OPTIMAL NONLINEAR FEEDBACK CONTROL OF SPACECRAFT ATTITUDE MANEUVERS

by

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(ABSTRACT)

Polynomial feedback controls are developed for large angle, nonlinear spacecraft attitude maneuvers. Scalar and two-state systems are presented as simple examples to demonstrate the method, and several systems of state variables to parameterize spacecraft motion are considered. Both external and internal control torques are treated; in the latter, attention is restricted to momentum transfer maneuvers that permit several order reductions. Several stability theorems with their application to polynomial feedback systems are discussed.

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Chapter I

INTRODUCTION

Rapid large angle attitude maneuvers have become increasingly important to the success of many current and future spacecraft missions. These maneuvers are characterized by nonlinear behavior, however, resulting in a control problem that is likewise nonlinear.

Feedback control of nonlinear systems has traditionally been attempted to linearization. A nonlinear, open loop, "nominal" maneuver is first determined by integration, and then linear feedback control is used on the departure motion. Such an approach is rather difficult to implement within the constraints of on-board, real-time computation; each feedback controlled maneuver must be preceded by calculation of a nonlinear open loop control history to define the nominal motion. Other linearizing methods for feedback control of nonlinear motion include "gain scheduling" in which the control history is divided into segments, each determined by its own set of linear gains. A more attractive approach is control of the entire nonlinear maneuver by a single set of gains.

An early formulation of the nonlinear control problem for nonlinear state equations is presented by Lukes [1], in which the Hamilton-Jacobi-Bellman equation is derived in partial differential form. Solution methods have been presented by Rhoten and Mulholland [2], Willemstein [3], Dabbous and Ahmed [4], and Dwyer [5]-[7]. In these approaches the cost-to-go functional is expanded as a polynomial in the states, and the

Hamilton-Jacobi-Bellman equation solved recursively, as summarized in [8]. Other solution methods include transforming the nonlinear problem to a linear one, as in [9], or development of nonoptimal but stable control laws as in [10] and [36]. For certain nonlinear systems and performance indices, the optimal control may be linear, as shown by Debs and Athans in [11].

In the approach presented here [12], [13], the costates are expanded as polynomials in the states and substituted into the costate equations. The polynomial coefficients are determined by equating coefficients of like powers, with linear terms providing the familiar Riccati equation of linear feedback analysis. The higher order coefficients are determined recursively, and a suboptimal control law is generated by truncation.

Chapter II presents a general formulation of polynomial feedback, with a discussion of solution methods for state equations without linear terms. A discussion of nonlinear feedback control via Lyapunov functions is also included, which will be used in Chapter VI to generate stable control laws.

The next four chapters provide examples to demonstrate the methods presented in Chapter II. Chapter III presents simple scalar and two-state systems which can be formulated and solved without an algebraic manipulator. Spacecraft attitude control systems are treated in Chapters IV through VI. The choice of Euler parameters as state variables provide polynomial state equations that make polynomial feedback control feasible; Euler angle kinematics produce transcendental equations for which this method is not suitable. Euler parameters and the conjugate angular momenta of Chapter VI are redundant variables, however, that pose a particular challenge to polynomial feedback laws.

Chapter IV treats single and three-axis maneuvers of a spacecraft using external control torques. Attitude and angular velocity equations are presented, and the optimal control problem is formulated. Numerical examples are provided for both single and three-axis maneuvers, with a discussion of Euler parameter boundary conditions.

Three-axis momentum-transfer maneuvers are considered in Chapter V, in which internal control torques are provided by four reaction wheels. A special inertial angular momentum frame of reference is introduced to reduce the number of feedback states, and corresponding equations of motion are derived. The numerical examples progress from state equations with negligible gyroscopic terms to those with dominant nonlinearities. A comparison is made of maneuvers using several combinations of wheels, and the problems associated with redundant state variables are discussed.

Chapter VI treats polynomial feedback control using the generalized angular momentum variables recently developed by Morton [14]. The problems of two redundant variables are discussed, and a time-dependent polynomial feedbakc law is developed that circumvents these difficulties. Stable nonlinear control laws are derived from Lyapunov functions, which can be made optimal by adjusting constants.

Several stability theorems and their applications to certain polynomial feedback systems are discussed in Chapter VII. Chapter VIII summarizes the results and makes recommendations for future study.

Chapter II

GENERAL FORMULATION

2.1 State Equations with Linear Terms

Consider nonlinear state equations of the form

$$\dot{\underline{x}} = A\underline{x} + \underline{F}(\underline{x}) + B\underline{u} \tag{2.1}$$

where $\underline{x}$ is the state vector, A is a constant coefficient matrix forming the linear part of the state equations, $\underline{F}(\underline{x})$ is the nonlinear part, and $\underline{u}$ is the control vector to be determined. If $\underline{F}(\underline{x})$ can be expressed as a polynomial in $\underline{x}$, then the state equations may be written in indicial notation as

$$\dot{x}_i = a_{ij}x_j + c_{ijk}x_jx_k + \ldots + b_{ij}u_j \quad (i = 1,2,\ldots,n) \tag{2.2}$$

We consider the optimal control problem of finding a feedback control law that brings the states to zero while minimizing a quadratic performance index

$$J = \frac{1}{2}\int_{t_o}^{t_f} \{q_{ij}x_ix_j + r_{ij}u_iu_j\}dt \tag{2.3}$$

The Hamiltonian for this system is

$$\mathcal{H} = \frac{1}{2}\{q_{ij}x_ix_j + r_{ij}u_iu_j\} + \lambda_i\dot{x}_i \ , \tag{2.4}$$

where it is understood that $\dot{x}_i$ is symbolic for the right-hand side of Eq. (2.1). The necessary conditions for a minimum provide the state equations, Eq. (2.1)

$$\dot{x}_i = \frac{\partial \mathcal{H}}{\partial \lambda_i} \tag{2.5}$$

and the costate equations

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \tag{2.6}$$

For unbounded control

$$\frac{\partial H}{\partial u_i} = 0 \tag{2.7}$$

which implies

$$u_i = -r_{ij}^{-1} b_{kj} \lambda_k \tag{2.8}$$

where r_{ij}^{-1} represents the elements of the matrix inverse of r_{ij}. The terminal boundary conditions for the costates are

$$\lambda_i(t_f) = 0 \tag{2.9}$$

By assuming the costates can be expressed as a polynomial in the states,

$$\lambda_i = k_{ij}x_j + d_{ijk}x_jx_k + \ldots \tag{2.10}$$

a nonlinear feedback control law is determined in which the $k_{ij}(t)$, $d_{ijk}(t)$ are the control gains we seek. The terminal boundary conditions on the costates, Eq. (2.9), are satisfied since the states go to zero at t_f. By substituting Eq. (2.10) into Eq. (2.6) and carrying out the ensuing algebra, we are led to n homogeneous polynomial equations of the form

$$[\alpha]x_j + [\beta]x_jx_k + \ldots = 0 \tag{2.11}$$

where

$$[\alpha] = \text{function}(A,B,Q,R,\dot{K},K) \tag{2.12}$$

$$[\beta] = \text{function}(A,B,C,R,K,\dot{D},D) \tag{2.13}$$

and K and D are arrays whose elements are the gains k_{ij} and d_{ijk}. Since Eq. (2.11) must hold at every point in the state space, we conclude that the functions in brackets must vanish independently, so we obtain

$$[\alpha(A,B,Q,R,\dot{K},K)] = 0 \tag{2.14}$$

$$[\beta(A,B,C,R,K,\dot{D},D)] = 0 \tag{2.15}$$

Eqs. (2.14) are differential equations determining the linear feedback gains; upon carrying through the details, we find that the scalar equations of Eqs. (2.14) are precisely the elements of the matrix Riccati equation [15], [16], which generates the optimal feedback control if all nonlinear terms in the state equation are absent. The solution of the matrix Riccati equation can be determined by Potter's method [17], [18], or Turner's method [19], in which an associated eigenvalue problem is solved and matrix exponentials are used. (See Appendix B.)

The quadratic feedback gains are determined by Eqs. (2.15), which can be rearranged into a set of linear differential equations of the form

$$\dot{d}_{ijk} = [\eta]d_{lmr} + [\gamma] \tag{2.16}$$

where

$$[\eta] = \text{function}(A,B,C,R,K) \tag{2.17}$$

$$[\gamma] = \text{function}(A,B,C,R,K) \tag{2.18}$$

The boundary condition for Eq. (2.16) is

$$d_{ijk}(t_f) = 0 \tag{2.19}$$

which is obtained from the transversality condition for a quadratic performance index. Upon solving the Riccati equation for the linear gains $k_{\ell m}(t)$, Eqs. (2.16) provide nonautonomous, nonhomogeneous, but linear equations which determine the quadratic gains $d_{ijk}(t)$. For the steady state case we can solve Eqs. (2.14) and (2.16) algebraically for the constant feedback gains, subject to

$$\dot{k}_{\ell m} = \dot{d}_{\ell mk} = 0 \tag{2.20}$$

The $k_{\ell m}$ are solutions of the algebraic Riccati equation, and the $d_{\ell mr}$ are obtained by setting $\dot{d}_{ijk} = 0$ in Eq. (2.16) and solving the linear algebraic system. In the numerical examples with linear terms in the state equations we will restrict our attention to the constant gain case.

Since, for n states, there are $n^2(n+1)/2$ equations in Eqs. (2.16), we will include the algebra for only the simplest systems considered. The emphasis of the present discussion is the following generalization: After solving the Riccati equation for the linear gains, one is led to sets of linear differential equations, of the functional form showm in Eqs. (2.16), which can be solved sequentially to obtain the quadratic gains, the cubic gains, and so on, up to any desired order. The differential equations for the gains of each order are linear in those gains, and the coefficients at each order depend upon the lower order gains. To illustrate the above, we will consider one and two dimensional examples in detail.

2.2 State Equations Without Linear Terms

When no linear terms are present in the state equations, the algebraic Riccati equation may in some instances be degenerate, so that no solution can be found for the constant gain case. Such a situation has been found to occur if there are fewer controls than states and no linear terms in the state equations. When the Riccati equation becomes degenerate, two remedies can be considered. The first is to transform the system equations into state equations containing linear terms. This may be done if the final states are not all zero by using departure motion differences as state variables. If $\underline{z}$ is the state vector in which there are no linear terms in the state equations, then we can define a new state vector $\underline{x} = \underline{z} - \underline{z}(t_f)$, and linear terms involving the constant coefficient $\underline{z}(t_f)$ are introduced into the new state equations. Constant linear gains may now be determined from the algebraic Riccati equation, and higher order gains from the subsequent linear algebraic equations.

The other remedy is to determine time-dependent gains for the original nonlinear state equations. The differential equations determining the linear and zeroth order gains are in general simple when there are no linear terms in the state equations, so that closed-form solutions may be found by integration. When the desired final states are not all zero, the performance index should minimize the difference between the current state and the desired final state, and a final state penalty can be imposed. The boundary conditions specified by this performance index are now satisfied by the time-dependent gains in the polynomial expansion for the costates.

Both of these methods will be used, since the state equations of Chapter IV have only quadratic terms, those of Chapter V are quadratic and cubic, and those of Chapter VI have only cubic terms (i.e., many practical dynamical systems do not have linear terms in the state equations).

2.3 Feedback Control from Lyapunov Functions

Stable feedback control laws may be determined for autonomous systems from Lyapunov's second method [20]. To apply this method, we first transform the state equations so that the target state is the original, which may be accomplished by defining new states $\underline{x} = \underline{z} - \underline{z}(t_f)$ as discussed in the previous section. If we define a scalar function $V(\underline{x})$ that is positive definite, and find a control law that makes the total time derivative of $V(\underline{x})$ nonpositive, then that control law is stable and drives the system to the origin. Furthermore, if $V(\underline{x})$ is positive definite and its total time derivative is negative definite, then the control law is asymptotically stable.

Consider the polynomial state equations

$$\dot{x}_i = a_{ij}x_j + c_{ijk}x_jx_k + \ldots + b_{ij}u_j \quad (i = 1,2,\ldots,n) \tag{2.21}$$

and a Lyapunov function

$$V = x_1^2 + x_2^2 + \ldots + x_n^2 \tag{2.22}$$

Note that $V > 0$ for all $\underline{x} \neq 0$ and $V = 0$ when $\underline{x} = 0$. Now consider the total time derivative of V

$$\dot{V} = 2(x_1\dot{x}_1 + x_2\dot{x}_2 + \ldots + x_n\dot{x}_n) \tag{2.23}$$

By substituting Eqs. (2.21) into $\dot{V}$, we get a polynomial in x_i

$$\dot{V} = 2(a_{ij}x_i x_j + c_{ijk}x_i x_j x_k + \dots + b_{ij}x_i u_j) \tag{2.24}$$

If a control law u_i can be determined so that $\dot{V} \leq 0$ for all x_i, the ensuing controlled motion is stable. One solution is to define the controls so that the nonnegative definite terms in $\dot{V}$ are cancelled and then add a nonpositive function, so that

$$u_j = - b_{ij}^{-1} (a_{ik}x_k + c_{ik\ell}x_k x_\ell + \dots) - Kf_j(x_k) \tag{2.25}$$

where K is a positive constant and f_j is defined so that $b_{ij}x_i f_j \geq 0$. For example, we could define f_j as

$$f_j = b_{ij}^{-1} x_i \tag{2.26}$$

when b_{ij} is positive definite. A suboptimal control could then be determined by adjusting the gain K until a given performance index is minimized.

2.4 Conclusions

When nonlinear state equations can be expressed as polynomials in the states, the optimal nonlinear control problem may be solved in polynomial feedback form and a suboptimal control law determined by truncation. If the state equations contain linear terms, and there is at most only one zero eigenvalue, then linear constant gains may be determined by Potter's method. Constant higher-order gains are then determined recursively from linear algebraic equations.

If the state equations do not contain linear terms and there are more states than control variables, or the linear system has more than one zero eigenvalue, then constant linear gains cannot be determined by Potter's method. In these cases linear terms may be introduced by changing the state variables, or time-dependent gains can be determined.

Nonlinear control laws may also be determined by use of a Lyapunov function. Polynomial state equations are not required, and an asymptotically stable control law is determined. Although the control is not optimal per se, constant coefficients can be introduced which may be adjusted to minimize a specified performance index.

Several simple examples for scalar and two-state systems will now be considered.

Chapter III

SCALAR AND TWO-STATE SYSTEMS

3.1 State Equations with Linear Terms

Consider the optimal control problem of minimizing the following performance index

$$J = \frac{1}{2}\int_{t_o}^{t_f} \{qx^2 + ru^2\}dt \tag{3.1}$$

subject to the state equation

$$\dot{x} = a_1x + a_2x^2 + \ldots + a_nx^n + bu \tag{3.2}$$

The costate equation is

$$\dot{\lambda} = - qx - \lambda\{a_1 + 2a_2x + \ldots + na_nx^{n-1}\} \tag{3.3}$$

and the control is

$$u = - \frac{b}{r}\lambda \tag{3.4}$$

Assuming the costate as a polynomial in the state x

$$\lambda = k_1x + k_2x^2 + k_3x^3 + \ldots \tag{3.5}$$

then the coefficient differential equations corresponding to Eqs. (2.14) and (2.16) and higher order terms are

$$\dot{k}_1 + 2a_1k_1 - \frac{b}{r}k_1^2 + q = 0$$

$$\dot{k}_2 + 3(a_1 - \frac{b}{r}k_1)k_2 = -3k_1a_2$$

$$\dot{k}_3 + 4(a_1 - \frac{b}{r}k_1)k_3 = -4k_1a_3 - 4k_2a_2 + 2\frac{b}{r}k_2^2$$

$$\dot{k}_4 + 5(a_1 - \frac{b}{r}k_1)k_4 = -5k_1a_4 - 5k_2a_3 - 5k_3a_2 + 5\frac{b}{r}k_2k_3$$

$$\dot{k}_5 + 6(a_1 - \frac{b}{r}k_1)k_5 = -6k_1a_5 - 6k_2a_4 - 6k_3a_3 - 6k_4a_2$$
$$+ 6\frac{b}{r}k_2k_4 + 3\frac{b}{r}k_3^2 \qquad (3.6)$$

$$\vdots$$

$$\dot{k}_n + (n+1)(a_1 - \frac{b}{r}k_1)k_n = -(n+1)\{k_1a_n + k_2a_{n-1} + \dots + k_{n-1}a_2\}$$
$$+ \begin{cases} (n+1)\frac{b}{r}\{k_2k_{n-1} + k_3k_{n-2} + \dots \\ \qquad + k_{n/2}\,k_{n/2+1}\} \text{ for n even} \\ (n+1)\frac{b}{r}\{k_2k_{n-1} + k_3k_{n-2} + \dots \\ \qquad + k_{(n-1)/2}\,k_{(n+3)/2} + \frac{1}{2}k_{(n+1)/2}^2\} \text{ for n odd} \end{cases}$$

Making the change of variable from time t to time-to-go $\tau = t_f - t$ and assuming a solution of the form

$$k_1 = k_{1SS} + z_1^{-1}$$

$$k_2 = z_1^{-3}z_2$$

$$k_3 = z_1^{-4}z_3 \qquad (3.7)$$

$$\vdots$$

$$k_n = z_1^{-(n+1)}z_n$$

where k_{1SS} is the steady state solution for k_1, we obtain the following equations

$$
\begin{aligned}
&\frac{dz_1}{d\tau} + 2(a_1 - \frac{b}{r} k_{1SS})z_1 = \frac{b}{r} \\
&\frac{dz_2}{d\tau} + 3(a_1 - \frac{b}{r} k_{1SS})z_2 = 3a_2(k_{1SS}z_1^3 + z_1^2) \\
&\frac{dz_3}{d\tau} + 4(a_1 - \frac{b}{r} k_{1SS})z_3 = 4a_3(k_{1SS}z_1^4 + z_1^3) \\
&\qquad\qquad\qquad\qquad + 4a_2z_1z_2 - 2\frac{b}{r} z_1^{-2} z_2^2 \qquad (3.8) \\
&\frac{dz_4}{d\tau} + 5(a_1 - \frac{b}{r} k_{1SS})z_4 = 5a_4(k_{1SS}z_1^5 + z_1^4) \\
&\qquad\qquad\qquad\qquad + 5a_3z_1z_2 + 5a_2z_1z_3 - 5\frac{b}{r} z_1^{-2} z_2z_3 \\
&\vdots \\
&\frac{dz_n}{d\tau} + (n+1)(a_1 - \frac{b}{r} k_{1SS})z_n = (n+1)\{a_n(k_{1SS}z_1^{n+1} + z_1^n) \\
&\qquad + a_{n-1} z_1^{n-2} z_2 + \dots + a_2z_1z_{n-1}\} \\
&\qquad - \begin{cases} (n+1)\frac{b}{r} z_1^{-2}\{z_2z_{n-1} + z_3z_{n-2} + \dots \\ \qquad + z_{n/2}\, z_{n/2+1}\} \text{ for } n \text{ even} \\ (n+1)\frac{b}{r} z_1^{-2}\{z_2z_{n-1} + z_3z_{n-2} + \dots \\ \qquad + z_{(n-1)/2}\, z_{(n+3)/2} + \frac{1}{2} z_{(n+1)/2}^2\} \text{ for } n \text{ odd} \end{cases}
\end{aligned}
$$

The variable changes of Eqs. (3.7) are generalizations of and were motivated by Refs. [19] and [21].

Eqs. (3.8) are easily solved, subject to specification of the boundary conditions; e.g., $k_1(\tau) = k_2(\tau) = \ldots = k_n(\tau) = 0$ at $\tau = 0$. Substitution of the solution for $z_i(\tau)$ into Eqs. (3.7) and then Eq. (3.5) yields a polynomial feedback control law with time-dependent coefficients.

A numerical example is considered for the state equation

$$\dot{x} = -x + \varepsilon x^2 + u \tag{3.9}$$

that minimizes the performance index

$$J = \frac{1}{2}\int_{t_o}^{t_f} \{x^2 + u^2\}dt \tag{3.10}$$

The costate equation is

$$\dot{\lambda} = -x + \lambda - 2\varepsilon\lambda x \tag{3.11}$$

and the control is

$$u = -\lambda \tag{3.12}$$

The performance indices are given in Table 3.1, and state variable and control histories given in Fig. 3.1 for the values $\varepsilon = .01$ and t_f = 5 sec. The time-dependent linear through quintic gains are plotted in Fig. 3.2. Fifth order polynomial feedback has essentially converged to the optimal control for this scalar problem.

3.2 State Equations Without Linear Terms

When no linear terms are present in the state equations, but the number of controls is the same as the number of states, then the algebraic Riccati equation may be used to determine constant linear

Table 3.1

PERFORMANCE INDICES FOR SCALAR EQUATION

$$\dot{x} = -x + \varepsilon x^2 + u$$

Feedback Order	Performance Index
1	4314.8
2	3796.4
3	3725.5
4	3717.2
5	3716.8

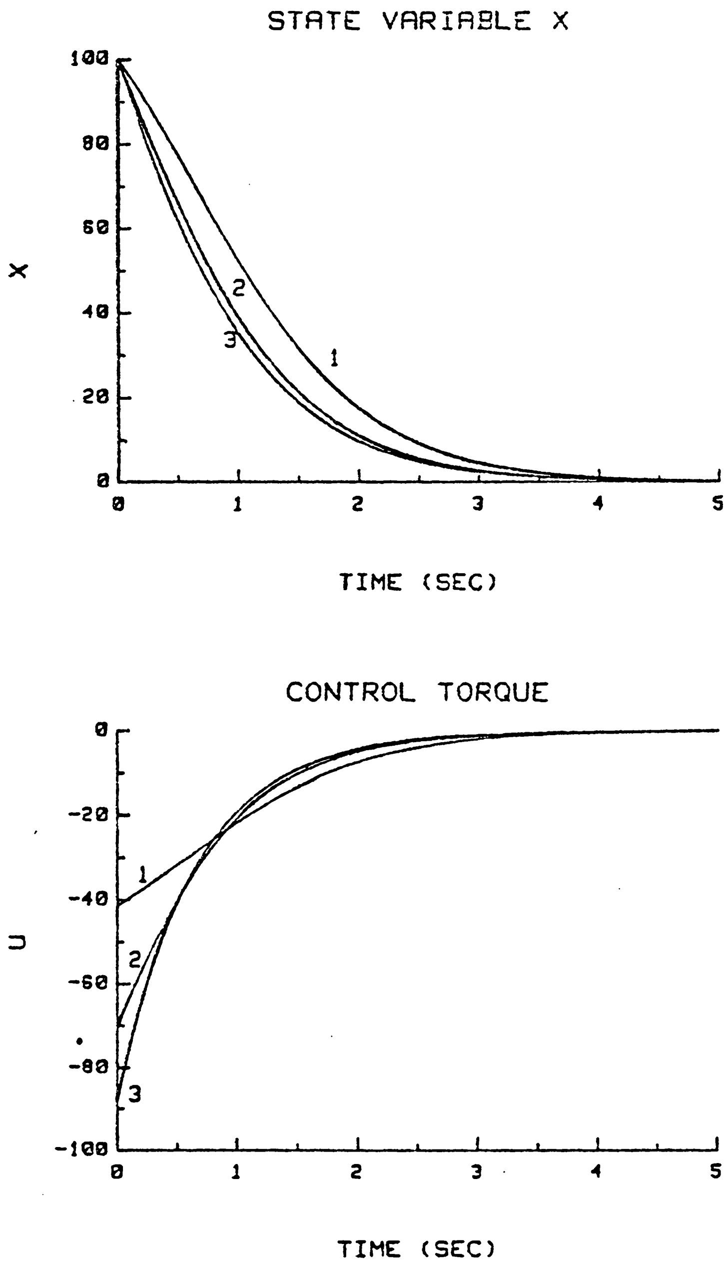


Fig. 3.1: State variable and control histories for scalar example.

1 linear feedback
2 linear plus quadratic feedback
3 linear through cubic feedback

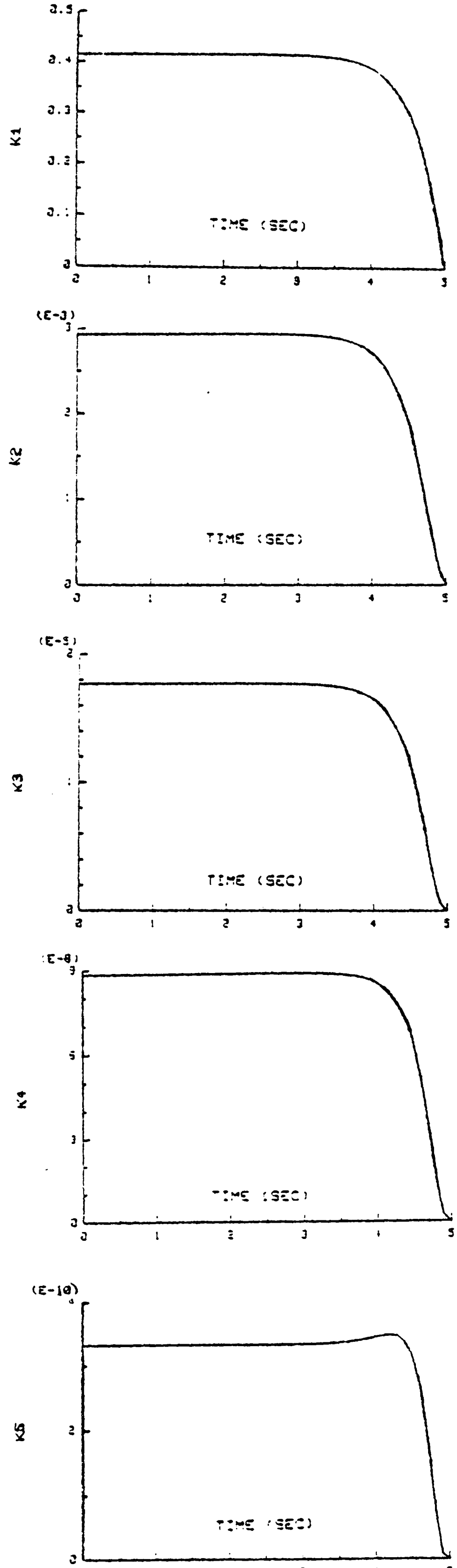


Fig. 3.2: Time dependent feedback gains for scalar example.

gains. Consider the two state, two control system

$$\dot{x}_1 = - x_1x_2 + u_1$$
$$\dot{x}_2 = - x_1^2 + u_2 \tag{3.13}$$

and find the controls that minimize a quadratic performance index

$$J = \frac{1}{2}\int_{t_o}^{t_f} \{q_1x_1^2 + q_2x_2^2 + u_1^2 + u_2^2\}dt \tag{3.14}$$

The costate equations obtained from the necessary conditions for a minimum are

$$\dot{\lambda}_1 = - q_1x_1 + \lambda_1x_2 + 2\lambda_2x_1$$
$$\dot{\lambda}_2 = - q_2x_2 + \lambda_1x_1 \tag{3.15}$$

with

$$u_1 = - \lambda_1$$
$$u_2 = - \lambda_2 \tag{3.16}$$

By assuming a polynomial form for the costates

$$\lambda_1 = k_1x_1 + k_2x_2 + d_1x_1^2 + d_2x_1x_2 + d_3x_2^2 + \ldots$$
$$\lambda_2 = k_2x_1 + k_3x_2 + d_4x_1^2 + d_5x_1x_2 + d_6x_2^2 + \ldots \tag{3.17}$$

and substituting into Eqs. (3.15), the following equations are obtained from the linear terms

$$k_1^2 + k_2^2 = q_1$$

$$k_2^2 + k_3^2 = q_2 \tag{3.18}$$

$$k_2(k_1 + k_3) = 0$$

The last equation in Eqs. (3.18) has two solutions, either $k_2 = 0$ or $k_1 = - k_3$. Since the second of these solutions implies $q_1 = q_2$, we consider the first, so that the following constant linear gains are determined

$$k_1 = \sqrt{q_1} \qquad k_2 = 0 \qquad k_3 = \sqrt{q_2} \tag{3.19}$$

The quadratic terms that arise from substituting Eqs. (3.17) into Eqs. (3.15) may be written in matrix form

$$[n]\underline{d} = - \underline{\gamma} \tag{3.20}$$

where

$$\underline{d} = \{d_1 \quad d_2 \quad d_3 \quad d_4 \quad d_5 \quad d_6\}^T \tag{3.21}$$

and

$$\underline{\gamma} = \{3k_2 \quad 2(k_1+k_3) \quad k_2 \quad k_1+k_3 \quad 2k_2 \quad 0\}^T \tag{3.22}$$

The matrix [n] is

$$\begin{bmatrix} 3k_1 & k_2 & 0 & k_2 & 0 & 0 \\ 2k_2 & 2k_1+k_3 & 2k_2 & 0 & k_2 & 0 \\ 0 & k_2 & 2k_3+k_1 & 0 & 0 & k_2 \\ k_2 & 0 & 0 & 2k_1+k_3 & k_2 & 0 \\ 0 & k_2 & 0 & 2k_2 & 2k_3+k_1 & 2k_2 \\ 0 & 0 & k_2 & 0 & k_2 & 3k_3 \end{bmatrix} \tag{3.23}$$

The linear gains, Eqs. (3.19), are substituted into Eqs. (3.22) and (3.23), and Eq. (3.20) is then solved for $\underline{d}$. Matrix [n] will be non-singular whenever full-state feedback is used.

