PROPERTIES OF COCONTINUOUS FUNCTIONS AND COCOMPACT SPACES

by

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Not for direction or assistance, but for love, he especially thanks his girls.

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Chapter I

Introduction

The concept of cotopologies originated from a seminar in 1964 under the direction of J. de Groot and J. M. Aarts at the Mathematical Centre in Amsterdam. Although the original idea was de Groot's, it should be noted that the first text, Colloquium Co-topologie, was written by J. M. Aarts. The initial idea was to study the properties of a given topological space by introducing a special weaker topology, (i.e., cotoplogy). The general background and earliest results concerning cotopologies can be found in [1], [2], and [3]. Later results associating cotopologies with the theory of absolutely closed and minimal spaces can be found in the dissertations of G. Strecker [19] and G. Viglino [22]. This paper will deal mainly with the concepts of cocompactness and cocontinuity as they originally appeared in [1]. In particular we investigate cocompactness as a local property, and we provide more results concerning cocontinuous functions.

In Chapter II we define a closed base for a topological space and use this property in the development of a cotopology for the space. Initially we show under what conditions subspace topologies have inherited cotopologies. We then prove that a space is locally compact if and only if the family of all compact closed sets is a closed base. Consequently, we have exhibited a particular closed base for a locally compact space. We also show that for any topological space the collection of all regular-closed subsets is a closed base. These two results will be beneficial in the study of cocontinuous functions in Chapter III. Finally, we prove that if A and B are closed subsets of a space with closed bases $\mathcal{C}$ and $\mathcal{D}$, respectively, then $\mathcal{K} = \{C \mid C \in \mathcal{C}\} \cup \{D \mid D \in \mathcal{D}\} \cup \{(C \cup D) \mid C \in \mathcal{C} \text{ and } D \in \mathcal{D}\}$ is a closed base for $(A \cup B)$. This result will be used to prove the main result of Chapter IV, namely, that the union of a cocompact closed subset of (X,T) and a compact closed subset of (X,T) is cocompact.

Chapter III is concerned with the notion of cocontinuous functions. After developing some basic concepts of cocontinuous functions we obtain as a corollary the known result, that if $f:(X,T) \rightarrow (Y,U)$ is a one to one continuous function from a compact space onto a T_2 space, then f is a homeomorphism. The next principal result is that if (Y,U) is a locally compact T_2 space, then $f:(X,T) \rightarrow (Y,U)$ is cocontinuous if and only if the inverse image of every open subset O of Y such that $(Y - O)$ is compact is open in X. Finally we relate the concept of cocontinuity to various weaker forms of continuous functions and provide counterexamples whenever necessary.

Chapter IV is divided into two parts. In Part I we give a partial answer to a question posed by J. M. Aarts as to when the union of cocompact subsets of a space is cocompact. The partial answer is that if A is a closed cocompact subset of (X,T), and B is a closed compact subset of (X,T), then (A ∪ B) is a cocompact subset of (X,T). In Part II cocompactness is investigated as a local property, and we show that local cocompactness is a stronger property for a space than property L, which was introduced by R. A. McCoy in [7].

Finally, we ask the following question: Is a function that is cocontinuous at each point a cocontinuous function? We also indicate how this question is related to an earlier problem of Albert Verbeek-Kroonenberg in [21]. He asked, "In the case that there exists no smallest cotopology, do there exist two cotopologies on X whose intersection is not a cotopology?" We conclude with several remarks concerning an unpublished paper by P. Fletcher that will bear upon several of the results found in Chapter IV, Part II.

Chapter II

Closed Bases and Cotopologies

In this chapter we study the notions of a closed base and a cotopology for a topological space. We initially show under what conditions a subspace inherits a cotopology from the original space. We also show that if A and B are closed subsets with closed bases $\mathcal{C}$ and $\mathcal{D}$ respectively, then $\mathcal{K} = \{C | C \in \mathcal{C}\} \cup \{D | D \in \mathcal{D}\} \cup \{(C \cup D) | C \in \mathcal{C} \text{ and } D \in \mathcal{D}\}$ is a closed base for $(A \cup B)$. We conclude by showing that the collection of all regular-closed subsets of any topological space is a closed base and indicate several results concerning the cospace obtained from this closed base. The results of this chapter will be significant to the study of cocontinuous functions in Chapter III and the main result, Theorem 4.7, of Chapter IV.

Definition 2.1: A closed base for a topological space (X,T) is a family $\mathcal{B}$ of closed subsets of (X,T) such that for each point $x \in X$ and each $O \in T$ containing x there exists some $B \in \mathcal{B}$ such that $x \in \operatorname{int} B \subset B \subset \operatorname{Cl} O$.

Definition 2.2: Let (X,T) be a topological space and let $\mathcal{B}$ be a closed base for (X,T). The cotopology of T relative to $\mathcal{B}$ is the topology on X having the family $\{X - B | B \in \mathcal{B}\}$ as a subbase for its open sets. The cotopology of T relative to $\mathcal{B}$ will be denoted by $T_{\mathcal{B}}$, and $(X,T_{\mathcal{B}})$ will be called a <u>cospace</u> of (X,T).

<u>Convention</u>: If (X,T) is a topological space and T^* is a topology on X, then the phrase "T^* is a cotopology of T" will mean there exists some closed base $\mathcal{B}$ for $(X,T_{\mathcal{B}})$ such that $T^* = T_{\mathcal{B}}$.

<u>Notation</u>: Let (X,T) be a topological space, let $A \subset X$, and let $\mathcal{B}$ be a closed base for (X,T). Then T_A will represent the subspace topology on A, and $T_{\mathcal{B}}$ will represent the cotopology of T relative to $\mathcal{B}$. No confusion should arise from these notations since the script letter will always indicate a collection of subsets of the space. Also, when indicating the closure or interior of a set A, if there is a question as to which topology the closure or interior is taken with respect to, we will indicate this with the notation $Cl_T A$, the closure of A with respect to T, or $int_T A$, the interior of A with respect to T. If it is clear from the context as to how the closure or interior is taken, we will simply use $Cl\ A$ or $int\ A$.

<u>Example 2.3</u>: In obtaining a cotopology for a given topological space, it should be pointed out that there are both advantages and disadvantages. The real numbers with the usual topology is a T_2 space that is not compact. To obtain a compact cospace consider the collection of all closed intervals as a closed base. Although the cospace is compact, it is not T_2.

<u>Remark 2.4</u>: Definition 2.1 and Definition 2.2 originally appeared in [1], whereas the following theorem is found in [3] as an equivalent definition of a cotopology. The proof of this theorem is included since it did not appear in either [1] or [3].

<u>Theorem 2.5</u>: For any space (X,T) a toplogy T^* on X is a cotopology of T if and only if

i) $T^* \subset T$ and

ii) For each point $x \in X$ and each closed neighborhood V of x in (X,T) there exists some neighborhood U of x in (X,T) such that $U \subset V$ and U is closed in (X,T^*).

<u>Proof</u>: Suppose that T^* is a topology on X satisfying (i) and (ii). Let $\mathcal{B}$ be the collection of all closed subsets of (X,T^*). By (i) $\mathcal{B}$ is a family of closed subsets of (X,T). For each point $x \in X$ and each $O \in T$ containing x, $Cl_T O$ is a closed neighborhood of x. Therefore, by (ii) there exists some neighborhood U of x in (X,T) such that $x \in U \subset Cl_T O$ and U is closed in (X,T^*). Thus $U \in \mathcal{B}$ and $x \in int_T U \subset U \subset Cl_T O$. Thus $\mathcal{B}$ is a closed base for (X,T), and it is clear from the choice of $\mathcal{B}$ that $T_{\mathcal{B}} = T^*$.

Suppose that T^* is a cotopology of T relative to the closed base $\mathcal{B}$. By the indicated convention $T^* = T_{\mathcal{B}}$, and it is clear that $T^* \subset T$. Let $x \in X$ and let V be a closed neighborhood of x in (X,T). Since V is a closed neighborhood of x, there exists some $O \in T$ such that $x \in O \subset V$, and there exists some $B \in \mathcal{B}$ such that $x \in \mathrm{int}_T B \subset B \subset \mathrm{Cl}_T O \subset V$. Thus B is a neighborhood of x, $B \subset V$, and B is closed in (X,T^*).

<u>Remark 2.6</u>: Note that if (X,T) is regular, then $\mathcal{B}$ is a closed base for (X,T) if and only if for each point $x \in X$ and each $O \in T$ containing x there is a $B \in \mathcal{B}$ such that $x \in \mathrm{int}\, B \subset B \subset O$.

<u>Theorem 2.7</u>: If (X,T^*) is a cospace of (X,T) and $A \subset X$, then $(A,(T^*)_A)$ is a cospace of (A,T_A).

<u>Proof</u>: By Theorem 2.5, $T^* \subset T$. Therefore $(T^*)_A \subset T_A$. Let $x \in A$ and let V be a closed neighborhood of x in (A,T_A). There exists some closed neighborhood W of x in (X,T) such that $V = W \cap A$. Since T^* is a cotopology of T, there exists a neighborhood U of x in (X,T) such that $U \subset W$ and U is closed in (X,T^*). Therefore, $O = U \cap A$ is a neighborhood of x in (A,T_A) such that $O \subset V$ and O is closed in $(A,(T^*)_A)$. By Theorem 2.5, $(T^*)_A$ is a cotopology of T_A and $(A,(T^*)_A)$ is a cospace of (A,T_A).

<u>Theorem 2.8</u>: Let (X,T) be a space, let $\mathcal{B}$ be a closed base for (X,T), let $A \subset X$, and let $\mathcal{C} = \{B \cap A \mid B \in \mathcal{B}\}$.

i) If A is open in (X,T), then $\mathcal{C}$ is a closed base for (A,T_A).

ii) If (X,T) is regular, then $\mathcal{C}$ is a closed base for (A,T_A).

iii) If A is dense in X, then $\mathcal{C}$ is a closed base for (A,T_A).

Proof: Since $\mathcal{B}$ is a closed base for (X,T), for any $x \in A$ there exists a $B' \in \mathcal{B}$ such that $x \in \text{int } B' \subset B' \subset X$. Therefore, $B' \cap A \neq \emptyset$ and $\mathcal{C}$ contains nonempty sets. Also, since each $B \in \mathcal{B}$ is closed, $\mathcal{C}$ is a collection of closed subsets of (A,T_A).

i) Let $x \in A$ and let $O \in T_A$ containing x. Since $O \in T_A$, there exists some $V \in T$ such that $x \in O = V \cap A$. Also, since $\mathcal{B}$ is a closed base for (X,T), there exists some $B \in \mathcal{B}$ such that $x \in \text{int } B \subset B \subset \text{Cl } V$. Clearly we have $x \in \text{int}_{T_A}(B \cap A) \subset (B \cap A)$ where $(B \cap A) \in \mathcal{C}$. Let $y \in (B \cap A)$, and let $y \in U \in T_A$. Since A is open, we have that U is an open set in (X,T) containing y. Therefore, $(U \cap V) \neq \emptyset$, and since $U \subset A$, we conclude that $U \cap O \neq \emptyset$. It follows that $x \in \text{int}_{T_A}(B \cap A) \subset (B \cap A) \subset \text{Cl}_{T_A} O$. From Definition 2.1 we have that $\mathcal{C}$ is a closed base for (A,T_A).

ii) Let $x \in A$ and let $x \in O \in T_A$. Since $O \in T_A$, there exists some $V \in T$ such that $x \in O = V \cap A$. Also, since $\mathcal{B}$ is a closed base and (X,T) is regular, there exists a $B \in \mathcal{B}$ such that $x \in \text{int } B \subset B \subset V$. It is clear that $x \in \text{int}_{T_A}(B \cap A) \subset (B \cap A) \subset (V \cap A) = O$, where $(B \cap A) \in \mathcal{C}$.

iii) Let $x \in A$ and let $x \in O \in T_A$. Since $O \in T_A$ and since $\mathcal{B}$ is a closed base, there exists some $V \in T$ and some $B \in \mathcal{B}$ such that $x \in \text{int } B \subset B \subset \text{Cl } V$, where $O = V \cap A$. Also, since $V \in T$ and A is dense in (X,T), we have $\text{Cl } (V \cap A) = \text{Cl } V$. Therefore, $x \in ((\text{int}_T B) \cap A) \subset (B \cap A) \subset ((\text{Cl } V) \cap A) \subset \text{Cl } V = \text{Cl } (V \cap A) = \text{Cl } O$. It follows that $\mathcal{C}$ is a closed base for (A,T_A).

Theorem 2.9: Let (X,T) be a space, let $\mathcal{B}$ be a closed base for (X,T), and let $A \subset X$. If $\mathcal{C} = \{B \cap A | B \in \mathcal{B}\}$ is a closed base for (A,T_A), then $(T_A)_{\mathcal{C}} = (T_{\mathcal{B}})_A$.

Proof: Let $U \in (T_{\mathcal{B}})_A$. There exists a $V \in T_{\mathcal{B}}$ such that $U = V \cap A$, and hence $V = \bigcup_{\gamma} (\bigcap_{i=1}^{n} (X - B_i))_{\gamma}$ where $B_i \in \mathcal{B}$ for $i=1, \ldots, n$, and γ is in some indexing set, Γ. It follows that $U = \bigcup_{\gamma} (\bigcap_{i=1}^{n} (X - B_i))_{\gamma} \cap A = \bigcup_{\gamma} ((\bigcap_{i=1}^{n} (X - B_i))_{\gamma} \cap A) = \bigcup_{\gamma} (\bigcap_{i=1}^{n} A - (B_i \cap A))_{\gamma}$. Since $\{A - (B \cap A) | B \in \mathcal{B}\}$ is a subbase for $(T_A)_{\mathcal{C}}$, we have $U \in (T_A)_{\mathcal{C}}$. By a similar argument one can show that $(T_A)_{\mathcal{C}} \subset (T_{\mathcal{B}})_A$.

Corollary 2.10: Let (X,T) be a space, let $\mathcal{B}$ be a closed base for (X,T), and let $A \subset X$.

i) If A is open in (X,T), then $(T_{\mathcal{B}})_A = (T_A)_{\mathcal{C}}$, where $\mathcal{C} = \{B \cap A | B \in \mathcal{B}\}$.

ii) If (X,T) is regular, then $(T_{\mathcal{B}})_A = (T_A)_{\mathcal{C}}$, where $\mathcal{C} = \{B \cap A | B \in \mathcal{B}\}$.

iii) If A is dense in (X,T), then $(T_{\mathcal{B}})_A = (T_A)_{\mathcal{C}}$ where $\mathcal{C} = \{B \cap A | B \in \mathcal{B}\}$.

Proof: The corollary follows directly from Theorem 2.8 and Theorem 2.9.

The following theorem indicates a closed base for an open subset of a regular space that is different from the closed base found in Theorem 2.8. The example following the theorem will then show that the respective cotopologies may also be different.

Theorem 2.11: Let (X,T) be a regular space, let $\mathcal{B}$ be a closed base for (X,T), and let O be an open subset of (X,T).

i) $\mathcal{B}' = \{B \in \mathcal{B} \mid B \subset O\}$ is a closed base for (O,T_O).

ii) $(T_O)_{\mathcal{B}'} \subset (T_{\mathcal{B}})_O = (T_O)_{\mathcal{C}}$, where $\mathcal{C} = \{B \cap O \mid B \in \mathcal{B}\}$.

Proof: i) Let $x \in U \in T_O$. Since $O \in T$, we have $U \in T$ and there exists some $B \in \mathcal{B}$ such that $x \in \mathrm{int}_T B \subset B \subset U$. Therefore, $B \in \mathcal{B}'$ and $x \in \mathrm{int}_{T_O} B \subset B \subset U$. It follows that $\mathcal{B}'$ is a closed base for (O,T_O).

ii) By Corollary 2.10, part (i), it follows that $(T_{\mathcal{B}})_O = (T_O)_{\mathcal{C}}$. If $B \in \mathcal{B}'$, then $B \in \mathcal{B}$ and $B \subset O$. Thus we have that $B = (B \cap O) \in \mathcal{C}$. Since $\mathcal{B}' \subset \mathcal{C}$, $(T_O)_{\mathcal{B}'} \subset (T_O)_{\mathcal{C}} = (T_{\mathcal{B}})_O$.

