GRAPHICAL SEQUENCES

by

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PRELIMINARY

The statement of definitions, elementary results and notation which will be used throughout the thesis is given below.

A *graph* or *ordinary graph*, G, is a pair (V,E) where V is a finite non-empty set and E is a set disjoint from V consisting of two element subsets of V. An element of V is called a *point* or *vertex* of G. An element of E, say, $\{x,y\}$ where $x,y \in V$ is an *edge* of G and for convenience we write xy for $\{x,y\}$. To indicate the dependence on G we often use V(G) or VG and E(G) or EG instead of V and E respectively. For the following let G be an arbitrary but fixed graph.

For a set S, $|S|$ denotes the number of elements of S. Thus if $|VG| = p$ and $|EG| = q$ we say G is a (p,q) graph. A (1,0) graph is called a trivial graph.

Two points x and y of VG are *adjacent* if $xy \in EG$. Two distinct edges are *adjacent* if they have a point in common. A point x is *incident* to an edge if the edge has the form xy. The *neighborhood* of $x \in VG$ is the set

$$N_x = \{y \mid xy \in EG\}$$

The *degree* of a vertex $x \in VG$, denoted by degx, is $|N_x|$. If $|N_x| = 0$ then x is called an *isolated vertex*.

A *subgraph* H of G is a graph H such that $VH \subseteq VG$ and $EH \subseteq EG$. For $S \subsetneq VG$ the *point-induced* or *induced* subgraph of G, denoted by $<S>$ is defined by $V<S> = S$ and $xy \in E<S>$ if and only if $x,y \in S$ and $xy \in EG$. For $S \leq EG$ the *edge-induced* subgraph of G is denoted by $<S>$and defined by $E<S> = S$, $V<S> = \bigcup_{s \in S} s$. For $x \in VG$, $G - x$ denotes the subgraph

$\langle V(G) - \{x\}\rangle$. Similarly for e EG, G - e denotes $\langle EG - \{e\}\rangle$.

Two graphs G_1 and G_2 are isomorphic, $G_1 \cong G_2$, if there is a one-to-one onto map f of VG_1 into VG_2 such that xy EG, implies f(x)f(y) EG_2.

Suppose that the points $VG = \{x_1,\cdots,x_p\}$ are indexed in such a way that deg $x_1 \geq$ deg $x_2 \geq\cdots\geq$ deg x_p. Then the sequence (deg $x_1,\cdots,$deg x_p) is the *degree sequence* of G. Sometimes we set deg $x_i = d_i$ or deg $x_i = s_i$, $i = 1,\cdots,p$, for simplicity. Conversely if $s = (d_1,\cdots,d_p)$ is a non-increasing sequence of non-negative integer so that s is a degree sequence for some graph then s is called a *graphical* sequence. If s is the degree sequence of G then G is said to *realize* s or *be a realization* of s or to *belong* to s, and we write G s. If s is a graphical sequence, $|s|$ denotes the number of non-isomorphic realizations of s. If G realizes $s = (d_1,\cdots d_p)$ then we set $d_1 = \Delta G$ and $d_p = \delta G$.

<u>P1</u>. If G belongs to $s = (d_1,\cdots,d_p)$ then $d_1+\cdots+d_p = 2q$ where $|EG| = q$.

For x,y VG an *x-y walk* is a sequence $a_0a_1,\cdots,a_r$ with $x = a_0$, $a_1,\cdots,y = a_r$ VG, $aa_1,\cdots,a_{r-1}a_r$ EG. An *x-y walk* is a walk such that no two members of the sequence $a_0a_1\cdots a_r$ are the same. The vertices x and y are called *terminal* or *endpoints* of the path or walk. A walk $a_0a_1\cdots a_r$ is said to be of *length* r--the number of edges in the walk. A circuit is a walk with the same terminal points and a *cycle* is a walk in which the endpoints are the only ones that occur twice in the sequence.

We now single out some special kinds of graphs because of their common occurence. A *complete* graph on p vertices, K , is a graph such that $|VK_p| = p$ and $|EK_p| = p(p-1)/2$. A *complete bipartite* graph G has the form $VG = S_1 \quad S_2$, $S_1 \quad S_2 =$ and xy EG if and only if x S_1 and y S_2. If $|S_1| = n$ and $|S_2| = m$ then G is denoted by $K_{n,m}$. The graph $K_{1,n}$ is called a *star* graph or *n-star*. A *bipartite* graph is a subgraph of some complete bipartite G.

P2. A graph G is bipartite if and only if every cycle of G is of even length.

If all the degrees of a graph are the same it is said to be *regular*. The *complement* of G, G^c, is defined by $VG^c = VG$ and $xy \in EG^c$ if and only if $xy \notin G$. If a graph on n points is itself a cycle or a path then it is denoted by C_n or P_n respectively. If a graph is itself a circuit it is called *eulerian*. If a graph has exactly one cycle it is called *unicyclic*. If G and H are graphs, $VG \cap VH = \phi$, then $G \cup H$ is called the *union* of G and H and is defined by $V(G \cup H) = VG \cup VH$ and $xy \in E(G \quad H)$ and only if $xy \in EG \cup EH$.

G is *connected* if for any $x,y \in VG$ there is an x-y path. A *component* of G is a maximally connected subgraph of G. A point $x \in VG$ is a *cut-point* if G-x has more components than G. Similarly a *bridge* of G is an edge $e \in EG$ such that G - e has one more component than G. G is *separable* if it has a cut point, and it is connected. A *block* of G is a maximal non-separable subgraph of G.

P3. If $|VG| \geq 3$ then G is non-separable (or itself a block) if and only if every two vertices of G lie on a common cycle.

A connected graph is called a *tree* if it has no circuits. A *forest* is a graph without circuits (acyclic).

P4. The following are equivalent for a connected graph G.

(a) G is a tree

(b) $|VG| = p$, $|EG| = p - 1$

(c) for any pair of distinct points $x,y \in VG$ there is a unique

x-y path.

P5. Let F be a forest with $|F| = p$ and c components. Then $|E(F)| = p - c$.

The following concepts relate graphs to the idea of distance. For $x,y \in VG$, $x \neq y$, $d(x,y)$ denotes the shortest x-y path, if such exists. If $x = y$ we set $d(x,y) = 0$. For a connected graph G, the *diameter* of G, dG, is $\sup \{d(x,y) | x,y \in VG\}$. The *eccentricity* of a point y in a connected graph G is $e(y) = \sup\{d(x,y) | x \in G\}$. For a connected graph G, the *radius* of G, $r(G)$, is $\min\{e(y) | y \in G\}$. Again for G connected the *center* of G is $\{x | x \in VG, e(x) = r(G)\}$.

For a graph G, $VG = \{x_1, \cdots, x_p\}$ the *adjuncency matrix* of G is the pxp matrix whose i,j entry is one if $x_i x_j \in EG$, zero otherwise.

I. GENERAL FEATURES OF GRAPHICAL SEQUENCES

A. Two Fundamental Properties of Graphical Sequences

Perhaps the most useful concept for studying graphical sequences is that of the 'transfer' or degree invariant transformation.

Definition. Let G be a graph and assume x, y, z, w are four distinct points of V(G) such that xy, zw $\in$ E(G) but xz, yw $\notin$ E(G). A transfer of G, denoted by t, is the replacement of the edges xy and zw in E(G) by the edges xz and yw. The new graph so obtained is denoted by tG. In tG replacing the edges xz and yw by xy and zw is also a transfer and is denoted by t^{-1}.

First, it is clear that both G and tG belong to the same graphical sequence. Second, note that $G = t^{-1}tG$.

We now establish that if two graphs belong to the same graphical sequence that they 'differ' by a finite number of transfers. This latter result has been obtained for directed multi-graphs but not ordinary graphs. The proof given below for ordinary graphs is modeled after Berge's result for directed s-graphs (See [1, p. 89-90]).

Theorem 1.1. Let G and H be graphs belonging to same graphical sequence. There exists a finite number of transfers $t_1, \cdots, t_r$ such that $H \cong (t_r \circ \cdots \circ t_1)G$.

Proof. Assume that $V(G) = V(H) = \{x_1, \cdots, x_p\}$. Let A, B be the adjacency matrices of G and H respectively. Since G, H belong to the same degree sequence we have

$$\sum_{i=1}^{p} A_{ij} = \sum_{i=1}^{p} B_{ij} = d_j = \deg x_j,\ j \in \{1,\cdots,p\} \tag{1}$$

Now let $C = A - B$. Then for any $i, j \in \{1,\cdots, p\}$ we have C_{ij} $\{-1,0,1\}$ and

$$\sum_{j=1}^{p} C_{ij} = \sum_{i=1}^{p} C_{ij} = 0 \tag{2}$$

by [1] above.

If $\sum_{ij}|C_{ij}| = 0$ then $A = B$ and we are done. Hence, assume $\sum_{ij}|C_{ij}| > 0$. We now show that there is a transfer t of either G or H, say G, so that if A_t is the adjacency matrix of tG and $C^1 = A_t - B$ then

$$\sum_{ij}|C^1_{ij}| < \sum_{ij}|C_{ij}|$$

Assuming $\sum_{ij}|C^1_{ij}| > 0$ yields immediately by (2) that there exist $i, j, k \in \{1,\cdots, p\}$ such that $C_{ij} = 1$, $C_{kj} = -1$. Again by (2) there is an $\ell \in \{1,\cdots, p\}$ such that $C_{k\ell} = 1$. Again, find $h \in \{1,\cdots, p\}$ such that $C_{h\ell} = -1$. We proceed in this fashion constructing a sequence of the form $C_{ij}C_{kj}C_{k\ell}C_{h\ell}C_{km}\cdots$ where the terms are alternately +1 and -1.

We also require in constructing the sequence that all terms but the first and the last be distinct i.e., if C_{rs} and C_{uv} are not both end terms then $(r, s) \neq (u, v)$. Notice now that C can be regarded as the adjacency matrix of a directed graph, with vertex set $\{x_1,\cdots,x_p\}$, where each node has as many arcs entering as leaving. Hence any cycle starting at the node x must end at x . Thus we can assume that the next to last term of the sequence has the form $c_{ir} = -1$.

Now consider the 'rectangle' C_{ij}, C_{kj}, $C_{k\ell}$ and $C_{i\ell}$. There are three cases according as whether $C_{i\ell}$ is -1, 0, or -1.

Case 1. $C_{i\ell} = -1$. In this case x_ix_j, x_kx_ℓ are in $E(G)$ but not in $E(H)$ and x_kx_j, x_ix_ℓ are in $E(H)$ but not in $E(G)$. Hence the transfer t, in G, of x_ix_j and x_kx_ℓ for x_kx_j and x_ix_ℓ yields a matrix $C^1 = A_t - B$ where C^1 is the same as C except the entries C^1_{ij}, C^1_{kj}, $C^1_{k\ell}$, $C^1_{i\ell}$, C^1_{ji}, C^1_{jk}, $C^1_{\ell h}$, $C^1_{\ell i}$ are all zero. Hence $\sum_{ij}|C^1_{ij}| < \sum_{ij}|C_{ij}|$.

Case 2. $C_{i\ell} = 0$. If also $A_{i\ell} = B_{i\ell} = 0$ then the same transfer as per case 1 can be made since $x_ix_\ell \notin E(G)$ and now C^1 is same as C except $C^1_{ij} = C^1_{kj} = C^1_{k\ell} = C^1_{ji} = C^1_{jk} = C^1_{\ell k} = 0$ and $C^1_{i\ell} = C^1_{\ell i} = 1$. If on the other hand $A_{i\ell} = 1$, $B_{i\ell} = +1$ then we make a transfer t in H by replacing x_ix_ℓ and x_kx_j by x_ix_j and x_kx_ℓ. If $C^1 = A - B_t$ then this C^1 is exactly the same as C^1 obtained in the first part of this case.

Case 3. $C_{i\ell} = +1$. In this case we can replace the sequence $C_{ij}C_{kj}C_{k\ell}C_{h\ell} \cdots C_{ir}C_{ij}$ by the shorter sequence $C_{i\ell}C_{h\ell}\circ\cdots\circ C_{ir}C_{i\ell}$. We now apply method of case 1 and case 2 to the new sequence to obtain the desired C^1 or a shorter sequence. Hence it is clear we get the desired C^1 at some stage--if the sequence is shortened to five terms then $C_{i\ell}$ must be -1.

Now consider C^1. If $\sum_{ij}|C^1_{ij}| = 0$ then $G = tH$ or $tG = H$. If $\sum_{ij}|C^1_{ij}| > 0$ we operate on C^1 as we did on C. It follows that at some point we get a C^n such that $\sum_{ij}|C^n_{ij}| = 0$ and this means that there exists transfers $t_1, \cdots, t_N, s_1, \cdots, s_M$ such that $(t_N\circ\cdots\circ t_1)G = (s_M\circ\cdots\circ s_1)H$. But then $(s_1^{-1}\circ\cdots\circ s_M^{-1}\circ t_N\circ\cdots\circ t_1)\, G = H$. This gives the theorem.

The next result is a generalization of a theorem by Havel and Hakimi.

Definition. Let $S = (d_1, \cdots, d_p)$ be a graphical sequence. For $i = 1, \cdots, p$ the sequence S_i is a sequence of $p - 1$ terms obtained from S by (1) deleting the term d_i, (2) subtracting one from the first d_i remaining terms and (3) arranging the latter integers in their natural order.

Note that for each i, $i = 1,\cdots, p$, the sequence S_i is indeed a sequence of non-negative integers.

<u>Theorem 1.2</u>. If $S = (d_1, \cdots, d_p)$ is a graphical sequence then so is each S_i, $i = 1,\cdots, p$. Conversely if there is some $i \in \{1,\cdots, p\}$ for which S_i is graphical then S is graphical.

<u>Proof</u>. If S_i is graphical for some $i \in \{1,\cdots, p\}$ then let H be a realization of S_i. If d_i is inserted in the sequence S_i it is clear which integers of the sequence S_i must be increased by one in order to obtain the sequence S. If T is the set of points of V(H) whose degrees are the latter integers then the graph obtained from H by adding new point adjacent to each point of T is a realization of S so that S is graphical.

Now assume that S is graphical. Let G be a realization of S, let $i \in \{1,\cdots, p\}$ and let x_i be of degree d_i. If x_i is adjacent to the points of highest degree in $V(G) - \{x_i\}$ then the graph $G - x_i$ is a realization of S . If x is not adjacent to vertices of highest degree then there exists x_j, $x_k \in V(G)$ with $d_j > d_k$, $x_ix_k \in E(G)$ and $x_ix_j \notin E(G)$. It follows that there is $x_\ell \in V(G)$ such that $x_\ell x_j \in E(G)$ but $x_\ell x_j \notin E(G)$ since $d_j > d_k$. Hence the transfer t of x_ix_j and $x_\ell x_j$ for x_ix_j and $x_\ell x_k$ can be made. If N_{x_i} is set of neighbors of x_i in G and $N^t_{x_i}$ is set of neighbors of x_i in tG then

$$\sum_{x \in N_{x_i}} \deg(x) < \sum_{x \in N^t_{x_i}} \deg(x).$$

Hence if x_i is not adjacent to points of largest degree then a transfer can be made so as to increase $\sum_{x \in Nx_i} \deg(x)$. Thus after a finite number of transfers we obtain a realization H of S such that x_i is adjacent to points of highest degree. This means $H - x_i$ is a realization of S_i. Since i was arbitrary we obtain the result.

B. Characterization of Types of Graphs by Graphical Sequences

Various kinds of graphs can be characterized by properties of degree sequences. Perhaps the first result in this direction is the following well-known proposition about trees.

Proposition 1.3. The sequence $S = (d_1, \cdots, d_p)$ is realized by a tree if and only if $d_p \geq 1$ and $\sum_{i=1}^{p} d_i = 2(p - 1)$.

Proof. The proof proceeds by induction on p using the fact that $d_p = 1$.

Note that the above proposition does not require S to be a graphical sequence. The following is a generalization of proposition 1.3.

Proposition 1.4. A forest with k nontrivial components belongs to a sequence $S = (d_1, \cdots, d_p)$ if and only if $\sum_{i=1}^{p} d_i = 2(p - k)$ where k such that $1 \leq 2k \leq p$ and $d_p \geq 1$.

Proof. The proof is by induction on k with $p \geq 2$. For $k = 1$ the proposition becomes that of 1.3.

Suppose that the proposition is true for $k \leq j$, with $1 \leq 2j \leq p$. Further suppose that $2(j + 1) \leq p$ and that $S = (d_1, \cdots, d_p)$ is a sequence for which $\sum_{i=1}^{p} d_i = 2(p - (j + 1))$ and $d_p \geq 1$. First note that we must have $d_{p-1} = d_p = 1$. For if not then the smallest $\sum_{i=1}^{p} d_i$ can be is $2 + \cdots + 2 + 1 = 2(p - 1) + 1$. But this latter is greater than $2(p - (j + 1))$ for any $j = 1,2,\cdots$ and this contradicts the assumption on $\sum_{i=1}^{p} d_i$.

