Estimating Partial Group Delay

by

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(ABSTRACT)

Partial group delay is a spectral parameter, which measures the time lag between two time series in a system after the spurious effects of the other series in the system have been eliminated.

For weakly-stationary processes, estimators for partial group delay are proposed based on indirect and direct approaches. Conditions for weak consistency and asymptotic normality of the proposed estimators are obtained. Applications to a multiple test of partial group delay are investigated.

The time lag interpretation of partial group delay is justified, which provides insight into the nature of linear relationships among weakly-stationary processes.

Extensions are made to group delay estimation and partial group delay estimation for non-stationary "oscillatory" processes.

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1.0 INTRODUCTION

The group delay at frequency λ between two univariate time series represents the time lag between the harmonic component at that frequency of the first series behind the corresponding harmonic component of the second series. Three examples follow.

Think of a furnace where a time series of fuel input observations is a "leading indicator" of the time series of furnace temperature readings. The group delay between these series measures the time lag between changes in the input series and corresponding changes in the temperature output series.

Two univariate series correspond to two spatially separated recorders. Each recorder receives a signal plus noise which is independent of the signal. The noises between the two recorders are also independent. The speed of propagation of the signal might be frequency dependent, as in a wave transmitted through a dispersive medium. Then the group delay between these two series will measure the lag at a certain frequency, in the receipt of the signal at the second site as compared to its reception at the first (see Hannan and Thomson (1973))

The simplest case of group delay arises when $x_1(t)=z(t)+u(t)$ and $x_2(t)=z(t+\tau)+v(t)$, where $u(t)$ and $v(t)$

are mutually independent noise components. In this case the group delay identically equals τ. Thus it seems that group delay is physically meaningful, since it has the direct interpretation as a time lag parameter.

The estimation of group delay is treated in Akaike and Yamanouchi (1963), Cleveland and Parzen (1975) and in Hannan and Thomson (1971, 1973). Foutz (1980) extended the procedure of Hannan and Thomson (1973) to the case of two multiple time series.

Deaton (1979) extended the concept of group delay to multivariate time series by defining the partial group delay between two series within a system of three or more time series. For example, an economic system might be monitored by an interest rate time series, a time series of stock prices, a series of unemployment rates,...etc. Partial group delay measures the time lag between two of the series in the system after the spurious effects of the other series in the system have been eliminated. Thus, if an apparent time lag between two series is due solely to a strong relationship between each of the two series to a third series in the system, then the partial group delay will be zero.

No statistical estimators for partial group delay are available in the literature. The aim of this research is to develop procedures for estimating partial group delay.

Chapter 2 introduces concepts from spectral analysis. The mathematical definitions of group delay and partial group delay are given. Important results from the literature are described for use in later chapters.

Chapter 3 suggests an "indirect approach" for estimating partial group delay. Weak consistency and asymptotic normality are obtained in this case. As an application, multiple tests for partial group delay are considered. For this, the asymptotic covariance matrix for a vector of partial group delay estimators is derived.

Chapter 4 introduces a "direct approach" for estimating partial group delay. Limiting properties of the direct approach, such as weak consistency and asymptotic distribution, are obtained.

Chapter 5 gives a numerical example to illustrate the estimation procedure.

Chapter 6 considers partial group delay and time lag relationships among multiple time series. Extensions of the result of Deaton and Foutz (1980) to a multiple time series are obtained.

Chapter 7 considers extensions of the concepts of group delay and partial group delay to non-stationary processes.

Chapter 8 gives a brief summary.

2.0 PRELIMINARIES

2.1 INTRODUCTION TO SPECTRAL ANALYSIS

A stochastic process $X=\{x(t);\ t\varepsilon T\}$ is said to be weakly stationary if

(1) $E[x(t)]=\mu$ for all $t\varepsilon T$ (assume throughout that $\mu=0$),

(2) $E[x(t+s)\ x(t)]=R(s)$ for all $t\varepsilon T$, $t+s\varepsilon T$

(3) $Var[x(t)]=R(0)<\infty$.

Similarly, a multivariate process $X=\{[x_1(t)\ldots x_p(t)]'\ t\varepsilon T\}$ is said to be weakly stationary if (1)——(3) hold for each component x_i and

(4) $E[x_i(t+s)\ x_j(s)]=R_{ij}(s)$ for all $i,j=1,\ldots p$
for all $t,t+s\varepsilon T$.

If the index set T is countable, the process is called a discrete process. When discussing discrete parameter processes, T is taken to be the set of integers. If T is uncountable, the process is called a continuous parameter process, and T is taken to be the real line R. In this research, we will assume that all processes are continuous time processes but observations on all processes are made at $t=1,...,N$.

If X is a p-dimensional zero-mean stationary process, then X has a spectral representation of the form (Rozanov, 1967 p.18):

$$X(t)=\int_{-\infty}^{\infty} e^{i\lambda t} dZ_x(\lambda) \qquad \text{for all } t\varepsilon(-\infty, \infty). \tag{2.1.1}$$

The process $Z_x(\lambda)$ is a p-dimensional process with orthogonal increments and is called the random spectral measure of the processes X.

The above representation affords a decomposition of X into frequency components. The component of X for frequency λ in a band Λ is

$$X_\Lambda(t)=\int_\Lambda e^{i\lambda t} dZ_x(\lambda). \tag{2.1.2}$$

Let $Z_x^*(\lambda)$ be the conjugate transpose of $Z_x(\lambda)=[Z_1(\lambda),...Z_p(\lambda)]'$ given in (2.1.2). Define the pXp matrix by

$$F(\lambda)=E[Z_x(\lambda)\ Z_x^*(\lambda)]=\{F_{jk}(\lambda)\} \qquad j,k=1,...p,$$

It can be shown that

$$E[x_j(t+s)\ x_k(t)]=\int_{-\infty}^{\infty} e^{i\lambda s} dF_{jk}(\lambda) \qquad j,k=1,...p,$$

(see Koopmans, 1974, p.43).

The matrix $F(\lambda)$ is called the spectral distribution matrix of X. If each element in $F(\lambda)$ is absolutely continuous with respect to the Lebesgue measure on R, then the Radon-Nikodym derivative of F with respect to the Lebesgue measure is a matrix function $f(\lambda)$ which is called the spectral density matrix of X. Write $f(\lambda)=[f_{ij}(\lambda)]$. We call $f_{ii}(\lambda)$ the auto-spectral density or spectral density of x_i. Sometimes we denote it by $f_i(\lambda)$. We call $f_{ij}(\lambda)$ the cross-spectral density of x_i and x_j when $i \neq j$.

An important cross-spectral parameter is the group delay parameter. To define this parameter, for simplicity, we assume p=2. The function $f_{12}(\lambda)/[f_{11}(\lambda)\ f_{22}(\lambda)]^{1/2}$ is written in polar form as follows

$$f_{12}(\lambda)/[f_{11}(\lambda)\ f_{22}(\lambda)]^{1/2}=\sigma(\lambda)e^{i\theta(\lambda)} \tag{2.1.3}$$

We call $\sigma(\lambda)$ the coherence spectrum between x_1 and x_2 and $\theta(\lambda)$ the phase spectrum between x_1 and x_2.

Under the assumption that $\theta(\lambda)$ is differentiable , the group delay, denoted by $\tau(\lambda)$, is defined to be the phase derivative. That is $\tau(\lambda) = d\theta(\lambda)/d\lambda$.

If $\theta(\lambda)=\tau\lambda$, for all λ and $g(\lambda)$ is a constant for all λ, then it is known that (see Deaton, 1979, p.43)

$$x_2(t)=gx_1(t-\tau)+\varepsilon(t),$$

where $\varepsilon(t)$ is orthogonal to the linear space spanned by $\{x(t)\}$. Thus for this special case, τ is the time lag of $x_1(t)$ behind $x_2(t)$.

2.2 LINEAR FILTERS AND MULTIVARIATE CORRELATION ANALYSIS

2.2.1 LINEAR FILTERS

A linear filter is a time-invariant linear transform. It transforms time series into new time series where the term "time series" can be interpreted in the broadest sense as meaning any numerical function of time whether continuous or discrete, random or nonrandom. A linear filter can be defined in the following way: A linear filter L transforms a time series $\{x(t)\}$, the input, into an output time series $\{y(t)\}$,

$$\{y(t)\}=L(\{x(t)\}),$$

where the transform L has the following properties:

(1) scale preservation:

$$L(\alpha\{x(t)\})=\alpha L(\{x(t)\}) \text{ for any complex constant } \alpha$$

(2) the superposition principle:

$$L(\{x(t)\}+\{y(t)\})=L(\{x(t)\})+L(\{y(t)\});$$

(3) time invariance: If $L(\{x(t)\})=\{y(t)\}$, then

$$L(\{x(t+h)\})=\{y(t+h)\}$$

for every number h, where $\{x(t+h)\}$ and $\{y(t+h)\}$ are the time series whose values at time t are $x(t+h)$ and $y(t+h)$, i.e., the time series obtained from $\{x(t)\}$ and $\{y(t)\}$ by shifting the time origin by the amount h.

From Koopmans (1974) p.82-83, we know that each linear filter L transforms $e^{i\lambda t}$ back into $e^{i\lambda t}$ multiplied by a factor $B(\lambda)$, i.e.,

$$L(e^{i\lambda t})=B(\lambda)e^{i\lambda t}.$$

$B(\lambda)$ is called the transfer function of the filter.

For weakly-stationary stochastic process $\{x(t)\}$ the spectral representation (2.1.1) yields

$$x(t)=\int e^{i\lambda t}dZ(\lambda).$$

Then from Koopmans (1974) p.86,

$$y(t)=L(x(t))=\int e^{i\lambda t}B(\lambda)dZ(\lambda) \qquad (2.2.1.1)$$

and

$$dF_y(\lambda)=|B(\lambda)|^2 dF_x(\lambda), \qquad (2.2.1.2)$$

where $F_y(\lambda)$ and $F_x(\lambda)$ are the spectral distribution function of $\{y(t)\}$ and $\{x(t)\}$. (see Koopmans, 1974, p.86)

Thus for the spectral density function we have

$$f_{yy}(\lambda)=|B(\lambda)|^2 f_{xx}(\lambda). \qquad (2.2.1.3)$$

Further if $x_1(t)$ and $x_2(t)$ are independently passed through linear filters with transfer function $B_1(\lambda)$ and $B_2(\lambda)$, and if $y_1(t)$ and $y_2(t)$ are the outputs, then for the cross-spectral densities $f_{x_1x_2}(\lambda)$ and $f_{y_1y_2}(\lambda)$ we have

$$f_{y_1y_2}(\lambda)=B_1(\lambda)B_2(\lambda)^c f_{x_1x_2}(\lambda). \qquad (2.2.1.4)$$

(see Koopmans, 1974, p.138),

We use "c" to denote the complex conjugate throughout the whole thesis.

If we write $f_{xy}(\lambda)/f_{xx}(\lambda)$ in polar form, i.e.,

$$f_{xy}(\lambda)/f_{xx}(\lambda)=g(\lambda)e^{i\theta(\lambda)},$$

then $g(\lambda)$ is called the gain function of the filter.

2.2.2 MULTIVARIATE CORRELATION ANALYSIS

If $X(t)$ is a multivariate weakly-stationary process with $p>2$ components, then it is often important to account for the interaction among several of the components when the association between two of the components is to be assessed.

Here we account for the influence of components $x_{m_1}(t)$, $x_{m_2}(t),\dots,x_{m_q}(t)$ on $x_j(t)$ by constructing the linear function of these series which best approximates $x_j(t)$. That is if

$$y_j(t)=L(x_{m_1}(t),\dots,x_{m_q}(t)) \qquad (2.2.2.1)$$

is a multivariate linear filter with q inputs and one output, then the filter which best approximates $x_j(t)$ is one which minimizes

$$E[x_j(t)-y_j(t)]^2. \qquad (2.2.2.2)$$

Let $x^{\neg}_{j.m}(t)$ be the output of the minimizing filter where $m=(m_1,m_2,\dots,m_q)$. Then (see Koopmans, 1974, p.153) the partial coherence is the coefficient of coherence of the residual processes $x_j(t)-x^{\neg}_{j.m}(t)$ and $x_k(t)-x^{\neg}_{k.m}(t)$.

Using the spectral representation (2.1.1)

$$x_{m_r}(t)=\int_{-\infty}^{\infty}e^{it\lambda}dZ^{x}_{m_r}(\lambda) \qquad r=1,\dots,q.$$

From Koopmans, 1974, p.130), we can write

$$y_j(t)=\Sigma_{r=1}^{q}\int e^{i\lambda t}B_{j,r}(\lambda)dZ_{m_r}^{x}(\lambda) \qquad (2.2.2.3)$$

where $B_{j,r}(\lambda)$ is the transfer function of the linear filter from $x_{m_r}(t)$ to $y_j(t)$.
Then for the spectral measure of $y_j(t)$,

$$dZ_j^y(\lambda)=\Sigma_{r=1}^{q}B_{j,r}(\lambda)dZ_{m_r}^{x}(\lambda)$$

where the 1Xq vector $B_j(\lambda)=[B_{jr}(\lambda)]$ is the transfer function of L. Thus (2.2.2.2) can be written as

$$\begin{aligned}&E[x_j(t)-y_j(t)]^2\\ &=\int E|dZ_j^x(\lambda)-\Sigma_{r=1}^{q}B_{jr}(\lambda)dZ_{m_r}^{x}(\lambda)|^2.\end{aligned} \qquad (2.2.2.4)$$

Thus, the minimum of this expression occurs at the transfer function $B_j^{\neg}(\lambda)$ which determines the projection of $dZ_j^x(\lambda)$ on the linear subspace generated by $dZ_{m_1}^x(\lambda),\ldots,dZ_{m_q}^x(\lambda)$ for each λ. By minimizing (2.2.2.2), we obtain (see Koopmans, 1974, p.154)

$$\Sigma_{r=1}^{q}B_{j,r}^{\neg}(\lambda)f_{m_r m_s}^{x}(\lambda)=f_{jm_s}^{x}(\lambda) \qquad (2.2.2.5)$$

$s=1,\ldots,q$, where $f^x_{m_r m_s}(\lambda)$ is the cross-spectral density of x_{m_r} and x_{m_s} and $f^x_{jm_s}(\lambda)$ is the cross-spectral density of x_j and x_{m_s}.

We can write (2.2.2.5) in vector form

$$B_j^{\neg}(\lambda)\ f^x_m(\lambda)\ = f^x_{jm}(\lambda) \qquad (2.2.2.6)$$

where $f^x_m(\lambda)$ and $f^x_{jm}(\lambda)$ are the qXq matrix and 1Xq vector of spectral densities indicated in (2.2.2.5). Thus when the inverse exists

$$B_j^{\neg}(\lambda) = f^x_{jm}(\lambda)\ f^x_m(\lambda)^{-1}. \qquad (2.2.2.7)$$

Based on (2.2.2.6) (see Koopmans, 1974, P.154-155) we can derive

$$f^{x\neg}_{j,j.m}(\lambda) = f^x_{jm}(\lambda) f^x_m(\lambda)^{-1}\ f^x_{jm}(\lambda)^*, \qquad (2.2.2.8)$$

where $f^{x\neg}_{j,j.m}(\lambda)$ is the spectral density function of $x^{\neg}_{j.m}(t)$ and $f^x_{jm}(\lambda)^*$ is the complex conjugate transpose of $f^x_{jm}(\lambda)$.

Further if we denote by $f^U_{j,k.m}(\lambda)$ the cross-spectral density of the residual process $x_j(t)-x^{\neg}_{j.m}(t)$, and $x_k(t)-x^{\neg}_{k.m}(t)$, we obtain

$$f^U_{j,k.m}(\lambda) = f^x_{jk}(\lambda) - f^x_{jm}(\lambda)^{-1}\ f^x_{km}(\lambda)^*. \qquad (2.2.2.9)$$

We also call $f^{U}_{j,k.m}(\lambda)$ the partial cross-spectral density between $x_j(t)$ and $x_k(t)$ adjusted for $x_{m_1}(t)$, $x_{m_2}(t),\ldots,x_{m_q}(t)$. The (p-q)X(p-q) matrix $f^{U}_{m}(\lambda)$ with elements given by (2.2.2.8) as j and k range over the indices $1,2,\ldots,p$ excluding $m_1,m_2,\ldots,m_q$ is called the residual spectral matrix.

2.3 CUMULANT AND CUMULANT SPECTRUM

Consider for the present a r-dimensional random vector $(y_1,\ldots,y_r)$ with $E|y_j|^r<\infty$ where y_j is real, for $j=1,\ldots,r$.

The rth order joint cumulant, $\mathrm{Cum}(y_1,\ldots,y_r)$ of $(y_1,\ldots,y_r)$ is defined as

$$\mathrm{Cum}(y_1,\ldots,y_r)=\Sigma(-1)^{p-1}(p-1)!E(\Pi_{j\varepsilon\nu_1}y_j)\ldots.E(\Pi_{j\varepsilon\nu_p}y_j)$$

where the summation extends over all partitions $(\nu_1,\ldots,\nu_p)$, $p=1,\ldots r$, of $(1,\ldots,r)$.

Now given the r vector valued time series $X(t)$, $t=0,\pm1,\ldots$ with components $x_j(t)$, $j=1,\ldots,r$ and $E[x_j(t)]$ with $E|x_j(t)|^k<\infty$, define

$$C_{j_1\ldots j_k}(t_1,\ldots,t_k)=\mathrm{Cum}(x_{j_1}(t_1),\ldots,x_{j_k}(t_k))$$
$$=C_{x_{j1}\ldots x_{jk}}(t_1,\ldots,t_k)$$

for $j_1\ldots j_k=1,\ldots,r$ and $t_1,\ldots,t_k=0,\pm1,\ldots$

Such a function is called a joint cumulant function of order k of the series t=0,±1,....

Suppose that the series x(t), t=0,±1,... is stationary and that its span of dependence is small enough that

$$\Sigma|C_{j_1,\ldots,j_k}(u_1,\ldots,u_{k-1})|<\infty,$$

where the summation is over all $u_1,\ldots,u_k$. Then the kth order cumulant spectrum

$f_{j_1,\ldots,j_k}(\lambda_1,\ldots\lambda_{k-1})\equiv f_{x_{j1},x_{jk}}(\lambda_1,\ldots,\lambda_{k-1})$ is defined by

$$C_{j_1,\ldots,j_k}(u_1,\ldots,u_{k-1})$$
$$=\int_{-\pi}^{\pi}\ldots\int_{-\pi}^{\pi}\exp\{i\Sigma_{j=1}^{k-1}\lambda_j\mu_j\}f_{j_1\ldots j_k}(\lambda_1,\ldots,\lambda_{k-1})d\lambda_1\ldots d\lambda_{k-1}.$$

2.4 RESULTS FROM THE LITERATURE

2.4.1

Deaton and Foutz (1980) showed that whenever the phase $\theta(\lambda)$ in (2.1.3) is not linear or the gain $g(\lambda)$ in (2.1.4) is not constant for all λ, the group delay can still be thought of as the lead time of the frequency components of x_1 over those of x_2. Let Λ be the frequency band, $d(\lambda)$ the length of the interval Λ, and let $\|x\|$ be the standard deviation of the random variable x. Then the following theorem holds.

THEOREM 2.4.1.1 (Deaton and Foutz, 1980)

Let $X=\{(x_1(t),x_2(t)\};\ -\infty<t<\infty\}$ be a bivariate weakly-stationary stochastic process with mean zero, having absolutely continuous spectrum, and spectral densities $f_{11}(\lambda)$, $f_{22}(\lambda)$, and $f_{12}(\lambda)$ that are nonzero and boundedly differentiable in a frequency band Λ containing the frequency λ_o. Then with $\tau(\lambda)$ the group delay of x_1 behind x_2, and with $\alpha(\lambda)=\exp[i\lambda\tau(\lambda)]f_{12}(\lambda)/f_{22}(\lambda)$, there exist zero-mean weakly-stationary stochastic processes ε_Λ and O_Λ that give a presentation of the frequency component $x_{1\Lambda}$ as

$$x_{1\Lambda}(t)=\alpha(\lambda_o)x_{2\Lambda}(t-\tau(\lambda_o))+\varepsilon_\Lambda(t)+O_\Lambda(t)\quad(-\infty<t<\infty)\qquad(2.4.1.1)$$

where the process ε_Λ is uncorrolated with each of the processes $x_{2\Lambda}$ and O_Λ. When Λ is a narrow frequency band with band length $d(\lambda)$, and for $\sigma(\lambda)$ the coherence between x_1 and x_2, the relative importance of the terms in (2.4.1.1) is indicated by the limits

$$\lim_{d(\Lambda)\to 0}\|O_\Lambda(t)\|/\|x_{1\Lambda}(t)\|=0;\qquad(2.4.1.2)$$

$$\lim_{d(\Lambda)\to 0}\|\alpha(\lambda_o)x_{2\Lambda}(t-\tau(\lambda_o))\|/\|x_{1\Lambda}(t)\|=\sigma(\lambda_o);\qquad(2.4.1.3)$$

$$\lim_{d(\Lambda)\to 0}\|\varepsilon_\Lambda(t)\|/\|x_{1\Lambda}(t)\|=\{1-\sigma^2(\lambda_o)\}^{1/2}.\qquad(2.4.1.4)$$

In the proof of the theorem, $O_\Lambda(t)$ is defined as $O_\Lambda(t)=I_1(t)+I_2(t)$, where,

$$I_1(t)=\int_\Lambda \alpha(\lambda_o)\exp\{i\lambda[t-\tau(\lambda_o)]\}$$
$$[\exp\{-i(\tau(\lambda^{**})-\tau(\lambda_o)(\lambda-\lambda_o)\}-1]dz_2(\lambda), \qquad (2.4.1.5)$$
$$I_2(t)=\int_\Lambda (d|g(\lambda)|/d\lambda|_{\lambda^*})(\lambda-\lambda_o)$$
$$\exp[-i\{\theta(\lambda_o)-\lambda_o\tau(\lambda_o)+\lambda\tau(\lambda_o)$$
$$+(\tau(\lambda^{**})-\tau(\lambda_o)(\lambda-\lambda_o)-\lambda t\}]dz_2(t), \qquad (2.4.1.6)$$

where λ is an interior point of Λ and with $\lambda^*$and λ^{**} each between λ and λ_o, and $g(\lambda)=f_{12}(\lambda)/f_{22}(\lambda)$.

Here we notice that if $\tau(\lambda)$ and $g(\lambda)$ are constants in Λ, then $O_\Lambda(t)\equiv 0$.

From that proof, we also know

$$E[x_{1\Lambda}(t)]^2=O(d(\Lambda)) \qquad (2.4.1.7)$$
$$E[O_\Lambda(t)]^2=O[d(\Lambda)^3]. \qquad (2.4.1.8)$$

2.4.2

The results of Hannan and Robinson (1973) are applied in Chapter 3 to establish the indirect approach to estimating partial group delay. These results treat the model

$$y(t)=\alpha_o+\beta_o z(t-\theta_o)+x(t) \qquad -\infty<t<\infty,$$

where x(t) and z(t) are uncorrelated stationary processes, x(t) having zero mean and z(t) having mean μ_z.

The aim of that paper is to estimate α_o, β_o and the lead or lag θ_o on the basis of observations on $y(t)$, $z(t)$ made at the points $t=1,2,\dots,N$.

The procedure is to estimate β_o, θ_o by minimizing

$$Q_N(\beta, \theta)$$
$$=N^{-1}\Sigma' |w_y(s)-\beta\exp(i\theta\omega_s)w_z(s)|^2\phi(\omega_s). \qquad (2.4.2.1)$$

Here the function $\theta(\lambda)$ will be defined in Conditions A later and the summation is over s such that $\omega_s=2\pi s/N\varepsilon B$(which will also be defined later) but excluding ω_0, and w_y, w_z are finite Fourier transform of y and z, i.e.

$$w_y(s)=(2\pi N)^{-1/2}\Sigma_1^N y(n)e^{in\omega_s}$$
$$w_z(s)=(2\pi N)^{-1/2}\Sigma_1^N z(n)e^{in\omega_s}.$$

We denote the estimators by $\beta^{\#}$ and $\theta^{\#}$.

The limiting properties of the estimators in Hannan and Robinson (1973) are summarized in the following theorems.

To begin with, we state three sets of conditions.

(I) conditions A

(1) $x(t)$ and $z(t)$ are uncorrelated stationary processes; $x(t)$ has zero mean; and $z(t)$ has mean μ_z.

(2) The spectral densities of $f_x(\lambda)$ and $f_z(\lambda)$ are continuous.

(3) The set B lies wholly within the set

$\{\lambda \mid |\lambda|<\pi-\delta\}$, made up of a finite number of disjoint interval and symmetric about λ_o.

(4) θ_o is restricted by $|\theta_o|\le a$, $0<a<\infty$.

(5) the function $\phi(\lambda)$ is even, positive and continuous.

(II) condition (*)

$$\int_B |\beta\exp(i\theta\lambda)-\beta_o\exp(i\theta_o\lambda)|^2\phi(\lambda)f_z(\lambda)>0, \qquad (\beta,\ \theta)\ne(\beta_o,\ \theta_o).$$

(III) conditions B

1) $x(t)$ satisfies a strong mixing condition (Rozanov, 1967, p.80, formula (9.6)). A stationary process X satisfies the strong mixing condition if

$$\alpha(\tau)=\sup_{A\varepsilon U_{-\infty}^{t},\,B\varepsilon U_{t+\tau}^{\infty}} |P(AB)-P(A)P(B)|\to 0,$$

as $\tau\to\infty$. Here U_u^v denotes the σ-algebra of ω sets generated by the variables $x_j(t)$ $u\le t\le v$, i.e., U_u^v is the algebra generated by sets of the form

$$\{\omega:\ x_{k_1}(t_1)\varepsilon\Gamma_1,\ldots,x_{k_N}(t_N)\varepsilon\Gamma_N\}$$

where $u\le t_k\le v$ for $k=1,\ldots N$, $\Gamma_1,\ldots\Gamma_N$ are intervals in R^1. This condition means that, in the course of time, events concerning the "future" of the process become almost independent of events in the past.

2) If $C_x(p,q,r)$ is the fourth cumulant between $x(m)$, $x(m+p)$, $x(m+q)$, $x(m+r)$, then

$$\Sigma_p \Sigma_q \Sigma_r |C_x(p,q,r)| < \infty .$$

3) $z(t)$ has finite fourth cumulants and the fourth cumulant between $z(t_1)$, $z(t_2)$, $z(t_3)$, $z(t_4)$, namely $C_z(t_2-t_1,\ t_3-t_1,\ t_4-t_1)$, satisfies

$$\begin{aligned} &C_z(t_2-t_1, t_3-t_1, t_4-t_1) \\ &= \int\int\int\int_{\Sigma\lambda_j=0} \exp\{i\Sigma t_j\lambda_j\} f(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \Pi_j d\lambda_j , \end{aligned}$$

where f is continuous on the plane $\Sigma_j\lambda_j=0$.

4) θ_o is an interior point of the set of points $|\theta|\le a$.

THEOREM 2.4.2.1. (Hannan and Robinson, 1973)

Let conditions A and (*) be satisfied, and let $x(t)$ and $z(t)$ be ergodic. Then $\theta^{\#}$ and $\beta^{\#}$ converge almost surely to θ_o and β_o.

THEOREM 2.4.2.2. (Hannan and Robinson, 1973)

Let conditions A, B, and (*) be satisfied, then $N^{1/2}(\theta^{\#}-\theta_o)$ and $N^{1/2}(\beta^{\#}-\beta_o)$ are asymptotically independent and each is asymptotically normally distributed.

2.4.3

Hannan and Thomson (1973) propose an estimation procedure of the group delay between processes $\{x(t)\}$ and $\{y(t)\}$. The finite Fourier transform of $x_j(n)$ for $n=1,\dots,N$ and $j=1,2$ is

$$w_j(s)=(2\pi N)^{1/2}\Sigma_{n=1}^{N}x_j(n)e^{in\omega_s}\ (\omega_s=2\pi s/N,\ 0<s\leq[N/2])$$

We define

$$p^{\#}(\tau)=(1/m)\Sigma_o I_{12}(s)e^{-i\tau\omega_s}, \qquad (2.4.3.1)$$

where $I_{12}(s)=w_1(s)w_2(s)^c$. Σ_o is the summation over a band of frequencies $\omega_s=2\pi s/N$ such that

$$\omega_s\varepsilon B=\{\lambda\,|\,\lambda_o-\pi/2M<\lambda<\lambda_o+\pi/2M\}.$$

Here we have $2mM=N$, so that B contains m fundemantal frequencies ω_s.

We state the following conditions.

(I) conditions C

1) $\{x_j(t)\}$ are ergodic stationary processes for $j=1,2$ and are purely nondeterministic in the sense that they contain no component that is perfectly predictable, linearly or nonlinearly.

2) $\{x_j(t)\}$ $(j=1,2)$ have zero means and finite variances and have absolutely continuous spectra with boundedly differentable spectral densities.

3) The phase $\theta(\lambda)$ is twice boundedly differentiable and the coherence $\sigma(\lambda)$ is positive.

(II) conditions D

1) $x_j(t)$ $(j=1,2)$ have finite fourth moments.

2) Let $C_{ijkl}(m,n,p,q)$ be the fourth cumulant between $x_i(m)$, $x_j(n)$, $x_k(p)$ and $x_l(q)$ $(i,j,k,l=1,2)$ then assume C_{ijkl} is a Fourier transform, namely,

$$C_{ijkl}(m,n,p,q)$$
$$=\iiiint_{\Sigma\lambda_j=0}\exp\{i(m\lambda_1+n\lambda_2+p\lambda_3+q\lambda_4\}\ f_{ijkl}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\Pi d\lambda_j.$$

Here $|\lambda_j|\le\pi$ $(i=1,\ldots,4)$, and f is a boundedly differentable function on the plane $\Sigma\lambda_j=0$. This is basically a condition on the rate at which dependence falls off as the lag increases.