Now consider the same two-state system without linear terms, but with only one control variable.

$$\begin{aligned} \dot{x}_1 &= - x_1 x_2 + u \\ \dot{x}_2 &= - x_1^2 \end{aligned} \tag{3.24}$$

To find the control that minimizes

$$\begin{aligned} J = \frac{1}{2} \{h_1 x_1^2(t_f) + 2h_2 x_1(t_f) x_2(t_f) + h_3 x_2^2(t_f)\} \\ + \frac{1}{2} \int_{t_o}^{t_f} \{q_1 x_1^2 + q_2 x_2^2 + u^2\} dt \end{aligned} \tag{3.25}$$

where here we have included final state penalties, the costate equations of Eq. (3.15) and the following control are determined

$$u = - \lambda_1 \tag{3.26}$$

We again assume the polynomial expansions for the costates in Eqs. (3.17), and substitute them into the costate equations. The linear terms give the following equations

$$k_1^2 = q_1$$

$$k_2^2 = q_2 \tag{3.27}$$

$$k_1 k_2 = 0$$

The last equation of Eqs. (3.27) implies either $k_1 = 0$ or $k_2 = 0$, which contradicts the other two equations if q_1 and q_2 are nonzero. Hence the algebraic equations to determine the linear gains are degenerate, and either linear terms should be introduced into the state equations, or time-dependent gains should be considered. If the final states are zero or if the linearized equations still produce a degenerate Riccati equation, then the second solution is used.

For time-dependent coefficients, the costates become

$$\begin{aligned}\lambda_1 &= k_1(t)x_1 + k_2(t)x_2 + d_1(t)x_1^2 + d_2(t)x_1x_2 + d_3(t)x_2^2 + \dots \\ \lambda_2 &= k_2(t)x_1 + k_3(t)x_2 + d_4(t)x_1^2 + d_5(t)x_1x_2 + d_6(t)x_2^2 + \dots\end{aligned} \tag{3.28}$$

and substitution into the costate equations give the following Riccati equations

$$\dot{k}_1 - k_1^2 = - q_1$$

$$\dot{k}_2 - k_1 k_2 = 0 \tag{3.29}$$

$$\dot{k}_3 - k_2^2 = - q_2$$

To solve these equations, we assume a solution of the form

$$k_1 = k_{1SS} + z_1^{-1}$$

$$k_2 = z_1^{-1}\, z_2 \qquad (3.30)$$

$$k_3 = z_1^{-1}\, z_2^2 + z_3$$

where $k_{1SS} = \sqrt{q_1}$ from the steady state solution for k_1. By making the change of variable from t to $\tau = t_f - t$ where t_f is the final time, we obtain the following equations

$$\frac{dz_1}{d\tau} - 2k_{1SS}\, z_1 - 1 = 0$$

$$\frac{dz_2}{d\tau} - k_{1SS}\, z_2 = 0 \qquad (3.31)$$

$$\frac{dz_3}{d\tau} - q_2 = 0$$

These equations are easily solved, and with the boundary conditions that $k_i = h_i$ at $t = t_f$, we obtain the following solutions

$$k_1 = k_{1SS} + \frac{2k_{1SS}}{(2k_{1SS}A_1 e^{2k_{1SS}(t_f-t)} - 1)}$$

$$k_2 = \frac{2k_{1SS}A_2 e^{k_{1SS}(t_f-t)}}{(2k_{1SS}A_1 e^{2k_{1SS}(t_f-t)} - 1)} \qquad (3.32)$$

$$k_3 = \frac{2k_{1SS}A_2^2 e^{k_{1SS}(t_f-t)}}{(2k_{1SS}A_1 e^{2k_{1SS}(t_f-t)} - 1)} + q_2(t_f-t) + A_3$$

where the integration constants are

$$A_1 = \frac{k_{1SS} + h_1}{(2k_{1SS}h_1 - 2k_{1SS}^2)}$$

$$A_2 = \frac{h_2(2k_{1SS}A_1 - 1)}{2k_{1SS}} \qquad (3.33)$$

$$A_3 = h_3 - \frac{2k_{1SS}A_2^2}{(2k_{1SS}A_1 - 1)}$$

Hence a time-dependent solution may be found for systems with no linear terms and fewer control variables than states.

3.3 Conclusions

The polynomial feedback control law proposed in Chapter II has been used for scalar and two-state nonlinear systems. A time-dependent solution has been presented for an n-th order scalar equation, and two-state systems without linear terms are examined. If the number of states and control variables is the same, the algebraic Riccati equations may be solved for constant linear gains. When there are fewer controls than state variables, the algebraic equations cannot be solved and time-dependent gains should be used.

Chapter IV

EXTERNAL TORQUE MANEUVERS

4.1 System Model and Optimal Control Problem

The attitude control problem for a rigid spacecraft is governed by a set of kinematic equations defined from the orientation of the body with respect to an inertial frame of reference, and a set of dynamic equations representing rotational motion. The latter set is defined by equating the time rate of change of angular momentum to the external torque on the spacecraft, which may be provided by rockets or thrusters, magnets interacting with the earth's magnetic field, or by other external means. This external torque is precisely the optimal control to be determined, such that the desired orientation and motion is achieved while a particular performance measure is minimized.

4.1.1 Spacecraft Orientation

The orientation of a spcecraft body-fixed reference frame $\{\underline{\hat{b}}\}$ to an inertial frame $\{\underline{\hat{n}}\}$ is given by the projection

$$\{\underline{\hat{b}}\} = [C]\{\underline{\hat{n}}\} \tag{4.1}$$

where [C] is the direction cosine matrix. Euler angles may be used to parameterize the elements of [C]; the 3-2-1 set of Euler angles are defined by a positive rotation ψ about the $\hat{n}_3$-axis, a positive rotation θ about the new 2-axis, and then a positive rotation ϕ about the latest 1-axis. The direction cosine matrix is then

$$[C] = \begin{bmatrix} c\psi c\theta & s\psi c\theta & -s\theta \\ -s\psi c\phi + c\psi s\theta s\phi & c\psi c\phi + s\psi s\theta s\phi & c\theta s\phi \\ s\psi s\phi + c\psi s\theta c\phi & -c\psi s\phi + s\psi s\theta c\phi & c\theta c\phi \end{bmatrix} \tag{4.2}$$

where c represents cosine and s represents sine. The disadvantage of this three parameter system is that if $\theta = \pm \pi/2$, the angles ψ and ϕ are undefined, and the attitude computation becomes singular.

Instead of Euler angles, four variables known as Euler parameters or quaternions may be used to describe orientation [22], [23]. They can be defined from Euler's Theorem, which states that the orientation of a body with respect to a given reference frame can be accomplished by a single rotation through an angle Φ about a principal vector $\underline{\ell}$.

The Euler parameters are

$$\begin{aligned} \beta_0 &= \cos \frac{\Phi}{2} \\ \beta_i &= \ell_i \sin \frac{\Phi}{2} \quad (i = 1,2,3) \end{aligned} \tag{4.3}$$

Since rotational motion has three degrees of freedom, four parameters are once redundant; the Euler parameters satisfy the constraint

$$\sum_{i=0}^{3} \beta_i^2 = 1 \tag{4.4}$$

The direction cosine matrix can now be parameterized [22]

$$[C] = \begin{bmatrix} \beta_0^2 + \beta_1^2 - \beta_2^2 - \beta_3^2 & 2(\beta_1\beta_2 + \beta_0\beta_3) & 2(\beta_1\beta_3 - \beta_0\beta_2) \\ 2(\beta_1\beta_2 - \beta_0\beta_3) & \beta_0^2 - \beta_1^2 + \beta_2^2 - \beta_3^2 & 2(\beta_2\beta_3 + \beta_0\beta_1) \\ 2(\beta_1\beta_3 + \beta_0\beta_2) & 2(\beta_2\beta_3 - \beta_0\beta_1) & \beta_0^2 - \beta_1^2 - \beta_2^2 + \beta_3^2 \end{bmatrix} \tag{4.5}$$

By substituting [C] into the following kinematic equation for direction cosines [23]

$$[\dot{C}][C]^T = - [\tilde{\omega}] \tag{4.6}$$

where

$$[\tilde{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{4.7}$$

then we obtain the following equations relating the Euler parameters to the body-frame components ω_i of the spacecraft angular velocity $\underline{\omega}$

$$\begin{aligned} \dot{\beta}_0 &= - \frac{1}{2} (\beta_1\omega_1 + \beta_2\omega_2 + \beta_3\omega_3) \\ \dot{\beta}_1 &= \frac{1}{2} (\beta_0\omega_1 - \beta_3\omega_2 + \beta_2\omega_3) \\ \dot{\beta}_2 &= \frac{1}{2} (\beta_3\omega_1 + \beta_0\omega_2 - \beta_1\omega_3) \\ \dot{\beta}_3 &= - \frac{1}{2} (\beta_2\omega_1 - \beta_1\omega_2 - \beta_0\omega_3) \end{aligned} \tag{4.8}$$

Eqs. (4.8) are the kinematic equations relating orientation to spacecraft motion. Note that the constraint Eq. (4.4) is an integral of Eqs. (4.8), and so any set of Euler parameters that satisfy the four kinematic equations with admissible boundary conditions automatically satisfy the constraint. This will be important later in defining the Hamiltonian for the optimal control problem.

4.1.2 Equations of Motion

The rotational equations of motion are derived by equating the time rate of change of the angular momentum $\underline{H}$ to the external torque $\underline{u}$ applied to the spacecraft, all with respect to an inertial frame of reference [24], [25]. Given

$$\underline{u} = \dot{\underline{H}} \tag{4.9}$$

and an inertial frame $\{\hat{\underline{n}}\}$, body frame $\{\hat{\underline{b}}\}$,

$$\underline{u} = \dot{\underline{H}}_b + \underline{\omega} \times \underline{H} \tag{4.10}$$

where $\dot{\underline{H}}_b$ is the time rate of change of $\underline{H}$ relative to the body frame, and $\underline{\omega}$ is the angular velocity of $\{\hat{\underline{b}}\}$ with respect to $\{\hat{\underline{n}}\}$. The system angular momentum is

$$\underline{H} = I\underline{\omega} \tag{4.11}$$

where I is the moment of inertia matrix. In the body frame of reference, the angular momentum's time derivative is

$$\dot{\underline{H}}_b = I\dot{\underline{\omega}} \tag{4.12}$$

and upon choosing a principal axis system for the body frame, $I = \text{diag}\ \{I_1, I_2, I_3\}$, and we obtain from Eq. (4.10) Euler's equations in the classical form

$$\begin{aligned} u_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \\ u_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 \\ u_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \end{aligned} \tag{4.13}$$

These equations may be rewritten as

$$\dot{\omega}_1 = - \mathcal{I}_1\omega_2\omega_3 + u_1/I_1$$

$$\dot{\omega}_2 = - \mathcal{I}_2\omega_1\omega_3 + u_2/I_2 \qquad (4.14)$$

$$\dot{\omega}_3 = - \mathcal{I}_3\omega_1\omega_2 + u_3/I_3$$

where u_1, u_2, u_3 are the external control torques to be determined, and the inertia ratios are defined below.

$$\mathcal{I}_1 = (I_3-I_2)/I_1 \quad , \quad \mathcal{I}_2 = (I_1-I_3)/I_2 \quad , \quad \mathcal{I}_3 = (I_2-I_1)/I_3 \qquad (4.15)$$

The system model is defined by the four kinematic equations, Eqs. (4.8), and the spacecraft dynamic equations, Eqs. (4.14).

4.1.3 State Space Formulation

Since we have three controls and seven state variables, linear terms need to be introduced into the state equations if constant gains are to be determined. To this end, let the Euler parameter differences and the angular velocities be the state variables

$$\underline{x} = \{\omega_1 \ \ \omega_2 \ \ \omega_3 \ \ \tilde{\beta}_0 \ \ \tilde{\beta}_1 \ \ \tilde{\beta}_2 \ \ \tilde{\beta}_3\}^T \qquad (4.16)$$

where the attitude departure variable $\tilde{\beta}_i$ is defined as the difference between the current Euler parameter and its desired state at the final time t_f

$$\tilde{\beta}_i = \beta_i - \beta_i(t_f) \qquad (i = 0,1,2,3) \qquad (4.17)$$

The state equations may now be written as

$$\dot{x}_1 = - \mathcal{I}_1 x_2 x_3 + u_1/I_1$$

$$\dot{x}_2 = - \mathcal{I}_2 x_1 x_3 + u_2/I_2$$

$$\dot{x}_3 = - \mathcal{I}_3 x_1 x_2 + u_3/I_3$$

$$\dot{x}_4 = - \frac{1}{2} (x_1 \beta_1(t_f) + x_2 \beta_2(t_f) + x_3 \beta_3(t_f)) - \frac{1}{2} (x_1 x_5 + x_2 x_6 + x_3 x_7)$$

$$\dot{x}_5 = \frac{1}{2} (x_1 \beta_o(t_f) - x_2 \beta_3(t_f) + x_3 \beta_2(t_f)) + \frac{1}{2} (x_1 x_4 - x_2 x_7 + x_3 x_6)$$

$$\dot{x}_6 = \frac{1}{2} (x_1 \beta_3(t_f) + x_2 \beta_o(t_f) - x_3 \beta_1(t_f)) + \frac{1}{2} (x_1 x_7 + x_2 x_4 - x_3 x_5)$$

$$\dot{x}_7 = - \frac{1}{2} (x_1 \beta_2(t_f) - x_2 \beta_1(t_f) - x_3 \beta_o(t_f)) - \frac{1}{2} (x_1 x_6 - x_2 x_5 - x_3 x_4) \tag{4.18}$$

Note that linear terms now appear in the fourth through seventh equations above.

4.1.4 Formulation of the Optimal Control Problem

Consider the optimal control problem of finding a feedback control law that brings the states to zero, so that the angular velocities ω_i and the attitude differences $\tilde{\beta}_i$ go to zero, while minimizing a quadratic performance index

$$J = \frac{1}{2} \int_{t_o}^{t_f} \{\underline{x}^T Q \underline{x} + \underline{u}^T \underline{u}\} dt \tag{4.19}$$

The Hamiltonian $\mathcal{H}$ for this system of equations is

$$\mathcal{H} = \frac{1}{2} \underline{x}^T Q \underline{x} + \frac{1}{2} \sum_{i=1}^{3} u_i^2 + \underline{\lambda}^T \underline{\dot{x}} \tag{4.20}$$

Note that the Euler parameter constraint, Eq. (4.4), is not explicitly included in the Hamiltonian. As discussed before, this constraint is an exact integral of the Euler parameter equations, and hence is implicitly enforced at each instant of time when the state equations are satisfied. Furthermore, differentiation of the Hamiltonian with respect to the states is valid even though the Euler parameters are not independent, since the constraint is implicitly satisfied (see Bryson and Ho [26] for a discussion of dependent states with equality constraints). The costates associated with the Euler parameters are also redundant, but no similar equality constraint can be derived for them when state weights are present in the performance index. For a discussion of Euler parameter costate constraints, see Vadali, Kraige, and Junkins [27] and Junkins and Turner [28].

The necessary conditions for optimality result in the following costate equations

$$\begin{aligned}
\dot{\lambda}_1 &= -q_{1j}x_j + \lambda_2 I_2 x_3 + \lambda_3 I_3 x_2 + (\lambda_4\beta_1(t_f) - \lambda_5\beta_0(t_f) \\
&\quad -\lambda_6\beta_3(t_f) + \lambda_7\beta_2(t_f) + \lambda_4x_5 - \lambda_5x_4 - \lambda_6x_7 + \lambda_7x_6)/2 \\
\dot{\lambda}_2 &= -q_{2j}x_j + \lambda_1 I_1 x_3 + \lambda_3 I_3 x_1 + (\lambda_4\beta_2(t_f) + \lambda_5\beta_3(t_f) \\
&\quad -\lambda_6\beta_0(t_f) - \lambda_7\beta_1(t_f) + \lambda_4x_6 + \lambda_5x_7 - \lambda_6x_4 - \lambda_7x_5)/2 \\
\dot{\lambda}_3 &= -q_{3j}x_j + \lambda_1 I_1 x_2 + \lambda_2 I_2 x_1 + (\lambda_4\beta_3(t_f) - \lambda_5\beta_2(t_f) \\
&\quad +\lambda_6\beta_1(t_f) - \lambda_7\beta_0(t_f) + \lambda_4x_7 - \lambda_5x_6 + \lambda_6x_5 - \lambda_7x_4)/2 \\
\dot{\lambda}_4 &= -q_{4j}x_j - (\lambda_5x_1 + \lambda_6x_2 + \lambda_7x_3)/2 \\
\dot{\lambda}_5 &= -q_{5j}x_j + (\lambda_4x_1 + \lambda_6x_3 - \lambda_7x_2)/2 \\
\dot{\lambda}_6 &= -q_{6j}x_j + (\lambda_4x_2 - \lambda_5x_3 + \lambda_7x_1)/2 \\
\dot{\lambda}_7 &= -q_{7j}x_j + (\lambda_4x_3 + \lambda_5x_2 - \lambda_6x_1)/2
\end{aligned} \qquad (4.21)$$

The costates are assumed to have the form

$$\lambda_i = k_{ij}x_j + d_{ijk}x_jx_k + \ldots \tag{4.22}$$

and by substituting into costate equations above, ordinary differential equations for the gains k_{ij} and d_{ijk} can be found. With the exception of low order systems, however, the algebra to determine these equations is extensive.

4.2 Single Axis Maneuvers

4.2.1 State Equations

For maneuvers that require rotation about only one body axis, say the i-th axis, the system model of section 4.1 reduces to the following equations

$$\begin{aligned}
\dot{\omega}_i &= u/I_i \\
\dot{\beta}_o &= -\frac{1}{2}\beta_i\omega_i \\
\dot{\beta}_i &= \frac{1}{2}\beta_o\omega_i
\end{aligned} \tag{4.23}$$

and the state equations can be written as

$$\begin{aligned}
\dot{x}_1 &= u/I \\
\dot{x}_2 &= -\frac{1}{2}\{\beta_i(t_f)x_1 + x_1x_3\} \\
\dot{x}_3 &= \frac{1}{2}\{\beta_o(t_f)x_1 + x_1x_2\}
\end{aligned} \tag{4.24}$$

where

$$\underline{x} = \{\omega_i \quad \tilde{\beta}_o \quad \tilde{\beta}_i\}^T \tag{4.25}$$

4.2.2 The Optimal Control Problem

The single axis maneuver may also be parameterized by the angle of rotation ϕ, in which case the system model becomes

$$\ddot{\phi} = u/I \tag{4.26}$$

and the state equations become

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u/I \end{aligned} \tag{4.27}$$

where

$$\underline{x} = \{\phi \quad \dot{\phi}\}^T \tag{4.28}$$

To minimize a performance index

$$J = \frac{1}{2} \int_{t_o}^{t_f} \{q\phi^2 + u^2\}dt \tag{4.29}$$

we form the Hamiltonian

$$H = \frac{1}{2} \{qx_1^2 + u^2\} + \lambda_1 x_2 + \lambda_2 \frac{u}{I} \tag{4.30}$$

and from the necessary conditions for optimality we obtain the costate equations

$$\begin{aligned} \dot{\lambda}_1 &= - q \\ \dot{\lambda}_2 &= - \lambda_1 \end{aligned} \tag{4.31}$$

and

$$u = - \lambda_2/I \tag{4.32}$$

Parameterization by angle of rotation results in a linear system of equations, Eqs. (4.27), so that the optimal feedback control law is also linear. Only two gains would be determined, for feedback on ϕ and $\dot{\phi}$, so that in practice this choice of variables would be used for single axis maneuvers. To examine convergence of the polynomial feedback control law, however, we will use Euler parameter kinematics.

Given state equations Eqs. (4.23), minimize the following performance index

$$J = \frac{1}{2}\int_{t_0}^{t_f} \{\underline{x}^T Q \underline{x} + \underline{u}^T \underline{u}\} dt \tag{4.33}$$

where the $\underline{x}$ above is given by Eq. (4.25). The costate equations are

$$\begin{aligned}
\dot{\lambda}_1 &= - q_{1j}x_j + (\lambda_2\beta_i(t_f) - \lambda_3\beta_o(t_f) + \lambda_2 x_3 - \lambda_3 x_2)/2 \\
\dot{\lambda}_2 &= - q_{2j}x_j - \lambda_3 x_1/2 \\
\dot{\lambda}_3 &= - q_{3j}x_j + \lambda_2 x_1/2
\end{aligned} \tag{4.34}$$

and the control is

$$u = - \lambda_1/I \tag{4.35}$$

The costates are assumed to include linear through quartic terms in the states

$$\lambda_i = k_{ij}x_j + d_{ijk}x_j x_k + f_{ijk\ell}x_j x_k x_\ell + h_{ijk\ell m}x_j x_k x_\ell x_m \tag{4.36}$$

4.2.3 Simulation Results

Every orientation of the body has two corresponding points in Euler parameter space, either plus or minus Eq. (4.3). Although the kinematic equations, Eq. (4.8), do not change with the choice of sign, the feedback control terms involving odd powers of the Euler parameters will change sign. Hence there are four possible combinations of boundary conditions corresponding to one physical maneuver, two for the initial conditions and two for the final conditions. The Euler parameters are points on a four-dimensional unit sphere, and the four combinations will give four paths on the surface of this sphere (see Fig. 4.1). The proper combination to choose is the one with the lowest value for the performance index.

Two cases were studied for a single-axis rotation about the yaw axis. The state penalty weights for both cases was

$$Q = \begin{bmatrix} .5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & .5 \end{bmatrix} \qquad (4.37)$$

and the moment of inertia was $I_1 = 1.00\ \mathrm{kg.m^2}$.

Both cases corresponded to a 90° rotation about the one-axis with initial and final Euler parameter states and angles given in Table 4.1. Steady-state values determined by Potter's method for the linear gains, and solution of the linear algebraic equations for quadratic, cubic and quartic gains, are given in Table 4.2, where the final states were plus β_0 and β_1.

Cases 4A and 4B

A "rest-to-rest" maneuver with a 90° rotation about the one axis is considered, corresponding to zero initial and final angular velocity. Fig. 4.2, case 4A, shows the response using linear, quadratic and cubic feedback control and positive initial and final Euler parameter values

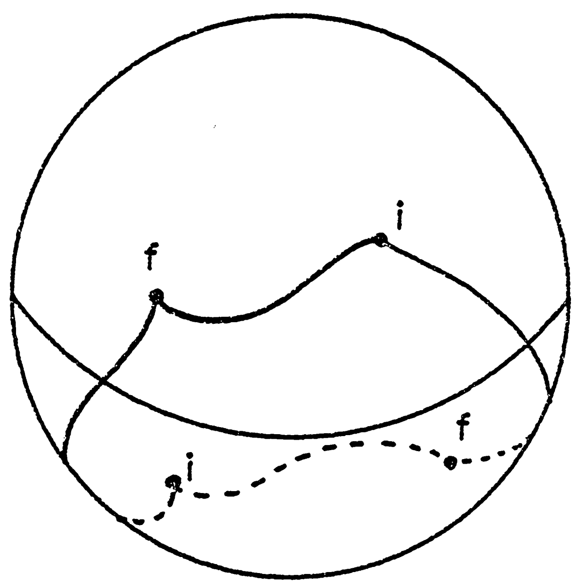


Fig. 4.1: Initial and final Euler parameter boundary conditions.

Table 4.1

BOUNDARY CONDITIONS FOR SINGLE AXIS MANEUVERS

	Initial State	Final State
β_0	1 or -1	0.707107 or -0.707107
β_1	0 or 0	0.707107 or -0.707107
ϕ	0	$\pi/2$

Table 4.2

SINGLE AXIS GAINS

Linear		Quadratic ($x10^{-2}$)	
k_{11}	1.000	d_{111}	-5.556
k_{12}	0	d_{112}	35.355
k_{13}	0.707	d_{113}	3.928
		d_{122}	0
		d_{123}	0
		d_{133}	0

Cubic ($x10^{-2}$)				Quartic ($x10^{-2}$)			
f_{1111}	-1.321	f_{1233}	0	h_{11111}	-0.226	h_{11223}	4.253
f_{1112}	4.365	f_{1333}	0	h_{11112}	0.970	h_{11233}	-3.430
f_{1113}	-5.538			h_{11113}	-0.752	h_{11333}	-1.492
f_{1122}	-6.250			h_{11122}	-3.429	h_{12222}	0
f_{1123}	-4.475			h_{11123}	3.180	h_{12223}	0
f_{1133}	3.839			h_{11133}	1.896	h_{12233}	0
f_{1222}	0			h_{11222}	2.210	h_{12333}	0
						h_{13333}	0

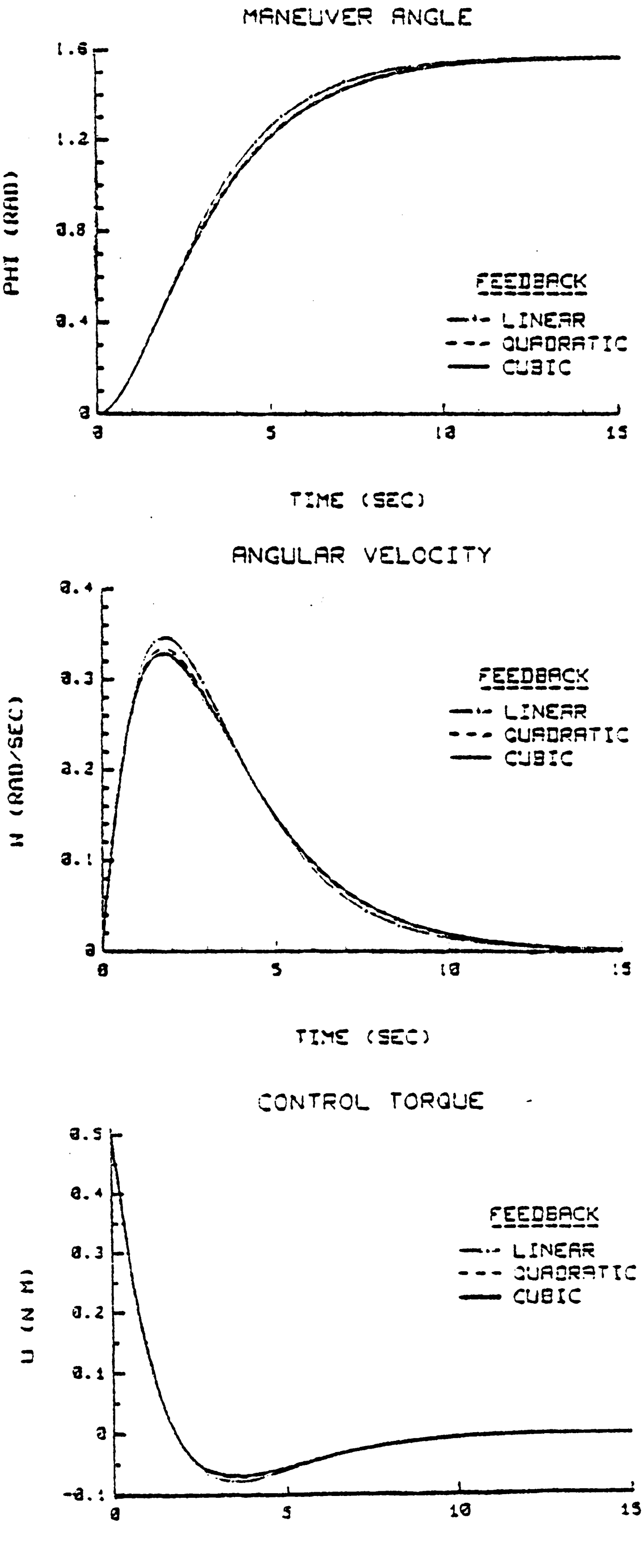


Fig. 4.2: Case 4A single-axis rest-to-rest maneuver.