Example 2.12: Let $(X,T) = ([0,4], U_{[0,4]})$ where U is the usual topology for the real numbers. Since every space is a cospace of itself with respect to the collection $\mathcal{B}$ of all closed subsets, let $(X,T_{\mathcal{B}}) = (X,T)$. If $O = (0,1)$, then by part (i) of Theorem 2.11, $\mathcal{B}' = \{B \in \mathcal{B} \mid B \subset O\}$ is a closed base for (O,T_O). Since (X,T) is compact, we have from [1, 1.6, Proposition 1] that $(T_O)_{\mathcal{B}'}$ is a compact cotopology of T_0. But $(T_{\mathcal{B}})_0 = T_0$, and (O,T_0) is not compact. It follows that $(T_0)_{\mathcal{B}'} \neq (T_{\mathcal{B}})_0$.

The following result concerning closed bases will be necessary to the proof of a result in Chapter IV, Theorem 4.7.

Theorem 2.13: Let A and B be closed subsets of the space (X,T), let $\mathcal{C}$ be a closed base for (A,T_A), and let $\mathcal{D}$ be a closed base for (B,T_B). Then $\mathcal{K} = \{C \mid C \in \mathcal{C}\} \cup \{D \mid D \in \mathcal{D}\} \cup \{C \cup D \mid C \in \mathcal{C}, D \in \mathcal{D}\}$ is a closed base for $(A \cup B, T_{A \cup B})$.

Proof: Since each of A, B, and $(A \cup B)$ is closed in (X,T), $\mathcal{K}$ is a family of closed subsets of $(A \cup B, T_{A \cup B})$.

a) Let $x \in (A \cup B)$ such that $x \in A$ and $x \notin B$, and let $O \in T_{A \cup B}$ containing x. Since $O \in T_{A \cup B}$, there exists a set $O' \in T$ containing x such that $O = O' \cap (A \cup B) = (O' \cap A) \cup (O' \cap B)$. Since $(O' \cap A) \in T_A$ and contains x, there is a $C \in \mathcal{C}$ such that $x \in \mathrm{int}_{T_A} C \subset C \subset \mathrm{Cl}_{T_A}(O' \cap A)$. Let $y \in C$ and let $W_y \in T_{A \cup B}$ containing y. Since $y \in C$, we have $y \in A$, and since $W_y \in T_{A \cup B}$ there exists some $Z_y \in T$ such that $W_y = Z_y \cap (A \cup B) = (Z_y \cap A) \cup (Z_y \cap B)$. Therefore, $y \in (Z_y \cap A) \in T_A$, and since $y \in C \subset \mathrm{Cl}_{T_A}(O' \cap A)$, $(Z_y \cap A) \cap (O' \cap A) \neq \emptyset$. It follows that $W_y \cap (O' \cap A) \neq \emptyset$. Therefore, $y \in \mathrm{Cl}_{T_{A \cup B}}(O' \cap A)$ and $C \subset \mathrm{Cl}_{T_{A \cup B}}(O' \cap A)$. Also, since $(O' \cap A) \subset O$, $\mathrm{Cl}_{T_{A \cup B}}(O' \cap A) \subset \mathrm{Cl}_{T_{A \cup B}} O$, we have that $C \in \mathcal{K}$ such that $\mathrm{int}_{T_{A \cup B}} C \subset C \subset \mathrm{Cl}_{T_{A \cup B}}(O' \cap A) \subset \mathrm{Cl}_{T_{A \cup B}} O$.

To finish the argument we need only show that $x \varepsilon \text{int}_{T_{AUB}} C$. Since $x \varepsilon \text{int}_{T_A} C$, there exists some $Z_x \varepsilon T$ such that $\text{int}_{T_A} C = Z_x \cap A$. Since the set $(Z_x - B)$ is open in (X,T) and contains x, $x \varepsilon (Z_x - B) \cap (A \cup B)$, which is open in $(A \cup B)$. Also, $(Z_x - B) \cap (A \cup B) = ((Z_x - B) \cap A) \cup ((Z_x - B) \cap B) = (Z_x - B) \cap A \subset (Z_x \cap A) = \text{int}_{T_A} C \subset C$. We therefore conclude that there is a $C \varepsilon \mathcal{K}$ such that $x \varepsilon \text{int}_{T_{A \cup B}} C \subset C \subset \text{Cl}_{T_{A \cup B}} O$.

b) For $x \varepsilon (A \cup B)$ such that $x \varepsilon B$ and $x \not\varepsilon A$, an argument similar to part (a) will prove the desired results.

c) Let $x \varepsilon (A \cup B)$ such that $x \varepsilon A$ and $x \varepsilon B$, and let $O \varepsilon T_{A \cup B}$ containing x. Since $O \varepsilon T_{A \cup B}$, there exists some $O' \varepsilon T$ containing x such that $O = O' \cap (A \cup B) = (O' \cap A) \cup (O' \cap B)$. Since $x \varepsilon A$ and $x \varepsilon B$, we have that $x \varepsilon (O' \cap A) \varepsilon T_A$ and that $x \varepsilon (O' \cap B) \varepsilon T_B$. Therefore, there exists some $C \varepsilon \mathcal{C}$ and some $D \varepsilon \mathcal{D}$ such that $x \varepsilon \text{int}_{T_A} C \subset C \subset \text{Cl}_{T_A} (O' \cap A)$, and $x \varepsilon \text{int}_{T_B} D \subset D \subset \text{Cl}_{T_B} (O' \cap B)$. Notice also, that as in part (a) since $C \subset \text{Cl}_{T_A} (O' \cap A)$, $C \subset \text{Cl}_{T_{A \cup B}} O$ and similarly $D \subset \text{Cl}_{T_{A \cup B}} O$.

We now have that $x \varepsilon (C \cup D)$, $(C \cup D) \varepsilon \mathcal{K}$, and that $\text{int}_{T_{A \cup B}} (C \cup D) \subset (C \cup D) \subset \text{Cl}_{T_{A \cup B}} O$. We need only show that $x \varepsilon \text{int}_{T_{A \cup B}} (C \cup D)$. Since $x \varepsilon \text{int}_{T_A} C$ and since $x \varepsilon \text{int}_{T_B} D$, there exist sets V_x and W_x in T such that $\text{int}_{T_A} C = V_x \cap A$ and $\text{int}_{T_B} D = W_x \cap B$. Consider the set $(V_x \cap W_x) \cap (A \cup B)$ which contains x and is open in $(A \cup B)$. We have that $(V_x \cap W_x) \cap (A \cup B) = ((V_x \cap W_x) \cap A) \cup ((V_x \cap W_x) \cap B)$ $\subset (V_x \cap A) \cup (W_x \cap B) = \mathrm{int}_{T_A} C \cup \mathrm{int}_{T_B} D \subset (C \cup D)$. Therefore, we have that there exists a $(C \cup D) \in \mathcal{H}$ such that $x \in \mathrm{int}_{T_{A \cup B}} (C \cup D) \subset (C \cup D) \subset \mathrm{Cl}_{T_{A \cup B}} O$, and we conclude that $\mathcal{H}$ is a closed base for $(A \cup B, T_{A \cup B})$.

We will now indicate a particular closed base for a locally compact space, and a particular closed base for any topological space. The cotopologies determined by these closed bases will be necessary to the study of cocontinuous functions in Chapter III.

<u>Definition 2.14</u>: A space (X,T) is <u>locally compact</u> provided that for each point $x \in X$ there exists some $O \in T$ containing x such that $\mathrm{Cl}\ O$ is compact.

<u>Theorem 2.15</u>: A space (X,T) is locally compact if and only if the family of all compact closed sets is a closed base for (X,T).

<u>Proof</u>: Suppose (X,T) is locally compact. The conclusion follows directly from the proof of Proposition 1, Section 1.6 of [1].

Suppose the collection $\mathcal{B}$ of all compact closed sets in (X,T) is a closed base. For each $x \in X$ there exists some $B \in \mathcal{B}$ such that $x \in \mathrm{int}\ B \subset B \subset X$. Since B is closed and compact, $\mathrm{int}\ B$ is an open set containing x such that Cl $(\mathrm{int}\ B)$ is compact.

Definition 2.16: Let (X,T) be a space. A subset $C \subset X$ is said to be a regular-closed (regular-open) set if $C = \mathrm{Cl}(\mathrm{int}\ C)$. $(C = \mathrm{int}\ (\mathrm{Cl}\ C))$.

Theorem 2.17: Let (X,T) be a space. A subset C of X is a regular-closed set if and only if $(X - C)$ is a regular-open set.

Proof: Suppose C is a regular-closed set in X. Since for any set A, $\mathrm{int}\ A = X - \mathrm{Cl}\ (X - A)$, and since $C = \mathrm{Cl}(\mathrm{int}\ C)$, we have that $\mathrm{int}\ (\mathrm{Cl}(X - C)) = X - \mathrm{Cl}\ (X - \mathrm{Cl}(X - C)) = X - \mathrm{Cl}(\mathrm{int}\ C) = X - C$. It follows that $(X - C)$ is a regular-open set.

Suppose $(X - C)$ is a regular-open set. Then $C = X - (X - C) = X - \mathrm{int}(\mathrm{Cl}\ (X - C)) = X - (X - \mathrm{Cl}\ (X - \mathrm{Cl}\ (X - C))) = X - (X - \mathrm{Cl}\ (\mathrm{int}\ C)) = \mathrm{Cl}\ (\mathrm{int}\ C)$. Therefore, C is a regular-closed set.

Definition 2.18: Let (X,T) be a space. The topology generated by the regular-open sets is called the semiregularization of T.

Theorem 2.19: Let (X,T) be a space.

i) The collection $\mathcal{R}$ of all regular-closed subsets of (X,T) is a closed base for (X,T).

ii) The semiregularization of T is $T_{\mathcal{R}}$.

Proof: i) Since $\mathcal{R}$ is the collection of all regular-closed sets, $\mathcal{R} = \{\mathrm{Cl}\ O \mid O \in T\}$, and it is clear that $\mathcal{R}$ is a closed base for (X,T).

ii) Since the cotopology $T_{\mathcal{R}}$ has $\{X - R | R \in \mathcal{R}\}$ as a subbase, by Theorem 2.17 $T_{\mathcal{R}}$ is generated by the regular-open sets. Therefore, $T_{\mathcal{R}}$ is the semiregularization of T.

Convention: In reference to Definition 2.18 and Theorem 2.19 the expression "the semiregularized cospace of (X,T)" will mean the cospace $(X,T_{\mathcal{R}})$ of (X,T).

Remark 2.20: It should be noted that $(X,T_{\mathcal{R}})$ is indiscrete if and only if (X,T) has only $\emptyset$ and X as regular-closed sets. Furthermore, $(X,T_{\mathcal{R}})$ is indiscrete is equivalent to the definition of a D-space given in Definition 3.68.

Remark 2.21: In an effort to relate different cotopologies on a space (X,T), J. M. Aarts indicates in [1, Section 3.1] that if (X,T^*) is a cospace of (X,T) and T' is any topology on X such that $T^* \subset T' \subset T$, then (X,T') is a cospace of (X,T), but it is not necessarily true that (X,T^*) is a cospace of (X,T'). He also notes that the cospace relation is neither symmetric nor transitive. The following theorem shows that we do get some of the aforementioned ideas if one works with the semiregularized cospace of (X,T).

Lemma 2.22: Let $(X,T_{\mathcal{R}})$ be the semiregularized cospace of (X,T). If O is a regular-open set in (X,T), then O is a regular-open set in $(X,T_{\mathcal{R}})$.

Proof: In [19] Strecker showed that for any $O \in T$, $Cl_T O = Cl_{T_{\mathcal{R}}} O$. Thus if O is a regular-open set in (X,T), then $\mathrm{int}_{T_{\mathcal{R}}}$ (Cl O) = int (Cl_TO) $\subset$ $\mathrm{int}_T(\mathrm{Cl}_T O)$ = O. Also, since O is a regular-open set in (X,T), O ε $T_{\mathcal{R}}$ and O $\subset$ $\mathrm{int}_{T_{\mathcal{R}}}$ ($\mathrm{Cl}_{T_{\mathcal{R}}}$O). It follows that O is a regular-open set in (X,$T_{\mathcal{R}}$).

<u>Theorem</u> <u>2.23</u>: Let (X,T) be a space and let (X,$T_{\mathcal{R}}$) be the semiregularized cospace of (X,T).

i) If T' is any topology on X such that $T_{\mathcal{R}} \subset T' \subset T$, then (X,T') is a cospace of (X,T) and (X,$T_{\mathcal{R}}$) is a cospace of (X,T').

ii) If T' is any topology on X such that $T' \subset T_{\mathcal{R}} \subset T$, then (X,T') is a cospace of (X,$T_{\mathcal{R}}$) if and only if (X,T') is a cospace of (X,T).

<u>Proof</u>: (i) Since $T_{\mathcal{R}}$ is the cotopology of T relative to the closed base $\mathcal{R}$ = {Cl O|O ε T} and since $T_{\mathcal{R}} \subset T' \subset T$, it is possible to show $\mathcal{R}$ is a closed base for (X,T') and conclude that (X,$T_{\mathcal{R}}$) is a cospace of (X,T'). We can first conclude that $\mathcal{R}$ is a family of closed subsets of (X,T') since each element in $\mathcal{R}$ is closed in (X,$T_{\mathcal{R}}$) and $T_{\mathcal{R}} \subset T'$. Let x ε O ε T'. Notice that Cl_TO ε $\mathcal{R}$ and that $\mathrm{int}_T(\mathrm{Cl}_T O)$ is a regular-open set in (X,T). Therefore, x ε $\mathrm{int}_T(\mathrm{Cl}_T O)$ which by Lemma 2.22 is in $T_{\mathcal{R}}$ and thus in T'. Hence x ε $\mathrm{int}_{T'}$ $(\mathrm{Cl}_T O) \subset \mathrm{Cl}_T O \subset \mathrm{Cl}_{T'} O$, and $\mathcal{R}$ is a closed base for (X,T'). It follows that (X,$T_{\mathcal{R}}$) is a cospace of (X,T').

ii) Suppose (X,T') is a cospace of (X,$T_{\mathcal{R}}$) and let $\mathcal{C}$ be a closed base for (X,$T_{\mathcal{R}}$) determining (X,T'). Let

$x \in O \in T$. By Lemma 2.22, $\mathrm{int}_T(\mathrm{Cl}_T O) \in T_{\mathcal{R}}$ so there exists some $C \in \mathcal{C}$ such that $x \in \mathrm{int}_{T_{\mathcal{R}}} C \subset C \subset \mathrm{Cl}_{T_{\mathcal{R}}}(\mathrm{int}(\mathrm{Cl}_T O)) = \mathrm{Cl}_T(\mathrm{int}_T(\mathrm{Cl}_T O)) = \mathrm{Cl}_T O$. Since $T_{\mathcal{R}} \subset T$, $x \in \mathrm{int}_T C \subset C \subset \mathrm{Cl}_T O$ and $\mathcal{C}$ is a closed base for (X,T). We conclude that (X,T') is a cospace of (X,T).

Suppose (X,T') is a cospace of (X,T) relative to the closed base $\mathcal{D}$. Let $x \in O \in T_{\mathcal{R}}$. Since $O \in T$, there exists some $D \in \mathcal{D}$ such that $x \in \mathrm{int}_T D \subset D \subset \mathrm{Cl}_T O$. Since $\mathrm{int}_T D$ is a regular-open set, by Lemma 2.22 we have that $\mathrm{int}_T D \in T_{\mathcal{R}}$ and $x \in \mathrm{int}_{T_{\mathcal{R}}} D \subset D \subset \mathrm{Cl}_T O = \mathrm{Cl}_{T_{\mathcal{R}}} O$. Hence $\mathcal{D}$ is a closed base for $(X,T_{\mathcal{R}})$, and it follows that (X,T') is a cospace of $(X,T_{\mathcal{R}})$.

We conclude Chapter II by introducing two weaker forms of compactness, namely cocompact and nearly compact, and we show how these properties are related to the semiregularized cospace. The definition of a cocompact space originally appeared in [1], and the definition of a nearly compact space originally appeared in [16].

Definition 2.24: A space (X,T) is cocompact provided that it has a compact cospace.

Definition 2.25: A space (X,T) is nearly compact if and only if every family of regular-closed sets having the finite intersection property has a non-empty intersection.

Theorem 2.26: If a space (X,T) is nearly compact, then it is cocompact.

