. Consider the external suspension ΣD, which is a $\mathbb{Z}[\pi \times T^2]$-module quadratic Poincaré complex on $B_\pi \times \mathbb{R}^{j-1}$. We claim that ΣC is cobordant to ΣD. Note that both ΣC and ΣD has a splitting along D: $\Sigma C \simeq (\Sigma C_+) \cup_D (\Sigma C_-)$ and $\Sigma D = (\Sigma_+ D) \cup_D (\Sigma_- D)$. $\Sigma C \oplus (-\Sigma D)$ is cobordant to

$$((\Sigma C_+) \cup_D (-\Sigma_+ D)) \oplus ((\Sigma C_-) \cup_D (-\Sigma_- D)).$$

Notice that $(\Sigma C_+) \cup_D (-\Sigma_+ D)$ (resp. $(\Sigma C_-) \cup_D (-\Sigma_- D)$) lies over $B_\pi \times \mathbb{R}^j \times [0,\infty)$ (resp. $B_\pi \times \mathbb{R}^j \times (-\infty,0]$), then the following lemma implies that these are cobordant to 0, and hence ΣC is cobordant to ΣD. Repeat this process until one gets a $\mathbb{Z}[\pi\times T^{2j}]$-module quadratic Poincaré complex. By the construction this represents an element of $L_n^{(1-2j)}(\pi)$ and hence an element of $L_n^{-\infty}(\pi)$.

3.2.5. Lemma Any quadratic Poincaré complex on $B_\pi \times \mathbb{R}^j \longrightarrow *$ which lies over $B_\pi \times \mathbb{R}^{j-1} \times [0,\infty)$ is cobordant to zero.

Proof: Let F be such a complex, and let t denote a parallel translation of $B_\pi \times \mathbb{R}^{j-1} \times \mathbb{R}$ defined by $t(x,y,z) = (x,y,z+1)$. Then

$$F \oplus (-tF) \oplus (t^2F) \oplus (-t^3F) \oplus \ldots.$$

gives the desired cobordism.

Let us go back to the proof of 3.2.4. We will prove the injectivity of θ. Pick an element x in the kernel of θ. We may assume that its image is cobordant to 0. Apply the same argument to this cobordism as in the onto part. This will show that x = 0.

This justifies the following notation:

3.2.6. Notation $\pi_n \mathbb{L}_0(B_\pi) = L_n^{-\infty}(B_\pi)$.

4. Crystallographic groups

4.1. Preliminaries on crystallographic groups

We begin this section by reviewing some work on crystallographic groups by Farrell and Hsiang in [FH2]. (See also [Ch], [F], and [Wo].) A group Γ is crystallographic if it is a discrete co-compact subgroup of $E(n)$, the group of rigid motions of Euclidean n-space. Identify $\mathbb{R}^n$ with the group of translations of $\mathbb{R}^n$, then $E(n) = \mathbb{R}^n \rtimes O(n)$. The intersection of Γ and $\mathbb{R}^n$ is the maximal abelian subgroup of Γ with finite index, which is denoted by A and is called the translation subgroup of Γ. The finite factor group Γ/A, called the holonomy group of Γ, is denoted by G. The rank of Γ is the rank of A. For any positive integer s, $\Gamma_s = \Gamma/sA$ and $A_s = A/sA$. T and T_n denote the infinite cyclic group and the finite cyclic group of order n respectively.

4.1.1. Examples (1) D_∞ denote the ∞-dihedral group; i.e., D_∞ $E(1)$ is the subgroup generated by $x \longmapsto x+1$ and $x \longmapsto -x$ (where $x \in \mathbb{R}$).

(2) See [FH2,§4] for the definition of 2-dimensional crystallographic groups of type 1, 2, and 3.

The following is a structure theorem of crystallographic groups.

4.1.2. <u>Theorem</u> Let Γ be a crystallographic group of rank $\ell(\geq 2)$. Then, either

(1) $\Gamma = \Gamma' \rtimes T$ where Γ' is a crystallographic subgroup of rank $\ell-1$; or

(2) Γ maps onto some crystallographic group $\Gamma^\wedge$ of rank $m(\geq 1)$ with holonomy group $G^\wedge$, and there are a crystallographic group $\Gamma^\sim$ of rank $n(\geq 1)$ and an infinite set of positive integers s each of which is relatively prime to $|G^\wedge|$ such that if H is a maximal hyperelementary subgroup of $\Gamma^\wedge_s$ and $|G^\wedge|$ divides $|H|$, then there exists a crystallographic group Π together with a group monomorphism $\theta\colon \Pi \longrightarrow \Gamma^\wedge$ and a group surjection $\eta\colon \Pi \longrightarrow \Gamma^\sim$ satisfying $\theta(\Pi) = q^{-1}(H')$, where $H' \subseteq \Gamma^\wedge_s$ is conjugate to H and $q\colon \Gamma^\wedge \longrightarrow \Gamma^\wedge_s$ is the canonical projection. Furthermore there exist a θ-equivariant bijection $g\colon \mathbb{R}^m \longrightarrow \mathbb{R}^m$ and an η-equivariant affine surjection $h\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$ such that

$$|d(hg^{-1})(X)| \leq (2/\sqrt{s})|X|$$

for each tangent vector X of $\mathbb{R}^m$.

<u>Proof</u>: This is implicitly proved by Farrell and Hsiang in [FH2]. Their Theorem 1.1 has three cases. Their case (i) is our (1). In case (ii), there is an epimorphism from Γ to a non-trivial crystallographic group $\Gamma^\wedge$ with holonomy group $G^\wedge$ and there is an infinite sequence of positive integers $s \equiv 1 \bmod |G^\wedge|$ such that

any hyperelementary subgroup of $\Gamma^\wedge_s$ which projects onto $G^\wedge$ projects isomorphically onto $G^\wedge$. The proof of Theorem 4.4 in [FH2] gives an s-expansive endomorphism $\theta: \Gamma^\wedge \longrightarrow \Gamma^\wedge$ and a θ-equivariant diffeomorphism $g: \mathbb{R}^m \longrightarrow \mathbb{R}^m$ such that $\theta(\Gamma^\wedge) = q^{-1}(H')$ where H' is a subgroup of $\Gamma^\wedge_s$ conjugate to H. Therefore let $\Gamma^\sim = \Pi = \Gamma^\wedge$, $\eta = 1$, and $h = 1$. Then

$$|d(hg^{-1})(X)| = (1/s)|X| \leqslant (2/\sqrt{s})|X|.$$

In case (iii), Γ maps onto a 2-dimensional crystallographic group $\Gamma^\wedge$ of type 1, 2, or 3, and Theorem 4.2 of [FH2] gives exactly what we want. D_∞ is used as $\Gamma^\sim$.