3) Let m_n be the subfield generated by $x_j(m)$ $(m\le n;\ j=1,2)$, and let $m_{-\infty}$ be the intersection of all m_n. Let $H_n=H(m_n)$ be the Hilbert space of random variables measurable with respect to m_n and of finite mean square. Consider $y_{jkp}(n)=x_j(n)x_k(n+P)$ $(j,k=1,2;\ p=0,1,\ldots,s)$.

If we arrange this as a vector of 4s+3 components, $Y(n)$, then $Y(n)$ is stationary with finite covariances and with mean vectors, γ, having components $\gamma_{jk}(p)$. Let $U_r(n)$ be the vector of projections of the components of $Y(n)$ on the orthogonal complement of H_{n-r} in H_n.
We require, following Gordin (1969), that

$$\inf_r \limsup_{N\to\infty} \| \Sigma_{n=1}^{N} \{Y(n)-\gamma-U_r(n)\} \|^2=0,$$

where by $|X|^2$ we mean $E(X^*X)$.
This condition is also a condition on the rate at which dependence falls off and is not independent of the condition on the fourth order cumulant in 2). of Conditions D.

Let $\tau^\#$ be the value of τ maximizing $q^\#(\tau)=|p^\#(\tau)|^2$. For this estimate, the following results in Hannan and Thomson (1973) will be applied in this research later.

LEMMA 2.4.3.1 (Hannan and Thomson, 1973,)

The convergence of $|p^\#(\tau)-(M/\pi)p(\tau)|\to 0$ when $N\to\infty$ and holding M fixed, is almost sure and uniform for all τ, where $p^\#(\tau)$ is defined in (2.4.3.1) and

$$p(\tau)=\int_B f_{12}(\lambda)e^{-i\tau\lambda}d\lambda. \qquad (2.4.3.2)$$

LEMMA 2.4.3.2. (Hannan and Thomson, 1973)

If $t_M(N)$ $(M=1,2,\ldots;N=1,2,\ldots)$ is a family of random variables satisfying $\lim_{N\to\infty} t_M(N)=t_M$, almost surely, and $\lim_{M\to\infty} t_M=t$, almost surely, then there is a sequence $M(N)$ increasing monotonically with N so that almost surely $\lim_{N\to\infty} t_{M(N)}(N)=t$.

THEOREM 2.4.3.3. (Hannan and Thomson, 1973)

Under the conditions C, there is a sequence M(N) increasing with N such that $\tau^{\#}$ converges almost surely to τ_o.

The main point of that proof is as follows:

Let τ_M be the value of τ maximizing $|p(\tau)|^2$ for fixed M, it is shown that when $M\to\infty$, then $\tau_M\to\tau_o$. Then by Lemma 2.4.3.1 and Lemma 2.4.3.2 $\tau^{\#}\to\tau_o$.

THEOREM 2.4.3.4. (Hannan and Thomson, 1973)

Under the conditions of Theorem 2.4.3.3 and conditions D, there is a sequence M(N) increasing with N, such that $N^{-1}m^{3/2}(\tau^{\#}-\tau_M)$ is asymptotically normally distributed with mean zero and variance $3\{1-\sigma^2(\lambda_o)\}/\{2\pi\sigma^2(\lambda_o)\}$, where τ_M is the value of τ maximizing $|p(\tau)|^2$ for fixed M.

The main steps of that proof are as follows:

1) When N is big enough and M is fixed,

$$N^{1/2}q^{\#'}(\tau_M)$$
$$=-N^{1/2}(\tau^{\#}-\tau_M)q^{\#''}(\tau^{o})$$
$$\simeq 2(M/\pi)2N^{1/2}\mathrm{Re}\{p^{\#'}(\tau_M)p(\tau_M)^{c}+p'(\tau_M)p^{\#}(\tau_M)^{c}\},$$

where $q^{\#'}(\tau)$ is the derivative of $q^{\#}(\tau)$ etc., and $|\tau^{o}-\tau_M|\le(\tau^{\#}-\tau_M)$.
From Lemma 2.4.3.1 when M is fixed

$$q^{\#''}(\tau^{o})\to(M/\pi)^2 q''(\tau_M) \text{ when } N\to\infty.$$

2) Define
$$\phi(\lambda)=(M/\pi)^2\int_B i(\theta-\lambda)e^{i\tau_M(\theta-\lambda)}f_{12}(\theta)^{c}d\theta$$

for $\lambda\varepsilon B$, and let $\phi(\lambda)$ be zero on the complement of B in $[0, \pi]$ and let $\phi(-\lambda)=\phi(\lambda)$, then

$$\int_{-\pi}^{\pi}\phi(\lambda)f_{12}(\lambda)d\lambda=0.$$

3) $N^{1/2}q^{\#'}(\tau_M)$
$$\simeq N^{1/2}\int_{-\pi}^{\pi}\phi(\lambda)[I_{12}(\lambda)-E\{I_{12}(\lambda)\}]d\lambda.$$

$E[N^{1/2}q^{\#}(\tau_M)]^2$ converges to

$$2\pi\int_{-\pi}^{\pi}\{|\phi(\lambda)|^2 f_1(\lambda)f_2(\lambda)+\phi^2(\lambda)f_{12}^2(\lambda)\}d\lambda$$
$$+2\pi\int\int_{-\pi}^{\pi}\phi(\lambda)\phi(\mu)^{c}f_{1212}(-\lambda,\lambda,\mu,-\mu)d\lambda d\mu.$$

The derivation of the above formula does not depend on any special properties of $\phi(\lambda)$ other than piecewise continuity. On B,

$$\phi(\lambda) \simeq (-i)(M/\pi)(\lambda-\lambda_o)f_{12}(\lambda_o)^c.$$

4) Let $\phi^*(\lambda)=\phi(\lambda)$ on the support of $\phi(\lambda)$, but it differs from it on a set of measure ε and is continuous and periodic. Let $\phi_s^*(\lambda)$ be the Cesaro sum to s terms of the Fourier series of $\phi^*(\lambda)$, i.e.,

$$\phi_s^*(\lambda)=\Sigma_{u=-s}^{s}(1-|u|/s)\phi^*(\lambda)e^{-iu\lambda}.$$

Choose s so that $\sup|\phi_s^*(\lambda)-\phi^*(\lambda)|<\varepsilon$.

5) Keeping M fixed, then

$$N^{1/2}\int_{-\pi}^{\pi}\phi_s^*(\lambda)[I_{12}(\lambda)-E\{I_{12}(\lambda)]d\lambda$$
$$=\Sigma_{u=-s}^{s}\delta(u)(1-|u|/s)N^{1/2}[c_{12}(u)-\gamma_{12}(u)],$$

where $\delta(u)$ is the uth Fourier coefficient of $\phi^*(\lambda)$ and

$$c_{12}(u)=N^{-1}\Sigma_{n=1}^{N-u}x_1(n)x_2(n+u)=c_{21}(-u).$$
$$\gamma_{12}(u)=E[c_{12}(u)].$$

Now the central limit theorem holds if it holds for the vector of quantities $N^{1/2}[c_{12}(u)-\gamma_{12}(u)]$; the central limit theorem holds for this vector by an obvious vector generalization of Theorem 1 of Gordin (1969). Then as

$$\Sigma_{u=-s}^{s}\delta(u)(1-|u|/s)N^{1/2}[c_{12}(u)-\gamma_{12}(u)]$$

is a linear combination of the components of the vector the asymptotic normality of $N^{1/2}q^{\#'}(\tau_M)$ holds.

6) By using Lemma 2.4.3.2, the asymptotic normality of $(m/N)^{3/2}N^{1/2}(\tau^{\#}-\tau_M)$ is obtained.

7) For the fixed M, the asymptotic variance of $N^{-1}m^{3/2}(\tau^{\#}-\tau_M)$ can be obtained based on the limit result of 1) and 3) as follows:

$$\begin{aligned}&(2M)^{-3}\{(M/\pi)^2q''(\tau_M)\}^{-2}2\pi[\int_{-\pi}^{\pi}\{|\phi(\lambda)|^2\\ &\quad f_1(\lambda)f_2(\lambda)+\phi(\lambda)^2f_{12}(\lambda)^2\}d\lambda\\ &\quad +\int\int_{-\pi}^{\pi}\phi(\lambda)\phi(\mu)^c f_{1212}(-\lambda,\lambda,\mu,-\mu)d\lambda d\mu].\end{aligned}$$

It is evident that

$$4\pi M\int\int_{-\pi}^{\pi}\phi(\lambda)\phi(\mu)^c f_{1212}(-\lambda,\lambda,\mu,-\mu)d\lambda d\mu$$

converges to zero as M increases, since the integrand is bounded and zero except on a set whose area is $O(M^{-2})$. We consider next

$$4\pi M\int_{-\pi}^{\pi}|\phi(\lambda)|^2 f_1(\lambda)f_2(\lambda)d\lambda.$$

On B

$$\phi(\lambda)\simeq(-i)(M/\pi)(\lambda-\lambda_o)f_{12}(\lambda_o)^c\{1+o(1)\}.$$

Since the $f_j(\lambda)$ $(j=1,2)$ are boundedly differentiable we may replace them in the integral by $f_j(\lambda)$. Evaluating the integral, we obtain

$$\lim_{M\to\infty}4\pi M\int_{-\pi}^{\pi}|\phi(\lambda)|^2 f_1(\lambda)f_2(\lambda)d\lambda$$
$$=(2/3)\pi^2|f_{12}(\lambda_o)|^2 f_1(\lambda_o)f_2(\lambda_o).$$

In the same way

$$\lim_{M\to\infty}4\pi M\int_{-\pi}^{\pi}\phi(\lambda)^2 f_{12}(\lambda)^2 d\lambda$$
$$=(-2/3)\pi^2|f_{12}(\lambda_o)|^4,$$
$$\lim_{M\to\infty}(4M^4/\pi^2)q''(\tau_M)=(2\pi^2/3)|f_{12}(\lambda_o)|^2.$$

Using these formulae we obtain the variance stated in the theorem.

Hannan and Thomson (1973) also consider the problem of wide band estimation. That is the case where the group delay is constant over a band B. The band shall now be held fixed when $N \to \infty$. In that case the weighting of individual frequencies is considered. Instead of $p^{\#}(\tau)$ and $p(\tau)$ in (2.4.3.1) and (2.4.3.2) we have

$$p_{\psi}^{\#}(\tau)=(1/N)\Sigma_{o}I_{12}(s)\psi(\omega_{s})e^{-i\tau\omega_{s}},$$

and

$$p_{\psi}(\tau)=(1/2\pi)\int_{B}f_{12}(\lambda)\psi(\lambda)e^{-i\tau\lambda}d\lambda.$$

Here ψ is an even real function. Choose $\tau^{\#}$ so as to maximize $|p_{\psi}(\tau)|^2$, then the following theorems are established.

THEOREM 2.4.3.5.(Hannan and Thomson, 1973)

If $\theta(\lambda)\equiv\tau_o$ $\lambda\varepsilon B$, and $|p_{\psi}^{\#}(\tau)|^2$ has a unique maximum at $\tau=\tau_o$, then under condition C, $\tau^{\#}\to\tau_o$ almost surely.

THEOREM 2.4.3.6. (Foutz, 1980)

If the conditions of Theorem 2.4.3.3 and conditions D are satisfied and

$$\exp\{-i\tau_o(\lambda-\mu)\}f_{1212}(-\lambda,\lambda,\mu,-\mu)$$

is an even function of λ and μ, then $N^{1/2}(\tau^{\#}-\tau_o)$ is asymptotically normally distributed with zero mean as N increases. The variance of the limiting distribution is bounded by

$$8\pi\int_{\pm B}|\phi^2(\lambda)f_{11}(\lambda)f_{22}(-\lambda)d\lambda \ +\int_{\pm B}\phi^2(\lambda)f_{12}^2(\lambda)d\lambda$$

$$+\{1/q''(\tau_o)\}^2\iint_{\pm B}\phi_{12}(\lambda)\phi_{12}(\mu)\ f_{1212}(-\lambda,\lambda,\mu,-\mu)d\lambda d\mu,$$

where $\phi(\lambda)=-(1/2\pi)\psi(\lambda)\exp[-i\tau_o\lambda]p_{\psi}(\tau_o)$.

2.4.4

In chapters 3 and 4, we apply the following results from Brillinger (1975)

THEOREM 2.4.4.1 (Brillinger, 1975)

Suppose $F_j(y_1,\ldots,y_m;z_1,\ldots,z_n)$ $j=1,\ldots,m$ are holomorphic functions of m+n variables in a neighborhood of $(u_1,\ldots,u_m;v_1,\ldots,v_n)\varepsilon C^{m+n}$, the m+n dimensional complex space. If $F_j(u_1,\ldots,u_m;v_1,\ldots,v_n)=0$ $j=1,\ldots,m$, while the determinant of the Jacobian matrix

$$\frac{\partial(F_1,\ldots,F_m)}{\partial(y_1,\ldots,y_m)}$$

is nonzero at $(u_1,\ldots,u_m;v_1,\ldots,v_n)$ then the equations $F_j(y_1,\ldots,y_m;z_1,\ldots,z_n)=0$ $j=1,\ldots,m$ have a unique solution $y_j=y_j(z_1,\ldots,z_n)$ $j=1,\ldots,m$ which is holomorphic in a neighborhood of $(v_1,\ldots,v_n)$.

ASSUMPTION 2.4.4.2 (Brillinger, 1975, assumption 2.6.1 and (4.3.10))

$x(t)$ is a strictly stationary r vector-valued series with components $x_j(t)$, $j=1,\ldots,r$ all of whose moments exist, and satisfy

$$\Sigma^{\infty}_{u_1,\ldots,u_{k-1}=-\infty}[1+|u_j|]\ |C_{a_1,\ldots,a_k}(u_1,\ldots u_{k-1})|<\infty$$

for $a_1,\ldots a_k=1,\ldots,r$ and $k=2,3\ldots.$ and $j=1,\ldots,k-1$.

THEOREM 2.4.4.3. (Brillinger, 1975)

Let $Y^{(N)}$, $N=1,2,\ldots$ be a sequence of r-dimensional vectors, with complex components, and such that all cumulants of the variate $[Y_1^{(N)},Y_1^{(N)^c},\ldots,Y_r^{(N)},Y_r^{(N)^c}]$ exist and tend to the corresponding cumulants of a variate $[Y_1,Y_1^c,\ldots,Y_r,Y_r^c]$ that is determined by its moments. Then $Y^{(N)}$ tends in distribution to a variate having components $Y_1,\ldots,Y_r$.

THEOREM 2.4.4.4 (Brillinger, 1975, Theorem 4.3.2 and (4.3.15) p.93)

Let $x(t)$, $t=0,\pm1,\ldots$ be a stationary r vector-valued series satisfying the condition

$$\Sigma_{u_1}\ldots\Sigma_{u_{k-1}}[1+|u_j|]C_{a_1\ldots a_k}(u_1\ldots u_{k-1})|<\infty \quad j=1,\ldots,k-1$$

where

$$C_{a_1\ldots a_k}(u_1\ldots u_{k-1}) =\mathrm{Cum}\{x_{a_1}(t+u_1),\ldots,x_{a_{k-1}}(t+u_{k-1}),x_{a_k}(t)\}.$$

Then

$$\mathrm{Cum}(w_{a_1}(\lambda_1),\ldots,w_{a_k}(\lambda)\}$$
$$=(2\pi)^{(k/2-1)}N^{-k/2}\Delta^{(N)}(\Sigma_1^k\lambda_j)f_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_{k-1})+O(N^{-k/2}),$$

where $w_{a_i}(\lambda_i)$ is the finite Fourier transform of x_{a_i} at λ_i for $i=1,\ldots,k$ and $\Delta^{(N)}(\lambda)=N$ for $\lambda\equiv 0 \pmod{2\pi}$, $\Delta^{(N)}(2\pi s/N)=0$ for s an integer with $s\equiv 0 \pmod N$. The error term is uniform in $\lambda_1,\ldots,\lambda_k$.

THEOREM 2.4.4.5. (Brillinger, 1975)

Let $x(t)$ $t=0,\pm 1,\ldots$ be an r vector value series satisfying Assumption 2.4.4.2. Let $K_j(N)$ be an integer with $\lambda_j(N)=2\pi K_j(N)/N\to\lambda_j$ as $N\to\infty$ for $j=1,\ldots,J$, suppose $2\lambda_j(N)$, $\lambda_j(N)\pm\lambda_k(N)\not\equiv 0 \pmod{2\pi}$ for $1\le j\le k\le J$. Let

$$w_x(\lambda)=(1/2\pi N)^{-1/2}\Sigma_{n=1}^{N}x(n)e^{-in\lambda} \qquad \text{for } -\infty<\lambda<\infty.$$

Then $w_x(\lambda_j)$ $j=1,\ldots,J$ are asymptotically independent $N_r^C(0,f_x(\lambda_j))$ variates respectively.

THEOREM 2.4.4.6. (Brillinger, 1975)

Let $x(t)$, $t=0\pm 1,\ldots$ be an r dimensional vector series satisfying Assumption 2.4.4.1. and

$$f_{xx}^{\#}(\lambda)=(1/[2m+1])\Sigma_{s=-m}^{m}I_{xx}(2\pi[K(N)+s\}/N)$$

be the r dimensional vector smoothed periodogram with K(N) being an integer and $2\pi K(N)/N \to \lambda$ as $N \to \infty$. Then $f^{\#}_{xx}(\lambda)$ is asymptotically distributed as $(1/[2m+1])W^{c}_{r}(2m+1, f_{xx}(\lambda))$, i.e., as $(1/[2m+1])$ times a matrix with a complex Wishart distribution of dimension r and degrees of freedom 2m+1, if $\lambda \not\equiv 0 (\mathrm{mod}\,\pi)$ and as $(1/2m)W_{r}(2m, f_{xx}(\lambda))$ if $\lambda \equiv 0 (\mathrm{mod}\,\pi)$. Also $f^{\#}_{xx}(\lambda_j)$ $j=1,\ldots J$ are asymptotically independent if $\lambda_k \pm \lambda_j \equiv 0 (\mathrm{mod}\,\pi)$ for $1 \le j < k \le J$.

Now let the (r+s) vector-valued series [X(t) Y(t)] $t=0,\pm 1,\ldots$ satisfy Assumption 2.4.4.2 and suppose its values are available for $t=1,\ldots,N$. We consider forming the matrix of statistics

$$\begin{bmatrix} f^{\#}_{xx}(\lambda) & f^{\#}_{xy}(\lambda) \\ f^{\#}_{yx}(\lambda) & f^{\#}_{yy}(\lambda) \end{bmatrix}$$

where $f^{\#}_{xy}(\lambda)$, $f^{\#}_{yx}(\lambda)$ and $f^{\#}_{yy}(\lambda)$ are defined in the same way as $f^{\#}_{xx}(\lambda)$ in Theorem 2.4.4.6. Let

$$g^{\#}_{\varepsilon}(\lambda)=[(2m+1)/(2m+1-r)][f^{\#}_{yy}(\lambda) - f^{\#}_{yx}(\lambda)f^{\#}_{xx}(\lambda)^{-1}f^{\#}_{xy}(\lambda)].$$

Then we have the following theorem.

THEOREM 2.4.4.7. (Brillinger, 1975, p.305-306)

Let the (r+s) vector-valued series $[(X(t)\ Y(t)]'$ satisfies Assumption 2.4.4.2, then $g_{\varepsilon}^{\#}(\lambda)$ is asymptotically $(2m+1-r)^{-1}W_s^c(2m+1-r, f_{\varepsilon\varepsilon}(\lambda))$ if $\lambda \not\equiv 0(\mathrm{mod}\pi)$ and asymptotically $(2m-r)^{-1}W_s(2m-r, f_{\varepsilon}(\lambda))$ if $\lambda \equiv 0(\mathrm{mod}\pi)$, where $f_{\varepsilon\varepsilon}(\lambda)=f_{yy}(\lambda)-f_{yx}(\lambda)f_{xx}(\lambda)^{-1}f_{xy}(\lambda)$.

2.4.5

The following results will be applied in Chapter 4.

THEOREM 2.4.5.1 (Billingsley, 1968)

The probability measures on (R^{∞}, B^{∞}) converge weakly if and only if all the corresponding finite-dimensional distributions converge weakly.

THEOREM 2.4.5.2 (Billingsley, 1968)

If $X_n \to X$ in distribution and $P\{X \varepsilon D_h\}=0$, where D_h is the set of discontinuties of the continuous mapping, then $h(X_n) \to h(X)$.

THEOREM 2.4.5.3. (Billingsley, 1968)

If $X_n \to X$ in distribution and $\rho(X_n, Y_n) \to 0$ in probability, then $Y_n \to X$ in distribution, where $\rho(X_n\ Y_n)$ is the distance between X_n and Y_n.

2.5 PARTIAL GROUP DELAY

Partial group delay can be defined in terms of partial phase. For simplicity, we consider the weakly-stationary processes $\{X(t)\}$,$\{Y(t)\}$ and $\{Z(t)\}$ and assume that all of these have absolutely continuous spectra with continuous densities.

Let $\{Z(t)\}$ be a series with a continuous spectrum and a spectral density, which is bounded from above and away from zero (see Koopmans (1974) (A 6.1) p.205). Then we firstly remove the influence of $\{Z(t)\}$ on $\{X(t)\}$ and $\{Y(t)\}$ by considering the processes

$$\varepsilon_x(t)=X(t)-\Sigma\ b_1(u)Z(t-u) \tag{2.5.1}$$

$$\varepsilon_y(t)=Y(t)-\Sigma\ b_2(u)Z(t-u) \tag{2.5.2}$$

where $b_1(u),b_2(u)$ are determined by minimizing $E[\varepsilon_1(t)]^2$ and $E[\varepsilon_2(t)]^2$ or alternatively, in general, by using the following Hilbert space technique. Let $H(Z)$ be the closed linear manifold generated by $\{Z(t);-\infty<t<\infty\}$ and let $\Pi(X(t)|H(z))$ and $\Pi(Y(t)|H(Z))$ be the projections of $X(t)$ and $Y(t)$ on $H(Z)$.

If $Z(t)$ has a continuous spectrum and a spectral density function which is bounded from above and away from zero (see Koopmans (A 6.1 p.205)), then the residuals are

$$\varepsilon_x(t)=X(t)-\Pi(X(t)|H(Z)) \tag{2.5.3}$$

$$\varepsilon_y(t)=Y(t)-\Pi(Y(t)|H(Z)). \tag{2.5.4}$$

Assuming that the cross-spectral density function of $\varepsilon_x(t)$ and $\varepsilon_y(t)$ is $f_{\varepsilon_x\varepsilon_y}(\lambda)$, we call $f_{\varepsilon_x\varepsilon_y}(\lambda)$ the partial cross-spectral density function of $\{X(t)\}$ and $\{Y(t)\}$ adjusted for $\{Z(t)\}$ and also denote it by $f_{xy.z}(\lambda)$.

From Koopmans (1974) (5.74) p.156,

$$f_{xy.z}(\lambda)=f_{xy}(\lambda)-f_{xz}(\lambda)f_{zz}(\lambda)^{-1}f_{zy}(\lambda). \tag{2.5.5}$$

The partial coherence is defined as

$$\sigma_{\varepsilon_x\varepsilon_y}(\lambda)=\frac{|f_{\varepsilon_x\varepsilon_y}(\lambda)|}{\{f_{\varepsilon_x}(\lambda)f_{\varepsilon_y}(\lambda)\}^{1/2}}$$

where $f_{\varepsilon_x}(\lambda)$ and $f_{\varepsilon_y}(\lambda)$ are auto-spectral density functions of ε_x and ε_y separately, and the partial phase spectrum is similarly defined as

$$\Phi_{xy.z}(\lambda)=\arg\left\{\frac{f_{\varepsilon_x\varepsilon_y}(\lambda)}{\{f_{\varepsilon_x}(\lambda)f_{\varepsilon_y}(\lambda)\}^{1/2}}\right\}.$$

Assuming that $\Phi_{xy.z}(\lambda)$ is differentiable w.r.t. λ, then the partial group delay is defined as the simple group delay between $\varepsilon_x(t)$ and $\varepsilon_y(t)$,

i.e, $$\tau_{xy.z}(\lambda)=d\Phi_{XY.z}(\lambda)/d\lambda.$$

The notion of partial cross-spectra and coherence were introduced by Tick (1963), and developed by Koopmans (1964, 1974), Akaike (1965), Goodman (1965) and Parzen (1967). Related material can also be found in Priestley (1981).

3.0 ESTIMATING PARTIAL GROUP DELAY ——INDIRECT APPROACH

3.1 ESTIMATING PROCEDURE

To simplify the problem, we assume that the given processes X(t), Y(t) and Z(t), are bounded in probability, i.e., $P(|X(t)|=\infty)=0$.

In Chapter 2 we introduced the frequency component of X(t) in a frequency band Λ as

$$X_\Lambda(t)=\int_\Lambda e^{i\lambda t}dZ_x(\lambda).$$

We can write it as $X_\Lambda(t)=\int 1_\Lambda(\lambda)e^{i\lambda t}dZ_x(\lambda)$, where $1_\Lambda(\lambda)$ is the indicator function of the band Λ. From Section 2.2.1, we can think of $\{X_\Lambda(t)\}$ as the output of a linear filter with $\{X(t)\}$ an input and transfer function $1_\Lambda(\lambda)$. Therefore we can assume that the frequency components in any band Λ, $X_\Lambda(t)$, $Y_\Lambda(t)$ and $Z_\Lambda(t)$ are known.

We assume that for these processes, the spectral densities $f_{xx}(\lambda)$, $f_{yy}(\lambda)$ and $f_{zz}(\lambda)$ are nonzero and boundedly differentiable in Λ, which is centered at λ_o.

We then have the following lemma.

LEMMA 3.1.1

The partial group delay between X and Y adjusted for Z at λ_o is equal to the partial group delay between X_Λ and Y_Λ adjusted for Z_Λ at λ_o when λ_o is an interior point of Λ.

PROOF

We shall derive the relationship among spectral densities of X, Y, Z and X_Λ, Y_Λ, Z_Λ.

Without loss of generality, we consider the cross-spectral densities $f_{xx}(\lambda)$ and $f_{xy}(\lambda)$. From (2.2.1.3) we have

$$f_{xx,\Lambda}(\lambda)=|1_\Lambda(\lambda)|^2 f_{xx}(\lambda), \tag{3.1.1}$$

where $f_{xx,\Lambda}(\lambda)$ is the autospectral density of $X_\Lambda(\lambda)$; and $1_\Lambda(\lambda)$, the indicator function of the frequency band Λ, is the transfer function corresponding to the band filter.

Similarly we have

$$f_{yy,\Lambda}(\lambda)=|1_\Lambda(\lambda)|^2 f_{yy}(\lambda) \tag{3.1.2}$$

$$f_{zz,\Lambda}(\lambda)=|1_\Lambda(\lambda)|^2 f_{zz}(\lambda). \tag{3.1.3}$$

From (2.2.1.4), the cross-spectral density between X_Λ and Y_Λ is

$$f_{xy,\Lambda}(\lambda)=|1_\Lambda(\lambda)|^2 f_{xy}(\lambda). \tag{3.1.4}$$

Also we have similar formulas for f_{xy}, f_{zy} etc. Thus from expresssion (2.5.5), the partial cross-spectral density between X_Λ and Y_Λ adjusted for Z_Λ is

$$f_{xy.z,\Lambda}(\lambda)=f_{xy,\Lambda}(\lambda)-f_{xz,\Lambda}(\lambda)f_{zz,\Lambda}^{-1}(\lambda)f_{zy,\Lambda}(\lambda).$$

On substituting (3.1.1)——(3.1.4) into the above expressions, when $\lambda\varepsilon\Lambda$ we have

$$f_{xy.z,\Lambda}(\lambda)=f_{xy.z}(\lambda).$$

By definition

$$\theta_{xy.z}(\lambda)=\arg\{f_{xy.z}(\lambda)\}$$

and

$$\theta_{xy.z,\Lambda}(\lambda)=\arg\{f_{xy.z,\Lambda}(\lambda)\}.$$

Hence we have

$$\theta_{xy.z,\Lambda}(\lambda)=\theta_{xy.z}(\lambda).$$

Then by definition, $\tau_{xy.z,\Lambda}(\lambda)=\tau_{xy.z}(\lambda)$ when $\lambda\varepsilon\Lambda$.

Based on this lemma to estimate the partial group delay between X and Y adjusted for Z, is equivalent to estimating the partial group delay between X_Λ and Y_Λ adjusted for Z_Λ.

From (2.5.3) and (2.5.4), $X_\Lambda(t)$ and $Y_\Lambda(t)$ can be decomposed as

$$X_\Lambda(t)=\Pi[X_\Lambda(t)|H(Z_\Lambda)]+\varepsilon_{x\Lambda}(t), \tag{3.1.5}$$

$$Y_\Lambda(t)=\Pi[Y_\Lambda(t)|H(Z_\Lambda)]+\varepsilon_{y\Lambda}(t), \tag{3.1.6}$$

where $\Pi[X_\Lambda(t)|H(Z_\Lambda)]$ and $\Pi[Y_\Lambda(t)|H(Z_\Lambda)]$ are the projections of $X_\Lambda(t)$ and $Y_\Lambda(t)$ on $H(Z_\Lambda)$, the closed linear manifold generated by $\{Z_\Lambda\}$. By definition, the group delay between X_Λ and Y_Λ adjusted for Z_Λ is just the group delay between $\varepsilon_{x\Lambda}$ and $\varepsilon_{y\Lambda}$.