Proof: Consider the semiregularized cospace $(X,T_{\mathcal{R}})$ of (X,T). It follows from Definition 2.25 and Remark 4.2, part (iv), that $(X,T_{\mathcal{R}})$ is a compact cospace of (X,T).

Corollary 2.27: A space (X,T) is nearly compact if and only if its semiregularized cospace is compact.

Proof: Suppose (X,T) is nearly compact. The proof of Theorem 2.26 shows that the semiregularized cospace is compact.

Suppose the semiregularized cospace $(X,T_{\mathcal{R}})$ is compact. Since $\mathcal{R}$ is the collection of all regular-closed sets, it follows from Definition 2.25 and Remark 4.2, part (iv), that (X,T) is nearly compact.

Example 2.28: As indicated in Example 2.3, the reals with the usual topology are cocompact, but it is clear that the reals are not nearly compact.

Chapter III

Cocontinuous Functions

In this chapter we study the concept of cocontinuous functions which originally appeared in the text Colloquium Co-topologie by J. de Groot and J. M. Aarts in 1964. Initially we develop some basic results concerning cocontinuous functions from which we obtain as a corollary the known result, that if $f:(X,T)\longrightarrow(Y,U)$ is a one to one continuous function from a compact space onto a T_2 space, then f is a homeomorphism. We then introduce the concept of a cocontinuous function at a point, and prove that if (Y,U) is a locally compact T_2 space, then $f:(X,T)\longrightarrow(Y,U)$ is cocontinuous if and only if f is cocontinuous at each point. The next principal result is that if (Y,U) is a locally compact T_2 space, then $f:(X,T)\longrightarrow(Y,U)$ is cocontinuous if and only if $f^{-1}(O)$ is open for each subset O of Y with compact complement. Finally we relate the concept of cocontinuity to various weaker forms of continuous functions and provide counterexamples whenever necessary.

Definition 3.1: Let (X,T) and (Y,U) be spaces. A function $f:(X,T)\longrightarrow(Y,U)$ is cocontinuous (cotopological) relative to cospaces (X,T^*) and (Y,U^*) if the induced map $f:(X,T^*)\longrightarrow(Y,U^*)$ is continuous (homeomorphic). A function $f:(X,T)\longrightarrow(Y,U)$ is cocontinuous (cotopological) if it is cocontinuous (cotopological) with respect to some cospaces (X,T^*) and (Y,U^*).

Remark 3.2: In regard to examples that appear throughout Chapter III it is useful to note that every space is a cospace of itself with respect to the collection of all closed subsets of the space. Also, if (X,T) is a T_1 space and D is the discrete topology on X, then (X,T) is a cospace of (X,D) with respect to the collection of all closed subsets of (X,T).

Example 3.3: A cocontinuous function that is not continuous.

Construction: Let (R,T) be the reals with the usual topology, let (R,D) be the reals with the discrete topology, and let $i:(R,T) \longrightarrow (R,D)$ be the identity map. If $\mathcal{B} = \{B | B$ is closed in $(R,T)\}$ then $\mathcal{B}$ is a closed base for (R,T), $\mathcal{B}$ is a closed base for (R,D), and $T_{\mathcal{B}} = T = D_{\mathcal{B}}$. Therefore, (R,T) is a cospace of itself and (R,T) is a cospace of (R,D). It follows that i is cocontinuous with respect to (R,T), and it is clear that i is not continuous.

Remark 3.4: Several results concerning cocontinuous functions are found in [1] and are listed here for reference.

i) A one to one map f from (X,T) onto (Y,U) is cotopological if and only if f and f^{-1} are both cocontinuous.

ii) A cocontinuous map into a compact T_2 space is continuous.

iii) A one to one cocontinuous map from a compact space onto a T_2 space is cotopological.

iv) If $f:(X,T) \to (Y,U)$ is continuous and $g:(Y,U) \to (Z,S)$ is cocontinuous, then $gf:(X,T) \to (Z,S)$ is cocontinuous.

Theorem 3.5 [1]: A function $f:(X,T) \to (Y,U)$ is cocontinuous if and only if there is a cotopology U^* of U such that $f:(X,T) \to (Y,U^*)$ is continuous.

Proof: The necessary part is clear since f's being cocontinuous implies there exists cotopologies $T^* \subset T$ and $U^* \subset U$ such that $f:(X,T^*) \to (Y,U^*)$ is continuous.

Suppose that there is a cotopology U^* of U such that $f:(X,T) \to (Y,U^*)$ is continuous. As is indicated in Remark 3.2, since every space is a cospace of itself, f is cocontinuous relative to (X,T) and (Y,U^*).

Corollary 3.6 [1]: A function $f:(X,T) \to (Y,U)$ is cocontinuous if and only if there is a closed base $\mathcal{B}$ for (Y,U) such that $f^{-1}(B)$ is closed for all $B \in \mathcal{B}$.

Theorem 3.7: Let $f:(X,T) \to (Y,U)$ be a function.

i) If f is cocontinuous and $A \subset X$, then the restriction of f to A is cocontinuous.

ii) If f is cocontinuous and $f(X)$ is taken with the subspace topology, then $f:(X,T) \to (Y,U_{f(X)})$ is cocontinuous.

Proof: i) Since f is cocontinuous, there are cospaces (X,T^*) and (Y,U^*) such that $f:(X,T^*) \to (Y,U^*)$ is continuous. From Theorem 2.7 $(A,(T^*)_A)$ is a cospace of (A,T_A). Therefore, since $f:(X,T^*) \to (Y,U^*)$ is continuous, we have that $f_{|A}:(A,(T^*)_A) \to (Y,U^*)$ is continuous. It follows that the restriction of f to A is a cocontinuous function.

ii) Since f is cocontinuous, there are cospaces (X,T^*) and (Y,U^*) such that $f:(X,T^*) \to (Y,U^*)$ is continuous. From Theorem 2.7 $(f(X),U^*_{f(X)})$ is a cospace of (Y,U^*). Therefore, since $f:(X,T^*) \to (f(X),U^*_{f(X)})$ is continuous, we have that $f:(X,T) \to (f(X),U_{f(X)})$ is cocontinuous.

Remark 3.4, part (iv), indicates that a continuous function followed by a cocontinuous function is cocontinuous. The following example shows that a cocontinuous function followed by a continuous open onto function is not necessarily cocontinuous.

Example 3.8: Let $X = [0,1]$, let U be the usual topology on X and let D be the discrete topology on X. Let $Y = \{0,1\}$ and let T be the discrete topology on Y. Define $f:(X,U) \to (X,D)$ as the identity map and define $g:(X,D) \to (Y,T)$ by $g([0,1)) = 0$ and $g(1) = 1$. Since (X,U) is a cospace of (X,D) we have that f is cocontinuous. It is also clear that g is continuous open and onto. Therefore, since (Y,T) is compact and since gf is not continuous, it follows from Remark 3.4 that gf is not cocontinuous.

<u>Theorem 3.9</u>: Let $f:(X,T) \to (Y,U)$ be an open onto function, and let $g:(Y,U) \to (Z,S)$. If gf is cocontinuous, then g is cocontinuous.

<u>Proof</u>: Suppose that gf is cocontinuous. Therefore, there is a cospace (Z,S^*) of (Z,S) such that $gf:(X,T) \to (Z,S^*)$ is continuous. If $0 \in S^*$, then $(gf)^{-1}(0) = f^{-1}(g^{-1}(0))$ is in T. Hence $f((gf)^{-1}(0)) = g^{-1}(0)$ is open in (Y,U), and $g:(Y,U) \to (Z,S^*)$ is continuous. By Theorem 3.5 we conclude that g is cocontinuous.

<u>Corollary 3.10</u>: Let $f:(X,T) \to (Y,U)$ be an open continuous onto function, and let $g:(Y,U) \to (Z,S)$. Then gf is cocontinuous if and only if g is cocontinuous.

<u>Proof</u>: The result follows from Theorem 3.9 and Remark 3.4, part (iv).

<u>Theorem 3.11</u>: Let $h:(X,T) \to (Y,U)$ be a homeomorphism.

i) If $\mathcal{B}$ is a closed base for (X,T), then $\mathcal{C} = \{h(B) \mid B \in \mathcal{B}\}$ is a closed base for (Y,U).

ii) If $\mathcal{B}$ is a closed base for (X,T), then $h:(X,T_{\mathcal{B}}) \to (Y,U_{\mathcal{C}})$ is a homeomorphism.

<u>Proof</u>: The results are clear from the definition of a closed base.

<u>Theorem 3.12</u>: Let $g:(Y,U) \to (Z,S)$ be a homeomorphism, and let $f:(X,T) \to (Y,U)$. Then gf is cocontinuous if and only if f is cocontinuous.

Proof: The result follows from Theorem 3.11.

Example 3.13: If $f:(X,T) \to (Y,U)$ is cocontinuous, then $f:(X,T) \to (Y,U^*)$ need not be cocontinuous for every cospace (Y,U^*) of (Y,U).

Construction: Let X be an uncountable set, let T be the finite complement topology on X, and let U be a topology on X given by the following: if $p \in X$, then $O \in U$ if and only if $X - O$ is countable or $p \in (X - O)$. Consider the identity map $i:(X,T) \to (X,U)$, and note that Example 3.44 shows that i is not cocontinuous. Let D be the discrete topology on X. Then, as noted in Remark 3.2, (X,T) is a cospace of (X,D) and (X,U) is a cospace of (X,D). Therefore, $i:(X,T) \to (X,D)$ is cocontinuous, but as indicated (X,U) is a cospace of (X,D) and $i:(X,T) \to (X,U)$ is not cocontinuous.

Convention: If $f:(X,T) \to (Y,U)$, then the expression "f is cocontinuous with respect to the cospace (Y,U^*)" will mean (Y,U^*) is a cospace of (Y,U) and $f:(X,T) \to (Y,U^*)$ is continuous.

Theorem 3.14: If $f:(X,T) \to (Y,U)$ is cocontinuous with respect to the semiregularized cospace $(Y,U_{\mathcal{R}})$ of (Y,U), then $f:(X,T) \to (Y,U^*)$ is cocontinuous for any cospace (Y,U^*) such that $U^* \subset U_{\mathcal{R}}$ or $U_{\mathcal{R}} \subset U^*$.

Proof: The result is clear if $U^* \subset U_{\mathcal{R}}$. Suppose that $U_{\mathcal{R}} \subset U^*$. From Theorem 2.23 we have that $(Y,U_{\mathcal{R}})$ is a co-space of (Y,U^*), and since $f:(X,T)\rightarrow(Y,U_{\mathcal{R}})$ is continuous, it follows that $f:(X,T)\rightarrow(Y,U^*)$ is cocontinuous.

Example 3.15: If $f:(X,T)\rightarrow(Y,U)$ is a one to one, onto, continuous function, then $f^{-1}:(Y,U)\rightarrow(X,T)$ need not be co-continuous.

Construction: Let X,T and U be as in Example 3.13. Then the identity map $i:(X,U)\rightarrow(X,T)$ is one to one, onto and continuous, but from Example 3.52 i^{-1} is not cocontinuous.

Theorem 3.16: If $f:(X,T)\rightarrow(Y,U)$ is a one to one continuous function from a locally compact space onto a T_2 space, then f^{-1} is cocontinuous.

Proof: Let $\mathcal{B} = \{f^{-1}(K) \mid K \text{ is closed in } (Y,U)\}$ and let $x \in O \in T$. Since X is locally compact, there exists some $U \in T$ such that $x \in U$ and Cl U is compact. Now $x \in (U \cap O) \in T$, and since $\text{Cl}\,(U \cap O) \subset \text{Cl}\,U$, we have that $\text{Cl}\,(U \cap O)$ is compact. Therefore $f(\text{Cl}\,(U \cap O))$ is a compact subset of a T_2 space and is closed. Since $f^{-1}(f(\text{Cl}\,(U \cap O))) = \text{Cl}\,(U \cap O) \in \mathcal{B}$, we have that $x \in (U \cap O) \subset \text{Cl}\,(U \cap O) \subset \text{Cl}\,O$. It follows that $\mathcal{B}$ is a closed base for (X,T). Consider $f^{-1}:(Y,U)\rightarrow(X,T)$. For each $B \in \mathcal{B}$, $f(B) = f(f^{-1}(K)) = K$, where K is a closed set in (Y,U). Therefore, by Corollary 3.6, f^{-1} is cocontinuous.

The following corollary shows how cocontinuous functions can be used to prove a known result concerning continuous functions.

Corollary 3.17: If $f:(X,T) \to (Y,U)$ is a one to one continuous function from a compact space onto a T_2 space, then f is a homeomorphism.

Proof: First note that from the hypothesis we also know that (X,T) is a T_2 space. From Theorem 3.16 we have that f^{-1} is cocontinuous, and from Remark 3.4, part (ii), we conclude that f^{-1} is continuous. Therefore, f is a homeomorphism.

Corollary 3.18: Let (X,T) be a locally compact space. If $f:(X,T) \to (Y,U)$ is one to one onto and cocontinuous with respect to a T_2 cospace, then f is cotopological.

Proof: Let (Y,U^*) be a T_2 cospace of (Y,U) such that $f:(X,T) \to (Y,U^*)$ is continuous. From Theorem 3.16 the map $f^{-1}:(Y,U^*) \to (X,T)$ is cocontinuous. Therefore, by Theorem 3.5 there exists a cotopology T^* of T such that $f^{-1}:(Y,U^*) \to (X,T^*)$ is continuous. From Definition 3.1 we have that $f^{-1}:(Y,U) \to (X,T)$ is cocontinuous, and it follows from Remark 3.4, part (i), that f is cotopological.

Theorem 3.19: Let $\{(Y_\alpha,T_\alpha): \alpha \in \mathcal{Q}\}$ be a family of spaces, let $Y = \prod_{\alpha} Y_\alpha$, let T be the product topology on Y, and let $f:(X,U) \to (Y,T)$. If $\pi_\gamma f$ is cocontinuous for each $\gamma \in \mathcal{Q}$, then f is cocontinuous.

Proof: By hypothesis for each $\gamma \in \mathcal{a}$ there exists a cotopology T_γ^* of T_γ such that $\pi_\gamma f:(X,U) \longrightarrow (Y_\gamma, T_\gamma^*)$ is continuous. By [1] $\prod_\gamma T_\gamma^* = T^*$ is a cotopology of T, so consider $f:(X,U) \longrightarrow (Y,T^*)$. For each subbasic open set $O \in T^*$ we have $O = \pi_\gamma^{-1}(O_\gamma)$, where $O_\gamma \in T_\gamma^*$ for some $\gamma \in \mathcal{a}$. Thus $f^{-1}(O) = f^{-1}(\pi_\gamma^{-1}(O_\gamma)) = (\pi_\gamma f)^{-1}(O_\gamma)$. Since $(\pi_\gamma f)$ is cocontinuous with respect to (Y_γ, T_γ^*), $f^{-1}(O) \in U$ and $f:(X,U) \longrightarrow (Y,T^*)$ is continuous. It follows that f is cocontinuous.

Theorem 3.20: For each $\gamma \in \Gamma$ let $f_\gamma:(X_\gamma,T_\gamma) \longrightarrow (Y_\gamma,U_\gamma)$ be a cocontinuous function. Then $\prod_\gamma f_\gamma: \prod_\gamma X_\gamma \longrightarrow \prod_\gamma Y_\gamma$ defined by $(x_\gamma) \longrightarrow (f_\gamma(x_\gamma))$ is cocontinuous.

Proof: Let $\alpha \in \Gamma$ and consider $\pi_\alpha(\prod_\gamma f_\gamma)$ where $\pi_\alpha: \prod_\gamma Y_\gamma \longrightarrow Y_\alpha$. For $(x_\gamma) \in \prod_\gamma X_\gamma$ we have $(\pi_\alpha (\prod_\gamma f_\gamma))((x_\gamma)) = \pi_\alpha((f_\gamma(x_\gamma))) = f_\alpha(x_\alpha) = f_\alpha \pi_\alpha'((x_\gamma))$ where $\pi_\alpha': \prod_\gamma X_\gamma \longrightarrow X_\alpha$.

Since $f_\alpha \pi_\alpha'$ is a continuous function followed by a cocontinuous function, we have that $f_\alpha \pi_\alpha' = \pi_\alpha(\prod_\gamma f_\gamma)$ is cocontinuous for each $\alpha \in \Gamma$. Therefore, it follows from Theorem 3.19 that $\prod_\gamma f_\gamma$ is cocontinuous.