Since the proposition is true for $p = 2, 3$ we can assume $p \geq 4$ so that $p - 2 \geq 2$. In addition we have $\sum_{i=1}^{p-2} d_i = 2(p - (j + 1)) - 2 = 2[(p - 2) - j]$ and $2(j + 1) \leq p$ implies $2j \leq p - 2$. Hence the sequence $S^* = (d_1, \cdots, d_{p-3}, d_{p-2})$ satisfies the induction assumption and so there is a forest F with j non-trivial components which belongs to S^*. Hence $F \cup K_2$ is a forest realizing S which has $j + 1$ non-trivial components. This proves the proposition.

Note again that the sequence S in the above proposition is not assumed to be graphical.

<u>Proposition 1.5</u>. Let $S = (d_1, \cdots, d_p)$ be a sequence with $d_p \geq 1$ and $p \geq 3$. Then a unicyclic graph belongs to S if and only if

$$(*) \quad \sum_{i=1}^{p} d_i = 2p$$

<u>Proof</u>. Since any unicyclic graph can be obtained from some tree by adding one edge it is clear that (*) is necessarily satisfied.

We prove the converse by induction. For $p = 3$ it is readily verified that the only sequence $S = (d_1, d_2, d_3)$ which satisfies (*) is (2, 2, 2) and this latter has K_3 as its only realization.

Assume the proposition true for all integers greater than 2 and less than k. Let $S = (d_1, \ldots, d_k)$ satisfy $\sum_{i=1}^{k} d_i = 2k$ and $d_k > 0$.

<u>Case 1</u>. If $d_1 = \cdots = d_k = 2$ the S is realized by C_k.

<u>Case 2</u>. If $d_i \neq 2$ for some $i \in \{1, \cdots, k\}$ then $\sum_{i=1}^{k} d_i = 2k$, $d_k > 0$ implies $d_k = 1$. The partition S_k then satisfies the induction assumption so that there is a unicyclic graph H which realizes S_k.

If G is obtained from H by adding a point of degree one adjacent to a point of degree $d_1 - 1$ of H then G is unicyclic and realizes S. This proves the theorem.

Again notice that the above proposition does not require that the sequence be graphical. Except for this latter point both propositions 1.3 and 1.5 are consequences of the following.

Proposition 1.6. Let $S = (d_1, \cdots, d_p)$ be a graphical sequence with $d_p > 0$ and $p \geq 2$. Then S has a connected realization if and only if

$$(*) \quad \sum_{i=1}^{p} d_i \geq 2(p-1)$$

Proof. Necessity is clear, since a connected realization of S has a spanning tree. The proof of sufficiency proceeds by induction on p. For $p = 2$ the only graphical sequence satisfying $d_1 + d_2 \geq 2$ is $(1, 1)$ which is realized by K_2.

Assume the proposition is true for p with $2 < p < k$. Let $S = (d_1, \cdots, d_k)$ be a graphical sequence with $d_k > 0$ satisfying (*). Let $S_k = (d_1', \cdots, d_{k-1}')$ be the sequence obtained from S by deleting d_k and subtracting one from $d_1, \cdots, d_{d_k}$ and rearranging the new terms in their natural order. By Proposition 1.2, S_k is a graphical sequence. There are two cases.

Case 1. $d_k = 1$. Then $d_1 > 1$ and

$$\sum_{i=1}^{k-1} d_i' \geq 2(p - 1) - 2 = 2(p - 2)$$

and $d_{k-1}' > 0$. Hence S_k satisfies (*) and has a connected realization H. The graph obtained from H by adding a point of degree one adjacent to a point of degree $d_1 - 1$ of H yields a connected realization of S.

Case 2. $d_k \geq 2$. Here it is clear $d'_j > 0$ for all $j \in \{1,\ldots, k-1\}$. Also

$$\sum_{i=1}^{k} d_i \geq kd \text{ implies } \sum_{i=1}^{k-1} d'_i \geq (k-2)d$$

But then $(k-2)\, d_k \geq 2(k-2)$ so that S again satisfies (*) and has a connected realization H. To get a connected realization of S we add a point of degree k to H; this gives the proposition.

We now consider 'arbitrarily traceable' graphs.

Definition. A graph G is *arbitrarily traceable from a point* $x \in V(G)$ if the following procedure always results in an eulerian trail: Start at x by traversing any line incident to x; on arriving at a point depart by traversing any incident line not yet used and continue until no new lines remain.

Following Ore[7, p. 74-77] we can state the following propositions.

(a) If G an arbitrarily traceable graph from a point x then G can be constructed from a forest by adding apoint--which will be the point x--and adjoining this point to the forest so that the resulting graph will have even degrees. The new point will be a point of maximal degree.

(b) If G is arbitrarily traceable from two distinct points x and y then G 'consists' of an even number of paths such that x and y are the terminal points of each path and any two of these paths have only the points x and y in common.

Proposition 1.7. A sequence $S = (d_1,\cdots, d_p)$ has an arbitrarily traceable realization if and only if

(i) For $i = 1,\cdots, p$, d_i is even, and

(ii) There is an integer k with $1 \leq 2k \leq d_1$ and

(a) $d_{p-2k+1} = \cdots = d_p = 2$

(b) $\sum_{i=2}^{p} d_i - d_1 = 2(p - 1 - k)$

Proof. Let S be a sequence satisfying (i) and (ii) above and form a new sequence S' from S by subtracting one from each of $d_{(p - d_1 + 1)}$, $\cdots$, d_p, deleting d_1 and arranging them in their natural order. If $S' = (d_1', \ldots, d_{p-1}')$ we have $d_{p-1}' > 0$ and

$$\sum_{i=1}^{p-1} d_i' = \sum_{i=2}^{p} d_i - d_1 = 2(p - 1 - k)$$

Hence by Proposition 1.4 there is a forest F which realizes S'. Adding a point x, to the forest, of degree d and having it adjacent to points of degree $d_{(p-d_1+1)} - 1, \cdots, d_p - 1$ in F yields a graph G which by (a) above is arbitrarily traceable from the point x.

For the converse let G be a graph which is arbitrarily traceable from a point $x \in V(G)$ and let G belong to the sequence $S = (d_1, \cdots, d_p)$. We may assume deg $x = d_1$ and that G - x is a forest. Let k be the number of component trees of G - x. Then since a tree has at least two points of degree one it follows that G - x has at least 2k points of degree one. Thus again by Proposition 1.4 it is clear that (ii) in the statement of proposition is satisfied. Because G is necessarily eulerian (i) is satisfied.

We now give a sequence of lemmas and propositions that leads to a proof of a result characterizing blocks by sequences. Most of the lemmas and propositions will also be used for other theorems which is why they are singled out.

First a few remarks are in order about a 'block-cut-point' graph, $BC(G)$, of a graph G. For $V[BC(G)]$ we take a set in 1 - 1 correspondence with the set of cut vertices and blocks of G. For x, $y \in V[BC(G)]$ we have $xy \in E[BC(G)]$ if x corresponds to a cut-point, y to a block (or vice-versa) and the block contains the cut point. It is known that (1) $BC(G)$ is a tree and that (2) if G has distinct cut-points then $BC(G)$ is not a star graph and hence $BC(G)$ has two pendant vertices which are of a distance of at least three apart. This means G has "end blocks" which are not adjacent i.e., blocks which contain only one cut-point and the cut-point contained by one block is not contained by the other. (See [2, p. 64-67]).

Lemma 1.8. Let G be a connected graph without pendant vertices. If G has more than one cut-point then there is a degree preserving transformation t such that $t(G)$ has fewer blocks than G.

Proof. By the above remarks, if G has distinct cut points x,y then it has two end blocks B_1, B_2 such that $x \in VB_1$ and $y \in VB_2$ and $V(B_1)$ $V(B_2)$ is empty. Then because G has no pendant vertices it follows that there exist edges $e_1 = x_1x_2 \in EB_1$ and $e_2 = y_1y_2 \in EB_2$ such that e_1 is incident to no point of $\{x\} \cup VB_2$ and e_2 is incident to no point of $\{y\} \in VB_1$. It now follows that the transfer t of x_1x_2 and y_1y_2 for x_1y_1 and x_2y_2 is defined. The effect of the transfer is to reduce the number of blocks by at least one (it is in fact exactly one) since the points $VB_1 \cup VB_2$ are contained in a block of tB and no other blocks of G are affected by the transfer. To see the latter it suffices to verify that any two points of $VB_1 \cup VB_2$ lie on a cycle in TG, since for any three points of B_1 or of B_2, there is a path beginning and ending at any two containing the third.

Lemma 1.9. Let G be a connected graph without pendant vertices. Let S be the degree sequence of G. Then there is a realization H of S such that H is connected and has at most one cut-point.

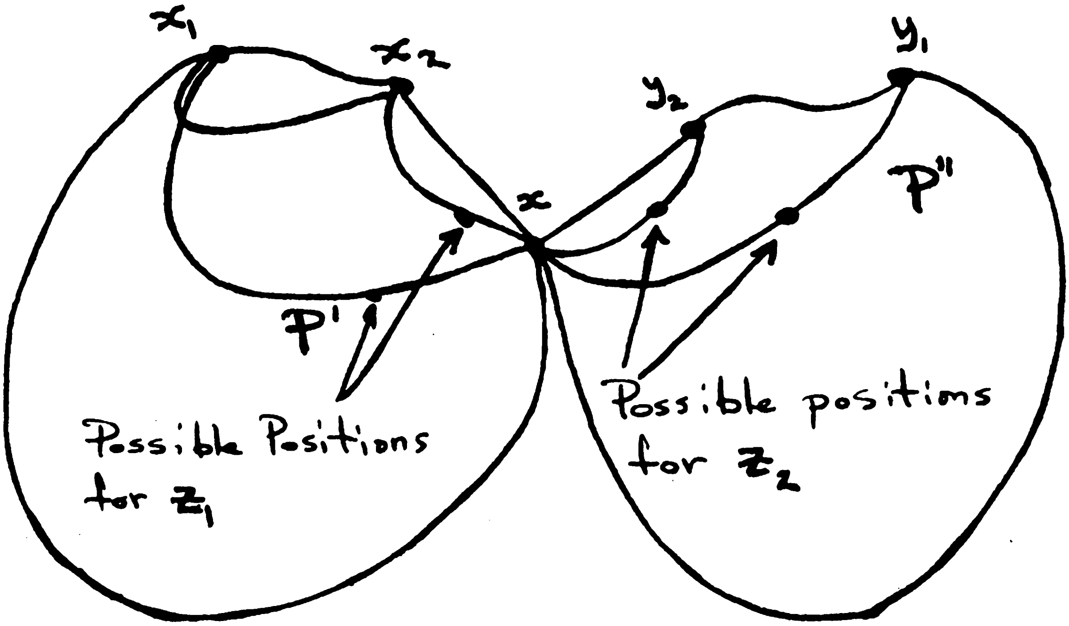
Proof. This follows from Lemma 1.8 since if G has more than one cut-point then there is a G_1 belonging to S which has at least one fewer block than G. Again if G_1 has distinct cut points we apply lemma again and so on. It is clear that at some stage we must obtain a graph H which is either a block or has exactly one cut point and which belongs to S.

Lemma 1.10. Let G be a connected graph with no pendant vertices, having exactly one cut point $x \in VG$. If G contains a cycle which does not contain x then there is a transfer t such that t(G) has fewer blocks than G does.

Proof. If C is a cycle of G which does not contain x then there is a block B of G such that $VC \subset VB$. Let B' be any other block of G and let $e = x_1x_2$ be an edge of C and let $f = y_1y_2$, $x \notin \{y_1, y_2\}$, be an edge of B'. It is clear that the transfer t of x_1x_2 and y_1y_2 for x_1y_1 and x_2y_2 is defined. The effect of the transfer is that $\langle VB \cup VB' \rangle$ is a block in tG. We verify this by showing that any two points of $VB \cup VB'$ lie on a cycle in tG. Let $z_1, z_2 \in VB \cup VB'$; there are three cases.

Case 1. $z_1, z_2 \in VB$. Since B is a block in G then z_1, z_2 lie on a cycle C_1 in G. If EC_1 does not contain $e = x_1x_2$ then z_1, z_2 lie on C_1 in tG. If EC_1 contains e let P denote the path from x_1 to x_2 along C_1. Assume for the moment that neither of z_1 or z_2 is x. Then since B'is a block there is a path P' in B' from y_1 to y_2 using x but not

The Graph G



The Graph tG

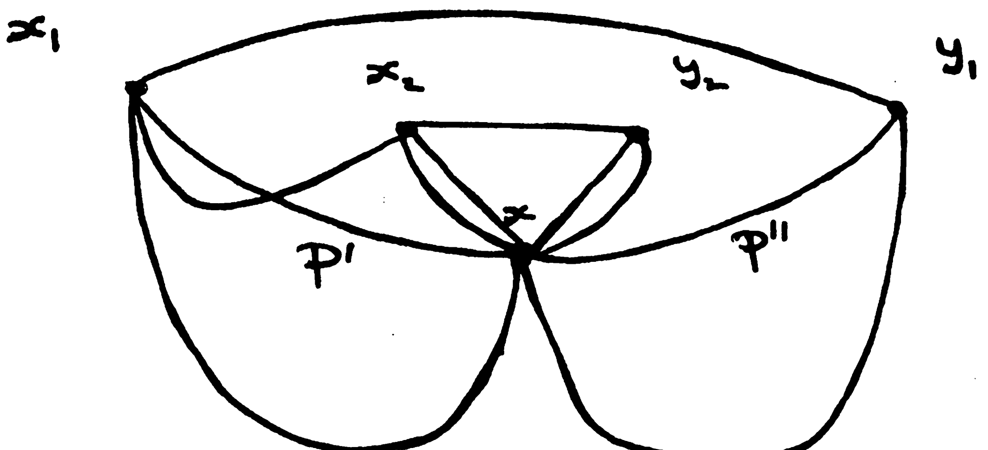


Figure 1

contained in a cycle so that by Lemma 1.10 the number of blocks can be reduced by a transfer t. If tH is a block we are done. If not we apply Lemma 1.10 again, and so on. At some stage we obtain a graph G having the required properties since the number of blocks of G is finite.

Lemma 1.12. If T is a tree belonging to the sequence $S = (d_1, \cdots, d_p)$ then the number of points of degree one of V(T) is greater than or equal to d_1.

Proof. Let $x \in VT$ be a point of degree d_1 and let $N = \{y_1, \cdots, y_{d_1}\}$ be the set of neighbors of x. For $i = 1, \cdots, d_1$ we define P_i to be a path in T containing the edge xy_i, no edge of the form xy_j, $j \in \{1, \cdots d_1\}$, $j \neq i$ and maximal with respect to the latter two properties. The following are true of the P_i: (a) for $i \neq j$, $i, j \in \{1, \cdots, d_1\}$, $VP_i \cap VP_j = \{x\}$ since otherwise T would have a cycle and (b) each P_i, $i \in \{1, \cdots, d_1\}$, 'ends' at a point of degree one, or better contains one point of degree one. For $i = 1, \cdots, d_1$ we associate the point y_i with the point of degree one in P_i. This gives the result.

Lemma 1.13. If $S = (d_1, \cdots, d_p)$ is graphical with

(i) $d_1 = d_2$

(ii) $d_p \geq 2$

then S is realized by a block.

Proof. By Proposition 1.11, since S satisfies (ii) above, S is realized by a block or by a connected graph G such that there is an $x \in VG$ common to each cycle of G. However the latter violates (i) above. To see this note that G - x is a forest containing trees $T_1, \cdots, T_r$. If r_1 is a point of T_1 of maximal degree then it is easy to see that, by Lemma 1.12, deg r_1 is less than or equal to one plus the number of points of degree one in T--the 'one' accounts for possibility that rx may be an edge of G. Since x is adjacent to all points of degree one of $T_1 \cup \cdots \cup T_r$ and $r \geq 2$ it follows that $\deg x > \deg y$ for all $y \neq x$, $y \in VG$. Since (i) is thus violated it follows that S is realized by a block.

Lemma 1.14. Let G be a graph with a single cut point x_1 such that every cycle of G contains x_1. Let G realize $S = (d_1,\cdots,d_p)$ with $V(G) = \{x_1,\cdots,x_p\}$ and and $\deg x_i = d_i \geq 2$ for $i = 1,\cdots,p$. Then there is H which realizes S that has $VH = VG$, a single cut point x_1, and blocks $B_1,\cdots,B_\ell$ such that $x_2 \in V(B_1),\cdots,x_{\ell+1} \in V(B_\ell)$.