We will also need the following result of Farrell and Hsiang.

4.1.3. <u>Lemma</u> ([FH2, Lemma 1.2]) Let $\phi: \Gamma \longrightarrow \Gamma^\wedge$ be an epimorphism between crystallographic groups $\Gamma \subset E(\ell)$ and $\Gamma^\wedge \subset E(m)$. Then there exists a ϕ-equivariant affine surjection $F: \mathbb{R}^\ell \longrightarrow \mathbb{R}^m$.

Let us consider the action on $\mathbb{R}^\ell$ of a crystallographic group Γ of rank ℓ, with holonomy group G. The action may not be free, since Γ may have torsion, but its translation subgroup A acts on $\mathbb{R}^\ell$ freely and the orbit space is the flat torus T^ℓ. The finite group G acts on T^ℓ as a group of isometries such that $\mathbb{R}^\ell/\Gamma = T^\ell/G$, where T^ℓ is given the natural induced metric.

Therefore we can use the argument for a finite group action on a smooth manifold. If (H) is the conjugacy class of the subgroup H of G, then $Y_{(H)}$ denotes the subset of T^ℓ/G consisting of the points x such that the isotropy subgroup of a point in T^ℓ lying in the orbit x is in (H). $\{Y_{(H)}\}$ gives a stratification. Enumerate the conjugacy classes of subgroups of G: (H_0), (H_1), ..., (H_N), such that

(1) $H_0 = G$,

(2) $H_N = 1$,

(3) if $(H_i) \subseteq (H_j)$, i.e., H_i is a subgroup of some member of (H_j), then $i \geqslant j$.

Define closed sets X_i of T^ℓ/G by

$$X_i = \bigcup_{(H) \supseteq (H_i)} Y_{(H)},$$

then $T^\ell/G = X_N \supseteq X_{N-1} \supseteq \dots \supseteq X_0$ is a closed filtration of $T^\ell/G = \mathbb{R}^\ell/\Gamma$ such that $X_i - X_{i-1} = Y_{(H_i)}$. Furthermore, for each i, $X_i - X_{i-1}$ has a neighborhood U_i in $T^\ell/G - X_{i-1}$ together with the cone bundle structure:

$$U_i \longrightarrow X_i - X_{i-1},$$

such that for $j > i$, $U_i \cap (X_j - X_{j-1})$ are sub-cone-bundles and these sub-cone-bundles together form the cone bundle.

Let W_Γ denote a free contractible Γ-space. Γ acts freely on $\mathbb{R}^\ell \times W_\Gamma$ diagonally.

4.1.4. Proposition The projection $p: (\mathbb{R}^{\ell} \times W_{\Gamma})/\Gamma \longrightarrow \mathbb{R}^{\ell}/\Gamma$ is a stratified system of fibrations.

Proof: For each orbit $x \in \mathbb{R}^{\ell}/\Gamma$, define Γ_x to be the isotropy subgroup of a point of $\mathbb{R}^{\ell}$ which is in the orbit x. Γ_x is well-defined up to conjugacy. The map p has point inverses $p^{-1}(x) = W_{\Gamma}/\Gamma_x$, which are classifying spaces for Γ_x. This proposition will be proved if one can show that (Γ_x) is constant on each stratum $Y_{(H)}$. Let x be a point in $Y_{(H)}$, and pick up a point $y \in \mathbb{R}^{\ell}$ in the preimage of x. Let $y' \in T^{\ell}$ denote the orbit by A to which y belongs, and assume H is the isotropy subgroup of y'. π denotes the quotient map: $\Gamma \longrightarrow G$. Obviously $\Gamma_x \subseteq \pi^{-1}(H)$, and the index $[\pi^{-1}(H) : A]$ is finite. Let $\pi^{-1}(H) = A_1 \cup A_2 \cup \cdots \cup A_m$ be the coset decomposition by A. Each coset A_i contains a unique element γ_i of Γ_x. Thus $\Gamma_x = \{\gamma_1, \ldots, \gamma_m\}$ and $\pi|\Gamma_x: \Gamma_x \longrightarrow H$ is an isomorphism. Let r denote the projection $\mathbb{R}^{\ell} \longrightarrow \mathbb{R}^{\ell}/\Gamma$, then Γ acts uniformly on $r^{-1}(Y_{(H)})$, and $r^{-1}(Y_{(H)}) \longrightarrow Y_{(H)}$ is a covering space. For any x' in the same component of $Y_{(H)}$ as x, we can actually show that $\Gamma_x = \Gamma_{x'}$, by carefully choosing the lifting $y' \in \mathbb{R}^{\ell}$ of x'.

Next let us see what good thing will happen if Γ satisfies (1) or (2) of 4.1.2, respectively. Suppose $\Gamma = \Gamma' \rtimes T$.

Then by 4.1.3, the epimorphism $\phi : \Gamma \longrightarrow T$ induces a ϕ-equivariant affine surjection $F: \mathbb{R}^{\ell} \longrightarrow \mathbb{R}$, and hence a projection $\bar{F}: \mathbb{R}^{\ell}/\Gamma \longrightarrow \mathbb{R}/T = S^1$. It is easily seen that F is a fibre bundle with fibre $\mathbb{R}^{\ell-1}/\Gamma'$. Suppose Γ satisfies (2), and let ϕ be the epimorphism: $\Gamma \longrightarrow \Gamma^{\wedge}$. Let s and H be as in (2), and we assume $H = H'$ for simplicity. Define a crystallographic group $C(\subset \Gamma)$ of rank ℓ by $C = \phi^{-1}(q^{-1}(H))$. The epimorphism ϕ induces a ϕ-equivariant affine surjection $f: \mathbb{R}^{\ell} \longrightarrow \mathbb{R}^m$. The maps f, g^{-1}, and h together induce a surjection:

$$\alpha : \mathbb{R}^{\ell}/C \longrightarrow \mathbb{R}^m/q^{-1}(H) \longrightarrow \mathbb{R}^m/\Pi \longrightarrow \mathbb{R}^n/\Gamma^{\sim}.$$