By Theorem 2.4.1, $\Pi[X_\Lambda(t)|H(Z_\Lambda)]$ and $\Pi[Y_\Lambda(t)|H(Z_\Lambda)]$ can be written as

$$\Pi[X_\Lambda(t)|H(Z_\Lambda)]=\alpha Z_\Lambda(t-\phi_1)+\varepsilon_{x\Lambda}(t)+O_{x\Lambda}(t)$$
$$\Pi[Y_\Lambda(t)|H(Z_\Lambda)]=\beta Z_\Lambda(t-\phi_2)+\varepsilon_{y\Lambda}(t)+O_{y\Lambda}(t)$$
$$(\infty<t<\infty).$$

Then from (3.1.5) and (3.1.6) we have

$$X_\Lambda(t)=\alpha Z_\Lambda(t-\phi_1)\ +\varepsilon_{x\Lambda}(t)+O_{x\Lambda}(t) \tag{3.1.7}$$

$$Y_\Lambda(t)=\beta Z_\Lambda(t-\phi_2)\ +\varepsilon_{y\Lambda}(t)+O_{y\Lambda}(t). \tag{3.1.8}$$

We have already stated that for the given X, Y and Z, we can assume that the frequency components X_Λ, Y_Λ and Z_Λ are also known. In (3.1.7) and (3.1.8) $O_{X\Lambda}(t)$ and $O_{Y\Lambda}(t)$ are unobservable. However by (2.4.1.2) $\|O_{x\Lambda}(t)\| = o(\|X_\Lambda(t)\|)$ when the band length $d(\Lambda)\to o$. Thus when the band length of Λ is small enough, we can approximate $X_\Lambda(t)$ and $Y_\Lambda(t)$ by

$$X_\Lambda(t) = \alpha Z_\Lambda(t-\phi_1) + \varepsilon_{x\Lambda}(t) \tag{3.1.9}$$
$$Y_\Lambda(t) = \alpha Z_\Lambda(t-\phi_2) + \varepsilon_{y\Lambda}(t). \tag{3.1.10}$$

Because of the approximations (3.1.9) and (3.1.10), we propose to estimate partial group delay in the following way.

(1) Suppose that λ_o is the frequency at which we want to estimate the partial group delay $\tau(\lambda_o)$. We choose a band of frequencies Λ which include λ_o as an interior point and whose length is relatively small. Based on model (3.1.9) and (3.1.10) apply the estimating procedure proposed by Hannan and Robinson (1973) and described in Section 2.4.2. We can obtain the α^+, β^+, ϕ_1^+, and ϕ_2^+, the estimates of α, β, ϕ_1, and ϕ_2. Then we define

$$\varepsilon^+_{x\Lambda}(t) = X_\Lambda(t) - \alpha^+ Z_\Lambda(t-\phi_1^+) \tag{3.1.11}$$
$$\varepsilon^+_{y\Lambda}(t) = Y_\Lambda(t) - \beta^+ Z_\Lambda(t-\phi_2^+). \tag{3.1.12}$$

(2) Based on $\varepsilon^+_{x\Lambda}(t)$ and $\varepsilon^+_{y\Lambda}(t)$, $t=1,\ldots N$, apply the estimating procedure of Hannan and Thomson (1973), which is described in Section 3.4.3. We define the finite Fourier transforms

$$w^+_{\varepsilon_x}(s)=\Sigma^N_{n=1}\varepsilon^+_{X\Lambda}(n)e^{in\omega s}, \tag{3.1.13}$$

$$w^+_{\varepsilon_y}(s)=\Sigma^N_{n=1}\varepsilon^+_{Y\Lambda}(n)e^{in\omega s}, \tag{3.1.14}$$

where $\omega_s=2\pi s/N$, $0<s\leq[N]/2$.

Then we define the cross periodogram

$$I^+_{\varepsilon_x\varepsilon_y}(s)=w^+_{\varepsilon_x}(s)w^+_{\varepsilon_y}(s)^c. \tag{3.1.15}$$

(3) Choose a band B of frequency λ, that is

$$B=\{\lambda\,|\,\lambda_o-\pi/2M<\lambda<\lambda_o+\pi/2M\}.$$

Here we have in mind the relation 2mM=N. Then when M is big enough B can always be a subset of Λ.

(4) Define

$$p^+(\tau)=(1/m)\Sigma_o I^+_{\varepsilon_x\varepsilon_y}(s)e^{-i\tau\omega_s}, \tag{3.1.16}$$

where the summation is over the m fundamental frequencies $\omega_s=2\pi s/N$ in B. We call τ^+ the value of τ that maximizes $|p^+(\tau)|^2$. This τ^+ is the estimate of the partial group delay between X and Y adjusted for Z at $\lambda=\lambda_o$.

To obtain the weak consistency of our estimate τ^+, we return to (3.1.7) and (3.1.8),

$$X_\Lambda(t)=\alpha Z_\Lambda(t-\phi_1)+\varepsilon_{x\Lambda}(t)+O_{x\Lambda}(t) \qquad (-\infty<t<\infty)$$
$$Y_\Lambda(t)=\alpha Z_\Lambda(t-\phi_2)+\varepsilon_{x\Lambda}(t)+O_{y\Lambda}(t) \qquad (-\infty<t<\infty).$$

Alternatively we can write those in the following way,

$$X_\Lambda(t)-O_{x\Lambda}(t)=\alpha Z_\Lambda(t-\phi_1)+\varepsilon_{x\Lambda}(t) \tag{3.1.17}$$
$$Y_\Lambda(t)-O_{y\Lambda}(t)=\alpha Z_\Lambda(t-\phi_2)+\varepsilon_{y\Lambda}(t). \tag{3.1.18}$$

Now we assume temporarily that we can observe $O_{x\Lambda}(t)$ and $O_{y\Lambda}(t)$. Thus if we use the estimating procedure of Section 2.4.2, but based on $X_{x\Lambda}(t)-O_{x\Lambda}(t)$ and $Z(t)$ $(t=1,\ldots N)$, we can obtain α^* and ϕ_1^*. We write

$$\varepsilon^*_{x\Lambda}(t)=X_\Lambda(t)-\alpha^* Z_\Lambda(t-\phi_2^*)-O_{x\Lambda}(t). \tag{3.1.19}$$

Similarly we have

$$\varepsilon^*_{y\Lambda}(t)=Y_\Lambda(t)-\beta^* Z_\Lambda(t-\phi_2^*)-O_{y\Lambda}(t). \tag{3.1.20}$$

Now based on $\varepsilon^*_{x\Lambda}(t)$ and $\varepsilon^*_{y\Lambda}(t)$, $t=1,2,\ldots N$ we can apply the estimating procedure in Section 2.4.3, and obtain τ^* by replacing $\varepsilon_{x\Lambda}(t)$ and $\varepsilon_{y\Lambda}(t)$ by $\varepsilon^*_{x\Lambda}(t)$ and $\varepsilon^*_{y\Lambda}(t)$ respectively.

Section 3.2 will give the weak consistency of τ^*.

3.2 WEAK CONSISTENCY OF THE ESTIMATE

In this section we consider the models (3.1.17) and (3.1.18). We have the following lemma.

LEMMA 3.2.1.

Assume that the conditions of Theorem 2.4.2.1 hold for $X_\Lambda(t)-O_{x\Lambda}(t)$ and $Z_\Lambda(t)$. We also assume that $Z_\Lambda(t)$ satisfies the following condition: for any $\varepsilon>0$ there exists an $L_\varepsilon>0$ such that

$$P\{|Z_\Lambda(t_1)-Z_\Lambda(t_2)|>L_\varepsilon|t_1-t_2|\}<\varepsilon \text{ for any } t_1 \text{ and } t_2.$$

Then for $\varepsilon_{x\Lambda}$ and $\varepsilon^*_{x\Lambda}$ defined in (3.1.5) and (3.1.19), $\varepsilon_{x\Lambda}(t)-\varepsilon^*_{x\Lambda}(t)\to 0$ in probability and uniformly for all t.

PROOF

By definition

$$\varepsilon_{x\Lambda}(t)-\varepsilon^*_{x\Lambda}(t)=\alpha^* Z_\Lambda(t-\phi^*_1)-\alpha Z_\Lambda(t-\phi_1).$$

For any $\varepsilon>0$ and $\delta>0$,

$$\begin{aligned}
&P\{|\alpha^* Z_\Lambda(t-\phi^*_1)-\alpha Z_\Lambda(t-\phi_1)| >\varepsilon\}\\
\le &P\{|\alpha^* Z_\Lambda(t-\phi^*_1)-\alpha Z_\Lambda(t-\phi^*_1)|>\varepsilon/2\}\\
&+P\{|\alpha Z_\Lambda(t-\phi^*_1)-\alpha Z_\Lambda(t-\phi_1)|>\varepsilon/2\}\\
= &P_1+P_2.
\end{aligned}$$

As $Z_\Lambda(t)$ is bounded in probability, for this δ, there exists a M_δ such that $P\{|Z_\Lambda(t)|>M_\delta\}<\delta/4$. Then

$$P_1=P\{|\alpha^*-\alpha||Z_\Lambda(t-\phi_1^*)>\varepsilon/2,\ |Z_\Lambda(t-\phi_1^*)|>M_\delta\}$$
$$+P\{|\alpha^*-\alpha||Z_\Lambda(t-\phi_1^*)|>\varepsilon/2,\ |Z_\Lambda(t-\phi_1^*)|\le M_\delta\}.$$

Hence $P_1<\delta/4+P(|\alpha^*-\alpha|M\delta>\varepsilon/2)$. As $\alpha^*\rightarrow\alpha$ a.s from Theorem 2.4.2.1, there exists a N_o such that when $N>N_o$, $P(|\alpha^*-\alpha|M_\delta >\alpha/2)<\delta/4$. Thus $P_1<\delta/2$. Without loss of generality, by assuming $|\alpha|<L$, we have

$$P_2\le P\{|Z_\Lambda(t-\phi_1^*)-Z_\Lambda(t-\phi_1)|>\varepsilon/2L\}.$$

By the assumption of the lemma, for this δ, there exists a R_δ such that

$$P\{|Z_\Lambda(t-\phi_1^*)-Z_\Lambda(t-\phi_1)|>R_\delta|\phi_1^*-\phi_1|\}<\delta/4.$$

Then

$$P_2\le P\{|Z_\Lambda(t-\phi_1^*)-Z_\Lambda(t-\phi_1)|>\varepsilon/2L,\ |Z_\Lambda(t-\phi_1^*)-Z_\Lambda(t-\phi_1)|>R_\delta|\phi_1^*-\phi_1|\}$$
$$+P\{|Z_\Lambda(t-\phi_1^*)-Z_\Lambda(t-\phi_1)|>\varepsilon/2L,\ |Z_\Lambda(t-\phi_1^*)-Z_\Lambda(t-\phi_1)|\le R_\delta|\phi_1^*-\phi_1|\}$$
$$\le\delta/4+P(|\phi_1^*-\phi_1|>\varepsilon/2LR_\delta).$$

We know that $\phi_1^*\rightarrow\phi$ a.s from Theorem 2.4.2.1, thus there exists a N_1 such that when $N>N_1$,

$$P(|\phi_1^*-\phi_1|>\varepsilon/2LR_\delta)<\delta/4.$$

Let $N'=\max(N_o,N_1)$, when $N>N'$ we have

$$P\{|\varepsilon_{x\Lambda}(t)-\varepsilon^*_{x\Lambda}(t)|>\varepsilon\}<\delta,$$

i.e., $\varepsilon_{x\Lambda}(t)-\varepsilon^*_{x\Lambda}(t)\to 0$ in probability, and uniformly for all t. Q.E.D.

Now define the finite Fourier transforms:

$$w_{\varepsilon_x}(s)=(1/2\pi)^{1/2}\Sigma_{n=1}^{N}\varepsilon_{x\Lambda}(n)e^{in\omega_s} \tag{3.2.1}$$

$$w^*_{\varepsilon_x}(s)=(1/2\pi)^{1/2}\Sigma_{n=1}^{N}\varepsilon^*_{x\Lambda}(n)e^{in\omega_s} \tag{3.2.2}$$

and

$$w_{\varepsilon_y}(s)=(1/2\pi)^{1/2}\Sigma_{n=1}^{N}\varepsilon_{y\Lambda}(n)e^{in\omega_s} \tag{3.2.3}$$

$$w^*_{\varepsilon_y}(s)=(1/2\pi)^{1/2}\Sigma_{n=1}^{N}\varepsilon^*_{y\Lambda}(n)e^{in\omega_s} \tag{3.2.4}$$

where $\omega_s=2\pi s/N$ and s is an integer.

We have the following Corollary.

COROLLARY 3.2.2.

Under the condition of Lemma 3.2.1, $\varepsilon_{x\Lambda}(t)-\varepsilon^*_{x\Lambda}(t)=o_p(N^{-1/2})$ when $N\to\infty$.

PROOF

From Theorem 2.4.2.2, we know that $\sqrt{N}(\alpha^*-\alpha)$ and $\sqrt{N}(\phi_1^*-\phi_1)$ are asymptotically normally distributed. We also know that

$$\alpha^* \to \alpha \quad \text{a.s. and } \phi_1^* \to \phi_1 \quad \text{a.s. when } N \to \infty.$$

Then

$$\alpha^*-\alpha = O_p(N^{-1/2}) \text{ and } \phi_1^*-\phi_1 = O_p(N^{-1/2}).$$

Hence in Lemma 3.2.1, if we replace $U_N(t)=\varepsilon^*_{x\Lambda}(t)-\varepsilon_{x\Lambda}(t)$ by $U_N(t)/N^{-1/2+\delta}$ for any $\delta>0$, then since $[\alpha^*-\alpha]/N^{-1/2+\delta} \to 0$ and $[\phi_1^*-\phi_1]/N^{-1/2+\delta} \to 0$ in probability when $N\to\infty$, we still have $[\varepsilon^*_{x\Lambda}(t)-\varepsilon_{x\Lambda}(t)]/N^{-1/2+\delta} \to 0$ in probability. Thus we have

$$\varepsilon^*_{x\Lambda}(t)-\varepsilon_{x\Lambda}(t) = O_p(N^{-1/2}). \qquad \text{Q.E.D.}$$

Based on Lemma 3.2.1 and Corollary 3.2.2 we have the following lemma.

LEMMA 3.2.3.

Suppose that the conditions of Lemma 3.2.1 are satisfied and $\varepsilon_{x\Lambda}(t)$ and $\varepsilon^*_{x\Lambda}(t)$ satisfy Assumption 2.4.4.2 and the covariance function of $U_N(t)=\varepsilon_{x\Lambda}(t)-\varepsilon^*_{x\Lambda}(t)$, $C_k^{(U)}$, satisfies $\Sigma_{k=-\infty}^{\infty}|C_k^{(U)}|<\infty$. Then the convergence $|w_{\varepsilon_x}(s)-w^*_{\varepsilon_x}(s)| \to 0$ is in probability for all $\omega_s=2\pi s(N)/N$ where $s(N)$ is an integer and $\omega_s \to \lambda_0 \neq 0,\pi$ when $N\to\infty$. The convergence is also uniform for all such ω_s. Further we have $w_{\varepsilon_x}(s)-w^*_{\varepsilon_x}(s)=O_p(1/N)$.

PROOF

Let $U_N(t)=\varepsilon_{x\Lambda}(t)-\varepsilon^*_{x\Lambda}(t)$, then $U_N(t)\to 0$ in probability and uniformly for all t by Lemma 3.2.1. We denote the corresponding finite Fourier transform of $U_N(t)$ at frequency ω_s by $w_{U_N}(s)$.

We know that $E(w_{U_N}(s))=0$, since $E(\varepsilon_{x\Lambda})=E(\varepsilon_{y\Lambda})=0$. From Theorem 2.4.4.4

$$\mathrm{COV}\{w_{U_N}(s_1)\ w_{U_N}(s_2)\}=(1/N)\Delta^{(N)}(2\pi(s_1-s_2)/N)f_{U_N}(\omega_{s_1})+O(1/N), \tag{3.2.5}$$

where $f_{U_N}(\omega_{s_1})$ is the spectral density of $U_N(s_1)$, and the function $\Delta^{(N)}(\lambda)=N$ for $\lambda\equiv 0(\mathrm{mod}2\pi)$ and $\Delta^{(N)}(2\pi s/N)=0$ for s an integer with $s\equiv 0(\mathrm{mod}\ N)$. Hence

$$\mathrm{COV}\{w_{U_N}(s_1)\ w_{U_N}(s_2)\}=f_{U_N}(w_{s_1})+o(1) \text{ or } o(1). \tag{3.2.6}$$

As $U_N(t)\to 0$ in probability for any t when $N\to\infty$, thus

$$C_k^{(U)}=E[U_N(t)\ U_N(t+k)]\to 0 \text{ for any k when } N\to\infty. \tag{3.2.7}$$

Then

$$|f_{U_N}(\lambda)|=(1/2\pi)|\Sigma_{k=-\infty}^{\infty}C_k^{(U)}e^{-i\lambda k}|\le(1/2\pi)\Sigma_{-\infty}^{\infty}|C_k^{(U)}|. \tag{3.2.8}$$

Under the condition $\Sigma_{-\infty}^{\infty}|C_k^{(U)}|<\infty$, from (3.2.7) and (3.2.8) we know that $f_{U_N}(\lambda)\to 0$ uniformly for all λ when $N\to\infty$. Combining this with (3.2.6), we have

$$\mathrm{COV}(w_{U_N}(s),\ w_{U_N}(t))\to 0 \text{ when } N\to\infty.$$

Finally from Theorem 2.4.4.4

$$\mathrm{CUM}\{w_{U_n}(\omega_{s_1}),\ldots w_{U_N}(\omega_{s_k})\}$$
$$=N^{-k/2}(2\pi)^{k/2-1}\Delta^{(N)}(\omega_{s_1}+\ldots+\omega_{s_k})f_{U_N}(\omega_{s_1},\ldots,\omega_{s_{k-1}})+O(N^{-k/2}).$$

Since $\Delta^{(N)}(\lambda)=0$ or N and $f_{U_N}(\lambda)\to 0$ when $N\to\infty$, it follows that the above cumulant $\to 0$ when $k>2$ and $N\to\infty$.

Hence by Theorem 2.4.4.3, we know that $w_{U_N}(s)\to 0$ in distribution for $\omega_s=2\pi s/N$, $\omega_s\to\lambda_o\equiv 0(\mathrm{mod}\pi)$. Equivalently $w_{U_N}(s)\to 0$ in probability, or $w_{\varepsilon_x}(s)-w^*_{\varepsilon_x}(s)\to 0$ in probability.

For the rate of convergence, from (3.2.7) and $U_N(t)=O_p(N^{-1/2})$, we have $C_s^{(U)}=O_p(1/N)$. Then from (3.2.8) $f_{U_N}(\lambda)=O_p(1/N)$ and from (3.2.6) $E|w_{U_N}(\lambda)|^2=O(1/N)$. By the Chebyshev inequality

$$P(|w_{U_N}(s)|>\delta)<E[w_{U_N}(s)]^2/\delta^2.$$

Hence $w_{\varepsilon_x}(s)-w^*_{\varepsilon_x}(s)=O_p(1/N)$.

Q.E.D.

LEMMA 3.2.4.

Suppose that the conditions of Lemma 3.2.3 are satisfied and $\varepsilon_{x\Lambda}(t)$, $\varepsilon^*_{x\Lambda}(t)$ as well as $\varepsilon_{y\Lambda}(t)$, and $\varepsilon^*_{y\Lambda}(t)$. satisfy Assumption 2.4.4.2. Then the convergence $I_{\varepsilon_x\varepsilon_y}(s)-I^*_{\varepsilon_x\varepsilon_y}(s)\to 0$ is in probability and uniformly for all $\omega_s=2\pi s(N)/N$ with $\omega_s\to\lambda_o$, when $N\to\infty$, where $I_{\varepsilon_x\varepsilon_y}(s)=w_{\varepsilon_x}(s)w_{\varepsilon_y}(s)^c$ and $I^*_{\varepsilon_x\varepsilon_y}(s)=w^*_{\varepsilon_x}(s)w^*_{\varepsilon_y}(s)^c$. Further $I_{\varepsilon_x\varepsilon_y}(s)-I^*_{\varepsilon_x\varepsilon_y}(s)=O_p(1/N)$.

PROOF

$$
\begin{aligned}
&|I_{\varepsilon_x\varepsilon_y}(s)-I^*_{\varepsilon_x\varepsilon_y}(s)|\\
&=|w_{\varepsilon_x}(s)w_{\varepsilon_y}(s)^c-w^*_{\varepsilon_x}(s)w^*_{\varepsilon_y}(s)^c|\\
&\le|w_{\varepsilon_x}(s)|\,|w_{\varepsilon_y}(s)^c-w^*_{\varepsilon_y}(s)^c|\\
&\quad+|w^*_{\varepsilon_y}(s)^c|\,|w_{\varepsilon_x}(s)-w^*_{\varepsilon_x}(s)|.
\end{aligned}
$$

By Theorem 2.4.4.5 $w_{\varepsilon_x}(s)$ and $w^*_{\varepsilon_y}(s)$ are asymptotically normally distributed; thus they are asymptotically bounded in probability. Obviously $I_{\varepsilon_x\varepsilon_y}(s)=w_{\varepsilon_x}(s)w_{\varepsilon_y}(s)^c\to 0$ in probability uniformly for all ω_s by Lemma 3.2.2. Since $w_{\varepsilon_x}(s)-w^*_{\varepsilon_x}(s)=O_p(1/N)$, it follows that $I_{\varepsilon_x\varepsilon_y}(s)-I^*_{\varepsilon_x\varepsilon_y}(s)=O_p(1/N)$.

Q.E.D.

We can now prove Theorem 3.2.5.

THEOREM 3.2.5.

For any fixed band Λ under the conditions of Lemma 3.2.4, we have $\tau^*-\tau^\#\to 0$ in probability when $N\to\infty$, where $\tau^\#$ and τ^* are the estimates obtained by using the procedure of Section 2.4.3, based on $\varepsilon_{x\Lambda}$ and $\varepsilon_{y\Lambda}$, $\varepsilon^*_{x\Lambda}$ and $\varepsilon^*_{y\Lambda}$ respectively. Further $\tau^*-\tau^\#=O_p(1/N)$.

PROOF

Let $p^\#(\tau)=(1/m)\Sigma_o(I_{\varepsilon_x\varepsilon_y}(s))e^{-i\tau\omega_s}$, with Σ_o the sum over the m frequencies ω_s in the band $B=\{\lambda|\lambda_o-m\pi/N<\lambda<\lambda_o+m\pi/N\}$ and let $\tau^\#$ be the value of τ maximizing $q^\#(\tau)=|p^\#(\tau)|^2$.

As $e^{-i\tau\omega_s}$ is holomorphic for τ, thus $p^\#(\tau)$ is holomorphic for τ and $q^\#(\tau)$ is a holomorphic function of τ.

Similarly

$$G(I_{\varepsilon_x\varepsilon_y}(s),\tau)=\partial q^\#(\tau)/\partial\tau$$
$$=[\partial p^\#(\tau)/\partial\tau]p^\#(\tau)^c+p^\#(\tau)[\partial p^\#(\tau)^c]/\partial\tau$$

is a holomorphic function of τ. Obviously $G(I_{\varepsilon_x\varepsilon_y}(s),\tau)$ is a holomorphic function of $I_{\varepsilon_x\varepsilon_y}(s)$ for s=1,...m. Since $\tau^\#$ maximizies $q^\#(\tau)$, $\partial G/\partial\tau=\partial^2q^\#(\tau)/\partial\tau^2$ evaluated at $\tau^\#$ is negative.

Hence by Theorem 2.4.4.1, $\tau^\#$ as a function of $I_{\varepsilon_x\varepsilon_y}(s)$ or, equivalently, of $(1/m)I_{\varepsilon_x\varepsilon_y}(s)$, is continuous. The partial derivatives exist and are also continuous. Then by the multivariate Taylor expansion

$$\tau^{*}-\tau^{\#}=(1/m)\Sigma_{o}A(s)(I_{\varepsilon_x\varepsilon_y}(s)-I^{*}_{\varepsilon_x\varepsilon_y}(s)), \tag{3.2.9}$$

where A(s) is uniformly bounded in probability in the neighborhood of λ_o. From Lemma 3.2.3 $I_{\varepsilon_x\varepsilon_y}(s)-I^{*}_{\varepsilon_x\varepsilon_y}(s)\to 0$ in probability and uniformly in ω_s. Thus we have $\tau^{*}-\tau^{\#}\to 0$ in probability when $N\to\infty$. Since $I_{\varepsilon_x\varepsilon_y}(s)-I^{*}_{\varepsilon_x\varepsilon_y}(s)=O_p(1/N)$ in (3.2.9), it follows that $\tau^{*}-\tau^{\#}=O_p(1/N)$. Q.E.D.

COROLLARY 3.2.6.

Assume that the conditions of Theorem 2.4.3.3 are satisfied for $\varepsilon_{x\Lambda}(t)$ and $\varepsilon_{y\Lambda}(t)$. Then under the conditions of Theorem 3.2.5 for any fixed band Λ, we have $\tau^{*}\to\tau_o$ in probability when $N\to\infty$.

The proof is easy to obtain when we combine Theorem 3.2.5 with the fact that $\tau^{\#}\to\tau_o$ almost surely by Theorem 2.4.4.3.

Until now we consider only the case that the frequency band is fixed. However to obtain the asymptotic property of the estimate τ^{+} in Section 3.1, we need to consider a sequence of bands Λ_k with the band length $d(\Lambda_k)$ tending to zero.

We give the following lemma for later use.

LEMMA 3.2.7.

If $t_M(N)$ $(M=1,2,\ldots;N=1,2,\ldots)$ is a family of random variables satisfying $\lim_{N\to\infty} t_M(N)=t_M$, in probability, and $\lim_{M\to\infty} t_M=t$, in probability, then there is a sequence $M(N)$ increasing monotonically with N so that $\lim_{N\to\infty} t_{M(N)}(N)=t$.

This lemma is similar to Lemma 2.4.3.2. The proof can be obtained easily by using Lemma 2.4.3.2.

Now we have the following theorem.

THEOREM 3.2.8.

Under the conditions of Corollary 3.2.6, there is a sequence $K(N)$ with $K(N)\uparrow\infty$. Correspondingly there is a sequence of bands $\Lambda_{K(N)}$ with $d(\Lambda_{K(N)})\to 0$ when $N\to\infty$. Based on $\Lambda_{K(N)}$, applying the estimating procedure of Section 2.4.3 the corresponding estimate τ^* which is denoted by $\tau^*_{\Lambda_{K(N)}}$, converges to τ_o in probability when $N\to\infty$.

PROOF

From Corollary 3.2.6, we know that for each fixed Λ, we have $\tau^*\to\tau_o$. Next we want to consider a sequence of bands $\{\Lambda_k\}$, with each band centered at λ_o and the band length $d(\Lambda_k)\to 0$ when $K\to\infty$.

Thus for each fixed Λ_k, we have a sequence of estimates $\tau^*_{\Lambda_k 1}$, $\tau^*_{\Lambda_k 2},\ldots\tau^*_{\Lambda_k N}$ that converges to τ_o in probability where $\tau^*_{\Lambda_k N}$ represents the estimate τ^* based on the band sequence $\{\Lambda_k\}$ and its sample size is N. We can express the convergence in the following way:

$$
\begin{array}{ll}
\Lambda_1 & \tau^*_{\Lambda_1 1}, \tau^*_{\Lambda_1 2}, \ldots \to \tau_o \\
\Lambda_2 & \tau^*_{\Lambda_2 1}, \tau^*_{\Lambda_2 2}, \ldots \to \tau_o \\
\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
\Lambda_k & \tau^*_{\Lambda_k 1}, \tau^*_{\Lambda_k 2}, \ldots \to \tau_o .
\end{array}
$$

Then by applying Lemma 3.2.5 and by letting $t_M(N)$ in that lemma be $\tau^*_{\Lambda_k N}$ and by letting t_M be τ_o, we complete the proof.

Q.E.D.

3.3 WEAK CONSISTENCY OF THE ESTIMATE

In Section 3.2 τ^* is based on $\{X_\Lambda(t)-O_{x\Lambda}(t)\}$, $\{Y_\Lambda(t)-O_{y\Lambda}(t)\}$, and $\{Z_\Lambda(t)\}$, $t=1,2,\ldots N$.. However $O_{x\Lambda}(t)$ and $O_{y\Lambda}(t)$ are unobservable. Thus we return to Section 3.1 and the estimate τ^+ which is motivated by the the following model approximation.

$$
\begin{aligned}
X_\Lambda(t) &= \alpha Z_\Lambda(t-\phi_1)+\varepsilon_{x\Lambda}(t) \\
Y_\Lambda(t) &= \beta Z_\Lambda(t-\phi_2)+\varepsilon_{y\Lambda}(t).
\end{aligned}
$$

By Theorem 3.2.8 we have chosen a sequence of bands $\Lambda_{K(N)}$ with $d(\Lambda_{K(N)})\to 0$ when $N\to\infty$. Based on this band sequence $\{\Lambda_{K(N)}\}$ correspondingly we have $\tau^+_{\Lambda_{K(N)}}$. To simplify the notation, in this section we denote $\Lambda_{K(N)}$ by Λ_k

We have the following lemma.

LEMMA 3.3.1.

The random variable $O_\Lambda(t)$ in (2.4.1.1) converges in probability to zero at the rate

$$O_\Lambda(t)=o_p(d^{3/2}(\Lambda)) \text{ when } d(\Lambda)\to 0.$$

PROOF

We consider the case that there is a sequence of bands $\{\Lambda_k)$ with the band length $d(\Lambda_k)\to 0$ when $k\to\infty$ From (2.4.1.8), $\|O_\Lambda(t)\|\to 0$ when $d(\Lambda)\to 0$, also $E[O_\Lambda^2(t)]=O(d^3(\Lambda))$. Then there exists a M and K such that when k>K

$$E[O_{\Lambda_k}^2(t)/d^3(\Lambda_k)]<M.$$

Then for this M and the given δ, there exists an L>0 such that $M/L<\delta$. By Markov's inequality, when k>K,

$$P(|O_{\Lambda_k}^2(t)/d^3(\Lambda_k)|>L)\le\{E[O_{\Lambda_k}^2(t)/d^3(\Lambda_k)]/L\}<M/L<\delta,$$

i.e.,

$$O_{\Lambda_k}(t)=o_p(d^{3/2}(\Lambda_k)). \qquad \text{Q.E.D.}$$

In this section we write

$$w_x^*(s)=\Sigma_{n=1}^{N}[X_{\Lambda_k}(n)-O_{x\Lambda_k}(n)]e^{in\omega_s} \qquad (3.3.1)$$

and

$$w_x^+(s)=\Sigma_{n=1}^{N}X_{\Lambda_k}(n)e^{in\omega s}, \qquad (3.3.2)$$

where $X_{\Lambda_k}(n)$ is the frequency component in the band $\Lambda_{K(N)}$ which we have chosen in Theorem 3.2.8. The frequency component $O_{x\Lambda_k}$ is defined similarly. We have the following lemma.