Proof. Since $G - x_1$ is a forest it follows that $\deg x_1 = d_1$ since x_1 must be adjacent to all points of degree one of $G - x_1$, (see Proof of 1.13). The point x_2 belongs to a block, call it B_1. If x_3 belongs to a block B_2, and $B_2 \neq B_1$ we consider x_4. If $x_3 \in VB_1$ and there is a block B different from B_1 we make the following kind of transfer. In $G - x$ there is a unique path from x_2 to x_3. Let $e = y_1y_2$ be an edge on that path and let $f = z_1z_2$ be an edge of B not incident to x_1. The transfer t of y_1y_2 and z_1z_2 for y_1z_1 and y_2z_2 has the effect of disconnecting x_2 and x_3 in $G - x$. This means that x_2 and x_3 lie in different blocks of tG. Notice that the transfer does not affect any blocks other than B_1 or B of G i.e., $\langle V(G) - (VB_1 \cup VB)\rangle = \langle V(tG) - (VB_1 \cup VB)\rangle$ --the former generated in G, the latter in tG. Now consider x_4. If $x_4 \notin VB_1 \cup VB_2$ then label as B_3 the block in tG which contains it. If $x_4 \in VB_1 \cup VB_2$ and there are no more blocks we are done. If $x_4 \in VB_1$, say, and there is a block B, $B \neq B_1$, $B \neq B_2$ then we can make a transfer as before yielding three distinct blocks, each containing exactly one x_i, $i = 1,2,3$. It is clear that we can continue this procedure until all of the blocks of G have been used. This yields the lemma.

Remark. Note that in the above proof that the relative size of the degrees of $x_2,\cdots,x_{\ell+1}$ was immaterial i.e., the fact that $d_2\geq\cdots\geq d_{\ell+1}$ was not used and in fact one could have chosen the required ℓ points arbitrarily from $V(G) - \{x_1\}$.

Theorem 1.14. Let $S = (d_1,\cdots,d_p)$ be a graphical sequence such that

(i) $d_p \geq 2$

(ii) $\sum_{i=1}^{p} d_i - 2d_1 \geq 2(p-2)$

where $p \geq 3$. Then S is realized by a block.

Proof. The proof proceeds by induction on p. For $p = 3$ the only sequence that satisfies (i) and (ii) is (2,2,2) which is realized by the block $K_3 = C_3$.

Assume that the proposition is true for $p = k \geq 3$ and let $S = (d_1,\cdots,d_{k+1})$ be a sequence which satisfies (i) and (ii). By Lemma 1.13 and (i) it follows that if $d_1 = d_2$, S is realized by a block. Thus we may assume $d_1 > d_2 \geq 2$. In fact $d_2 > 2$ for if $d_2 =\cdots= d_{k+1} = 2$

$$\begin{aligned}\sum_{i=1}^{k+1} d_i - 2d_1 &= 2k - 2d_1\\ &= 2(k-d_1)\\ &< 2(k-1) \text{ since } d_1 > 1\end{aligned}$$

and so (ii) is not satisfied. Thus $d_1 > 2$ implies $d_2 > 2$.

Assume first that $d_{k+1} = 2$. Form the sequence $S_{k+1} = (d_1',\cdots,d_k')$ by deleting d_{k+1} from S, subtracting one from d_1 and d_2 and arranging $d_1 - 1$, $d_2 - 1$, $d_3,\cdots,d_k$ in their natural order. We claim that S_{k+1}

satisfies (i) and (ii). Since $d_1 - 1 \geq 2$ and $d_2 - 1 \geq 2$, it follows that $d_i' \geq 2$ for all $i \in \{1,\cdots,k\}$. Also $d_1 > d_2$ implies $d_1' = d_1 - 1$ so that

$$\sum_{i=1}^{k} d_i' - 2d_1' = \sum_{i=1}^{k} d_i - 2 - 2(d_1 - 1)$$

or

$$(*) \quad \sum_{i=1}^{k} d_i' - 2d_1' = \sum_{i=1}^{k} d_i - 2d_1$$

But by assumption $d_{k+1} = 2$ and

$$(\ast\ast) \quad \sum_{i=1}^{k+1} d_i - 2d_1 \geq 2(k-1)$$

It now follows from (*) and subtracting two from both sides of $(\ast\ast)$ that

$$\sum_{i=1}^{k} d_i' - 2d_i' \geq 2(k-2)$$

Or, S_{k+1} satisfies the induction assumption. Hence there is a block H realizing S_{k+1} and to obtain a block realization of S simply add a point of degree two to VH in such a way that it is adjacent to points of degree $d_1 - 1$ and $d_2 - 1$ in VH.

Now assume that $d_{k+1} \geq 3$. Form the sequence $S_1 = (d_1',\cdots,d_k')$ by putting the numbers $d_2 - 1,\cdots, d_{d_1+1}-1, d_{d_1+2},\cdots,d_{k+1}$ in their natural order. S_1 is graphical and $d_i' \geq 2$ for all $i \in \{1,\cdots,k\}$. By Lemma 1.11 and Lemma 1.14 S_1 is either realized by a block or by a graph with a single cut point x and blocks $B_1,\cdots,B_\ell$ where each block B_j contains a point y_j of degree $d_{2+j}-1$ for $j = 1,\cdots,\ell$. (Notice that the required ℓ points exist since deg x must be larger than the number of blocks and $d_1 > \deg(x) > \ell$.) In either case we adjoin a point to the resulting graph making it adjacent to points of degree $d_2 - 1,\cdots,d_{d_1+1}-1$ and making sure in the latter case that the new point is adjacent to $y_1,\cdots,y_\ell$. It is easily verified that the graph obtained is a block realizing S.

Corollary 1.15. Let $S = (d_1,\cdots,d_p)$ be a graphical sequence with $p \geq 3$. Then S is realized by a block if and only if

(i) $d_p \geq 2$ and

(ii) $\sum_{i=1}^{p} d_i - 2d_1 \geq 2(p-2)$

Proof. Theorem 1.14 gives sufficiency. Necessity follows from Proposition 1.6 and the fact that if any point of a block is deleted, (and hence any point of maximal degree), the graph is still connected.

Remark. Hakimi [4] has proven the results given in proposition 1.6 and Theorem 1.14 in the case of multi-graphs, where parallel edges and loops are permitted. His proof for 1.6 can be carried over easily to ordinary graphs while his proof of 1.14 depends on the following fact for multi-graphs: If S is a graphical sequence, $S = (d_1,\cdots,d_p)$, $d_p \geq 2$, then so is $(d_1 - 2,\cdots,d_p - 2)$ and so S has a realization with a hamiltonian circuit. To see that this latter does not hold for *ordinary* graphs consider the following.

Definition. A *theta-r* graph, denoted by θ_r, is obtained from a $K_{1,r}$ by adding a new point adjacent to each point of $K_{1,r}$. Thus θ_r is a realization of $S = (r+1, r+1,2,\cdots,2)$ where the sequence S has r points of degree two.

To see why Hakimi's proof does not carry over to ordinary graphs it suffices to notice that, with $r > 2$, for the sequence $S = (d_1,\cdots,d_p)$, realized by a theta-r graph, we have $(d_1 - 2, \cdots, d_p - 2) = (r - 1, r - 1, 0, \cdots, 0)$ which is not (ordinary) graphical. Further note that, for $r > 0$, θ_r is a block.

C. Extending Graphical Sequences

In this section we give a new method for obtaining new graphical sequences from a given graphical sequence. To do this we need another definition.

Definition. Let G be a graph. A subset M of EG is said to be a *matching* if no two edges of M are adjacent.

For any graph G, with $S \subset VG$ let $C_0(S)$ denote the number of components of odd order of $\langle VG - S \rangle$. The following is a generalization by Berge of a result by Tutte (See [1, pp. 181-182]).

Theorem [T-B]. Let G be a connected graph. If

$$\xi = \max_{S \; VG} [C_0(S) - |S|]$$

then

$$\frac{1}{2}(p - \xi)$$

is the number of edges in a maximum matching, where $p = |VG|$.

Theorem 1.15. Let $S = (d_1, \cdots, d_p)$ be a graphical sequence. The sequence $S' = (d_1, \cdots, d_p, d_{p+1})$ is graphical if and only if d_{p+1} is an even integer, non-negative, less than or equal to d_p.

Proof. If S is graphical and S' is graphical then $\sum_{i=1}^{p} d_i$ and $\sum_{i=1}^{p+1} d_i$ are even so that d_{p+1} is even.

Conversely assume that $d_{p+1} = 2k$ where

$$0 \leq 2k \leq d_p.$$

Let G be a realization of S. Suppose that in G one can find k non-adjacent edges (a matching): $x_1y_1, \cdots, x_ky_k$. We can add a new point z to G, of degree 2k, while maintaining the degrees of the vertices of G in the following manner. Delete the edges $x_1y_1, \cdots, x_ky_k$ and add the edges $x_1z, y_1z, \cdots, x_kz, y_kz$. The graph so obtained is a realization G' of S' so that S' is graphical. The requirement that the k edges by non-adjacent prevents the occurence of multiple edges in the above construction.

The Tutte-Berge result is now used to guarantee that one can always find the k non-adjacent edges.

If d_p is zero or one then we have $d_{p+1} = 0$ and the required G* is obtained by adding an isolated point to G. If $d_p \geq 2$ then we may assume that the G realizing S is connected by Proposition 1.6 since

$$\sum_{i=1}^{p} d_i \geq 2p > 2(p-1)$$

Thus the theorem of Tutte and Berge applies and the number of edges in a maximum matching of G is $\frac{1}{2}(p - \xi)$, where $p = |VG|$ and $\xi = \max_{S \in VG} [C_0(S) - |S|]$. To get the result we need $\frac{1}{2}(p - \xi) \geq k$ where $0 \leq 2k \leq d_p$.

Now given $S \subset V(G)$ the largest $C_0(S) - |S|$ can be is $(p - |S|) - |S| = p - 2|S|$. Hence for some $S^* \subset VG$

$$\frac{1}{2}(p - \xi) = \frac{1}{2}(P - [C_0(S^*) - |S^*|])$$

$$\geq \frac{1}{2}(p - (p - 2|S^*|)$$

$$= |S^*|$$

Hence if $|S^*| \geq k$ we have $\frac{1}{2}(p - \xi) \geq k$ and thus the result follows.

Assume now that $|S^*| < k$. Consider the size of the components of $\langle VG - S^* \rangle$. If a point belongs to a component then it has degree at least d_p in G so that in $\langle VG - S^* \rangle$ it has degree at least $d_p - |S^*|$. Since $k > |S^*|$, $d_p - k < d_p - |S^*|$. Hence a component must have at least $d_p - k + 2$ points. Hence

$$|S^*| + (d_p - k+2)C_0(S^*) \leq p \tag{1}$$

If $\frac{1}{2}(p - \xi) < k$, and thus the theorem were false, we would have

$$\frac{1}{2}(p - (C_0(S^*)) - |S^*|) < k \tag{2}$$

and multiplying (2) by 2 and adding the result to (1) we have

$$(d_p - k+1)C_0(S^*) + 2|S^*| < 2k.$$

But $d_p \geq 2k$ implies

$$(k+1)C_0(S^*) + 2|S^*| < 2k. \tag{3}$$

Thus if $C_0(S^*) \geq 2$ (3) is violated so that (2) leads to a contradiction. On the other hand if $C_0(S^*) \leq 1$ it follows that $\xi = C_0(S^*) - |S^*| \leq 1$. Hence in this case

$$\frac{1}{2}(p - \xi) \geq \frac{1}{2}(p - 1) \geq \frac{1}{2}d_p \geq \frac{1}{2}k$$

This latter gives desired result.

The ways in which the above result can be generalized are not clear. For example, given a graphical sequence $S = (d_1,\cdots,d_p)$, one might attempt to insert an even number, where possible, between d_i and d_{i+1} for some $i \in \{1,\cdots,p\}$. Because of the following it is clear that there must be some restrictions. Consider the graph θ_7 which realizes (8,8,2,2,2,2,2,2,2). Inserting a six yields (8,8,6,2,2,2,2,2,2,2). Now the latter by Hakimi's result is graphical if and only if (7,5,2,1,1,1,1,1 1) is graphical if and only if (4,1,1,0,0,0,0,0) is graphical. Now the latter is clearly now graphical. We can however say the following.

<u>Corollary 1.16</u>. Let $S = (d_1,\cdots,d_p)$ be graphical. Let d be any even positive integer such that $\frac{1}{2}(p - \xi) \geq d$ where $\xi = \max\limits_{S\,VG} \{C_0(S) - |S|\}$ for some G realizing S. Then the sequence S* obtained by arranging the numbers $\{d_1,\cdots,d_p,d\}$ in their natural order is graphical.

<u>Proof</u>. The result follows directly from the method used in above proof.

II. SIMPLE GRAPHS

A graphical sequence may have, up to isomorphism, only one realization. If a graph realizes such a sequence then it is determined up to isomorphism by the degrees of its vertices. Hence the importance of 'simple' graphs and sequences.

Definition. Let G be a graph and S a graphical sequence realized by G. If given any graph H belonging to S we have $G \cong H$ then G is called a *simple graph* and S a *simple sequence*.

A. Elementary Results on Simple Graphs

Proposition 2.1. A graph G is simple if and only if given any transfer t of G we have

$$tG \cong G$$

Proof. We use Theorem 1.1. Suppose that $G \cong tG$ for all transfers t of G. If S is the degree sequence of G then by 1.1 any realization of S has the form $t_r \circ \cdots \circ t_1 G$ where $t_1, \cdots, t_r$ are transfers. Hence it is immediate that $t_r \circ \cdots \circ t_1 G \cong G$ and thus any realization of S is isomorphic to G. Conversely given that G is simple it is obvious that $G \cong tG$ for all transfers t since tG belongs to the same sequence that G does.

Proposition 2.2. Let G be a graph. G is simple if and only if the complement of G, G^c, is simple. If $S = (d_1, \cdots, d_p)$ is a graphical sequence then S is simple if and only if $(p-1 - d_p, \cdots, p-1-d_1)$ is simple.

Proof. First note that if f is an isomorphism of graphs G and H then f is also an isomorphism of G^c and H^c. Let G be simple and let H' belong to the same sequence as G^c. Then $(H')^c$ belongs to the same sequence as G does so that $G \cong (H')^c$. Hence by the above $G^c \cong H'$ so that G^c simple. A similar argument shows that G^c simple implies that G is. The proof is completed by noting that G belongs to $(d_1,\cdots,d_p)$ if and only if G^c belongs to $(p-1-d_p,\cdots,p-1-d_1)$.

<u>Proposition 2.3</u>. $S = (d_1,\cdots,d_p)$ is a graphical sequence if and only if $S' = (p-1-d_p,\cdots,p-1-d_1)$ is and $|S| = |S'|$.

<u>Proof</u>. The proof is similar to the above proposition.

<u>Proposition 2.4</u>. The sequences $(d_1,\cdots,d_p)$, $(p,d_1+1,\cdots,d_p+1)$ and $(d_1,\cdots,d_p,0)$ have the same number of realizations.

Proof. It is clear that $|(d_1,\cdots,d_p)| = |(d_1,\cdots,d_p,0)|$.

Applying 2.3 twice we have

$$\begin{aligned} |(d_1,\cdots,d_p)| &= |(p-1-d_1,\cdots,p-1-d_p)| \\ &= |(p-1-d_1,\cdots,p-1-d_p,0)| \\ &= |(p,d_1+1,\cdots,d_p+1)| \end{aligned}$$

It should be clear that any realization of $(p,d_1+1,\cdots,d_p+1)$, if there are any, can be obtained from a realization of $(d_1,\cdots,d_p)$ by adding a new point adjacent to all other points of the latter realization.

<u>Corollary 2.5</u>. The sequences $(d_1,\cdots,d_p)$, $(p,d_1+1,\cdots,d_p+1)$, $(d_1,\cdots,d_p,0)$ are either all simple or all not simple.

<u>Remark</u>. We now give some examples of simple graphs and sequences. First note that K_p and the star graph $K_{1,p}$ are simple because no elemen-

tary transfers can be made (defined). Since a θ_p graph is obtained from a $K_{1,p}$ by adding a point adjacent to all others it follows from 2.4 that θ_p is simple. A cycle C_p is simple if and only if $p \leq 5$ since if $p \geq 6$ a single transfer can be made on C_p to create a disconnected graph. One can verify that all graphs on four or less points are simple. The next proposition guarantees the existence of many examples of simple graphs.

Proposition 2.6. For any positive integer p and any integer q such that $0 \leq q \leq p\,(p-1)/2$ there is a simple (p,q) graph.

Proof. The proof is by induction on p. Since the trivial graph is simple the result is true for $p = 1$. Assume the proposition is true for $p \leq k$ and let q be an integer such that

$$0 \leq q \leq (k + 1)\ k/2.$$

If $q \leq k(k - 1)/2$ then the result follows from the induction hypothesis since there is a simple (k,q) graph from which a simple $(k + 1,q)$ graph is obtained by adding a point of degree zero (Corollary 2.5).