we will denote the composition $hg^{-1}f: \mathbb{R}^{\ell} \longrightarrow \mathbb{R}^n$ by $\alpha^{\sim}$. For an orbit $x \in \mathbb{R}^n/\Gamma^{\sim}$, define a subgroup Δ_x of C by $\Delta_x = (\eta\theta^{-1}\phi|C)^{-1}(\Gamma_x^{\sim})$; i.e., Δ_x is the set of elements of C which leaves $\alpha^{\sim -1}(y)$ invariant for some $y \in \mathbb{R}^n$ on the orbit x. Δ_x is well-defined up to conjugacy. Then the point inverse $\alpha^{-1}(x)$ is $\mathbb{R}^{\ell-n}/\Delta_x$. If we consider the projection p_H: $(\mathbb{R}^{\ell} \times W_{\Gamma})/C \longrightarrow \mathbb{R}^{\ell}/C$, then the composition αp_H: $(\mathbb{R}^{\ell} \times W_{\Gamma})/C \longrightarrow \mathbb{R}^n/\Gamma^{\sim}$ has point-inverses $(\alpha p_H)^{-1}(x) = (\mathbb{R}^{\ell-n} \times W_{\Gamma})/\Delta_x$. Notice that αp_H is a stratified system of fibrations, and that we can use the same filtration and the neighborhoods of strata as those for the projection $\mathbb{R}^n \longrightarrow \mathbb{R}^n/\Gamma^{\sim}$, since Δ_x depends on $\Gamma_x^{\sim}$. The good thing with α is that, if we transfer things on $\mathbb{R}^{\ell}/\Gamma$ to things on $\mathbb{R}^{\ell}/C$ without changing the radius a lot, then by choosing s sufficiently large, we can make the transfer image very small on $\mathbb{R}^n/\Gamma^{\sim}$.

4.2. Induction theory

In this section we review Dress's induction theory[D], which will be necessary in the proof of the main theorem. [tD, §6] of tom Dieck is also a good reference. Let G be a finite group and let G-set be the category of finite G-sets and G-maps. Let Ab denote the category of abelian groups. A bi-functor

$$M = (M^*, M_*): \text{G-set} \longrightarrow Ab$$

is a pair of functors, where M^* is contravariant and M_* is covariant. We require these to coincide on objects, and we write $M(S) = M^*(S) = M_*(S)$ for a finite G-set S. If $f: S \longrightarrow T$ is a morphism, the notation $f^* = M^*(f)$ and $f_* = M_*(f)$ will be often used.

4.2.1. Definition A bi-functor $M = (M^*, M_*)$: G-set —— Ab is a Mackey functor if it satisfies

(1) for any pullback diagram in G-set

$$\begin{array}{ccc} U & \xrightarrow{F} & S \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle h} \\ T & \xrightarrow[f]{} & V \end{array}$$

the diagram

$$\begin{array}{ccc} M(U) & \xrightarrow{F_*} & M(S) \\ {\scriptstyle H^*}\uparrow & & \uparrow{\scriptstyle h^*} \\ M(T) & \xrightarrow[f_*]{} & M(V) \end{array}$$

commutes, and

(2) the two embeddings $S \longrightarrow S \amalg T \longleftarrow T$ into the disjoint union induce an isomorphism

$$M^*(S \amalg T) \longrightarrow M^*(S) \oplus M^*(T).$$

Let M and N be bi-functors. A natural transformation of bi-functors $X: M \longrightarrow N$ consists of a family of maps $X(S): M(S) \longrightarrow N(S)$, indexed by the objects of G-set, such that this family is a natural transformation $M^* \longrightarrow N^*$ and $M_* \longrightarrow N_*$.

Let M be a Mackey functor and S a G-set. We can define a new Mackey functor M_S by

$$M_S(T) = M(S \times T)$$

$$M_S{}^*(f) = M^*(id_S \times T)$$

$$M_{S*}(f) = M_*(id_S \times T),$$

and the projection map $p: S \times T \longrightarrow T$ define a natural transformation of bi-functors $\theta^S: M \longrightarrow M_S$ by

$$\theta^S(T) = p^* : M(T) \longrightarrow M_S(T) = M(S \times T).$$

4.2.2. Definition A Mackey functor M is S-injective if θ^S is split injective as a natural transformation of bi-functors.

Let S be a G-set. We define S^0 to be a point and $S^k =$

$S \times \ldots \times S$ (k copies). $p_i: S^{k+1} \longrightarrow S^k$ denotes the projection which omits the i-th factor, $0 \leqslant i \leqslant k$. The important property of an S-injective Mackey functor is the following.

4.2.3. Proposition [D, Proposition 1.1'] If a Mackey functor M is S-injective, then the following two sequences are exact:

$$1. \quad 0 \longrightarrow M(S^0) \xrightarrow{d^0} M(S^1) \xrightarrow{d^1} M(S^2) \xrightarrow{d^2} \ldots$$

$$2. \quad 0 \longleftarrow M(S^0) \xleftarrow[d_0]{} M(S^1) \xleftarrow[d_1]{} M(S^2) \xleftarrow[d_2]{} \ldots ,$$

where $d^k = \sum_{i=0}^{k} (-1)^i p_i^*$, $d_k = \sum_{i=0}^{k} (-1)^i p_{i*}$.

We introduce another notation. If H is a subgroup of G, let G/H denote the set of (right) cosets of H. G/H is naturally a transitive G-set. Actually any finite G-set is G-isomorphic to a disjoint union of these. We often write M(H) instead of M(G/H). If $H \subseteq K \subseteq G$ and $f: G/H \longrightarrow G/K$ is the canonical map, we will call the map

$$f^*: M(K) = M(G/K) \longrightarrow M(G/H) = M(H)$$

the <u>restriction</u> <u>from</u> K <u>to</u> H and denote it by res_H^K, and call the map

$$f_*: M(H) = M(G/H) \longrightarrow M(G/K) = M(K)$$

the <u>induction</u> <u>from</u> H <u>to</u> K and denote it by ind_H^K. If $K = G$, we write $res_H = res_H^G$ and $ind_H = ind_H^G$. We will rewrite the two exact sequences in 4.2.3 using this notation. Let S =

$\coprod_{H\in F} G/H$, where F is a family of subgroups of G. Then $M(S^0) = M(G)$ and $M(S) = \oplus_{H\in F} M(H)$. The projection $p\colon S \longrightarrow S^0$ is made up of the canonical maps $f_H\colon G/H \longrightarrow G/G$; therefore

$$d^0 = (\mathrm{res}_H)_{H\in F}\colon M(G) \longrightarrow \oplus_{H\in F} M(H)$$

$$d_0 = \sum_{H\in F} \mathrm{ind}_H : \oplus_{H\in F} M(H) \longrightarrow M(G).$$