LEMMA 3.3.2.

Assume that $O_{x\Lambda_k}(t)$ satisfies Assumption 2.4.4.2 and the covariance function of $O_{x\Lambda_k}(t)$, $C_k^{(O)}$ satisfies $\Sigma_{k=-\infty}^{\infty}|C_k^{(O)}|<\infty$. Then the convergence $w_x^*(s)-w_x^+(s)\to 0$ is in probability, when $N\to\infty$ where $\omega_s=2\pi s/N\to\lambda_o$ when $N\to\infty$. Further we have $w_x^*(s)-w_x^+(s)=O_p(d(\Lambda_K)^{3/2})$.

PROOF

When $N\to\infty$, the length of the frequency band $\Lambda_{K(N)}$ obtained in Theorem 3.2.7, goes to zero. Thus by Theorem 2.4.1.1 $O_{x\Lambda_K}(t)\to 0$ in probability when $d(\Lambda_k)\to 0$, or $N\to\infty$. The result now follows by the argument used to prove Lemma 3.2.2.

We denote the corresponding finite Fourier transform of $O_{x\Lambda_k}(t)$ by $w_{O_N}(s)$.

We know that $E(w_{O_N}(s))=0$ from Theorem 2.4.1.1. From Theorem 2.4.4.4

$$COV\{w_{O_N}(s)\ w_{O_N}(t)\}=(1/N)\Delta^{(N)}(2\pi(s-t)/N)f_{O_N}(\omega_s)+O(1/N), \tag{3.3.3}$$

where $f_{O_N}(\omega_s)$ is the spectral density of $O_{x\Lambda_k}$, and the function $\Delta^{(N)}(\lambda)=N$ for $\lambda\equiv 0(\text{mod}2\pi)$ and $\Delta^{(N)}(2\pi s/N)=0$ for s an integer with $s\equiv 0(\text{mod } N)$. Hence

$$COV\{w_{O_N}(s),\ w_{O_N}(t)\}=f_{O_N}(w_s)+o(1) \text{ or } o(1). \qquad (3.3.4)$$

From Theorem 2.4.1.1, $Var[O_{x\Lambda_k}(t)]\to 0$ for any t when $N\to\infty$, thus

$$C_k^{(O)}=E[O_N(t)\ O_N(t+k)]\to 0 \text{ for any } k \text{ when } N\to\infty. \qquad (3.3.5)$$

Then

$$|f_{O_N}(\lambda)|=(1/2\pi)|\Sigma_{-\infty}^{\infty}C_k^{(O)}\ e^{-i\lambda k}|\le(1/2\pi)\Sigma_{-\infty}^{\infty}|C_k^{(O)}|. \qquad (3.3.6)$$

Under the condition $\Sigma|C_s^{(O)}|<\infty$, from (3.3.5) we know that $f_{O_N}(\lambda)\to 0$ uniformly for all λ. Combining this with (3.3.4), we have

$$COV(w_{O_N}(s),\ w_{O_N}(t))\to 0 \text{ when } N\to\infty.$$

Finally from Theorem 2.4.4.4

$$CUM\{w_{O_N}(\omega_{s_1}),\dots w_{O_N}(\omega_{s_k})\}$$
$$=N^{-k/2}(2\pi)^{k/2-1}\Delta^{(N)}(\omega_{s_1}+\dots+\omega_{s_k})f_{O_N}(\omega_{s_1},\dots,\omega_{s_{k-1}})+O(N^{-k/2}).$$

Since $\Delta^{(N)}(\lambda)=0$ or N and $f_{O_N}(\lambda)\to 0$ when $N\to\infty$, it follows that the above cumulant $\to 0$ when $k>2$ and $N\to\infty$.

Hence by Theorem 2.4.4.3, we know that $w_{O_N}(s)\to 0$ in distribution for $\omega_s=2\pi s/N$, $\omega_s\to\lambda_o\equiv 0(\mathrm{mod}\pi)$. Equivalently $w_{O_N}(s)\to 0$ in probability, or $w_x^*(s)-w_x^+(s)\to 0$ in probability.

Further if we replace $O_{x\Lambda}(t)$ in $w_x^*(t)-w_x^+(t)$ by $O_{x\Lambda}(t)/d(\Lambda_{K(N)})^{3/2-\delta}$ for any $\delta>0$, $O_{x\Lambda}(t)/d(\Lambda_{K(N)})^{3/2-\delta}$ converges to zero in probability when $N\to\infty$. If we use the same argument in the above proof, then we know that $w_x^*(s)-w_x^+(s)=O_p(d(\Lambda_{K(N)})^{3/2})$. Q.E.D

LEMMA 3.3.3.

Under the conditions of Lemma 3.3.2, the convergence of $\varepsilon_{x\Lambda}^*(t)-\varepsilon_{x\Lambda}^+(t)\to 0$ is in probability and uniformly for all t when $N\to\infty$, where $\varepsilon_{x\Lambda}^*(t)$ and $\varepsilon_{x\Lambda}^+(t)$ are defined in (3.1.19) and (3.1.11). Further we have

$$\varepsilon_{x\Lambda}^*(t)-\varepsilon_{x\Lambda}^+(t)=O_p(d(\Lambda_{K(N)})^{3/2}).$$

PROOF

From Section 2.4.2 α^* and ϕ_1^* are the α and ϕ_1 maximizing

$$Q_N(\alpha,\phi_1)=\Sigma'|w_x(s)-\alpha e^{i\phi_1\omega_s}\, w_z(s)|^2\psi(\omega_s).$$

Similar to the proof of Theorem 3.2.4 by maximization, we obtain $\phi_1^*=V_1(w_x(s),\ w_z(s))$ and $\alpha^*=V_2(w_x(s),w_z(s))$, where V_1 and V_2 are some continuous functions.

Using Taylor sereies expansions we can write

$$\alpha^+-\alpha^*=(1/m)\Sigma_o B(s)[w_x^*(s)-w_x^+(s)]$$

where B(s) is uniformly bounded in probability for s. Then by Lemma 3.3.2 we know that $\alpha^{+}-\alpha^{*}\to 0$ in probability. Further as $w_x^{*}(s)-w_x^{+}(s)=O_p[d^{3/2}(\Lambda_k)]$, we have $\alpha^{+}-\alpha^{*}=O_p[d^{3/2}(\Lambda_K)]$ when $N\to\infty$. Similar results hold for $\phi_1^{+}-\phi_1^{*}$.

Now

$$\varepsilon^{*}_{x\Lambda_k}(t)-\varepsilon^{+}_{x\Lambda_k}(t)$$
$$=\alpha^{+}Z_{\Lambda_k}(t-\phi_1^{+})-\alpha^{*}Z_{\Lambda_k}(t-\phi_1^{*})+O_{x\Lambda_k}(t). \qquad (3.3.7)$$

Under the conditions of Lemma 3.2.1, and following the same argument used to prove Lemma 3.2.1, it is easy to show that

$$|\alpha^{+}Z_{\Lambda_k}(t-\phi_1^{+})-\alpha^{*}Z_{\Lambda_k}(t-\phi_1^{*})|=O_p(d^{3/2}(\Lambda_k)) \text{ when } N\to\infty.$$

From Lemma 3.3.2, we also know that $O_{x\Lambda_k}(t)=O_p(d^{3/2}(\Lambda_k))$. Substituting this in (3.3.7) completes the proof.

Q.E.D.

From (2.4.1.7), we know that $X_{\Lambda_k}(t)=O_p(d^{1/2}(\Lambda_K))$. Recall from (3.1.11) that

$$\varepsilon^{+}_{x\Lambda}(t)=X_{\Lambda_k}(t)-\alpha^{+}Z_{\Lambda_k}(t-\phi_1^{+}),$$

thus

$$\varepsilon^{+}_{x\Lambda}(t)=O_p(d(\Lambda_{K(N)})^{1/2}).$$

Also from Lemma 3.3.3, $\varepsilon^{*}_{x\Lambda}(t)-\varepsilon^{+}_{x\Lambda}(t)=O_p(d(\Lambda_{K(N)})^{3/2})$.

Hence $\varepsilon^{*}_{x\Lambda}(t)-\varepsilon^{+}_{x\Lambda}(t)/\varepsilon^{+}_{x\Lambda}(t)\to 0$ in probability when $N\to\infty$.

Let $I^{+}_{\varepsilon_x\varepsilon_y}(s)$ be the cross periodogram of $\varepsilon^{+}_{x\Lambda}(t)$ and $\varepsilon^{+}_{y\Lambda}(t)$ defined in (3.1.15) and $p^{+}(\tau)=(1/m)\Sigma I^{+}_{\varepsilon_x\varepsilon_y}(s)e^{-i\tau\omega_s}$ defined in (3.1.16). τ^{+} is the value of τ maximizing $|P^{+}(\tau)|^{2}$.

We have the following theorem.

THEOREM 3.3.4.

Suppose that $\varepsilon_{x\Lambda}(t)$ and $\varepsilon^{*}_{x\Lambda}(t)$ satisfy Assumption 2.4.4.2 and the covariance function of $V=\varepsilon_{x\Lambda}(t)-\varepsilon^{*}_{x\Lambda}(t)$, $C_k^{(V)}$, satisfies $\Sigma_{k=-\infty}^{\infty}C_k^{(V)}\cdot<\infty$. Also suppose that the above conditions are satisfied for $\varepsilon_{y\Lambda}(t)$ and $\varepsilon^{*}_{y\Lambda}(t)$. Then the convergence $\tau^{+}-\tau^{*}\to 0$ is in probability, where τ^{+} is the estimate based on $\varepsilon^{+}_{x\Lambda_k}$ and $\varepsilon^{+}_{y\Lambda_k}$ defined in (3.1.11) and (3.1.12) and based on the band $\Lambda_{K(N)}$ with $K(N)\uparrow\infty$ where $d(\Lambda_{K(N)})\to 0$ when $N\to\infty$. Further the rate of convergence is $\tau^{+}-\tau^{*}=O_p[d^{3/2}(\Lambda_{K(N)})]$.

PROOF

The proof is similar to the proofs of Lemma 3.2.2——Lemma 3.2.5. Thus we summarize the main steps of these proofs.

(1) Similar to Lemma 3.2.2, based on

$\varepsilon^{+}_{x\Lambda}(t)-\varepsilon^{*}_{x\Lambda}(t)\to 0$ in probability when $N\to\infty$,

which is obtained from Lemma 3.3.3, we have

$$w^{+}_{\varepsilon_x}(s)-w^{*}_{\varepsilon_x}(s)\to 0 \tag{3.3.8}$$

in probability when $N\to\infty$. Similarly,

$$w^{+}_{\varepsilon_y}(s)-w^{*}_{\varepsilon_y}(s)\to 0 \qquad (3.3.9)$$

in probability when $N\to\infty$ The rate of convergence is

$$w^{+}_{\varepsilon_y}(s)-w^{*}_{\varepsilon_y}(s)=O_p[d^{3/2}(\Lambda_{K(N)})].$$

(2) Similar to Lemma 3.2.3, based on (3.3.8) and (3.3.9) we have

$$I^{+}_{\varepsilon_x\varepsilon_y}(s)-I^{*}_{\varepsilon_x\varepsilon_y}(s)\to 0 \qquad (3.3.10)$$

in probability when $N\to\infty$.

The rate of convergence is

$$I^{+}_{\varepsilon_x\varepsilon_y}(s)-I^{*}_{\varepsilon_x\varepsilon_y}(s)=O_p[d^{3/2}(\Lambda_{K(N)})].$$

(3) As τ^{+} is the value of τ maximizing $|p^{+}(\tau)|^2$, it follows from the proof of Lemma 3.2.4, that τ^{+} is a function of $(1/m)I^{+}_{\varepsilon_x\varepsilon_y}(s)$ for $s=1,\ldots m$ and has partial derivative with respect to $I^{+}_{\varepsilon_x\varepsilon_y}(s)$ for $s=1,\ldots m$. Hence by the multivariate Taylor series expansion we have

$$\tau^{+}-\tau^{*}=(1/m)\Sigma_0 C(s)[I^{+}_{\varepsilon_x\varepsilon_y}(s)-I^{*}_{\varepsilon_x\varepsilon_y}(s)],$$

where $C(s)$ is uniformly bounded in probability in the neighborhood of λ_o. Thus by (3.3.10) we have $\tau^* - \tau^+ \to 0$ in probability when $N \to \infty$.

(4) Furthermore the rate of convergence is

$$\tau^* - \tau^+ = O_p(d^{3/2}(\Lambda_{k(N)})),$$

since

$$I^+_{\varepsilon_x \varepsilon_y}(s) - I^*_{\varepsilon_x \varepsilon_y}(s) = O_p[d^{3/2}(\Lambda_{K(N)})].$$

Q.E.D.

Now we can establish the weak consistency of the estimate τ^+.

THEOREM 3.3.5.

Under the conditions of Theorem 3.3.4 and Theorem 3.2.8, $\tau^+ \to \tau_o$ in probability when $N \to \infty$ where τ^+ is based on the frequency components $x_{\Lambda_{K(N)}}$, $y_{\Lambda_{K(N)}}$ and $z_{\Lambda_{K(N)}}$.

PROOF

From Theorem 3.3.4 we have $\tau^+ - \tau^* \to 0$ in probability when $N \to \infty$. On the other hand $\tau^* \to \tau_o$ in probability from Theorem 3.2.8. Combining these two results completes the proof.

Q.E.D.

3.4 CENTRAL LIMIT THEOREM

This section deals with the asymptotic distribution of the estimate τ^+. To begin with, we recall that in Section 2.4.3 $p(\tau)$ is defined as

$$p(\tau)=\int_B f_{\varepsilon_x \varepsilon_y}(\lambda)e^{-i\tau\lambda}d\lambda,$$

with $B=\{\lambda\,|\,\lambda_o-\pi/M<\lambda<\lambda_o+\pi/M\}$ and $2mM=N$. Let τ_M be the value of τ maximizing $|p(\tau)|^2$ for fixed M. Then we have the following theorem.

THEOREM 3.4.1

Assume Conditions D of Section 3.4.3 and the conditions of Theorem 3.2.7. There is a sequence $M=M(N)$ and there is a sequence $K(N)$ increasing with N and a corresponding sequence of bands $\Lambda_{K(N)}$ with $d(\Lambda_{K(N)})\to 0$ when $N\to\infty$ such that the estimate τ^+ which is based on $\Lambda_{K(N)}$, satisfies

$$N^{-1}m^{3/2}(\tau^+-\tau_M)\to N(0,3(1-\sigma^2(\lambda_o))/2\pi^2\sigma^2(\lambda_o))$$

in distribution when $N\to\infty$.

PROOF

From Theorem 3.3.4, we know that $\tau^+-\tau^*=O_p(m^{3/2}(\Lambda_{K(N)}))$. We also know from Theorem 2.4.3.4, that

$$N^{-1}m^{3/2}(\tau^{\#}-\tau_M)\to N(0,3\{1-\sigma^2(\lambda_o)\}/\{2[\pi\sigma(\lambda_o)]^2\}. \quad (3.4.1)$$

The convergence is in distribution, where $\tau^{\#}$ is the estimate based on $\varepsilon_{x\Lambda}$ and $\varepsilon_{y\Lambda}$ by using the procedure of Section 2.4.3 and maximizing the square of the norm of $p^{\#}(\tau)=(1/m)\Sigma_o I_{\varepsilon_x\varepsilon_y}(s)e^{-i\tau\omega_s}$.

From (3.4.1) $m=N^{(2+\delta)/3}$ for some δ such that $0<\delta<1$, thus

$$N^{-1}m^{3/2}[d^{3/2}(\Lambda_{K(N)})]=N^{\delta/2}[d(\Lambda_{K(N)})]^{3/2}.$$

In the proof of Theorem 3.2.8 we can have a sequence $\{\Lambda_k\}$ such that $d(\Lambda_k)\to 0$ as fast as we want. Thus we can have the sequence $\Lambda_{K(N)}$ such that

$$\{d(\Lambda_{K(N)})\}^{3/2}N^{\delta/2}\to 0 \text{ when } N\to\infty. \quad (3.4.2)$$

Write

$$N^{-1}m^{3/2}(\tau^{+}-\tau_M)=N^{-1}m^{3/2}(\tau^{+}-\tau^{*}+\tau^{*}-\tau^{\#}+\tau^{\#}-\tau_M). \quad (3.4.3)$$

From Theorem 3.3.4

$$N^{-1}m^{3/2}(\tau^{+}-\tau^{*})=O_p[N^{\delta/2}d^{3/2}(\Lambda_{K(N)})].$$

By (3.4.2) we have

$$N^{-1}m^{3/2}(\tau^{+}-\tau^{*})\to 0 \text{ in probability when } N\to\infty. \quad (3.4.4)$$

From Theorem 3.2.5, $\tau^{*}-\tau^{\#}=O_p(1/N)$. Thus in (3.4.3)

$$N^{-1}m^{3/2}(\tau^{*}-\tau^{\#})=O_p(N^{\delta/2-1}).$$

We have

$$N^{-1}m^{3/2}(\tau^{*}-\tau^{\#})\to 0 \text{ in probability when } N\to\infty. \qquad (3.4.5)$$

Combining (3.4.2)——(3.4.5) and using Theorem 3.4.5.2 we complete the proof. Q.E.D.

3.5 APPLICATION

As discussed in Chapters 1 and 2, for three given weakly-stationary stochastic processes $\{x_1(t)\}$, $\{x_2(t)\}$ and $\{x_3(t)\}$ the partial group delay $\tau_{12.3}(\lambda_o)$ measures the time-lag of $\{x(t)\}$ behind $\{y(t)\}$ adjusted for $\{z(t)\}$. Based on the results in Section 3.2 an immediate application of the asymptotic distribution of the estimator of partial group delay is in testing hypotheses about time lag relationships.

Theorem 3.2.1 gives the asymptotic distribution of $N^{-1}m^{3/2}(\tau^{+}_{12.3}-\tau_{12.3,M})$ where $\tau^{+}_{12.3}$ is the estimate of partial group delay between X_1 and X_2 adjusted for X_3 and $\tau_{12.3,M}$ is the value of τ maximizing $|p(\tau)|^2$ defined in (2.4.3.2) and Theorem 2.4.3.3. The obvious defect of that result is that $\tau_{12.3,M}$ may not equal the partial group delay $\tau_{12.3}(\lambda_o)$. Thus strictly speaking we can not readily apply the result to test the hypothesis

$$H_o\colon\ \tau_{12.3}(\lambda_o)=\tau_o.$$

However from the proof of Theorem 1 of Hannan and Thomson (1973) (see the statement after Theorem 2.4.3.3) when M (the length of $B=\pi/M$) increases to infinity, $\tau_M\to\tau_o$. Thus when the band length is small enough, it is feasible to approximate $\tau_{12.3}(\lambda_o)$ by the corresponding $\tau_{12.3,M}(\lambda_o)$, so that the result is applicable.

Further if we want to know whether X_1 lags behind X_2 after both series have been adjusted for X_3 and at the same time we want to know whether X_1 lags behind X_3 after both series have been adjusted for X_2 at λ_o, then we need to test of the two hypotheses

$$H_o\colon\ \tau_{12.3}(\lambda_o)=\tau_{o3}$$
$$H_o\colon\ \tau_{13.2}(\lambda_o)=\tau_{o2}.$$

The aim is to find the joint distribution of $\overset{+}{\tau}_{12.3}$ and $\overset{+}{\tau}_{13.2}$, which are the estimators defined in Section 3.1 step (4). To do this we first extend the result of Hannan and Thomson (1973) (see Theorem 2.4.3.4) to the case of four processes X_1, X_2, X_3 and X_4.

Now we have the following theorem.

THEOREM 3.5.1.

For weakly-stationary time series $\{X_1(t)\}$, $\{X_2(t)\}$, $\{X_3(t)\}$ and $\{X_4(t)\}$, if the conditions C and conditions D in Section 2.4.3 are satisfied for X_1, X_2, X_3, and X_4, then there is a sequence M(N) increasing with N, such that the vector with components $N^{-1}m^{3/2}(\tau^{\#}_{12}-\tau_{12,M})$ and $N^{-1}m^{3/2}(\tau^{\#}_{34}-\tau_{34,M})$ has a distribution converging to the bivariate normal distribution with zero mean and variances

$$3\{1-\sigma_i^2(\lambda_o)\}/\{2\pi^2\sigma_i^2(\lambda_o)\} \qquad (3.5.1)$$

and covariance

$$\frac{3f_{31}(\lambda_o)f_{24}(\lambda_o)f_{12}(\lambda_o)f_{43}(\lambda_o)-|f_{14}(\lambda_o)|^2|f_{32}(\lambda_o)|^2}{2\pi^2\ |f_{12}(\lambda_o)|^2\ |f_{34}(\lambda_o)|^2},$$

where $f_{ij}(\lambda)$ is the cross-spectral density of X_i and X_j for $i,j=1,2,3,4$, and where σ_i^2 is the partial coherence between X_i and X_{i+1} for $i=1$ and 3.

PROOF

We start by considering the asymptotic bivariate normality of the vector statistic

$$\{N^{-1}m^{3/2}(\tau^{\#}_{12}-\tau_{12,M}),\ N^{-1}m^{3/2}(\tau^{\#}_{34}-\tau_{34,M})\}. \qquad (3.5.2)$$

where we denote the estimator of group delay between X_1 and X_2 by $\tau^{\#}_{12}$. Similarly define $\tau^{\#}_{34}$. Write $\tau_{ij,M}$ for the value of τ maximizing

$$q_i=|\int_B f_{ij} e^{-i\tau\lambda} d\lambda|^2 \qquad (i,j)=(1,2) \text{ or } (3,4).$$

We repeat the argument of the proof of Theorem 2.4.3.4 for $N^{-1}m^{3/2}(\tau^{\#}_{12}-\tau_{12,M})$ and $N^{-1}m^{3/2}(\tau^{\#}_{34}-\tau_{34,M})$ individually from step 1) up to 4). Then in step 5), instead of considering the vector with components

$$\{N^{1/2}(c_{12}(-s)-\gamma_{12}(-s)),\ldots,N^{1/2}(c_{12}(s)-\gamma_{12}(s)\},$$

we consider the following vector with 2s+2 components,

$$N^{1/2}\{(c_{12}(-s)-\gamma_{12}(-s)),\ldots,(c_{12}(s)-\gamma_{12}(s)),$$
$$(c_{34}(-s)-\gamma_{34}(-s)),\ldots,(c_{34}(s)-\gamma_{34}(s))\}. \tag{3.5.3}$$

Then by using exactly the same argument as in the proof of Theorem 2.4.3.4 step (5), by using the vector generalization of Theorem 1 of Gordin (1969), the vector in (3.5.3) has an asymptotic multivariate normal distribution, as each component of the bivariate statistic in (3.5.2) is a linear combination of the components of the vector statistic in (3.5.3). Thus the bivariate statistic in (3.5.2) has a distribution that converges to the bivariate normal distribution, with zero mean and with each variance of the form in Theorem 2.4.3.4.

We have found the asymptotic normality and the asymptotic variances. Now we shall find the asymptotic covariance. Just as in the proof of Theorem 2.4.3.4 (see step (1) there)

$$\begin{aligned} &N^{1/2}(\tau^{\#}_{ij}-\tau_{ij,M}) \\ &=-N^{1/2}q^{\#'}_{i}(\tau_{ij,M})/q^{\#''}_{i}(\tau^{o}_{i}), \end{aligned}$$

where τ^{o}_{i} is some value such that $|\tau^{o}_{i}-\tau_{ij,M}|\le|\tau^{\#}_{ij}-\tau_{ij,M}|$ for (i,j)=(1,2) and (3,4).

Then

$$\begin{aligned} &E[m^{3/2}N^{-1}(\tau^{\#}_{12}-\tau_{12,M}),\ m^{3/2}N^{-1}(\tau^{\#}_{34}-\tau_{34,M})] \\ &=m^{3}N^{-3}E\left[\frac{N^{1/2}q^{\#'}_{1}(\tau_{12,M})\ N^{1/2}q^{\#'}_{3}(\tau_{34,M})}{q^{\#''}_{1}(\tau^{o}_{1})\quad q^{\#''}_{3}(\tau^{o}_{3})}\right]. \end{aligned} \tag{3.5.4}$$

When $N\to\infty$, we know from Lemma 2.4.3.1, that

$$\begin{aligned} &q^{\#''}_{1}(\tau^{o}_{1})\to(M/\pi)^{2}q_{1}''(\tau_{12,M}) && \text{a.s.} \\ &q^{\#''}_{3}(\tau^{o}_{3})\to(M/\pi)^{2}q_{3}''(\tau_{34,M}) && \text{a.s..} \end{aligned}$$

Thus instead of the mean square of $N^{1/2}q^{\#'}(\tau_{M})$ in step 3) in Theorem 2.4.3.4, (3.5.4) can be approximated by

$$=m^3N^{-3}E\left[\frac{N^{1/2}q_1^{\#'}(\tau_{12,M})\ N^{1/2}q_3^{\#'}(\tau_{34,M})}{(M/\pi)^2q_1''(\tau_{12,M})(M/\pi)^2q_3''(\tau_{34,M})}\right]. \qquad (3.5.5)$$

Substituting $m=N/2M$, (3.5.5) becomes

$$=(2M)^3E\left[\frac{N^{1/2}q_1^{\#'}(\tau_{12,M})\ N^{1/2}q_3^{\#'}(\tau_{34,M})}{(M/\pi)^4q_1''(\tau_{12,M})q_3''(\tau_{34,M})}\right]. \qquad (3.5.6)$$

Following step 2) and step 3) in Theorem 2.4.3.4 $N^{1/2}q_i^{\#'}(\tau_{ij,M})$ can be replaced by (in the sense of approximation)

$$N^{1/2}\int_{-\pi}^{\pi}\phi_i(\lambda)[I_{ij}(\lambda)-EI_{ij}(\lambda)]d\lambda \quad (i,j)=(1,2) \text{ and } (3,4).$$

Now similar to step 2) and step 3) in Theorem 2.4.3.4 when M is fixed and $N\to\infty$, the numerator in (3.5.6), i.e., $E[N^{1/2}q_1^{\#'}(\tau_{12,M})\ N^{1/2}q_3^{\#'}(\tau_{34,M})]$ converges to

$$\begin{aligned}&2\pi\int_{-\pi}^{\pi}[f_{31}(\lambda)f_{24}(\lambda)\phi_1(\lambda)^c\phi_3(\lambda)\\&\qquad+f_{14}(\lambda)f_{32}(\lambda)\phi_1(\lambda)\phi_2(\lambda)]d\lambda\\&+2\pi\iint_{-\pi}^{\pi}\phi_1(\lambda)\phi_3(\mu)^cf_{1234}(-\lambda,\lambda,\mu,-\mu)d\lambda d\mu,\end{aligned} \qquad (3.5.7)$$

where

$$\phi_i(\lambda)=(M/\pi)^2\int_B i(\theta-\lambda)e^{i\tau_{ij,M}(\theta-\lambda)}f_{ij}(\theta)^cd\theta$$

for $\lambda\varepsilon B$ and $\phi_i(\lambda)$ is zero on the complement of B in $(0,\pi)$ and $\phi_i(-\lambda)=\phi_i(\lambda)$.

Hence by (3.5.4)——(3.5.7) the covariance of $m^{3/2}N^{-1}(\tau^{\#}_{12}-\tau_{12,M})$ and $m^{3/2}N^{-1}(\tau^{\#}_{34}-\tau_{34,M})$ when M is fixed and $N\to\infty$, converges to

$$
\begin{aligned}
&(2M)^{-3}(M/\pi)^{-4}[q_1''(\tau_{12,M})q_3''(\tau_{34,M})]^{-1}\\
&\quad 2\pi\{\textstyle\int_{-\pi}^{\pi}[f_{31}(\lambda)f_{24}(\lambda)\phi_1(\lambda)^c\phi_3(\lambda)\\
&\qquad\qquad +f_{14}(\lambda)f_{32}(\lambda)\phi_1(\lambda)\phi_2(\lambda)]d\lambda\}\\
&\quad +2\pi\textstyle\iint_{-\pi}^{\pi}\phi_1(\lambda)\phi_3(\lambda)^c f_{1234}(-\lambda,\lambda,\mu,-\mu)d\lambda d\mu.
\end{aligned}
$$

Then just as in step 7) in Theorem 2.4.3.2,

$$\phi_i(\lambda)\simeq(-i)(M/\pi)(\lambda-\lambda_o)f_{ij}(\lambda_o)^c\{1+o(1)\}$$

for (i,j)=(1,2) and (3,4).

Since $f_{ij}(\lambda)$ is boundedly differentiable, we may replace it in the integral by $f_{ij}(\lambda_o)$. Evaluating the integral and let $M\to\infty$ we obtain

$$
\begin{aligned}
&4\pi M\textstyle\int_{-\pi}^{\pi}f_{31}(\lambda)f_{24}(\lambda)\phi_1(\lambda)^c\phi_3(\lambda)d\lambda\\
&\to (2\pi^2/3)f_{31}(\lambda_o)f_{24}(\lambda_o)f_{12}(\lambda_o)f_{43}(\lambda_o)
\end{aligned}
$$

and

$$
\begin{aligned}
&4\pi M\textstyle\int_{-\pi}^{\pi}f_{14}(\lambda)f_{32}(\lambda)\phi_1(\lambda)\phi_3(\lambda)d\lambda\\
&\to (-1)(2/3)\pi^2|f_{14}(\lambda_o)|^2|f_{32}(\lambda_o)|^2.
\end{aligned}
$$

It is evident that

$$\iint_{-\pi}^{\pi}\phi_1(\lambda)\phi_3(\lambda)^c f_{1234}(-\lambda,\lambda,\mu,-\mu)d\lambda$$

converges to zero as M increases, since the integrand is bounded and zero except on a set whose area is $O(M^{-2})$.