On the other hand if q satisfies

$$k(k - 1)/2 \leq q \leq (k + 1)\ k/2$$

then $q = \ell + k(k - 1)/2$ where $0 \leq \ell \leq k$. To get the desired simple graph we adjoin a new point to any ℓ points of K_k. The graph so obtained is simple because its complement is the union of $K_{1,k-\ell}$ and trivial graphs.

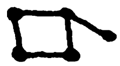
The next result goes in the opposite direction.

Proposition 2.7. Let p,q be positive integers such that $p \geq 5$

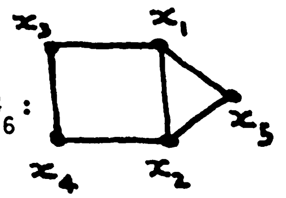
and $4 \leq q \leq p(p - 1)/2 - 4$. There is a (p,q) graph which is not simple.

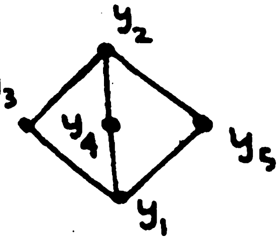
Proof. For $p = 5$ the possible values for q are four, five and six. Each pair of non-isomorphic graphs given below realizes the given sequence.

(a) (2, 2, 2, 1, 1) $G_4 = K_3 \;\; K_2$, $H_4 + P_5$

(b) (3, 2, 2, 2, 1) $G_5 =$  , $H_5 =$ 

(c) (3, 3, 2, 2, 2)

G_6: 

H_6: 

Note that each of G_4, G_5, G_6 have a triangle while none of H_4, H_5, H_6 does.

For $p > 5$ and j such that $4 \leq j \leq p(p - 1)/2 - 4$ we construct (p,j) graphs G_j and H_j as follows. For $j = 4, 5, 6$ we union the above G_j and H_j with trivial graphs to obtain the required results. For $j > 6$ let $VG_j = \{x_1,\cdots,x_p\}$, $VH_j = \{y_1,\cdots,y_p\}$, let $x_1, x_2,\cdots,x_5$ be related exactly as in G_6 above. Then adjoin x_6 to $x_1,\cdots,x_5$ in turn and x_7 to $x_1,\cdots,x_6$ in turn and so on until we have j edges. The graph so obtained is G_j. Let H_j be constructed from H_6 (above) in the same fashion. It is easily verified that (i) G_j and H_j belong to the same sequence, and (ii) at each stage of the construction there is a difference of at least one triangle between the two graphs. This yields the proposition.

The next sequence of lemmas leads to a theorem giving necessary and sufficient conditions for a regular graph to be simple.

Lemma 2.8. Let p be a positive integer, $p \geq 1$, $p = 2m + 1$.

(a) If m is odd the sequence of length p, $(m + 1, \cdots, m + 1)$ is not simple.

(b) If m even the sequence of length p, $(m, \cdots, m)$ is not simple.

Proof. Suppose m is odd. On $X = \{x_1, \cdots, x_{m+1}\}$ and $Y = \{y_1, \cdots, y_m\}$ construct a complete bipartite graph--let the edge set be $\{\{x_i y_j\} | 1 \leq i \leq m + 1, 1 \leq j \leq m\}$. Next add the edges $x_1 x_2$, $x_3 x_4$, $\cdots$, $x_m x_{m+1}$--which is possible since m is odd--and call the resulting graph G. Let t be the transfer of $x_1 y_1$ and $x_{m+1} y_m$ for $x_1 x_{m+1}$ and $y_1 y_m$. We compare the number of triangles in G and tG. Since a bipartite graph has no triangles the addition of the edges $x_i x_{i+1}$, $1 \leq i \leq m$ introduces the only triangles into G. Clearly each such $x_i x_{i+1}$ lies on precisely m triangles since we started with a complete bipartite graph. Hence G has a total of $(m + 1)m/2$ triangles. By similar reasoning tG has $\left\{\frac{1}{2}(m + 1)m - 2\right\} + (m - 2) + (m - 1)$ triangles. This proves (a).

Assume now m is even. As above construct the complete bipartite graph on the point sets $\{x_1, \cdots, x_m\}$ and $\{y_1, \cdots, y_m\}$. Then (i) delete the edges $x_1 y_1, \cdots, x_m y_m$, (ii) add the edges $x_1 x_2, \cdots, x_{m-1} x_m$ (possible since m even) and (iii) add a point y adjacent to $y_1, \cdots, y_m$. Call the resulting graph G and note that G belongs to the sequence $(m, \cdots, m)$ of length $2m + 1$. Let t be the transfer of $x_2 y_1$ and $x_{m-1} y_m$ for $x_2 x_{m-1}$ and $y_1 y_m$. Counting triangles we have G with $m(m - 2)/2$ and $t(G)$ with $m(m - 2)/2 + 2(m - 4) + 1$. This proves (b).

Lemma 2.9. For $p = 2m \geq 6$ the sequence $(m, \cdots, m)$ of length p is not simple.

Proof. $K_{m,m}$ has no triangles but clearly an elementary transfer of it does.

Remark. Let $S = (d_1, \cdots, d_p)$ be a sequence of non-negative integers such that $d_1 + \cdots + d_p$ is even. Then S is graphical if and only if for each positive integer r, $1 \leq r \leq p-1$

$$(*)\ d_1 + \cdots + d_r < r(r-1) + \min\{r, d_{r+1}\} + \cdots + \min\{r, d_p\}.$$

This result was proven by Erdos and Gallai and can be found in Harary ([6, p. 59-62]).

Lemma 2.10. For p,m, positive integers, $0 \leq m \leq p - 1$ the sequence $S = (m, \cdots, m)$ of length p is graphical provided m is even when p is odd. At least one realization of S is connected when $m \geq 2$.

Proof. If r is less than m then (*) above becomes, for S,

$$m \cdot r \leq r(r - 1) + m(p - r).$$

If r greater than or equal to m then (*) becomes

$$r \cdot m \leq r(r - 1) + r(p - r) = r(p - 1).$$

Hence in either case (*) is satisfied so that S must be graphical. Finally if $m \geq 2$ it follows immediately from proposition 1.6 that S has at least one connected realization.

Proposition 2.11. Let p,r be positive integers such that $p \geq 6$ and $2 \leq r \leq [p/2] - 1$. Then the sequence $S = (m, \cdots, m)$ of length p has both a connected and disconnected realization, provided m is even when p is odd.

Proof. It suffices to exhibit the disconnected realization. If

$p = 6$ the only permissible value for r is 2 and $K_3 \cup K_3$ is the required graph. Now assume $p \geq 7$. There are two cases. (a) $p = 2m$. Here $[p/2] - 1 = m - 1$. If $r = m - 1$, $K_m \cup K_m$ suffices. For $r, 2 \leq r \leq m - 2$ we have two more cases. If m is even the union of two regular graphs of degree r will do. If m is odd then the union of two regular graphs of degree r on $m + 1$ and $m - 1$ points will do.

(b) $p = 2m + 1 = m + (m + 1)$. Here $[p/2] - 1 = m - 1$. Since p odd r can only assume even values and thus regardless of the parity of m one can construct the desired graph by taking the union of regular graphs of degree r on $m+1$ and m points.

Theorem 2.12. If G is a regular graph of degree r on p points then G is simple if and only if $r \in \{0, 1, p-2, p-1\}$.

Proof. Since G is simple if and only if G^c is the result follows from 2.8, 2.9, 2.10 and 2.11.

B. Simple Graphs Which Are Not Blocks

In this section the simple graphs which are not blocks are characterized. To this end we define the 'Giap' graphs.

Definition. Let p be a positive integer, $p \geq 2$, and let $S = (m, n, 1, \cdots, 1)$ where $p = m+n$. Then any realization of S is a *Giap* graph or an $[m,n]G$, or an $[m,n]$-Giap graph.

Lemma 2.13. Giap graphs are simple.

Proof. Let p be a positive integer, $p \geq 2$ and let $S = (m,n,1,\cdots,1)$, $p = m + n$. One realization G of S can be defined as follows. Let $V(G) = \{x_1, \cdots, x_p\}$, where $\deg x_1 = m$, $\deg x_2 = n, x_1x_2 \in E(G)$, x_1 is adjacent to

$m - 1$ points of degree one and x_2 is adjacent to $n - 1$ points of degree one. G can be constructed by connecting a $K_{1,m-1}$ with a $K_{1,n-1}$ by an edge at their points of maximal degree. The only transfer that is *defined* on G is of the following type. Let $y_1, y_2 \in V(G)$, $y_1 \neq x_2$, $y_2 \neq x_1$ with $y_1x_1 \in EG$, $y_2x_2 \in EG$. Then the transfer t of y_1x_1 and y_1x_2 for y_2x_1 and y_2x_1 yields a graph isomorphic to G, and is the only possible kind of transfer since $x_1x_2 \in EG$. (Note: the transfer t amounts to interchanging points of degree one adjacent to x_1 and x_2.)

Theorem 2.14. A tree is simple if and only if it is a Giap graph.

Proof. Since lemma 2.13 says Giap graphs are simple and the proof of 2.13 shows that Giap graphs are trees, sufficiency is clear.

Now let T be a simple tree. We first show that T has no part of length four and hence has no path of length greater than three. Suppose the contrary i.e., there exist $y_0, y_1, \cdots, y_4 \in VT$ for which $y_0y_1y_2y_3y_4$ is a path in T. Since T is a tree neither y_0y_4 nor y_1y_3 are in ET so that the transfer t of y_0y_1 and y_3y_4 for y_0y_3 and y_1y_3 is defined and $tT \neq T$ since tT has a triangle. Hence T simple implies that $dT \leq 3$, the diameter of T is less than or equal to 3.

Now let x VT with deg $x > 1$. If each point of N_x has degree one then T is a star graph, i.e., [p,1] Giap graph. If N_x has two points x', x" of degree greater than one then these exist a', a" $\in$ VT with a'x', a"x" ET, a' $\neq$ x, a" $\neq$ x. Since T has no cycles a' $\neq$ a" and thus a'x'xx"a" is a path of length four, contradicting the above. Hence N_x has at most one point, say x', of degree greater than one. Reasoning as above it follows that $N_{x'}$ has only one point of degree greater than one which must be x. Since T is connected, VT consists of x, x' and points of degree one adjacent to either x or x'. Hence T is a Giap graph.

Observe that the above argument also shows that if T is a tree with $dT \leq 3$ then T is a Giap graph. That is, the simplicity of T was used to derive $dT \leq 3$ and then from the *latter* the structure of T was derived. This yields the following corollary.

Corollary 2.15. Let T be a tree. T is simple if and only if $dT \leq 3$.

Proposition 2.16. Let G be a disconnected graph with non-trivial components. G is simple if and only if

$$(*)\ G = sK_2 \cup t[m,n]G$$

where s is a non-negative integer, $t \in \{0,1\}$ and $s \neq 0$ implies either $m = 1$ or $n = 1$.

Proof. It is easy to verify that a graph of the form (*) is simple. Conversely assume that G is simple and disconnected. First it is shown that no component of G (of which there are at least two) contains a cycle. Let C be a component and suppose to the contrary that C contains a cycle and $x,y \in VC$ with xy on the cycle. Let $x',y' \in VC'$, where C' is another component of G, with $x'y' \in EC'$. The transfer t of xy and x'y' for xx' and yy' is defined and $G \neq tG$ since tG has one fewer component than G. This latter contradicts the fact that G is simple. It now follows that G is a forest and since each component of G must be simple the component of G must be Giap graphs or trivial graphs.

Now suppose that $[n,m]G$ and $[k,\ell]G$ are components of G. We show this is impossible unless at least three of the four numbers n,m,k,ℓ, are one. If $n \geq 2$, $m \geq 2$ and $k = \ell = 1$, then a transfer can be yielding a $K_{1,n}$ and $K_{1,m}$. If $n \geq 2$, $k \geq 2$ then a transfer exists which creates an additional K_2. Hence in either case a transfer exists which creates different components so that the graph cannot be simple. This gives the result.

Remark. Since G is simple if and only if G^c simple 2.16 yields at once a criterion for a simple graph to have a disconnected complement.

The above characterizes disconnected simple graphs. We now turn to connected simple graphs with cut points. For the present we consider graphs without vertices of degree one (without pendant vertices).

Proposition 2.17. Let G be a connected graph without pendant vertices. If G has two or more cut points then G cannot be simple.

Proof. By lemma 1.9 we conclude that such a graph G belongs to a sequence S such that S contains a graph with one cut point. Thus G cannot be simple.

The above proposition shows that to characterize connected simple graphs which are not blocks we need consider only those connected graphs with one cut point.

Definition. Let T be a tree and p points. A *1-cone* of T is a graph obtained from T by adding a point to T which is adjacent to at least the points of degree one of T, and perhaps other points of T.

Lemma 2.18. Let T be a tree on two or more points. If G is a 1-cone of T then G is a block.

Proof. Let $x,y \in VT$. It is clear that x and y belong to a path in T where end points are points of degree one. In G these latter points are adjacent to the same point and hence x and y lie on a circuit. If $z \in VG - VT$ then one can show in a similar fashion that for any $x \in VT$, z and x lie on a circuit also, yielding the result.

Lemma 2.19. Let T be a tree which is not simple. Then any 1-cone of T is not simple.

Proof. If T is not simple it follows by corollary 2.15 that there is a path P of length greater than or equal to four. As above one may assume that the end points x,y have degree one. If the path is $xz_1 \cdots z_r y$ then the transfer t of xz_1 and $z_r y$ for $z_1 z_r$ and xy is defined--since $z_1 \neq z_r$ and $z_1 z_r \notin ET$. Hence tT has a component K_2. If G is any 1-cone of T the same transfer t is defined on G and $G \neq tG$ since tG has a K_3 as a block and by Lemma 2.18 G is itself a block.

Lemma 2.20. Let T be a Giap graph with $T \neq P_3$, $T \neq P_4$. Then a 1-cone G of T is simple if and only if the point of VG - VT is adjacent to each point of VT.

Proof. Let T be a Giap graph and $z \in VG - VT$ where G is a 1-cone of T. If z is adjacent to every point of VT it follows from proposition 2.4 that G is simple.

Now suppose G is simple. If $T = K_2 = P_2$ then $G = K_3$ and z is adjacent to each point of K_2. Thus assume T is an [n,m]G different from P_2, P_3, P_4. Then at least one of n and m, say n is greater than two. By lemma 1.12 the new point z has degree greater than or equal to max {n,m}.

Let n = max{n,m} and suppose z is not adjacent to either x_1 or x_2 in T where deg x_1 = n, deg x_2 = m and $x_1 x_2 \in E$. First assume m = 1. Then $zx_1 \notin EG$ but $zx_2 \in EG$. Now $n \geq 2$ implies there is an x such that $x_1 x_3$, $zx_3 \in EG$ and $x_1 x_2 \in EG$. The transfer t of $x_2 x_1$ and $x_3 z$ for $x_2 x_3$ and $x_1 z$ changes the adjacency relations of G i.e., tG has more points of highest degree adjacent. Hence $tG \neq G$.

Next assume deg x_2 = m > 1, and let $x_4 \in VT$ with $x_2 x_4 \in ET$. Since deg $x_4 = 1$ we have $zx_4 \in EG$ and $x_1x_4 \notin E(G)$. If z is not adjacent to x_1 then the transfer t of zx_3 and x_1x_2 for zx_1 and x_2x_3 again changes adjancy relations. Finally suppose that $zx_1 \in EG$ but $zx_2 \notin EG$. If deg x_2 is greater than two we can transfer as per above to get z adjacent to x_2, leaving z adjacent to x_1. If deg $x_2 = 2$ then the transfer of zx_1 and x_2x_4 for x_1x_4 and zx_2 yields a graph in which the points of highest degree are not adjacent. Thus in any case G is not simple. This proves the lemma.

Lemma 2.21. If $T = P_3$ or $T = P_4$ then any 1-cone of T is simple.

Proof. The proof follows from looking at the four possible cases.

Definition. Let G_1, G_2 be graphs. Let $VG_1 \cap VG_2 = \phi$ and x_1, y_1 be points of maximal degree of VG_1, VG_2 respectively. By $G_1{*}G_2$ we mean the graph obtained by identifying x_1 and y_1 assuming that the construction is independent of the points of maximal degree chosen.

Lemma 2.22. Let S_1 be the set of points of degree one in a tree T_1. Let G_1 be a cone of T_1. Let G_2 be a cone of a tree T_2. If $|VT_1 - S_1| \geq 2$ then $G_1{*}G_2$ is defined and not simple.

Proof. If G_1 is not simple then neither is $G_1{*}G_2$. Hence assume G_1 is simple.

Case 1. $T_2 = K_2$. Here one block of $G_1{*}G_2$ is a triangle and T_1 is, by assumption and lemma 2.19, an $[n,m]G$ with $n \geq 2$, $m \geq 2$. Let xy be the edge of $[n,m]G$ with deg $x = n$, deg $y = m$ and let x_1y_1 be the edge of K_2. Then the transfer of xy and x_1y_1 for xx_1 and yy_1 yields a graph without a triangle as a block.