These maps are also called (the product of) the restriction map(s) and (the sum of) the induction map(s) respectively. Finally $M(S^2)$ can be written as $\oplus_{H,K,g} M(H \cap (gKg^{-1}))$, where for a fixed pair $(H, K) \in F \times F$, g runs over the double coset representatives of H, K. In our application we are only interested in these, and we can restate 4.2.3 as follows:

4.2.3! Proposition Let $S = \coprod_{H\in F} G/H$. If a Mackey functor M is S-injective, then we have the following exact sequences:

$$0 \longrightarrow M(G) \longrightarrow \oplus_{H\in F} M(H) \longrightarrow \oplus_{H,K,g} M(H \cap (gKg^{-1}))$$

$$0 \longleftarrow M(G) \longleftarrow \oplus_{H\in F} M(H) \longleftarrow \oplus_{H,K,g} M(H \cap (gKg^{-1})).$$

Next we consider some sufficient condition for a Mackey functor to be S-injective. Let M, N, and L be bi-functors G-set $\longrightarrow$ Ab. A pairing $M \times N \longrightarrow L$ is a family of bilinear maps

$$M(S) \times N(S) \longrightarrow L(S);\ (x,y) \longmapsto x\cdot y,\ S \in \mathrm{Ob}(G\text{-set})$$

such that for any morphism $f\colon S \longrightarrow T$, one has:

$$L^*f(x\cdot y) = (M^*fx)\cdot(N^*fy), \quad x \in M(T),\ y \in N(T)$$

$$x\cdot(N_*fy) = L_*f((M^*fx)\cdot y), \quad x \in M(T),\ y \in N(S)$$

$$(M_*fx)\cdot y = L_*f(x\cdot(N^*fy)), \quad x \in M(S),\ y \in N(T).$$

A Green functor $U\colon G\text{-set} \longrightarrow Ab$ is a Mackey functor U together with a pairing $U \times U \longrightarrow U$. If U is a Green functor, then a left U-module is a Mackey functor together with a pairing $U \times M \longrightarrow M$ such that via this pairing $M(S)$ becomes a left $U(S)$-module.

The following proposition is very important in the theory of S-injectivity.

4.2.4. Proposition [D, Proposition 1.2] Let $U\colon G\text{-set} \longrightarrow Ab$ be a Green functor, and let S be a G-set. Then the following assertions are equivalent.

(1) The sum of the induction maps $U(S) \longrightarrow U(S^0)$ is surjective.

(2) U is S-injective.

(3) All U-modules are S-injective.

4.2.5. Example [D, Theorm 2] Let $GW(G, \mathbb{Z})$ be the equivariant Witt ring of Dress [D, p.293]. $GW(-, \mathbb{Z})$ induces a Green functor and

$$\oplus_{H \in F} GW(H, \mathbb{Z}) \otimes A \longrightarrow GW(G, \mathbb{Z}) \otimes A$$

is surjective in any of the following cases:

1. F = family of cyclic subgroups of G, A = Q(the rationals),
2. F = family of p-elementary subgroups of G, p odd, $A = \mathbb{Z}[\frac{1}{2}]$,
3. F = family of 2-hyperelementary subgroups of G,

$$A = \mathbb{Z}_{(2)} = \mathbb{Z}[1/3, 1/5, \ldots],$$

4. F = union of the families in 2 and 3 above, $A = \mathbb{Z}$.

As was mentioned before, any finite G-sets S is G-isomorphic to a disjoint union G/H_i of homogeneous G-sets G/H_i. If M is a Mackey functor, $M(S) \cong \oplus M(G/H_i) = \oplus M(H_i)$. In fact we can redefine a Mackey functor M to be a bi-functor from $\underline{G}$ to Ab, where $\underline{G}$ is the category of subgroups of G whose morphisms $H \longrightarrow K$ are triples (H,g,K) such that $g \in G$ and gHg^{-1} is a subgroup of K, satisfying:

1. for any isomorphism $f: H \longrightarrow K$, f^*f_* is the identity $M(H) \longrightarrow M(H)$,
2. for any inner conjugation $f = (H,h,H)$, $h \in H$, f^* and f_* are identity : $M(H) \longrightarrow M(H)$, and
3. the double coset formula holds: Let L and L' be subgroups of the subgroup H of G. Suppose H has a double codet decomposition $H = \coprod_{i=1}^{n} Lg_iL'$, $g_i \in H$. Then

$$(L,e,H)^*(L',e,H)_* = \sum_{i=1}^{n} (L\ (g_iL'g_i^{-1}),e,L)_*(L\ (g_iL'g_i^{-1}),g_i^{-1},L')^*.$$

Green functors and modules over a Green functor are redefined in the obvious way. Given a Mackey functor in this sense, we can construct a Mackey functor in the original sense.

4.3. <u>Induction theorems</u>

In this section we construct two Mackey functors which are modules over Dress's equivariant Witt ring (4.2.5), and, by applying the facts in §4.2, we obtain two exact sequences (4.3.5).

Let Γ be a crystallographic group of rank ℓ satisfying 4.1.2 (2), and consider an exact sequence $1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$, where $G = \Gamma_s^{\wedge}$. For a subgroup H of G, $C = C_H$ denotes the subgroup $\phi^{-1}(q^{-1}(H))$ of Γ and p_H denotes the projection $(\mathbb{R}^\ell \times W_\Gamma)/C \longrightarrow \mathbb{R}^\ell/C$ as before.

4.3.1. <u>Definition</u> Let $H_j(\mathbb{R}^\ell/C; \mathbb{L}(p_H)) = \pi_j\ \mathbb{H}(\mathbb{R}^\ell/C; \mathbb{L}(p_H))$, where $\mathbb{H}(\ ;\ )$ is the homology spectrum defined in 3.2.1.

Let us define a Mackey functor M: $\underline{G} \longrightarrow Ab$ as follows. For a subgroup H of G, define $M(H)$ to be $H_j(\mathbb{R}^\ell/C_H; \mathbb{L}(p_H))$. Suppose $f = (H,g,K)$ is a morphism from H to K, and let γ be any element of Γ such that $q\phi(\gamma) = g$. Then f induces a map $f_\#: p_H \longrightarrow p_K$ between stratified systems of fibrations; i.e., we have a commutative diagram

$$\begin{tikzcd}
(\mathbb{R}^{\ell} \times W_{\Gamma})/C_H \arrow[r, "f_{\#}"] \arrow[d, "p_H"'] & (\mathbb{R}^{\ell} \times W_{\Gamma})/C_K \arrow[d, "p_K"] \\
\mathbb{R}^{\ell}/C_H \arrow[r, "\bar{f}_{\#}"] & \mathbb{R}^{\ell}/C_K
\end{tikzcd}$$

where $f_{\#}$ and $\bar{f}_{\#}$ are maps induced by the action of γ on $\mathbb{R}^{\ell} \times W_{\Gamma}$ and $\mathbb{R}^{\ell}$ respectively. $f_{\#}$ is a finite covering. We have the following two operations corresponding to $f_{\#}$.