We now proceed in a natural way to define a function to be cocontinuous at a point. We prove some theorems concerning functions that are cocontinuous at a point, and indicate how cocontinuity and cocontinuity at a point are related.

Definition 3.21: A function $f:(X,T) \longrightarrow (Y,U)$ is <u>cocontinuous at a point x</u> $\in X$ if there exists some cospace (Y,U^*)

of (Y,U) such that $f:(X,T) \to (Y,U^*)$ is continuous at x.

<u>Theorem 3.22</u>: If $f:(X,T) \to (Y,U)$ is cocontinuous, then f is cocontinuous at each $x \in X$.

Although it is not known if cocontinuity at each point implies cocontinuity, Theorem 3.24 does give a partial answer. We first indicate several results of Verbeek-Kroonenberg that are needed in the proof of Theorem 3.24.

<u>Theorem 3.23</u> [21]: Let (X,T) be a regular T_1 space.

i) A subset A of X is closed in each cotopology if and only if A is compact in (X,T).

ii) The space (X,T) is locally compact if and only if the intersection of all cotopologies of T is a cotopology of T.

<u>Theorem 3.24</u>: Let (Y,U) be a locally compact T_2 space. Then $f:(X,T) \to (Y,U)$ is cocontinuous if and only if f is cocontinuous at each $x \in X$.

<u>Proof</u>: Suppose that f is cocontinuous at each $x \in X$. Then for each $x \in X$ there is a cotopology U_x of U such that $f:(X,T) \to (Y,U_x)$ is continuous at x. Since Y is locally compact T_2, from Theorem 3.23 we have that S, the intersection of all cotopologies of U, is a cotopology. Let $x \in X$ and let $f(x) \in O \in S$. Then $O \in U_x$ and there exists an open set $W \in T$ containing x such that $f(x) \in f(W) \subset O$. It follows that f is a cocontinuous function.

Theorem 3.25: Let $f:(X,T)\to(Y,U)$ and let $x \in X$. If there exists an open set O containing x such that f restricted to O is cocontinuous at x, then f is cocontinuous at x.

Proof: Suppose that $f_{|O}$ is cocontinuous at x, where O is an open set in (X,T). Then there exists a cospace (Y,U^*) of (Y,U) such that $f_{|O}:(O,T_O)\to(Y,U^*)$ is continuous at x. Thus $f:(X,T)\to(Y,U^*)$ is continuous at x, and f is cocontinuous at x.

Corollary 3.26: Let $f:(X,T)\to(Y,U)$ where (Y,U) is a locally compact T_2 space. If $\{O_\gamma \mid \gamma \in \Gamma\}$ is an open cover of X such that $f_{|O_\gamma}$ is cocontinuous at each point of O_γ for each $\gamma \in \Gamma$, then f is cocontinuous.

Proof: The result follows from Theorem 3.25 and Theorem 3.24.

Theorem 3.27: Let $f:(X,T)\to(Y,U)$ where (Y,U) is a locally compact T_2 space. If $\{C_i \mid i=1, \ldots, n\}$ is a closed cover of X such that $f_{|C_i}$ is cocontinuous at each point of C_i for each $i=1, \ldots, n$, then f is cocontinuous.

Proof: Let (Y,U^*) be the cospace of (Y,U) where U^* is the cotopology of U relative to the closed base $\mathcal{B}$ of all compact subsets of (Y,U). From Theorem 3.23 each $D \in \mathcal{B}$ is closed in each cotopology of U, and thus $f_{|C_i}(D)$ is closed in C_i for each $i=1, \ldots, n$. Since each C_i is closed in

(X,T), $\bigcup_{i=1}^{n} f|_{C_i}^{-1}(D) = f^{-1}(D)$ is a closed set in (X,T). By Corollary 3.6 f is cocontinuous.

<u>Theorem 3.28</u>: Let $f:(X,T)\to(Y,U)$ where (Y,U) is a locally compact T_2 space. If $\{C_\gamma | \gamma \in \Gamma\}$ is a locally-finite closed cover of (X,T) such that $f|_{C_\gamma}$ is cocontinuous for each $\gamma \in \Gamma$, then f is cocontinuous.

<u>Proof</u>: Let $x \in X$. By hypothesis there exists an $O \in T$ containing x such that O intersects only finitely many C_γ, say C_{γ_i}, $i = 1, \ldots, n$. Consider the collection $\{O \cap C_{\gamma_i} | i = 1, \ldots, n\}$. This is a finite collection of closed subsets of O that covers O, and since each $f|_{C_{\gamma_i}}$ is cocontinuous, Theorem 3.7 implies $f|_{(C_{\gamma_i} \cap O)}$ is cocontinuous. By Theorem 3.27 $f|_O$ is cocontinuous at $x \in O$ and by Theorem 3.25 f is cocontinuous at x. It follows from Theorem 3.24 that f is cocontinuous.

In order to learn more about cocontinuous functions various weaker forms of continuity are considered. The following definition of a c-continuous function is found in [9]. We will investigate how c-continuous functions and cocontinuous functions are related.

<u>Definition 3.29</u>: Let $f:(X,T)\to(Y,U)$ and let $p \in X$. Then f is said to be c-<u>continuous</u> <u>at</u> <u>p</u> provided if U is an open subset of Y containing f(p) such that $(Y - U)$ is compact,

then there is an open subset V of X containing p such that $f(V) \subset U$. The function f is c-<u>continuous</u> provided f is c-<u>continuous</u> at each point of X.

<u>Example 3.30</u> [9]: A function f that is c-continuous but not continuous.

<u>Construction</u>: Let R be the reals with the usual topology and define $f: R \rightarrow R$ by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$$

Then f is c-continuous but not continuous.

<u>Theorem 3.31</u> [9]: A function $f:(X,T) \rightarrow (Y,U)$ is c-continuous if and only if $f^{-1}(O)$ is open for each open subset O of Y with compact complement. Also, if $f^{-1}(C)$ is closed for each compact subset C of Y then f is c-continuous; and if Y is T_2 and f is c-continuous, then $f^{-1}(C)$ is closed for each compact subset C of Y.

<u>Definition 3.32</u>: A function $f:(X,T) \rightarrow (Y,U)$ is <u>compact</u> provided that the inverse image of every compact subset of Y is compact in X.

<u>Theorem 3.33</u>: If $f:(X,T) \rightarrow (Y,U)$ is a compact function from a T_2 space, then f is c-continuous.

<u>Proof</u>: Let $p \in X$ and let $O \in U$ such that $f(p) \in O$ and $(Y - O)$ is compact. Then $f^{-1}(Y - O) = (X - f^{-1}(O))$ is

compact in X, so that $f^{-1}(O) \varepsilon T$. Therefore, $f^{-1}(O) \varepsilon T$ such that $p \varepsilon f^{-1}(O)$ and $f(f^{-1}(O)) \subset O$. It follows that f is c-continuous.

<u>Theorem 3.34</u>: If $f:(X,T) \rightarrow (Y,U)$ is a c-continuous function from a compact space to a T_2 space, then f is compact.

<u>Proof</u>: Let K be a compact subset of Y. Since Y is T_2, it follows that $(Y - K)$ is an open subset of Y with compact complement. Thus since $f^{-1}(Y - K) = X - f^{-1}(K)$ is open in X, we have that f^{-1} is compact.

Although the composition of c-continuous functions is not c-continuous, [9, Example 3], we do have the following result.

<u>Theorem 3.35</u>: If $f:(X,T) \rightarrow (Y,U)$ is c-continuous and $g:(Y,U) \rightarrow (Z,S)$ is compact and c-continuous, then gf is c-continuous.

<u>Proof</u>: Let $O \varepsilon S$ such that $(Z - O)$ is compact. Then $g^{-1}(O) \varepsilon U$ and $g^{-1}(Z - O) = Y - g^{-1}(O)$ is compact. Therefore, $f^{-1}(g^{-1}(O)) \varepsilon T$ and gf is c-continuous.

<u>Corollary 3.36</u>: If $f:(X,T) \rightarrow (Y,U)$ is c-continuous, if $g:(Y,U) \rightarrow (Z,S)$ is compact, and if (Y,U) is T_2, then gf is c-continuous.

Proof: The corollary follows from Theorem 3.33 and Theorem 3.35.

Example 3.37: if $f:(X,T) \rightarrow (Y,U)$ is c-continuous and (Y,U^*) is a cospace of (Y,U), then $f:(X,T) \rightarrow (Y,U^*)$ is not always c-continuous.

Construction: Let R be the reals, let U be the usual topology on R, let C be the finite complement topology on R, and let D be the discrete topology on R. Then (R,U) is a cospace of (R,D), and the identity map $i:(R,C) \rightarrow (R,D)$ is c-continuous since every open subset in (R,D) having compact complement must be a set 0 such that $(R - 0)$ is finite. Hence $i^{-1}(0) = 0$ is open, and i is c-continuous. But $i:(R,C) \rightarrow (R,U)$ is not c-continuous since $0 = (-\infty, 1) \cup (2,\infty)$ is open in (R,U) such that $(R - 0)$ is compact, but $i^{-1}(0)$ is not open. Hence $i:(R,C) \rightarrow (R,U)$ is not c-continuous.

Theorem 3.38: Let (X,T) be a regular T_1 space. If 0 is any open subset of X such that $(X - 0)$ is compact, then for any cospace (X,T^*) of (X,T), $0 \in T^*$ and $(X - 0)$ is compact in (X,T^*).

Proof: Suppose that (X,T^*) is a cospace of (X,T) and let $0 \in T$ such that $(X - 0)$ is compact in (X,T). Then clearly $(X - 0)$ is compact in (X,T^*) and by Theorem 3.23 $(X - 0)$ is closed in (X,T^*). It follows that $0 \in T^*$.

Theorem 3.39: Let $f:(X,T) \to (Y,U)$ where (Y,U) is a regular T_1 space. If $f:(X,T) \to (Y,U^*)$ is c-continuous for any cospace (Y,U^*) of (Y,U), then $f:(X,T) \to (Y,U)$ is c-continuous.

Proof: Let $O \in U$ such that $(Y - O)$ is compact in (Y,U). From Theorem 3.38, $O \in U^*$ and $(Y - O)$ is compact in (Y,U^*). It follows that $f^{-1}(O) \in T$ and f is c-continuous.

Remark 3.40: Note that Theorem 3.39 says that for a function to be c-continuous it needs only to be c-continuous on any cospace. This result therefore implies the following corollary.

Corollary 3.41: Let (Y,U) be a regular T_1 space. Every cocontinuous function $f:(X,T) \to (Y,U)$ is c-continuous.

Proof: Since $f:(X,T) \to (Y,U)$ is cocontinuous, there exists some cospace (Y,U^*) such that $f:(X,T) \to (Y,U^*)$ is continuous, hence c-continuous. It follows from Theorem 3.39 that $f:(X,T) \to (Y,U)$ is c-continuous.

Corollary 3.42: Let (Y,U) be a regular T_1 space and let $f:(X,T) \to (Y,U)$ be cocontinuous.

i) If U is an open subset of Y with compact complement, then $f^{-1}(U)$ is an open subset of X.

ii) If C is a compact subset of Y, then $f^{-1}(C)$ is a closed subset of X.

We now use Corollary 3.42 to prove a statement indicated in Remark 3.4.

<u>Theorem 3.43</u>: If $f:(X,T) \to (Y,U)$ is cocontinuous and (Y,U) is a compact T_2 space, then f is continous.

<u>Proof</u>: Since any closed subset of (Y,U) is compact, Corollary 3.42 implies its inverse image is closed, and therefore f is continuous.

We have that every cocontinuous function into a regular T_1 space is c-continuous. We now give an example to show that the converse is false, and then we prove the equivalence of c-continuity and cocontinuity if the range space is locally compact and T_2.

<u>Example 3.44</u>: Let X be an uncountable set, let T be the finite complement topology on X, and let U be the topology on X given by the following: if $p \varepsilon X$, then $O \varepsilon U$ if and only if $(X - O)$ is countable or $p \varepsilon (X - O)$. From [18] it follows that (X,U) is not locally compact but is regular and T_1. Consider the identity map $i:(X,T) \to (X,U)$.

i) Let $O \varepsilon U$ such that $(X - O)$ is compact. Then $(X - O)$ is finite and $i^{-1}(X - O) = X - i^{-1}(O)$ is finite and therefore closed. We conclude that $i^{-1}(O)$ is open, and that i is c-continuous.

ii) If i is cocontinuous, then there exists some closed base $\mathcal{B}$ for (X,U) such that $i^{-1}(B)$ is closed in (X,T)

for each $B \in \mathcal{B}$. For (X,U) the only closed sets such that $i^{-1}(B)$ is closed are the finite sets and X. Thus the closed base must be a collection of these sets. The following will show that a collection of these sets is not a closed base, and hence that i is not cocontinuous. Let $x \neq p$ be in X. Since (X,U) is T_2, there are open sets U_x of x and U_p of p such that $U_p \cap U_x = \emptyset$. Since $p \in U_p \in U$, there exists some $B \in \mathcal{B}$ such that $p \in \text{int } B \subset B \subset \text{Cl } U_p$. Note that B must be the set X or B must be finite. Since U_x is an open subset of x not intersecting U_p, it follows that $\text{Cl } U_p \neq X$ and therefore $B \neq X$. If B is finite, then int B is finite; since int B is open and $p \in \text{int } B$, we have that $(X - \text{int } B)$ is countable. Therefore, $X = (\text{int } B) \cup (X - \text{int } B)$ is countable, and this is a contradiction. Thus B is not finite, and it follows that i is not cocontinuous.

Theorem 3.45: Let (Y,U) be a locally compact space. If $f:(X,T) \to (Y,U)$ is c-continuous, then f is cocontinuous.

Proof: Suppose that f is c-continuous and let $\mathcal{B} = \{K \mid K \text{ is a closed compact subset of } Y\}$. From Theorem 2.15 $\mathcal{B}$ is a closed base for (Y,U) determining the cospace $(Y,U_{\mathcal{B}})$. Consider $f:(X,T) \to (Y,U_{\mathcal{B}})$. Let $K \in \mathcal{B}$ and consider $(Y - K)$. Since $\mathcal{B}$ is a closed base for (Y,U), we have that $(Y - K) \in U$ such that its complement is compact. It follows from Theorem 3.31 that $f^{-1}(Y - K) = X - f^{-1}(K)$ is open in (X,T).

Corollary 3.46: Let $f:(X,T) \to (Y,U)$ is a compact map from a T_2 space into a locally compact space, then f is cocontinuous.

Proof: The result follows from Theorem 3.33 and Theorem 3.45.

Corollary 3.47: Let (X,T) be a T_2 space and let (Y,U) be locally compact and T_2. If $f:(X,T) \to (Y,U)$ is one to one, onto and compact, then f is cotopological.

Proof: From Corollary 3.46 we have that f is cocontinuous, and from a result by Whyburn in [23] we have that f is closed. Therefore, f^{-1} is continuous and by Remark 3.4, f is cotopological.

Corollary 3.48: Let (Y,U) be a locally compact T_2 space. A function $f:(X,T) \to (Y,U)$ is cocontinuous if and only if f is c-continuous.

Proof: The equivalence follows from Theorem 3.45 and Corollary 3.41.

Corollary 3.41 and Corollary 3.48 prove the following three results which are corollaries to theorems obtained by Hoyle and Gentry in [9].

Corollary 3.49: Let X be a Baire space and let Y be a regular T_1 space which is the countable union of compact sets. Then every cocontinuous function from X to Y is continuous on a dense subset of X.

<u>Definition 3.50</u>: A function $f:(X,T)\rightarrow(Y,U)$ is <u>connected</u> provided that for each connected subset C of X, $f(C)$ is connected in (Y,U).

<u>Corollary 3.51</u>: Let (X,T) be locally connected and let (Y,U) be metrizable such that every closed and bounded set is compact. If $f:(X,T)\rightarrow(Y,U)$ is a cocontinuous connected function, then f is continuous.

<u>Corollary 3.52</u>: Let (X,T) be a space and let (Y,U) be metrizable such that every closed and bounded set is compact. If $\{f_n\}_1^\infty$ is a sequence of cocontinuous functions from X to Y which converges uniformly to f, then f is cocontinuous.