As step 7) in the proof of Theorem 2.4.3.4, when $M\to\infty$

$$(4M^4/\pi^2)q_i''(\tau_{ij,M})\to(2\pi^2/3)|f_{ij}(\lambda_o)|^2.$$

Therefore based on the choice of m(N)

$$E[m^{3/2}N^{-1}(\tau_{12}^{\#}-\tau_{12,m})m^{3/2}N^{-1}(\tau_{34}^{\#}-\tau_{34,m})]$$

approaches

$$\frac{3f_{31}(\lambda_o)f_{24}(\lambda_o)f_{12}(\lambda_o)f_{43}(\lambda_o)-|f_{14}(\lambda_o)|^2|f_{32}(\lambda_o)|^2}{2\pi^2\ |f_{12}(\lambda_o)|^2\ |f_{34}(\lambda_o)|^2}.$$

We note that $f_{31}(\lambda_o)f_{24}(\lambda_o)f_{12}(\lambda_o)f_{34}(\lambda_o)^c$ takes real value. This completes the proof. Q.E.D.

We now return to the original problem of the multiple test:

$$H_o:\quad \tau_{12.3}=\tau_{o3}$$
$$\tau_{13.2}=\tau_{o2}.$$

We define the residual processes as in section 2.5, by

$$\varepsilon_1(t)=X_1(t)-\Pi(X_1(t)|X_3(t)),$$
$$\varepsilon_2(t)=X_2(t)-\Pi(X_2(t)|X_3(t)),$$
$$\varepsilon_3(t)=X_1(t)-\Pi(X_1(t)|X_2(t)),$$
$$\varepsilon_4(t)=X_3(t)-\Pi(X_3(t)|X_2(t)).$$

Then $\tau_{12.3}$ is the group delay between ε_1 and ε_2, $\tau_{13.2}$ is the group delay between ε_3 and ε_4. Hence from Theorem 3.5.1 $\{N^{-1}m^{3/2}(\tau^{\#}_{12.3}-\tau_{12.3,M}),\quad N^{-1}m^{3/2}(\tau^{\#}_{13.2}-\tau_{13.2,M})\}$, is asymptotically bivariate normally distributed with the covariance matrix stated in Theorem 3.5.1. If we replace $\tau^{\#}_{12.3}$ by $\tau^{+}_{12.3}$ and $\tau^{\#}_{13.2}$ by $\tau^{+}_{13.2}$, then the asymptotic result still holds by following the argument in the proof of Theorem 3.2.1. Therefore we can do multiple tests as desired.

4.0 ESTIMATING PARTIAL GROUP DELAY——DIRECT APPROACH

4.1 THE ESTIMATING PROCEDURE

In Chapter 3, we discussed an indirect approach to estimating partial group delay. The estimator is weakly consistent and asymptotically normally distributed.

But for this approach there are the following disadvantages:

1. For actual data processes, the computation of the estimate is rather lengthy. We need to compute the frequency components of X, Y, Z, the estimates, $\varepsilon^{+}_{x\Lambda}$ and $\varepsilon^{+}_{y\Lambda}$ of $\varepsilon_{x\Lambda}$ and $\varepsilon_{y\Lambda}$, and then the estimate of partial group delay from $\varepsilon^{+}_{x\Lambda}$ and $\varepsilon^{+}_{y\Lambda}$;

2. We estimate $\varepsilon^{+}_{x\Lambda}(t)$ by first estimating α and ϕ_1 by α^{+} and ϕ^{+}_1; then we let

$$\varepsilon^{+}_{x\Lambda}(t)=X_\Lambda-\alpha^{+}Z_\Lambda(t-\phi^{+}_1). \tag{4.1.1}$$

Similarly we let

$$\varepsilon^{+}_{y\Lambda}(t)=Y_\Lambda-\beta^{+}Z_\Lambda(t-\phi^{+}_2). \tag{4.1.2}$$

In cases where $Z_\Lambda(t)$ is observed only at integral times $t=0,\pm1,\ldots$ and in the case that ϕ_1^+ and ϕ_2^+ are not integers, $Z_\Lambda(t-\phi_1^+)$ and $Z_\Lambda(t-\phi_2^+)$ are not available observations for computing (4.1.1) and (4.1.2). Hence in that case the indirect approach is not available for estimating partial group delay.

Therefore we shall look for some other approach.

We still consider the weakly-stationary processes $\{x(t)\}$, $\{y(t)\}$ and $\{z(t)\}$ with mean zero, and assume that all of these have absolutely continuous spectra with continuous densities.

As defined in Chapter 3 we have the finite Fourier transforms of $\{X(t)\}$, $\{Y(t)\}$, and $\{Z(t)\}$ as follows:

$$w_x(s)=(2\pi N)^{-1/2}\Sigma_{n=1}^{N}x(n)e^{in\omega}s,$$

$$w_y(s)=(2\pi N)^{-1/2}\Sigma_{n=1}^{N}y(n)e^{in\omega}s,$$

$$w_z(s)=(2\pi N)^{-1/2}\Sigma_{n=1}^{N}z(n)e^{in\omega}s,$$

where $\omega_s=2\pi s/N$ for $s\le[N/2]$.

Correspondingly we have the periodograms as follows:

$$I_{xy}(s)=w_x(s)w_y(s)^c,$$

$$I_{xz}(s)=w_x(s)w_z(s)^c,$$

$$I_{zy}(s)=w_z(s)w_y(s)^c,$$

$$I_{zz}(s)=w_z(s)w_z(s)^c.$$

We also write $I_{xy}(s)$ as $I_{xy}(\omega_s)$ etc..

Our aim is to estimate the partial group delay between X and Y adjusted for Z at frequency $\lambda=\lambda_o$.

We consider a band of frequencies centered at λ_o and containing m fundamental frequencies ω_s, i.e., $B=\{\lambda|\lambda_o-\pi/2M<\lambda<\lambda_o+\pi/2M\}$. Here we still have the relation 2mM=N. We also consider two other bands B' and B'' which are the location shift of B, i.e., $B'=\{\lambda|\lambda_o<\lambda<\lambda_o+2\pi/2M\}$ and $B''=\{\lambda|\lambda_o-2\pi/2M<\lambda<\lambda_o\}$.

For each $\omega_s \varepsilon B$ define

$$f^*_{xx}(\omega_s)=(1/3)[I_{xx}(\omega_s)+I_{xx}(\omega_s+\pi/M)+I_{xx}(\omega_s-\pi/M)]. \tag{4.1.3}$$

Similarly we also have $f^*_{xz}(\omega_s)$, $f^*_{zy}(\omega_s)$, $f^*_{zz}(\omega_s)$ and $f^*_{zz}(\omega_s)$. With those we define

$$f^*_{xy.z}(\lambda)=f^*_{xy}(\lambda)-f^*_{xz}(\lambda)f^{*^{-1}}_{zz}(\lambda)f^*_{zy}(\lambda), \tag{4.1.4}$$

and let

$$p^-(\tau)=(1/m)\Sigma_o f^*_{xy.z}(\omega_s)e^{-i\tau\omega_s} \tag{4.1.5}$$

where the summation is over the m frequencies $\omega_s=2\pi s/N\varepsilon B$.

Let τ^- be the value of τ maximizing $|p^-(\tau)|^2$; this τ^- is the estimate we propose.

4.2 WEAK CONSISTENCY OF THE ESTIMATE

From Section 2.5 x(t) and y(t) can be decomposed as

$$x(t)=\Pi(x(t)|H(z))+\varepsilon_x(t)$$

and

$$y(t)=\Pi(y(t)|H(z))+\varepsilon_y(t),$$

where H(z) is the closed linear manifold generated by {z(t)} and $\Pi(x(t)|H(z))$, $\Pi(y(t)|H(z))$ are the projections of x(t) and Y(t) onto H(z).

As we defined in Section 2.5, the partial group delay is just the simple group delay between $\varepsilon_x(t)$ and $\varepsilon_y(t)$. However, we do not know $\varepsilon_x(t)$ and $\varepsilon_y(t)$. We only know x(t), y(t) and z(t) for t=1,..,N.

First we want to find the relationship between an estimate of partial group delay based on $\varepsilon_x(t)$, $\varepsilon_y(t)$ and an estimate based on x(t), y(t) and z(t).

Let u(t)=(x(t),y(t))'. Then

$$u(t)=\begin{bmatrix}\Pi(x(t)|H(z))\\ \\ \Pi(y(t)|H(z))\end{bmatrix}+\begin{bmatrix}\varepsilon_x(t)\\ \\ \varepsilon_y(t)\end{bmatrix}.$$

We represent $(\varepsilon_x(t)\ \varepsilon_y(t))'$ by $\varepsilon(t)'$ and define

$$f^{\#}_{xx}(\lambda)$$
$$=(1/[2m+1])\Sigma^{m}_{s=-m}w_x\{[2\pi K(N)+s]/N\}w_x\{[2\pi K(N)+s]/N\}^{c} \qquad (4.2.1)$$

where $K(N)$ is an integer depending on N with $2\pi K(N)/N\to\lambda$ when $N\to\infty$. We also write

$$f^{\#}_{xy}(\lambda)$$
$$=(1/[2m+1])\Sigma^{m}_{s=-m}w_x\{[2\pi K(N)+s]/N\}w_y\{[2\pi K(N)+s]/N\}^{c} \qquad (4.2.2)$$

$$f^{\#}_{uu}(\lambda)$$
$$=(1/[2m+1])\Sigma^{m}_{s=-m}w_u\{[2\pi K(N)+s]/N\}w_u\{[2\pi K(N)+s]/N\}^{c} \qquad (4.2.3)$$

and

$$f^{\#}_{uz}(\lambda)$$
$$=(1/[2m+1])\Sigma^{m}_{s=-m}w_u\{[2\pi K(N)+s]/N\}w_z\{[2\pi K(N)+s]/N\}^{c}. \qquad (4.2.4)$$

Then as defined in the statement before Theorem 2.4.4.7,

$$g^{\#}_{\varepsilon}(\lambda)=C(m,r)\{f^{\#}_{uu}(\lambda)-f^{\#}_{uz}(\lambda)f^{\#}_{zz}(\lambda)^{-1}f^{\#}_{zu}(\lambda)\} \qquad (4.2.5)$$

when $\lambda \not\equiv 0(\mathrm{mod}\pi)$ and $C(m,r)=(2m+1)/(2m+1-r)$. In (4.2.3), we can write the smoothed periodogram $f^{\#}_{uu}(\lambda)$ as

$$f^{\#}_{uu}(\lambda)=(1/[2m+1])\Sigma^{m}_{s=-m}I_{uu}(2\pi K(N)+s/N), \tag{4.2.6}$$

where $I_{uu}(s)=w_u(s)w_u(s)^c$ is the periodogram. Sometimes we also denote it by $I_{uu}(\omega_s)$ for $\omega_s=2\pi s/N$. In matrix form, it is written

$$I_{uu}(s)=\begin{bmatrix} w_x(s)w_x(s)^c & w_x(s)w_y(s)^c \\ w_y(s)w_x(s)^c & w_y(s)w_y(s)^c \end{bmatrix}.$$

$$=\begin{bmatrix} I_{xx}(s) & I_{xy}(s) \\ I_{yx}(s) & I_{yy}(s) \end{bmatrix}.$$

From Theorem 2.4.4.7 with r=1 s=2, $g^{\#}_{\varepsilon}(\lambda)$ is asymptotically distributed as the complex Wishart distribution, i.e.,

$$g^{\#}_{\varepsilon}(\lambda)\rightarrow (1/2m)W^{c}_{2}(2m,f_{\varepsilon\varepsilon}(\lambda)) \text{ in distribution} \tag{4.2.7}$$

when $\lambda \not\equiv 0(\mathrm{mod}\pi)$ and $N\rightarrow\infty$, where $f_{\varepsilon\varepsilon}(\lambda)$ is the spectral density matrix of $\varepsilon(t)$.

On the other hand, based on $\varepsilon(t)$ we define

$$f^{\#}_{\varepsilon\varepsilon}(\lambda)=(1/[2m+1])\Sigma^{m}_{s=-m}I_{\varepsilon\varepsilon}([2\pi K(N)+s]/N). \qquad (4.2.8)$$

By Theorem 2.4.4.6,

$$f^{\#}_{\varepsilon\varepsilon}(\lambda)\rightarrow (1/[2m+1])W^{c}_{2}(2m+1,\ f_{\varepsilon\varepsilon}(\lambda)) \qquad (4.2.9)$$

in distribution, when $\lambda\not\equiv 0(\mathrm{mod}(\pi)$ and $N\rightarrow\infty$.

Now in (4.2.8), let m=0. We obtain the periodogram $f^{\#}_{\varepsilon\varepsilon}(\lambda)=I_{\varepsilon\varepsilon}(\lambda)$. Thus from (4.2.9), $I_{\varepsilon\varepsilon}(\lambda)\rightarrow W^{2}_{2}(1,\ f_{\varepsilon\varepsilon}(\lambda))$. In (4.2.7) let m=1 then,

$$g^{\#}_{\varepsilon}(\lambda)\rightarrow(1/2)W^{c}_{2}(2,f_{\varepsilon\varepsilon}(\lambda)) \qquad (4.2.10)$$

in distribution, when $\lambda\equiv 0(\mathrm{mod}\ \pi)$ and $N\rightarrow\infty$

We can write $I_{\varepsilon\varepsilon}(\lambda)$ and $g^{\#}_{\varepsilon}(\lambda)$ in partitioned form as

$$I_{\varepsilon\varepsilon}(\lambda)= \begin{bmatrix} I_{\varepsilon_x\varepsilon_x}(\lambda) & I_{\varepsilon_x\varepsilon_y}(\lambda) \\ I_{\varepsilon_y\varepsilon_x}(\lambda) & I_{\varepsilon_y\varepsilon_y}(\lambda) \end{bmatrix}$$

and

$$g^{\#}_{\varepsilon}(\lambda)=(3/2)\begin{bmatrix} f^{\#}_{xx}-f^{\#}_{xz}f^{\#\,-1}_{zz}f^{\#}_{zx} & f^{\#}_{xy}-f^{\#}_{xz}f^{\#\,-1}_{zz}f^{\#}_{zy} \\ f^{\#}_{yx}-f^{\#}_{yz}f^{\#\,-1}_{zz}f^{\#}_{zx} & f^{\#}_{yy}-f^{\#}_{yz}f^{\#\,-1}_{zz}f^{\#}_{zy} \end{bmatrix}.$$

Now for the band B defined in Section 4.1, there are m fundamental frequencies $\omega_s=2\pi s/N$ in B and as $M\to\infty$ $\omega_s\to\lambda_o$ for s=1,...m. Define

$$I^*_{\varepsilon\varepsilon}(\omega_s)=(1/2)[I_{\varepsilon\varepsilon}(\omega_s)+I_{\varepsilon\varepsilon}(\omega_s+(\pi/M))] \tag{4.2.11}$$

as the average of the frequencies in B and the corresponding frequencies in B', which is defined in Section 4.1 and is a location shift of B that contains the frequencies of $2\pi(s+2m)/N$, s=1,...m.

For later use, we give the following lemma.

LEMMA 4.2.1.

Let X(t), t=0,±1,... be a r-dimensional vector series satisfying Assumption 2.4.4.2. Let $K_j(N)$ be an integer with $\lambda_j(N)=2\pi K_j(N)/N\to\lambda$ for j=1,2, as $N\to\infty$. Suppose $K_1(N)\pm K_2(N)\not\equiv 0$ (mod N), then the finite Fourier transforms $w_x(\lambda_1(N))$ and $w_x(\lambda_2(N))$ are asymptotically independent $N^C_r(0,f_x(\lambda))$ variates respectively.

PROOF

$E[w_x(\lambda_j(N))]=0$ since $E[x(t)]=0$. From Theorem 2.4.4.4

$$\begin{aligned}&COV\{w_x(\lambda_j(N)),w_x(\lambda_k(N))\}\\&=(1/N)\Delta^{(N)}\{2\pi[K_j(N)-K_k(N)]/N\}f_{xx}(\lambda_j(N))+O(1/N),\end{aligned}$$

where $\Delta^{(N)}(\lambda)=N$ if $\lambda\equiv 0 \pmod{\pi}$ and $\Delta^{(N)}(2\pi s/N)=0$ if s is an integer. Thus $COV\{w_x(\lambda_j(N), w_x(\lambda_k(N)\}$ tends to zero if $K_j(N)-K_k(N)\not\equiv 0 \pmod{N}$ and $Var(w_x(\lambda_j(N))$ tends to $f_{xx}(\lambda)$.

Finally, again from Theorem 2.4.4.4,

$$\begin{aligned}
&Cum(w_x(\lambda_1),\ldots,w_x(\lambda_k)) \\
&=(2\pi)^{k/2-1}N^{-k/2}\Delta^{(N)}(\Sigma_1^k\lambda_j)f_{x\ldots x}(\lambda_1,\ldots\lambda_k)+O(N^{-k/2}).
\end{aligned}$$

This last tends to 0 as $N\to\infty$ if $k>2$ by the definition of $\Delta^{(N)}(.)$.

Putting the above results together, we see that the cumulants of the variates at issue, and the conjugates of those variates, tend to the cumulants of a normal distribution. The conclusion of the lemma now follows from Theorem 2.4.4.3 since the normal distribution is determined by its moments Q.E.D.

Based on the above lemma we have the following lemma.

LEMMA 4.2.2

Let $X(t)$, $t=0,\pm1,\ldots$ be a r-dimensional vector series satisfying Assumption 2.4.4.2. Let $K_j(N)$ be an integer with $\lambda_j(N)=2\pi K_j(N)/N\to\lambda$ for $j=1,2$ as $N\to\infty$ let $I_{xx}(\lambda)=w_x(\lambda)w_x(\lambda)^*$ for $-\infty<\lambda<\infty$. Suppose $K_1(N)\pm K_2(N)\not\equiv 0 \pmod{N}$. Then $I_{xx}(\lambda_j)$ $j=1,2$ are asymptotically independent complex Wishart distributed $W_r^C(1,f_{xx}(\lambda))$, $j=1,2$.

PROOF

From Lemma 4.1.1, $w_x(\lambda_j)$ j=1,2 are asymptotically independent $N_r^C(0,f_{xx}(\lambda))$ variates. By the definition of the complex Wishart distribution (see Brillinger (1975) p.90) and Theorem 2.4.4.6, the lemma is completed. Q.E.D.

We assume throughout this chapter that $\varepsilon(t)$ satisfies Assumption 2.4.4.2.

Thus by Lemma 4.2.2, $I_{\varepsilon\varepsilon}(\omega_s)$ and $I_{\varepsilon\varepsilon}(\omega_s+2\pi m/N)$ are asymptotically independent and each of those is asymptotically distributed as $W_2^C(1,f_{\varepsilon\varepsilon}(\lambda))$ when $N\to\infty$ and $M\to\infty$. Thus from (4.2.11)

$$I^*_{\varepsilon\varepsilon}(\omega_s)\to(1/2)W_2^C(2,f_{\varepsilon\varepsilon}(\lambda)) \tag{4.2.12}$$

in distribution when $N\to\infty$.

Now, instead of

$$p^{\#}(\tau)=(1/m)\Sigma_o I_{\varepsilon_x\varepsilon_y}(\omega_s)e^{-i\tau\omega_s}, \tag{4.2.13}$$

consider

$$p^*(\tau)=(1/m)\Sigma_o I^*_{\varepsilon_x\varepsilon_y}(\omega_s)e^{-i\tau\omega_s}, \tag{4.2.14}$$

where the summation is over the m frequencies $\omega_s \varepsilon B$ and $I^*_{\varepsilon_x\varepsilon_y}(\omega_s)$ is the corresponding element in matrix $I^*_{\varepsilon\varepsilon}(\omega_s)$, i.e.,

$$I^*_{\varepsilon_x\varepsilon_Y}(\omega_s)=(1/2)[I_{\varepsilon_x\varepsilon_Y}(\omega_s)+I_{\varepsilon_x\varepsilon_Y}(\omega_s+\pi/M)]. \tag{4.2.15}$$

From Lemma 2.4.3.1 for fixed M,

$$|p^{\#}(\tau)-(M/\pi)p(\tau)|\to 0 \quad \text{almost surely}$$

and uniformly for all τ when $N\to\infty$, where

$$p(\tau)=\int_B f_{\varepsilon_x\varepsilon_y}(\lambda)e^{-i\tau\lambda}d\lambda. \tag{4.2.16}$$

Now let

$$p^{+}(\tau)=(1/2)[\int_B f(\lambda)e^{-i\tau\lambda}d\lambda+\int_{B'} f(\lambda)e^{-i\tau\lambda}d\lambda e^{i\tau\pi/M}]. \tag{4.2.17}$$

Here $f(\lambda)$ stands for $f_{\varepsilon_x\varepsilon_y}(\lambda)$. Now we give the following lemma.

LEMMA 4.2.3.

For fixed M, $p^{*}(\tau)-(M/\pi)p^{+}(\tau)\to 0$ a.s. when $N\to\infty$ and uniformly for τ in any bounded interval V.

PROOF

We shall omit the subscripts from $I_{\varepsilon_x\varepsilon_y}$ during the proof. From (4.2.14)

$$p^{*}(\tau)=(1/2m)\Sigma_o I(\omega_s)e^{-i\tau\omega_s}+(1/2m)\Sigma_o I(\omega_s+\pi/M)e^{-i\tau\omega_s}$$
$$=(1/2m)\Sigma_o I(\omega_s)e^{-i\tau\omega_s}+(1/2m)\Sigma_{o'} I(\omega_s)e^{-i\tau\omega_s}\, e^{i\tau\pi/M}$$

where the summation o' is over all ω_s in B'. By Lemma 2.4.3.1 when M fixed and $N\to\infty$, we have

$$(1/m)\Sigma_o I(\omega_s)e^{-i\tau\omega_s}-(M/\pi)\int_B f_{12}(\lambda)e^{-i\tau\lambda}d\lambda\to 0 \quad \text{a.s..}$$

Similarly when we replace B by B',

$$(1/m)\Sigma_{o'} I(\omega_s)e^{-i\tau\omega_s}-(M/\pi)\int_{B'} f_{12}(\lambda)e^{-i\tau\lambda}d\lambda\to 0 \quad \text{a.s..}$$

Thus $p^*(\tau)-(M/\pi)p^+(\tau)\to 0$ a.s.. Q.E.D.

It is easy to establish the following lemma.

LEMMA 4.2.4.

If $u_n-v_n\to 0$ when $n\to\infty$, and if either the sequence u_n or the sequence v_n is bounded and if $t_n\to 1$ when $n\to\infty$, then $t_n u_n-v_n\to 0$ when $n\to\infty$.

Let $p^+(\tau)$ be defined by (4.2.17) and $p(\tau)$ be defined by (4.2.16), we also can prove the following lemma.

LEMMA 4.2.5.

$(M/\pi)[p^+(\tau)-p(\tau)]\to 0$ when $M\to\infty$ and uniformly for any bounded interval of τ.

PROOF

We consider $(M/\pi)[\int_B f(\lambda)e^{-i\tau\lambda}d\lambda-\int_{B'} f(\lambda)e^{-i\tau\lambda}d\lambda]$, and define the real part and imaginary part of $g(\lambda)=f(\lambda)e^{-i\tau\lambda}$ by g_R and g_I. Then

$$\int_B f(\lambda)e^{-i\tau\lambda}d\lambda-\int_{B'}f(\lambda)^{-i\tau\lambda}d\lambda,$$
$$=|\int_B(g_R+ig_I)d\lambda-\int_{B'}(g_R+ig_I)d\lambda|$$
$$\le|\int_B g_R d\lambda-\int_{B'}g_R d\lambda|+|\int_B g_I d\lambda-\int_{B'}g_I d\lambda|. \tag{4.2.18}$$

By using the mean value theorem,

$$(M/\pi)[\int_B g_R d\lambda-\int_{B'}g_R d\lambda] =g_R(\lambda^*)-g_R(\lambda^{**}).$$

where $\lambda^*\varepsilon B$ and $\lambda^{**}\varepsilon B'$ with $(\lambda^*-\lambda^{**})\to 0$ when $M\to\infty$. Since $g_R(\lambda)$ is a continuous function of λ, the above difference converges to zero when $M\to\infty$. A similar result holds for $(M/\pi)\int_B g_I d\lambda-\int_{B'}g_I d\lambda$.

Thus from (4.2.18) we have

$$(M/\pi)\int_B f(\lambda)e^{-i\tau\lambda}d\lambda-\int_{B'}f(\lambda)e^{-i\tau\lambda}d\lambda\to 0, \tag{4.2.19}$$

when $M\to\infty$ and uniformly for τ in any bounded interval.

Since $p^+(\tau)-p(\tau)$

$$=(1/2)[\int_{B'}f(\lambda)e^{-i\tau\lambda}d\lambda e^{i\tau\pi/M} -\int_B f(\lambda)e^{-i\tau\lambda}d\lambda],$$

also since $e^{-i\tau\pi/M}\to 1$ and uniformly for all bounded τ when $M\to\infty$ and by applying Lemma 4.2.3 we complete the proof.

Further we have the following lemma.

LEMMA 4.2.6.

Let τ_M^* be the value maximizing $|p^+(\tau)|^2$ for fixed M, then $\tau_M^* \to \tau_o = \theta'(\lambda_o)$ when $M \to \infty$.

PROOF

By definition, for fixed M τ_M^* maximizes $q^+(\tau)=|p^+(\tau)|^2$ and τ_M maximizes $q(\tau)=|p(\tau)|^2$. Equivalently τ_M^* maximizes $R^+(\tau)=(M/\pi)^2 q^+(\tau)$ and τ_M maximizes $R(\tau)$.

From Lemma 4.2.5 as $R(\tau)$ is bounded,

$$R^+(\tau)-R(\tau)\to 0 \quad \text{a.s.} \tag{4.2.20}$$

when $M\to\infty$ and uniformly for bounded τ. From Theorem 3.4.4.3 $\tau_M \to \tau_o$ when $M\to\infty$. Thus for any given $\eta>0$, there exists an integer $M_o>0$ such that when $M>M_o$, τ_M is an interior point of the interval $(\tau_o-\eta,\ \tau_o+\eta)$. We can assume that $\tau_M \varepsilon (\tau_o-\eta,\ \tau_o)$.

Since τ_M maximizes $R(\tau)$, in a half neighborhood of τ_M, say, $(\tau_M-\eta,\ \tau_M)$, $R(\tau)$ is an increasing function of τ. Then in that interval there exists a unique inverse function $\tau=V(R)$. τ is continuous in R. Thus for the above given η there exists a $\varepsilon>0$ such that when $|R(\tau)-R(\tau_M)|<\varepsilon$,

$$|\tau-\tau_M|<\eta. \tag{4.2.21}$$

From $R^+(\tau)-R(\tau)\to 0$ when $M\to\infty$, in (4.2.20) for that ε, there exists a M_1 such that when $M>M_1$

$$R(\tau_M)\le R^+(\tau_M)+\varepsilon/2. \tag{4.2.22}$$

Without loss of generality, we assume that $\tau_M^*<\tau_M$. Since τ_M^* maximizes $R^+(\tau)$,

$$R^+(\tau_M)\le R^+(\tau_M^*). \tag{4.2.23}$$

Again from (4.2.20) when $M>M_1$,

$$R^+(\tau_M^*)\le R(\tau_M^*)+\varepsilon/2. \tag{4.2.24}$$

Combining (4.2.22)——(4.2.24) we have $R(\tau_M)\le R(\tau_M^*)+\varepsilon$.

Thus from (4.2.21) when $M>M_2=\max\{M_o, M_1\}$, we have $|\tau_M-\tau_M^*|<\eta$ and complete the proof as we know that $\tau_M\to\tau_o$ when $M\to\infty$.

Now let τ^* be the value of τ maximizing $|p^*(\tau)|^2$ which is defined in (4.2.14). Then based on Lemma 4.2.6, we have the following lemma.

LEMMA 4.2.7.

Assume that $\varepsilon_x(t)$, $\varepsilon_y(t)$ satisfy Assumption 2.4.4.2 and conditions C in Section 2.4.3. Then there is a sequence M(N) increasing with N, such that τ^* converges almost surely to τ_o.

PROOF

Choosing an interval of τ which is big enough, say V, it follows from Lemma 4.2.3 when M is fixed and $N\to\infty$,

$$p^*(\tau)-(M/\pi)p^+(\tau)\to 0. \qquad \text{a.s.}$$

i.e., $$p^*(\tau)\to(M/\pi)p^+(\tau) \qquad \text{a.s.}$$

Hence we have

$$q^*(\tau)\to(M/\pi)^2q^+(\tau) \qquad \text{a.s.} \tag{4.2.25}$$

and uniformly for all $\tau\varepsilon V$ where $q^*(\tau)=|p^*(\tau)|^2$ and $q^+(\tau)=|p^+(\tau)|^2$. Thus from (4.2.25) for any $\varepsilon>0$ and for N sufficiently large almost surely we have

$$(M/\pi)^2q^+(\tau_M^*)-\varepsilon\le q^*(\tau_M^*). \tag{4.2.26}$$

However by definition τ^* is the value of τ maximizing $q^*(\tau)$,

$$q^*(\tau_M^*)\le q^*(\tau^*). \tag{4.2.27}$$

Again from (4.2.25) almost surely,

$$q^*(\tau^*)\le(M/\pi)^2q^+(\tau^*)+\varepsilon. \tag{4.2.28}$$

Then combining (4.2.26)——(4.2.28), we know that for fixed M as $N\to\infty$, τ^* must almost surely converge to τ_M^*. Then by using Lemma 2.4.3.2, Lemma 4.2.5, and letting t_M in Lemma 2.4.3.2 to be τ_M^*, t to be τ_o for each M and $t_M(N)=\tau^*$, we complete the proof. Q.E.D.