1.(functorial image) $f_*: \mathbb{P}(\mathbb{R}^{\ell}/C_H; p_H) \longrightarrow \mathbb{P}(\mathbb{R}^{\ell}/C_K)$ is defined as follows. If M is a geometric $\mathbb{Z}C_H$-module on p_H generated by $\{x_\alpha\} \subseteq (\mathbb{R}^{\ell} \times W_{\Gamma})/C_H$, then f_*M is the geometric $\mathbb{Z}C_K$-module on p_K generated by $\{f_{\#}(x_\alpha)\}$. As the fixed lifting of $f_{\#}(x_\alpha)$ in $\mathbb{R}^{\ell} \times W_{\Gamma}$, $\gamma\bar{x}_\alpha$ will be used. f_*M is a direct sum of copies of M, when viewed as a $\mathbb{Z}C_H$-module. Abstractly it is just a tensor product. A $\mathbb{Z}C_H$-module homomorphism $h: M \longrightarrow M'$ naturally induces a $\mathbb{Z}C_K$-module homomorphism $f_*h: f_*M \longrightarrow f_*M'$. (If $g = 1$ and $\gamma = 1$, then $f_*h = 1 \otimes h : f_*M = \mathbb{Z}C_K \otimes_{\mathbb{Z}C_H} M \longrightarrow \mathbb{Z}C_K \otimes_{\mathbb{Z}C_H} M' = f_*M'$.) By the natural identification $f_*(M^*) = f_*(M^*)$, f_* operates on quadratic Poincaré complexes ([R3,§2.2]), and it induces the desired $f_*: \mathbb{P}(\mathbb{R}^{\ell}/C_H; p_H) \to \mathbb{P}(\mathbb{R}^{\ell}/C_K; p_K)$.

2.(pullback) $f^* : \mathbb{P}(\mathbb{R}^\ell/C_K;p_K) \to \mathbb{P}(\mathbb{R}^\ell/C_H;p_H)$ is defined as follows. If M is a geometric $\mathbb{Z}C_K$-module on p_K generated by $\{x_\alpha\} \subseteq (\mathbb{R}^\ell \times W_\Gamma)/C_K$, then f^*M is a geometric $\mathbb{Z}C_H$-module on p_H generated by $f_\#^{-1}\{x_\alpha\}$; i.e., f^*M is a free abelian group generated by the points in $\gamma^{-1}(q^{-1}(\{x_\alpha\}))$, where q is the projection $\mathbb{R}^\ell \times W_\Gamma \longrightarrow (\mathbb{R}^\ell \times W_\Gamma)/C_K$, and the $\mathbb{Z}C_H$-module structure is obtained by forgetting the action by elements in $\gamma^{-1}C_K\gamma - C_H$. For a $\mathbb{Z}C_K$-homomorphism h, f^*h is defined by $f^*h = \gamma^{-1}h\gamma$. These induce the desired f^*.

Obviously, f_* does not increase the radius. On the other hand, f^* may increase the radius; but the result has radius at most $2|G_K|$ times as large as the original radius, where G_K is the holonomy of C_K. Therefore f_* and f^* induce maps $\mathbb{L}(\mathbb{R}^\ell/C_H;p_H) \longrightarrow \mathbb{L}(\mathbb{R}^\ell/C_K;p_K)$ and $\mathbb{L}(\mathbb{R}^\ell/C_K;p_K) \longrightarrow \mathbb{L}(\mathbb{R}^\ell/C_H;p_H)$ respectively. By the characterization theorem, these induce the desired maps

$$f_*: M(H) \longrightarrow M(K) \qquad \text{and} \qquad f^*: M(K) \longrightarrow M(H).$$

Obviously M is a bifunctor, and satisfies the first two conditions of Mackey functor.

4.3.2. <u>Proposition</u> M satisfies the double coset formula, and hence is a Mackey functor.