Case 2. $T_2 \neq K_2$. Again by assumption T_1 cannot be a K_2 so that $G_1{*}G_2$ does not have a triangular block. Let $y_1 \in VT$, $z_1 \in VT_2$ with

$\deg y_1 = \deg z_1 = 1$ and let $y \in VT_1$, $z \in VT_2$ such that $yy_1 \in ET$, $zz_1 \in ET_2$. Then the transfer of y_1y and zz_1 for yz and y_1z_1 creates a block which is a triangle. Hence in either case $G_1 * G_2$ is not simple.

It is easy to verify that $G_1 * G_2$ is defined whenever G_1 and G_2 are 1-cones.

<u>Lemma 2.23</u>. If $T_1, \cdots, T_\ell$ are trees having $G_1, \cdots, G_\ell$ as 1-cones and $G_1 * \cdots * G_\ell$ is simple then (re-ordering if necessary) we have either

(a) $G_1 = C_4$ and $T_2 = \cdots = T_\ell = K_2$

or

(b) $G_1 = \theta_r$ and $T_2 = \cdots = T_\ell = K_2$

<u>Proof</u>. First note that $G_1 * \cdots * G_\ell$ is defined. By lemma 2.22 each T_i, $i = 1, \cdots, \ell$ is a star graph. Thus by lemmas 2.20 and 2.21 each $G_i = \theta_r$ or $G_i = C_4$ for $i = 1, \cdots, \ell$. Now it is easy to verify that for $r, s \geq 2$ the graphs $C_4 * \theta_r$, $\theta_r * \theta_s$ and $C_4 * C_4$ are not simple. The only remaining possibilities are given in (a) and (b).

<u>Lemma 2.24</u>. Let G be a graph with a single cut point x having no pendant vertices. If some cycle of G does not contain x then G is not simple.

<u>Proof</u>. By lemma 1.10 there is a transfer of G which reduces the number of blocks of G so that G cannot be simple.

<u>Theorem 2.25</u>. Let G be a graph without pendant vertices and having a single cut point. Then G is simple if and only if G has one of the two following forms.

(a) $C_4 * C_3 * \cdots * C_3$

(b) $\theta_r * C_3 * \cdots * C_3$

<u>Proof</u>. To see that the graphs in (a) and (b) are simple it suffices

to note that removing a point of degree $|VG| - 1$ in (a) or $|VG| - 2$ in (b) results in a graph of the form $K_{1,n} \cup mK_2$ which by proposition 2.16 is simple, and so the original graphs, by corollary 2.5, are simple.

For the converse note that by lemma 2.24 the single cut point x lies on each cycle of G so that G - x is a forest. Since there are no pendant vertices in G the point x must be adjacent to each point of degree one of G - x. This means that G has to be of the form of the graph of lemma 2.23 a 'product' of 1-cones and hence the result is true.

Corollary 2.26. Let G be a connected graph $\delta(G) \geq 2$, which is not a block. Then G is simple if and only if $G = H*C_3*\cdots*C_3$ where H is either C_4 or θ_r.

Proof. This result follows from Proposition 2.17 together with Theorem 2.25.

Notice now that all simple graphs which are not blocks and which do not have pendant vertices have been characterized. We now turn our attention to the case when G has pendant vertices. We need the following definition.

Definition. Let G be a graph with $x,y \in VG$ and $e = xy \in EG$. Then the *subdivision of* G *at* e, $S_e(G)$, is the graph obtained by adding a new point z to VG and taking $(\{xz,yz\} \cup EG) - \{xy\}$ as the edge set.

Remark. If a graph G has a point of degree two that does not lie on a triangle then it is clear that $G = S_e(H)$ for some H. It is also evident that if $e_1, e_2 \in EG$ then $S_{e_1}(G)$ and $S_{e_2}(B)$ belong to the same degree sequence. This means that in order to show $S_e(G)$ is not simple we need only find some $e' \in EG$ such that $S_e(G) \not\to S_{e'}(G)$.

We now proceed to characterize all graphs G for which $S_e(G)$ is simple.

Lemma 2.27. Let p be a positive integer, $p \geq 6$. If G is a regular graph of degree r, then $S_e(G)$ is simple if and only if $r = 0$, $r = 1$ or $r = p - 1$.

Proof. If $r = 0$ or $r = 1$ the result is clear.

Case 1. Suppose r such that $2 \leq r < [p/2] - 1$. By Proposition 2.11 there are disconnected and connected graphs which are regular of degree r and subdividing an edge in each of these graphs gives the result.

Case 2. Suppose r such that, $[p/2] \leq r \leq p - 3$. Since $p - 1 \geq r \geq 2$ it follows that $(r,\cdots,r)$ has two realizations G_1 and G_2 so that G_1^c is a block and G_2^c is disconnected. Further, G_2^c has at least one component which is not complete since if each component of G_2^c is complete it would follow that G_2 is a complete k-partite graph contradicting the fact that G is regular of degree $r > [p/2]$. From this it follows that there are $x,y \in VG$ such that x, and y belong to a component of G_2^c but $xy \notin EG_2^c$, and $xy \in EG_2$. But now the complement of $S_{xy}(G_2)$ has a cut point. Since the complement of G_1 is a block so is the complement of $S_e(G_1)$ where e is any edge of G. Since $S_{xy}(G_2)$ and $S_e(G_1)$ belong to the same degree sequence this gives the result.

Case 3. $r = p - 2$. In this case $[S_e(G)]^c = C_5 * K_3 * \cdots * K_3$ belongs to the same sequence as $[S_e(G)]^c$.

Case 4. $r = [p/2]$. If $p = 2r$ note that lemma 2.9 gives two realizations of $(r,\cdots,r)$, one with no triangles and one with a triangle. Now subdivide each one and in the latter choose an edge not on the triangle to subdivide each one and in the latter choose an edge not on the triangle to subdivide. Hence the resulting subdivision graphs maintain the difference in the number of triangles. If $p = 2r + 1$ then as in Lemma 2.8

we construct a graph G as follows: Let $VG = \{x_1,\cdots,x_r\} \cup \{y_1,\cdots,y_r\} \cup \{z\}$, $EG = \{x_i y_i \mid i \neq j,\ 1 \le i,\ j \le r\} \cup \{x_{2i-1}x_{2i} \mid i = 1,\cdots,r/2\} \cup \{y_i z \mid i \in \{1,\cdots,r\}\}$. Then the graphs $S_{x_1y_2}(G)$ and $S_{x_1y_3}(G)$ have different numbers of triangles. Note that we can, as we have, assume r even since p is odd.

<u>Corollary 2.28</u>. Let G be a connected regular graph of degree r on p points. Then S_eG is simple if and only if $G = K_p$ or $G = C_4$.

<u>Proof</u>. For $p \ge 6$ the corollary follows by lemma 2.27 and for $p \le 5$ the corollary follows by checking the six possible cases.

<u>Theorem 2.29</u>. Let G be a connected graph, $e \in EG$. $S_e(G)$ is simple if and only if G is of one of the following forms:

(a) C_4

(b) $K_{1,n}$

(c) $C_3 * \cdots * C_3$

(d) K_n.

<u>Proof</u>. Let $e, e' \in E(G)$, $x,y,x',y' \in VG$ with $e = xy$, $e' = x'y'$. We consider the two sets of numbers $S_e = \{\deg x, \deg y\}$, $S_{e'} = \{\deg x', \deg y'\}$. Now if the latter two sets are distinct and at most two of the numbers deg x, deg y, deg x, deg x' are equal to two then $S_e(G) \neq S_{e'}(G)$ since they have different adjacency relations. There remain two cases.

<u>Case 1</u>. $S_e = S_{e'}$ for all $e, e' \in EG$. Here there are two subcases.

(i) deg x = deg y or G regular

(ii) deg x = deg x', deg y = deg y'

In case (i) we can use Corollary 2.28 to obtain $G = K_p$ or $G = C_4$. In case (ii) we have a bipartite graph. If deg y = 1, G is a star graph and $S_e(G)$ is simple since it is a Giap graph. Now assume $\deg y \ge 2$; G is not regular implies deg x $\neq$ deg y, say, deg x > deg y. Notice that G is bipartite and thus has no triangles. But because deg x > deg y $\geq$ 2 we can always make a transfer, t, to get a triangle and if e_0 is on an edge not on the triangle then $S_{e_0}(tG)$ has a triangle but $S_e(G)$ has no triangles.

Case 2. $S_e \neq S_{e'}$ and deg x' = deg y' = deg y = 2 and deg x = k > 2. If there is another point z of deg k it cannot be adjacent to x since then we would have the case considered at the outset. Now z and x must have points of degree two separating them, so let P be a path from z to x. Note there is an edge $e_0 \in EG$ not on P. Remove the points of degree two from P(by "un-subdividing") and put them on e_0 (by repeated subdivision). This process yields a graph H belonging to the same degree sequence but with points of degree k adjacent. If e_1 is an edge incident with two points of degree k and if e_2 is an edge incident with two points of degree two then $S_{e_1}H \not\cong S_{e_2}H$. Thus we may assume that x is the only point of degree k so that it must be a cut-point of G and is also the only cut point of $S_e(G)$. Hence $S_e(G)$ is either $C_4 * C_3 * \cdots * C_3$ or $\theta_r * C_3 * \cdots * C_3$. But the latter cannot be a subdivision graph since all points of degree two lie on a triangle. Thus $S_e(G) = C_4 * C_3 * \cdots * C_3$ so that $G = C_3 * \cdots * C_3$. This gives the theorem.

Corollary 2.30. Let G be a graph with a point of degree two that does not lie on a triangle with non-trivial components. G is simple if an and only if G is C_4, C_5, $C_4 * C_3 * \cdots * C_3$, $[2,m]G$, $S_e(K_n)$ or $sK_2 \cup t[1,2]G$, $t \in \{0,1\}$.

Proof. The proof follows immediately from Theorem 2.29 and Proposition 2.16.

We now apply the latter results to characterize simple graphs with cut points and pendant vertices.

<u>Proposition 2.31</u>. If G is a simple graph and x a pendant vertex at G then G - x is also simple.

<u>Proof</u>. Let x G, deg x = 1. Let S' be the degree sequence of G - x and suppose that H_1 and H_2 belong to S'. We can assume that $VG - \{x\} = VH_1 = VH_2$, and that $H_1 = tH_2$ so that the degree of any point in $VG - \{x\}$ is the same in H_1 as in H_2. Let $VG \in x'$ be the point to which x is adjacent in G, let G_1, G_2 be the graphs obtained from H_1, H_2, respectively, by attaching x to x'. Then G simple implies $G_1 \cong G_2$. Hence let f: $VG_1 \to VG_2$ be the isomorphism. If f(x') = x' then f(x) is x or some other point of degree one adjacent to x'. If f(x) = x then $f|_{VH_1}$ is an isomorphism of H_1 onto H_2. Let $N_{x'} \cap S$ denote the set of points of degree one adjacent to x'. Then f permutes the points of this set. So let f*: VG → VG be such that f(y) = y for $y \in VG - (N_{x'} \cap S)$ and on $N_{x'} \cap S$ f* is the inverse of f. Then f*f: VG → VG is an isomorphism of G such that (f*f)(x) = x, so that $f^*f|_{VH}$ is an isomorphism. Thus if f(x') = x' we are done. So suppose y', z' such that f(y') = x', f(x') = z'. Since f is an isomorphism each of x', y' and z' are adjacent to the same number of pendant vertices. Now if $t \in VG$ is adjacent to any of x', y', z' it must be to all. To see this latter suppose the contrary, that $wx' \in EG$, $wy' \in EG$ and y" a point of degree one adjacent to y'. Then the transfer t of wx' and y'y" for wy' and x'y" is defined and $G \neq tG$ since tG has fewer vertices adjacent to $|N_{x'} \cap S_1|$ points of degree one then G does, contradicting the simplicity of G. Because of this latter property the function f' defined by:

$$f'(z) = z \quad z \in N_{x'} \cap S$$

$$f'(x') = x'$$

$$f'(y') = z' = f(f(y'))$$

$$f'(s) = f(f(s)) \text{ if } sy' \in EG$$

$$f'(z) = f(z), \quad z \in VG - \{(N_{x'} \cap S) \cup \{x', y'\}\}$$

is an isomorphism mapping x' to itself so that by the first part of the proof we have $H_1 \cong H_2$,

Corollary 2.32. Let G be a simple graph and let S be the set of pendant vertices of G. Then for any subset S of S_1, $<VG - S>$ is simple.

Proof. To obtain the corollary, apply the previous proposition repeatedly $|S|$ times.

Proposition 2.33. Let G be a simple graph with non-trivial components and let $S_1 \leq VG$ be the set of pendant vertices of G then $<VG - S_1>$ is either a block or is a graph of the form K_2 or $C_4 * C_3 * \cdots * C_3$ or $\theta_r * C_3 * \cdots * C_3$.

Proof. If G is not connected and $S_1 \neq VG$ then $<VG - S_1>$ simple and from proposition 2.16 it is clear that $<VG - S_1>$ is K_2.

Now assume that $<VG - S_1>$ is connected. If it is not a block it has a cut point. Now $<VG - S_1>$ has no pendant vertices or $<VG - S_1> = K_2$. To see this, let $x \in <VG - S_1>$ be a pendant vertex. Since $x \notin S_1$ in G we have deg $x > 1$ so that x was adjacent to some $y \in S_1$. Then $<(VG - S_1) \cup \{y\}>$ has a point of degree two not lying on any triangle (namely x) and is connected. Hence $<(VG - S_1) \cup \{y\}>$ is a subdivision graph so that by Corollary 2.30 we have $<(VG - S_1) \cup \{y\}> = [2,m]G$. But G is simple and the difference between G and $<(VG - S_1) \cup \{y\}>$ consists of points of degree one, so G is a giap graph and hence

$< VG - S > = K_2$.

Now assume that G is not a giap graph, so that $\delta<VG - S_1> \geq 2$, and $<VG - S_1>$ is not a block. By Corollary 2.26 $<VG - S_1>$ is $H*C_3*\cdots*C_3$ where $H = \theta_r$ or C_4. This gives the result.

Corollary 2.34. Let G be a simple graph with pendant vertices. Then G connected implies that G has one of the following forms:

(a) $[n,m]G$

(b) $H*C_3*\cdots*C_3*K_2*\cdots*K_2$ where $H = \theta_r$ or $H = C_4$

(c) $<VG - S_1>$ is a simple block, where S_1 is set of pendant vertices of G.

Proof. This corollary follows from proposition 2.33 and the fact that the only way to add a pendant vertex to a graph of the form $H*C_3*\cdots*C_3$ is at the vertex of highest degree if we want to preserve simplicity.

Theorem 2.35. Summary.

Let G be a simple graph which is not a block. Then G must be one of the following graphs:

(a) $sK_2 \cup t\,[n,m]G$ where r,s are nonnegative integers, $t \in \{0,1\}$ and $s \neq 0$ implies $n = 1$ or $m = 1$.

(b) $H*C_3*\cdots*C_3*K_2*\cdots*K_2$ where $H = C_4$ or $H = \theta_r$.

(c) $<VG - S_1>$ is a simple block where S_1 is set of pendant vertices of $G, S_1 \neq VG$.

(d) the union of graphs from (a), (b) and (c) with any number of trivial graphs.

Remark. Notice that in view of theorem 2.35 any simple graph with pendant vertices which is not a block must realize one of the following type sequences, if $\langle VG - S_1\rangle$ is not a simple block

(a) $(n,m,1,\cdots,1,0,\cdots,0)$

(b) $(m,1,\cdots,1,0,\cdots,0)$

(c) $(m,2,\cdots,2,1,\cdots,1,0,\cdots,0)$

(d) $(m,r+1,2,\cdots,2,1,\cdots,1,0,\cdots,0)$

with suitable restrictions on the number of two's and one's using (a) and (b) of 2.35.

C. Some Necessary Conditions for a Graph to be Simple.

With theorem 2.35 the problem of characterizing simple graphs reduces to that of characterizing simple blocks. To this end we develop fairly general necessary conditions that must be satisfied by simple graphs with respect to girth, diameter and radius. We first show the connection between critical and minimal blocks and simplicity.

Definition. Let B be a block. B is a *minimal* block if given any $e \in EB$, $B - e$ is not a block; B is a *critical* block if, given any $x \in VG$, $B - x$ is not a block.

Lemma 2.36. Let G be a block, $|VG| \geq 4$. For any $x \in VG$, x belongs to a cycle of length greater than or equal to four.

Proof. Let $x \in VG$. We have $\deg x \geq 2$ since G is a block so that there exist $x_1, x_2 \in VG$ with $xx_1, xx_2 \in EG$. Since x is not a cut point there is a path in G from x_1 to x_2 not using x. If such a path is of length greater than one the desired cycle has been found. Suppose the edge x_1x_2 is the only such path. Since $|VG| \geq 4$, let $x_3 \in VG - \{x,x_1,x_2\}$ Then since G a block there is a cycle C containing x_1x_2 and x_3. If x is on C we are done, if not delete x_1x_3 from C and add the edges x_1x and

x_2x which could not have been on C if x was not.