Recalling (4.2.12), we know that,

$$I_{\varepsilon\varepsilon}^*(\omega_s)\to(1/2)W_2^C(2,f_{\varepsilon\varepsilon}(\lambda_o)) \tag{4.2.29}$$

in distribution, when $\lambda \not\equiv 0 \pmod{\pi}$ and $N\to\infty$ and $M\to\infty$.

As we defined in Section 4.1,

$$f_{xx}^*(\omega_s)=(1/3)[I_{xx}(\omega_s)+I_{xx}(\omega_s+\pi/M)+I_{xx}(\omega_s-\pi/M)] \tag{4.2.30}$$

with $\omega_s=2\pi s/N$ $s=1,2,\ldots m$. Similarly, we also have

$$f_{xz}^*(\omega_s)=(1/3)[I_{xz}(\omega_s)+I_{xz}(\omega_s+\pi/M)] +I_{xz}(\omega_s-\pi/M],$$

$$f_{zy}^*(\omega_s)=(1/3)[I_{zy}(\omega_s)+I_{zy}(\omega_s+\pi/M) +I_{zy}(\omega_s-\pi/M)],$$

$$f_{zz}^*(\omega_s)=(1/3)[I_{zz}(\omega_s)+I_{zz}(\omega_s+\pi/M) +I_{zz}(\omega_s-\pi/M)],$$

and

$$f_{xy}^*(\omega_s)=(1/3)[I_{xy}(\omega_s)+I_{xy}(\omega_s+\pi/M) +I_{xy}(\omega_s-\pi/M)].$$

With these, we form a special case of $g_\varepsilon^\#(\omega_o)$ as

$$g^*_\varepsilon(\lambda)=(3/2)\begin{bmatrix} f^*_{xx}-f^*_{xz}f^{*^{-1}}_{zz}f^*_{zx} & f^*_{xy}-f^*_{xz}f^{*^{-1}}_{zz}f^*_{zy} \\ f^*_{yx}-f^*_{yz}f^{*^{-1}}_{zz}f^*_{zx} & f^*_{yy}-f^*_{yz}f^{*^{-1}}_{zz}f^*_{zy} \end{bmatrix}. \tag{4.2.31}$$

As defined in(4.1.4)

$$f^*_{xy.z}(\lambda)=f^*_{xy}(\lambda)-f^*_{xz}(\lambda)f^{*^{-1}}_{zz}(\lambda)f^*_{zy}(\lambda)$$

and in (4.1.5)

$$p^-(\tau)=(1/m)\Sigma_o f^*_{xy.z}(\omega_s)e^{-i\tau\omega_s},$$

when τ^- is the value of τ maximizing $|p^-(\tau)|^2$, then we have the following theorem.

THEOREM 4.2.8.

Assume the conditions of Theorem 4.2.7, and assume that x(t), y(t) and z(t) satisfy Assumption 2.4.4.2 and Conditions C in Section 2.4.3. The estimator of partial group delay $\tau^-(\lambda)$ is a weak consistent estimator of $\tau_o=\tau(\lambda_o)$.

PROOF

We know from Lemma 4.2.2 that $f^*_{xx}(\omega_{s_1})$ and $f^*_{xx}(\omega_{s_2})$ are asymptotically independent when $\omega_{s_1}\neq\omega_{s_2}$ for $\omega_{s_j}=2\pi k_j/N$ (j=1,2) with k_j an integer and $w_{s_j}\to\lambda_o$. when N and M tend to infinity. Similar results also hold for $f^*_{xz}(\omega_s)$, $f^*_{zy}(\omega_s)$, $f^*_{zz}(\omega_s)$, and $f^*_{xy}(\omega_s)$. Then $g^*_\varepsilon(\omega_{s_1})$ and $g^*_\varepsilon(\omega_{s_2})$ are also asymptotically independent when $\omega_{s_1} \neq \omega_{s_2}$, and $g^*_\varepsilon(\omega_s) \rightarrow (1/2)W^C_2(2, f_{\varepsilon\varepsilon}(\lambda_o))$ for $\omega_{s_j} \rightarrow \lambda_o$ when $N \rightarrow \infty$ and $M \rightarrow \infty$.

We know from (4.2.12),

$$I^*_{\varepsilon\varepsilon}(\omega_s) \rightarrow (1/2)W^C_2(2, f_{\varepsilon\varepsilon}(\lambda_o)) \quad \text{in distribution.}$$

and from the above,

$$g^*_\varepsilon(\omega_s) \rightarrow (1/2)W^C_2(2, f_{\varepsilon\varepsilon}(\lambda_o)) \qquad \text{in distribution}$$

when $N \rightarrow \infty$ and $M \rightarrow \infty$. Since $I_{\varepsilon_x \varepsilon_y}(\omega_s)$ and $(2/3)f^*_{xy.z}(\omega_s)$ are the first row and second column elements in the matrix $I^*_{\varepsilon\varepsilon}(\omega_s)$ and the matrix $g^*_\varepsilon(\omega_s)$, by Theorem 2.4.5.2 $I^*_{\varepsilon_x \varepsilon_y}(\omega_s)$ and $(2/3)f^*_{xy.z}(\omega_s)$ have the same limiting distribution.

Now let $\omega_{s_j} = 2\pi k_j/N$ with k_j an integer $(j=1,\ldots,m)$ be the frequencies in the band B. From the above $I^*_{\varepsilon_x \varepsilon_y}(\omega_{s_1})$ and $I^*_{\varepsilon_x \varepsilon_y}(\omega_{s_2})$ are asymptotically independent and $(2/3)f^*_{xy.z}(\omega_{s_1})$ and $(2/3)f^*_{xy.z}(\omega_{s_2})$ are also asymptotically independent. Hence

$$\{I^*_{\varepsilon_x \varepsilon_y}(\omega_{s_1}),\ I^*_{\varepsilon_x \varepsilon_y}(\omega_{s_2})\}$$

and

$$(2/3)\{f^*_{xy.z}(\omega_{s_1}),\ f^*_{xy.z}(\omega_{s_2})\}$$

have the same joint limiting distribution. Note that pairwise independence is equivalent to joint independence for normal distributions as well as for Wishart distributions. Hence

$$\{I^*_{\varepsilon_x\varepsilon_y}(\omega_{s_1}), \ldots . I^*_{\varepsilon_x\varepsilon_y}(\omega_{s_m})\}$$

and

$$(2/3)\{f^*_{xy.z}(\omega_{s_1}), \ldots, f^*_{xy.z}(\omega_{s_m})\}$$

have the same limiting distribution for any integer k.

Further we define the elements in R^∞

$$U_{1N}=\{I^*_{\varepsilon_x\varepsilon_y}(\omega_{s_1}), \ldots, I^*_{\varepsilon_x\varepsilon_y}(\omega_{s_m}), 0, 0, \ldots\}$$

and

$$U_{2N}=\{(2/3)f^*_{xy.z}(\omega_{s_1}), \ldots, (2/3)f^*_{xy.z}(\omega_{s_m}), 0, 0, \ldots\}.$$

As the corresponding finite dimensional distribution of U_{1N} and U_{2N} have the same limiting distribution, thus by Theorem 2.4.5.1, U_{1N} and U_{2N} have the same limiting distribution. We denote the common limiting distribution by W.

Now recall that $\tau^{\#}$ (defined in Section 2.4.3) is the value maximizing $q^{\#}(\tau)=|p^{\#}(\tau)|^2$, where as defined in (4.2.13)

$$p^{\#}(\tau)=(1/m)\Sigma_o I_{\varepsilon_x\varepsilon_y}(s)e^{-i\tau\omega_s}.$$

Instead of $I_{\varepsilon_x\varepsilon_y}(s)$, we replace it by $I^*_{\varepsilon_x\varepsilon_y}(s)$. In (4.2.14) we defined

$$p^*(\tau)=(1/m)\Sigma_o I^*_{\varepsilon_x\varepsilon_y}(s)e^{-i\tau\omega_s},$$

whose norm is maximized when $\tau=\tau^*$. Corresponding to the maximization, we obtain the equation

$$\Psi_N(U_{1N},\tau^*)=0,$$

where Ψ_N is some function which satisfies the conditions of Theorem 2.4.4.1 for F_j there. Then by Theorem 2.4.4.1

$$\tau^*=\Xi_N(U_{1N}).$$

We know that Ξ is continuous and differentiable for U_{1N}. Similarly after replacing $I_{xy}(s)$ by $(2/3)f^*_{xy.z}(\omega_s)$ defined in (4.2.31), we obtain the estimator

$$\tau^-=\Xi_N(U_{2N}).$$

From Lemma 4.2.7 $\tau^*=\Xi_N(U_{1N})\to\tau_o$ thus Ξ_N converges to some function Ξ.

For any Borel set $A\varepsilon R^1$, we consider

$$\begin{aligned}
&p(\Xi_N(U_{1N})\varepsilon A)-p(\Xi_N(U_{2N})\varepsilon A)\\
&=p(U_{1N}\varepsilon\Xi_N^{-1}(A))-p(U_{2N}\varepsilon\Xi_N^{-1}(A))\\
&=p(U_{1N}\varepsilon\Xi_N^{-1}(A))-p(W\varepsilon\Xi_N^{-1}(A))\\
&\quad+p(W\varepsilon\Xi_N^{-1}(A))-p(U_{2N}\varepsilon\Xi_N^{-1}(A))
\end{aligned}\tag{4.2.32}$$

where $\Xi_N(A)$ is the inverse image of A in R^∞.

Now consider

$$
\begin{aligned}
&p(U_{1N}\varepsilon\Xi_N^{-1}(A))-p(W\varepsilon\Xi_N^{-1}(A))\\
&=p(U_{1N}\varepsilon\Xi_N^{-1}(A))-p(U_{1N}\varepsilon\Xi^{-1}(A)\\
&+p(U_{1N}\varepsilon\Xi^{-1}(A)-p(W\varepsilon\Xi^{-1}(A))\\
&+p(W\varepsilon\Xi^{-1}(A))-p(W\varepsilon\Xi_N^{-1}(A)).
\end{aligned}
\tag{4.2.33}
$$

As $U_{1N}\to W$ in distribution when N and M $\to\infty$, the second difference in the last expression converges to zero as N and $M\to\infty$. The third difference also converges to zero when N and M $\to\infty$ since Ξ_N converges to Ξ. For the first difference

$$
\begin{aligned}
&p(U_{1N}\varepsilon\Xi_N^{-1}(A))-p(U_{1N}\varepsilon\Xi^{-1}(A))\\
&=p\{U_{1N}\varepsilon[\Xi_N^{-1}(A)\ \ (\Xi^{-1}(A)^c]\},
\end{aligned}
$$

where $\Xi^{-1}(A)^c$ is the complement of $\Xi^{-1}(A)$. As $\Xi_N\to\Xi$ the measure of the set of $\Xi_N^{-1}(A)\ \Xi^{-1}(A)$ converges to zero. Then the second difference in (4.1.33) $\to 0$ when N and M $\to\infty$.

Thus

$$p(U_{1N}\varepsilon\Xi_N^{-1}(A))-p(W\varepsilon\Xi_N^{-1}(A))\to 0 \quad \text{when N and M } \to\infty.$$

Similarly,

$$p(U_{2N}\varepsilon\Xi_N^{-1}(A))-p(W\varepsilon\Xi_N^{-1}(A))\to 0 \quad \text{when N and M } \to\infty.$$

Hence from (4.1.32) τ^* and τ^- have the same limiting distribution when N and M tend to infinity.

For the sequence of $M(N)$ chosen in Lemma 4.2.7 , we know that $M\to\infty$ when $N\to\infty$ and

$$\tau^* \to \tau_o \quad \text{a.s} \quad \text{when } N\to\infty.$$

Thus $\tau^*\to\tau_o$ in probability. Finally since τ^* and τ^- have the same limiting distribution, we obtain $\tau^-\to\tau_o$ in distribution i.e. $\tau^-\to\tau_o$ in probability when $N\to\infty$. Thus we complete the proof. Q.E.D.

4.3 CENTRAL LIMIT THEOREM

For the asymptotic distribution of the proposed estimate τ^- in Section 4.1, we still consider the frequency band $B=\{\lambda\,|\,\lambda_o+\pi/2M<\lambda<\lambda_o+\pi/2M\}$ which contains $m=N/2M$ frequencies $\omega_s=2\pi k_s/M$ $(s=1,\dots m)$ with k_s an integer. We begin with the following theorem.

THEOREM 4.3.1.

Assume that $x(t)$, $y(t)$, $z(t)$ and $\varepsilon_x(t)$, $\varepsilon_y(t)$ satisfy Assumption 2.4.4.2 and Conditions D in Section 2.4.3. Then there is a sequence $M(N)$ increasing with N such that $N^{-1}m^{3/2}(\tau^*-\tau^*_M)$ has an asymptotic normal distribution.

PROOF

We shall omit the subscripts from $I_{\varepsilon_x\varepsilon_y}(\lambda)$ and $f_{\varepsilon_x\varepsilon_y}(\lambda)$ in this proof.

Just as in the proof of Theorem 2.4.3.4, we keep M fixed. Then

$$N^{1/2}q^{*'}(\tau_M^*)=-N^{1/2}(\tau^*-\tau_M^*)q^{*''}(\tau^o)$$

where $|\tau^o-\tau_M^*|\le|\tau^*-\tau_M^*|$ and

$$q^{*''}(\tau^o)\to(M/\pi)^2q''(\tau_M^*).$$

Using the same argument as in step 1 of the proof of Theorem 2.4.3.4, $N^{1/2}q^{*'}(\tau_M^*)$ can be replaced approximately by

$$(M/\pi)2N^{1/2}\mathrm{Re}[p^{*'}(\tau_M^*)p^+(\tau_M^*)^c+p^{+'}(\tau_M^*)p^*(\tau_M^*)^c],$$

where $p^*(\tau)=(1/2m)\Sigma_o[I(\omega_s)+I(\omega_s+\pi/M)]e^{-i\tau\omega_s}$.

Thus $p^*(\tau)=(1/2m)\Sigma_oI(\omega_s)e^{-i\tau\omega_s}+(1/2m)\Sigma_oI(\omega_s+\pi/M)e^{-i\tau\omega_s}$
$=(1/2m)\Sigma_oI(\omega_s)e^{-i\tau\omega_s}+(1/2m)\Sigma_{o'}I(\omega_s)[e^{-i\tau\omega_s}]e^{i\tau\pi/M}$,

where $\Sigma_{o'}$ is the summation over all $\omega_s\varepsilon B'$.

Hence by Lemma 2.4.3.1,

$$p^*(\tau)\simeq(M/2\pi)\int_BI(\lambda)e^{-i\tau\lambda}d\lambda+(M/2\pi)\int_{B'}I(\lambda)[e^{-i\tau\lambda}]d\lambda\ e^{i\tau\pi/M}.$$

Similarly,
$dp^*(\tau)/d\tau=p^{*'}(\tau)$

$$\approx (M/2\pi)\{\int_B(-i\lambda)I(\lambda)e^{-i\tau\lambda}d\lambda \ +\int_{B'}(-i\lambda)I(\lambda)e^{-i\tau\lambda}d\lambda \ e^{i\tau\pi/M}$$
$$+\int_{B'}I(\lambda)e^{-i\tau\lambda}d\lambda(i\pi/M)e^{i\tau\pi/M}\}.$$

On the other hand by definition

$$p^{+}(\tau)=(1/2)\int_B f(\lambda)e^{-i\tau\lambda}d\lambda+(1/2)\int_{B'}f(\lambda)e^{-i\tau\lambda}d\lambda e^{i\tau\pi/M};$$

$$p^{+'}(\tau)=(1/2)\int_B f(\lambda)e^{-i\tau\lambda}(-i\lambda)d\lambda+(1/2)\int_{B'}f(\lambda)e^{-i\tau\lambda}(-i\lambda)e^{i\tau\pi/M}d\lambda$$
$$+(1/2)\int_{B'}f(\lambda)e^{-i\tau\lambda}(\pi i/M)d\lambda e^{i\tau\pi/M}.$$

Hence

$$(2\pi/M)\mathrm{Re}\{p^{*'}(\overset{*}{\tau}_M)p^{+}(\overset{*}{\tau}_M)^{c}+p^{+'}(\overset{*}{\tau}_M)p^{*}(\overset{*}{\tau}_M)^{c}\}$$
$$\approx\int_B I(\lambda)(-i\lambda)[e^{-i\overset{*}{\tau}_M\lambda}]p^{+}(\overset{*}{\tau}_M)^{c}d\lambda$$
$$+\int_B I(\lambda)^{c}(i\lambda)[e^{i\overset{*}{\tau}_M\lambda}]p^{+}(\overset{*}{\tau}_M)d\lambda$$
$$+\int_{B'}I(\lambda)(-i\lambda)[e^{-i\overset{*}{\tau}_M\lambda}]p^{+}(\overset{*}{\tau}_M)^{c}d\lambda e^{i\overset{*}{\tau}_M\pi/M}$$
$$+\int_{B'}I(\lambda)^{c}(i\lambda)[e^{i\overset{*}{\tau}_M\lambda}]p^{+}(\overset{*}{\tau}_M)d\lambda e^{-i\overset{*}{\tau}_M\pi/M}$$
$$+(\pi i/M)\int_{B'}I(\lambda)[e^{-i\overset{*}{\tau}_M\lambda}]p^{+}(\overset{*}{\tau}_M)^{c}d\lambda e^{i\overset{*}{\tau}_M\pi/M}$$
$$-(\pi i/M)\int_{B'}I(\lambda)^{c}[e^{-i\overset{*}{\tau}_M\lambda}]p^{+}(\overset{*}{\tau}_M)d\lambda e^{-i\overset{*}{\tau}_M\pi/M}$$
$$+\int_B I(\lambda)^{c}[e^{i\overset{*}{\tau}_M\lambda}]p^{+'}(\overset{*}{\tau}_M)d\lambda$$
$$+\int_B I(\lambda)[e^{-i\overset{*}{\tau}_M\lambda}]p^{+'}(\overset{*}{\tau}_M)^{c}d\lambda$$
$$+\int_{B'}I(\lambda)[e^{i\overset{*}{\tau}_M\lambda}]p^{+'}(\overset{*}{\tau}_M)d\lambda e^{-i\overset{*}{\tau}_M\pi/M}$$
$$+\int_{B'}I(\lambda)[e^{-i\overset{*}{\tau}_M\lambda}]p^{+'}(\overset{*}{\tau}_M)^{c}d\lambda e^{i\overset{*}{\tau}_M\pi/M}$$
$$=\Delta_1+\ldots+\Delta_{10}.$$

It is easy to see that

$$\Delta_1=\Delta_2{}^c,\ \Delta_3=\Delta_4{}^c\ldots,\ \text{and}\ \Delta_9=\Delta_{10}{}^c.$$

Further we substitute the expression of $p^+(\tau_M^*)$ and $p^{+'}(\tau_M^*)$ in $\Delta_1+\ldots+\Delta_{10}$.

After arranging the terms, we obtain

$$(4\pi/M)\mathrm{Re}\{p^{*'}(\tau_M^*)p^+(\tau_M^*)^c+p^{+'}(\tau_M^*)p^*(\tau_M^*)^c\}$$
$$\approx 2\mathrm{Re}(\Sigma_{i=1}^{8}A_i),$$

where $\Sigma_{i=1}^{8}A_i$

$$=\int_B\int_B I(\lambda)i(\theta-\lambda)e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c d\theta d\lambda$$
$$+\int_B\int_{B'} I(\lambda)i(\theta-\lambda)e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c e^{-i\tau_M^*\pi/M}d\theta d\lambda$$
$$+\int_{B'}\int_B I(\lambda)i(\theta-\lambda)e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c e^{i\tau_M^*\pi/M}d\theta d\lambda$$
$$+\int_{B'}\int_{B'} I(\lambda)i(\theta-\lambda)e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c d\theta d\lambda$$
$$+\int_B\int_{B'}(-i)(\pi/M)I(\lambda)e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c e^{-i\tau_M^*\pi/M}d\theta d\lambda$$
$$+\int_{B'}\int_B i(\pi/M)I(\lambda)e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c e^{i\tau_M^*\pi/M}d\theta d\lambda$$
$$+\int_{B'}\int_{B'} i(\pi/M)I(\lambda)e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c d\theta d\lambda$$
$$+\int_{B'}\int_{B'}(-i)(\pi/M)I(\lambda)e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c d\theta d\lambda.$$

Thus

$$(4\pi/M)\mathrm{Re}\{p^{*'}(\tau_M^*)^c+p^{+'}(\tau_M^*)p^+(\tau_M^*)^c\}$$
$$\approx 2\mathrm{Re}(\Sigma_{i=1}^{6}A_i).$$

Let

$$\phi_1(\lambda)=(1/2)(M/\pi)^2\int_B i(\theta-\lambda)\ e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c d\theta$$

on B, and $\phi_1(\lambda)=0$ on the intersection of the complement of B and $(0, \pi)$. Also let $\phi_1(-\lambda)=\phi_1(\lambda)^c$. Similarly, we have

$\phi_2(\lambda)=(1/2)(M/\pi)^2\int_{B'} i(\theta-\lambda)e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c e^{-i\tau_M^*\pi/M}d\theta$ on B', and $\phi_2(\lambda)=0$ on the intersection of the complement of B' and $(0, \pi)$. Also let $\phi_2(-\lambda)=\phi_2(\lambda)^c$.

$$\phi_3(\lambda)=(1/2)(M/\pi)^2\int_B i(\theta-\lambda)e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c e^{i\tau_M^*\pi/M}d\theta$$

on B', and $\phi_3(\lambda)=0$ on the intersection of the complement of B' and $(0, \pi)$. Also let $\phi_3(-\lambda)=\phi_3(\lambda)^c$.

$$\phi_4(\lambda)=(1/2)(M/\pi)^2\int_{B'} i(\theta-\lambda)e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c d\theta$$

on B', and $\phi_4(\lambda)=0$ on the intersection of the complement of B' and $(0, \pi)$. Also let $\phi_4(-\lambda)=\phi_4(\lambda)^c$.

$$\phi_5(\lambda)=(1/2)(M/\pi)^2\int_{B'}(-2i\pi/M)e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c e^{-i\tau_M^*\pi/M}d\theta$$

on B, and $\phi_5(\lambda)=0$ on the intersection of the complement of B and $(0, \pi)$. Also let $\phi_5(-\lambda)=\phi_5(\lambda)^c$.

$$\phi_6(\lambda)=(1/2)(M/\pi)^2\int_B i(2\pi/M)e^{i\tau_M^*(\theta-\lambda)}f(\theta)^c e^{i\tau_M^*\pi/M}d\theta$$

on B', and $\phi_6(\lambda)=0$ on the intersection of the complement of B' and $(0, \pi)$. Also let $\phi_6(-\lambda)=\phi_6(\lambda)^c$.

It is feasible to assume the above, because when N is big enough then B and B' are subsets of $(0, \pi)$.

Now let $\phi^*(\lambda)=\Sigma_{i=1}^{6}\phi_i(\lambda)$. We want to show that

$$\int_{-\pi}^{\pi}\phi^*(\lambda)f(\lambda)d\lambda=0.$$

Since τ_M^* maximizes $q^+(\tau)=|p^+(\tau)|^2$, thus from

$$(p^+(\tau_M^*)p^+(\tau_M^*)^c)'=0,$$

we have

$$p^{+'}(\tau_M^*)p^+(\tau_M^*)^c+p^+(\tau_M^*)p^{+'}(\tau_M^*)^c=0.$$

or

$$\mathrm{Re}[p^{+'}(\tau_M^*)p^+(\tau_M^*)^c]=0.$$

We further manipulate as follows:

$$\begin{aligned}
&p^{+'}(\tau_M^*)p^+(\tau_M^*)^c\\
&=(1/2)[\int_B f(\lambda)e^{-i\tau_M^*\lambda}(-i\lambda)d\lambda+\int_{B'}f(\lambda)e^{-i\tau_M^*\lambda}(-i\lambda)e^{i\tau_M^*\pi/M}d\lambda\\
&\qquad+(\pi/M)\int_{B'}f(\lambda)e^{-i\tau_M^*\lambda}(\pi i/M)e^{i\tau_M^*\pi/M}]\\
&\quad(1/2)[\int_B f(\theta)^c e^{i\tau_M^*\theta}d\theta+\int_{B'}f(\theta)^c e^{i\tau_M^*\theta}e^{-i\tau_M^*\pi/M}d\theta]\\
&=(1/4)\int_B\int_B f(\lambda)f(\theta)^c(-i\lambda)e^{i\tau_M^*(\theta-\lambda)}d\theta d\lambda\\
&\quad+(1/4)\int_{B'}\int_B f(\lambda)f(\theta)^c(-i\lambda)e^{i\tau_M^*(\theta-\lambda)}e^{i\tau_M^*\pi/M}d\theta d\lambda\\
&\quad+(1/4)\int_{B'}\int_B f(\lambda)f(\theta)^c(2\pi i/M)e^{i\tau_M^*(\theta-\lambda)}e^{i\tau_M^*\pi/M}d\theta d\lambda\\
&\quad+(1/4)\int_B\int_{B'}f(\lambda)f(\theta)^c(-i\lambda)e^{i\tau_M^*(\theta-\lambda)}e^{-i\tau_M^*\pi/M}d\theta d\lambda
\end{aligned}$$

$$+(1/4)\int_{B'}\int_{B'}f(\lambda)f(\theta)^c(-i\lambda)e^{i\tau_M^*(\theta-\lambda)}d\theta d\lambda$$

$$+(1/4)\int_{B'}\int_{B'}f(\lambda)f(\theta)^c(\pi/M)e^{i\tau_M^*(\theta-\lambda)}d\theta d\lambda.$$

We can write $p^{+'}(\tau_M^*)p^+(\tau_M^*)^c$ in an alternative way, i.e.,

$$p^{+'}(\tau_M^*)p^+(\tau_M^*)^c$$

$$=(1/4)[\int_B f(\lambda)^c e^{i\tau_M^*\lambda}d\lambda+\int_{B'}f(\lambda)^c e^{i\tau_M^*\lambda}e^{-i\tau_M^*\pi/M}d\lambda]$$

$$[\int_B f(\theta)e^{-i\tau_M^*\lambda}(-i\theta)d\theta$$

$$+\int_{B'}f(\theta)e^{-i\tau_M^*\theta}(-i\theta)e^{i\tau_M^*\pi/M}d\theta$$

$$+\int_{B'}f(\theta)e^{-i\tau_M^*\theta}(\pi i/M)e^{i\tau_M^*\pi/M}d\theta]$$

$$=(1/4)\int_B\int_B f(\lambda)^c f(\theta)e^{i\tau_M^*(\lambda-\theta)}(-i\theta)d\theta d\lambda$$

$$+(1/4)\int_{B'}\int_B f(\lambda)^c f(\theta)e^{i\tau_M^*(\lambda-\theta)}e^{-i\tau_M^*\pi/M}(-i\theta)d\theta d\lambda$$

$$+(1/4)\int_B\int_{B'}f(\lambda)^c f(\theta)e^{i\tau_M^*(\lambda-\theta)}e^{i\tau_M^*\pi/M}(-i\theta)d\theta d\lambda$$

$$+(1/4)\int_{B'}\int_{B'}f(\lambda)^c f(\theta)e^{i\tau_M^*(\lambda-\theta)}(-i\theta)d\theta d\lambda$$

$$+(1/4)\int_B\int_{B'}f(\lambda)^c f(\theta)e^{i\tau_M^*(\lambda-\theta)}e^{i\tau_M^*2\pi/M}(\pi i/M)d\theta d\lambda$$

$$+(1/4)\int_{B'}\int_{B'}f(\lambda)^c f(\theta)e^{i\tau_M^*(\lambda-\theta)}e^{i\tau_M^*\pi/M}(\pi i/M)d\theta d\lambda.$$

Hence

$$4\mathrm{Re}[p^{+'}(\tau_M^*)p^+(\tau_M^*)^c]$$

$$=\int_B\int_B f(\lambda)f(\theta)^c e^{i\tau_M^*(\theta-\lambda)}i(\theta-\lambda)d\theta d\lambda$$

$$+\int_{B'}\int_B f(\lambda)f(\theta)^c e^{i\tau_M^*(\theta-\lambda)}i(\theta-\lambda)d\theta d\lambda$$

$$+\int_B\int_{B'}f(\lambda)f(\theta)^c e^{i\tau_M^*(\theta-\lambda)}e^{-i\tau_M^*\pi/M}[i(\theta-\lambda)]d\theta d\lambda$$

$$+\int_{B'}\int_{B'}f(\lambda)f(\theta)^{c}e^{i\tau_M^*(\theta-\lambda)}[i(\theta-\lambda)]d\theta d\lambda$$
$$+\int_{B'}\int_{B}f(\lambda)f(\theta)^{c}e^{i\tau_M^*(\theta-\lambda)}e^{i\tau_M^*\pi/M}(\pi i/M)d\theta d\lambda$$
$$+\int_{B}\int_{B'}f(\lambda)f(\theta)^{c}e^{i\tau_M^*(\theta-\lambda)}e^{-i\tau_M^*\pi/M}(-\pi i/M)d\theta d\lambda$$

$$=(M/\pi)^{-2}2\int_{o}^{\pi}[\Sigma_1^6\phi_i^*(\lambda)]f(\lambda)d\lambda$$
$$=(M/\pi)^{-2}\int_{-\pi}^{\pi}[\Sigma_1^6\phi_i^*(\lambda)]f(\lambda)d\lambda$$
$$=(M/\pi)^{-2}\int_{-\pi}^{\pi}\phi^*(\lambda)f(\lambda)d\lambda.$$

Since $\mathrm{Re}[p^{+'}(\tau_M^*)p^{+}(\tau_M^*)^{c}]=0$, we have

$$\int\phi^*(\lambda)f(\lambda)d\lambda=\mathrm{Re}[p^{+'}(\tau_M^*)p^{+}(\tau_M^*)^{c}]=0.$$

Hence we have

$$\int_{-\pi}^{\pi}\phi^*(\lambda)f(\lambda)d\lambda=0,$$

which corresponds exactly to step 2 of the proof of Theorem 2.4.3.4.