Observe that results similar to those of Corollary 3.41, Theorem 3.45, and Corollary 3.48 will follow when working with cocontinuity at a point and c-continuity at a point. Note also that if $f:X\rightarrow Y$, where Y is regular and T_1, is cocontinuous at each point, then f is c-continuous; but Example 3.44 shows that the converse does not hold.

Other weaker forms of continuity also contribute to the study of cocontinuous functions. We now relate cocontinuity to several other weaker forms of continuity, namely weakly continuous and almost continuous.

<u>Definition 3.53</u> [17]: A function $f:(X,T)\rightarrow(Y,U)$ is <u>weakly continuous</u> provided that for every $O \in U$, $f^{-1}(O) \subset \text{int}(f^{-1}(\text{Cl } O))$.

Definition 3.54 [17]: A function $f:(X,T) \rightarrow (Y,U)$ is almost continuous provided that the inverse image of every regular-closed (regular-open) set is closed (open).

Note that every continuous function is almost continuous, and that every almost continuous function is weakly continuous. Also, if the range space of a function f is regular, then f is weakly continuous if and only if f is continuous. Whereas, if the range space of a function f is semi-regular, then f is almost continuous if and only if f is continuous.

Theorem 3.55: If $f:(X,T) \rightarrow (Y,U)$ is almost continuous, then f is cocontinuous.

Proof: Let $(Y,U_{\mathcal{R}})$ be the semiregularized cospace of (Y,U). Let $R \varepsilon \mathcal{R}$. Since R is regular-closed we have that $f^{-1}(R)$ is closed in X. Therefore from Corollary 3.6, f is cocontinuous.

Corollary 3.56: A function $f:(X,T) \rightarrow (Y,U)$ is almost continuous if and only if f is cocontinuous with respect to the semiregularized cospace of (Y,U).

Proof: The result follows from Theorem 3.55 and Definition 3.54.

Corollary 3.57: The almost continuous image of a compact space is cocompact.

<u>Proof</u>: From Corollary 3.56 we see that the semiregularized cospace of the range space is compact. Therefore, the range space is cocompact.

<u>Example 3.58</u>: A cocontinuous function that is not weakly continuous and hence not almost continuous.

<u>Construction</u>: Let R be the reals, let U be the usual topology on R and let D be the discrete topology on R. Let i be the identity function from (R,U) onto (R,D). Then i is cocontinuous with respect to the cospace (R,U) of (R,D). Let $O = (0,1) \cup \{5\}$ in (R,D). Then O is open and $i^{-1}(O) = i^{-1}(\mathrm{Cl}\ O) = (0,1) \cup \{5\}$. Since $\mathrm{int}(i^{-1}(\mathrm{Cl}\ O) = (0,1)$, it follows that i is not weakly continuous.

<u>Example 3.59</u>: A weakly continuous function that is not cocontinuous.

<u>Construction</u>: Let R be the reals and let T consist of ϕ, R and the complements of all countable subsets of R. Let $X = \{a, b, c\}$ and let U be the topology given by $U = \{\phi, X, \{a\}, \{c\}, \{a,c\}\}$. Define $f:(R,T) \to (X,U)$ by

$$f(x) = \begin{cases} a \text{ if } x \text{ is rational} \\ b \text{ if } x \text{ is irrational} \end{cases}$$

From [17, Example 2.3] we get that f is weakly continuous and not almost continuous, so it is also not continuous. The following includes all closed subsets of X: ϕ , X, $\{b,c\}$, $\{a,b\}$, $\{b\}$. Note that $\{\phi, X\}$ is not a closed base for (X,U) and also that no combination of closed sets is a closed base for (X,U). Therefore (X,U) represents the only cospace, and since f is not continuous, f is not cocontinuous.

<u>Remark 3.60</u>: Suppose that $f:(X,T) \to (Y,U)$. Differing from c-continuous functions if $f:(X,T) \to (Y,U^*)$ is almost continuous for some cospace of (Y,U), then f need not be almost continuous. Example 3.58 gives a function that is almost continuous on a cospace but the function is not almost continuous. The same remark also holds for weakly continuous functions.

<u>Theorem 3.61</u>: If $f:(X,T) \to (Y,U)$ is almost continuous, then $f:(X,T) \to (Y,U^*)$ is almost continuous for any cospace (Y,U^*) such that $U^* \subset U_{\mathcal{R}}$ or $U_{\mathcal{R}} \subset U^*$.

<u>Proof</u>: Since $f:(X,T) \to (Y,U)$ is almost continuous, we have from Corollary 3.56 that $f:(X,T) \to (Y,U_{\mathcal{R}})$ is continuous. Therefore, if $U^* \subset U_{\mathcal{R}}$, then $f:(X,T) \to (Y,U^*)$ is not only almost continuous but also continuous. Suppose that $U_{\mathcal{R}} \subset U^*$ and let O be a regular-open subset of (Y,U^*). Since $O \in U^*$, we have that $O \in U$ and $Cl_U O \subset Cl_{U^*} O \subset Cl_{U_{\mathcal{R}}} O$. As noted in the proof of Lemma 2.22 we have $Cl_U O = Cl_{U_{\mathcal{R}}} O$, and therefore $Cl_U O = Cl_{U^*} O = Cl_{U_{\mathcal{R}}} O$. Since $int_{U_{\mathcal{R}}}(Cl_{U_{\mathcal{R}}} O) = int_{U_{\mathcal{R}}}(Cl_{U^*} O) \subset int_{U^*}(Cl_{U_{\mathcal{R}}} O)$, we have that $int_{U_{\mathcal{R}}}(Cl_{U_{\mathcal{R}}} O) = O$ and $O \in U_{\mathcal{R}}$. Therefore, $f^{-1}(O) \in T$ and $f:(X,T) \to (Y,U^*)$ is almost continuous.

<u>Theorem 3.62</u>: Let $f:(X,T) \to (Y,U)$ be a weakly continuous open function. Then for every closed subset C of Y, $\mathrm{int}(f^{-1}(C)) = f^{-1}(\mathrm{int}\ C)$.

<u>Proof</u>: Suppose that C is a closed subset of (Y,U) and let $x \in \mathrm{int}(f^{-1}(C))$. Then there is an open set O containing x such that $f(x) \in f(O) \subset C$. Since f is open, $f(x) \in \mathrm{int}\ C$ and $x \in f^{-1}(\mathrm{int}\ C)$. Let $x \in f^{-1}(\mathrm{int}\ C)$. Then $\mathrm{int}\ C \neq \emptyset$, and since f is weakly continuous, $f^{-1}(\mathrm{int}\ C) \subset \mathrm{int}(f^{-1}(\mathrm{Cl}(\mathrm{int}\ C))) \subset \mathrm{int}(f^{-1}(C))$. Therefore, $x \in \mathrm{int}(f^{-1}(C))$, and $\mathrm{int}(f^{-1}(C)) = f^{-1}(\mathrm{int}\ C)$.

<u>Theorem 3.63</u>: If $f:(X,T) \to (Y,U)$ is weakly continuous and open, then f is cocontinuous.

<u>Proof</u>: Consider the cospace $(Y,U_{\mathcal{R}})$ of (Y,U). Since $\mathcal{R}$ is the collection of regular-closed sets in (Y,U), by Theorem 2.17 $(Y - B)$ is a regular-open set for each $B \in \mathcal{R}$. Letting $B \in \mathcal{R}$ we have by Theorem 3.62 that $\mathrm{int}(f^{-1}(\mathrm{Cl}(Y - B))) = f^{-1}(\mathrm{int}(\mathrm{Cl}(Y - B))) = f^{-1}(Y - B)$, since $(Y - B)$ is a regular-open set. It follows that $f^{-1}(Y - B)$ is open in (X,T) and we conclude that $f^{-1}(B)$ is closed and by Corollary 3.6 that f is cocontinuous.

Note that in Theorem 3.63 we have actually proved that f is cocontinuous with respect to the semiregularized cospace. Hence by Corollary 3.56 we can actually conclude that f is almost continuous. However, it should be noted that this result had already been proved in [17], and is included here for its cotopological approach.

<u>Corollary 3.64</u>: The weakly continuous open image of a compact space is cocompact.

<u>Theorem 3.65</u>: Let (X,T) be compact and let (Y,U) be T_2. If $f:(X,T) \to (Y,U)$ is one to one, onto and almost continuous, then f is cotopological.

<u>Proof</u>: The result follows from Theorem 3.55 and Remark 3.4.

<u>Theorem 3.66</u>: If $f:(X,T) \to (Y,U)$ is a weakly continuous function such that for every regular-open set O of (Y,U), $f^{-1}(\text{Bdy } O)$ is closed, then f is cocontinuous.

<u>Proof</u>: Consider the semiregularized cospace $(Y,U_{\mathcal{R}})$ of (Y,U). Let $B \in \mathcal{R}$. By Theorem 2.17 $(Y - B)$ is a regular-open set. Let $x \in f^{-1}(Y - B)$. Since f is weakly continuous $x \in f^{-1}(Y - B) \subset \text{int } (f^{-1}(\text{Cl}(Y - B)))$, and hence there exists some $V \in T$ such that $x \in f^{-1}(Y - B) \subset V \subset f^{-1}(\text{Cl}(Y - B))$. Since $(Y - B) \in U$ we note that $\text{Bdy}(Y - B) = \text{Cl}(Y - B) - (Y - B)$. Since $f(x) \in (Y - B)$ we have that $f(x) \notin \text{Bdy}(Y - B)$, and hence $x \notin f^{-1}(\text{Bdy}(Y - B))$. By hypothesis $f^{-1}(\text{Bdy}(Y - B))$ is closed and therefore $x \in V - f^{-1}(\text{Bdy}(Y - B))$ which is open in (X,T). Let $y \in V - f^{-1}(\text{Bdy}(Y - B))$. Since $V \subset f^{-1}(\text{Cl}(Y - B))$ we have that $f(y) \in \text{Cl}(Y - B)$; but $y \notin f^{-1}(\text{Bdy}(Y - B))$ implies $f(y) \notin \text{Bdy}(Y - B)$, and therefore $f(y) \in (Y - B)$ or $y \in f^{-1}(Y - B)$. Thus we have that $x \in V - f^{-1}(\text{Bdy}(Y - B)) \subset f^{-1}(Y - B)$ and conclude that $f^{-1}(Y - B)$ is open in (X,T). It follows that

$f^{-1}(B)$ is closed and by Corollary 3.6 that f is cocontinuous.

Corollary 3.67: A function $f:(X,T) \to (Y,U)$ is weakly continuous such that for every regular-open set O of (Y,U), $f^{-1}(\text{Bdy } O)$ is closed if and only if f is cocontinuous with respect to the semiregularized cospace.

Proof: Suppose that f is weakly continuous such that $f^{-1}(\text{Bdy } O)$ is closed for each regular-open subset of (Y,U). From the proof of Theorem 3.66 we obtain the desired result.

Suppose that f is cocontinuous with respect to $(Y,U_{\mathcal{R}})$. From Corollary 3.56 we can conclude that f is weakly continuous. Let O be a regular-open subset of (Y,U). Then by Lemma 2.22 $O \varepsilon U_{\mathcal{R}}$, and since f is cocontinuous with respect to $(Y,U_{\mathcal{R}})$, we have that $f^{-1}(O) \varepsilon T$. Also, $\text{Cl } O \varepsilon \mathcal{R}$ and by Corollary 3.6, $f^{-1}(\text{Cl } O)$ is closed in (X,T). Thus, since $\text{Bdy } O = (\text{Cl } O - O$, $f^{-1}(\text{Bdy } O) = f^{-1}(\text{Cl } O) - f^{-1}(O)$ is a closed sbuset of (X,T).

Finally we conclude Chapter III with some results concerning cocontinuous functions that were obtained by placing restrictions on the domain or the range of the function.

Definition 3.68 [12]: A space (X,T) is a D-space if and only if every non-empty open subset of X is dense in X.

Definition 3.69 [10]: A space (X,T) is said to be saturated if and only if any intersection of open sets is itself an open set.

Lemma 3.70: If $f:(X,T) \to (Y,U)$ is cocontinuous from a saturated space to a regular space, then the inverse image of every open (closed) set is closed (open).

Proof: Suppose that $f:(X,T) \to (Y,U)$ is cocontinuous with respect to the cospace $(Y,U_{\mathcal{B}})$. Let $O \in U$. For each $y \in O$ there is some $B_y \in \mathcal{B}$ such that $y \in \text{int } B_y \subset B_y \subset O$. Thus $O = \cup B_y$ where $y \in O$ and $f^{-1}(O) = f^{-1}(\cup B_y) = \cup f^{-1}(B_y)$. Since f is cocontinuous with respect to $(Y,U_{\mathcal{B}})$, by Corollary 3.6 $f^{-1}(B_y)$ is closed for each B_y. Since (X,T) is saturated, $X - f^{-1}(O) = X - \cup(f^{-1}(B_y)) = \cap (X - f^{-1}(B_y))$ is open so that $f^{-1}(O)$ is closed. A similar argument shows that the inverse image of a closed set is open.

Lemma 3.71: If $f:(X,T) \to (Y,U)$ is such that the inverse image of every closed set is open, then f is weakly continuous.

Proof: Let $O \in U$. Since $f^{-1}(O) \subset f^{-1}(\text{Cl } O)$ and since $f^{-1}(\text{Cl } O)$ is open, we have that $f^{-1}(O) \subset \text{int}(f^{-1}(\text{Cl } O))$. It follows that f is weakly continuous.

Theorem 3.72: If f is a cocontinuous function from a saturated space to a regular space, then f is continuous.

Proof: The result follows from Lemma 3.70, Lemma 3.71 and the paragraph following Definition 3.54.

Theorem 3.73: If $f:(X,T) \to (Y,U)$ is a function into a D-space, then f is almost continuous, hence weakly continuous and cocontinuous.

Proof: From Remark 2.20 it follows that f is almost continuous. Therefore f is also weakly continuous and cocontinuous.

Chapter IV

Cocompact and Locally Cocompact

Part I - Cocompact

In [1] J. M. Aarts formulated several problems relating to the invariance of cocompactness. Among these problems were the following:

i) If A and B are open cocompact subsets of (X,T), is $A \cup B$ cocompact?

ii) If A and B are cocompact subsets of (X,T), is $A \cap B$ cocompact?

In the same work, he provided an example which answered (ii) in the negative. Although the answer to (i) is still unknown, the following results were obtained while investigating these ideas. It will first be beneficial to list a few known results for cocompactness which are found in [1].

Definition 4.1: A collection $\mathcal{C}$ of sets is a centered system if and only if every non-empty finite subcollection of $\mathcal{C}$ has a non-empty intersection.

Remark 4.2: As indicated, the following results are found in [1].

i) In a regular space, every open subset of a cocompact space is cocompact.

ii) The topological union of cocompact spaces is cocompact.

iii) The product of cocompact spaces is cocompact.

iv) A space (X,T) is cocompact if and only if there is a closed base $\mathcal{B}$ such that every centered system of $\mathcal{B}$ has a non-empty intersection.

From these known results we get the following two theorems.

Theorem 4.3: Let (X,T) be a regular space, let $A \subset X$ be cocompact, and $B \subset X$ be open. Then $A \cap B$ is cocompact.

Proof: Since $B \subset X$, we have that $B \cap A$ is an open subset of (A,T_A). Therefore, by Remark 4.2, $B \cap A$ is cocompact as a subset of (A,T_A). But since $T_{A|A \cap B} = T_{A \cap B}$, it follows that $(A \cap B, T_{A \cap B})$ is cocompact.

Theorem 4.4: Let (X,T) be a regular space, let $A \subset X$ be open and cocompact, and let $B \subset X$ be closed. Then $A - B$ is cocompact.

Proof: Since A is open and B is closed, we have that $A - B$ is open in (X,T) and (A,T_A). From Remark 4.2, $A - B$ is cocompact in (A,T_A). Since $T_{A|A - B} = T_{A - B}$ we have that $(A - B, T_{A - B})$ is cocompact.

As noted above it is not known if the union of two open cocompact subsets is cocompact, but the following results do give some indication as to when the union of given subsets of a space will be cocompact.

Lemma 4.5: If A and B are open cocompact subsets of (X,T) such that $A \cap B = \phi$, then $A \cup B$ is cocompact.

Proof: The result follows from Remark 4.2 and [6, Theorem 2.22].