<u>Proposition 2.37</u>. Let G be a critical block, $|VG| \geq 6$, then G cannot be simple.

<u>Proof</u>. A critical block contains a point x of degree two (see Chartrand, [1, p. 30-32]). Let $N_x = \{y,z\}$ be the set of neighbors of x. Suppose first that $yz \in EG$. This contradicts the fact that G - x is not a block. To see this note that if $z_1, z_2 \in VG - \{x,y,z\}$ then z_1 and z_2 lie on a cycle in G - x. It is also clear that this is true for $z_2 = y$ or $z_2 = z$. Finally y and z lie on a cycle in G - x, for, by Lemma 2.36, x lies on a cycle of length of at least four in G. Since y and z must also lie on this cycle it is clear that by using the edge yz we have y and z on a cycle in G - x. Hence every pair of points in G - x lie on a cycle, contradicting the fact that G is critical.

Now suppose that $yz \in EG$. This means that x is a point of degree two that does not lie on a triangle. Hence $G = S_{yz}(H)$ for some H. But by Theorem 2.29, since H is a block if G is a block, it follows that $H = K_n$, $n \geq 5$. But this contradicts the fact that G is a critical block.

Hence in either case we get a contradiction so that G cannot be simple, yielding the result.

Note we *must* have $|VG| \geq 6$ for C_5 is a critical block which is also simple.

<u>Proposition 2.38</u>. If G is a minimal block with $|VG| \geq 6$ then G is not simple.

<u>Proof</u>. Again by Lemma 2.36 if G is a block then G contains a cycle C of length at least four. Since G is simple G is not a cycle and hence $G \neq C$. By Theorem 2, C of [8] contains a point x of degree two which,

by Corollary 1b, of [8] cannot lie on a triangle. If x is adjacent to y and z in VG consider the graph H with $VH = VG - \{x\}$, $EH = (EG \cup \{yz\}) - \{yx, zx\}$. Then H is a block and $S_{yz}H = G$. If G is simple then again by Theorem 2.29 G cannot be a minimal block since $H = K_p$, $p \geq 5$.

Roughly speaking the above two theorems generalize the fact that C_n, $n \geq 6$ is not simple. Or, in some sense simple blocks cannot be to 'thin'. The next proposition says, essentially, that a simple block must be 'thick' enough to contain a triangle.

Proposition 2.39. Let G be a connected graph, $|VG| \geq 6$. If S is the degree sequence of G then S contains a graph with a triangle, provided G has a cycle.

Proof. Let C be a cycle of shortest length of G, so that C has no diagonals--each point of the cycle is adjacent to exactly two other points of the cycle.

Case 1. Suppose $|VC| \geq 6$. Let $C = x_1 x_2 x_3 \cdots x_{n-2} x_{n-1} x_n$--where $3 \neq n-2$ since $n \geq 6$. Then the transfer of $x_{n-1}x_{n-2}$ and x_2x_3 for $x_{n-1}x_2$ and $x_{n-2}x_3$ is defined and yields a graph with a triangle.

Case 2. $|VC| = 5$. Let $C = x_1x_2x_3x_4x_5$. Since $|VG| \geq 6$ we must have x_1, say, adjacent to an $x \in VG$ which is not on C. Note that x is not adjacent to x_4 for then G would have a cycle of length four contradicting the definition of C. Thus the transfer of x_3x_4 and x_1x for x_3x_1 and x_4x is defined and yields a triangle.

Case 3. $|VC| = 4$. Let $C = x_1x_2x_3x_4$. Then as above suppose that x is not on C and adjacent to x_1. Now x cannot be adjacent to x_4 for that would again contradict the definition of C as shortest cycle. Thus the transfer of x_3x_4 and x_1x for x_3x_1 and x_4x is defined and creates a triangle.

Corollary 2.40. Let G be a connected simple graph, $|VG| \geq 6$. Then G is either a giap graph or G has a triangle i.e., the girth of G is three.

Proof. Let G be connected and simple. If G has no cycles then it must be a giap graph. If G has a cycle then the conditions of Proposition 2.39 are fulfilled so that G has a triangle.

We now consider the diameter of simple graphs; recall the diameter of G, d(G) is $\max\{d(x,y) \mid x,y \in VG\}$, with G connected.

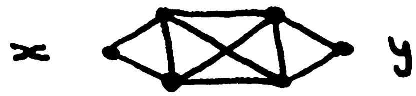
Theorem 2.41. If G is simple and connected then $d(G) \leq 3$.

Proof. Assume $d(G) \geq 4$, G simple. We can suppose that there are $x,y \in VG$ with $d(x,y) = k \geq 4$ and $z_1,\cdots,z_{k-1} \in VG$ so that $xz_1\cdots z_{k-1}y$ is the shortest xy path. In particular this means that z_1z_3 and xz_4 are not in EG.

Case 1. Either xz_1 or z_3z_4 lie on a triangle. If z^* is adjacent to both x and z_1 then $z^* \neq z_2,\cdots,z_{k-1}$ and $z^*z_i \notin EG$ for $i = 3,4,\ldots,k-1$ for otherwise $d(x,y) = k$ would be contradicted. Consider the transfer t of xz_1 and z_3z_4 for xz_3 and z_1z_4. Now if in tG xz_3 and z_1z_4 lie on triangles then this, in G, would contradict $d(x,y) = k$. Hence the 'new' edges do not introduce new triangles and the removal of xz_1 removes one triangle so that $G \neq tG$. In exactly the same way if z_3z_4 lie on a triangle then the same transfer reduces the number of triangles. Thus, either way, we get a contradiction.

Case 2. Neither xz_1 nor z_3z_4 lie on a triangle. The transfer t of xz_1 and z_3z_4 for z_1z_3 and xz_4 yields a graph tG with at least one more triangle than G and again $G \neq tG$. This gives the result.

Remark. There are simple graphs of diameters one, two and three: K_p, θ_r, [n,m]G respectively, where $n,m \geq 2$. At first sight simple blocks appear to have diameter of at most two, but consider the following graph:



The above graph is simple since it is the complement of [3,3]G, and the points x and y are clearly a distance three apart.

We now consider the radius of simple graphs. Remember for a connected graph G, $\text{rad}(G) = \min\{e(y) | y \in VG\}$ where $e(y) = \max\{d(x,y) | x \in VG\}$. The number e(y) is usually called the 'eccentricity' of y. The next sequence of lemmas and propositions will be used to show that the radius of a connected graph is not greater than two. Again, note that $\text{rad}(K_p) = 1$ and $\text{rad}(\theta_r) = 2$ so that the result is the best possible.

Proposition 2.42. Let $S = (d_1, \cdots, d_p)$ be a graphical sequence. For any j, $2 \leq j \leq p$, for which $d_j > 0$ there is a realization of S with a point of degree d_j adjacent to a point of degree d_1.

Proof. Let H be a realization of S. Let $x,y \in VH$ such that deg x = d_1, deg y = d_j, where $d_j > 0$, $1 \leq j \leq p$. Suppose that y is not adjacent to any point of degree d_1. Then y is adjacent to a point z, deg z < d_1. Now $xy \notin EH$ implies that x is adjacent to some point, $w, w \in \{z\} \cup N_z$. Then the transfer t of xw and yz for xy and zw is defined and the graph tG has the required property.

Corollary 2.43. Let G be a simple graph with degree sequence $(d_1, \cdots, d_p)$. If G has a single point of degree d_1 then that point is adjacent to points of any given positive degree, d_j, $d_j \neq d_1$.

Remark. A natural generalization of the above would be the following: If given a graphical sequence $(d_1,\cdots,d_p)$ we have for any $i,j \in \{1,\cdots,p\}$, $i \neq j$, that there is a realization G of $(d_1,\ldots,d_p)$ in which there are points of degree d_i adjacent to points of degree d_j. But the latter is false since (4,4,3,2,2,1) has two realizations and in neither is there a point of degree one adjacent to a point of degree two. Even if we restrict attention to simple sequences the above generalization fails since in the graph realizing (5,5,4,4,4,2) no point of degree two is adjacent to a point of degree four.

Lemma 2.44. Let G be a simple connected graph, having degree sequence $(d_1,\cdots,d_p)$. Then there is a point $x \in VG$ such that

(1) $\deg x = d_1$

(2) Each point of degree d_1 of VG is within a distance two of x.

(3) The point x is adjacent to points of degree $d_2,\cdots,d_{d_1+1}$.

Proof. By theorem 1.2, (the generalization of Havel and Hakimi) there is a point x satisfying (1) and (3) above. If not all points of N_x are of degree d_1 then (2) also holds. Thus we may assume that each point of N_x is of degree d_1. Let y be another point of degree d_1 for which $d(x,y) > 2$, or better by Theorem 2.41, $d(x,y) = 3$. We proceed to show the following: (a) if there are two points of N_x that are adjacent it follows that there is a transfer moving y to within two of x and not moving any other points of degree d_1 further away from x thus increasing the number of points of degree d_1 within two of x and (b) no pair of N_x being adjacent contradicts assumed simplicity of G. This shows that after a finite number of transfers every point of degree d_1 is within two of x.

First we assume that no two points of N_x are adjacent and show that G is not simple. There are four cases.

Case (i). Assume that there are $r,s \in V(G)$ of degree less than d_1 which are adjacent. Then r is not adjacent to some $x' \in N_x$ so that the transfer of rs and xx' for rx' and sx decreases the number pairs of points of degree d_1 which are adjacent (remember x is adjacent to points of degree d_1 only).

Case (ii). There are at least two points of degree less than d_1 no pair of which are adjacent. If r and s are such points they can be adjacent only to points of degree d_1. First assume deg r > 1. It then follows that there exist $u,v \in VG$ with $u \neq v$, ru, sv $\in EG$ and d_1 = deg u = deg v. If $uv \notin E(G)$ the transfer of ru and sv for rs and uv changes the number of pairs of points of degree d_1 that are adjacent. If $uv \in E(G)$ then one of u and v, say v does not belong to N_x (we are assuming $\langle N_x \rangle = d_1K_1$ i.e., no pair in N_x is adjacent). Further by a degree argument there is an $x' \in N_x$ such that $ux' \notin E(G)$. Then the transfer of uv and x'x for ux' and vx takes us back to the above case where $uv \notin EG$.

Now assume deg r = deg s = 1. If $u \in VG$ is adjacent to both r and s then there is an $x' \in N_x$ such that $x'u \in EG$. Then the transfer of xx' and ur for xr and x'u increases the number of vertices of degree d_1 that are adjacent to a single pendant vertex. Suppose now that r,s have distinct neighbors u,v; then we proceed exactly as above when we assumed deg r > 1.

Case (iii). Suppose there is only one point, r, of degree less than d_1. *First* assume deg r > 1. Since r is the unique point of degree less than d_1 then exhibiting a transfer that changes the number of neighbors of r that are adjacent suffices to show that G is not simple. So let $r',r'' \in N_r$ with say $r'r'' \in EG$. Then not both r',r'' are in N_x, say r'. Also (by a degree argument again) r'' is not adjacent to some $x' \in N_x$. The transfer, t, of $r'r''$ and xx' for xr' and $x'r''$ reduces the number of pairs of neighbors of r that are adjacent, if $x'r \notin EG$. If $x'r \in EG$ then note that $d(x,y) = 3$ implies $x'y \notin EG$ and also that there is a $y' \in N_y$ such that $y'r \notin EG$. Consider the transfer t_0 of yy' and rx' for yx' and $y'r$. If t G changes the number of adjacent pairs of points in N_r we are done; otherwise, $t_0(tG)$ does. Now assume that no pair in N_r are adjacent. In this case it is clear that there must exist two points r',r'' in N_r which have non-adjacent neighbors i.e., there are $z' \in N_{r'}$, and $z'' \in N_{r''}$ for which $z'z'' \notin EG$. In this case the transfer of $r'z'$ and $r''z''$ for $r'z''$ and $z'z''$ suffices--makes a pair in N_r adjacent.

Now assume deg r = 1. Here again there are several subcases but in each we use the fact that r can be made by a transfer, adjacent to y in such a way that $d(x,r) = 4$ contradicting $dG = 3$.

Case (iv). G is a regular graph--all points have degree d_1. This is impossible since by Theorem 2.12 the only regular connected simple graphs are K_p and $(mK_2)^c$ which have diameter one and two respectively--we are assuming $d(x,y) = 3$.

Hence N_x contains an adjacent pair--say x',x''. It is clear that x' cannot be adjacent to each point of N_y--say $x'y' \notin EG$, $y' \in N_y$. Then the transfer of yy' and $x'x''$ for $x'y'$ and yx'' increases the number of points of degree d_1 within two of x.

Lemma 2.45. Let G be simple and connected. If $x \in VG$ is not a point of highest degree then it is within a distance of two of a point highest degree.

Proof. Let d_1 denote the highest degree of a vertex in VG. Let $x \in VG$, $\deg x < d_1$. By Proposition 2.42 there are $y, z \in VG$ with $\deg y = d_1$, $k = \deg z = \deg x$ and $yz \in EG$. We want to show that $d(x,y) \leq 2$. So we may assume that $x \neq z$, $xy, xz \notin EG$, for otherwise the result holds. As in the last lemma it is assumed that $d(x,y) > 2$, or $d(x,y) = 3$, from which a contradiction of the simplicity of G is derived.

First we show that each $r \notin N_x$ has degree k. Suppose otherwise; there is a $w \in N_x$ such that $\deg w \neq k$. Now $d(x,y) = 3$ implies $yw \in EG$ so that the transfer of zy and xw ~~and~~ xy alters the adjacency relations of G, contradicting the fact that G is simple.

Second, all points adjacent to y must be of degree k also. Suppose otherwise; there is a $w \in N_y$, $\deg w \neq k$. In this case the transfer yw and xr, $r \in N_x$, for zr and xw is possible and again alters the relations of G, contradicting fact that G is simple.

Third, the degree sequence of G must be $(d_1, k, \cdots, k)$ i.e., each point different from y has degree k. To prove this let $w \notin EG$ $\deg w \neq k$ $w \neq y$. Then w is a distance at least two from each of x and y. Let $w' \in N_w$ lie on a shortest w-y path. By a degree there is some $y' \in N_y$, argument, such that $w'y' \in EG$. If $\deg w' \neq d_1$ then the transfer ww' and yy' for wy and w'y' changes adjacency relations. Assume $\deg w' = d_1$. Then $d(y,w) = 3$ and $wy' \notin EG$ and if $\deg w \neq d_1$ the transfer of ww' and yy' for wy' and w'y changes adjacency relations. Finally if $\deg w' = \deg w = d_1$ let $w'' \in N_{w'} \cap N_y$. Then w'' is not adjacent to some $y'' \in N_y$ (again by a degree argument and fact that $w' \notin N_y$). Then the transfer of yy'' and $w''w'$ for $w'y$ and $y''w''$ is possible and alters the adjacency relations since w'' has degree k. Hence by being simple G must belong to the sequence $(d_1, k, \cdots, k)$.

Now assume yt_1t_2x is a shortest x-y path in G. There are three cases. *First*, assume t_2 is adjacent to $y' \in N_y$, $y' \neq t_1$. Now t_1 is not adjacent to some $x' \in N_x$. Then the transfer t_2y' and xx' for $y'x$ and t_2x' is defined and increases the number of points within two of the unique point y of degree d_1. *Second* if $y'y'' \in EG$ for some $y', y'' \notin N_y$ then y' is not adjacent to some $x' \in N_x$ and the transfer of $y'y''$ and xx' for $x'y'$ and xy'' has the same effect as above. *Third* assume neither of the above hold. Let $y' \in N_y$, $y' \neq t_1$. The transfer of xt_2 and yy' for yt and xy' changes the number of pairs of neighbors of y that are adjacent. Hence in all three cases the simplicity of G is contradicted. Thus we are forced to conclude $d(x,y) \leq 2$.

Theorem 2.46. [The Nathaniel Turner Theorem] If G is a simple connected graph then the radius of G is less than or equal to two.

Proof. Let $x \in VG$ have the following properties as guaranteed by Lemma 2.44: (a) x is of maximal degree d_1, (b) x is adjacent to points of degree $d_2, \cdots, d_{d_1+1}$ where G has degree sequence $S = (d_1, \ldots, d_p)$ and (c) x is within two of every other point of degree d_1.

We shall assume that there is a point $y \in VG$ with $d(x,y) = 3$ and $\deg y = k < d_1$ and prove that this contradicts the simplicity of G.

By Lemma 2.45 there is apoint $x' \in VG$ with $\deg x' = d_1$ and $(x',y) \leq 2$. No matter whether $x' \in N_x$ or $d(x,x') = 2$ it follows from property (b) that there is a point of degree d_1 adjacent to x.

First, the point y cannot be adjacent to a point y', $\deg y' < d_1$.