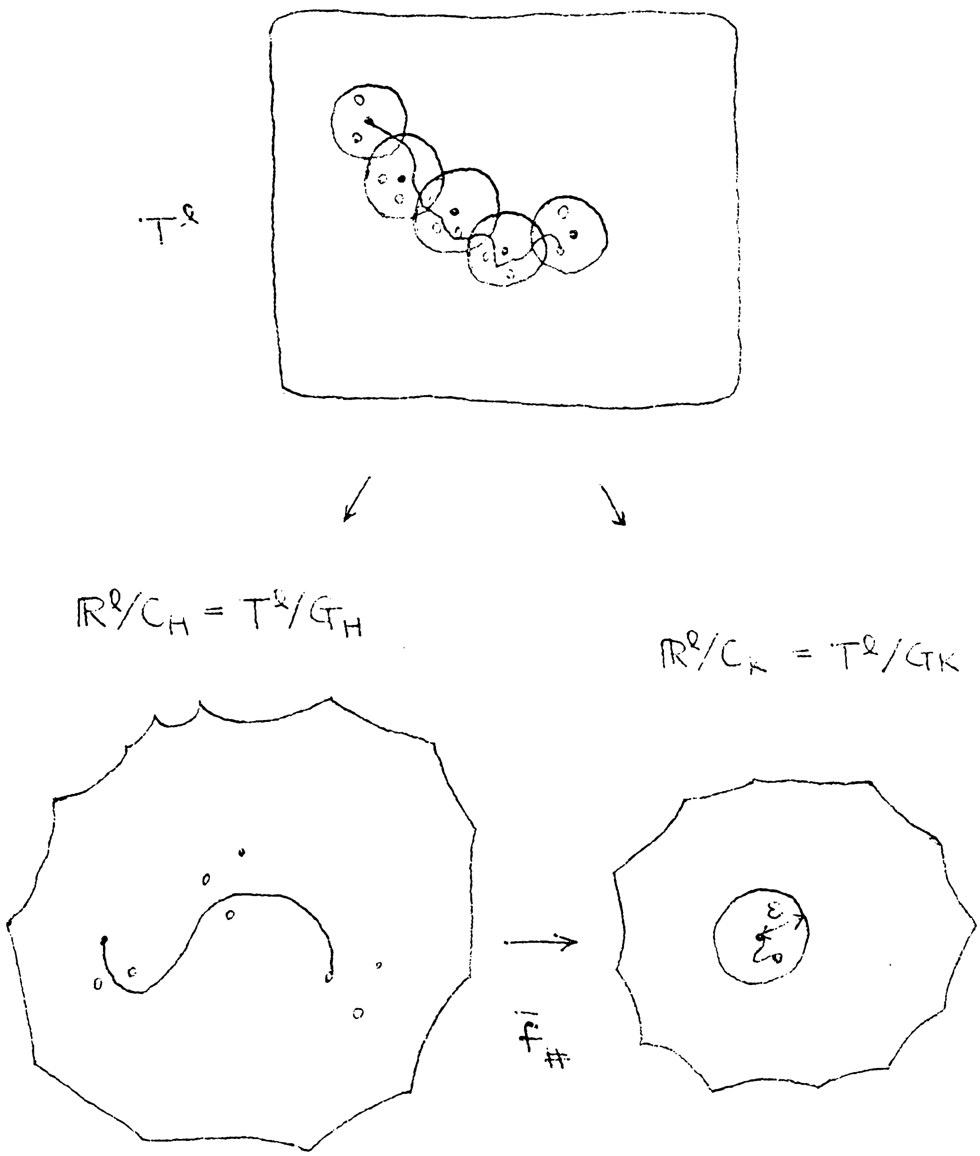


Figure 2. The change of radius by restriction.(g = 1)

Proof: Let L and L' be subgroups of a subgroup H of G, and suppose H has a double coset decomposition $H = \bigcup_{i=1}^{n} Lg_iL'$, $g_i \in H$. Let $C_H = \bigcup_{i=1}^{n} C_L\bar{g}_iC_{L'}$ be a corresponding double coset decomposition of C_H, where $\bar{g}_i \in C_H$ such that $q\phi(\bar{g}_i) = g_i$. Let P be the pullback:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & (\mathbb{R}^{\ell} \times W_{\Gamma})/C_{L'} \arrow[d, "(L',e,H)_{\#}"] \\ (\mathbb{R}^{\ell} \times W_{\Gamma})/C_L \arrow[r, "(L,e,H)_{\#}"'] & (\mathbb{R}^{\ell} \times W_{\Gamma})/C_H \end{tikzcd}$$

i.e.,

$$P = \{([x], [y]) \in (\mathbb{R}^{\ell} \times W_{\Gamma})/C_L \times (\mathbb{R}^{\ell} \times W_{\Gamma})/C_{L'} \mid [x]_{C_H} = [y]_{C_H} \in (\mathbb{R}^{\ell} \times W_{\Gamma})/C_H\}.$$

Then it is easily verified that the following map

$$\bigcup_{i=1}^{n} (\mathbb{R}^{\ell} \times W_{\Gamma})/C_{L \cap g_iL'g_i^{-1}} \to P;$$

$$[x]_{C_{L \cap g_iL'g_i^{-1}}} \longmapsto ([x], [\bar{g}_i^{-1}x])$$

is a C_H-isomorphism, where $x \in \mathbb{R}^{\ell} \times W_{\Gamma}$ and $[\ ]$ is the corresponding orbit. Therefore we obtain a pullback diagram:

$$\begin{tikzcd}
\coprod_{i=1}^{n} (\mathbb{R}^{\ell} \times W_{\Gamma})/C_{L \cap g_i L' g_i^{-1}} \arrow[r, "(L\cap(g_iL'g_i^{-1}), g_i^{-1}, L')_{\#}"] \arrow[d, "(L\ (g_iL'g_i^{-1}), e, L)_{\#}"] & (\mathbb{R}^{\ell} \times W_{\Gamma})/C_{L'} \arrow[d, "(L', e, H)_{\#}"'] \\
(\mathbb{R}^{\ell} \times W_{\Gamma})/C_{L} \arrow[r, dash, "(L,e,H)_{\#}"'] & (\mathbb{R}^{\ell} \times W_{\Gamma})/C_{H}
\end{tikzcd}$$

and the double coset formula for modules and chain complexes are easily derived from this. Since direct products of Poincaré complexes correspond to glueing along empty (i.e. 0) boundary, this establishes the desired double coset formula for M.

$GW(H, \mathbb{Z})$ acts on $M(H)$ by tensor product. Recall that $GW(H, \mathbb{Z})$ is constructed using H-spaces. An H-space is a $\mathbb{Z}$-free (left) $\mathbb{Z}H$-module N together with a symmetric H-invariant non-singular form $f: N \times N \longrightarrow \mathbb{Z}$. Let $N^* = \mathrm{Hom}(M, \mathbb{Z})$, then N^* is also a (left) $\mathbb{Z}H$-module. An element $h \in H$ acts on N^* by $h \cdot \alpha(y) = \alpha(h^{-1} \cdot y)$ for $\alpha \in N^*$, $y \in N$. By letting $(C_N)_0 = N^*$ and $(C_N)_i = 0$ for $i \neq 0$, we have a $\mathbb{Z}$-module chain complex C_N. We define a 0-dimensional symmetric Poincaré structure $\phi_f: N \longrightarrow N^*$ by $\phi_f(x) = f(x,-)$. By assumption ϕ_f is an isomorphism and H-invariant. If $\{e_1, \ldots, e_n\}$ is a basis of N^* and M is a free $\mathbb{Z}C_H$-module with a basis $\{\sigma_1, \ldots, \sigma_m\}$, then $N^* \otimes_{\mathbb{Z}} M$ is a free $\mathbb{Z}C_H$-

module with a basis $\{e_i \otimes \sigma_j\}$. Here we use the diagonal action of $\mathbb{Z}C_H$. C_H acts on N^* via H. In $(\mathbb{R}^\ell \times W_\Gamma)/C_H$, the generator $e_i \otimes \sigma_j$ corresponds to the same point as σ_j, and, in $\mathbb{R}^\ell \times W_\Gamma$, the lifting of $e_i \otimes \sigma_j$ is chosen to be the same point as the lifting of σ_j. Tensor products with (C_N, ϕ_f) described in [R3, §1.9] induce the desired action of $GW(H, \mathbb{Z})$ on $M(H)$.

4.3.3. <u>Theorem</u> M is a $GW(-, \mathbb{Z})$-module.

<u>Proof</u>: Same as the proof of [FH1, Theorem 2.3].

Another Mackey functor $M' : \underline{G} \longrightarrow Ab$ can be defined by setting $M'(H) = L_j^{-\infty}((\mathbb{R}^\ell \times W_\Gamma)/C_H)$. Notice that $(\mathbb{R}^\ell \times W_\Gamma)/C_H$ is a classifying space of C_H. f_* and f^* are defined in the same way as before. This time we do not have to worry about the radius. We can prove the following in the same way.