Theorem 4.6: If A and B are open cocompact subsets of X such that B is also closed, then $A \cup B$ is cocompact.

Proof: Since $A \subset X$ is cocompact and open, and since $X - B$ is open, we have from Theorem 4.3 that $A \cap (X - B)$ is open and cocompact. Since $A \cap (X - B)$ and B are disjoint, open and cocompact, it follows from Lemma 4.5 that $(A \cap (X - B)) \cup B = A \cup B$ is cocompact.

Theorem 4.7: If A is a closed cocompact subset of (X,T) and B is a closed compact subset of (X,T), then $A \cup B$ is a cocompact subset of (X,T).

Proof: Since (A,T_A) is cocompact, by Remark 4.2, (A,T_A) has a closed base $\mathcal{C}$ such that every centered system of sets in $\mathcal{C}$ has a non-empty intersection. Notice also that since A is closed each $C \in \mathcal{C}$ is closed in (X,T). Since (B,T_B) is compact, it is also cocompact and hence it has a closed base $\mathcal{D}$ such that every centered system of sets in $\mathcal{D}$ has a non-empty intersection. Again, each D $\varepsilon \mathcal{D}$ is closed in (X,T). Let $\mathcal{L} = \{C \cup D | C \varepsilon \mathcal{C}, D \varepsilon \mathcal{D}$ and $(C \cup D) \not\varepsilon \mathcal{C}$, $(C \cup D) \not\varepsilon \mathcal{D}\}$, and consider $\mathcal{K} = \{C | C \varepsilon \mathcal{C}\} \cup \{D | D \varepsilon \mathcal{D}\} \cup \{C \cup D | (C \cup D) \varepsilon \mathcal{L}\}$. By Theorem 2.13, $\mathcal{K}$ is a closed base for A U B. To show A U B is cocompact we will show that every centered system of sets in $\mathcal{K}$ has a non-empty intersection. Several cases will be considered.

i) Let $\mathcal{K}'$ be a centered system of sets from $\mathcal{K}$ such that each K $\varepsilon \mathcal{K}'$ is from $\mathcal{C}$. Then $\mathcal{K}'$ is a centered system of sets from $\mathcal{C}$ and as noted above $\mathcal{K}'$ has a non-empty intersection. Similarly, if $\mathcal{K}'$ consists of only elements from $\mathcal{D}$, then again it has a non-empty intersection.

ii) Let $\mathcal{K}'$ be a centered system of sets from $\mathcal{K}$ such that $\mathcal{K}' \cap \mathcal{C} \neq \phi \neq \mathcal{K}' \cap \mathcal{D}$ and $\mathcal{K}' \cap \mathcal{L} = \phi$. Consider the collection $\mathcal{W} = \{C \cap B | C \varepsilon (\mathcal{C} \cap \mathcal{K}')\} \cup \{D | D \varepsilon (\mathcal{D} \cap \mathcal{K}')\}$. If there exists some $C^* \varepsilon \mathcal{K}'$ such that $C^* \cap B = \phi$, then $C^* \subset (A - B)$, and since each D $\varepsilon \mathcal{D}$ is a subset of B we have that $C^* \cap D = \phi$. This contradicts $\mathcal{K}'$ being a centered system, hence $C \cap B \neq \phi$ for every C $\varepsilon \mathcal{C}$ such that C $\varepsilon \mathcal{K}'$. Thus we have that $\mathcal{W}$ is a collection of non-empty closed subsets of B. We will show that $\mathcal{W}$ has the finite intersection property. Let $\mathcal{W}'$ be a non-empty, finite subcollection of $\mathcal{W}$.

a) Suppose $\mathcal{W}'$ consists of D_i, i=1, . . . , n. Since $\mathcal{K}'$ is a centered system of sets and each $D_i \varepsilon \mathcal{K}'$, we have that $\bigcap_{i=1}^{n} D_i \neq \phi$.

b) Suppose $\mathcal{W}'$ consists of $C_i \cap B$ for i=1, . . . , n. Since $\mathcal{K}' \cap \mathcal{D} \neq \phi$ there exists some $D \varepsilon \mathcal{D}$ such that $D \varepsilon \mathcal{K}'$. Since $\mathcal{K}'$ is a centered system of sets it follows that $(\bigcap_{i=1}^{n} C_i) \cap D \neq \phi$. Thus $\phi \neq ((\bigcap_{i=1}^{n} C_i) \cap D) \subset ((\bigcap_{i=1}^{n} C_i) \cap B) = \bigcap_{i=1}^{n} (C_i \cap B)$.

c) Suppose $\mathcal{W}'$ consists of $C_i \cap B$ for i=1, . . . , n, and D_j for j=1, . . . , m. Then $(\bigcap_{i=1}^{n} (C_i \cap B)) \cap (\bigcap_{j=1}^{m} D_j) = (\bigcap_{i=1}^{n} C_i) \cap B \cap (\bigcap_{j=1}^{m} D_j) = (\bigcap_{i=1}^{n} C_i) \cap (\bigcap_{j=1}^{m} D_j) \neq \phi$, since $\mathcal{K}'$ is a centered system. Therefore, $\mathcal{W}$ is a collection of closed subsets of B having the finite intersection property. Since B is compact we have that $\phi \neq \cap\{W | W \varepsilon \mathcal{W}\} = (\cap \{C \cap B | C \varepsilon (\mathcal{C} \cap \mathcal{K}')\}) \cap (\cap \{D | D \varepsilon (\mathcal{D} \cap \mathcal{K}')\}) \subset (\cap\{C | C \varepsilon (\mathcal{C} \cap \mathcal{K}')\}) \cap (\cap \{D | D \varepsilon (\mathcal{D} \cap \mathcal{K}')\}) = \cap \{K | K \varepsilon \mathcal{K}'\}$.

iii) Let $\mathcal{K}'$ be a centered system of sets from $\mathcal{K}$ such that $\mathcal{K}' \cap \mathcal{C} = \phi = \mathcal{K}' \cap \mathcal{D}$ and $\mathcal{K}' \cap \mathcal{E} \neq \phi$. Consider the collection $\mathcal{W} = \{(C \cup D) \cap B | (C \cup D) \varepsilon \mathcal{K}'\}$. This is a collection of non-empty closed subsets of B.

a) If $\mathcal{W}$ has the finite intersection property, then since B is compact, $\phi \neq \cap \{W | W \varepsilon \mathcal{W}\} = \cap\{(C \cup D) \cap B | (C \cup D) \varepsilon \mathcal{K}'\} \subset \cap \{C \cup D | (C \cup D) \varepsilon \mathcal{K}'\} = \cap \{K | K \varepsilon \mathcal{K}'\}$.

b) Suppose $\mathcal{W}$ does not have the finite intersection property. Then there is a finite subcollection of $\mathcal{K}'$ say $C_i \cup D_i$ for i=1, . . . , n, such that $\bigcap_{i=1}^{n}((C_i \cup D) \cap B) = \phi$.

Since $\bigcap_{i=1}^{n} (C_i \cup D_i) \neq \phi$, $\bigcap_{i=1}^{n} (C_i \cup D_i) \subset (A - B)$. Let $\mathcal{C}' = \{C \mid (C \cup D) \in \mathcal{K}'$ for any $D \in \mathcal{D}\}$. Let C^j for $j=1, \ldots, m$, be any finite subcollection of $\mathcal{C}'$ and suppose that $\bigcap_{j=1}^{m} C^j = \phi$. For each C^j, $j=1, \ldots, m$, let $C^j \cup D^j$ be one such element in $\mathcal{K}'$ having C^j as its first part. Therefore $\{(C^j \cup D^j)$ for $j=1, \ldots, m\}$ is a finite subcollection of $\mathcal{K}'$, hence $\bigcap_{j=1}^{m} (C^j \cup D^j) \neq \phi$. But since $\bigcap_{j=1}^{m} C^j = \phi$, it follows that $\phi \neq \bigcap_{j=1}^{m} (C^j \cup D^j) \subset B$. Hence $\{C_i \cup D_i \mid i=1, \ldots, n\} \cup \{C^j \cup D^j \mid j=1, \ldots, m\}$ is a finite subcollection of $\mathcal{K}'$ and therefore has a non-empty intersection. But $\bigcap_{i=1}^{n} (C_i \cup D_i) \subset (A - B)$ and $\bigcap_{j=1}^{m} (C^j \cup D^j) \subset B$, and therefore $(\bigcap_{i=1}^{n} (C_i \cup D_i)) \cap (\bigcap_{j=1}^{m} (C^j \cup D^j)) = \phi$. This is a contradiction, and hence any finite subcollection of $\mathcal{C}'$ has a non-empty intersection. Since A is cocompact relative to $\mathcal{C}$ and since $\mathcal{C}'$ is a centered system of sets from $\mathcal{C}$, it follows from Remark 4.2 that $\bigcap \{C \mid C \in \mathcal{C}'\} \neq \phi$. Therefore, $\phi \neq \bigcap \{C \mid C \in \mathcal{C}'\} \subset \bigcap \{C \cup D \mid (C \cup D) \in \mathcal{K}'\} = \bigcap \{K \mid K \in \mathcal{K}'\}$.

iv) Let $\mathcal{K}'$ be a centered system of sets from $\mathcal{K}$ such that $\mathcal{K}' \cap \mathcal{C} = \phi$, $\mathcal{K}' \cap \mathcal{D} \neq \phi$ and $\mathcal{K}' \cap \mathcal{E} \neq \phi$. Consider the collection $\mathcal{W} = \{D \mid D \in (\mathcal{K}' \cap \mathcal{D})\} \cup \{(C \cup D) \cap B \mid (C \cup D) \in (\mathcal{K}' \cap \mathcal{E})\}$. This is a collection of non-empty closed subsets of B. We will show that $\mathcal{W}$ has the finite intersection property. Let $\mathcal{W}'$ be a non-empty finite subcollection of $\mathcal{W}$.

a) Suppose $\mathcal{W}'$ consists of D_i for $i = 1, \ldots,$ n. Since $\mathcal{K}'$ is a centered system of sets and since each $D_i \in \mathcal{K}'$ we have that $\bigcap_{i=1}^{n} D_i \neq \phi$.

b) Suppose $\mathcal{W}'$ consists of $(C_i \cup D_i) \cap B$ for $i = 1, \ldots, n$. Since $\mathcal{K}' \cap \mathcal{D} \neq \phi$ there exists some $D \in \mathcal{D}$ such that $D \in \mathcal{K}'$ and $\bigcap_{i=1}^{n} (C_i \cup D_i) \cap D \neq \phi$. Therefore, $\phi \neq (\bigcap_{i=1}^{n} (C_i \cup D_i) \cap D) \subset (\bigcap_{i=1}^{n} (C_i \cup D_i) \cap B) = \bigcap_{i=1}^{n} (C_i \cup D_i) \cap B)$.

c) Suppose $\mathcal{W}'$ consists of D_i, $i=1, \ldots, n$, and $C^j \cup D^j$, $j=1, \ldots, m$. Then $(\bigcap_{i=1}^{n} D_i) \cap (\bigcap_{j=1}^{m} (C^j \cup D^j) \cap B) = (\bigcap_{i=1}^{n} D_i) \cap (\bigcap_{j=1}^{m} (C^j \cup D^j)) \cap B = (\bigcap_{i=1}^{n} D_i) \cap (\bigcap_{j=1}^{m} (C^j \cup D^j)) \neq \phi$, since $\mathcal{K}'$ is a centered system of sets. Now, since B is compact and since $\mathcal{W}$ is a collection of closed subsets of B having the finite intersection property then $\phi \neq \bigcap \{W | W \in \mathcal{W}\} = (\bigcap \{D | D \in \mathcal{K}'\}) \cap (\bigcap \{(C \cup D) \cap B | (C \cup D) \in \mathcal{K}'\}) \subset (\bigcap \{D | D \in \mathcal{K}'\}) \cap (\bigcap \{C \cup D | (C \cup D) \in \mathcal{K}'\}) = \bigcap \{K | K \in \mathcal{K}'\}$.

v) Let $\mathcal{K}'$ be a centered system of sets from $\mathcal{K}$ such that $\mathcal{K}' \cap \mathcal{C} \neq \phi$, $\mathcal{K}' \cap \mathcal{D} = \phi$ and $\mathcal{K}' \cap \mathcal{E} \neq \phi$. Consider the collection $\mathcal{W} = \{C \cap B | C \in (\mathcal{C} \cap \mathcal{K}')\} \cup \{(C \cup D) \cap B | (C \cup D) \in (\mathcal{E} \cap \mathcal{K}')\}$. Let $\mathcal{C}' = \{C | C \in (\mathcal{C} \cap \mathcal{K}')$ or $(C \cup D) \in (\mathcal{E} \cap \mathcal{K}')$ for some $D \in \mathcal{D}\}$.

a) Suppose there is a $C^* \in \mathcal{K}'$ such that $C^* \cap B = \phi$. Let $\mathcal{C}''$ be any finite subcollection of $\mathcal{C}'$.

i) Suppose $\mathcal{C}''$ consists of C_i, $i=1, \ldots, n$, such that $C_i \in (\mathcal{C} \cap \mathcal{K}')$. Then since $\mathcal{K}'$ is a centered system, $\bigcap_{i=1}^{n} C_i \neq \phi$.

ii) Suppose $\mathcal{C}''$ consists of C_i, $i=1, \ldots, n$, such that $(C_i \cup D_i) \in (\mathcal{L} \cap \mathcal{K}')$ for some $D_i \in \mathcal{D}$. Since $\mathcal{K}'$ is a centered system of sets, $(\bigcap_{i=1}^{n} (C_i \cup D_i)) \cap C^* \neq \phi$. Also, since $C^* \cap B = \phi$ we have that $\phi \neq (\bigcap_{i=1}^{n} (C_i \cup D_i)) \cap C^* = ((\bigcap_{i=1}^{n} C_i) \cup (\bigcap_{i=1}^{n} D_i)) \cap C^* = ((\bigcap_{i=1}^{n} C_i) \cap C^*) \cup ((\bigcap_{i=1}^{n} D_i) \cap C^*) = (\bigcap_{i=1}^{n} C_i) \cap C^* \subset (\bigcap_{i=1}^{n} C_i)$.

iii) Suppose $\mathcal{C}''$ consists of C_i, $i=1, \ldots, n$, such that $C_i \in (\mathcal{C} \cap \mathcal{K}')$ and C^j, $j=1, \ldots, m$, such that $(C^j \cup D^j) \in (\mathcal{L} \cap \mathcal{K}')$ for some $D^j \in \mathcal{D}$. As before $\phi \neq ((\bigcap_{i=1}^{n} C_i) \cap (\bigcap_{j=1}^{m} (C^j \cup D^j)) \cap C^*) = ((\bigcap_{i=1}^{n} C_i) \cap (\bigcap_{j=1}^{m} C^j) \cap C^*)$ $\subset ((\bigcap_{i=1}^{n} C_i) \cap (\bigcap_{j=1}^{m} C^j))$. Therefore, if there is a C^* such that $C^* \cap B = \phi$, then $\mathcal{C}'$ is a centered system of sets from $\mathcal{C}$ and since (A,T_A) is cocompact relative to $\mathcal{C}$, it follows from Remark 4.2 that $\phi \neq \bigcap \{C \mid C \in \mathcal{C}'\} \subset (\bigcap \{C \mid C \in (\mathcal{C} \cap \mathcal{K}')\}) \cap (\bigcap \{C \cup D \mid (C \cup D) \in (\mathcal{L} \cap \mathcal{K}')\}) = \bigcap \{K \mid K \in \mathcal{K}'\}$.

b) Suppose $C \cap B \neq \phi$ for all $C \in (\mathcal{C} \cap \mathcal{K}')$. The set $\mathcal{W} = \{C \cap B \mid C \in (\mathcal{C} \cap \mathcal{K}')\} \cup \{(C \cup D) \cap B \mid (C \cup D) \in (\mathcal{L} \cap \mathcal{K}')\}$ is therefore a collection of non-empty closed subsets of B.

i) If $\mathcal{W}$ has the finite intersection property, then since B is compact we have that $\phi \neq \cap \{W | W \in \mathcal{W}\} = (\cap \{(C \cap B) | C \in (\mathcal{C} \cap \mathcal{K}')\}) \cap (\cap \{(C \cup D) \cap B | (C \cup D) \in (\mathcal{E} \cap \mathcal{K}')\}) \subset (\cap \{C | C \in (\mathcal{C} \cap \mathcal{K}')\}) \cap (\cap \{(C \cup D) | (C \cup D) \in (\mathcal{E} \cap \mathcal{K}')\}) = \cap \{K | K \in \mathcal{K}'\}$.

ii) Suppose there exists a finite subcollection $(C_i \cup D_i) \cap B$, $i=1, \ldots, n$, such that $\bigcap_{i=1}^{n}((C_i \cup D_i) \cap B) = \phi$. Then by an argument similar to that in part (iii) - (b) we get $\cap \{K | K \in \mathcal{K}'\} \neq \phi$.

iii) If there is a finite subcollection $C_i \cap B$, $i=1, \ldots, n$, such that $\bigcap_{i=1}^{n} (C_i \cap B) = \phi$, then by an argument similar to that in part (v) - (a) will show $\cap \{K | K \in \mathcal{K}'\} \neq \phi$.

iv) Suppose there is a finite subcollection of $\mathcal{W}$ say $C_i \cap B$, $i=1, \ldots, n$, where $C_i \in (\mathcal{C} \cap \mathcal{K}')$ and $(C^j \cup D^j) \cap B$, $j=1, \ldots, m$, where $(C^j \cup D^j) \in (\mathcal{E} \cap \mathcal{K}')$ such that $(\bigcap_{i=1}^{n} (C_i \cap B)) \cap (\bigcap_{j=1}^{m} ((C^j \cup D^j) \cap B)) = \phi$. Since $\mathcal{K}'$ is a centered system of sets and since $(\bigcap_{i=1}^{n} (C_i \cap B)) \cap (\bigcap_{j=1}^{m} ((C^j \cup D^j) \cap B)) = (\bigcap_{i=1}^{n} C_i) \cap (\bigcap_{j=1}^{m} (C^j \cup D^j)) \cap B = \phi$, we have that $\phi \neq (\bigcap_{i=1}^{n} C_i) \cap (\bigcap_{j=1}^{m} (C^j \cup D^j)) \subset (A - B)$. Let $\mathcal{C}''$ be any finite subcollection of $\mathcal{C}'$.