To see this suppose that each point of N_x has degree d_1. It is clear that y' is not adjacent to some $x'' \in N_x$ and the transfer of yy' and xx'' for xy and y'x'' changes adjacency relations. Now suppose that some points adjacent to x are not of degree d_1 so that by (b), again *all* points of degree d_1 are adjacent to x. Hence $x' \in N_x$ and there is a y' such that xx'y'y is a shortest x-y path. The transfer of xx' and yy' for xy' and yx' is thus possible and changes adjacency relations again.

Now we have that each point in N_y is of degree d_1. This implies that (by (b) again) each point of N_x is of degree d_1. Let $r \in VG$, $d(x,r) = 2$ and let xzr be a shortest x-r path. It follows that deg $r = d_1$. If not, note that there is a $z' \in N_x$ such that $z'z \notin EG$ (since degree x is d_1 and z is adjacent to some point not in N_x). Also $y' \in VG$ with $yy' \in EG$. The following pair of transfers can be made (a) rz and xz' for zz' and rx and (b) yy' and rx for ry and y'x. This results in a change of adjacency relations since by above paragraph more points of degree less than d_1 are adjacent.

We now have all points within two of x being of degree d_1 while those that are of a distance three from x are of degree less than d_1 and none of these are adjacent.

Let $z \in VG$, deg $z < d_1$, $z \neq y$. Let zz'x'x be a shortest z - x path. Let x'' be a point of N_x such that $z'x'' \in EG$--such a point exists since deg z' = deg x and $z' \in N_z$. Make the following transfers: (a) zz' and x''x for zx and z'x'' and (b), $y' \in N_y$, yy' and zx for yz and y'x. The effect, again, of these transfers is to make points of degree less than d_1 adjacent. Thus it follows that y must be the *only* point of degree less than d_1.

Assume deg $y > 1$. Let $y',y'' \in N_y$ with $y'y''$ EG. As above y' is not adjacent to some $x' \in N_x$. Then the transfer of $y'y''$ and xx' for $x'y'$ and xy'' changes the number of pairs of adjacent neighbors of the unique point of degree less than d_1. Suppose no points of N_y are adjacent. Then let $x',x'' \in N_x$ $y',y'' \in N_y$ and if $x'x'' \in EG$ we transfer yy', yy'', xx',xx'' for yx',yx'', yx',xy'' which has same effect as last transfer. Assume now that no pair in either N_x or N_y are adjacent. Here let $x' \in N_x$, $y' \in N_y$ where $x'y' \in EG$ (possible since $d(x,y) = 3$). There is $y'' \in N_y$, $y'' \neq y'$ and $y''\ z \in EG$, $z \neq x'$. If z N_x then we transfer zy'' and $y'x'$ for $y'y''$ and zx'. The latter follows since x' is adjacent to point, y', that y'' is not and so deg x' = deg $y'' = d_1$ implies y'' adjacent to a z which x' is not. Then the transfer of zy'' and $y'x'$ for $y'y''$ and zx' changes adjacency relations among the neighbor of y, the unique point of degree less than d_1.

Final Case. Assume deg $y = 1$, $xx'y'y$ is shortest x-y path. If any pair of N_x is adjacent we can make a transfer to get y within two of x keeping other distances fixed. Suppose now that no pair of N_x is adjacent. Since there is only one point of degree one it follows that the number of pairs of $N_{y'}$ that are adjacent is also an invariant. Suppose $y'',y''' \in N_{y'}$ and $y''y''' \in EG$. Then there is an $x'' \in N_x$ such that $x'' \notin Ny'$. The transfer of xx'' and yy' for xy and $x''y'$ now puts the point of degree one next to a point with no adjacent neighbors which again contradicts the simplicity of G. Now suppose no pair of N_x or $N_{y'}$ are adjacent. It is then clear that $N_{x'} \cap N_{x'} = N_{x'} \cap N_{y'} \neq \phi$. Assume now that $N_x \cap N_{y'} \neq \{x'\}$ i.e., there is a $y'' \in N_y$ with $y''x \in EG$. Since $d_1 \neq 2$

there is an $x'' \in N_{x'}$, $x'' \neq x$, $x'' \neq y'$. Then the transfer of xx'' and $y'y''$ for $y''x'$ and $x''x$ takes us back to last case. Hence $N_x \cap N_{y'} \neq \{x'\}$ that is each point of $N_{y'}$ is exactly two away from x. Thus let $y'' \in N_{y'}$ and $y''x''x$ be a shortest y''-x path. Let $x''' \in N_x$. The transfer of xx''' and $y'y''$ for xy'' and $x'''y'$ is thus defined and again takes us back to an above case. This exhausts all possibilities when deg $y = 1$ and thus in each case $d(x,y) = 3$ leads to a contradiction. This proves the theorem.

Corollary 2.47. Let G be a simple connected graph. Then if $p = |VG|$

(a) G^2 has a point of degree $p - 1$.

(b) $G^3 = K_p$.

Proof. Part (a) is immediate from 2.46 since G simple implies that there is a point with two of every other point. Part (b) follows from 2.41 since $d(x,y) \leq 3$ in G implies $d(x,y) \leq 1$ in G^3.

Chapter III

COUNTING REALIZATIONS OF GRAPHICAL SEQUENCES

This chapter concerns itself mainly with enumeration problems. Many such problems present themselves. For example, given a graphical sequence how many non-isomorphic graphs realize the sequence? Or, for a given positive integer n is there a graphical sequence with n non-isomorphic realizations? Or again, what kind of bounds can be put on the number of realizations of graphical sequences? This chapter presents some methods for attacking these questions.

For convenience we introduce the following notation. If $s = (s_1, \cdots s_p)$ is graphical sequence then $|s| = (s_1, \cdots, s_p)$ denotes the number of non-isomorphic realizations of s. If s is not graphical we set $|s| = 0$. Also if $r_i = |\{s_j | s_j = i, 1 \le j \le p\}|$ for $i = 0, \cdots, p-1$ then s can also be written $0^{r_0} 1^{r_1} 2^{r_2} \cdots$ meaning that s has r_0 zero entries, r_1 one entries,.... For example if s is a sequence of length p such that $s = (2, \cdots, 2)$ then $s = 2^p$. This latter notation is commonly used in connection with 'partitions.'

Throughout this chapter the concept of 'ordinary generating function' (o.g.f.) is used freely and especially in connection with the theory of partitions. Here the basic reference is Riordan ([9], Chapter 6). One ordinary generating function that is used frequently in the sequel is $P(x) = 1/(1-x)(1-x^2)(1-x^3)\cdots$ where the coefficient of x^n is p_n, the number of partitions of n.

A. Counting Realizations for Elementary Graphical Sequences

We first find ordinary generating functions which enumerate the number of realizations of sequences of rather simple structure. These latter are then used to construct the ordinary generating functions for sequences that are somewhat more complicated.

Proposition 3.1. The number of realizations of $s = (2,\cdots,2) = 2^p$ is the coefficient c_p of x^p in

$$(*) \qquad C(x) = 1/(1-x^3)(1-x^4)\cdot\cdots$$

Proof. For p a natural number consider any partition $(p_1,\ldots,p_r)$ of p where $p \geq\cdots\geq p_r \geq 3$, $p +\cdots+ p_r = p$. It is clear that the graph $C_p \cup \cdots \cup C_{p_r}$ is a realization of $2^p = (2,\cdots,2)$.

Conversly consider any realization G of 2^p. Since each point of VG has degree two G is the union of disjoint cycles--say $C_{p_1},\cdots,C_{p_r}$ where $p_i \geq 3$ for $i = 1,\cdots,r$ and $p_1+\cdots+p_r = p$.

Thus there is a one-to-one correspondence between partitions of p, each part greater than or equal to three and the realizations of 2^p. Now by the same reasoning that $P(x) = 1/(1 - x)(1 - x^2)(1 - x^3)\cdots$, it follows that $1/(1 - x^3)(1 - x^4)\cdots$ has as coefficient of x^n, $n > 0$, equal to $(2,\cdots,2) = 2^n$. For convenience we say that $c_o = 1$ so that now

$$C(x) = \sum_{i=0}^{\infty} c_i x^i = 1/(1 - x^3)(1 - x^4)\cdot\cdots.$$

Proposition 3.2. The number of realizations of the sequence $(2,\cdots,2,1,1)$ of length p is the coefficient of x^p in

$$C(x) \cdot \frac{x}{1-x}$$

where C(x) is as above.

Proof. Let d_p denote the number of realizations of the p-part sequence $(2,\cdots,2,1,1)$. Any realization of $(2,\cdots,2,1,1)$ must have the following form: $c_{p_1},\cdots c_{p_t}, c_{p_{t+1}}$ where $p+1 = p_1 + \ldots + p_t + p_{t+1}$ If c_i is the coefficient of x^i, $i \geq 0$, in $\sum_{i=0}^{\infty} c_i x^i$ then there are c_{p-2} realizations of $(2,\cdots,2,1,1)$ that have a path of length one, c_{p-3} realizations that have a path of length two, and so on. It follows that

$$d_n = c_{n-2} + c_{n-3} + \cdots + c_3 + 1$$

$$= c_{n-2} + c_{n-3} + \cdots + c_3 + c_2 + c_1 + c_0$$

$$(\text{since } c_2 = c_1 = 0 \text{ and } c_0 = 1)$$

$$= \sum_{i=0}^{n} c_i - c_n - c_{n-1}$$

where $n \geq 2$. Setting $d_1 = 0$, $d_0 = 0$ we have

$$d_n x^n = \sum_{i=0}^{n} c_i x^n - c_n x^n - c_{n-1} x^n, \; n \geq 1$$

and thus

$$\sum_{i=1}^{\infty} d_i x^i = \sum_{i=1}^{\infty} \sum_{j=1}^{i} c_j x^i - \{c(x) - 1\} - x \sum_{i=0}^{\infty} c_i x^i$$

$$= \sum_{i=1}^{\infty} \sum_{j=0}^{i} c_j \; x^i - c(x)(1+x) + 1$$

Thus since $d_0 = 0$ and $c_0 = 1$ we have

$$\sum_{i=0}^{\infty} d_i x^i = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} c_j \right) x^i - c(x)(1+x)$$

$$= C(x)(1-x)^{-1} - C(x)(1+x)$$

$$= C(x)\left[\frac{1}{1-x} - (1+x)\right]$$

$$= C(x) \cdot \frac{x^2}{1-x}$$

Proposition 3.3. The sequence $(4,2,\cdots,2)$ of length n, has p_{n-5} realizations, $n \geq 5$, where p_i, $i = 1,2,\cdots$, is the coefficient of x^i in $P(x) = 1/(1-x)(1-x^2)\cdots$. Or, the ordinary generating function of $(4,2,\cdots 2) = 42^{n-1}$ is $x^5P(x)$.

Proof. First it is clear that any realization consists of the union of a 'figure eight' (a connected graph with one point of degree four, the rest of degree two) with disjoint cycles. In the same spirit as Proposition 3.2 we count the number of realizations of $(4,2,\cdots 2)$ by counting those with a figure eight with five points, six points,$\cdots$. To this end we first find the ordinary generating function (o.g.f.) for $\langle f_n \rangle$ where f_n is number of figure eights on n points. If a figure eight has n points then it has $n - 1$ points distributed on two loops. Hence there is a one-to-one correspondence between figure eights on n points and partitions of $n - 1$ into two parts each part greater than or equal to two, $n \geq 5$. It is easy to verify that the values of f_n for $n = 1,\cdots,12$ are $0,\cdots,0,1,1,2,2,3,4,4$. It is easily shown by induction that the latter pattern continues. We first find the o.g.f. for $\langle 1,1,2,2,3,3,\cdots \rangle$. The open form for the generating function is

$$\sum_{k=0}^{\infty} (k+1)x^{2k} + \sum_{k=0}^{\infty} (k+1)x^{2k+1}$$

$$= \sum_{k=0}^{\infty} x^k + \sum_{k=1}^{\infty} kx^{2k+1} + \sum_{k=1}^{\infty} kx^{2k}$$

$$= \sum_{k=1}^{\infty} x^k + x\sum_{k=0}^{\infty} ky^k + \sum_{k=0}^{\infty} ky^k \quad y = x^2$$

$$= \sum_{k=0}^{\infty} x^k + x\sum_{k=0}^{\infty} ky^k + \sum_{k=0}^{\infty} ky^k \quad y = x^2$$

$$= \frac{1}{1-x} + (1+x)\ [y/(1-y)^2], y = x^2$$

$$= \frac{1}{1-x} + (1+x)\ \frac{x}{(1-x^2)^2}$$

$$= \frac{1}{(1-x)(1-x^2)}$$

To obtain the ordinary generating function for the f_n we merely multiply the above by x^5:

$$F(x) = \sum_{n=0}^{\infty} f_n x^n = x^5/(1-x)(1-x^2).$$

Back to computing $a_p = |(4,2,\cdots,2)|$, where $(4,2,\cdots,2)$ is of length p. Let $C(x) = \sum c_i x^i$ be as per Proposition 3.1. Then by the remark at the beginning of the proof, for $p \geq 5$,

$$a_p = c_0 \cdot f_p + f_{p-1} \cdot c_1 + \cdots + f_0 \cdot c_p$$
$$= \sum_{k=0}^{p} f_k c_{p-k}$$

Defining $a_0 = \cdots = a_4 = 0$ we have

$$a_p = \sum_{k=0}^{p} f_k c_{p-k} \quad \text{for} \quad p = 0,1,2,\cdots$$

Hence the o.g.f. for a_p must be

$$
\begin{aligned}
&F(x)\ C(x)\\
&\quad = [x^5/(1-x)(1-x^2)]\ [1/(1-x^3)(1-x^4)\cdots]\\
&\quad = x^5/(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots\\
&\quad = x^5\ P(x)
\end{aligned}
$$

We now give a sequence a lemmas which will yield a generalization of Proposition 3.3. The proofs depend on 'Ferrers' graphs (see [7,p.114-117]) and I doubt that these lemmas are original.

Lemma 3.4. There is a one-to-one correspondence between partitions of n-1 into m parts each part being greater than or equal to two and the partitions of n-1-2m having no part greater than m.

Proof. It will suffice to consider the Ferrers graphs and their conjugates. The conjugate of a partition of n-1 into m parts each part greater than equal to two will have m dots in each of the first two rows. The remaining rows of the conjugate will have m or fewer dots, since the original partition had m or fewer dots in each column. Finally it is clear that the third, fourth,$\cdots$, etc, rows of the conjugate uniquely determine the original and vice versa. The result follows.

Lemma 3.5. The number of partitions of n-1 into m parts, each part being greater than or equal to two is the coefficient of x^n in the expansion of $x^{2m+1}(1-x)^{-1}\circ\cdots\circ(1-x^m)^{-1}$.

Proof. The number of partitions of n into parts no greater than m is the coefficient of x^m in $(1-x)^{-1}\circ\cdots\circ(1-x^m)^{-1}$. Hence the coefficient of x^n in $x^{2m+1}(1-x)^{-1}\circ\cdots\circ(1-x^m)^{-1}$ is number of partitions of $n-2m+1 = (n-1)-$

2m with parts no greater than m. By Lemma 3.4 the result follows.

Lemma 3.6. The number of realizations of the n-part sequence $(2m,2,\cdots,2)$ which are connected is the coefficient of x^n in $x^{m+1}/(1-x)^{-1}\circ\cdots\circ(1-x^m)^{-1}$.

Proof. Any connected realization of $(2m,2,\cdots,2)$ consists of m cycles 'joined' at a single point. If the cycles are $Cr_1,\cdots,Cr_m$ then $r_i \geq 3$ for $i = 1,\cdots, m$ and $(r_1 - 1) +\cdots+ (r_m-1) = n - 1$. Thus the number of non-isomorphic graphs of this type is clearly the number of partitions of n-1 into m-1 parts each part greater than or equal to two. The result now follows from the previous lemmas.

Remark. A connected realization of the n-part $(2m)2^{n-1} = (2m,2,\cdots,2)$ is called a *rose* on n points with m *petals*. Thus the o.g.f. $R^m(x)$ of m-petaled roses given by $\sum_{n=}^{\infty} r_n^m x^n = x^{2m+1}/(1-x)\circ\cdots\circ(1-x^m)$.

Proposition 3.7. Let s_n be the n-part sequence $(2m,2,\cdots,2)$. Then the o.g.f. of $<|s_n|>$ is $C(x)\cdot R^m(x)$ when R^m and C are as above.

Proof. Again, in the spirit of previous proofs, it is clear that any realization of $s_n = (2m,2,\cdots,2)$ must be the union of a m-petaled rose and disjoint cycles. Reasoning as before we have for $n \geq 2m+1$, $|s_n| = r^m_{2m+1}\cdot c_{n-2m+1} +\cdots+ r^m_{n-3}\cdot c_3+r^m_n$. Since $r^m_j = 0$ for $0 \leq j \leq 2m$ we have $|s_n| = \sum_{k=0}^{n} r^m_k\cdot c_{n-k}$. But this latter shows that the o.g.f. for $<|s_n|>$ is the product of $R^m(x)$ and $C(x)$ and hence the result.