(the quartic gains resulted in curves undistinguishable from the cubic curves and so were not plotted). The same response is obtained when negative initial and final Euler parameter values are used. By using positive initial conditions and negative final conditions, or vice versa, the response in Fig. 4.3, case 4B, is obtained. Note that the final angle obtained is $5\pi/2$ in Fig. 4.3. The maneuver was accomplished in about 15 seconds. The performance indices for the maneuvers in Figs. 4.2 and 4.3 are tabulated in Table 4.3.

Case 4C

A "spin-down" maneuver corresponding to a reorientation about the one axis through 90° is performed, with an initial angular velocity of .5 rad/sec and zero final angular velocity. Fig. 4.4 shows the response to linear, quadratic and cubic feedback control with positive initial and final Euler parameters, where quartic gains again give no significant improvement from cubic gains. The performance indices for this maneuver, and the maneuver involving initial and final Euler parameters of opposite sign, are given in Table 4.4.

Discussion

Although there are four possible combinations for Euler parameter boundary conditions, there are only two different control histories, as seen in Figs. 4.2 and 4.3. The maneuver performed in Fig. 4.2 reaches the desired final angle of $\pi/2$, while that in Fig. 4.3 must go through an extra rotation to end at $5\pi/2$. Furthermore, adding more feedback terms to the maneuver with the extra rotation increases the performance index, so we can see that the optimum maneuver will not be

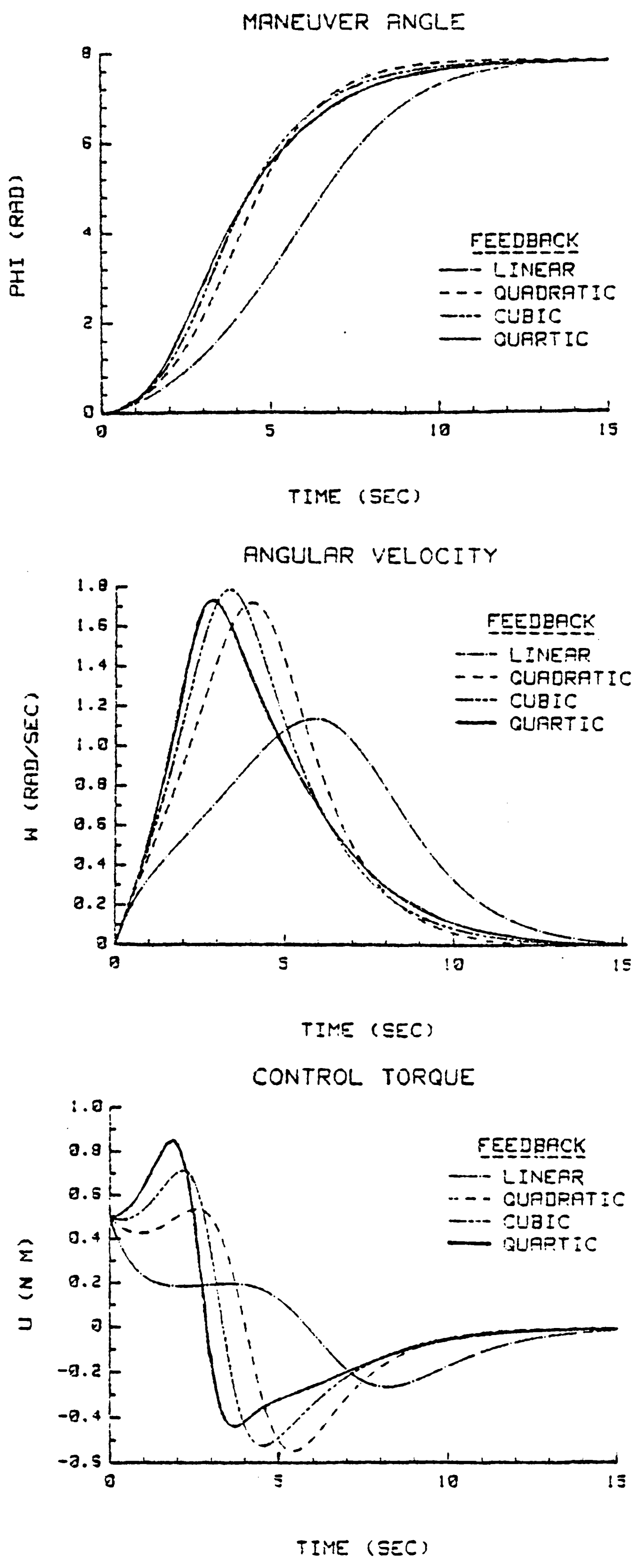


Fig. 4.3: Case 4B Single-axis rest-to-rest maneuver.

Table 4.3

PERFORMANCE INDICES FOR CASES 4A AND 4B

Feedback Gains	J_1	J_2
linear	0.396605	5.22423
quadratic	0.395830	5.35058
cubic	0.395784	5.39222
quartic	0.395783	5.38487

Table 4:4

PERFORMANCE INDICES FOR CASE 4C

Feedback Gains	J_1	J_2
linear	0.287457	5.03120
quadratic	0.286447	5.15163
cubic	0.286297	5.22255
quartic	0.286296	5.24521

J_1 corresponds to initial and final Euler parameters of the same sign.

J_2 corresponds to initial and final Euler parameters of opposite sign.

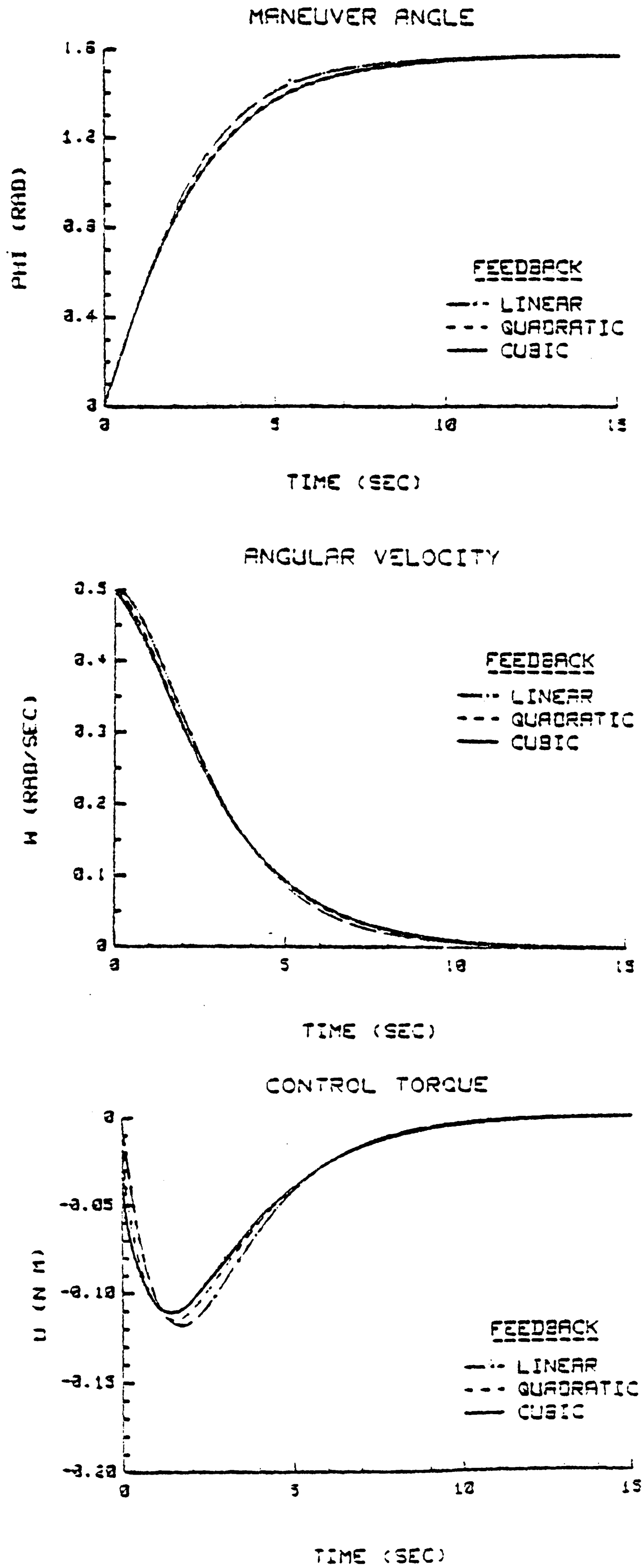


Fig. 4.4: Case 4C single-axis spin-down maneuver.

obtained from these boundary conditions. Note that the linear feedback performance index is much lower for the first maneuver, and hence a linear simulation will indicate the correct Euler parameter boundary conditions for convergence. It appears that cubic and quartic gains are unnecessary for these maneuvers; linear feedback performs the maneuver, with a slight improvement in the performance index for quadratic gains.

Although steady-state values for the gains were used in these examples, Case 4A was also executed using transient linear gains and the resulting transient quadratic gains determined by numerical integration. This simulation showed no significant deviation from the response using steady state values, which is expected in the case of linear feedback for a linear system with constant coefficients.

Several state weighting matrices were also used in Case 4A. For very small state weights, the problem became singular, as expected, and for larger weights, the linear, quadratic, cubic and quartic controls all gave the same response, indicating the linear control was very close to optimal.

An identity state weight matrix was also tried, which is harder to solve with Potter's method. This problem arises because the linear part of the state equations is not completely controllable, due to the redundancy of the Euler parameters, and so the system has a zero eigenvalue corresponding to β_0. When there are no state weights on β_0, the linear gains on β_0 are zero and the eigenvector associated with the zero eigenvalue is not needed; when the performance index requires the error in β_0 to be minimized, the eigenvector corresponding to the zero eigenvalue must be determined accurately enough to find nonzero linear gains. For the case of weights on ω and β_o and no penalty on β_1, gains were found that would drive the system to either $-\phi_f$ or $+\phi_f$. This problem is due to the evenness (and associated quadrant ambiguity) of the cosine function defining β_o.

4.3 Three-Axis Maneuvers

4.3.1 State Equations and Optimal Control Problem

The state equations for three-axis maneuvers are given in Eqs. (4.18), and the costate equations are given in Eqs. (4.21). For these maneuvers, linear plus quadratic feedback was considered, so that the costates were of the following form

$$\lambda_i = k_{ij}x_j + d_{ijk}x_jx_k \qquad (i = 1,\ldots,7) \tag{4.38}$$

4.3.2 Simulation Results

Two cases were studied for rotations involving all three axes. The state penalty weights in both cases was

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.39}$$

Table 4.5 contains the mass properties of the spacecraft. The algebra required to derive the gain equations for both the single-axis and three-axis maneuvers was performed on the IBM 370 algebraic manipulator

Table 4.5

SPACECRAFT INERTIA

Axis	Moments of Inertia (kg · m^2)
yaw	1.00
pitch	0.83
roll	0.92

Table 4.6

CASES 4D AND 4E BOUNDARY CONDITIONS

	Initial State			Final State		
ϕ	0			$\pi/2$		
θ	0			$\pi/3$		
ψ	0			$\pi/4$		
β_0	1		-1	-0.33141		0.33141
β_1	0	or	0	-0.46194	or	0.46194
β_2	0		0	-0.19134		0.19134
β_3	0		0	-0.80010		0.80010
ω_1	.01			0		
ω_2	0			0		
ω_3	0			0		

FORMAC. The initial and final Euler parameter states, angular velocities and the corresponding 3-1-3 angles are given in Tables 4.6 and 4.7. The linear gains found by Potter's method for positive final Euler parameters are given in Table 4.8, and the corresponding quadratic gains are given in Table 4.9.

Cases 4D and 4E

These cases involve a large angle nonlinear maneuver from pure spin about the one axis to zero angular velocity. The angular velocities, Euler parameters, and control torques are given in Figs. 4.5 and 4.6 for positive initial and final boundary conditions, Case 4D. Case 4E, the same maneuver but with negative initial boundary conditions, is shown in Figs. 4.7 and 4.8. The performance indices for this maneuver are given in Table 4.10.

Case 4F

Case 4F requires a large angle maneuver from tumbling to rest. Figs. 4.9 and 4.10 show the responses to the linear and quadratic gains. The performance indices for this maneuver are given in Table 4.11.

Discussion

The four possible Euler parameter boundary conditions result in two stable maneuvers determined by positive final Euler parameters, and two unstable maneuvers involving negative final Euler parameters. The importance of using the correct set of Euler parameter boundary conditions is demonstrated in Figs. 4.5 and 4.7. The angular velocities and control torques are much larger when negative initial boundary

Table 4.7

CASE 4F BOUNDARY CONDITIONS

	Initial State	Final State
ϕ	$-\pi/2$	$\pi/2$
θ	$-\pi/3$	$\pi/3$
ψ	$-\pi/4$	$\pi/4$
β_o	-0.33141	0.33141
β_1	0.46194	0.46194
β_2	-0.19134	0.19134
β_3	0.80010	0.80010
ω_1	-.5	0
ω_2	.3	0
ω_3	.1	0

Table 4.8

LINEAR GAINS FOR THREE-AXIS MANEUVERS

k_{11}	1.3509	k_{21}	-0.0217	k_{31}	-0.1007
k_{12}	-0.0217	k_{22}	1.1153	k_{32}	-0.0346
k_{13}	-0.1007	k_{23}	-0.0346	k_{33}	1.1144
k_{14}	0.0	k_{24}	0.0	k_{34}	0.0
k_{15}	0.4923	k_{25}	-0.6116	k_{35}	0.4262
k_{16}	0.8654	k_{26}	0.2963	k_{36}	-0.3235
k_{17}	0.0937	k_{27}	0.4764	k_{37}	0.7484

Table 4.9

QUADRATIC GAINS FOR THREE-AXIS MANEUVERS

d_{111}	-.0202	d_{211}	-.0062	d_{311}	-.0179
d_{112}	-.0123	d_{212}	-.0094	d_{312}	-.0107
d_{113}	-.0358	d_{213}	-.0107	d_{313}	-.0413
d_{114}	.1952	d_{214}	.0259	d_{314}	.1225
d_{115}	-.1191	d_{215}	-.0349	d_{315}	-.1213
d_{116}	.0967	d_{216}	-.0998	d_{316}	.0435
d_{117}	.2487	d_{217}	.0506	d_{317}	.0018
d_{122}	-.0469	d_{222}	-.0035	d_{322}	-.0061
d_{123}	-.0110	d_{223}	-.0122	d_{323}	-.0130
d_{124}	.0259	d_{224}	.0930	d_{324}	.0408
d_{125}	-.0349	d_{225}	.0652	d_{325}	.0025
d_{126}	-.0998	d_{226}	-.0421	d_{326}	-.1700
d_{127}	.0506	d_{227}	.1355	d_{327}	.0181
d_{133}	-.0207	d_{233}	-.0065	d_{333}	-.0444
d_{134}	.1225	d_{234}	.0408	d_{334}	.2935
d_{135}	-.1213	d_{235}	.0025	d_{335}	.2195
d_{136}	.0435	d_{236}	-.1700	d_{336}	.0409
d_{137}	.0018	d_{237}	.0181	d_{337}	-.0380
d_{144}	0.0	d_{244}	0.0	d_{344}	0.0
d_{145}	.6737	d_{245}	.1494	d_{345}	-.3952
d_{146}	-.3491	d_{246}	.7115	d_{346}	.0104
d_{147}	-.3154	d_{247}	-.2510	d_{347}	.2296
d_{155}	-.3459	d_{255}	.2291	d_{355}	.0658
d_{156}	-.2622	d_{256}	.2369	d_{356}	-.0662
d_{157}	.1357	d_{257}	.1045	d_{357}	-.4429
d_{166}	.0896	d_{266}	.1452	d_{366}	.3664
d_{167}	-.5572	d_{267}	.2272	d_{367}	.1263
d_{177}	-.3651	d_{277}	-.2517	d_{377}	.2007

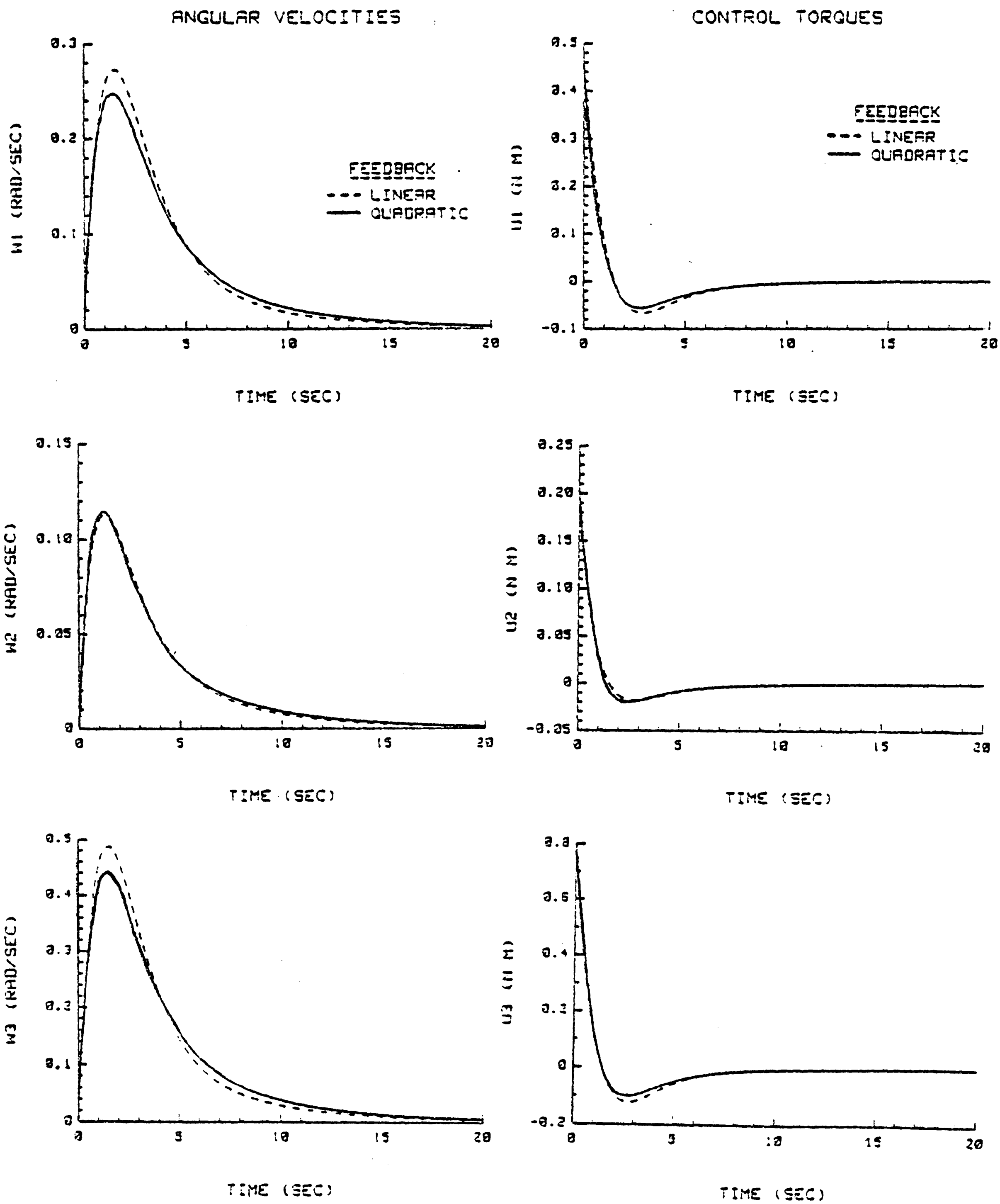


Fig. 4.5: Case 4D Three-axis maneuver.

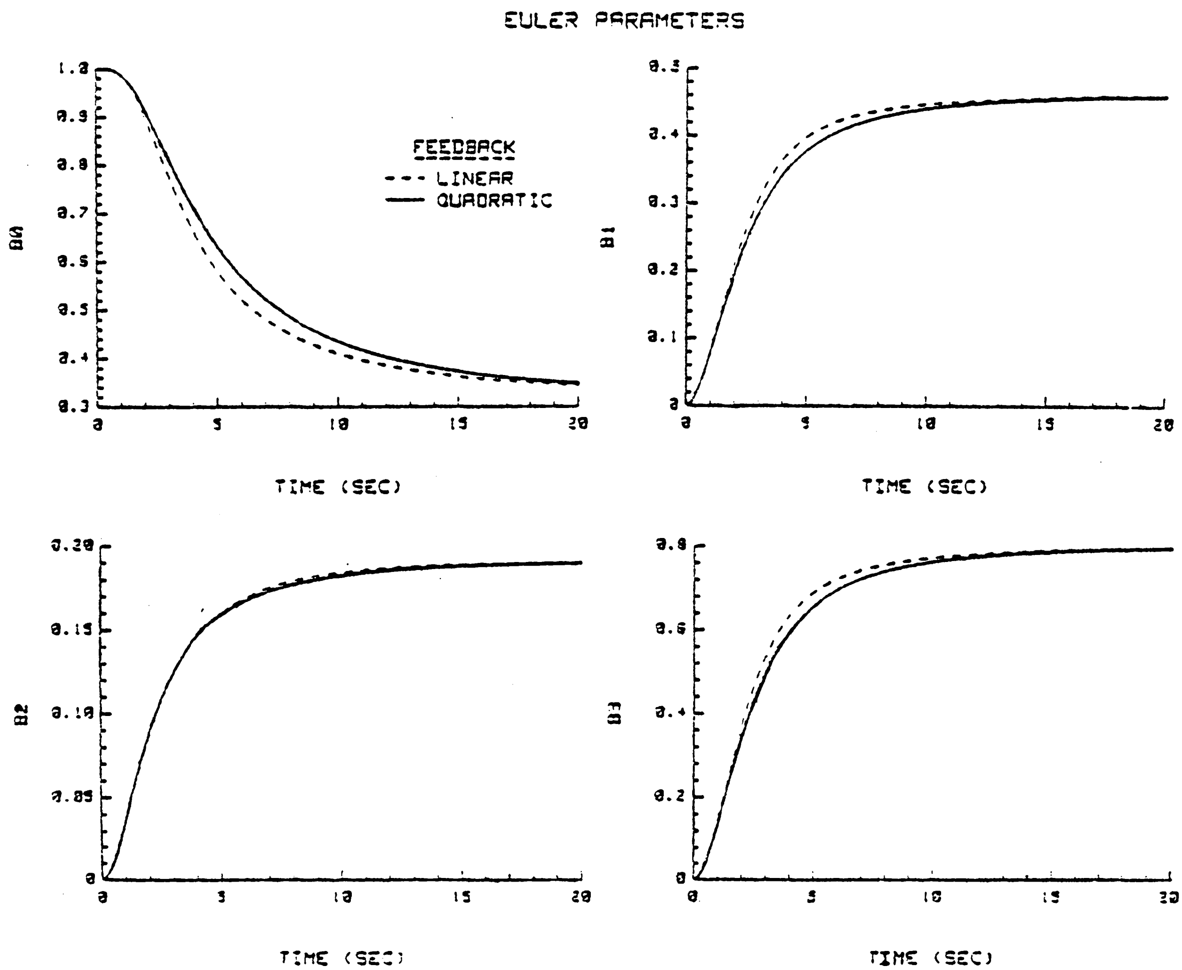


Fig. 4.6: Case 4D Three-axis maneuver.

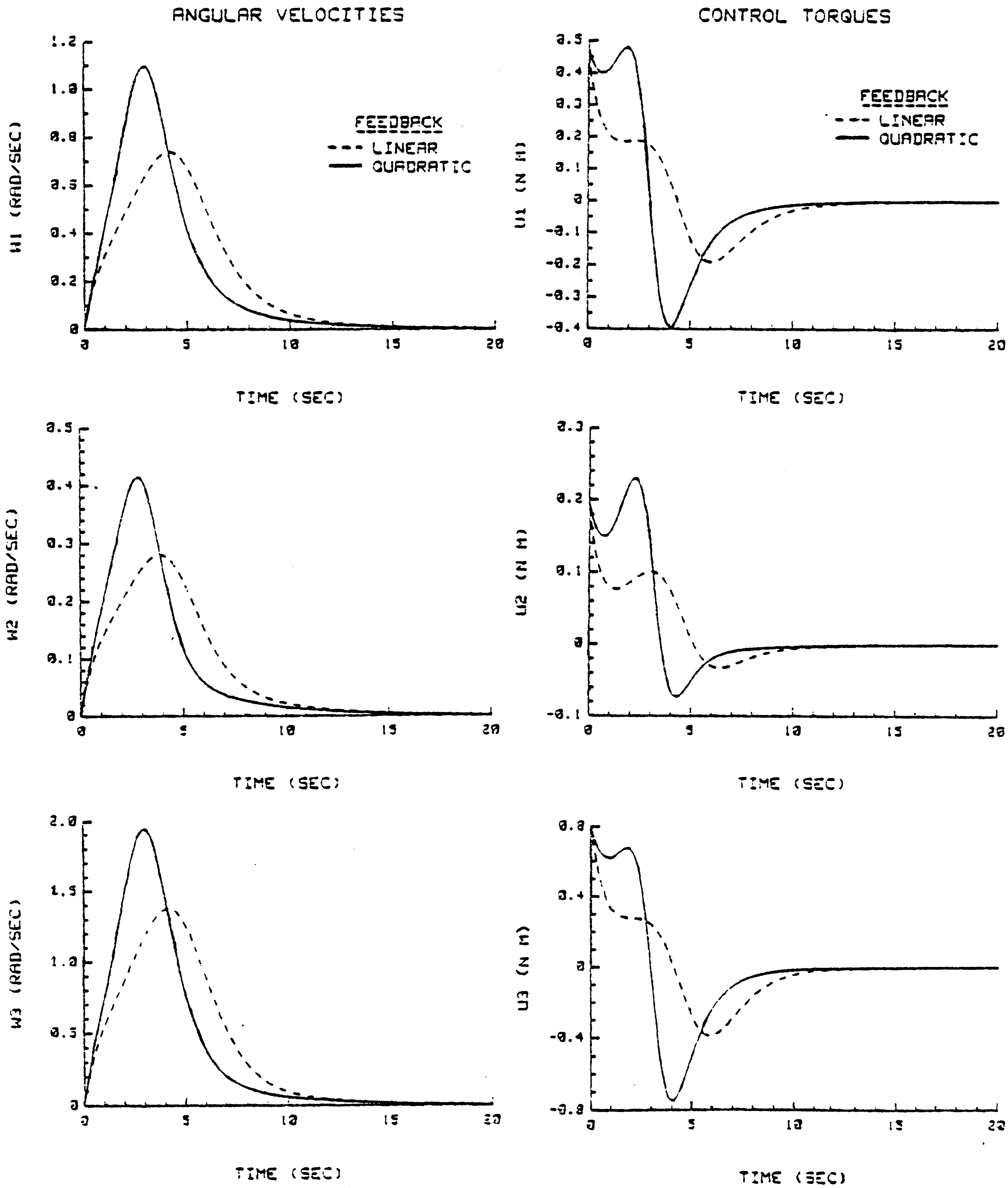


Fig. 4.7: Case 4E Three-axis maneuver.