1) If $\mathcal{C}''$ consists of C_k, $k=1, \ldots, s$, such that $C_k \in (\mathcal{C} \cap \mathcal{K}')$, then since $\mathcal{K}'$ is a centered system, $\bigcap_{k=1}^{s} C_k \neq \phi$.

2) Suppose $\mathcal{C}''$ consists of C_k, $k=1, \ldots, s$, such that $(C_k \cup D_k) \in (\mathcal{E} \cap \mathcal{K}')$ for some $D_k \in \mathcal{D}$ and suppose $\bigcap_{k=1}^{s} C_k = \phi$. Since $\mathcal{K}'$ is a centered system, $\phi \neq \bigcap_{k=1}^{s}$

$(C_k \cup D_k) = (\bigcap_{k=1}^{s} C_k) \cup (\bigcap_{k=1}^{s} D_k) = \bigcap_{k=1}^{s} D_k \subset B$. Therefore, since $(\bigcap_{i=1}^{n} C_i) \cap (\bigcap_{j=1}^{m} (C^j \cup D^j)) \subset (A - B)$ we have $(\bigcap_{i=1}^{n} C_i) \cap (\bigcap_{j=1}^{m} (C^j \cup D^j)) \cap (\bigcap_{k=1}^{s} (C_k \cup D_k)) = \phi$ which contradicts $\mathcal{K}'$ being a centered system. Thus $\bigcap_{k=1}^{s} C_k \neq \phi$.

3) Suppose $\mathcal{C}''$ consists of C_k, $k=1, \ldots, s$, where $C_k \in (\mathcal{C} \cap \mathcal{K}')$ and C_p, $p=1, \ldots, t$, such that $(C_p \cup D_p) \in (\mathcal{E} \cap \mathcal{K}')$ for some $D_p \in \mathcal{D}$. Suppose $(\bigcap_{k=1}^{s} C_k) \cap (\bigcap_{p=1}^{t} C_p) = \phi$, then $\phi \neq (\bigcap_{k=1}^{s} C_k) \cap (\bigcap_{p=1}^{t} (C_p \cup D_p)) = (\bigcap_{k=1}^{s} C_k) \cap (\bigcap_{p=1}^{t} C_p) \cup (\bigcap_{p=1}^{t} D_p) = \bigcap_{p=1}^{t} D_p \subset B$. Therefore the collection C_i, $i=1, \ldots, n$, $(C^j \cup D^j)$, $j=1, \ldots, m$, C_k, $k=1, \ldots, s$, and $(C_p \cup D_p)$, $p=1, \ldots, t$, have an empty intersection contradicting $\mathcal{K}'$ being a centered system. Hence $(\bigcap_{k=1}^{s} C_k) \cap (\bigcap_{p=1}^{t} C_p) \neq \phi$, and we conclude that $\mathcal{C}'$ is a centered system of sets from $\mathcal{C}$. Since (A, T_A) is cocompact relative to $\mathcal{C}$, we have from Remark 4.2 that $\phi \neq \bigcap \{C \mid C \in \mathcal{C}'\} \subset (\bigcap \{C \mid C \in (\mathcal{C} \cap \mathcal{K}')\}) \cap (\bigcap \{(C \cup D) \mid (C \cup D) \in (\mathcal{E} \cap \mathcal{K}')\}) = \bigcap \{K \mid K \in \mathcal{K}'\}$.

(vi) Let $\mathcal{K}'$ be a centered system of sets from $\mathcal{K}$ such that $\mathcal{K}' \cap \mathcal{C} \neq \phi$, $\mathcal{K}' \cap \mathcal{B} \neq \phi$ and $\mathcal{K}' \cap \mathcal{E} \neq \phi$. Consider the collection $\mathcal{W} = \{C \cap B | C \in (\mathcal{C} \cap \mathcal{K}')\} \cup \{D | D \in \mathcal{K}'\} \cup \{(C \cup D) \cap B | (C \cup D) \in (\mathcal{E} \cap \mathcal{K}')\}$. Since $\mathcal{K}' \cap \mathcal{B} \neq \phi$ and since $\mathcal{K}'$ is a centered system, we have that $C \cap B \neq \phi$ for all $C \in (\mathcal{C} \cap \mathcal{K}')$. Therefore, $\mathcal{W}$ is a collection of closed subsets of B. Consider the finite subcollection $C_i \cap B$, $i=1, \ldots, n$, D_j, $j=1, \ldots, m$, and $(C_k \cup D_k) \cap B$, $k=1, \ldots, s$, then $(\bigcap_{i=1}^{n} (C_i \cap B)) \cap (\bigcap_{j=1}^{m} D_j) \cap (\bigcap_{k=1}^{s} ((C_k \cup D_k) \cap B)) = (\bigcap_{i=1}^{n} C_i) \cap (\bigcap_{j=1}^{m} D_j) \cap (\bigcap_{k=1}^{s} (C_k \cup D_k)) \neq \phi$, since $\mathcal{K}'$ is a centered system of sets. For any other finite subcollection of $\mathcal{W}$ it has already been considered in the previous work. Since B is compact we have that $\cap \{K | K \in \mathcal{K}'\} \neq \phi$. Therefore $\mathcal{K}$ is a closed base for $A \cup B$ such that every centered system of sets in $\mathcal{K}$ has a non-empty intersection. By Remark 4.2 $(A \cup B, T_{A \cup B})$ is cocompact.

<u>Corollary 4.8</u>: If (X,T) contains a closed cocompact subset and a closed compact subset whose union is X, then (X,T) is cocompact.

<u>Corollary 4.9</u>: If (X,T) is locally compact at a point and every proper closed subset of X is cocompact, then (X,T) is cocompact.

Proof: Since X is locally compact at a point, there exists some $x \in X$ and some open set O containing x such that Cl O is compact. Since $(X - O)$ is closed, we have that $(X - O)$ is cocompact, and by Theorem 4.7 $X = (X - O) \cup \text{Cl } O$ is cocompact.

Part II - Locally Cocompact

In his original work on cospaces J. M. Aarts proved that every locally compact space was cocompact, and that every cocompact regular space was Baire and hence second category. In [7], R. McCoy proved that every locally compact T_2 space had property L, and that property L implied second category. From this paper by R. McCoy and from the suggestions of P. Fletcher this section will relate cocompact to property L, consider cocompact as a local property, note some equivalences of second category and a local property on Baire, and provide some counterexamples.

Definition 4.10 [7]: A collection $\mathcal{C}$ of subsets of a space X will be called locally finite somewhere if there is an open set U in X which intersects only finitely many members of $\mathcal{C}$.

Definition 4.11 [7]: A space X will have property L if every point finite open cover is locally finite somewhere.

Definition 4.12 [7]: A space X will have property L locally if for every open subset U and each point $x \in U$ there is an open subset V such that $x \in V \subset U$ and V has property L.

Note that as indicated in [7], if (X,T) has property L locally then (X,T) has property L.

Definition 4.13: A space (X,T) is somewhere cocompact if and only if there exists some non-empty open set O such that (O,T_O) is cocompact.

Definition 4.14: A space X is said to be locally cocompact if and only if for each $p \in X$ there is an open set U containing p such that U is cocompact.

Definition 4.15: A space X is said to be strongly locally cocompact if and only if for each $p \in X$ there is an open set U containing p such that Cl U is cocompact.

Definition 4.16: A space X is a Baire space if and only if every non-empty open subset is of second category in the space.

Definition 4.17: A space X is said to be somewhere Baire if and only if there is a non-empty open set O such that O is Baire.

Convention: By the expression "nowhere cocompact" (nowhere Baire) will be meant not somewhere cocompact (somewhere Baire).

The next four results are immediate from the definitions.

Theorem 4.18: Every cocompact space is both locally cocompact and strongly locally cocompact.

Theorem 4.19: Every locally cocompact space is somewhere cocompact.

Theorem 4.20: If (X,T) is a strongly locally cocompact regular space, then it is locally cocompact.

Proof: For each $x \in X$, by hypothesis, there is an open set O containing x such that $\mathrm{Cl}\ O$ is cocompact. Since $\mathrm{Cl}\ O$ is regular and since O is open, it follows that O is cocompact as a subspace of $\mathrm{Cl}\ O$. However, $T_O = (T_{\mathrm{Cl}\ O})_O$, and therefore (X,T) is locally cocompact.

Theorem 4.21: If (X,Υ) is a metric space, then (X,Υ) is locally cocompact if and only if (X,Υ) is strongly locally cocompact.

Proof: Suppose (X,Υ) is locally cocompact. For each $x \in X$ there is an open set O containing x such that O is cocompact. Since X is regular there exists some $V \in T$ such that $x \in V \subset \mathrm{Cl}\ V \subset O$. Since O is metrizable and cocompact, it follows that $\mathrm{Cl}\ V$ is cocompact. Thus (X,Υ) is strongly locally cocompact.

If (X,Υ) is strongly locally cocompact, then the result follows from Theorem 4.20.

<u>Theorem 4.22</u>: If (X,T) is a locally cocompact regular space, then every open subset of (X,T) is locally cocompact.

<u>Proof</u>: Let U be any non-empty open subset of (X,T) and let $p \in U$. Since X is locally cocompact, there is an open set V containing p such that V is cocompact. Thus $p \in V \cap U$ which is open in U. Also, since $V \cap U$ is open in V, $V \cap U$ is cocompact. Hence $V \cap U$ is open in (U,T_U), contains p, and is cocompact as a subset of U. Therefore, U is locally cocompact.

<u>Example 4.23</u>: A closed subspace of a locally cocompact space need not be locally cocompact.

<u>Construction</u>: From [6, p. 154], the rationals are homeomorphic to a closed subset of the Cartesian product of real-lines. From Remark 4.2, the product of a number of real-lines is cocompact, hence locally cocompact. Since the rationals are of first category, it follows from the diagram following Corollary 4.32 that the rationals are not locally cocompact. Therefore, a closed subspace of a locally cocompact space need not be locally cocompact.

<u>Theorem 4.24</u>: The topological union of locally cocompact spaces is locally cocompact.

Proof: Let $(X_\gamma, T_\gamma)_{\gamma \in \Gamma}$ be a collection of locally cocompact spaces, and let (X,T) represent their topological union. For each $x \in X$ there is an $\gamma \in \Gamma$ such that $x \in X_\gamma$, and hence there is an open set $U_\gamma^* \in T_\gamma$ such that U_γ^* is cocompact. By the definition of T, $U_\gamma^* \in T$. Also, since $T_{U_\gamma^*} = (T_\gamma)_{U_\gamma^*}$ it follows that $(U_\gamma^*, T_{U_\gamma^*})$ is cocompact and hence that X is locally cocompact.

Theorem 4.25: The finite product of locally cocompact spaces is locally cocompact.

Proof: Consider the collection (X_i, T_i), $i=1, \ldots,$ n, of locally cocompact spaces, and let (X,T) represent the product space. For each $x \in X$ there is a $U_i \in T_i$ such that $x_i \in U_i$ and U_i is cocompact. By Remark 4.2, $\prod_{i=1}^{n} U_i$ is cocompact and since it is an open set in (X,T) containing x, it follows that (X,T) is locally cocompact.

Theorem 4.26: If (X,T) has a locally compact cospace, then (X,T) is locally cocompact.

Proof: If (X,T) has a locally compact cospace, then, by [1, p. 4), (X,T) is cocompact, and, by Theorem 4.18, (X,T) is locally cocompact.

The following results relate cocompactness to property L. We first show that a space has property L if it has a non-empty open subset that is cocompact. We next show that if about each point in the space there exists an open subset that is cocompact, then the space has property L locally. We conclude with a diagram relating cocompact and property L and provide counterexamples whenever necessary.

Theorem 4.27: If (X,T) is a somewhere cocompact regular space, then (X,T) has property L.

Proof: Suppose (X,T) does not have property L. Then there is a point finite open cover $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ of X such that every open set in X intersects infinitely many members of $\mathcal{U}$. Since X is somewhere cocompact, there is a non-empty open set U such that (U,T_U) is cocompact. Let $(U,(T_U)')$ be a compact cospace of U determined by a closed base $\mathcal{B}$ for (U,T_U). Let $x_0 \in U$. Since $U \in T_U$, there is a $B_1 \in \mathcal{B}$ such that $x_0 \in \text{int } B_1 \subset B_1 \subset U$, and B_1 is closed in $(U,(T_U)')$ and hence compact. Let $O_1 = \text{int } B_1$. Since $O_1 \in T_U$ and since U is open, it follows that $O_1 \in T$ and there exists a $U_{\gamma_1} \in \mathcal{U}$ such that $O_1 \cap U_{\gamma_1} \neq \phi$. Let $x_1 \in O_1 \cap U_{\gamma_1}$. Since $O_1 \subset U$, $(O_1 \cap U_{\gamma_1}) \in T_U$. Therefore, there is a $B_2 \in \mathcal{B}$ such that $x_1 \in \text{int } B_2 \subset B_2 \subset (O_1 \cap U_{\gamma_1})$. Also, since B_2 is closed in $(U,(T_U)')$, B_2 is compact in $(U,(T_U)')$. Let $O_2 = \text{int } B_2$. Again $O_2 \in T$ and there is a $U_{\gamma_2} \in \mathcal{U}$ such that $U_{\gamma_2} \neq U_{\gamma_1}$ and $O_2 \cap U_{\gamma_2} \neq \phi$. Let $x_2 \in (O_2 \cap U_{\gamma_2})$. Thus for each $n > 1$ there is a $B_n \in \mathcal{B}$ such that $\text{int } B_n \neq \phi$ and

$B_n \subset (\text{int } B_{n-1} \cap U_{\gamma_{n-1}})$, where B_n is closed and compact in $(U,(T_U)')$ and $U_{\gamma_n} \in \mathcal{U}$ such that $(\text{int } B_n) \cap U_{\gamma_n} \neq \phi$ and $U_{\gamma_n} \neq U_{\gamma_i}$ for $i < n$. Also, note that $\ldots \subset ((\text{int } B_2) \cap U_{\gamma_2}) \subset \text{int } B_2 \subset B_2 \subset ((\text{int } B_1) \cap U_{\gamma_1}) \subset \text{int } B_1 \subset B_1 \subset U$. By the Cantor Product Theorem, we have $\bigcap_{n=1}^{\infty} B_n \neq \phi$, and since $B_n \subset U_{\gamma_{n-1}}$ for $n > 1$, it follows that $\bigcap_{n=1}^{\infty} U_{\gamma_n} \neq \phi$. This contradicts $\mathcal{U}$ being a point finite open cover and proves the theorem.