4.3.4. <u>Theorem</u> M' is a $GW(-, \mathbb{Z})$-module.

As an immediate consequence of 4.3.3 and 4.3.4, we have the following theorem.

4.3.5. <u>Theorem</u> Let F denote the family of conjugacy classes of maximal hyperelementary subgroups of $\Gamma_s^\wedge$. Then the following sequences are exact.

$$0 \longrightarrow H_j(\mathbb{R}^\ell/\Gamma;\, \mathbb{L}(p)) \xrightarrow{(\mathrm{res}_H)} \oplus_{H\in F} H_j(\mathbb{R}^\ell/C_H;\, \mathbb{L}(p_H))$$

$$\longrightarrow \bigoplus_{H,K,g} H_j(\mathbb{R}^\ell/C_{H\cap gKg^{-1}};\, \mathbb{L}(p_{H\cap gKg^{-1}})),$$

$$0 \longrightarrow L_j^{-\infty}((\mathbb{R}^\ell \times W_\Gamma)/\Gamma) \xrightarrow{(\mathrm{res}_H)} \bigoplus_{H\in F} L_j^{-\infty}((\mathbb{R}^\ell \times W_\Gamma)/C_H)$$

$$\longrightarrow \bigoplus_{H,K,g} L_j^{-\infty}((\mathbb{R}^\ell \times W_\Gamma)/C_{H\cap gKg^{-1}}) .$$

There are similar exact sequences for induction maps.

4.4. Calculation of surgery groups

The following is the main result.

4.4.1. Theorem Let Γ be a crystallographic group of rank ℓ with no 2-torsion, and p the projection $(\mathbb{R}^\ell \times W_\Gamma)/\Gamma \longrightarrow \mathbb{R}^\ell/\Gamma$, where W_Γ is a contractible free Γ-space. Then there is a natural isomorphism

$$a : H_j(\mathbb{R}^\ell/\Gamma;\, \mathbb{L}(p)) \longrightarrow L_j^{-\infty}((\mathbb{R}^\ell \times W_\Gamma)/\Gamma).$$

Proof: The map is induced by the following composition:

$$\mathbb{H}_{-j}(\mathbb{R}^{\ell}/\Gamma;\mathbb{L}(p)) \xrightarrow{A_{-j}} \mathbb{L}_j(\mathbb{R}^{\ell}/\Gamma;p) \xrightarrow{F} \mathbb{L}_j((\mathbb{R}^{\ell}\times W_{\Gamma})/\Gamma)$$

where A_{-j} is the assembly map and F is the restriction map to 0; i.e., if ρ is a k-simplex of $\mathbb{H}_{-j}(\mathbb{R}^{\ell}/\Gamma;\mathbb{L}(p))$, then $FA_{-j}(\rho) = (A_{-j}\rho)(0)$. We will prove the isomorphism inductively on the size $(\mathrm{rank}(\Gamma), |G|)$ of Γ using the lexicographic order.

If $\ell = 0$, then $\Gamma = 1$ and $(\mathbb{R}^{\ell}\times W_{\Gamma})/\Gamma$ is a single point. Since $\mathbb{H}_{-j}(*;\mathbb{L}(p)) \simeq \mathbb{L}_j(*;p)$ ($= \mathbb{L}_j(*)$) by 3.2.2, the theorem is obvious in this case. If $\ell = 1$, then $\Gamma = T$ and hence the argument below for crystallographic groups satisfying 4.1.2 (1) with rank > 1 can be applied.

Now assume that $\ell \geqslant 2$. First suppose that Γ satisfies (1) of 4.1.2. See Diagram 2. The first row is the exact sequence for the pair $(\mathbb{R}^{\ell}/\Gamma, \mathbb{R}^{\ell-1}/\Gamma')$; notice that $(\mathbb{R}^{\ell}/\Gamma)/(\mathbb{R}^{\ell-1}/\Gamma') = \Sigma(\mathbb{R}^{\ell-1}/\Gamma')^{+}$, where Σ is the reduced suspension. p' denotes the restriction of p over $\mathbb{R}^{\ell-1}/\Gamma'$; i.e., p' is the projection $(\mathbb{R}^{\ell-1}\times W_{\Gamma})/\Gamma' \longrightarrow \mathbb{R}^{\ell-1}/\Gamma'$. The second row is the well-known exact sequence

$$\longrightarrow L_j^{-\infty}(\Gamma') \longrightarrow L_j^{-\infty}(\Gamma'\rtimes\mathbb{Z}) \longrightarrow L_{j-1}^{-\infty}(\Gamma') \longrightarrow L_{j-1}^{-\infty}(\Gamma') \longrightarrow$$

due to Wall, Shaneson, Farrell and Hsiang. By 5-lemma, a is proved to be an isomorphism.

Next suppose Γ satisfies (2) of 4.1.2. We first show that

$$
\begin{tikzcd}
{} \arrow[r] & H_j(\mathbb{R}^{\ell-1}/\Gamma'; \mathbb{L}(p')) \arrow[r] \arrow[d, "a'"] & H_j(\mathbb{R}^{\ell}/\Gamma; \mathbb{L}(p)) \arrow[r] \arrow[d, "a"] & H_{j-1}(\mathbb{R}^{\ell-1}/\Gamma'; \mathbb{L}(p')) \arrow[r] \arrow[d, "a'"] & H_{j-1}(\mathbb{R}^{\ell-1}/\Gamma'; \mathbb{L}(p')) \arrow[r] \arrow[d, "a'"] & {} \\
{} \arrow[r] & L_j^{-\infty}((\mathbb{R}^{\ell-1}\times W_\Gamma)/\Gamma') \arrow[r] & L_j^{-\infty}((\mathbb{R}^{\ell}\times W_\Gamma)/\Gamma) \arrow[r] & L_{j-1}^{-\infty}((\mathbb{R}^{\ell-1}\times W_\Gamma)/\Gamma') \arrow[r] & L_{j-1}^{-\infty}((\mathbb{R}^{\ell-1}\times W_\Gamma)/\Gamma') \arrow[r] & {}
\end{tikzcd}
$$

Diagram 2.