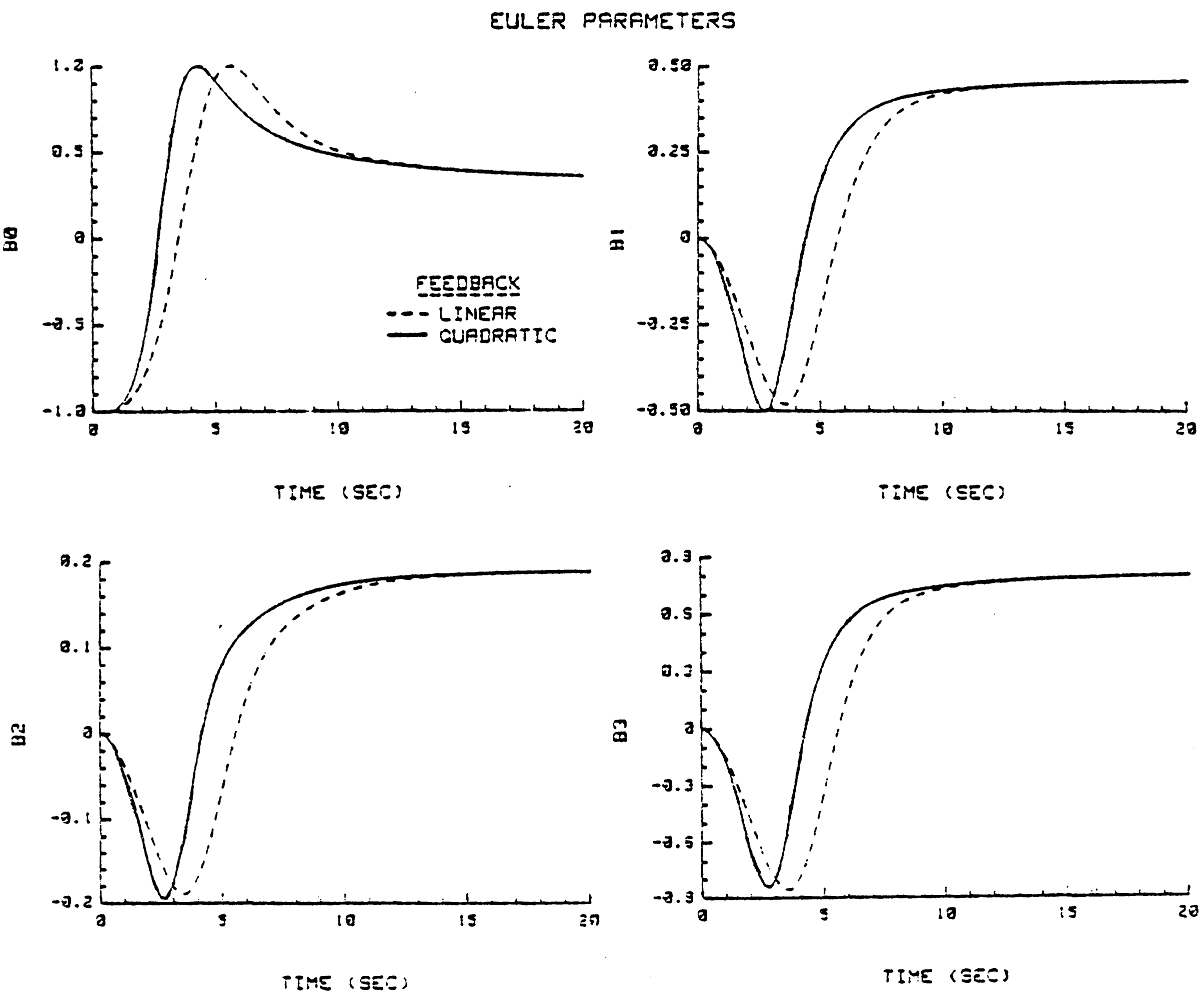


Fig. 4.8: Case 4E Three-axis maneuver.

Table 4.10

PERFORMANCE INDICES FOR CASES 4D AND 4E

Feedback Gains	J_1	J_2
linear	1.32873	12.0971
quadratic	1.31442	12.7792

J_1 corresponds to positive initial and final conditions.

J_2 corresponds to negative initial conditions and postive final conditions.

Table 4.11

PERFORMANCE INDICES FOR CASE 4F

Feedback Gains	J_1	J_2
linear	0.80214	6.27908
quadratic	0.67243	6.81439

J_1 corresponds to boundary conditions given in Table 4.7.

J_2 corresponds to negative initial conditions and positive final conditions.

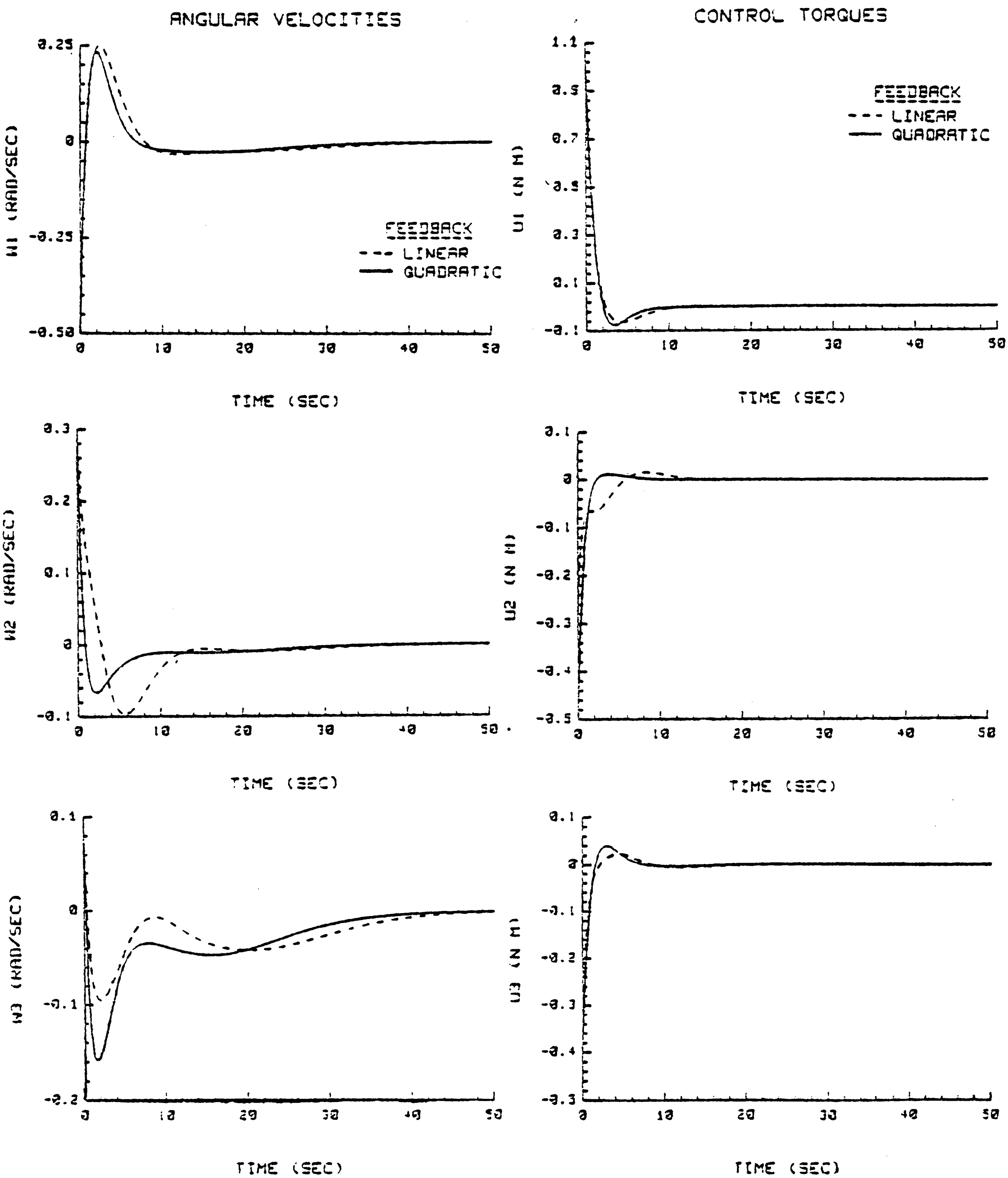


Fig. 4.9: Case 4F Three-axis maneuver.

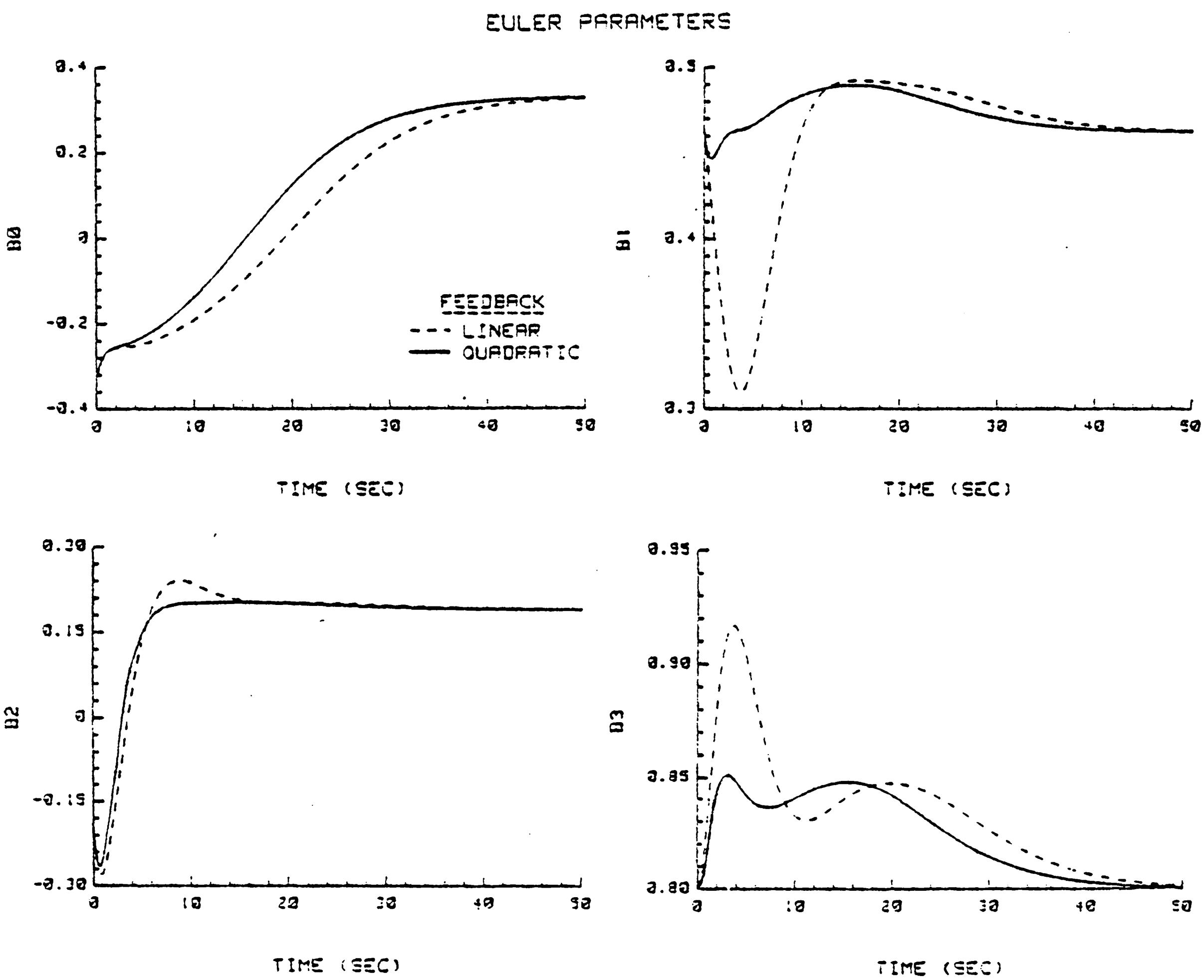


Fig. 4.10: Case 4F Three-axis maneuver.

conditions are used, and the addition of quadratic feedback terms produces even larger angular velocities and control torques. As we observed in the single-axis maneuvers, the linear feedback performance index indicates the proper boundary conditions to obtain the optimal maneuver.

Unstable behavior is observed when negative final Euler parameter boundary conditions were used. This instability occurs in the linear part of the system; some of the closed-loop poles are in the right-half plane. Hence an eigenvalue analysis will eliminate some boundary condition combinations, and a linear feedback simulation will indicate which of the remaining combinations give the minimum performance index. These boundary conditions will then result in higher order feedback that approaches the optimal solution.

4.4 Conclusions

The nonlinear large angle maneuver control problem has been solved in feedback form for both single axis and three-axis maneuvers. The polynomial feedback controls have been found to converge with relatively low order (linear in some cases) by formulating the kinematics in terms of Euler parameters. Although there are four combinations of Euler parameter boundary conditions, a linear analysis will indicate the correct combination for convergence to a minimum. The other solutions may be saddle points or maxima on the performance index manifold; they appear because Pontryagin's Principle generates only necessary conditions for optimality.

Unfortunately, the smooth external control torques determined in this chapter are not physically realizable in practice. Most external torquing mechanisms do not provide continuous torques; both rockets and magnets usually operate in a "bang-bang" mode, so that they are either on at a fixed value or off. Continuous control torques can be provided by internal means, however, and will be considered in the next chapter.

Chapter V

INTERNAL TORQUE MANEUVERS

5.1 System Model and Optimal Control Problem

Attitude control is considered for a five-body configuration consisting of an asymmetric spacecraft and four reaction wheels. We restrict our attention to the momentum transfer class of internal control torques that provides a reduction in the number of feedback states.

5.1.1 Spacecraft Orientation

A spacecraft body-fixed reference frame $\{\hat{\underline{b}}\}$ is related to an inertial frame $\{\hat{\underline{n}}\}$ by the direction cosine matrix $[C(\beta)]$,

$$\{\hat{\underline{b}}\} = [C(\beta)]\{\hat{\underline{n}}\} \tag{5.1}$$

where $[C(\beta)]$ is defined in Eq. (4.5) by the four Euler parameters $\beta_o,\beta_1,\beta_2,\beta_3$ of Eq. (4.3). These attitude variables are related to the body-frame components of the spacecraft angular velocity $\underline{\omega}$ by the kinematic differential equations of Eq. (4.8), which are repeated here for reference.

$$\begin{aligned}
\dot{\beta}_o &= -\frac{1}{2}(\beta_1\omega_1 + \beta_2\omega_2 + \beta_3\omega_3) \\
\dot{\beta}_1 &= \frac{1}{2}(\beta_o\omega_1 - \beta_3\omega_2 + \beta_2\omega_3) \\
\dot{\beta}_2 &= \frac{1}{2}(\beta_3\omega_1 + \beta_o\omega_2 - \beta_1\omega_3) \\
\dot{\beta}_3 &= -\frac{1}{2}(\beta_2\omega_1 - \beta_1\omega_2 - \beta_o\omega_3)
\end{aligned} \tag{5.2}$$

In addition to using an arbitrary, general inertial frame $\{\hat{\underline{n}}\}$, we introduce a special inertial angular momentum frame $\{\hat{\underline{h}}\}$, where $\hat{\underline{h}}_2$ is aligned with the system angular momentum $\underline{H}$, as Kraige, Vadali, and Junkins [29], [30] have discussed. The other two unit vectors $(\hat{\underline{h}}_1, \hat{\underline{h}}_3)$ are defined as the directions $\{\hat{\underline{n}}_1\}$ and $\{\hat{\underline{n}}_3\}$ assume after $\{\hat{\underline{n}}_2\}$ is rotated to coincide with $\underline{H}$ (see Fig. 5.1). The $\{\hat{h}\}$ reference frame can be considered inertial if the external torques are negligible during the maneuver and only internal torques are present. As is shown below, introducing this frame allows us to make use of the angular momentum integral to reduce the number of state variables.

The orientation of $\{\hat{\underline{b}}\}$ with respect to the momentum frame $\{\hat{\underline{h}}\}$ is given by the projection

$$\{\hat{\underline{b}}\} = [C(\delta)]\{\hat{\underline{h}}\} \tag{5.3}$$

where the 3x3 direction cosine matrix [C] is a function of four variable Euler parameters $(\delta_0, \delta_1, \delta_2, \delta_3)$. The inertial frame $\{\hat{\underline{n}}\}$ is projected onto $\{\hat{\underline{h}}\}$ by

$$\{\hat{\underline{n}}\} = [C(\alpha)]\{\hat{\underline{h}}\} \quad . \tag{5.4}$$

where α_0, α_1, α_2, α_3 are constant Euler parameters since both $\{\hat{\underline{n}}\}$ and $\{\hat{\underline{h}}\}$ are inertial. Using the inertial frame $\{\hat{\underline{n}}\}$ components of the system angular momentum

$$\underline{H} = H_{n1}\hat{\underline{n}}_1 + H_{n2}\hat{\underline{n}}_2 + H_{n3}\hat{\underline{n}}_3 \tag{5.5}$$

the constant α_i Euler parameters can be defined

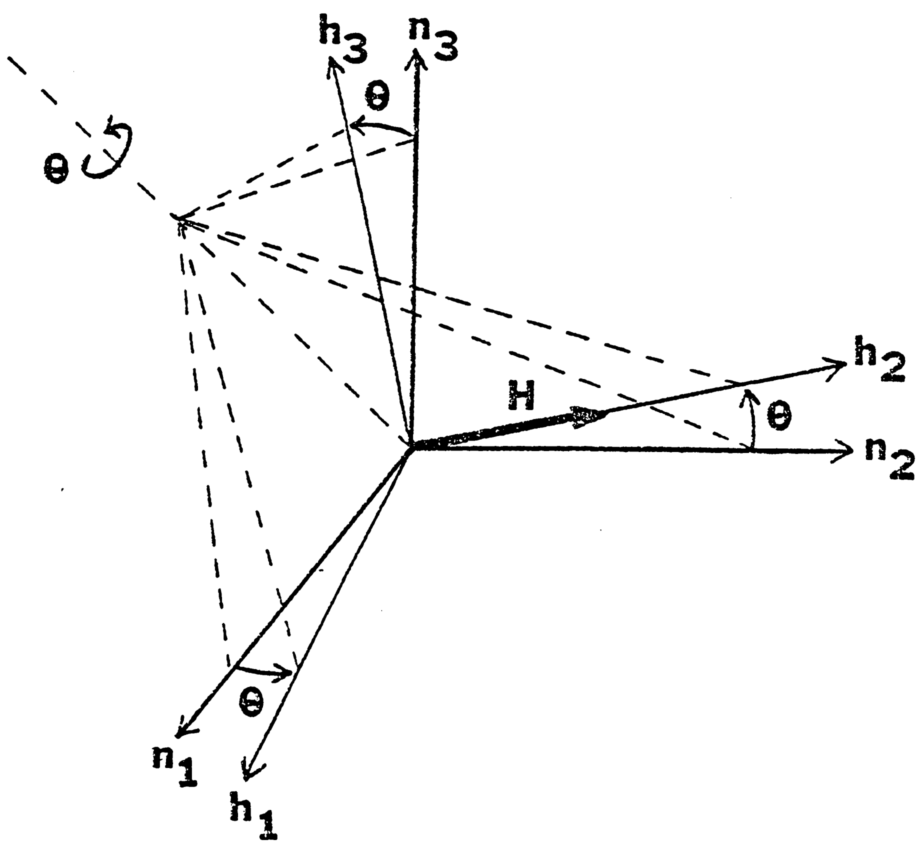


Fig. 5.1: Angular momentum inertial frame of reference.

$$\alpha_0 = \left[\frac{H + H_{n2}}{2H}\right]^{1/2}$$

$$\alpha_1 = -H_{n3}\left[\frac{H - H_{n2}}{2H(H_{n1}^2 + H_{n3}^2)}\right]^{1/2} \qquad (5.6)$$

$$\alpha_2 = 0$$

$$\alpha_3 = H_{n1}\left[\frac{H - H_{n2}}{2H(H_{n1}^2 + H_{n3}^2)}\right]^{1/2}$$

The δ_i Euler parameters are then related to the β_i parameters by the bilinear, orthogonal equation

$$\begin{Bmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{bmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix} \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} \qquad (5.7)$$

and are related to the body-frame components of $\underline{\omega}$ by the differential equations

$$\begin{aligned}
\dot{\delta}_0 &= -\frac{1}{2}(\delta_1\omega_1 + \delta_2\omega_2 + \delta_3\omega_3) \\
\dot{\delta}_1 &= \frac{1}{2}(\delta_0\omega_1 - \delta_3\omega_2 + \delta_2\omega_3) \\
\dot{\delta}_2 &= \frac{1}{2}(\delta_3\omega_1 + \delta_0\omega_2 - \delta_1\omega_3) \\
\dot{\delta}_3 &= -\frac{1}{2}(\delta_2\omega_1 - \delta_1\omega_2 - \delta_0\omega_3)
\end{aligned} \qquad (5.8)$$

We can show from Eq. (5.3) and using the algebraic expressions for $[C(\delta)]$ from Eq. (4.5) where δ's are used instead of β's, that

$$\underline{\hat{h}}_2 = 2(\delta_1\delta_2 + \delta_0\delta_3)\underline{\hat{b}}_1 + (\delta_0^2 - \delta_1^2 + \delta_2^2 - \delta_3^2)\underline{\hat{b}}_2 + 2(\delta_2\delta_3 - \delta_0\delta_1)\underline{\hat{b}}_3 , \tag{5.9}$$

so the body-frame components of the system angular momentum can now be written from $\underline{H} = H\underline{\hat{h}}_2$ as

$$\begin{aligned} H_1 &= 2(\delta_1\delta_2 + \delta_0\delta_3)H \\ H_2 &= (\delta_0^2 - \delta_1^2 + \delta_2^2 - \delta_3^2)H \\ H_3 &= 2(\delta_2\delta_3 - \delta_0\delta_1)H \end{aligned} \tag{5.10}$$

Thus we have an explicit relationship to eliminate the H_i in terms of the δ_i for the equations of motion below.

5.1.2 Spacecraft and Reaction Wheel Dynamics

An arbitrary asymmetric spacecraft with four reaction wheels in NASA standard configuration [31], [32] is considered (see Fig. 5.2). Three wheels are aligned along the orthogonal body-frame axes, and the fourth skewed wheel is aligned so that its axis is at equal angles to the body axes. Normal operations will use only the three orthogonal wheels for attitude control; the fourth wheel will be activated in the event of motor failure for one of the primary control wheels.

The system angular momentum $\underline{H}$ is the sum of the spacecraft and wheel angular momenta; the body components of $\underline{H}$ and $\underline{\omega}$ are related by

$$\underline{H} = [I^*]\underline{\omega} + [\tilde{C}]^T[J]\underline{\Omega} \tag{5.11}$$

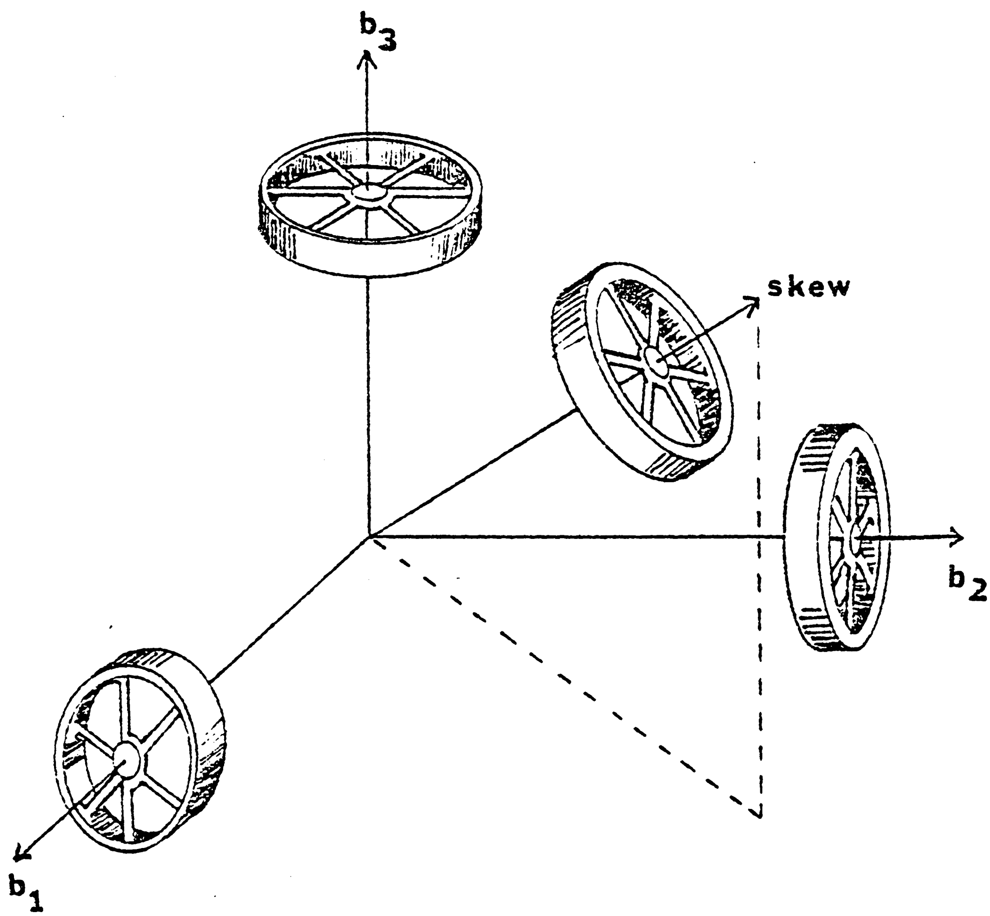


Fig. 5.2: NASA Standard four-reaction-wheel attitude control system.

where [I*] is the system inertia matrix with respect to the body frame $\{\hat{\underline{b}}\}$, $\underline{\Omega}$ is a vector of the four wheel angular velocities and [J] the wheel axial moment of inertia matrix defined by $[J] = \text{diag}\,\{J_{ai}\}$, $i = 1,2,3,4$. $[\tilde{C}]$ is a 4x3 matrix whose rows are the three orthogonal body-frame components of four unit vectors along the wheel spin axes.

Assuming negligible external torques, the time rate of change of the angular momentum is zero, and thus we obtain the Eulerian equation of motion

$$\dot{\underline{H}} = [I^*]\dot{\underline{\omega}} + [\tilde{C}]^T[J]\dot{\underline{\Omega}} + [\tilde{\omega}]\underline{H} = 0 \tag{5.12}$$

where

$$[\tilde{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{5.13}$$

The reaction wheel equations of motion are obtained by considering the angular momentum of each wheel about its center of mass, and equating its time rate of change to the torque applied to that wheel. The four wheel equations are

$$[J]\dot{\underline{\Omega}} + [J][\tilde{C}]\dot{\underline{\omega}} = \underline{u} \tag{5.14}$$

where the four components of the control vector $\underline{u}$ are the axial torques applied by the motors to their respective wheels. To eliminate the wheel angular velocities $\underline{\Omega}$ from the spacecraft equations of motion, Eq. (5.14) is multiplied by $[\tilde{C}]^T$ and then substituted into Eq. (5.12) to obtain

$$\underline{\dot{\omega}} = - [G]\{[\tilde{\omega}]\underline{H} + [\tilde{C}]^T\underline{u}\} \tag{5.15}$$

where $[G] = [I^* - \tilde{C}^T J\tilde{C}]^{-1}$, a constant matrix. Implicit in Eq. (5.15) is the substitution of Eqs. (5.10); thus $\underline{\dot{\omega}}$ is just a function of $\underline{\omega}$, $\underline{\delta}$, and $\underline{u}$. It is interesting to note that rest-to-rest maneuvers (characterized by $\underline{H} = 0$) remove all the gyroscopic terms in Eq. (5.15). Hence it is evident that for this class of maneuvers, angular velocity control is near trivial and attitude control is nonlinear only because of kinematic nonlinearities.

The three equations of motion in Eq. (5.15) and the four attitude equations in Eq. (5.8) will be used to determine the state equations. Note that if the external torques were not negligible, the system angular momentum would not be constant and the momentum frame $\{\underline{\hat{h}}\}$ would not be inertial. The system equations would then include the three spacecraft equations of Eq. (5.12), the four wheel equations of Eq. (5.14), and the four Euler parameter equations of Eq. (5.2).

5.1.3 State Equations

To obtain state equations of the form

$$\underline{\dot{x}} = A\underline{x} + \underline{F}(\underline{x}) + B\underline{u} \tag{5.16}$$

in which A is a constant coefficient matrix and $\underline{F}(\underline{x})$ is a vector function containing the nonlinear terms, let the state variables be the spacecraft angular velocities ω_i and the Euler parameter differences $\tilde{\delta}_i$; thus the seven element state vector is

$$\underline{x} = \{\omega_1 \;\; \omega_2 \;\; \omega_3 \;\; \tilde{\delta}_0 \;\; \tilde{\delta}_1 \;\; \tilde{\delta}_2 \;\; \tilde{\delta}_3\}^T \tag{5.17}$$

where

$$\tilde{\delta}_i = \delta_i - \delta_i(t_f) \qquad (i = 0,1,2,3) \tag{5.18}$$

These new state variables introduce linear terms into the dynamic and kinematic equations, Eq. (5.15) and Eq. (5.8), respectively.