Now consider

$$\int_{-\pi}^{\pi}\phi^*(\lambda)I(\lambda)d\lambda$$
$$=\int_{-\pi}^{o}+\int_{0}^{\pi}\phi^*(\lambda)I(\lambda)d\lambda$$
$$=-\int_{\pi}^{o}\phi^*(-\lambda)I(-\lambda)d\lambda+\int_{o}^{\pi}\phi^*(\lambda)I(\lambda)d\lambda$$
$$=2\int_{o}^{\pi}\mathrm{Re}[\phi^*(\lambda)I(\lambda)]d\lambda.$$

It is easy to see that

$$\int_{o}^{\pi}\phi^*(\lambda)I(\lambda)d\lambda=(1/2)(M/\pi)^2\Sigma_1^6A_i.$$

Thus

$$
\begin{aligned}
&\mathrm{Re}[\int_0^{\pi}\mathrm{Re}[\phi^*(\lambda)I(\lambda)]d\lambda] \\
&=(M/\pi)^2(1/2)\mathrm{Re}[\Sigma_1^6 A_i] \\
&\simeq(M/\pi)^2(1/2)(\pi/M)2\mathrm{Re}\{p^{*'}(\tau_M^*)^c+p^{+'}(\tau_M^*)p^*(\tau_M^*)^c\}.
\end{aligned}
$$

Following the argument of the proof of Theorem 2.4.3.4, or just copying the first 14 lines of that proof with replacing the corresponding notation, we conclude

$$
\begin{aligned}
&N^{1/2}\underline{q}^{*'}(\tau_M^*) \\
&\simeq(M/\pi)2N^{1/2}\mathrm{Re}\{p^{*'}(\tau_M^*)p^+(\tau_M^*)^c+p^{+'}(\tau_M^*)p^*(\tau_M^*)^c\} \\
&=(M/\pi)2N^{1/2}(\pi/M)\int_0^{\pi}\phi^*(\lambda)I(\lambda)d\lambda \\
&=N^{1/2}\int_{-\pi}^{\pi}\phi^*(\lambda)I(\lambda)d\lambda.
\end{aligned}
$$

As $f(\lambda)$ is differentiable, $E\{I(\lambda)-f(\lambda)\}=O(\log N/N)$. Hence

$$
\begin{aligned}
&N^{1/2}\underline{q}^{*'}(\tau_M^*) \\
&\simeq N^{1/2}\int_{-\pi}^{\pi}\phi^*(\lambda)\{I(\lambda)-E[I(\lambda)]\}d\lambda.
\end{aligned}
$$

Since $\phi^*(\lambda)$ is piecewise continuous, by the argument of step 4 of the proof of Theorem 2.4.3.4 we know that $N^{1/2}(\tau^*-\tau_M^*)$ is asymptotically normally distributed for fixed M.

To complete the proof, we cite Lemma 2 in Hannan and Thomson (1973) here.

LEMMA (Hannan and Thomson, 1973)

If $t_M(N)$ $(M=1,2\ldots;N=1,\ldots)$ is a sequence of random vectors whose distributions converge to F_M as $N\to\infty$ for each fixed M, and if F_M converges properly to F as $N\to\infty$, then there is a monotone increasing sequence M(N) such that the distributions of $t_{M(N)}(N)$ converge properly to F as $N\to\infty$

Thus we let $t_{M(N)}(N)$ be $(m/N)^{3/2}N^{1/2}(\tau^*-\tau_M^*)$. When M is fixed, i.e. m/N is fixed, $N^{-1}m^{3/2}(\tau^*-\tau_M^*)$ is asymptotically normally distributed from the above conclusion. Then by the cited lemma we complete the proof. Q.E.D.

Now we calculate the asymptotic variance of $m^{3/2}N^{-1}(\tau^*-\tau_M^*)$.

THEOREM 4.3.2.

Under the conditions of Theorem 4.2.1, the asymptotic variance of $m^{3/2}N^{-1}(\tau^*-\tau_M^*)$ when N and M $\to\infty$ is $12\{1-\sigma^2(\lambda_o)\}/\{\pi^2\sigma^2(\lambda_o)\}$, where $\sigma^2(\lambda_o)$ is the partial coherence $\sigma^2_{xy.z}$ at $\lambda=\lambda_o$.

PROOF

Just as in the proof of 2.4.3.4 step 1) to 3) the variance of $m^{3/2}N^{-1}(\tau^*-\tau_M^*)$ when M is fixed converges to

$$(2\pi)(2M)^{-3}\{(M/\pi)^2q^{+''}(\tau_M^*)\}^{-2}$$
$$[\int_{-\pi}^{\pi}|\phi^*(\lambda)|^2f_{\varepsilon_x}(\lambda)f_{\varepsilon_y}(\lambda)+\phi^*(\lambda)^2f_{\varepsilon_x\varepsilon_y}(\lambda)\}d\lambda$$

$$+\iint_{-\pi}^{\pi}\phi^*(\lambda)\phi^*(\mu)^c f_{\varepsilon_x\varepsilon_y\varepsilon_x\varepsilon_y}(-\lambda,\lambda,\mu,-\mu)d\lambda d\mu]. \quad (4.3.1)$$

Write the coefficient in (4.3.1) as

$$\frac{4\pi M}{(4M^4/\pi^2)^2\ q^{+''}(\tau_M^*)^2}.$$

Now the last term in (4.3.1)

$$4\pi M\iint_{-\pi}^{\pi}\phi^*(\lambda)\phi^*(\mu)^c f_{\varepsilon_x\varepsilon_y\varepsilon_x\varepsilon_y}(-\lambda,\lambda,\mu,-\mu)d\lambda d\mu\to 0$$

when $M\to\infty$ since the integrand is bounded and zero except on a set whose area is $O(M^{-2})$. For the first term in (4.3.1), as we defined in the proof of Theorem 4.3.1 and stated in step 3) of the proof of Theorem 2.4.3.4,

$$\begin{aligned}\phi_1(\lambda)&=(M^2/2\pi^2)\int_B i(\theta-\lambda)e^{\tau_M^*(\theta-\lambda)}f_{12}(\theta)^c d\theta\\ &\simeq(-i/2)(M/\pi)(\lambda-\lambda_o)f_{\varepsilon_x\varepsilon_y}(\lambda_o)^c \qquad (4.3.2)\end{aligned}$$

on B. Similarly let $\lambda_1=\lambda_o+(\pi/M)$, then as $f_{\varepsilon_x\varepsilon_y}(\lambda)$ is bounded differentiable,

$$\begin{aligned}\phi_2(\lambda)&=(M^2/2\pi^2)\int_{B'} i(\theta-\lambda)e^{-i\tau_M^*(\theta-\lambda)}f_{12}(\theta)^c e^{-i\tau_M^*(\pi/M)}d\theta\\ &\simeq(-i/2)(M/\pi)(\lambda-\lambda_1)f_{\varepsilon_x\varepsilon_y}(\lambda_o)^c \qquad (4.3.3)\end{aligned}$$

on B' when M is big enough.

We also have

$$\phi_3(\lambda)=(M^2/2\pi^2)\int_B i(\theta-\lambda)e^{i\tau_M^*(\theta-\lambda)}f_{\varepsilon_x\varepsilon_y}(\theta)^c e^{i\tau_M^*(\pi/M)}d\theta$$

$$\simeq(-i/2)(M/\pi)(\lambda-\lambda_o)f_{\varepsilon_x\varepsilon_y}(\lambda_o)^c \tag{4.3.4}$$

on B.

$$\phi_4(\lambda)=(M^2/2\pi^2)\int_B i(\theta-\lambda)e^{i\tau_M^*(\theta-\lambda)}f_{\varepsilon_x\varepsilon_y}(\theta)^c$$

$$\simeq(-i/2)(M/\pi)(\lambda-\lambda_o)f_{\varepsilon_x\varepsilon_y}(\lambda_o)^c \tag{4.3.5}$$

on B, and

$$\phi_5(\lambda)=(M/\pi)^2\int_{B'}(-i\pi/M)e^{i\tau_M^*(\theta-\lambda)}f_{\varepsilon_x\varepsilon_y}(\theta)^c e^{-i\tau_M^*(\pi/M)}d\theta$$

$$\simeq(-i/2)(1/\pi)(\lambda-\lambda_1)f_{\varepsilon_x\varepsilon_y}(\lambda_o)^c \tag{4.3.6}$$

on B' when M is big enough.

$$\phi_6(\lambda)=(M/\pi)^2\int_{B'}i(\pi/M)e^{i\tau_M^*(\theta-\lambda)}f_{\varepsilon_x\varepsilon_y}(\theta)^c e^{-i\tau_M^*(\pi/M)}d\theta$$

$$\simeq(i/2)(1/\pi)(\lambda-\lambda_1)f_{\varepsilon_x\varepsilon_y}(\lambda_o)^c \tag{4.3.7}$$

on B when M is big enough.

Any terms that include $\phi_5(\lambda)$ and $\phi_6(\lambda)$ in the first term of (4.3.1) will go to zero when $M\to\infty$. In this sense we can have $\phi^*(\lambda)\simeq\phi_1+\ldots+\phi_4$. Thus we have

$$\int_{-\pi}^{\pi}|\phi^*(\lambda)|^2 f_{\varepsilon_x}(\lambda)f_{\varepsilon_y}(\lambda)d\lambda$$
$$=2\int_{o}^{\pi}|\phi^*(\lambda)|^2 f_{\varepsilon_x}(\lambda)f_{\varepsilon_y}(\lambda)d\lambda$$
$$=2[\int_B|\phi_1+\phi_3|^2 f_{\varepsilon_x}f_{\varepsilon_y}d\lambda+\int_{B'}|\phi_2+\phi_4|^2 f_{\varepsilon_x}f_{\varepsilon_y}d\lambda$$

as ϕ_1, ϕ_3 are nonzero only on B and ϕ_2, ϕ_4 are nonzero only on B'. Thus by (4.3.2)——(4.3.5),

$$
\begin{aligned}
&\int_{-\pi}^{\pi}|\phi^*(\lambda)|^2 f_{\varepsilon_x}(\lambda) f_{\varepsilon_y}(\lambda)d\lambda \\
&\simeq 4\int_B (M^2/\pi^2)(\lambda-\lambda_o)^2 |f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^2 f_{\varepsilon_x}(\lambda) f_{\varepsilon_y}(\lambda)d\lambda \\
&=4(M/\pi)^2(1/8)|f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^2 f_{\varepsilon_x}(\lambda_o) f_{\varepsilon_y}(\lambda_o)(2/3)(\pi/M)^3 \\
&=(\pi/3M)|f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^2 f_{\varepsilon_x}(\lambda_o) f_{\varepsilon_y}(\lambda_o).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&(4\pi M)\int_{-\pi}^{\pi}|\phi^*(\lambda)|^2 f_{\varepsilon_x}(\lambda) f_{\varepsilon_y}(\lambda)d\lambda \\
&\simeq (4\pi^2/3)|f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^2 f_{\varepsilon_x}(\lambda_o) f_{\varepsilon_y}(\lambda_o).
\end{aligned}
$$

For the second term in (4.3.1),

$$
\begin{aligned}
&\int_{-\pi}^{\pi}[\phi^*(\lambda)]^2 f^2_{\varepsilon_x\varepsilon_y}(\lambda)d\lambda \\
&=2\int_{o}^{\pi}[\phi^*(\lambda)]^2 f^2_{\varepsilon_x\varepsilon_y}(\lambda)d\lambda \\
&\simeq 4\int_B[(-i)(M/\pi)(\lambda-\lambda_o) f_{\varepsilon_x\varepsilon_y}(\lambda_o)]^2 f^2_{\varepsilon_x\varepsilon_y}(\lambda)d\lambda \\
&=(-4)(M/\pi)^2\int_B(\lambda-\lambda_o)^2|f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^4 d\lambda \\
&=(-4)(M/\pi)^2(1/3)(1/4)(\pi/M)^3|f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^4 \\
&=(-1/3)(\pi/M)|f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^4.
\end{aligned}
$$

Thus when M approaches infinity

$$4\pi M\int_{-\pi}^{\pi}[\phi^*(\lambda)]^2 f^2_{\varepsilon_x\varepsilon_y}(\lambda)d\lambda \to (-3/4)\pi^2|f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^4.$$

Now we consider $q^{+''}(\tau_M^*)$. Since

$$p^+(\tau_M^*)=(1/2)\{\int_B f_{\varepsilon_x\varepsilon_y}(\lambda)[e^{-i\tau_M^*\lambda}]d\lambda$$
$$+\int_{B'} f_{\varepsilon_x\varepsilon_y}(\lambda)[e^{-i\tau_M^*\lambda}]d\lambda[e^{i\tau_M^*\pi/M}]\}$$

and

$$p^+(\tau_M^*)^c=(1/2)\{\int_B f_{\varepsilon_x\varepsilon_y}(\lambda)^c[e^{i\tau_M^*\lambda}]d\lambda$$
$$+\int_{B'} f_{\varepsilon_x\varepsilon_y}(\lambda)^c[e^{i\tau_M^*\lambda}]d\lambda[e^{-i\tau_M^*(\pi/M)}]\},$$

thus

$$q^+(\tau_M^*)=|p^+(\tau_M^*)|^2$$
$$=q_1(\tau_M^*)+\ldots+q_4(\tau_M^*),$$

where

$$q_1(\tau)$$
$$=(1/4)\int_B f_{\varepsilon_x\varepsilon_y}(\lambda)e^{-i\tau\lambda}d\lambda\int_B f_{\varepsilon_x\varepsilon_y}(\theta)^c e^{i\tau\theta}d\theta$$
$$=(1/4)\int\int_B f_{12}(\lambda)f_{\varepsilon_x\varepsilon_y}(\theta)^c e^{i\tau(\theta-\lambda)}d\theta d\lambda.$$

Hence

$$q_1''(\tau)=(1/4)\int\int_B(-1)(\theta-\lambda)^2 f_{\varepsilon_x\varepsilon_y}(\lambda)f_{\varepsilon_x\varepsilon_y}(\theta)^c e^{i\tau(\theta-\lambda)}d\theta d\lambda.$$

When M is big enough and in any bounded interval of τ, we have

$$\int_B(\theta-\lambda)^2 f_{\varepsilon_x\varepsilon_y}(\lambda)f_{\varepsilon_x\varepsilon_y}(\theta)^c e^{i\tau(\theta-\lambda)}d\theta$$
$$\approx(\pi/M)(\lambda_o-\lambda)^2 f_{\varepsilon_x\varepsilon_y}(\lambda_o)f_{\varepsilon_x\varepsilon_y}(\lambda_o)^c.$$

Thus $q_1''(\tau_M^*)$ can be approximated by

$$(-1/4)\int_B(\pi/M)(\lambda_o-\lambda)^2 f_{\varepsilon_x\varepsilon_y}(\lambda_o)\bar{f}_{\varepsilon_x\varepsilon_y}(\lambda_o)d\lambda$$
$$=(-1/4)(\pi/M)|f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^2(1/3)(\pi^3/4M^3).$$

Also $q_2''(\tau_M^*)$, $q_3''(\tau_M^*)$ and $q_4''(\tau_M^*)$ have the same approximation. Thus

$$\lim_{M\to\infty}(4M^4/\pi^2)q^{\#''}(\tau_M^*)$$
$$=(-1/3)\pi^2|f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^2.$$

Now the variance is

$$\frac{(4\pi^2/3)[|f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^2 f_{\varepsilon_x}(\lambda_o)f_{\varepsilon_y}(\lambda_o)-|f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^4]}{(\pi^2/3)^2\ |f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^4}$$

$$=\frac{12\pi^2 f_{\varepsilon_x}(\lambda_o)f_{\varepsilon_y}(\lambda_o)-|f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^2}{|f_{\varepsilon_x\varepsilon_y}(\lambda_o)|^2}.$$

By definition $\sigma^2(.)=|f_{\varepsilon_x\varepsilon_y}(.)|^2/f_{\varepsilon_x}(.)f_{\varepsilon_y}(.)$.
Thus the limiting variance equals

$$(12/\pi^2)\{[1-\sigma^2(\lambda_o)]/\sigma^2(\lambda_o)\}.$$

Q.E.D.

Based on Theorem 4.2.1 and Theorem 4.2.2 , we have the following central limit theorem.

THEOREM 4.2.3

For the estimator τ^{-}(see Section 4.1.), there is a sequence M(N) increasing with N, such that $N^{-1}m^{3/2}(\tau^{-}-\tau^{*}_{M})$ has a distribution converging to the normal distribution with zero mean and variance $12[1-\sigma^2(\lambda_o)]/[\pi^2\sigma^2(\lambda_o)]$.

PROOF

From Theorem 4.2.1 $N^{-1}m^{3/2}(\tau^{*}-\tau^{*}_{M})$ has an asymptotic normal distribution. Now we want to show that $N^{-1}m^{3/2}(\tau^{*}-\tau^{*}_{M})$ and $N^{-1}m^{3/2}(\tau^{-}-\tau^{*}_{M})$ have the same asymptotic distribution.

As we defined in (4.2.14)

$$p^{*}(\tau)=(1/m)\Sigma_o I^{*}_{\varepsilon_x\varepsilon_y}(\omega_s)e^{-i\tau\omega_s}.$$

We can write the above as

$$p^{*}(\tau)$$
$$=(1/m)\Sigma_o I^{*}_{\varepsilon_x\varepsilon_y}(\omega_s)\exp\{[-iN^{-1}m^{3/2}(\tau-\tau^{*}_{M})\omega_s]/[N^{-1}m^{3/2}]-i\tau^{*}_{M}\omega_s\}.$$

Similar to the proof of Theorem 4.2.8 corresponding to the maximization of $|p^{*}(\tau)|^2$, we obtain the equation

$$\Phi_N(U_{1N},\ N^{-1}m^{3/2}(\tau^{*}-\tau^{*}_{M}))=0$$

where U_{1N} is defined in the proof of Theorem 4.2.8, i.e.,

$$U_{1N}=(I^{*}_{\varepsilon_x\varepsilon_y}(\omega_{s_1}),\ldots,\ I^{*}_{\varepsilon_x\varepsilon_y}(\omega_{s_m}),0,0,\ldots)$$

and Φ_N is some function which satisfies the conditions of Theorem 2.4.4.2 for F_j there. Then by Theorem 2.4.4.2

$$N^{-1}m^{3/2}(\tau^*-\tau_M^*)=\Omega_N(U_{1N}).$$

Similarly we have

$$N^{-1}m^{3/2}(\tau^- -\tau_M^*)=\Omega_N(U_{2N}).$$

where U_{2N} is defined in the proof of Theorem 4.2.8, i.e.,

$$U_{2N}=((2/3)f^*_{xy.z}(\omega_{s_1}),\ldots,\ (2/3)f^*_{xy.z}(\omega_{s_m}),0,0,\ldots).$$

From Theorem 4.2.1 $N^{-1}m^{3/2}(\tau^*-\tau_M^*)$ has an asymptotic normal distribution. Thus Ω_N converges to some function Ω. Then using the argument in the proof of Theorem 4.2.8, we know that $N^{-1}m^{3/2}(\tau^*-\tau_M^*)$ and $N^{-1}m^{3/2}(\tau^- -\tau_M^*)$ have the same asymptotic distribution. Hence $N^{-1}m^{3/2}(\tau^- -\tau_M^*)$ has the asymptotic distribution as stated in the theorem. Q.E.D.

5.0 NUMERICAL EXAMPLE

In this chapter the proposed procedure for estimating partial group delay is illustrated with simulated time series data that are designed to satisfy the conditions required by Theorem 4.2.8.

For the simulation let $\alpha(t)$, $\beta(t)$, $\eta(t)$ and $\gamma(t)$ be mutually independent zero mean, normal random variables for $t=0,1,\ldots.$ The variances of $\alpha(t)$ and $\beta(t)$ are taken to be 0.04 and the variance of $\eta(t)$ is taken to be 3.0. The variance of $\gamma(t)$ is taken to be 0.06. Define the time series $s(t)$ to satisfy the relation

$$s(t)=\eta(t)+0.75\eta(t-1); \tag{5.1}$$

and for $t=1,2,\ldots$ construct $\varepsilon_x(t)$ and $\varepsilon_y(t)$ as

$$\varepsilon_x(t)=s(t)+\alpha(t) \tag{5.2}$$

and

$$\varepsilon_y(t)=s(t+3)+\beta(t). \tag{5.3}$$

Define the time series $z(t)$ to satisfy the relation

$$z(t)=0.5z(t-1)+\gamma(t). \tag{5.4}$$

Now construct

$$x(t)=0.8z(t+1)+\varepsilon_x(t) \tag{5.5}$$

and

$$y(t)=0.6z(t+2)+\varepsilon_y(t). \tag{5.6}$$

Thus by the definition the partial group delay between x(t) and y(t) adjusted for z(t) is the group delay between $\varepsilon_x(t)$ and $\varepsilon_y(t)$. The conditions of Theorem 4.2.8 may be verified for these time series. The autospectra for $\varepsilon_x(t)$ and $\varepsilon_y(t)$ are

$$f_{\varepsilon_{xx}}(\lambda)=f_{\varepsilon_{yy}}(\lambda)$$
$$=(2\pi)^{-1}[|1+0.75e^{i\lambda}|^2\ 0.04+3]$$

and the coherence between $\varepsilon_x(t)$ and $\varepsilon_y(t)$ is

$$\sigma(\lambda)=[|1+0.75e^{i\lambda}|^2\ 0.04]/[|1+0.75e^{i\lambda}|^2\ 0.04+3].$$

Since

$$f_{\varepsilon_x\varepsilon_y}(\lambda)=(2\pi)^{-1}|1+0.75e^{i\lambda}|^2 0.04e^{i3\lambda},$$

the group delay between $\varepsilon_x(t)$ and $\varepsilon_y(t)$ is $\tau_o=3$.

For the example $\varepsilon_x(t)$ and $\varepsilon_y(t)$ were simulated for t=1,...,500 according to (5.5), (5.6) and (5.4).

Choose $\lambda_o=\pi/2$ and center the band B at $\pi/2$. Let m=18, 24, 30 respectively. Then M=500/2m. Based on x(t), y(t) and z(t) for t=1,...,500, according to (4.1.3) construct

$$f^*_{zz}(\omega_s)=(1/3)[I_{zz}(\omega_s)+I_{zz}(\omega_s+\pi/M)+I_{zz}(\omega_s-\pi/M)].$$

where I_{zz} is the periodgram of z(t). As ω_s is in the form of $2\pi k_s/N$ with k_s an integer (s=1,...,m), and since ω_s belongs to B, thus $[N-2m/4]\le k_s\le[N+2m/4]$. Similarly we also define $f^*_{xy}(\omega_s)$, $f^*_{xz}(\omega_s)$ and $f^*_{zy}(\omega_s)$. Let

$$f^*_{xy.z}(\lambda)=f^*_{xy}(\lambda)-f^*_{xz}(\lambda)f^*_{zz}(\lambda)^{-1}f^*_{zy}(\lambda)$$

as defined in (4.1.4) and

$$p^-(\tau)=(1/m)\Sigma_o f^*_{xy.z}(\omega_s)e^{-i\tau\omega_s}$$

as defined in (4.1.5).

Then by maximizing $|p^-(\tau)|^2$ the estimate τ^- is obtained.

These exact steps were repeated 20 times for independent realizations of x(t), y(t) and z(t) (t=1,...,500) to give 20 corresponding estimates of τ.

Table 1 summarizes the result of 20 cases of this example. From table 1, the mean of τ^- when m=18, 24, 30 are 2.18, 2.52, and 2.71 respectively, which shows that the estimate is biased. From Theorem 4.2.3 the variance of $N^{-1}m^{3/2}(\tau^- -\tau^*_M)$ is approximated by $12(1-\sigma^2)/\pi^2\sigma^2$. Here for the partial coherence σ^2, we have

$$\sigma_{\varepsilon_x\varepsilon_y}(\lambda)=3|1+0.75e^{i\lambda}|^2/\{0.04|1+0.75e^{i\lambda}|^2+3\}.$$

When $\lambda=\pi/2$, $\sigma_{\varepsilon_x \varepsilon_y}(\pi/2)=0.92$. Thus $\mathrm{Var}\tau^{-} \approx 0.96$ which coincides with the result of 0.92 in table 1 for m=30.

Table 1. Summary of results of simulations.

No.	$\tau^{-}(m=18)$	$\tau^{-}(m=24)$	$\tau^{-}(m=30)$
1	3.7	2.7	1.6
2	1.6	2.1	2.6
3	2.4	2.0	4.8
4	2.8	3.4	3.8
5	1.4	1.5	2.8
6	1.3	1.8	4.9
7	1.4	1.5	2.8
8	4.9	4.0	1.2
9	3.0	1.8	2.3
10	3.1	2.0	1.9
11	1.7	2.8	2.1
12	1.8	4.2	2.5
13	0.1	1.9	3.5
14	3.3	1.1	2.7
15	3.5	4.5	1.9
16	3.3	1.5	1.5
17	1.8	1.4	2.6
18	0.2	1.6	2.7
19	2.1	4.7	2.8
20	0.1	3.9	3.2
average of τ^{-}	2.18	2.52	2.71
sample variance of τ^{-}	1.65	1.35	0.95

6.0 PARTIAL GROUP DELAY AND TIME LAG RELATIONSHIPS AMONG MULTIPLE TIME SERIES

Theorem 2.4.1.1 gives the result from Deaton and Foutz (1980) for the time lag relationship between two stochastic processes. A time-lag interpretation is given for the group delay between two continuous time weakly-stationary stochastic processes that can be summarized as

$$X_{1\Lambda}(t)=\alpha(\lambda_o)X_{2\Lambda}(t-\tau(\lambda_o))+\varepsilon_\Lambda(t)+O_\Lambda(t) \quad (-\infty<t<\infty).$$

The notations are defined in Theorem 2.4.1.1. That result shows that as the band Λ shrinks to a single frequency λ_o, the relationship between $X_{1\Lambda}$ and $X_{2\Lambda}$ approaches the simple form

$$X_{1\Lambda}(t)=\alpha X_{2\Lambda}(t-\tau)+\varepsilon_\Lambda(t).$$

Now we extend the result to the case of more than two stochastic processes.

THEOREM 6.1

Let $\{X(t),Y_1(t),\ldots Y_p(t);\ -\infty<t<\infty\}$ be a multivariate weakly stationary stochastic process. Assume the component processes have mean zero and have absolutely continuous spectra and spectral density matrix

$$\begin{bmatrix} f_{xx} & f_{x1} \cdots & f_{xp} \\ f_{1x} & f_{11} \cdots & f_{1p} \\ \vdots & \vdots & \vdots \\ f_{px} & f_{p1} \cdots & f_{pp} \end{bmatrix}$$

with each element nonzero and boundedly differentiable in a frequency band Λ containing the frequency λ_o. Further assume the spectral density matrix of $[Y_1(t),\ldots Y_p(t)]$ is Hermitian positive definite. Then with $\tau_i(\lambda)$ the partial group delay of X behind Y_i adjusted for $Y_1,\ldots Y_{i-1},\ldots Y_p$, and with

$$\alpha_i(\lambda)=\exp[i\lambda\tau_i\lambda]\{f_{xj.1,\ldots,p}(\lambda)/f_{jj.1,\ldots,p}(\lambda)\},$$

where $f_{xj.1,\ldots,p}(\lambda)$ is the partial cross-spectral density adjusted for $Y_1,\ldots Y_{j-1},Y_{j+1},\ldots Y_p$ and $f_{jj.1,\ldots,p}(\lambda)$ is the auto-spectral density adjusted for $Y_1,\ldots,Y_{j-1},Y_{j+1},\ldots,Y_p$, there exist zero-mean weakly-stationary stochastic processes ε_Λ and O_Λ that give a representation of the frequency component X_Λ as

$$X_\Lambda(t)=\Sigma_{i=1}^{p}\alpha_i(\lambda_o)Y_{i\Lambda}(t-\tau_i(\lambda_o))+\varepsilon_\Lambda(t)+O_\Lambda(t) \qquad (6.1)$$

where the process ε_Λ is uncorrelated with each of the processes $Y_{i\Lambda}$ ($i=1,\ldots p$); and O_Λ. When Λ is a narrow frequency band, and for $\sigma_i(\lambda)$ the partial coherence between X and Y_j

adjusted for $Y_1,\ldots Y_{j-1},Y_{j+1},\ldots Y_p$ the relative importance of the terms in (5.1) is indicated by the limits

$$\lim_{d(\Lambda)\to 0}\|O_\Lambda(t)\|/\|X_\Lambda(t)\|=0, \tag{6.2}$$

$$\lim_{d(\Lambda)\to 0}\|\alpha_i(\lambda_o)Y_i(t-\tau_i(\lambda_o)\|/\|X_\Lambda(t)\|=\sigma_i(\lambda_o), \tag{6.3}$$

$$\lim_{d(\Lambda)\to 0}\|\varepsilon_\Lambda(t)\|/\|X_\Lambda(t)\|=\{1-\Sigma_{i=1}^{p}\sigma_i(\lambda_o)^2\}^{1/2}. \tag{6.4}$$

PROOF

Let H(Y) be the closed linear manifold generated by $Y_1,Y_2,\ldots,Y_p$. It is well known and stated in Section 2.5 that the process X can be decomposed into a unique process in H(Y) plus a noise $\varepsilon=\{\varepsilon(t);\ -\infty<t<\infty\}$, that is uncorrelated with elements of H(Y), this decomposition takes the form

$$X(t)=\Pi(X|Y_1,Y_2,\ldots,Y_p)+\varepsilon(t).$$

If we consider the components of X(t) and $Y_i(t)$ $i=1,\ldots,p$ in the band Λ we have

$$X_\Lambda(t)=\Pi(X_\Lambda|Y_{1\Lambda},\ldots,Y_{p\Lambda})+\varepsilon_\Lambda(t).$$

From (2.2.2.3)

$$\Pi(X_\Lambda|Y_{1\Lambda},\ldots,Y_{p\Lambda})=\Sigma_{i=1}^{p}\int_\Lambda B_i(\lambda)e^{i\lambda t}dZ_i(\lambda),$$

where $B(\lambda)=[B_1(\lambda),\ldots,B_p(\lambda)]$ is the transfer function. From (2.2.2.7)

$$B(\lambda)=[B_1(\lambda),\ldots,B_p(\lambda)]$$

$$=(f_{x1}(\lambda),\ldots f_{xp}(\lambda))\begin{bmatrix} f_{11}(\lambda)\ldots\ldots f_{1p}(\lambda) \\ \\ f_{p1}(\lambda)\ldots\ldots f_{pp}(\lambda)\end{bmatrix},$$

where $f_{xi}(\lambda)$ is the cross-spectral density between X and Y_i and

$$f(\lambda)=\begin{bmatrix} f_{11}(\lambda)\ldots\ldots f_{1p}(\lambda) \\ \\ f_{p1}(\lambda)\ldots\ldots f_{pp}(\lambda)\end{bmatrix}$$

is the spectral density matrix of the vector process $(Y_1,Y_2,\ldots,Y_p)$. As we assumed before, $f(\lambda)$ is Hermitian positive definite.