the map being considered is injective. Suppose y is an element of the kernel. Represent it by a 0-cell ρ of $\mathbb{H}_{-j}(\mathbb{R}^{\ell}/\Gamma;\mathbb{L}(p))$. $A_{-j}\rho(0)$ represents the image $a(y)$ by a. Kan condition implies that there exists a 1-simplex σ of $\mathbb{L}_j((\mathbb{R}^{\ell}\times W_{\Gamma})/\Gamma)$ which connects $A_{-j}\rho(0)$ and 0. Let the radius of ρ be δ measured in $\mathbb{R}^{\ell}/\Gamma$. It is automatically finite, because $\mathbb{R}^{\ell}/\Gamma$ is compact. Choose a positive number s in (2) sufficiently small so that

$$4|G|\delta K/\sqrt{s} < \varepsilon$$

where K is the Lipschitz constant of the affine surjection $f\colon \mathbb{R}^{\ell}\longrightarrow\mathbb{R}^{m}$ induced by the epimorphism $\phi\colon \Gamma\longrightarrow\Gamma^{\wedge}$ and ε is the positive number posited in 3.2.3, when we consider $\mathbb{R}^{n}/\Gamma^{\sim}$. (ε depends only on $\mathbb{R}^{n}/\Gamma^{\sim}$, its filtration, the neighborhood system, and the dimension of the thing being considered.) Now we have a commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & H_j(\mathbb{R}^{\ell}/\Gamma;\mathbb{L}(p)) & \xrightarrow{(\mathrm{res}_H)} & \bigoplus_H H_j(\mathbb{R}^{\ell}/C;\mathbb{L}(p_H)) & \longrightarrow \\
 & \downarrow a & & \downarrow a_H & \\
0 \longrightarrow & L_j^{-\infty}((\mathbb{R}^{\ell}\times W_{\Gamma})/\Gamma) & \xrightarrow{(\mathrm{res}_H)} & \bigoplus_H L_j^{-\infty}((\mathbb{R}^{\ell}\times W_{\Gamma})/C_H) & \longrightarrow
\end{array}$$

where each exact row comes from the restriction maps corresponding to the maximal hyperelementary subgroups H of $\Gamma_s^{\wedge}$. For H such that $|G^{\wedge}|$ does not divide $|H|$, the size of $C=\phi^{-1}(q^{-1}(H))$ is smaller than the size of Γ. So by induction hypothesis, a_H is an isomorphism, and hence $\mathrm{res}_H(y) = 0$. For H such that $|G^\wedge|$ does divide $|H|$, we have a shrinking map $\alpha\colon \mathbb{R}^\ell/C \longrightarrow \mathbb{R}^n/\Gamma^\sim$, and $\mathrm{res}_H(\sigma)$ has radius less than ε on $\mathbb{R}^n/\Gamma^\sim$. Therefore the image of $\mathrm{res}_H(y)$ by

$$H_j(\mathbb{R}^\ell/C; \mathbb{L}(p_H)) \longrightarrow H_j(\mathbb{R}^n/\Gamma^\sim; \mathbb{L}(\alpha p_H)),$$

which is represented by $\mathrm{res}_H(A_{-j}\rho(0))$, is 0. We claim that the above map is an isomorphism. For the convenience of the proof, let us replace $|\mathbb{L}(\alpha p_H)|$ by

$$|\mathbb{L}(p_H)|' = \coprod_{\Delta\in\mathbb{R}^n/\Gamma^\sim} |\mathbb{L}((\alpha p_H)^{-1}(v_\Delta))| \times \Delta/\approx,$$

where v_Δ is a carefully chosen vertex of Δ. See [Q3, Proof of 8.6]. Notice that $\mathbb{L}((\alpha p_H)^{-1}(v_\Delta)) = \mathbb{L}((\mathbb{R}^{\ell-n} \times W_\Gamma)/\Delta_x)$, where $x = v_\Delta$, and that the size of Δ_x is strictly smaller than the size of Γ. Therefore, by induction, $\mathbb{L}((\alpha p_H)^{-1}(v_\Delta)) \simeq \mathbb{H}(\mathbb{R}^{\ell-n}/\Delta_x; \mathbb{L}(p_x))$. Here p_x is the projection $(\mathbb{R}^{\ell-n} \times W_\Gamma)/\Delta_x \longrightarrow \mathbb{R}^{\ell-n}/\Delta_x$. So,

$$\begin{aligned}
&\mathbb{H}(\mathbb{R}^n/\Gamma^\sim; \mathbb{L}(\alpha p_H)) \\
&\simeq \mathbb{H}(\mathbb{R}^n/\Gamma^\sim; \coprod |\mathbb{H}(\mathbb{R}^{\ell-n}/\Delta_x; \mathbb{L}(p_x))| \times \Delta) \\
&\simeq \lim_{i\to\infty} \Omega^i(\coprod(\lim_{k\to\infty} \Omega^k|\mathbb{L}_{-k-i}(p_x)|/(\mathbb{R}^{\ell-n}/\Delta_x)))/(\mathbb{R}^n/\Gamma^\sim) \\
&\simeq \lim_{j\to\infty} \Omega^j\ (\coprod|\mathbb{L}_{-j}(p_x)|)/(\mathbb{R}^\ell/C_H) \\
&= \lim_{j\to\infty} \Omega^j|\mathbb{L}_{-j}(p_H)|/(\mathbb{R}^\ell/C_H) \\
&= \mathbb{H}(\mathbb{R}^\ell/C_H; \mathbb{L}(p_H)).
\end{aligned}$$

Now this implies $res_H(y) = 0$. The restriction map $(res_H)_H$ is injective; therefore, y is 0; i.e., a is injective.

The onto part is similar. Pick any element in $L_j^{-\infty}(\mathbb{R}^\ell \times W_\Gamma)/\Gamma)$, represent it by a quadratic Poincaré complex with radius δ, and do the diagram chase as in the proof of 5-lemma. We need to use the next column this time, but it is already known to be isomorphic or at least injective. This completes the proof.

4.4.2. Remark In 4.4.1, we assumed that Γ has no 2-torsion. This is because 4.1.2 may not be true for $T \rtimes T_2$ and the induction step does not proceed. So the theorem is true even for Γ with 2-torsion if there appear no crystallographic groups of the form $T \rtimes T_2$ in the induction steps.

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SURGERY GROUPS OF CRYSTALLOGRAPHIC GROUPS

by

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(ABSTRACT)

Let Γ be a crystallographic group acting on the n-dimensional Euclidean space. In this dissertation, the surgery obstruction groups of Γ are computed in terms of certain sheaf homology groups defined by F. Quinn, when Γ has no 2-torsion. The main theorem is :

Theorem : If a crystallographic group Γ has no 2-torsion, there is a natural isomorphism

$$a : H_*(\mathbb{R}^n/\Gamma; \mathbb{L}(p)) \longrightarrow L_*^{-\infty}(\Gamma).$$