The elements of the A matrix for the linear part of the state equations are found to be

$$\begin{aligned}
a_{11} &= g_{12}H_3^o - g_{13}H_2^o & a_{12} &= g_{13}H_1^o - g_{11}H_3^o \\
a_{13} &= g_{11}H_2^o - g_{12}H_1^o & a_{21} &= g_{22}H_3^o - g_{23}H_2^o \\
a_{22} &= g_{23}H_1^o - g_{21}H_3^o & a_{23} &= g_{21}H_2^o - g_{22}H_1^o \\
a_{31} &= g_{32}H_3^o - g_{33}H_2^o & a_{32} &= g_{33}H_1^o - g_{31}H_3^o \\
a_{33} &= g_{31}H_2^o - g_{32}H_1^o & a_{41} &= -\ \delta_1(t_f)/2 \\
a_{42} &= -\ \delta_2(t_f)/2 & a_{43} &= -\ \delta_3(t_f)/2 \\
a_{51} &= \delta_0(t_f)/2 & a_{52} &= -\ \delta_3(t_f)/2 \\
a_{53} &= \delta_2(t_f)/2 & a_{61} &= \delta_3(t_f)/2 \\
a_{62} &= \delta_0(t_f)/2 & a_{63} &= -\ \delta_1(t_f)/2 \\
a_{71} &= -\ \delta_2(t_f)/2 & a_{72} &= \delta_1(t_f)/2 \\
a_{73} &= \delta_0(t_f)/2 & & \\
& a_{ij} = 0 \quad (i = 1,\ldots,7\ ; j = 4,\ldots,7) & &
\end{aligned} \tag{5.19}$$

where $\underline{H}^o$ is Eq. (5.10) evaluated at $\delta_i(t_f)$. Notice that the a_{ij} are explicit functions of the specific terminal state $\delta_i(t_f)$ and the magnitude of the system angular momentum H. B is a 7x4 matrix

$$B = \left[\begin{array}{c} -[G][\tilde{C}]^T \\ \hline 0 \end{array}\right] \tag{5.20}$$

and the vector $\underline{F}(\underline{x})$ contains quadratic and cubic terms in x_i.

5.1.4 Performance Indices

Two quadratic performance indices are considered

$$J_1 = \frac{1}{2}\int_{t_o}^{t_f} \{\underline{x}^T Q \underline{x} + \underline{u}^T R \underline{u}\} dt \tag{5.21}$$

and

$$J_2 = \frac{1}{2}\int_{t_o}^{t_f} \{\underline{x}^T Q \underline{x} + \underline{m}^T W \underline{m}\} dt \tag{5.22}$$

where

$$\underline{m} = [\tilde{C}]^T \underline{u} \tag{5.23}$$

is the 3x1 vector sum of the motor torques and where Q, R and W are positive semidefinite weighting matrices. In the developments below, we show that using J_2 has the advantage that an optimal $\underline{m}(t)$ can be determined uniquely and realized by an infinity of wheel torques if four wheels are used. This allows for a versatile control scheme with a built in provision for redundancy. In the numerical examples, we have used R = W = identity matrices and several choices are made for Q.

5.1.5 Feedback Control

For the performance index in Eq. (5.21), the optimal control is

$$\underline{u} = -R^{-1} B_1^T \underline{\lambda} \tag{5.24}$$

where B_1 is the B matrix of Eq. (5.20). The analogous development for the performance index of Eq. (5.22) uses the state equations in the form

$$\dot{\underline{x}} = A\underline{x} + \underline{F}(\underline{x}) + B_2\underline{m} \tag{5.25}$$

where B_2 is a 7x3 matrix

$$B_2 = \begin{bmatrix} -[G] \\ \hline 0 \end{bmatrix} \tag{5.26}$$

The optimal control $\underline{m}$ for J_2 is

$$\underline{m} = -W^{-1}B_2^T\underline{\lambda}$$

and the four wheel torques are obtained by inverting Eq. (5.23). Since $[\tilde{C}]$ is 4x3 rectangular matrix, the solution for $\underline{u}$ is not unique, but we can use a minimum norm criterion to obtain

$$\underline{u}^* = [\tilde{C}]([\tilde{C}]^T[\tilde{C}])^{-1}\underline{m} \tag{5.27}$$

Hence a unique optimal solution for $\underline{m}(t)$ is determined, but there are an infinite number of other wheel torque strategies that may be implemented, all constrained by Eq. (5.23). The numerical examples in this chapter do not use the minimum norm solution of Eq. (5.27); performance index J_1 is used for the four wheel case, and J_2 used for three wheels. In the latter case the 3x3 nonzero submatrix of $[\tilde{C}]$ is used in Eq. (5.23) to determine a unique control vector $\underline{u}$ for the three active wheels.

5.2 Numerical Examples

Several examples are considered for an asymmetric spacecraft with four reaction wheels as shown in Fig. 5.2. The moments of inertia of the spacecraft without the wheels and the wheel axial moment of inertia are given in Table 5.1; the inertia matrix [I*] and wheel geometry matrix $[\tilde{C}]$ are given in Table 5.2. The mass of the spacecraft is 500 kg, the mass of each wheel is 5 kg, and the distance from the center to

Table 5.1

MOMENTS OF INERTIA ($kg \cdot m^2$)

I_1	86.215
I_2	85.070
I_3	113.565
J_a	0.05

Table 5.2

INERTIA AND WHEEL GEOMETRY MATRICES

$$[I^*] = \begin{bmatrix} 87.212 & -0.2237 & -0.2237 \\ -0.2237 & 86.067 & -0.2237 \\ -0.2237 & -0.2237 & 114.562 \end{bmatrix}$$

$$[\tilde{C}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \end{bmatrix}$$

each wheel is 0.2 m. All examples were performed using linear and quadratic steady-state gains with free final time t_f.

Case 5A

A 3-d maneuver with zero initial wheel speeds is studied in which two state weights in the performance indices are considered

$$Q_1 = I$$

and (5.28)

$$Q_2 = I \text{ with } q_{44} = 0$$

The four-wheel maneuver uses performance index J_1 with R = I evaluated at t_f = 120 sec., and the three-wheel maneuvers are with performance index J_2, W = I, and t_f = 240 sec. The boundary conditions are given in Table 5.3 using the 3-1-3 Euler angles, and the performance indices are listed in Table 5.4. In all maneuvers the final wheel speeds were small. Fig. 5.3 shows the spacecraft and wheel angular velocities for the case of three orthogonal wheels (skew wheel off) and Q_2 in performance index J_2, and Fig. 5.4 shows the control torques required for this maneuver.

For the case when Q_1 is used in performance index J_2, with the skew wheel off, we obtain the spacecraft and wheel angular velocities shown in Fig. 5.5. When one of the primary wheels is turned off, and the skew wheel used for compensation, we obtain spacecraft angular velocity histories exactly like those of Fig. 5.3, and similar Euler parameter histories. The wheel angular velocities for each of these three cases is quite different, however, as can be seen in Figs. 5.6, 5.7, and 5.8.

Table 5.3

CASE 5A BOUNDARY CONDITIONS

	Initial States	Final States
ω_1	.0001	0.0
ω_2	.0001	0.0
ω_3	.0001	0.0
ϕ	$-\pi/2$	$\pi/2$
θ	$-\pi/3$	$\pi/3$
ψ	$-\pi/4$	$\pi/4$
δ_0	-.54611	-.30257
δ_1	.47921	-.13976
δ_2	.67687	.81747
δ_3	.11820	.46974
β_0	-.33141	.33141
β_1	.46194	.46194
β_2	-.19134	.19134
β_3	.80010	.80010
Ω_1	0.0	*
Ω_2	0.0	*
Ω_3	0.0	*
Ω_4	0.0	*

*Specific final boundary conditions for $\Omega_i(t_f)$ need not be formally enforced; these are determined implicitly because angular momentum is conserved; i.e., for $\underline{H}$ = constant and $\underline{\omega}(t_f)$ specified, $\underline{\Omega}(t_f)$ is implicitly constrained by Eq. (5.11).

Table 5.4

CASE 5A PERFORMANCE INDICES

Control Configuration and Performance Index	Linear Feedback	Linear Plus Quadratic Feedback
4 wheels (J_1)		
Q_1	6.15126	6.13691
Q_2	5.76886	5.62314
skew wheel off (J_2)		
Q_1	6.48600	6.47143
Q_2	5.92983	5.73828
3rd wheel off (J_2)		
Q_2	5.93077	5.76176
2nd wheel off (J_2)		
Q_2	5.92980	5.76077
1st wheel off (J_2)		
Q_2	5.92962	5.76071

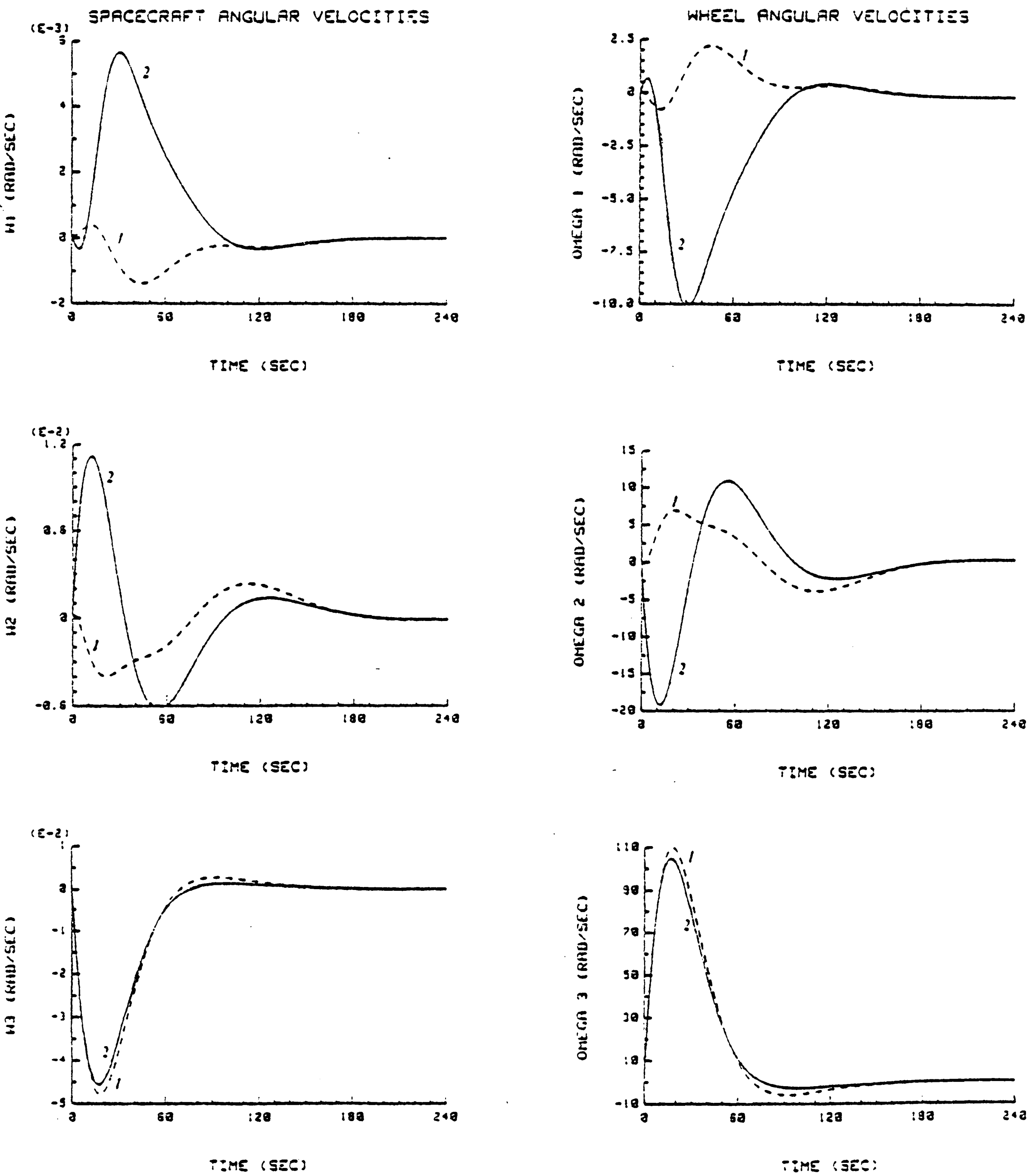
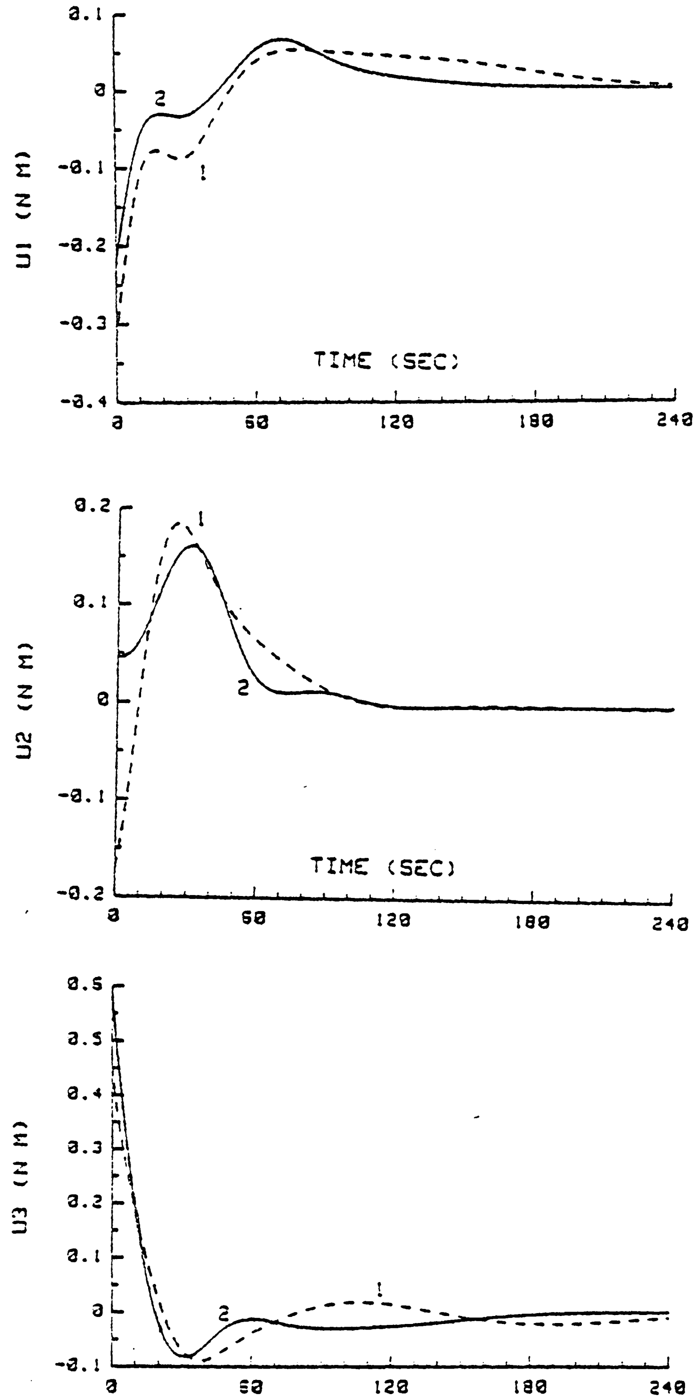


Fig. 5.3: Case 5A spacecraft and wheel angular velocities (Q_2 in J_2, skew wheel off).

1 Linear Feedback
2 Linear plus Quadratic Feedback



TIME (SEC)

Fig. 5.4: Case 5A control torques

(Q_2 in J_2, skew wheel off).

1 Linear Feedback

2 Linear plus Quadratic Feedback

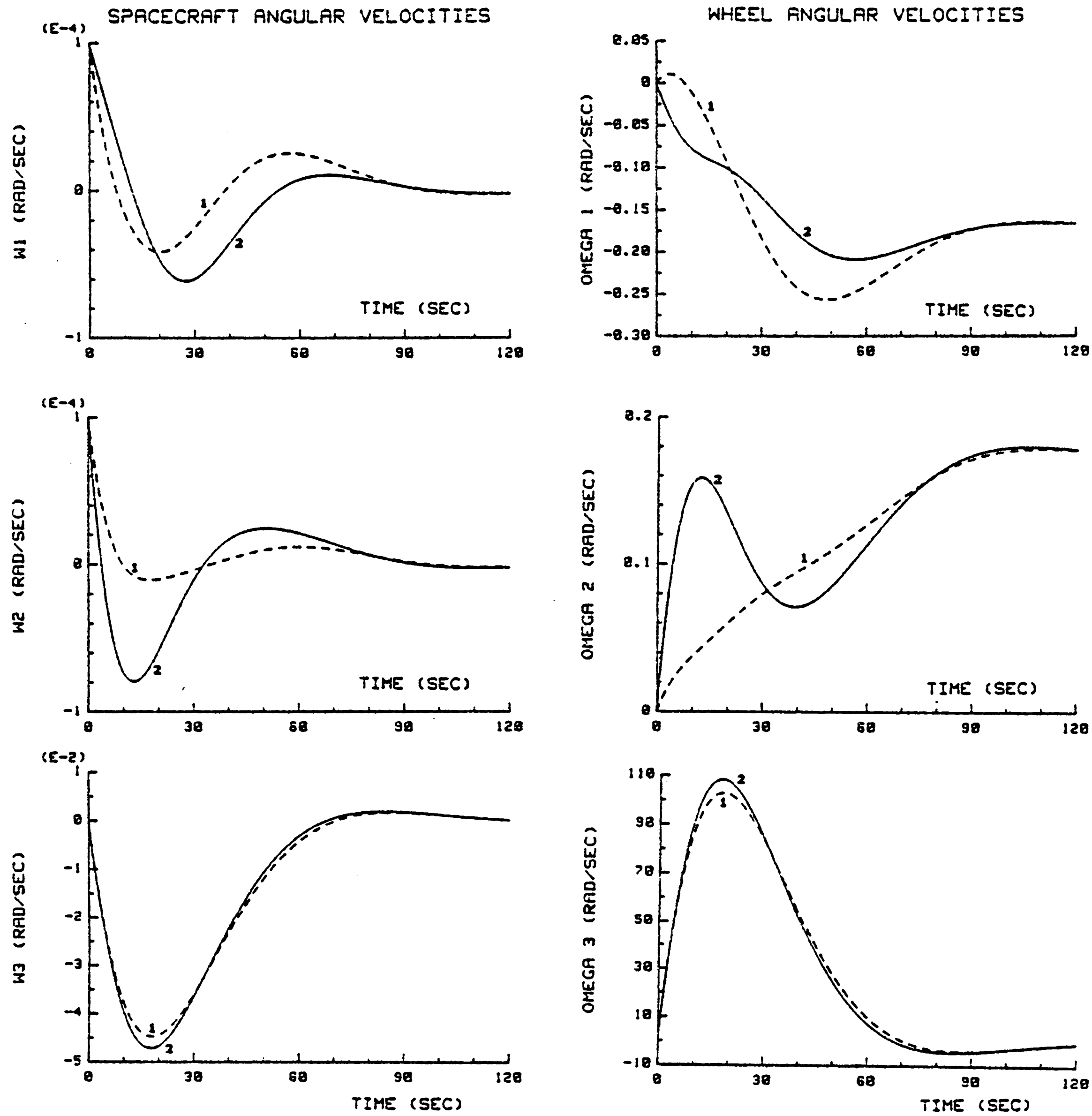


Fig. 5.5: Case 5A spacecraft and wheel angular velocities (Q_1 in J_2, skew wheel off).

1 Linear Feedback
2 Linear plus Quadratic Feedback

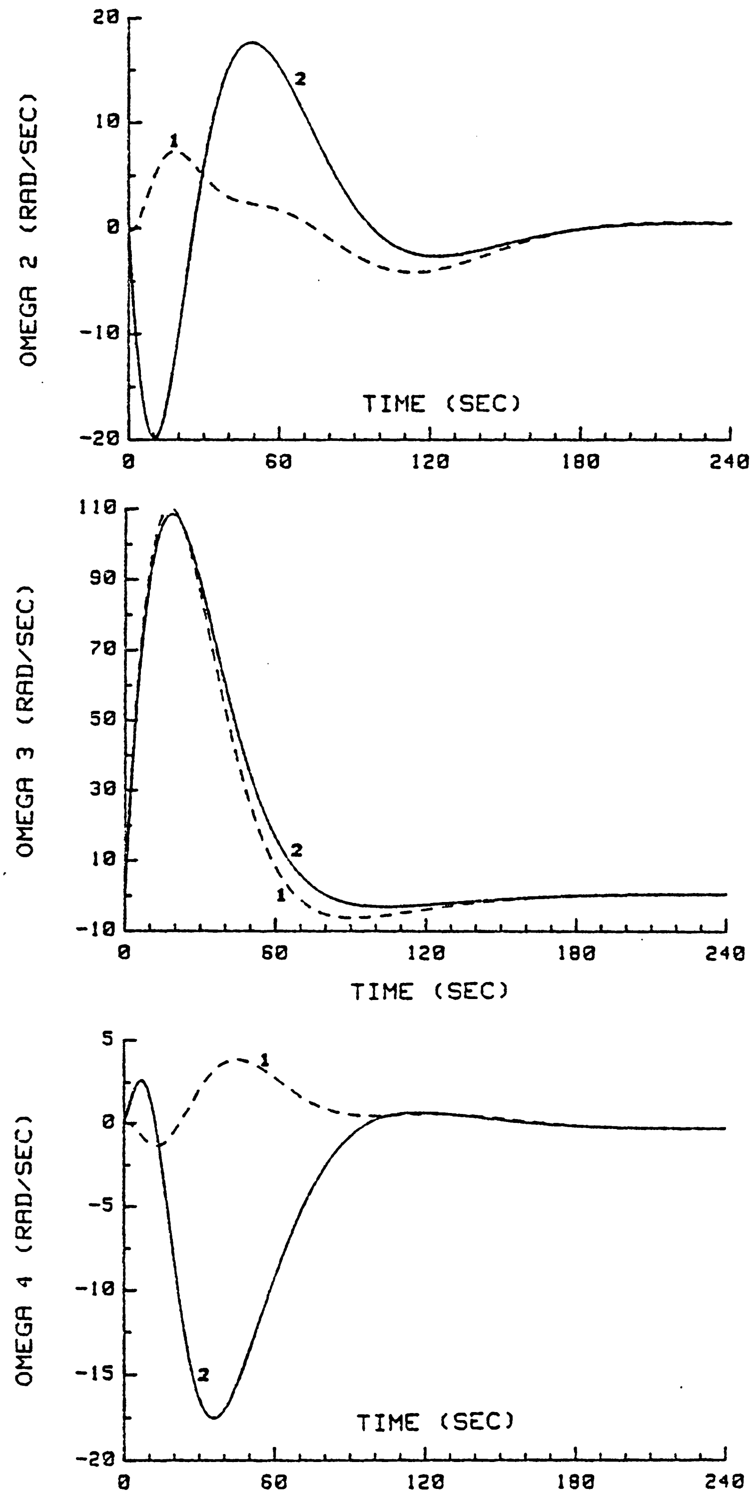


Fig. 5.6: Case 5A wheel angular velocities (first wheel off).

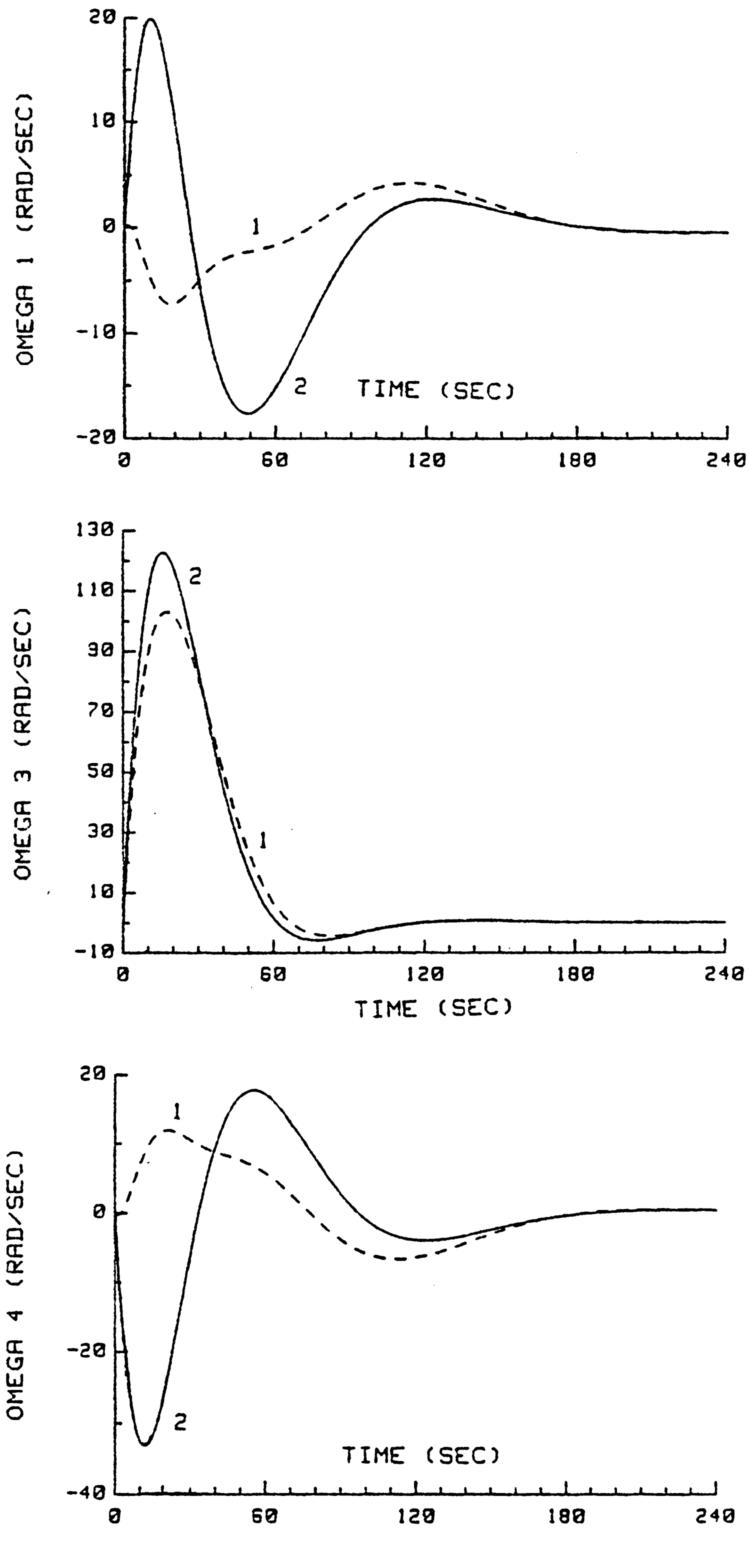


Fig. 5.7: Case 5A wheel angular velocities (second wheel off).

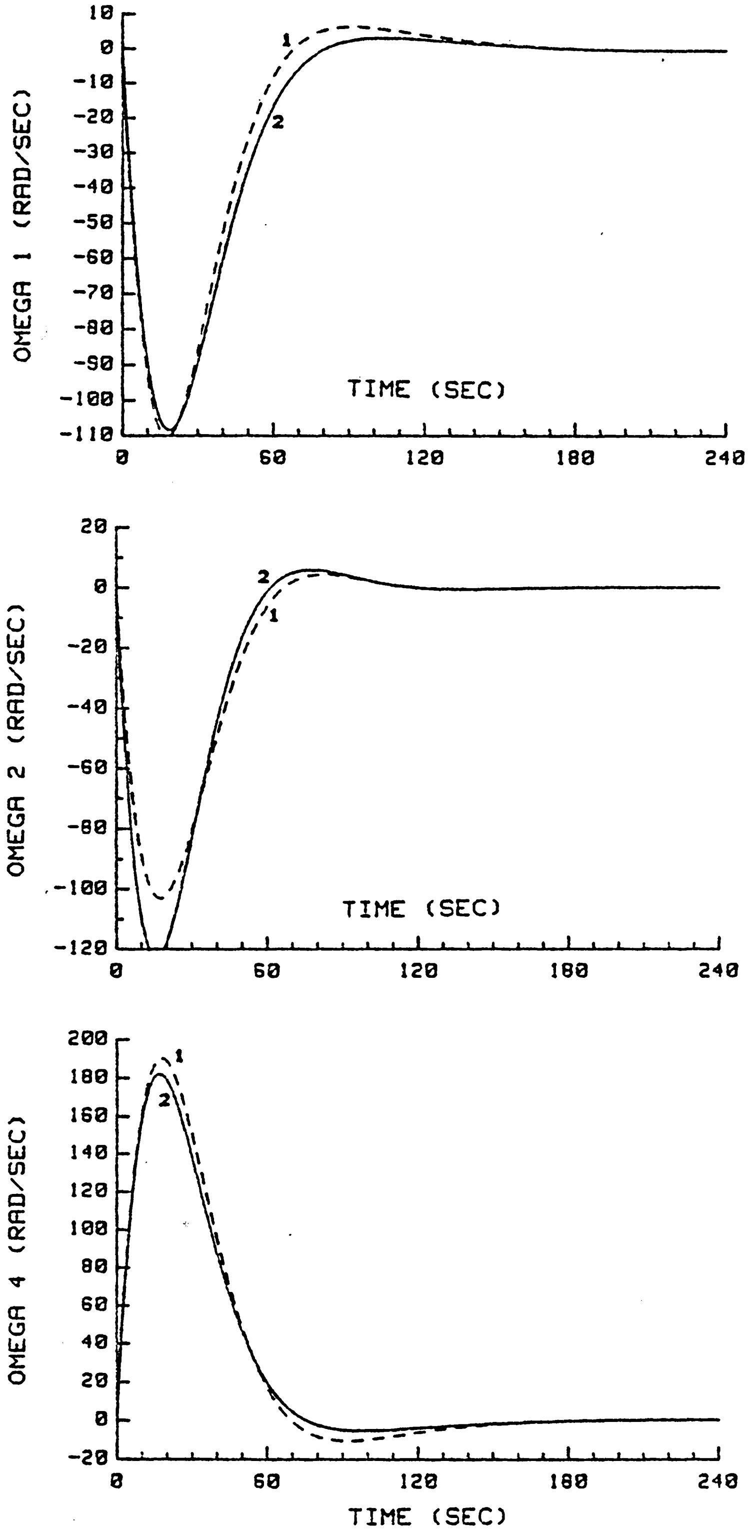


Fig. 5.8: Case 5A wheel angular velocities (third wheel off).