<u>Corollary</u> <u>4.28</u>: If X is a cocompact regular space, then X has property L.

<u>Corollary</u> <u>4.29</u>: If (X,T) is a locally cocompact regular space, then (X,T) has property L locally.

<u>Proof</u>: Let $U \in T$ and let $x \in U$. By hypothesis there is an open set V containing x such that V is cocompact. Consider $V \cap U$ which is open in X and contains x. Since $V \cap U$ is open in V, from Remark 4.2, $V \cap U$ is cocompact as a subset of V and is cocompact as a subset of X. From Corollary 4.28, we have that $V \cap U$ is an open set in X, containing x, that has property L. It follows that (X,T) has property L locally.

<u>Lemma</u> <u>4.30</u>: A space (X,T) has property L locally if and only if every non-empty open subset of (X,T) has property L.

Proof: Suppose (X,T) has property L locally. Let O be any non-empty open set, let $x \in O$, and let $\mathcal{C}$ be any point finite open cover of O. By hypothesis there is an open set V containing x such that $V \subset O$ and V has property L. Note that since $\mathcal{C}$ is a point finite open cover of O, $\mathcal{C}$ is a point finite open cover of V. Since V has property L, we have, from Definition 4.11, that $\mathcal{C}$ is locally finite somewhere with respect to (V, T_V), and therefore there is an open set W in (V, T_V) which intersects only finitely many members of $\mathcal{C}$. But since W is also open in O, $\mathcal{C}$ is locally finite somewhere with respect to (O, T_O). It follows that O has property L.

The converse is obvious.

Theorem 4.31: If a space (X,T) has property L locally, then it is a Baire space.

Proof: For each non-empty open set O, we have, by Lemma 4.30, that O has property L. From [7, Theorem 2], we conclude that O is of second category, and it follows from Definition 4.16 that (X,T) is a Baire space.

Corollary 4.32: If (X,T) is a locally cocompact regular space, then (X,T) is a Baire space.

Proof: The result follows from Corollary 4.29 and Theorem 4.31.

From the preceding material we have the following diagram relating cocompact and property L.

$$\begin{tikzcd}
\text{locally compact} \arrow[d, Rightarrow] & & & & \\
\text{cocompact} \arrow[r, Rightarrow] \arrow[ddr, Rightarrow, "+\ \text{regular}"'] & \text{locally cocompact} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "+\ \text{regular}"] & \text{somewhere cocompact} \arrow[r, Rightarrow, "+\ \text{regular}"] & \text{property L} \arrow[r, Rightarrow] & \text{second category} \\
 & \text{property L locally} \arrow[urr, Rightarrow] \arrow[d, Rightarrow] & & & \\
 & \text{Baire} \arrow[uurrr, Rightarrow] & & &
\end{tikzcd}$$

In reference to the above diagram, we provide the following examples.

i) A space with property L locally that is not locally cocompact. Therefore,

  a) A Baire space that is not locally cocompact.

  b) A space with property L that is not cocompact.

ii) A space with property L and therefore of second category that is not somewhere cocompact.

iii) A space that is somewhere cocompact and not locally cocompact.

iv) A space that is locally cocompact and not cocompact.

<u>Example 4.33</u>: A space with property L locally that is not locally cocompact.

<u>Construction</u>: (First, note from [1] that a metric space is cocompact if and only if it is topologically complete.)

Let R be the reals with the usual topology, let Ra be the rationals, and let Z be the integers. Let X be the subspace of (R x R) consisting of the points (r,o) where $r \in$ Ra, and the points $(\frac{k}{n},\frac{1}{n})$ where $n \geq 1$, $n \in Z$ and $k \in Z$. Then X as a subspace of (R x R) is metrizable.

i) We show that X has property L locally. Let O be any non-empty open set in X, and let $\mathcal{C}$ be any point finite open cover of O.

a) Suppose O contains no points of the form (r,o) where $r \in$ Ra. Let $x \in O$. Then there exists a set V, open in (R x R), containing x such that $V \cap X = \{x\}$. Thus $\{x\}$ is open in X and therefore open in O. Since $\mathcal{C}$ is a point finite open cover of O, we have that x is in only finitely many $C \in \mathcal{C}$. It follows that $\mathcal{C}$ is locally finite somewhere and hence O has property L.

b) Suppose O contains some points of the form (r,o) where $r \in$ Ra. Then O must contain some point above the x-axis, say $y \in O$. Then $\{y\}$ is open in O and intersects only finitely many $C \in \mathcal{C}$. It follows that O has property L. From Lemma 4.30, X has property L locally.

ii) We show X is not locally cocompact. Let (r,o) ε X, where r ε Ra, and let U be any open set in X such that (r,o) ε U. We show U is not toplogically complete, hence not cocompact. Suppose U is toplogically complete. Consider $S = \{(r,o) | r \varepsilon Ra\} \cap U$. Then S is a closed subset of U, since $\{(r,o) | r \varepsilon Ra\}$ is closed in X. Also, S is not of second category and therefore S is not topologically complete. This is a contradiction, since every closed subset of a complete metric space is complete. Therefore, U is not topologically complete and also not cocompact. It follows that there does not exist an open set of (r,o), where r ε Ra, that is cocompact, and hence X is not locally cocompact.

Example 4.34: A space with property L that is not somewhere cocompact.

Construction: The following will prove the existence of a space with property L that is not somewhere cocompact. In his dissertation (to appear), R. Haworth proved that there exists a separable metrizable Baire space such that each open subset is not toplogically complete. From the comment in Example 4.33 we conclude that there exists a separable metric Baire space such that each open subset is not cocompact. Thus the space is not somewhere cocompact, and from [7, Proposition 4] the space has property L.

Example 4.35: A space that is somewhere cocompact and not locally cocompact.

Construction: Let Ra be the rationals with the usual topology, let x be a point not in Ra, and let (X,T) represent the topological union of Ra and x. Since {x} is an open compact subset of (X,T), it follows that (X,T) is somewhere cocompact. Also, note that (X,T) is regular and T_2. Let $y \varepsilon X$ such that $y \neq x$ and let $O \varepsilon T$ containing y. Suppose O is cocompact and consider $O' = O - \{x\}$. Since O is cocompact and regular, $(O', T_{O'})$ is cocompact and thus of second category. But, since $O' \subset Ra \subset X$ and since both O' and Ra are open in X, we have that $(O', T_{O'})$ is of first category. This contradiction says that O is not cocompact, and therefore (X,T) is not locally cocompact.

Example 4.36: A space that is locally cocompact and not cocompact.

Construction: In [20] Tall develops a space that is Cech complete, Tychnoff, locally metrizable and not cocompact. He also notes that Cech completeness is inherited by open sets, and that in metric spaces cocompact, topologically complete, and Cech complete are equivalent. Thus we conclude that for each point in the space there is an open set containing the point that is metrizable and Cech complete, thus cocompact. It follows that the space is locally cocompact.

Since both somewhere Baire and somewhere cocompact give second category, the following results were obtained by R. McCoy and myself after investigating as to when second category gives somewhere Baire or somewhere cocompact. A note concerning these results is included in the appendix.

Lemma 4.37: If a space (X,T) contains an open dense subset of first category, then (X,T) is of first category.

Proof: Suppose Y is an open dense subset of X of first category in X. Then since Y is dense and $(X - Y)$ is closed, we have that any open set about a point in $X - Y$ intersects Y. Therefore, $\operatorname{int}(X - Y) = \emptyset$ and $X - Y$ is nowhere dense. Since Y is the countable union of nowhere dense sets, it follows that $X = Y \cup (X - Y)$ is of first category.

Remark 4.38: Lemma 4.37 says that the 1-point compactification of a space of first category is also of first category.

Theorem 4.39: If (X,T) is a separable, first countable, nowhere Baire space, then (X,T) has an open dense subset of first category.

Proof: Since (X,T) is separable and first countable we can let D be a countable dense subset of (X,T), and for each $x \varepsilon D$ let $\mathcal{B}^x$ be a countable base at x. Since the space is nowhere Baire, we have that for each $B_i^x \varepsilon \mathcal{B}^x$ there exists an open set of first category, U_i^x, such that $U_i^x \subset B_i^x$ for each x. Let $Y^x = \bigcup_{i=1}^{\infty} U_i^x$ and let $Y = \bigcup_{x \varepsilon D} Y^x$. Therefore, Y is open, and since $D \subset Y$ and D is dense in X, we have that Y is dense in X. Since Y^x is the countable union of sets of first category, we have for each $x \varepsilon D$ that Y^x is of first category. Similarly, $Y = \bigcup_{x \varepsilon D} Y^x$ is of first category. Therefore, Y is an open dense subset of first category.

Corollary 4.40: If (X,T) is a separable, first countable, nowhere Baire space, then (X,T) is of first category.

Corollary 4.41: If (X,T) is a separable, first countable, second category space, then (X,T) is somewhere Baire.

Corollary 4.42: Let (X,T) be a separable, first countable space. The space (X,T) is second category if and only if it is somewhere Baire.

Theorem 4.43: If the space (X,T) is hereditarily Lindelof and nowhere Baire, then (X,T) has an open dense subset.

Proof: Consider the collection $\mathcal{B}$ of all open subsets of X. Since the space is nowhere Baire, we have for each $B \varepsilon \mathcal{B}$ an open subset of first category, U_γ, such that $U_\gamma \subset B$. Let $Y = \bigcup U_\gamma$, and note that Y is an open dense subset of X. Since the space is hereditarily Lindelof, Y is Lindelof and hence there is a countable collection of U_γ, say U_{γ_i} $i=1, 2, \ldots,$ such that $Y = \bigcup_{i=1}^{\infty} U_{\gamma_i}$. Then, since each U_{α_i} is of first category, we have that Y is an open dense subset of first category.

Corollary 4.44: Let (X,T) be a space that is hereditarily Lindelof. The space (X,T) is of second category if and only if it is somewhere Baire.

Theorem 4.45: If the space (X,T) is hereditarily paracompact and nowhere Baire, then (X,T) has an open dense subset of first category.

Proof: Consider the collection $\mathcal{B}$ of all open subsets of X. For each $B \in \mathcal{B}$ there is an open subset of first category, U_B, such that $U_B \subset B$. Let $Y = \bigcup_{B \in \mathcal{B}} U_B$ and note that Y is an open dense subset of X. Since X is hereditarily paracompact and since $\{U_B\}$ is an open covering of Y, there is an open, locally finite refinement of $\{U_B\}$, say $\{V_\gamma\}_{\gamma \in \Gamma}$. Since each U_B is of first category we have that each V_γ is of first category. For each $\gamma \in \Gamma$ represent V_γ by $V_\gamma = \bigcup_{i=1}^{\infty} Y_i^\gamma$ where Y_i^γ is nowhere dense in X. Let $Y_n = \bigcup_{\gamma \in \Gamma} Y_n^\gamma$ for n = 1, 2, . . ., and note that $Y = \bigcup_{n=1}^{\infty} Y_n$. Each $\{Y_n^\gamma : \gamma \in \Gamma\}$ is locally finite since $Y_n^\gamma \subset V_\gamma$ for each γ, and $\{V_\gamma\}_{\gamma \in \Gamma}$ is locally finite. Therefore, since a locally finite collection is closure preserving we have that int (Cl Y_n) = int (Cl $(\bigcup_\gamma Y_n^\gamma)$) = int $(\bigcup_\gamma (\mathrm{Cl}\ Y_n^\gamma))$. Suppose $x \in$ int (Cl Y_n). Then $x \in$ int $(\bigcup_\gamma (\mathrm{Cl}\ Y_n^\gamma))$, and without loss of generality there is an open set O containing x such that $x \in O \subset \bigcup_{j=1}^{k} Y_n^{\gamma_j}$. Now $\bigcup_{j=1}^{k} Y_n^{\gamma_j}$ is nowhere dense, since it is the finite union of nowhere dense sets. This says there does not exist an open set contained in Cl $(\bigcup_{j=1}^{k} Y_n^{\gamma_j}) = \bigcup_{j=1}^{k} (\mathrm{Cl}\ Y_n^{\gamma_j})$. Thus

int (Cl Y_n) $= \Phi$, and for each n, Y_n is nowhere dense and Y is therefore of first category. Thus, Y is an open dense subset of first category.

<u>Corollary</u> <u>4.46</u>: Let (X,T) be hereditarily paracompact. The space (X,T) is of second category if and only if (X,T) is somewhere Baire.

Note that Example 4.34 indicates the existence of a space that is separable, metrizable and Baire that is nowhere cocompact. Therefore, we conclude that analogous theorems to the above relating second category and somewhere cocompact are not true.

APPENDIX

From Theorem 3.24 we have that cocontinuity at each point is equivalent to cocontinuity provided the range space is locally compact and T_2. As previously indicated, if the range space is not locally compact, then the exact relation is not known. In regard to this notion consider the following remarks.

Proposition: Let (X,T) and (Y,U) be spaces such that $X = A \cup B$ where A and B are open subsets of X. If $f:(X,T) \to (Y,U)$ is a function such that $f_{|A}$ and $f_{|B}$ are cocontinuous with respect to (Y,U^*) and (Y,U') respectively, and if $U^* \cap U'$ is a cotopology on Y, then f is cocontinuous.

Proof: Let $O \in (U^* \cap U')$. Since $f_{|A}^{-1}(O)$ and $f_{|B}^{-1}(O)$ are open in (X,T), and since $f^{-1}(O) = f_{|A}^{-1}(O) \cup f_{|B}^{-1}(O)$, it follows that $f^{-1}(O) \in T$ and f is cocontinuous with respect to $(Y, U^* \cap U')$.

Now consider the following: Suppose $f:(X,T) \to (Y,U)$ is a function such that

i) (Y,U) is regular and T_1

ii) $X = A \cup B$ where A and B are open

iii) $f_{|A}$ and $f_{|B}$ are cocontinuous

iv) f is not cocontinuous.

If there exists such a function, then from Theorem 3.25 we can conclude that f is cocontinuous at each x, but f is not cocontinuous.

Also, Verbeek-Kroonenberg asked in [2], that if a regular T_1 space were not locally compact do there exist two cotopologies whose intersection is not a cotopology. In regard to this question and the suggested example, it follows from Theorem 3.24 that (Y,U) is not locally compact. Therefore from the proposition we could conclude that there are two cotopologies of U whose intersection is not a cotopology.

In a paper to appear by P. Fletcher and W. F. Lindgren entitled, A Note on Spaces of Second Category, they show that a space of second category is equivalent to a space having property L. This equivalence allows a direct proof of Theorem 4.27 and provides immediate examples of spaces that are not cocompact but do have property L.

Also, a preprint recently received by R. McCoy from J. C. Oxtoby entitled, The Banach-Mazur Game and Banach Category Theorem, provides a corollary that will allow a generalization of Corollaries 4.40, 4.42 and 4.44.

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PROPERTIES OF COCONTINUOUS FUNCTIONS

AND COCOMPACT SPACES

by

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(ABSTRACT)

In this paper we study the concept of cotopology in the areas of cocontinuous functions and cocompact spaces. Initially we investigate and provide needed results concerning closed bases for a topological space. We then study cocontinuous functions by relating them to various other weaker forms of continuous functions, namely c-continuous, almost continuous and weakly continuous. We show that if (Y,U) is locally compact T_2, then $f:(X,T) \to (Y,U)$ is cocontinuous if and only if $f^{-1}(O) \in T$ for every $O \in U$ such that $(Y - O)$ is compact. We note that every almost continuous function is cocontinuous, and we provide conditions under which a weakly continuous function is cocontinuous. We also show that a cocontinuous function from a saturated space to a regular space is continuous.

In the area of cocompact spaces we first provide a partial answer to a question of J. M. Aarts as to when the union of cocompact subsets of a space is cocompact. We show that the union of a closed cocompact subset and a

closed compact subset is cocompact. We then introduce the properties, locally cocompact and somewhere cocompact, and relate them to property L which was introduced by R. McCoy. We show that every somewhere cocompact regular space has property L, and that every locally cocompact regular space has property L locally. We provide examples to show that neither cocompact nor locally cocompact is equivalent to property L.