Remark. Taking m=2 in 3.7 yields 3.3. The interest of Proposition 3.3 is the curious occurrence of $P(x)$.

Now consider realizations of the p-part partition $F_p = (2m,2n,2,\cdots,2)$ with $m > n > 1$. Let r^i_j and c_j denote the same quantities as in the previous propositions and let $p_k(j)$ denote the number of partitions of j into k parts. If G is a realization of F_p let $a,b, \in V(G)$ be the points of degree 2m and 2n respectively.

Any realization G of F_p may be classified according to the number of such paths is of the form 2ℓ, $0 \leq 2\ell \leq 2n$. There are two cases: (a) one of these paths consists of an edge and (b) each of the paths contain at least two edges.

Case (a). If there are 2 edge disjoint paths of G joining a and b then there are $m - \ell$ cycles at a not containing b and $n - \ell$ cycles at be not containing a. Suppose that $i + 1$ points of G lie on cycles adjacent to a and $j + 1$ points of G lie on cycles containing b. Let $p_{t,2}$ $p_{t,2}(j)$ denote a number of partitions of j into t parts, each part greater than or equal to two. For the purpose that there are k points (other than and a b) distributed on the $2\ell - 1$ disjoint paths (one path consists simply of the edge ab). We then have

$$p_{n-\ell,2}(i)\cdot p_{m-\ell,2}(j)\cdot p_{2\ell-1}(k)\cdot c_t, \quad i+k+j+t=p-2$$

such graphs G which realize F_p. Hence the total number is

$$(*) \quad \sum_{i+j+k+t=p-2} p_{n-\ell,2}(i)\cdot p_{m-\ell,2}(j)\cdot p_{2\ell-1}(k)\cdot c_t$$

where $0 \leq \ell \leq n$ and for the sake of consistency we have set $p_{-1}(k) = p_{0,2}(k) = 1$ for all k.

Case (b). Arguing the same way the number of realizations is

$$(*) \quad \sum_{i+j+k+t=p-2} p_{n-\ell,2}(i)\cdot p_{m-\ell,2}(j)\cdot p_{2\ell}(k)\cdot c_t$$

Next we set

$$P_{t,2} = P_{t,2}(x) = \sum_{i=0}^{\infty} p_{t,2}(i)\cdot x^i$$

$$P_t = P_t(x) = \sum_{i=0}^{\infty} P_t(i)x^i$$

Then letting g_p^{ℓ} be the sum of (*) and ($\ast\atop\ast$), multiplying both sides by x^p and summing for $p \geq 2$ we have

$$\sum_{p=2}^{\infty} g_p^{\ell}\, x^p = \sum_{p=2}^{\infty} \sum_{i+j+k+t=p-2} p_{n-\ell,t}^{(i)} \cdot p_{m-\ell,t}^{(j)}\; c_t(P_{2\ell}(k) + P_{2\ell-1}(k))\; x^p.$$

Setting $g_0^s = g_1^s = 0$ we have

$$(\overset{*}{\underset{*}{*}}) \quad \sum_{p=0}^{\infty} g_p^{\ell} x^p = x^2\, P_{n-\ell,2}\, P_{m-\ell,2}\, C\, [P_{2\ell} + P_{2\ell-1}]$$

<u>Proposition 3.8</u>. Let $F_p = (2m,2n,2,\cdots,2)$, $m > n \geq 2$. If $|F_p| = g_p$ the o.g.f. for $<g_{p_n}>$ is

$$x^2 \cdot \sum_{\ell=0}^{n} P_{n-\ell,2}\, P_{m-\ell,2} \cdot C \cdot [P_{2\ell} + P_{2\ell-1}].$$

<u>Proof</u>. The formula follows from ($\overset{*}{\underset{*}{*}}$) upon summing for $\ell = 0,\cdots, n$ where ℓ denoted fact that there were 2 edge disjoint paths joining a and b.

<u>Remark</u>. It is of interest to note that

$$P_{2s} = x^{2s}/(1 - x)\circ\cdots\circ(1-x^{2s})$$

$P_{m-\ell,2} = x^{2(m-s)}/(1-x)\circ\cdots\circ(1-x^{m-s})$ and that P_{n-s} has a similar expression. In spite of this proposition 3.8 does not admit of an obvious simplification. However it does follow that the o.g.f's in the previous proposition have form

$$P(x)\cdot[r(x)/s(x)]$$

where P is the o.g.f. for partitions, and r and s are polynomials.

Other sequences whose realizations can be enumerated without great difficulty are $(2,\cdots,2,1,\cdots,1)$ and $(2m+1, 2n+1,2,\cdots,2)$. Lastly one should note the relative complexity of the o.g.f. for relatively simple sequences.

B. Important Special Cases

In this section we give the o.g.f. for realizations of sequences which have a much different structure than those of Section A. The results will then be applied to the problem of finding bounds on the number of realizations of a graphical sequence.

Proposition 3.9. Let s_p be a sequence of length p^2 such that $d_1 =\cdots= d_p = p-1$ and $d_{p+1} =\cdots= d_{p^2} = 1$. Then the o.g.f. of $\langle|s_p|\rangle$ is the o.g.f. for the number of graphs on p points.

Proof. $K_{1,p-1}\cup\cdots\cup K_{1,p-1}$ is a realization of s_p. Let $v_1,\cdots,v_p$ be the centers of p $K_{1,p-1}$. Let G be any graph with $\{v_1',\cdots,v_p'\} = V(G)$. We construct an isomorphic copy of G on the centers $v_1,\cdots,v_p$ in $K_{1,p-1}\cdots K_{1,p-1}$ as follows. If $v_i'v_j' \in E(G)$ choose v_i^k, $v_j \in V(K_{1,p-1}\cdots K_{1,p-1})$ so that $v_i^k v_i$, $v_j^\ell v_j$ are $E(K_{1,p-1}\cup\cdots\cup K_{1,p-1})$. Then transfer $v_i^k v_i$ and $v_j^\ell v_j$ for $v_i v_j$ and $v_i^k v_j^\ell$. Since deg v_i = deg v_j^ℓ = p-1 the choice of v_i^k and v_j is always possible. After such a transfer is made for each e E(G) the resulting graph H has an isomorphic copy of G on $\{v_1,\cdots,v_p\}$ (or $<\{v_1,\cdots v_p\}> \cong G$); if deg $v_i' = t$ then in H, v_i is also adjacent to p - 1 - t vertices of degree one and, finally, there are $|E(G)|$ components of H which are K_2 graphs.

By Theorem 1.1 any realization of s_p can be obtained from $K_{1,p-1}\cup\cdots\cup K_{1,p-1}$ by a finite sequence of transfers. This implies that if H is such a realization then it consists of a graph on p points, $H' = <\{v_1,\cdots,v_p\}>$, where deg $v_i = k$ *in H'* implies v_i is adjacent to p - 1 - k points of degree one and H has $|EH'|$ components are K_2. Note that the realization H of s_p uniquely determines the H' uniquely and vice versa.

The above means that there is a one-to-one correspondence between realizations of s_p and graphs on p points. This gives the result.

Corollary 3.10. Let s be a sequence of k·p parts such that $d_1 = \cdots = d_p = k, d_{p+1} = \cdots = d_p = 1, 1 \leq k \leq p-1$. Then the number of realizations of s is the number of graphs on p points where each point has degree less than or equal to k (all p point graphs G with $\Delta(G) \leq k$).

Proof. The proof uses the same technique as Proposition 3.9.

Remark. The above proofs go through only because the star graphs used have the same size. Otherwise the graphs obtained from transfers are not uniquely determined by the adjacency of the centers of the star graph.

Proposition 3.11. Let s_p be a sequence on $p^2(p-1)+p$ parts such that

$$d_1 = (2p-1)(p-1), d_2 = (2(p-1)-1)(p-1), \cdots, d_p = 1\cdot(p-1)$$

and $d_{p+1} = \cdots = d_{p^2(p+1)+p} = 1$. Then the number of realizations of s_p is $2^{\binom{p}{2}}$, the number of labelled graphs on p points.

Proof. One realization of s_p consists of the p star graphs $K_{1,(2p-1)(p-1)}, \cdots, K_{1,p-1}$. Any other realization of s_p can be obtained by joining the centers of two of the above stars and then detaching two points of degree one, one each from the centers of the chosen stars and joining them--the same procedure given in 3.9. However because of the differing degrees of the centers of stars it follows that graph determined by the centers is a *labelled* graph, the labelling induced by the differing degrees of the centers. Again it can be verified that there is a one-to-one correspondence between labelled graphs on p points and realizations of s_p.

Remark. The above proposition says that the o.g.f. of s_p is

$$\sum_{i=0}^{\infty} 2^{\binom{i}{2}} x^i$$

The latter is of little interest since it converges only for $x = 0$ and consequently has no closed form. However the fact that there are 2 realizations of s_p will be used in the next section.

C. Bounds for the Number of Realizations of Graphical Sequences

We now apply some of the results of the previous two sections to the problem of estimating the number of realizations of a graphical sequence.

Lemma 3.12. Given a constant C there exists a natural number p for which the number of connected realizations of the p-part sequence $(2t,2,\cdots,2)$ is greater than Cp for at least C positive intergers t.

<u>Proof</u>. Given C choose p so that both $p/4 > C + 3$ and $p^2/48 - 1 > C \cdot p$ are satisfied.

By Lemma 3.6 the number of connected realizations of the p-part $(2t, 2, \cdots, 2)$ is the coefficient a_p of x^p in $x^{2t+1}/(1-x)\cdots(1-x^t)$. But the latter is also the o.g.f. of the number of partitions of $p - 1 - 2t$ into parts of t or less. Assume $3 \leq t \leq C+3$ and let $p_t = p - 1 - 2t$. Then the number of partitions of p_t into parts no greater than t is the same as the number of parititons of p_t into t or less parts, $t \geq 3$. Then by the way p was chosen we have

$$p/4 > C + 3 \geq t \geq 3$$

Hence $p/4 > t$ and this in turn implies that

$$(*) \quad p_t = (p = 1) - 2t \geq p/2.$$

Now if $p_3(n)$ denotes the number of partitions of n into three parts we have by Hall [5, p.30] that

$$(\overset{*}{*}) \quad p_3(n) > n^2/12 - 1.$$

Since the number of partitions of p_t into t or fewer parts, a_p, is greater than or equal to $p_3(p_t)$ we have

$$\begin{aligned} a_p &\geq p_3(p_t) \\ &\geq p_3(p/2), \text{ by } (*) \\ &> (p/2)^2/12 - 1 \text{ by } (\overset{*}{*}) \\ &> p^2/48 - 1 \end{aligned}$$

$$> C \cdot p \quad \text{by choice of } p.$$

Hence $a_p > C \cdot p$ for $t = 3, \cdots, C+3$. This yields the lemma.

Proposition 3.13. Let C be a natural number. There is a natural number p for which there is a graphical sequence s on p-parts with $|s| > p^C$.

Proof. Let C be given. By Lemma 3.12 there is a p_0 such that the number of connected realizations of the p_0-part $(2t, 2, \cdots, 2)$ is greater than $C \cdot p_0$ for at least C distinct values of t: $t_1, \cdots, t_c$. Now consider the graphs on $p_o \cdot C$ points obtained by taking the union of C connected graphs--one for each $(2t_i, 2, \cdots, 2)$, $i = 1, \cdots, C$. It is clear that these are more than $(C\ p_0)^C$ non-isomorphic ways of constructing such a union. Moreover each such union is a realization of the p_0C - part sequence $(2t_c, \cdots, 2t_1, 2, \cdots, 2)$. Hence we take s to be the latter sequence and $p = P_0C$ to obtain the result.

Corollary 3.14. For a given integer C there exists an integer p and a sequence s of length p such that the number of connected realizations, s_c, of s exceeds p^c.

Proof. Applying the previous proposition to 2c we get an m-part sequence which is realized by a union of 2c roses with $t_{2c}, \cdots, t_2$ and t_1 petals. To each such realization adjoin a new point adjacent to the points of degree $2t_{2c}, \cdots, 2t_1$. Hence there is an m+1 part sequence with more than m^{2c} distinct connected realizations. But for $m \geq 2$, $m^{2c} = (m^2)^c > (m+1)^c$. Taking $p = m+1$ gives the result.

Lemma 3.15. For $p \geq 6$ we have $\binom{p}{2}^2 > p^3 - p^2 + p$.

Proof. If the left hand side of above is squared out we get

$$\frac{1}{4}\ p^4 - \frac{1}{2}p^3 + \frac{1}{4}p^2 > p^3 - p^2 + p$$

or $p^3 - 6p^2 + 5p - 4 > 0$. Or $p(p - 5)(p - 1) - 4 > 0$. Thus the lemma is equivalent to the latter being true and if $p \geq 6$ the latter is trivially true.

Proposition 3.16. Let n be a positive integer of the form $p^3 - p^2 + p$, $p \geq 6$. There is a sequence of length n which has more than $2^{\sqrt{n}}$ realizations.

Proof. By Proposition 3.11 there is a sequence on $n = p^2(p - 1) + p = p^3 - p^2 + p$ parts which has $2^{\binom{p}{2}}$ realizations. By the lemma

$$\frac{1}{2}\ (p^2-p)^{\ 2} > p^3 - p^2 + p = n$$

or

$$\binom{p}{2} > p^3 - p^2 + p \ = \sqrt{n}$$

This gives the result.

The meaning of propositions 3.13 and 3.16 is not only that there are sequences with an abritrarily large number of realizations but also that the 'order' of largeness is itself large. Again, very roughly, there can be no useful bound b_n for an arbitrary positive integer n such that $|s| < b_n$ for sequence s on n parts.

Turning in another direction we look briefly at another question raised by Senior [10] in 1951: For which n does there exist a graphical sequence s such that $|s| = n$, such that s has n realizations. Senior raised the question with respect to multigraphs with loops and found no solution. The status of the problem has remained essentially the same and has not been raised with respect to ordinary graphs. It is conjectured here that the equation $|s| = n$ has solutions for all $n = 1,2,\cdots$. First note that the conjecture has been verified for $n = 1,\cdots,8$. We now give a few simple results in the direction of the conjecture.

Proposition 3.17. Let n be a positive integer. If there is one graphical sequence $s = (s_1,\cdots,s_p)$ such that

$$(*) \quad |s| = n$$

then there are an infinite number of graphical sequences which also satisfy (*).

Proof. If s satisfies (*) then by Proposition 2.4 the sequence $s' = (p, s_1+1,\cdots,s_p+1)$, also satisfies (*). Obviously this process of 'coming' can be continued indefinitely so that (*) must have an infinite number of solutions.

Notation. If s is a graphical sequence let $|s|$ denote the number of connected realizations of s.

Proposition 3.18. For any positive integer n the equation $|s|_c = n$ has a solution.

Proof. Consider the proof of 3.3. It was proven there that if f_n denotes the number of 'figure eights' on n-points then the o.g.f. for $\langle f_n \rangle$ is $x^5/(1-x)(1-x^2)$. It is easily verified that $f_n = \frac{n-5}{2} + 1$ if $n \geq 5$ and $fn = 0$ for $1 \leq n \leq 4$. Now notice that f_n assumes all non-negative integer values.

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GRAPHICAL SEQUENCES

by

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(ABSTRACT)

The motivating idea behind the thesis is the study of the relationship of the degrees of the vertices of a graph and the structure of a graph.

To each graph (indirected, without multiple edges) one can associate a *graphical sequence* by arranging the degrees of the vertices in their natural order. Conversely, an arbitrary sequence of numbers is *graphical* if it is a graphical sequence for some graph.

In the first chapter general properties of graphical sequences are studied. We give conditions under which a sequence can be lengthened or shortened and have the property of being graphical be preserved. The concept of a 'transfer' is introduced to show how all realizations of a graphical sequence can be obtained from a given realization. Also in chapter one we show how graphical sequences can be used characterize concepts like 'connected', 'block' and 'arbitrarily traceable'.

If a graphical sequence has one 'realization' up to isomorphism then the sequence and the graph are called *simple*. Since simple graphs are determined up to isomorphism by the degrees of the vertices it is hoped that simple graphs will reveal in some measure the effect of the degree sequence on the structure of a graph.

Thus, in chapter two, we attempt to characterize simple graphs--the

central problem of the thesis. Simple trees, simple disconnected graphs, and simple graphs with cut points and no pendant vertices are characterized. (This means that characterizing simple blocks will solve the problem). Probably the most useful result is that a connected, simple graph must be of radius ≤ two and diameter ≤ three

The third chapter is devoted to the problem of counting the number of non-isomorphic realizations of a given graphical sequence. Generating functions are used and several interesting special cases are given. These latter are in turn used to establish certain bounds on the number of realizations for sequences of a given length.