Case 5B

A spacecraft rest-to-rest maneuver with nonzero initial wheel speeds is performed with three orthogonal wheels (skew wheel off) and zero weight on δ_0 in performance index J_2. The boundary conditions are given in Table 5.5 and performance indices evaluated at $t_f = 240$ sec are given in Table 5.6. Since the initial and final Euler angles are the same as in Case 5A, the Euler parameters β_i are the same, but the system angular momentum is larger and hence the Euler parameters δ_i are different. In Figs. 5.9, 5.10, and 5.11 are the spacecraft and wheel angular velocities, Euler parameter histories, and the control torques. Note that the final wheel speeds are 75, 50, 100 rad/sec for Ω_1, Ω_2, and Ω_3, respectively.

Case 5C

A 3-d maneuver is considered for the three wheel configuration with large initial spacecraft angular velocities and the initial wheel speeds of Case 5B. The boundary conditions for this maneuver are given in Table 5.7. Performance index J_2 was used, once with equal weights on all Euler parameters δ_i and once with no weight on δ_0. Only linear feedback was used for both performance index weights. Figs. 5.12 and 5.13 give the time histories of the Euler parameters, spacecraft angular velocities, and wheel speeds for this maneuver.

Discussion

The maneuvers performed in Case 5A all resulted in the same spacecraft angular velocities and attitude histories when Q_2 was used in the performance index, indicating that attitude control is not

Table 5.5

CASE 5B BOUNDARY CONDITIONS

	Initial States	Final States
ω_1	0.0	0.0
ω_2	0.0	0.0
ω_3	0.0	0.0
ϕ	$-\pi/2$	$\pi/2$
θ	$-\pi/3$	$\pi/3$
ψ	$-\pi/4$	$\pi/4$
δ_0	-.12815	.37037
δ_1	.59459	.10026
δ_2	.45281	.74062
δ_3	.65192	.55159
Ω_1	50	*
Ω_2	-75	*
Ω_3	100	*
Ω_4	0.0	*

*See Table 5.3 footnote.

Table 5.6

CASE 5B PERFORMANCE INDICES

linear feedback	4.81211
linear plus quadratic feedback	4.29420

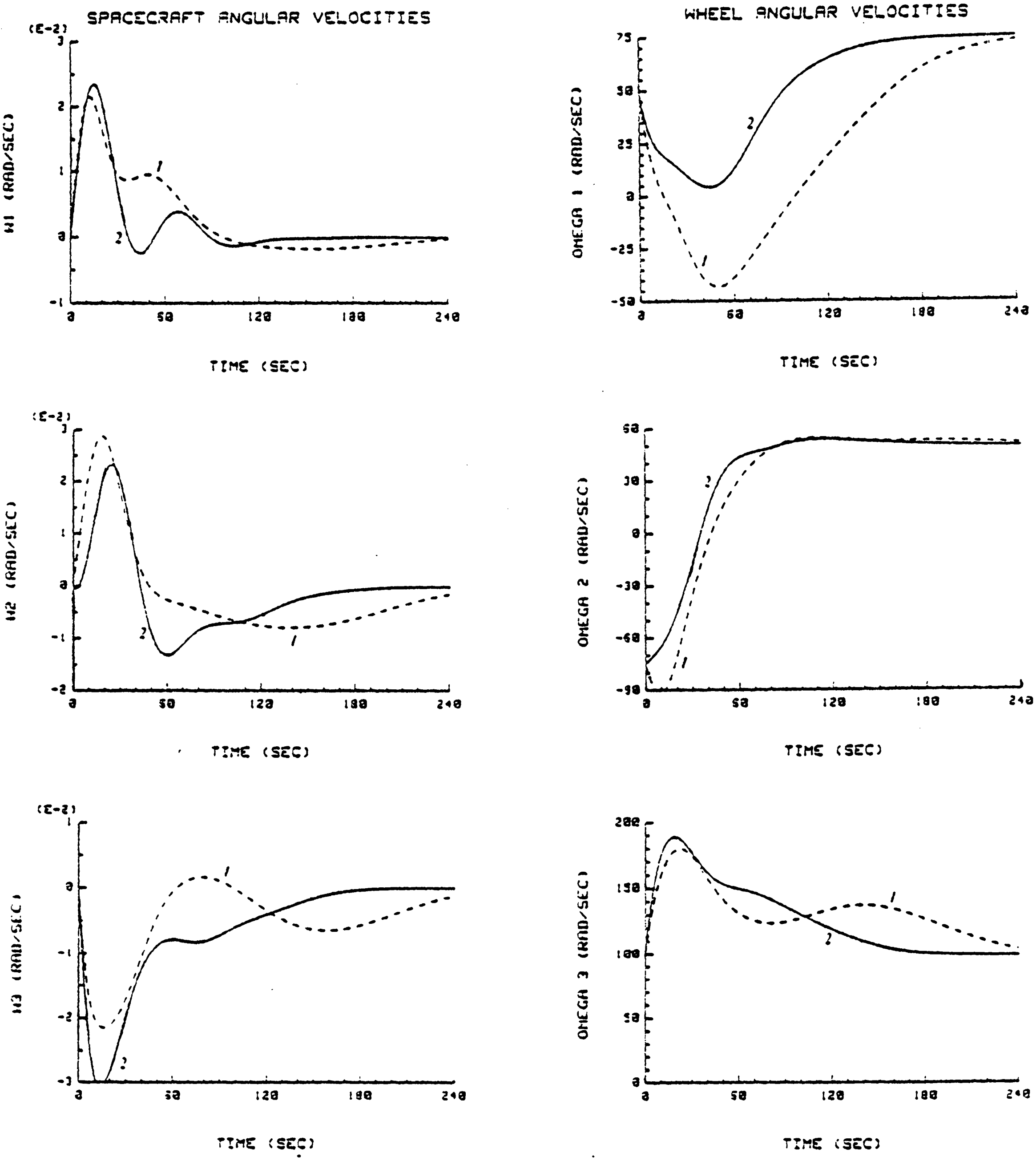


Fig. 5.9: Case 5B spacecraft and wheel angular velocities.

1 linear feedback
2 linear plus quadratic feedback

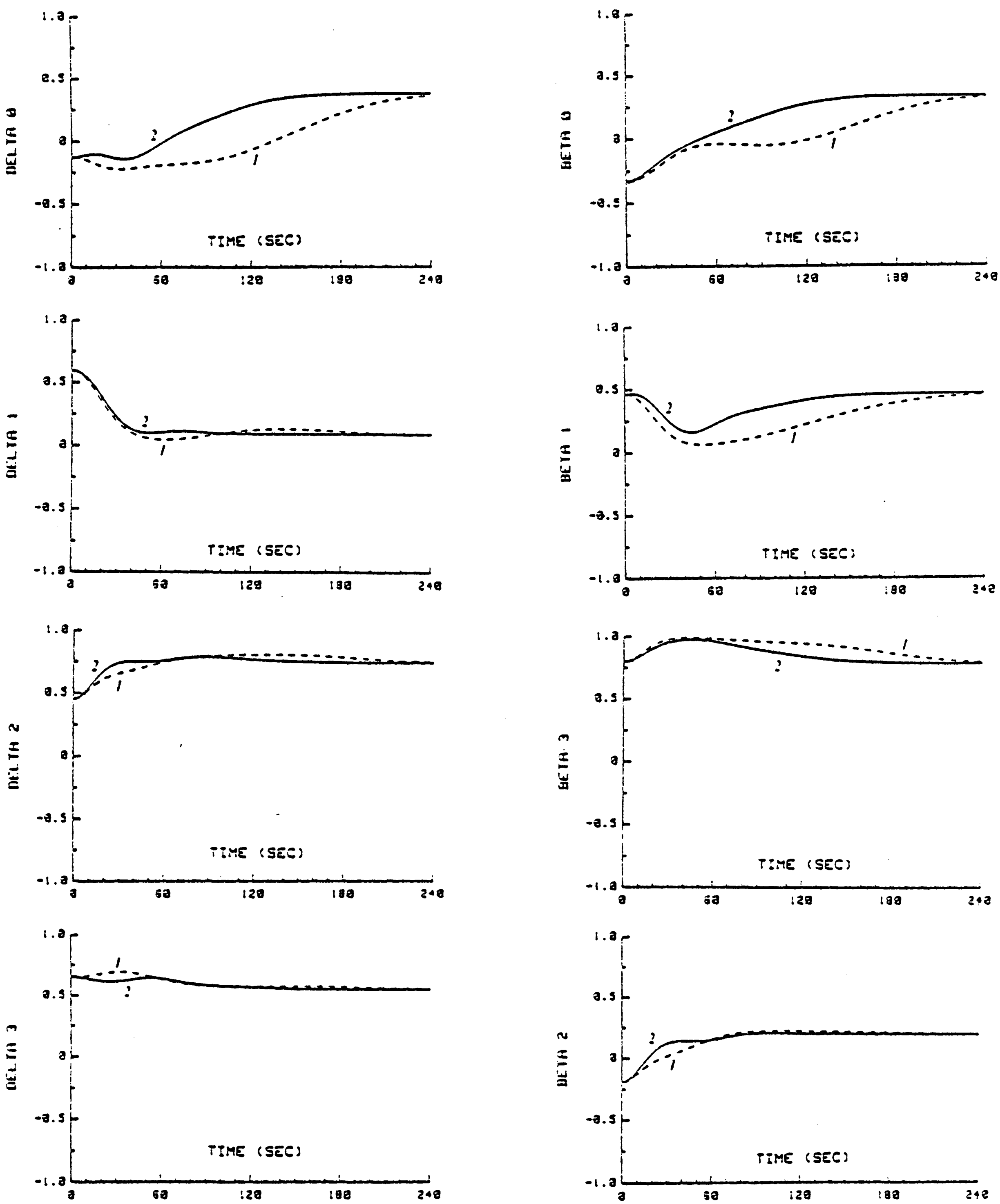


Fig. 5.10: Case 5B Euler parameter histories.

1 linear feedback
2 linear plus quadratic feedback

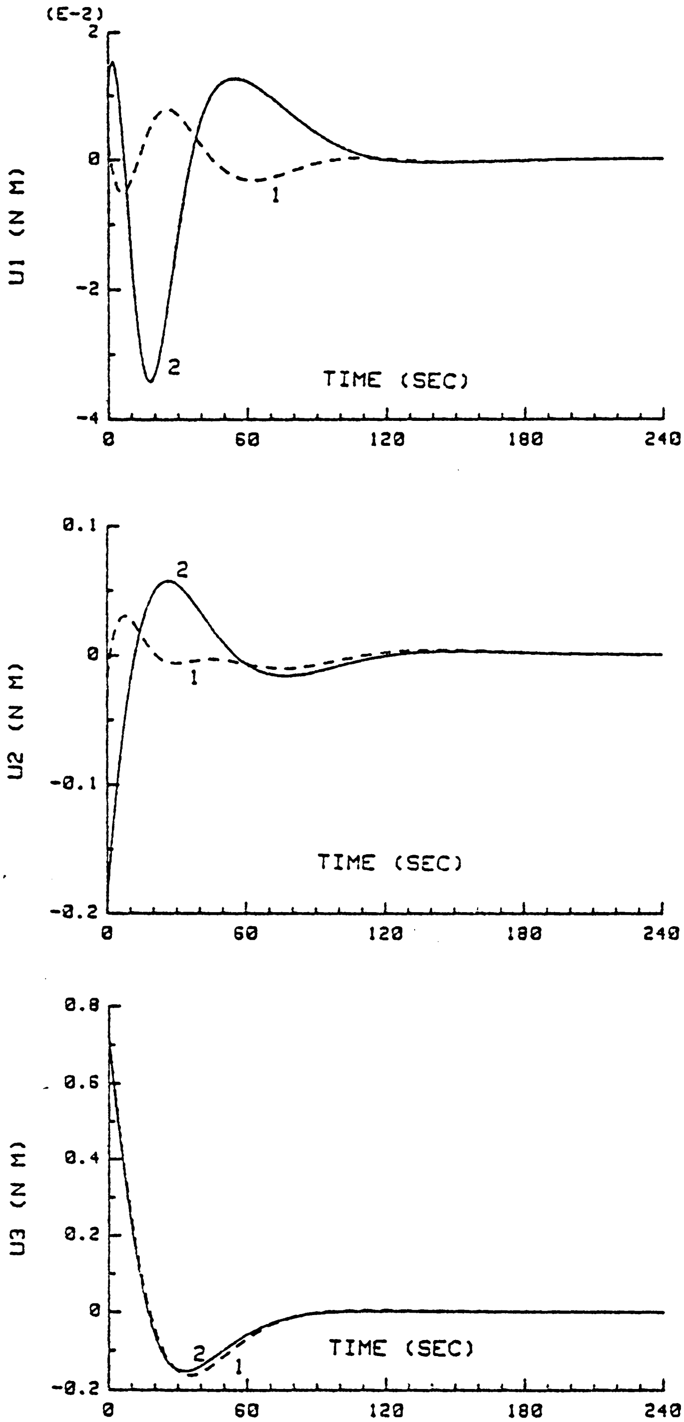


Fig. 5.11: Case 5B control torques.

1 linear feedback
2 linear plus quadratic feedback

Table 5.7

CASE 5C BOUNDARY CONDITIONS

	Initial States	Final States
ω_1	.05	0.0
ω_2	.1	0.0
ω_3	-.01	0.0
ϕ	$-\pi/2$	$\pi/2$
θ	$-\pi/3$	$\pi/3$
ψ	$-\pi/4$	$\pi/4$
δ_o	-.22769	-.07728
δ_1	.47213	-.25249
δ_2	.84335	.93018
δ_3	.11840	.24472
Ω_1	50	*
Ω_2	-75	*
Ω_3	100	*
Ω_4	0.0	*

*See Table 5.3 footnote.

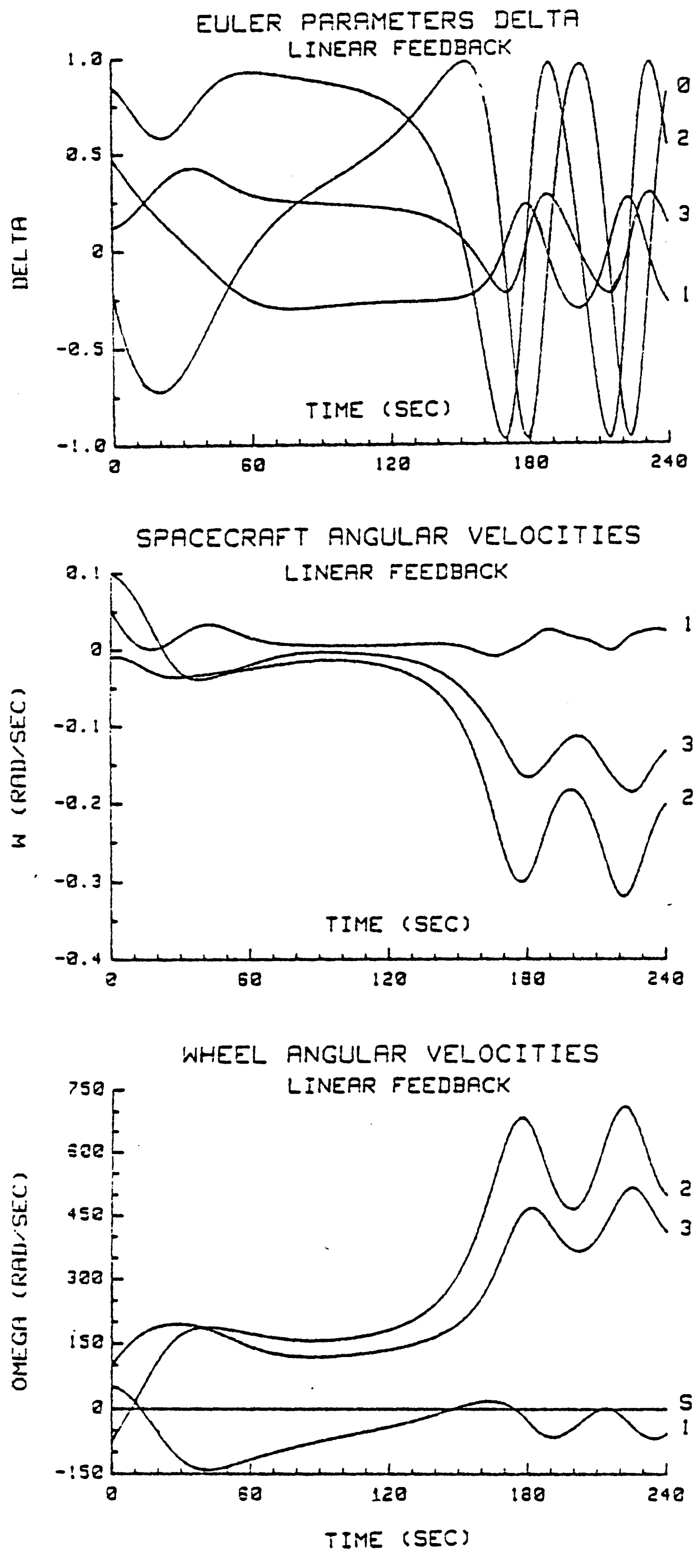


Fig. 5.12: Case 5C, δ_0 not constrained by performance index.

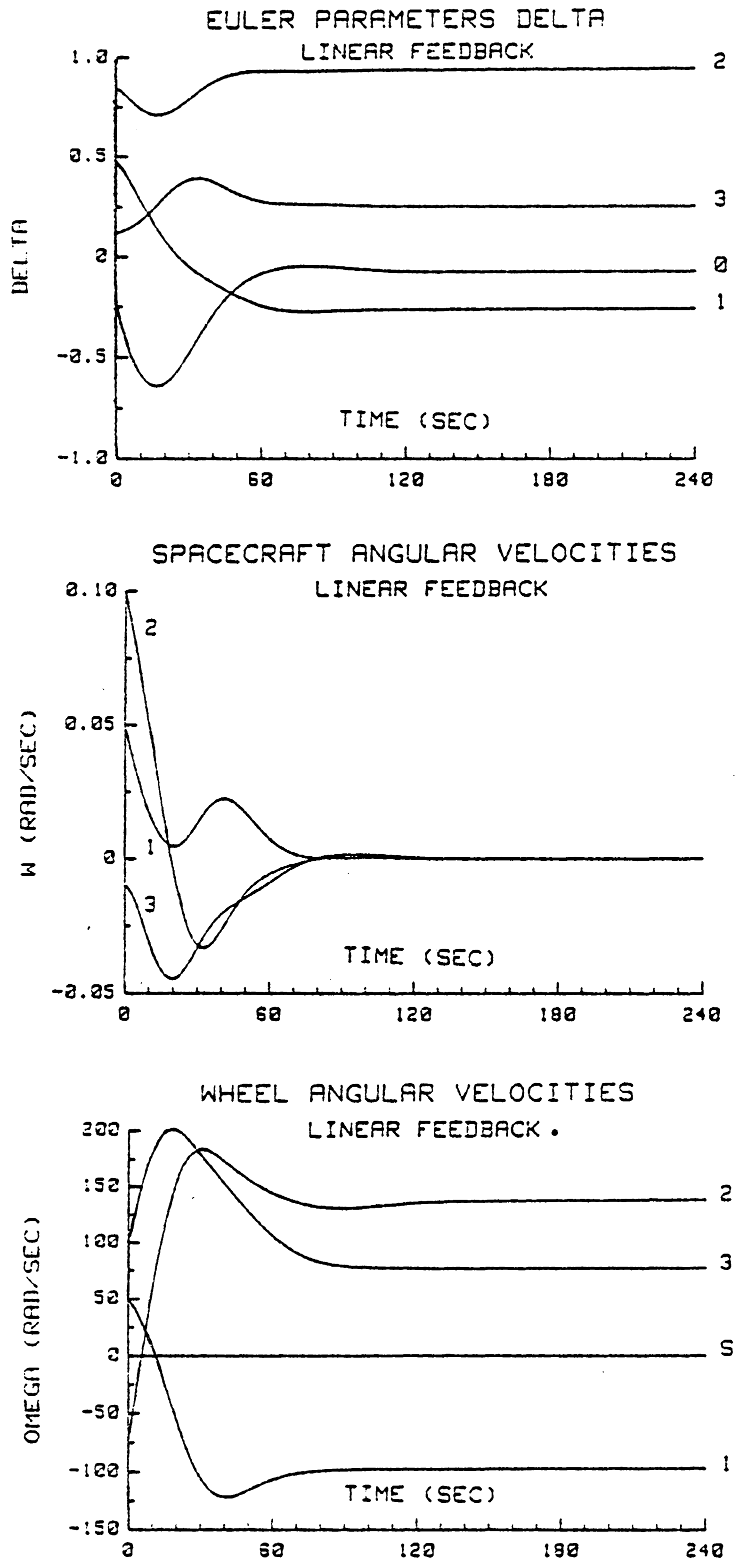
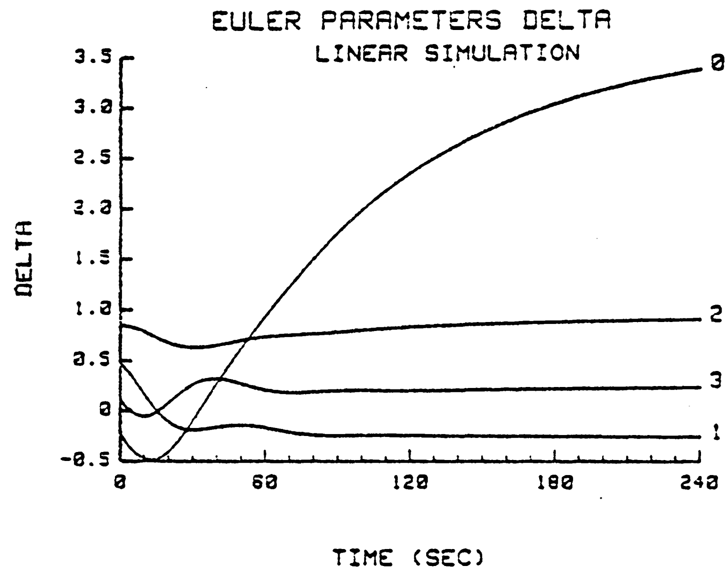


Fig. 5.13: Case 5C, δ_0 contrained by performance index.



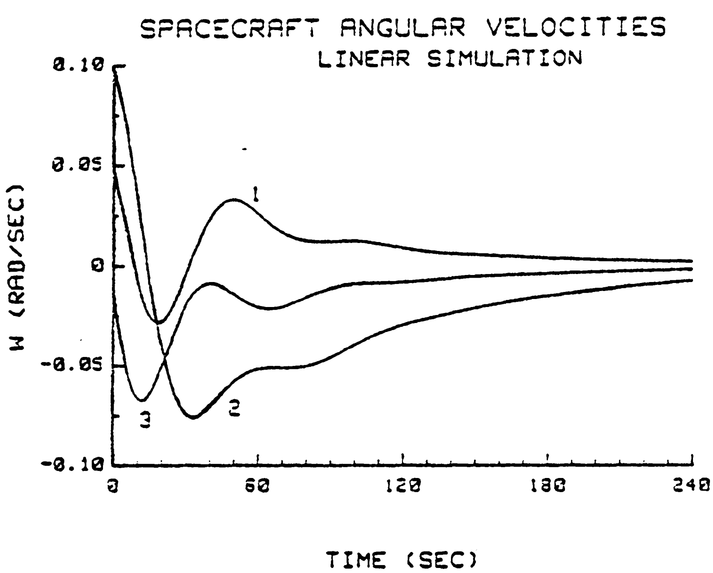


Fig. 5.14: Case 5C, linearized state equations with linear control.

degraded when the skew wheel compensates for a primary wheel failure. The wheel torques and angular velocities were different depending upon which wheel was off, which produce the performance index differences listed in Table 5.4. Note that the skew wheel angular velocity developed the same profile as the missing primary wheel when either wheels one or two were off. Since the third wheel normally develops large velocities for this maneuver, its failure required all three wheels for compensation.

The system angular momentum in Case 5A is quite small since the initial wheel speeds are zero and the initial spacecraft angular velocities are small. Hence we observe the axis-by-axis spacecraft-wheel opposition demonstrated in Fig. 5.3.

The change in performance index weights produces the two different maneuvers in Figs. 5.3 and 5.5. The magnitude of the spacecraft and wheel angular velocities is much smaller for the first two axes when β_0 excursions are penalized, but the performance index listed in Table 5.4 is larger since the error in β_0 is now measured. This maneuver may actually be more desirable than that produced when Q_2 is used, since in the latter the error in β_0 may be quite large but is not included in the performance index.

The initial wheel speeds in Case 5B are large, so that the system angular momentum is not small and we do not observe the spacecraft-wheel opposition that was seen in Case 5A. As in the first case, quadratic feedback gave spacecraft angular velocities and Euler parameters δ_i that remained closer to their desired values, and hence a reduction in the performance index occurred.

Case 5C has larger nonlinear terms in the state equations than either Case 5A or 5B. In Fig. 5.12, where no weight on δ_0 was used in the performance index, we see a highly undesirable tumbling behavior. By not explicitly enforcing the terminal boundary condition on δ_0 through the performance index, δ_0 is determined by the state equations and the boundary conditions on the other three Euler parameters. When only linear terms in the state equations are used in the simulation, δ_0 undergoes the large excursion of Fig. 5.14, i.e., it badly violates the quadratic constraint $\sum_{i=0}^{3} \delta_i^2(t) = 1$. If the full nonlinear state equations are used, as in Fig. 5.12, δ_0 is now constrained to be consistent with the rigorous kinematic equation, but since the linear motion incurs large errors, the nonlinear terms must necessarily introduce rather large departures from the linear approximation. Hence large torques and tumbling behavior ensues. Similar behavior is seen in Cases 5A and 5B, except that the quadratic terms change δ_0 by a relatively smaller perturbation since the gyroscopic terms are smaller, and so these corrections do not have the same destabilizing effect.

When the performance index includes weights on the departure motion of all four Euler parameters, δ_0 excursions are penalized, so it will very likely remain close to its desired final state when linear feedback is used. The quadratic terms in the state and feedback control equations do not then have to compensate for such large, obviously inadmissible excursions. As is evident in comparing Figs. 5.12 and 5.13, introducing the penalty on δ_0 departure motion immediately eliminates the tumbling motion and yields an attractive optimal maneuver.

This behavior is a consequence of the fact that we are using the once-redundant Euler parameters and there is no rigorous way to enforce the constraint $\Sigma\delta_i^2(t) = 1$ using the linearized departure motion differential equations. As is evident, however, a modest level of experimentation with the weight matrix leads to an attractive nonlinear control which is fully consistent with this constraint.

5.3 Conclusions

Polynomial feedback on angular velocities and Euler parameters has been used for nonlinear control of a spacecraft with four reaction wheels. A comparison of linear and quadratic control was made, with a reduction in the performance index for quadratic feedback. When using redundant attitude variables, care must be taken so that the linear gains (based upon linearized departure motion) result in modest violations of the implicit constraint(s). We have shown that practical optimal controls can be computed after experimentation with the weight matrices, which is invariably required anyway, even for strictly linear systems.

Chapter VI

GENERALIZED MOMENTA FEEDBACK

6.1 Generalized Momentum Variables and Equations of Motion

An alternate system of variables for rigid-body rotational dynamics that is conjugate to the Euler parameter kinematic variables has been developed recently by Morton [14]. These generalized angular momenta are defined from the rotational kinetic energy T as follows

$$p_i = \frac{\partial T}{\partial \dot{\beta}_i} \qquad i = 0,1,2,3 \tag{6.1}$$

where β_i are the four Euler parameters defined in Eq. (4.3). Four dynamic equations and four kinematic equations define the system (see Appendix A)

$$\{\dot{\underline{p}}\} = -\frac{1}{4}[Q(p)][I^{-1}]_4[Q(p)]^T\{\underline{\beta}\} + 2[Q(\beta)]\{\underline{u}\}_4 \tag{6.2}$$

$$\{\dot{\underline{\beta}}\} = \frac{1}{4}[Q(\beta)][I^{-1}]_4[Q(\beta)]^T\{\underline{p}\} \tag{6.3}$$

where $[Q(\beta)]$ and $[I^{-1}]_4$ are 4x4 matrices defined as follows

$$[Q(\beta)] = \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \tag{6.4}$$

and, for a principal-axis body-reference frame,

$$[I^{-1}]_4 = \text{diag} \ \{0, 1/I_1, 1/I_2, 1/I_3\} \tag{6.5}$$

The vector $\{\underline{u}\}_4$ is defined from the external control torques

$$\{\underline{u}\}_4 = \{0 \quad u_1 \quad u_2 \quad u_3\}^T \tag{6.6}$$

Note that Eqs. (6.2) and (6.3) are cubic polynomials in p's and β's, and the control influence matrix is a function of the states. These variables are also twice redundant, with the following constraints

$$\sum_{i=0}^{3} \beta_i^2 = 1 \tag{6.7}$$

$$\sum_{i=0}^{3} p_i^2 = 4H^2 \tag{6.8}$$

where H is the magnitude of the system angular momentum. Eqs. (6.7) and (6.8) are both integral properties of the system equations, however, and hence they do not need to be explicitly enforced when defining the optimal control problem.

6.2 Optimal Control Problem

Since there are no linear terms in Eqs. (6.2) and (6.3) and fewer controls than state variables, we either need to introduce linear terms in the state equations or solve for time-dependent gains. If we choose the first, then we can define new state variables as

$$\underline{x} = \{p_0 \quad p_1 \quad p_2 \quad p_3 \quad \tilde{\beta}_0 \quad \tilde{\beta}_1 \quad \tilde{\beta}_2 \quad \tilde{\beta}_3\}^T \tag{6.9}$$

where the departure motion Euler parameters are

$$\tilde{\beta}_i = \beta_i - \beta_i(t_f) \qquad i = 0,1,2,3 \tag{6.10}$$

Linear terms in $\underline{p}$ now appear in the kinematic equations, and constant terms appear in the control influence matrix. The state equations may now be written as

$$\{\dot{\underline{x}}\} = A\{\underline{x}\} + \underline{F}(\underline{x}) + B_1\{\tilde{\underline{u}}\}_4 + B_2(\underline{x})\{\underline{u}\}_4 \tag{6.11}$$

where $\{\underline{u}\}_4$ is defined in Eq. (6.6), $\underline{F}(\underline{x})$ contains quadratic and cubic terms in $\underline{x}$, and the 8x8 matrix A and 8x4 matrices B_1 and B_2 are defined as follows

$$A = \frac{1}{4}\left[\begin{array}{c|c} 0 & 0 \\ \hline [Q(\beta(t_f))][I^{-1}]_4[Q(\beta(t_f))]^T & 0 \end{array}\right] \tag{6.12}$$

$$B_1 = 2\left[\begin{array}{c} [Q(\beta(t_f))] \\ \hline 0 \end{array}\right] \qquad B_2 = 2\left[\begin{array}{c} [Q(\tilde{\beta})] \\ \hline 0 \end{array}\right] \tag{6.13}$$

To minimize a quadratic performance index

$$J = \frac{1}{2}\int_{t_0}^{t_f} \{\underline{x}^T Q\underline{x} + \underline{u}_4^T R\, \underline{u}_4\}dt \tag{6.14}$$

Potter's method using matrices A and B_1 was attempted. Unfortunately, we now have two zero poles, corresponding to β_0 and p_0, that produce two zero eigenvalues, and even the Schur method [33] does not determine the corresponding eigenvectors accurately enough to produce reasonable linear gains. Hence constant gains cannot be used for the optimal control problem, and time-dependent gains must be determined.

We may use the original cubic polynomial equations of Eqs. (6.2) and (6.3) if we assume time-dependent gains. The performance index to be minimized is

$$J = \frac{1}{2}\,\underline{z}^T(t_f)H\underline{z}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \{\underline{z}^T Q\underline{z} + \underline{u}_4^T R\, \underline{u}_4\}dt \tag{6.15}$$

where $\underline{z}$ is the difference between the state $\underline{x}$ and the target state $\underline{r}$

$$\underline{z} = \underline{x} - \underline{r}(t_f) \tag{6.16}$$

$$\underline{x} = \{p_0 \quad p_1 \quad p_2 \quad p_3 \quad \beta_0 \quad \beta_1 \quad \beta_2 \quad \beta_3\}^T \tag{6.17}$$

The necessary conditions for optimality produce the following costate equations

$$\dot{\lambda}_i = \frac{1}{4}\left[\frac{\partial c_{jk}(p)}{\partial p_i}\right]\lambda_j\beta_k - \frac{1}{4}c_{ji}(\beta)\gamma_j - q_{i\ell}z_\ell \tag{6.18}$$

$$\dot{\gamma}_i = \frac{1}{4}c_{ji}(p)\lambda_j - \frac{1}{4}\left[\frac{\partial c_{jk}(\beta)}{\partial \beta_i}\right]\gamma_j p_k - 2\left[\frac{\partial Q_{jk}(\beta)}{\partial \beta_i}\right]\lambda_j u_k - q_{i\ell}z_\ell$$

$$i,j,k = 0,\ldots,3$$
$$\ell = 1,\ldots,8 \tag{6.19}$$

where λ_i and γ_i are the costates corresponding to p_i and β_i respectively; the 4x4 matrix C is

$$C(p) = [Q(p)][I^{-1}]_4[Q(p)]^T \tag{6.20}$$

and the terms in brackets are Jacobians. The control is

$$\underline{u}_4 = -\,2\,R^{-1}Q^T(\beta)\underline{\lambda} \tag{6.21}$$

and the boundary conditions at $t = t_f$ on the costates λ_i and γ_i are

$$\begin{Bmatrix}\lambda_i\\ \gamma_i\end{Bmatrix} = H\underline{z}(t_f) \tag{6.22}$$

We now assume the costates are polynomials in the states

$$\begin{Bmatrix}\lambda_i\\ \gamma_i\end{Bmatrix} = s_i(t) + k_{ij}(t)x_j + d_{ijk}(t)x_jx_k + \ldots \tag{6.23}$$

Since the feedback is on the states rather than the errors in the states, the time-dependent coefficients must satisfy the boundary conditions of Eq. (6.22)

$$\begin{aligned} \underline{s}(t_f) &= - H\underline{r}(t_f) \\ K(t_f) &= H \end{aligned} \tag{6.24}$$

and all higher order coefficients go to zero at $t = t_f$.

By substituting Eq. (6.23) into the costate equations and equating coefficients of powers of x, we obtain the following sets of equations

$$\begin{aligned} \dot{s}_i &= q_{ij} r_j \\ \dot{k}_{ij} &= - q_{ij} + f_1(s_i) \\ \dot{d}_{ijk} &= f_2(s_i, k_{ij}) \\ &\vdots \end{aligned} \tag{6.25}$$

where f_1 and f_2 are functions of lower order coefficients. Eqs. (6.25) can be directly integrated, and the constants of integration determined by the boundary conditions in Eq. (6.24). Hence the feedback gains are polynomials in time.

6.3 Single-Axis Maneuvers

For maneuvers requiring only a rotation about one axis, Eqs. (6.2) and (6.3) reduce to the following equations

$$\dot{x}_1 = \frac{1}{4I}(x_1x_2x_4 - x_2^2x_3) - 2x_4u$$

$$\dot{x}_2 = \frac{1}{4I}(x_1x_2x_3 - x_1^2x_4) + 2x_3u$$

$$\dot{x}_3 = \frac{1}{4I}(x_1x_4^2 - x_2x_3x_4) \tag{6.26}$$

$$\dot{x}_4 = \frac{1}{4I}(x_2x_3^2 - x_1x_3x_4)$$

where $\underline{x} = \{p_o \quad p_1 \quad \beta_o \quad \beta_1\}^T$ (6.27)

and u is the control torque to be determined. The performance index to be minimized is

$$J = \frac{1}{2}\,\underline{z}^T(t_f)\,H\underline{z}(t_f) + \frac{1}{2}\int_{t_o}^{t_f}\{\underline{z}^TQ\underline{z} + u^2\}dt \tag{6.28}$$

where $\underline{z} = \underline{x} - \underline{r}$ (6.29)

and the target state is

$$\underline{r} = \{0 \quad 0 \quad 1 \quad 0\}^T \tag{6.30}$$

The costate equations are

$$\dot{\lambda}_1 = -q_{1j}z_j - \frac{1}{4I}(\lambda_1x_2x_4 + \lambda_2(x_2x_3 - 2x_1x_4) + \lambda_3x_4^2 - \lambda_4x_3x_4)$$

$$\dot{\lambda}_2 = -q_{2j}z_j + \frac{1}{4I}(\lambda_1(2x_2x_3 - x_1x_4) - \lambda_2x_1x_3 + \lambda_3x_3x_4 - \lambda_4x_3^2)$$

$$\dot{\lambda}_3 = -q_{3j}z_j + \frac{1}{4I}(\lambda_1x_2^2 - \lambda_2x_1x_2 + \lambda_3x_2x_4 + \lambda_4(x_1x_4 - 2x_2x_3))$$
$$- 2\lambda_2u \tag{6.31}$$

$$\dot{\lambda}_4 = -q_{4j}z_j - \frac{1}{4I}(\lambda_1x_1x_2 - \lambda_2x_1^2 + \lambda_3(2x_1x_4 - x_2x_3)$$
$$- \lambda_4x_1x_3) + 2\lambda_1u$$

and the control is

$$u = 2(x_4\lambda_1 - x_3\lambda_2) \tag{6.32}$$

The zeroth order equations are

$$\dot{s}_i = q_{i3} \tag{6.33}$$

which, with the boundary conditions, have the solution

$$s_i = - q_{i3}(t_f - t) - \delta_{i3}h_{33} \tag{6.34}$$

where δ_{ij} is the Kronecker delta. The linear equations are

$$\begin{aligned} \dot{k}_{33} &= - q_{33} + 4s_2^2 \\ \dot{k}_{34} = \dot{k}_{43} &= - q_{34} - 4s_1s_2 \\ \dot{k}_{44} &= - q_{44} + 4s_1^2 \end{aligned} \tag{6.35}$$

and $\dot{k}_{ij} = - q_{ij}$ where not otherwise specified. These equations have the solution

$$\begin{aligned} k_{33} &= q_{33}(t_f-t) - 4q_{23}^2 \{\tfrac{1}{3}(t_f^3-t^3) + t_ft^2 - t_f^2t\} + h_{33} \\ k_{34} = k_{43} &= q_{34}(t_f-t) + 4q_{13}q_{23} \{\tfrac{1}{3}(t_f^3-t^3) + t_ft^2 - t_f^2t\} + h_{34} \\ k_{44} &= q_{44}(t_f-t) - 4q_{13}^2 \{\tfrac{1}{3}(t_f^3-t^3) + t_ft^2 - t_f^2t\} + h_{44} \end{aligned} \tag{6.36}$$

and $k_{ij} = q_{ij}(t_f-t) + h_{ij}$ where not otherwise specified. Similar polynomials in time are found for the higher order gains.

A numerical example was executed with $I = 1.00$ kg·m^2 and t_f = 20 sec. The boundary conditions are listed in Table 6.1, and the performance indices in Table 6.2. The weights in the performance index were

Table 6.1

SINGLE AXIS BOUNDARY CONDITIONS

	Initial State	Final State
ϕ	$-\pi/2$	0
β_0	.70711	1
β_1	-.70711	0
ω	.1	0
p_0	.14142	0
p_1	.14142	0

Table 6.2

SINGLE AXIS PERFORMANCE INDICES

Feedback	Performance Index
$s_i + k_{ij}x_j$	.62819
$s_i + k_{ij}x_j + d_{ijk}x_jx_k$	.17741

$$H = Q = \begin{bmatrix} .1 & 0 & 0 & 0 \\ 0 & .1 & 0 & 0 \\ 0 & 0 & .1 & 0 \\ 0 & 0 & 0 & .1 \end{bmatrix} \tag{6.37}$$

Fig. 6.1 shows the conjugate angular momenta and Euler parameter histories, and Fig. 6.2 contains the maneuver angle, angular momentum, and control torque histories.

Note that linear feedback only results in angular momentum control, and that the addition of quadratic feedback is required for attitude control. Hence, although quadratic feedback initially increases the angular momentum, the performance is decreased due to a major reduction in attitude errors.

6.4 Lyapunov Control for Single Axis Maneuvers

A Lyapunov function to determine a stable feedback control law is applied to the single axis maneuver discussed in Sec. 6.3. The target state must first be transfored from $\{0 \quad 0 \quad 1 \quad 0\}^T$ to the origin by a change of variable

$$\underline{x} = \{p_0 \quad p_1 \quad \beta_{0-1} \quad \beta_1\}^T \tag{6.38}$$

and the state equations become

$$\begin{aligned} \dot{x}_1 &= \frac{1}{4I}\,(x_1x_2x_4 - x_2^2\,(1 + x_3)) - 2x_4u \\ \dot{x}_2 &= \frac{1}{4I}\,(x_1x_2\,(1+x_3) - x_1^2\,x_4) + 2(1+x_3)u \\ \dot{x}_3 &= \frac{1}{4I}\,(x_1x_4^2 - x_2(1+x_3)x_4) \\ \dot{x}_4 &= \frac{1}{4I}\,(x_2(1+x_3)^2 - x_1(1+x_3)x_4) \end{aligned} \tag{6.39}$$

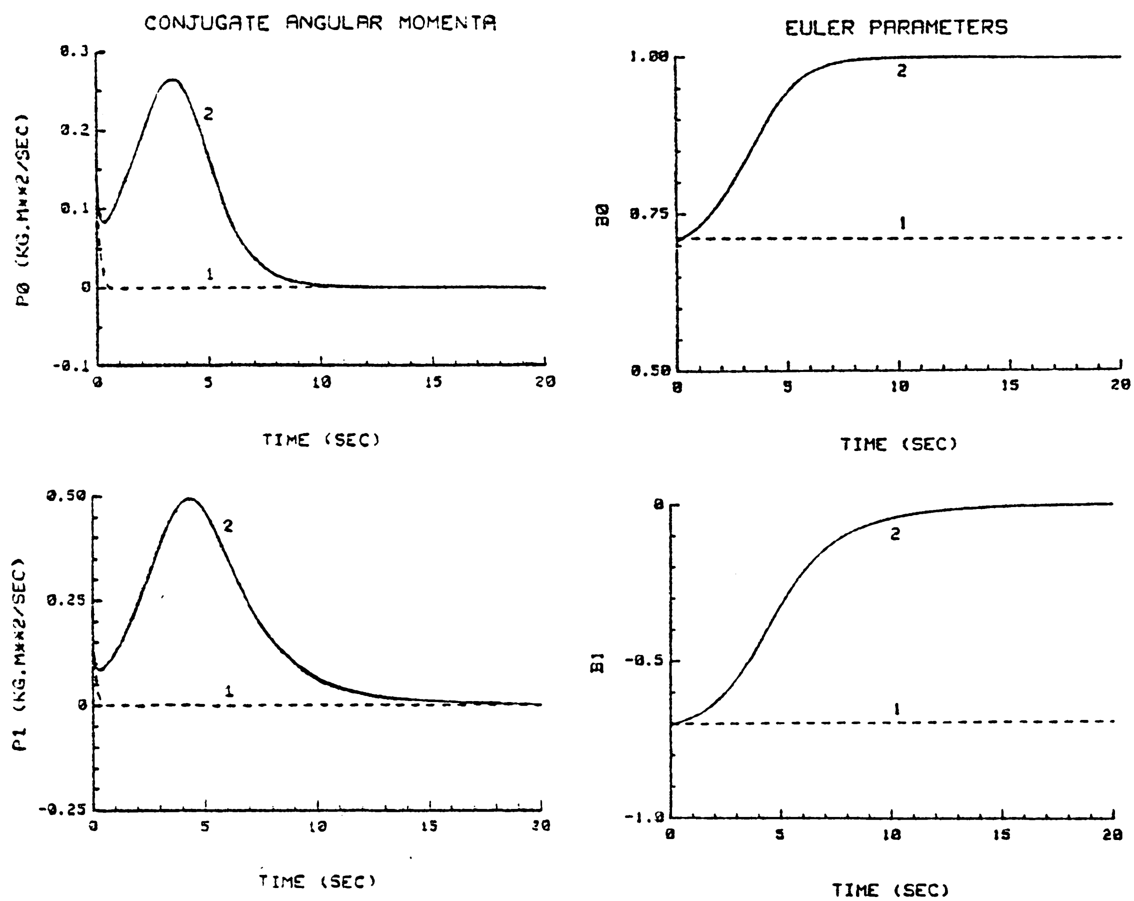


Fig. 6.1: Single-axis angular momenta and Euler parameters.

1 zeroth-order plus linear feedback
2 zeroth-order through quadratic feedback

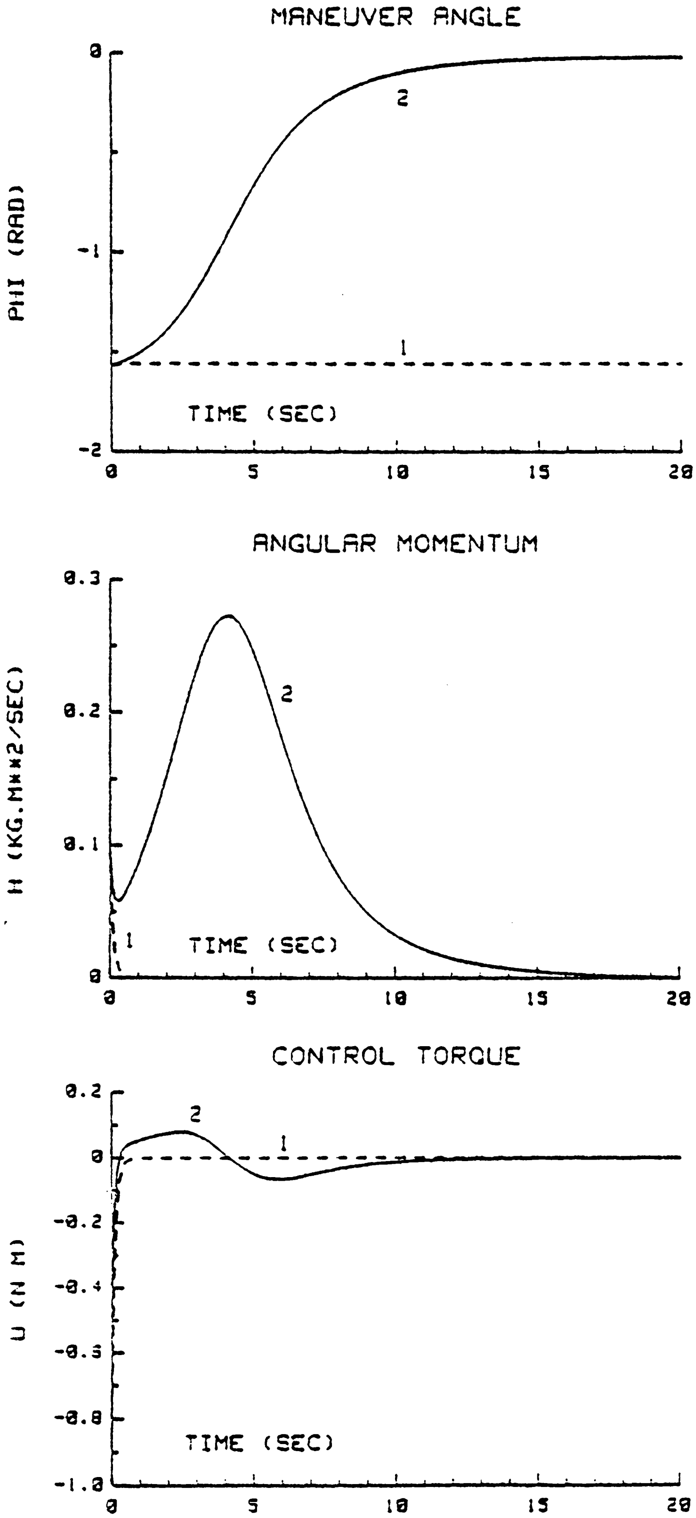


Fig. 6.2: Single-axis maneuver angle, angular momentum, and control torque.

1 zeroth-order plus linear feedback
2 zeroth-order through quadratic feedback

Consider the positive definite function

$$V = 2I(x_1^2 + x_2^2) + 2IK_1(x_3^2 + x_4^2) \tag{6.40}$$

where K_1 is a positive constant. Then

$$\dot{V} = 4I(x_1\dot{x}_1 + x_2\dot{x}_2) + 4IK_1(x_3\dot{x}_3 + x_4\dot{x}_4) \tag{6.41}$$

and by substituting Eqs. (6.39) into Eq. (6.41), we obtain

$$\dot{V} = 8I(-x_1x_4 + (1+x_3)x_2)u + K_1(x_2x_3x_4 + x_2x_4 - x_1x_4^2) \tag{6.42}$$

To determine an asymptotically stable control law, we need to find u such that $\dot{V}$ is negative definite. If we define

$$u = \frac{-K_1(x_2x_3x_4 + x_2x_4 - x_1x_4^2) - f(x_1,x_2,x_3,x_4)}{8I((1+x_3)x_2 - x_1x_4)} \tag{6.43}$$

then $\dot{V} = -f$. Here we have chosen f as a positive constant K_2. To find an optimal control, we may adjust the constants K_1 and K_2 to minimize the performance index given in Eq. (6.28).

Fig. 6.3 shows the maneuver angle responses for several values of K_2, with $K_1 = 1$. The corresponding performance indices are listed in Table 6.3 for weighting matrices

$$H = Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.44}$$

and $t_f = 30$ sec. Note that the minimum performance index corresponds to $K_2 = 1.5$, which is also the "critically damped" response in Fig. 6.3. For this $K_2 = 1.5$ case, Fig. 6.4 contains the Lyapunov function, angular momentum, and control torque histories, and Fig. 6.5 shows the

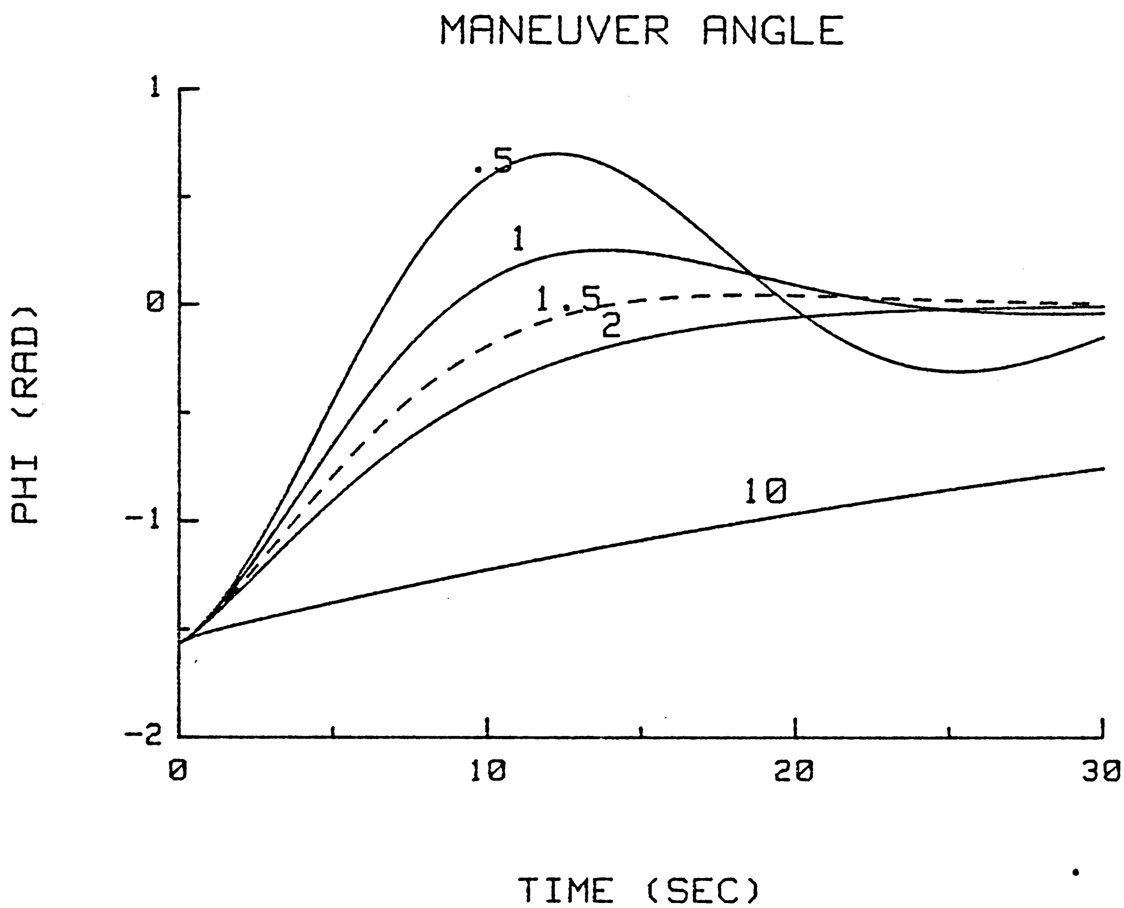


Fig. 6.3: Lyapunov control with $K_1=1$ and several values of K_2.

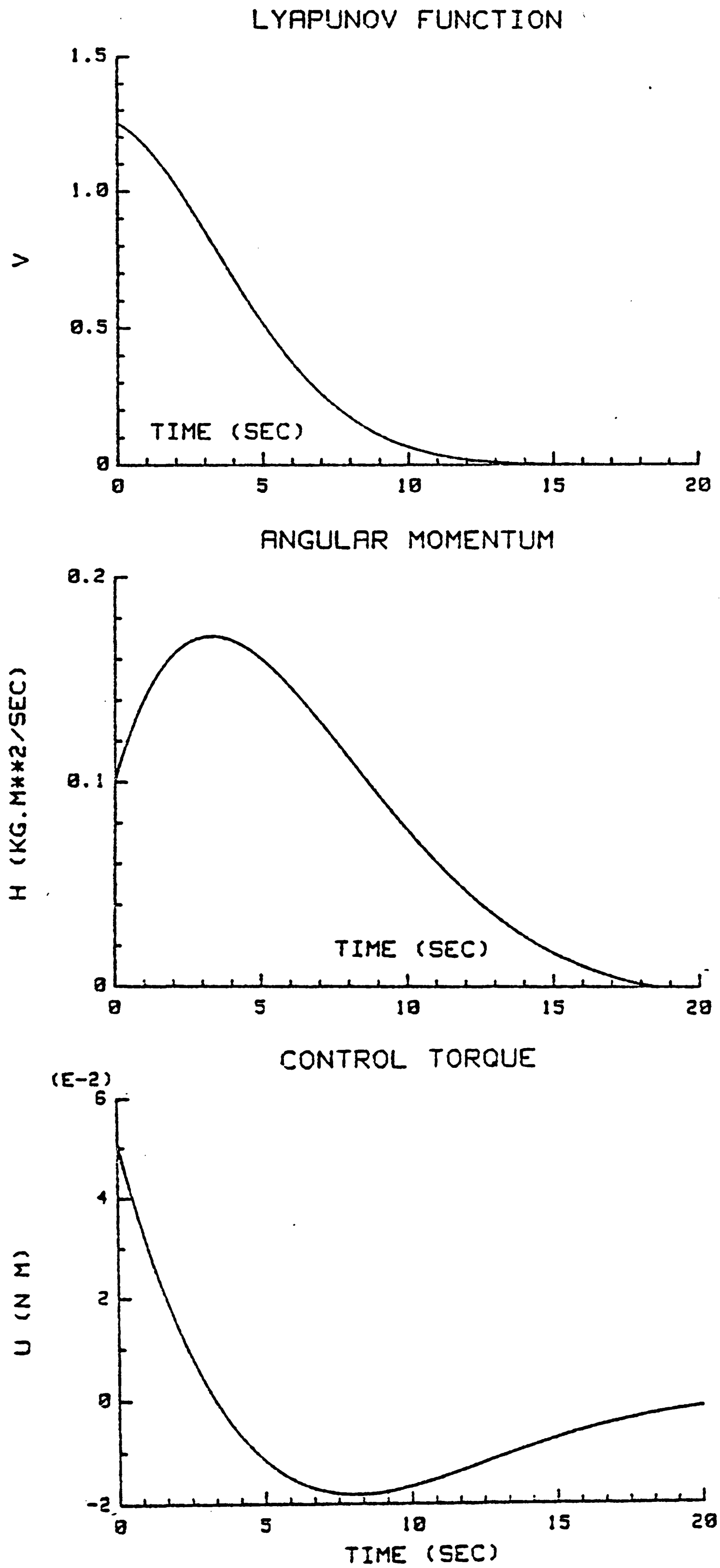


Fig. 6.4: "Critically damped" Lyapunov control (K_2=1.5).

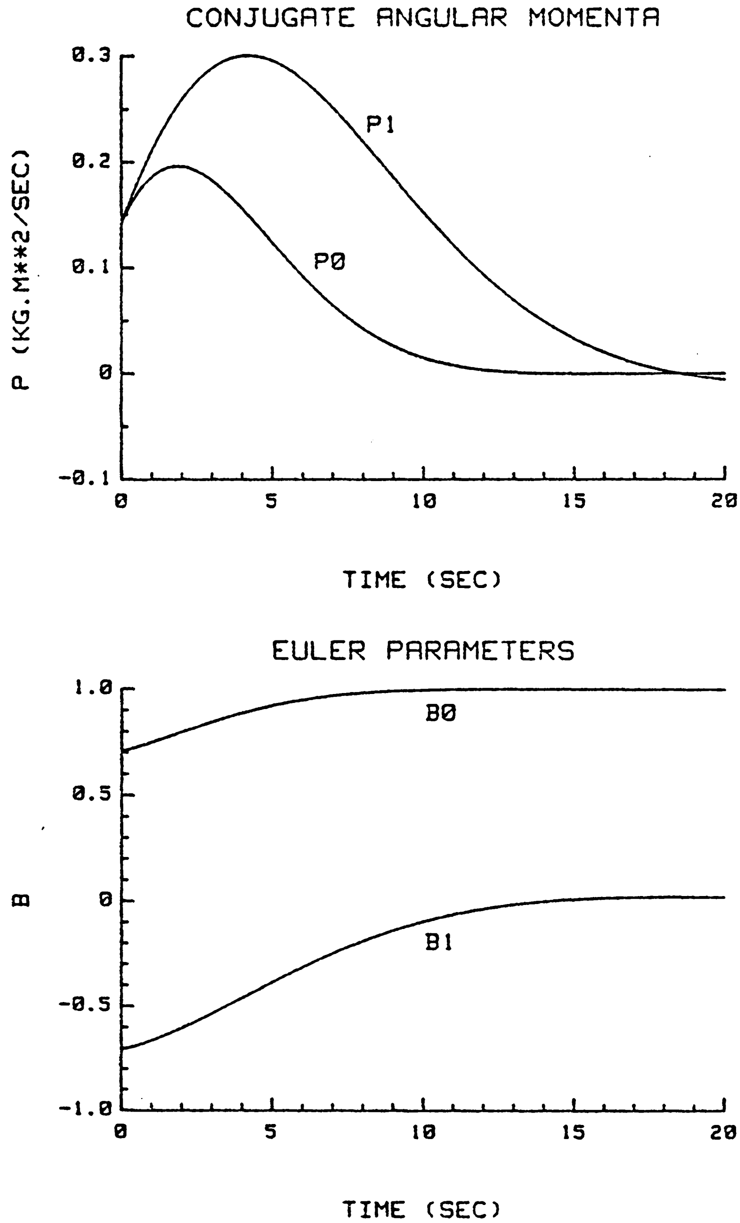


Fig. 6.5: "Critically damped" Lyapunov control (K_2=1.5).

Table 6.3

PERFORMANCE INDICES FOR LYAPUNOV CONTROL

$K_1 = 1$

K_2	Performance Index
.1	7.3638
.5	2.4864
1	1.5886
1.5	1.4813
2	1.5842
10	4.7048

conjugate angular momenta and Euler parameter curves.

The performance index was also calculated for Lyapunov feedback control with the same performance index weights as in Sec. 6.3. The minimum also occurred at $K_1 = 1$ and $K_2 = 1.5$, and at a value of .15011 it compared favorably with the performance indices of Table 6.2.

6.5 Three-Axis Maneuvers

Time-dependent gains similar to those calculated for the single-axis maneuvers can be determined for three-axis maneuvers, in which the gain equations are integrated directly to produce polynomials in t. Lyapunov feedback control can also be used if we assume a function of the form

$$V = \frac{1}{4}\,(x_1^2 + x_2^2 + x_3^2 + x_4^2) + 2K(x_5^2 + x_6^2 + x_7^2 + x_8^2) \tag{6.45}$$

Then
$$\dot{V} = \frac{1}{2}\,(x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3 + x_4\dot{x}_4) + 4K(x_5\dot{x}_5 + x_6\dot{x}_6 + x_7\dot{x}_7 + x_8\dot{x}_8) \tag{6.46}$$

The state equations, Eqs. (6.2) and (6.3), are transformed by a change of variable to the origin, so that the new states are

$$\underline{x} = \{p_0 \quad p_1 \quad p_2 \quad p_3 \quad \beta_0 - 1 \quad \beta_1 \quad \beta_2 \quad \beta_3\}^T \tag{6.47}$$

These new state equations are substituted into Eq. (6.46), which reduces to the following

$$\begin{aligned}\dot{V} = & (-x_6x_1 + (x_5+1)x_2 + x_8x_3 - x_7x_4)u_1 \\ & + (-x_7x_1 - x_8x_2 + (x_5+1)x_3 + x_6x_4)u_2 \\ & + (-x_8x_1 + x_7x_2 - x_6x_3 + (x_5+1)x_4)u_3 \\ & + \frac{K}{I_1}((x_5+1)x_2x_6 - x_1x_6^2 + x_3x_6x_8 - x_4x_6x_7) \\ & + \frac{K}{I_2}((x_5+1)x_3x_7 - x_1x_7^2 + x_4x_6x_7 - x_2x_7x_8) \\ & + \frac{K}{I_3}((x_5+1)x_4x_8 - x_1x_8^2 + x_2x_7x_8 - x_3x_6x_8)\end{aligned} \tag{6.48}$$

One choice for the controls is

$$\begin{aligned} u_1 &= \frac{-K_1I_1 - K((x_5+1)x_2x_6 - x_1x_6^2 + x_3x_6x_8 - x_4x_6x_7)}{I_1(-x_6x_1 + (x_5+1)x_2 + x_8x_3 - x_7x_4)} \\ u_2 &= \frac{-K_2I_2 - K((x_5+1)x_3x_7 - x_1x_7^2 + x_4x_6x_7 - x_2x_7x_8)}{(-x_7x_1 - x_8x_2 + (x_5+1)x_3 + x_6x_4)} \\ u_3 &= \frac{-K_3I_3 - K((x_5+1)x_4x_8 - x_1x_8^2 + x_2x_7x_8 - x_3x_6x_8)}{(-x_8x_1 + x_7x_2 - x_6x_3 + (x_5+1)x_4)} \end{aligned} \tag{6.49}$$

where K_1 through K_3 are positive constants; then $\dot{V} = -(K_1 + K_2 + K_3)$. As in the single-axis maneuvers, the coefficients K_1, K_2, K_3, and K may be adjusted so that a specific performance index is minimized to determine an optimal stable feedback control law.

6.6 Conclusions

Generalized momenta variables corresponding to the Euler parameters have been used to parameterize rigid-body rotational dynamics. The constant-gain polynomial feedback law could not be used for this system since p_0 and β_0 produce two zero eigenvalues, and their eigenvectors could not be determined accurately. Closed-form time-dependent gains could be determined, however, and Lyapunov feedback control laws were found for both the single and three-axis external torque maneuvers. By adjusting constants in the Lyapunov laws, an optimal control could be found to minimize a specific performance index.

The choice of angular velocities over the conjugate angular momenta as state and feedback variables is clear; not only are there fewer states with the angular velocities, but constant gains that are more easily implemented can be used.

Chapter VII

STABILITY

Stability is of some concern in the development of feedback control laws. A general and thorough treatment of stability for non-linear systems is beyond the scope of this dissertation; however, several theorems from Brauer and Nohel [20] will be presented with a discusssion of how they may be applied to certain systems. (For proofs of these theorems, see Ref. [20].)

A differential equation or system of differential equations of the form

$$\dot{\underline{y}} = \underline{f}(t,\underline{y}) \tag{7.1}$$

is said to be <u>stable</u> if for every $\varepsilon > 0$ and $t_0 \geq 0$ there is a $\delta > 0$ such that

(1) if $\underline{\phi}(t)$ is a solution of Eq. (7.1)

(2) $|\phi(t_0) - \underline{y}_0| < \delta$

then

(3) the solution $\underline{\phi}(t)$ exists for all $t \geq t_0$

(4) and $|\underline{\phi}(t) - \underline{\psi}(t,\underline{y}_0)| < \varepsilon$ for $t \geq t_0$

where $\underline{\psi}(t,\underline{y}_0)$ is the solution of Eq. (7.1) corresponding to $\underline{y}_0$. In other words, if $\underline{y}_0$ is a critical point, i.e. $\underline{f}(\underline{y}_0) = 0$, then any motion that begins in the neighborhood of $\underline{y}_0$ remains in that neighborhood. Furthermore, if $\underline{\phi}(t)$ approaches $\underline{\psi}(t,\underline{y}_0)$ as $t \to \infty$, i.e.

$$\lim_{t\to\infty} |\underline{\phi}(t) - \underline{\psi}(t,\underline{y}_0)| = 0 \tag{7.2}$$

then $\underline{\phi}(t)$ is asymptotically stable.

In the systems considered in this dissertation, the target states have been zero or a fixed constant. If we make a change of variables so that all the final states are zero, then the critical points for the systems we have studied are $\underline{y}_0 = 0$.

A simple stability test exists for linear systems with constant coefficients. Given the linear system

$$\dot{\underline{y}} = C\underline{y} \tag{7.3}$$

then the motion about $\underline{y}_0 = 0$ is stable if all the eigenvalues of C have nonpositive real parts and eigenvalues with zero real parts are simple. If all the eigenvalues have negative real parts, then $\underline{y}(t)$ asymptotically approaches $\underline{y}_0 = 0$ as $t \to \infty$. For linear constant-gain feedback control systems, we need to evaluate the closed-loop poles of the system, i.e., the eigenvalues of the matrix $C = A - BR^{-1}B^TK$.

If we consider linear time-dependent gains, then we must examine the system

$$\dot{\underline{y}} = (C + S(t))\underline{y} \tag{7.4}$$

where C is the constant part of the closed loop system and S(t) is the time-dependent part. If all the eigenvalues of C have negative real parts and S(t) is continuous with

$$\lim_{t\to\infty} S(t) = 0 \tag{7.5}$$

then solutions of Eq. (7.4) in the neighborhood of $\underline{y} = 0$ are asymptotically stable. Note that this theorem does not necessarily require that the uncontrolled system be stable; the feedback gains can be written as

$$K(t) = K_1 + K_2(t) \tag{7.6}$$

where K_1 is the constant, steady-state part of the gain and K_2 the time-dependent part. To apply the theorem, the matrix $C = A\text{-}BR^{-1}B^TK_1$ must have eigenvalues with negative real parts, and $S(t) = -\ BR^{-1}B^TK_2$ should satisfy condition (7.5).

Nonlinear state equations with nonlinear feedback control may be written as

$$\dot{\underline{y}} = A\underline{y} + \underline{F}(t,\underline{y}) - BR^{-1}B^T\{K\underline{y} + \underline{G}(\underline{y})\} \tag{7.7}$$

where the linear part of the system is

$$C\underline{y} = (A - BR^{-1}B^TK)\underline{y} \tag{7.8}$$

and the nonlinear part is

$$\underline{f}(t,\underline{y}) = \underline{F}(t,\underline{y}) - BR^{-1}B^T\underline{G}(\underline{y}) \tag{7.9}$$

To examine stability of this system, we first consider the linear part of the equations. All the eigenvalues of C must have negative real parts if the linear part of the system is asymptotically stable. Given this, we then examine the nonlinear part of the system. If $\underline{f}$ and $\partial\underline{f}/\partial\underline{y}$ are continuous for $|\underline{y}|$ bounded, and

$$\lim_{|\underline{y}|\to 0} \frac{|\underline{f}(t,\underline{y})|}{|\underline{y}|} = 0 \tag{7.10}$$

uniformly, then all solutions of Eq. (7.7) in some neighborhood of $\underline{y}_0 = 0$ are asymptotically stable. Hence, if the linear part of the closed-loop system is asymptotically stable, then quadratic and higher-order polynomial feedback of polynomial state equations is also stable, as long as the coefficients are bounded.

Note that the closed-loop poles of the linear system must be in the left-half plane; for systems with redundant variables we obtain zero poles, and these may lead to instability in the nonlinear system as shown in Case 5C. If, however, we constrain the state corresponding to the zero eigenvalue to approach its target state as $t \to \infty$, and all the controllable modes have negative real parts, then the linear part of the system will be asymptotically stable. In these cases, higher order polynomial feedback with constant bounded coefficients is continuous and satisfies condition (7.10). If time-dependent coefficients are used in higher-order feedback, those coefficients must be continuous and bounded to meet condition (7.10) [34].

In summary, when the linear part of the closed-loop system is asymptotically stable, the higher order polynomial terms will not destabilize the system as long as the gains are bounded, and continuous in the time-dependent case. Note that this applies only to situations where there are linear terms in the state equations; when time-dependent gains are used with quadratic or cubic equations, as in Chapter VI, these theorems do not apply. To guarantee stability, the Lyapunov control method should be used.

Chapter VIII

CONCLUSIONS, COMMENTS, AND RECOMMENDATIONS

A straight-forward method of polynomial feedback control for nonlinear plants with polynomial state equations has been developed from optimal control theory, and successfully demonstrated for several systems. The linear gains were found to satisfy the familiar Riccati equation of linear system theory, and sets of linear equations were solved sequentially to determine higher order gains. A suboptimal control law is generated by taking a finite number of terms in the series.

Some of the systems considered in this dissertation produced degenerate Riccati equations. In particular, when no linear terms are present in the state equations, and the number of controls is less than the number of state variables, then constant linear gains could not be found. To circumvent this problem, linear terms were introduced in the state equations by a change of variable, or else time-dependent gains were used. In the time-dependent case, closed-form polynomials in t were determined for the feedback coefficients.

The Euler parameter kinematics of Chapters IV through VI provide quadratic or cubic polynomial equations rather than the transcendental equations of Euler angle parameterizations. They also present several problems when used as feedback variables. Since two sets of Euler parameters correspond to one physical location, four combinations of boundary conditions are possible in the formulation of the optimal control problem. Certain combinations produce feedback polynomials that diverge as higher order terms are added, or that are unstable. However, the correct combination of boundary conditions can be easily identified by calculating the closed-loop poles of the linear system to eliminate unstable combinations, and then a linear simulation is made to identify the set that minimizes the performance index.

The Euler parameters are also redundant, so that a zero pole corresponding to β_0 (or δ_0) occurs in the linear analysis. Note that β_0 is not a physically uncontrollable mode; the Euler parameters are guaranteed to satisfy the quadratic constraint $\Sigma\beta_i^2 = 1$, if (i) proper initial conditions have been imposed, and (ii) they satisfy the Euler parameter kinematic equations. However, asymptotic stability of the linear system cannot be guaranteed since this nonlinear constraint cannot be rigorously enforced by a linear differential equation. This instability is observed in Case 5C, where β_0 (δ_0) penalties were not included in an example with dominant nonlinear terms; the instability was eliminated when β_0 (δ_0) was required to approach its target state by weighing it in the performance index. The linear system was then asymptotically stable, and the nonlinear terms did not have a destabilizing effect.

When only one redundant variable was present, Potter's method could be used for solving the algebraic Riccati equation. Unfortunately, when a system contains more than one redundant variable, the set of eigenvectors corresponding to multiple zero eigenvalues cannot be determined accurately enough to provide reasonable linear gains. By assuming time-dependent gains, however, the coefficients were found to be polynomials in time determined analytically by integration.

Several comments about this method need to be made. Although straight-forward, the algebra generated by this method becomes prohibitive for even a modest number of state variables and terms in the series. All the algebra for the three, seven, or eight state variable systems of Chapters IV through VI was performed on the IBM 370 algebraic manipulator FORMAC, which produced Fortran code for the elements of the quadratic coefficient matrix. As more terms are taken in the series, the linear systems determining the polynomial coefficients require huge regions of computer core. Hence the computational burden of this method limits the system size and polynomial order of the state equations.

Recommendations for future study include application of the time-dependent gain method developed in Chapter VI to the attitude control systems of Chapters IV and V. Stability of this method should also be examined, since the theorems of Chapter VII do not apply to systems without linear terms.

The computational storage limitations discussed above suggest that partial state feedback methods should be investigated. A subset of state variables could be used in the polynomial expansions, and a representative submatrix of the linear system generating the coefficients could be solved. One attractive possibility is to remove all linear Euler parameter feedback and replace the quadratic terms with three direction cosines. This substitution not only reduces the number of feedback terms, but also removes the sign ambiguity of the Euler parameter boundary conditions. A judicious choice of nonsingular equations to determine the coefficients would then need to be made.

This discussion concludes the presentation of polynomial feedback control for spacecraft attitude maneuvers.

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Appendix A

CONJUGATE ANGULAR MOMENTA

Recent developments of generalized variables for rotational dynamics have been presented by Morton [14]. A set of four generalized angular momentum variables that are conjugate to the four Euler parameters are defined, and the equations of motion corresponding to Euler's equations are derived.

The orientation of a spacecraft body-fixed reference frame to an inertial frame may be parameterized by the Euler parameters of Eq. (4.3). The kinematic equations relating the Euler parameters to the body-frame components ω_i of the spacecraft angular velocity are given by Eq. (4.8) and repeated below for reference.

$$\begin{Bmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \tag{A.1}$$

or

$$\{\dot{\underline{\beta}}\} = \frac{1}{2} [Q(\beta)]\{\underline{\omega}\}_4 \tag{A.2}$$

where $\{\underline{\omega}\}_4 = \{0 \quad \omega_1 \quad \omega_2 \quad \omega_3\}^T$ (A.3)

The rotational equations of motion were derived in Chapter IV from Eq. (4.9), and the use of a principal axis system for the body reference frame gave us Euler's equations, Eq. (4.13). They may be rewritten as four equations as follows

$$\begin{Bmatrix} 0 \\ \dot{H}_1 \\ \dot{H}_2 \\ \dot{H}_3 \end{Bmatrix} = - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_3 & \omega_2 \\ 0 & \omega_3 & 0 & -\omega_1 \\ 0 & -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ H_1 \\ H_2 \\ H_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{A.4}$$

where $H_i = I_i\omega_i$ (A.5)

and u_i are the external control torques. The rotational kinetic energy of the system is

$$T = \frac{1}{2} \sum_{i=1}^{3} I_i\omega_i^2 \tag{A.6}$$

As is done in Lagrangian and Hamiltonian mechanics, we can define generalized angular momentum variables as follows

$$p_i = \frac{\partial T}{\partial \dot{\beta}_i} \qquad i = 0,1,2,3 \tag{A.7}$$

Since $[Q(\beta)]$ is an orthogonal matrix [35], Eq. (A.2) may be easily inverted and substituted into Eq. (A.7) to yield four expressions

$$\{\underline{p}\} = 2[Q(\beta)]\{\underline{H}\}_4 \tag{A.8}$$

where $\{\underline{H}\}_4 = \{0 \quad H_1 \quad H_2 \quad H_3\}^T$ (A.9)

Morton [14] has presented the following properties of the conjugate angular momenta

$$\underline{\beta} \cdot \underline{p} = 0 \tag{A.10}$$

$$\dot{\underline{\beta}} \cdot \underline{p} = 2T \tag{A.11}$$

$$\underline{p} \cdot \underline{p} = 4H^2 \tag{A.12}$$

where H is the magnitude of the angular momentum vector. Since rotational motion has three degrees of freedom and we have four conjugate angular momentum variables, the p_i are once redundant and must satisfy the constraint Eq. (A.12).

To use the conjugate angular momenta as generalized variables, Euler's equations and the kinematic equations must be rewritten to remove the ω's in favor of the p's. Eq. (A.8) may be rewritten as

$$\{\underline{H}\}_4 = \frac{1}{2}\,[Q(\beta)]^T\{\underline{p}\} \tag{A.13}$$

and from Eq. (A.5) we obtain

$$\{\underline{\omega}\}_4 = [I^{-1}]_4\{\underline{H}\}_4 \tag{A.14}$$

where $[I^{-1}]_4 = \text{diag}\ \{0,\ 1/I_1,\ 1/I_2,\ 1/I_3\}$ (A.15)

When a principal axis system is not used, $[I^{-1}]_4$ is the 4x4 matrix with zeros in the first row and column and the inverse of the usual inertia matrix in the lower 3x3 submatrix. By substituting Eq. (A.13) into Eq. (A.14), we obtain $\{\underline{\omega}\}_4$ in terms of $\{\underline{\beta}\}$ and $\{\underline{p}\}$

$$\{\underline{\omega}\}_4 = \frac{1}{2}\,[I^{-1}]_4\ [Q(\beta)]^T\ \{\underline{p}\} \tag{A.16}$$

Eq. (A.16) is then used in Eq. (A.2) to obtain the following kinematic equations

$$\{\underline{\dot{\beta}}\} = \frac{1}{4}\,[Q(\beta)][I^{-1}]_4[Q(\beta)]^T\{\underline{p}\} \tag{A.17}$$

Euler's equations must also be expressed in terms of $\{\underline{p}\}$ and $\{\underline{\beta}\}$. If we differentiate Eq. (A.8) with respect to time, we obtain

$$\{\underline{\dot{p}}\} = 2[Q(\dot{\beta})]\{\underline{H}\}_4 + 2[Q(\beta)]\{\underline{\dot{H}}\}_4 \tag{A.18}$$

The kinematic equations, Eq. (A.2), express $\{\dot{\underline{\beta}}\}$ in terms of $\{\underline{\beta}\}$ and $\{\underline{\omega}\}$, so that

$$[Q(\dot{\beta})] = \frac{1}{2}\,[Q(\beta)][Q(\omega_4)] \tag{A.19}$$

Eq. (A.4) may be rewritten in matrix form as

$$\{\dot{\underline{H}}\}_4 = \frac{1}{2}\,[R(\omega_4) - Q(\omega_4)]\,\{\underline{H}\}_4 + \{\underline{u}\}_4 \tag{A.20}$$

where

$$[R(\omega_4)] = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{A.21}$$

By substituting Eqs. (A.19) and (A.20) into Eq. (A.18), we obtain

$$\{\dot{\underline{p}}\} = [Q(\beta)][R(\omega_4)]\,\{\underline{H}\}_4 + 2[Q(\beta)]\{\underline{u}\}_4 \tag{A.22}$$

But $\quad [R(\omega_4)]\{\underline{H}\}_4 = [Q(H_4)]\{\underline{\omega}\}_4 \qquad\qquad$ (A.23)

and $[Q(\beta)][Q(H_4)] = \frac{1}{2}\,[Q(p)] \qquad\qquad$ (A.24)

from Eq. (A.8), so that

$$\{\dot{\underline{p}}\} = \frac{1}{2}\,[Q(p)]\{\underline{\omega}\}_4 + 2[Q(\beta)]\{\underline{u}\}_4 \tag{A.25}$$

By rewriting Eq. (A.16) as

$$\{\underline{\omega}\}_4 = -\,\frac{1}{2}\,[I^{-1}]_4[Q(p)]^T\{\underline{\beta}\} \tag{A.26}$$

and substituting into Eq. (A.25), we obtain the following four equations of motion

$$\{\dot{\underline{p}}\} = -\frac{1}{4}[Q(p)][I^{-1}]_4[Q(p)]^T\{\underline{\beta}\} + 2[Q(\beta)]\{\underline{u}\}_4 \qquad \text{(A.27)}$$

Hence we have four kinematic equations, Eqs. (A.17), and four dynamic equations, Eqs. (A.27). It is interesting to note that when no external torques are present, Eqs. (A.17) and (A.27) with $\underline{u} = 0$ may be derived from Hamilton's canonical form

$$\begin{aligned} \dot{\beta}_i &= \frac{\partial T}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial T}{\partial \beta_i} \end{aligned} \qquad \text{(A.28)}$$

Appendix B

POTTER'S METHOD

The algebraic matrix Riccati equation is

$$Q + A^TK + KA - KBR^{-1}B^TK = 0 \tag{B.1}$$

where A, B, R and Q are constant nxn matrices and K is the unknown matrix to be determined. Potter's method first determines the eigenvalues and eigenvectors of a 2n x 2n matrix M

$$M = \left[\begin{array}{c|c} A^T & Q \\ \hline BR^{-1}B^T & -A \end{array}\right] \tag{B.2}$$

If Q and R are positive definite symmetric real matrices, then the eigenvalues of M fall into two groups, the eigenvalues $\lambda_1,\ldots,\lambda_n$ with positive real parts, and the eigenvalues $-\lambda_1,\ldots,-\lambda_n$. The n eigenvectors corresponding to $\lambda_1,\ldots,\lambda_n$ are put by columns into a 2nxn matrix, with the first n rows forming the matrix D and the lower n rows forming matrix C. Then the positive semidefinite solution to Eq. (B.1) is

$$K = DC^{-1} \tag{B.3}$$

To see that the solution matrix K is Eq. (B.3), we form a matrix G

$$G = BR^{-1}B^TK-A \tag{B.4}$$

Then, using Eq. (B.1),

$$KG = Q + A^TK \tag{B.5}$$

Let S transform G into Jordan canonical form

$$S^{-1}GS = J \tag{B.6}$$

It can be shown that J is diagonal (see [17]). Let

$$R = KS \tag{B.7}$$

Then $RJ = QS + A^TR$ (B.8)

from Eqs. (B.6) and (B.5), and

$$SJ = BR^{-1}B^TR - AS \tag{B.9}$$

from Eqs. (B.6) and (B.4). Eqs. (B.8) and (B.9) may be rewritten as

$$\begin{bmatrix} R \\ S \end{bmatrix} J = \left[\begin{array}{c|c} A^T & Q \\ \hline BR^{-1}B^T & -A \end{array}\right] \begin{bmatrix} R \\ S \end{bmatrix} \tag{B.10}$$

So J contains eigenvalues of M, with corresponding eigenvectors $\begin{bmatrix} R \\ S \end{bmatrix}$.
But from Eq. (B.7)

$$K = RS^{-1} \tag{B.11}$$

which is the same as Eq. (B.3). To show that the eigenvalues in J, which are the eigenvalues of matrix G, have positive real parts, we examine

$$G^TK + KG = Q + KBR^{-1}B^TK \tag{B.12}$$

which follows from Eqs. (B.4) and (B.1). If Q and R are symmetric positive definite matrices, then the right hand side of Eq. (B.12) is positive definite, and so G is positive definite. Hence the eigenvalues of G have positive real parts.

To show that $K = RS^{-1}$ is the positive semidefinite solution to Eq. (B.1), we form matrix

$$P = S^T R \tag{B.13}$$

Then $K = S^{-T} P S^{-1}$, and if we show that P is postive semidefinite, then so is matrix K. Define a 2nxn matrix

$$U(t) = [e^{-\lambda_1 t} a_1 \quad \ldots \quad e^{-\lambda_n t} a_n] \tag{B.14}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of M with positive real parts, and $a_1, \ldots, a_n$ are their corresponding eigenvectors (i.e., $a_1, \ldots, a_n$ are the columns of matrix $\begin{bmatrix} R \\ S \end{bmatrix}$). Then

$$\frac{d}{dt} U(t) = - MU(t) \tag{B.15}$$

and $$P = U^T(0) \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} U(0) \tag{B.16}$$

But since U(t) is made up of eigenvalues with positive real parts, then $U(t) \to 0$ as $t \to \infty$, and

$$P = - \int_0^\infty \frac{d}{dt} \left\{ U^T(t) \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} U(t) \right\} dt \tag{B.17}$$

or $$P = \int_0^\infty U^T(t) \left\{ M^T \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} M \right\} U(t)\, dt \tag{B.18}$$

by using Eq. (B.15). Examining Eq. (B.18)

$$M^T \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} M = \begin{bmatrix} B\ R^{-1} B^T & 0 \\ 0 & Q \end{bmatrix} \tag{B.19}$$

which is positive semidefinite, so the integrand in Eq. (B.18) is positive semidefinite, and $K = RS^{-1}$ is the positive semidefinite solution to Eq. (B.1).

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