We want to derive the expressions of $B_i(\lambda)$, $i=1,\ldots,p$. Without loss of generality we work on $B_p(\lambda)$, the others are similar.

Let $f(\lambda)$ be partitioned into the following way,

$$f(\lambda)=\begin{bmatrix} f_{11} \cdots\cdots & f_{1p-1} & f_{1p} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ f_{p-1\ 1} \cdots & f_{p-1\ p-1} & f_{p-1p} \\ f_{p1} \cdots\cdots & f_{p\ p-1} & f_{pp} \end{bmatrix}$$

and write it as

$$f(\lambda)=\begin{bmatrix} A & B \\ & \\ C & f_{pp} \end{bmatrix}$$

Then from Rao (1973) p.33, the inverse of $f(\lambda)$ can be written as

$$\begin{bmatrix} * & -A^{-1}B(f_{pp}-CA^{-1}B)^{-1} \\ & \\ ** & (f_{pp}-CA^{-1}B)^{-1} \end{bmatrix}$$

where "*" is a (p-1)X(p-1) matrix and "**" is a 1X(p-1) vector. As $f(\lambda)$ is assumed to be a Hermitian positive definite, so is A. Thus the inverse of A exists.

We also write the vector $(f_{x1}, \ldots f_{xp-1}\ f_{xp})$ as $(f_I\ f_{xp})$, where f_I is a 1X(p-1) vector.

Hence

$$B(\lambda)=(f_I\ f_{xp}) \begin{bmatrix} * & -A^{-1}B(f_{pp}-CA^{-1}B)^{-1} \\ ** & (f_{pp}-CA^{-1}B)^{-1} \end{bmatrix},$$

and so

$$B_p(\lambda)=(f_{xp}-f_I A^{-1}B)\ (f_{pp}-CA^{-1}B)^{-1}.$$

We know that $C=B^*$, the conjugate transpose matrix of B. Since A is Hermitian positive definite, there exists a non-singular $(p-1)X(p-1)$ matrix Q such that $A^{-1}=QQ^*$. Then $CA^{-1}B=CQQ^*C^*=(CQ)\ (CQ)^*$ show that $CA^{-1}B$ is a real scalar.

We decompose X and Y_p in the following way

$$X(t)=\Pi(X|Y_1,\ldots,Y_{p-1})+\varepsilon_x(t)$$
$$Y_p(t)=\Pi(Y_p|Y_1,\ldots,Y_{p-1})+\varepsilon_p(t).$$

Comparing with (2.2.2.9), $f_{xp}-f_I A^{-1}B$ is just the spectral density of the residual processes $X_\Lambda-\Pi(X_\Lambda|Y_{1\Lambda},\ldots Y_{p-1\Lambda})$ and $Y_{p\Lambda}-\Pi(Y_{p\Lambda}|Y_{1\Lambda},\ldots Y_{p-1\Lambda})$. Thus $f_{xp}-f_I A^{-1}B$ is the partial cross-spectral density between X and Y_p adjusted for $Y_1,\ldots Y_{p-1}$. We denote it by $f_{xp.1,\ldots,p-1}$.

Similarly $f_{pp}-CA^{-1}B=f_{pp.1,\ldots,p-1}$ is the partial spectral density of Y_p adjusted for $Y_1,\ldots Y_{p-1}$. Then $f_{xp}-f_I A^{-1}B$ is the cross spectral density between $\varepsilon_x(t)$ and $\varepsilon_p(t)$ and $f_{pp}-CA^{-1}B$ is the auto-spectral density of $\varepsilon_p(t)$.

Thus if we replace $f_{12}(\lambda)$ and $f_{22}(\lambda)$ in the proof of Theorem 2.4.1.1 (see Deaton and Foutz, 1980) by $f_{xp.1,...,p-1}(\lambda)$ and $f_{pp.1,...,p-1}(\lambda)$ respectively and follow exactly the same proof word by word we have

$$\int B_p(\lambda)e^{i\lambda t}dZ_p(\lambda)=\alpha_p(\lambda)Y_{p\Lambda}(t-\tau_p(\lambda_o))+O_{p\Lambda}(t),$$

where $\tau_p(\lambda_o)$ is the partial group delay of X behind Y_p at λ_o.

For other $\int B_i(\lambda)e^{i\lambda t}dZ_i(\lambda)$ $i=1,...,p-1$, we have similar expressions.

Let $O_\Lambda(t)=\Sigma_{i=1}^p O_{i\Lambda}(t)$. Obviously $O_\Lambda(t)$ is uncorrelated with $\varepsilon(t)$ as each $O_{i\Lambda}(t)$ is. Thus we obtain (6.1).

As to (6.2)——(6.4), these results can be obtained easily as proof of Theorem 2.4.1.1 (see Deaton and Foutz, 1980)

From (6.1) and the uncorrelateness of $\varepsilon_\Lambda(t)$ with all $\alpha_i(\lambda_o)Y_\Lambda(t-\tau_i(\lambda_o))+O_{j\Lambda}(t)$ and $O_{j\Lambda}(t)$ the variance of $\varepsilon_\Lambda(t)$ becomes

$$
\begin{aligned}
&\mathrm{Var}[\varepsilon_\Lambda(t)]\\
&=\mathrm{Var}[X(t)]-\mathrm{Var}[\Sigma_{i=1}^p\{\alpha_i(\lambda_o)Y_{i\Lambda}(t-\tau_i(\lambda_o))+O_{j\Lambda}(t)\}]\\
&=\mathrm{Var}[X(t)]-\Sigma_{i=1}^p\mathrm{Var}[\alpha_i(\lambda_o)Y_j(t-\tau_i(\lambda_o))]\\
&\quad-\Sigma_{i=1}^p\mathrm{Var}[O_{j\Lambda}(t)]\\
&\quad-\Sigma_{i=1}^p E[\alpha_i(\lambda_o)Y_i(t-\tau_j(\lambda_o))\ \alpha_k(\lambda_o)^c Y_k(t-\tau_j(\lambda_o))^c]\\
&\quad-\Sigma_{i=1}^p E[\alpha_i(\lambda_o)Y_i(t-\tau_i(\lambda_o))\ O_{k\Lambda}(t)^c].\\
&\quad-\Sigma_{i=1}^p E[O_{i\Lambda}(t)\ O_{k\Lambda}(t)^c],
\end{aligned}
$$

where the last three summations are over the j and k =1,...,p and $j \neq k$. Similar to the proof of 2.4.1.1 the last four terms are of order $d(\Lambda)^2$.

We also have

$$\mathrm{Var}[X(t)]=f_x(\lambda_o)d(\Lambda)+O[d(\Lambda)]^2,$$

$$\mathrm{Var}[\alpha_i(\lambda_o)Y_i(t-\tau_i(\lambda_o))]$$

$$=\frac{|f_{xj.1,\dots,p}(\lambda_o)|^2}{f_{jj.1,\dots,p}(\lambda_o)}\ d(\Lambda)+O[d(\Lambda)^2].$$

Thus we have

$$\mathrm{Var}[\varepsilon_\Lambda(t)]=[f_x(\lambda_o)-\ \Sigma_{j=1}^{p}\ \frac{|f_{xj.1,\dots,p}(\lambda_o)|^2}{f_{jj.1,\dots,p}(\lambda_o)}]\ d(\Lambda)+O[d(\Lambda)^2].$$

Hence we complete the proof. Q.E.D.

Interpretations of the results can be given in the geometry of the Hilbert space $L_2(P)$. The elements of this space are random variables on (Ω, B, P) having mean zero and finite variances, and the norm of each random variable in $L_2(P)$ is defined to be its deviation. Thus (6.2), (6.3) and (6.4) give the asymptotic relative sizes of the terms in (6.1) as Λ

shrinks to λ_o. Expression (6.1) show that $X_\Lambda(t)$ can be approximated by the the two uncorrelated processes as

$$X_\Lambda(t)=\Sigma_{i=1}^{p}\alpha_i(\lambda_o)Y_{i\Lambda}(t-\tau_i(\lambda_o))+\varepsilon_\Lambda(t) \quad (-\infty<t<\infty). \qquad (6.5)$$

with approximation error

$$O_\Lambda(t)=X_\Lambda(t)-\Sigma_{i=1}^{p}\alpha_i(\lambda_o)Y_{i\Lambda}(t-\tau_i(\lambda_o))-\varepsilon_\Lambda(t).$$

The importance of this theorem is that it justifies a time-lag interpretation of partial group delay: As Λ shrinks to λ_o and the relationship in (6.5) becomes precise, the partial group delay, $\tau_i(\lambda_o)$ is seen to be the time lag between X and Y_i in (6.5).

Furthermore, this theorem provides insight into the nature of linear relationships between weakly stationary processes. From this theorem, we know that as Λ shrinks to λ_o

$$\begin{aligned}&\Pi(X|Y_1,Y_2,\ldots,Y_p)\\&=\Sigma_{i=1}^{p}\alpha_i(\lambda_o)Y_{i\Lambda}(t-\tau_i(\lambda_o)).\end{aligned}$$

Thus it provides a frequency-by-frequency decomposition of the relationship characterized by $\Pi(X|Y_1,Y_2,\ldots,Y_p)$. It shows that in the frequency domain this relationship becomes a simple linear relationship between $X_\Lambda(t)$ and p time-lagged variables $Y_{i\Lambda}(t-\tau_i(\lambda_o))$.

7.0 EXTENSION TO NON-STATIONARY PROCESSES

7.1 UNIVARIATE NON-STATIONARY PROCESSES

Consider a continuous parameter stochastic process $X=\{x(t);\ -\infty<t<\infty\}$. We assume that $E[x(t)]=0$ and $E[x(t)]^2<\infty$. The covariance function is defined by

$$R(s,t)=E[x^c(s)\ x(t)].$$

If $\{x(t)\}$ is weakly-stationary, i.e. $R(s,t)$ is a function of $|x-t|$only, then we know $R(s,t)$ admits the representation

$$R(s,t)=\int_{-\infty}^{\infty}e^{-i(s-t)}dF(\lambda), \qquad (7.1.1)$$

where $F(\lambda)$ is the integrated spectrum. From Chapter 2, $X(t)$ admits the spectral representation

$$X(t)=\int_{-\infty}^{\infty}e^{i\lambda t}dZ(\lambda) \qquad (7.1.2)$$

where $Z(\lambda)$ is a process with orthogonal increments, and $E[|dZ(\lambda)|^2]=dF(\lambda)$.

It thus follows that if $X(t)$ is non-stationary, it cannot be represented in the form (7.1.2), and $R(s,t)$ cannot be represented in the form (7.1.1). However we know (cf.

Priestley, 1981, 4.11) that for a fairly general class of stochastic processes, R(s,t) can be represented in a form similar to (7.1.1), provided we replace the functions $\{e^{i\lambda t}\}$ by a more general "family" of functions $\{\phi_t(\lambda)\}$.

We now restrict attention to the class of processes for which there exists a family of functions $\{\phi_t\}$ defined on the real line, and a measure $\mu(\lambda)$ on the real line, such that for each s,t the covariance function R(s,t) admits a representation of the form

$$R(s,t)=\int_{-\infty}^{\infty}\phi_s(\lambda)^c\phi_t(\lambda)d\mu(\lambda). \tag{7.1.3}$$

In order for Var{x(t)} to be finite for all t, $\phi_t(\lambda)$ must be quadratically integrable with respect to the measure μ for each t.

It then follows, from the theorem of general orthogonal expansions (see Theorem 4.11.2 in Priestley, 1981, p.262), that whenever R(s,t) has the representation (7.1.3), the process {x(t)} admits a representation of the form

$$X(t)=\int_{-\infty}^{\infty} \phi_t(\lambda)\ dZ(\Lambda) \tag{7.1.4}$$

where $Z(\lambda)$ is an orthogonal process with

$$E[|dZ(\lambda)|^2]=d\mu(\lambda).$$

Suppose that, for each fixed λ, $\phi_t(\lambda)$ (considered as a function of t) possesses a generalized Fourier transform whose modulus has an absolute maximum at frequency $\theta(\lambda)$, say. Then we may regard $\phi_t(\lambda)$ as an amplitude modulated sine wave with frequency θ, and write $\phi_t(\lambda)$ in the form

$$\phi_t(\lambda)=A_t(\lambda)e^{i\theta t}. \tag{7.1.5}$$

Now we have the following definition:

DEFINITION 7.1.1.

The function of t, $\phi_t(\lambda)$ is called an oscillatory function if for some (necessarily unique) $\theta(\lambda)$, $\phi_t(\lambda)$ is of the form (7.1.5) where

$$A_t(\lambda)=\int_{-\infty}^{\infty}e^{it\theta}\,dK_\lambda(\theta).$$

with $|dK_\lambda(\theta)|$ having an absolute maximum at θ_o. We now can write (7.1.4) as

$$X(t)=\int_{-\infty}^{\infty} A_t(\lambda)e^{i\lambda t}dZ(\lambda), \tag{7.1.6}$$

and we also have

$$R(s,t)=\int_{-\infty}^{\infty}A_s^c(\lambda)A_t(\lambda)e^{i\lambda(t-s)}d\mu(\lambda). \tag{7.1.7}$$

DEFINITION 7.1.2.

If there exists a family of oscillatory functions $F \equiv \{\phi_t(\lambda)\} \equiv \{A_t(\lambda)e^{i\lambda t}\}$, in terms of which the process $\{X(t)\}$ has representation of the form (7.1.6), $\{X(t)\}$ will be called an oscillatory process.

For an oscillatory process, we can define the evolutionary power spectrum at time t with respect to the family F, $dF_t(\lambda)$, by

$$dF_t(\lambda)=|A_t(\lambda)|^2 \, d\mu(\lambda).$$

Thus from (7.1.7), we have

$$Var[X(t)]=\int_{-\infty}^{\infty} dF_t(\lambda).$$

When the measure $\mu(\lambda)$ is absolutely continuous with respect to the Lebesgue measure, we may write $dF_t(\lambda)=f_t(\lambda)d\lambda$ and call $f_t(\lambda)$ the evolutionary spectral density.

For estimating the evolutionary group delay, we are interested in the oscillatory processes whose non-stationary characterics are changing "slowly" over time. For the purpose of defining this "slowly changing" character Priestley (1965) introduces the semi-stationary process by specifying that the Fourier transform of the oscillatory function must be "highly concentrated" in the region of zero frequency.

For each family, define the function $B_F(\lambda)$ by

$$B_F(\lambda)=\int|\theta|dK_\lambda(\theta).$$

[note that $B_F(\lambda)$ is a measure of the "width" of $|dK_\lambda(\theta)|$]

DEFINITION 7.1.3.

A family of oscillatory functions will be termed semi-stationary if the function $B_F(\lambda)$ is bounded for all λ, and the constant B_F defined by

$$B_F=[\sup_\lambda\{B_F(\lambda)\}]^{-1}$$

will be termed the characteristic width of the family.

DEFINITION 7.1.4.

A semi-stationary process $\{x(t)\}$ is defined as one for which there exists a semi-stationary family in terms of which $\{x(t)\}$ has a representation of the form of (7.1.6).

For a particular semi-stationary process $\{x(t)\}$, consider the class **C** of semi-stationary families, in terms of each of which $\{x(t)\}$ admits a spectral representation. We define the characterstic width, B_X of the process $\{x(t)\}$ by

$$B_X = \sup_{F \varepsilon C} B_F$$

7.2 EVOLUTIONARY CROSS-SPECTRA FOR MULTIVARIATE OSCILLATORY PROCESS

Let $X(t)=\{(X_1(t),\ldots X_p(t))' \ t\varepsilon T\}$ be a vector process, in which each component is an oscillatory process. We can write

$$x_i(t)=\int e^{it\lambda}A_{t,x_i}(\lambda)dZ_{x_i}(\lambda) \qquad i=1,2,\ldots p \quad (7.2.1)$$

or

$$X(t)=\int e^{it\lambda}A_{t,X}(\lambda)dZ_X(\lambda), \qquad (7.2.2)$$

where

$$E[dZ_{x_i}(\lambda)\ dZ^c_{x_j}(\lambda')]=0 \text{ for } \lambda\neq\lambda' \ i=j \text{ or } i\neq j$$

$$E|dZ_{x_i}(\lambda)|^2=d\mu_{x_i x_j}(\lambda)$$

$$E[dZ_{x_i}(\lambda)dZ^c_{x_j}(\lambda)]=d\mu_{x_i x_j}(\lambda) \qquad i,j=1,2,\ldots,p.$$

We have the following definition.

DEFINITION 7.2.1.

Let $\{F_{x_1},\ldots,F_{x_p}\}$ be a vector family of oscillatory functions, and let $\{(x_1(t),\ldots,x_p(t)'\}$ be a multivariate oscillatory process with components admitting representations of forms (7.2.1) with respect to the family $\{F_{x_1},\ldots,F_{x_p}\}$. The evolutionary power cross-spectrum at time t with respect to the families F_{x_i} and F_{x_j}, $dF_{t,x_i x_j}(\lambda)$, is defined as $dF_{t,x_ix_j}(\lambda)=A_{t,x_i}(\lambda)A_{t,x_j}(\lambda)^c d\mu_{ij}(\lambda)$. When the measure $\mu_{ij}(\lambda)$ is absolutely continuous with respect to Lebesgue measure, we may write $dF_{t,x_ix_j}(\lambda)=f_{t,x_ix_j}(\lambda)d\lambda$ and call $f_{t,x_ix_j}(\lambda)$ the evolutionary cross-spectral density function of x_i and x_j when $i\neq j$, and the evolutionary spectral density function when $i=j$. The matrix

$$f_{t,X}(\lambda)=\begin{bmatrix} f_{t,x_1x_1}(\lambda)\cdots\cdots f_{t,x_1x_p}(\lambda) \\ \vdots \qquad\qquad \vdots \\ f_{t,x_px_1}(\lambda)\cdots\cdots f_{t,x_px_p}(\lambda) \end{bmatrix}$$

is termed correspondingly the evolutionary spectral density matrix.

Similar to the stationary case, we can now have the following definition.

DEFINITION 7.2.2

We call

$$\sigma_{x_ix_j}(\lambda)=\frac{|f_{t,x_ix_j}(\lambda)|}{[f_{t,x_ix_i}(\lambda)\ f_{t,x_jx_j}(\lambda)]^{1/2}}$$

the coherency between $\{x_i(t)\}$ and $\{x_j(t)\}$, and

$$\phi_{t,x_ix_j}(\lambda)=\arg\left\{\frac{f_{t,x_ix_j}(\lambda)}{[f_{t,x_ix_i}(\lambda)\ f_{t,x_jx_j}(\lambda)]^{1/2}}\right\}$$

the phase spectrum between $\{x_i(t)\}$ and $\{x_j(t)\}$.

We note from

$$dF_{t,x_ix_j}(\lambda)=E[A_{t,x_i}(\lambda)dZ_{x_i}(\lambda)\ A^c_{t,x_j}(\lambda)dZ^c_{x_j}(\lambda)],$$

that $\sigma_{x_ix_j}(\lambda)$

$$=\frac{|f_{t,x_ix_j}(\lambda)|}{[f_{t,x_ix_i}(\lambda)\ f_{t,x_jx_j}(\lambda)]^{1/2}}$$

$$=\frac{E[dZ_{x_i}(\lambda)dZ^c_{x_j}(\lambda)]}{\{E|dZ_{x_i}(\lambda)|^2\}^{1/2}\{E|dZ_{x_j}(\lambda)|^2\}^{1/2}}$$

Thus the coherency is independent of the parameter t. However the phase is not time invariant. Now assuming that $\phi_{t,x_ix_j}(\lambda)$ is differentiable with respect to λ, we define $\tau_t(\lambda)=d\phi_{t,x_ix_j}(\lambda)/d\lambda$ the group delay of $\{x_i(t)\}$ behind $\{x_j\}$ at frequency λ.

7.3 PARTIAL GROUP DELAY FOR OSCILLATORY PROCESSES

As in the case for stationary process, partial group delay between the oscillatory processes can be defined in terms of partial phase. However some difficulties will arise in defining the partial phase between the oscillatory processes.

We still consider the oscillatory processes $\{x_1(t)\}$, $\{x_2(t)\}$ and $\{x_3(t)\}$, and assume that all of these have absolutely continuous evolutionary spectral densities.

Similar to Chapter 2 we first remove the influence of $\{x_3(t)\}$ on $\{x_1(t)\}$ and $\{x_2(t)\}$ and consider the residual process

$$\varepsilon_1(t)=x_1(t)-\Pi(x_1|x_3)$$
$$\varepsilon_2(t)=x_2(t)-\Pi(x_2|x_3)$$

or alternatively, if $\{x_3(t)\}$ has a continuous spectrum and a spectral density, which is bounded from above and away from zero (see Koopmans, 1974), (A 6.1) p.205), then we can write $\varepsilon_1(t)$ and $\varepsilon_2(t)$ in the following way,

$$\varepsilon_1(t)=x_1(t)-\Sigma_{-\infty}^{\infty}b_{1t}(u)x_3(t-u)$$
$$\varepsilon_2(t)=x_2(t)-\Sigma_{-\infty}^{\infty}b_{2t}(u)x_3(t-u)$$

where $b_{1t}(u)$ and $b_{2t}(u)$ are determined by minimizing $E[\varepsilon_1(t)]^2$ and $E[\varepsilon_2(t)]^2$. We denote $\Pi(x_1|x_3)$ by $w_1(t)$ and $\Pi(x_2|x_3)$ by $w_2(t)$.

Here the question arises: Are $\varepsilon_1(t)$ and $\varepsilon_2(t)$ also oscillatory processes?

The answer is affirmative when certain conditions are met.

To begin with, we consider a linear open loop system in the form

$$y(t)=\Sigma_{-\infty}^{\infty} d_t(u)x(t-u)+\varepsilon(t),$$

where $\{x(t)\}$, $\{y(t)\}$, and $\{\varepsilon(t)\}$ are oscillatory processes. Assume that $\Sigma_{-\infty}^{\infty}|d_t(u)|<\infty$ for all t. Then we write for each fixed t

$$D_t(\lambda)=\Sigma_{-\infty}^{\infty} d_t(u)e^{-i\lambda u},$$

and term $D_t(\lambda)$ the time dependent transfer function corresponding to the (time-dependent) impulse response function $\{d_t(u)\}$. Further we assume that $D_t(\lambda)$ is "slowly" varying over time. To make the statement more precise, we need the following definition.

DEFINITION 7.3.1. (Subba Rao and Tong, 1972)

If for each fixed λ, $D_t(\lambda)$ considered as a function of t, possesses a generalized Fourier transform

$$D_t(\lambda)=\int_{-\infty}^{\infty} e^{it\theta}dL_\lambda(\theta)$$

such that $|dL_\lambda(\theta)|$ has an absolute maximum at $\theta=0$, then we define the bandwidth, B_L, of $dL_\lambda(\theta)$ by

$$B_L=\{\sup_\lambda \int_{-\infty}^{\infty}|\theta||dL_\lambda(\theta)|\}^{-1}.$$

From now on we assume that B_L is greater than zero. We also assume $B_L \geq B_x$.

Back to our question. If $\{x_1(t)\}$ is an oscillatory process and the conditions of definition 6.3.1 are satisfied for $\Sigma_{-\infty}^{\infty} b_{1t}(u)e^{-i\lambda u}$, then it follows from Tong (1972) or Subba Rao and Tong (1972) that $\{w_1(t)\}$is also an oscillatory process.

We further assume that each oscillatory process of $\{x_1(t)\}$ and $\{w_1(t)\}$ has evolutionary spectral representation with respect to the same function, $\phi_t(\lambda)=A_{t1}(\lambda)e^{i\theta t}$ say, that is just the case which Abdrabbo and Priestley (1969) considered.

Based on the above assumptions, $x_1(t)$ and $w_1(t)$ can be represented as

$$x_1(t)=\int A_{t1}e^{i\lambda t}dZ_{x_1}(\lambda)$$
$$w_1(t)=\int A_{t1}e^{i\lambda t}dZ_{w_1}(\lambda).$$

Thus

$$\begin{aligned}\varepsilon_1(t)&=x_1(t)-w_1(t)\\ &=\int A_{t1}(\lambda)e^{i\lambda t}(dZ_{x_1}(\lambda)-dZ_{w_1}(\lambda))\\ &=\int A_{t1}(\lambda)e^{i\lambda t}d[z_{x_1}(\lambda)-Z_{w_1}(\lambda)]\\ &=\int A_{t1}(\lambda)e^{i\lambda t}dZ_{v_1}(\lambda),\end{aligned}$$

where $dZ_{v_1}(\lambda)=d(Z_{x_1}(\lambda)-z_{w_1}(\lambda))$. From

$$\begin{aligned}
&E[dZ_{x_1}(\lambda)\ dZ^c_{x_1}(\lambda')] \\
&=E[dZ_{w_1}(\lambda)\ dZ^c_{w_1}(\lambda')] \\
&=E[dZ_{x_1}(\lambda)\ dZ^c_{w_1}(\lambda')] \\
&=0 \qquad\qquad \text{for } \lambda\neq\lambda',
\end{aligned}$$

we have

$$E[dZ_{v_1}(\lambda)\ dZ^c_{v_1}(\lambda')]=0 \qquad \text{for } \lambda\neq\lambda'$$

and

$$\begin{aligned}
&E|dZ_{v_1}(\lambda)|^2 \\
&=E|dZ_{x_1}(\lambda)|^2+E|dZ_{w_1}(\lambda)|^2 \\
&=d\mu_{v1}(\lambda),
\end{aligned}$$

i.e., $\{Z_{v_1}(\lambda)\}$ is an orthogonal increment process. By the definition, $\varepsilon_1(t)=\int A_{t,1}(\lambda)e^{i\lambda t}dZ_{v_1}(\lambda)$ is an oscillatory process with respect to the family. We denote its evolutionary spectral density by $f_{t,\varepsilon_1}(\lambda)$ on the condition that it exists.

Similarly we also have that $\varepsilon_2(t)$ is an oscillatory process with evolutionary spectral density $f_{t,\varepsilon_2}(\lambda)$.

We further assume that the evolutionary cross-spectral density between ε_1 and ε_2 is $f_{t,\varepsilon_1\varepsilon_2}(\lambda)$.

Now we define the evolutionary partial coherency between $\{x_1(t)\}$ and $\{x_2(t)\}$ adjusted for $\{x_3(t)\}$ as the coherence between $\{\varepsilon_1(t)\}$ and $\{\varepsilon_2(t)\}$, which is

$$\frac{|f_{t,\varepsilon_1\varepsilon_2}(\lambda)|}{[f_{t,\varepsilon_1}(\lambda)\ f_{t,\varepsilon_2}(\lambda)]^{1/2}}\ .$$

Similarly the corresponding partial phase is defined as

$$\phi_{t,x_1x_2.x_3}(\lambda)=\arg\left\{\frac{f_{t,\varepsilon_1\varepsilon_2}(\lambda)}{[f_{t,\varepsilon_1}(\lambda)\ f_{t,\varepsilon_2}(\lambda)]^{1/2}}\right\}$$

and the partial group delay is defined as $\tau_{t,x_1x_2.x_3}(\lambda)=d\phi_{t,x_1x_2.x_3}(\lambda)/d\lambda$.

7.4 ESTIMATING THE EVOLUTIONARY GROUP DELAY AND PARTIAL GROUP DELAY

(a) Estimating evolutionary group delay

Suppose we are given one vector sample record of the bivariate semi-stationary process $\{x(t), y(t)\}$, $t=1,\ldots N$ (see Definition 7.1.3). One method of estimating the evolutionary auto-spctral density $f_{t,x}(\lambda)$, $f_{t,y}(\lambda)$ and cross-spectral density $f_{t,xy}(\lambda)$ are proposed in Priestley (1965) and Priestley and Tong (1973) respectively.

In those papers, by choosing suitable filter and weight-function and using "double window" technique, an approximately unbiased estimate of the average value of $f_{t,x}(\lambda)$ in the neighborhood of t, $f^{\#}_{t,x}(\lambda)$ is obtained. Similarly we also obtain $f^{\#}_{t,y}(\lambda)$ and $f^{\#}_{t,xy}(\lambda)$.

The evolutionary spectral phase can be estimated by

$$\phi^{\#}_{t,xy}(\lambda)=\arg\left\{ \frac{f^{\#}_{t,xy}(\lambda)}{[f^{\#}_{t,x}(\lambda)f^{\#}_{t,y}(\lambda)]^{1/2}} \right\}. \tag{7.4.1}$$

Thus to estimate the evolutionary group delay between $\{x(t)\}$ and $\{y(t)\}$ at frequency λ_o, we can choose a small frequency interval centered at λ_o i.e. $[\lambda_o-\varepsilon,\ \lambda_o+\varepsilon]$, then the estimator of group delay can be

$$\tau^{\#}_{t,xy}(\lambda)=\{\phi^{\#}_{t,xy}(\lambda_o+\varepsilon)\ -\phi^{\#}_{t,xy}(\lambda_o-\varepsilon)\}/2\varepsilon. \tag{7.4.2}$$

(b) Estimating partial evolutionary group delay

Just as in the stationary cases , to estimate partial group delay, we consider the indirect and direct approaches. For the indirect approach, we need to approximate the residual process. However to obtain the approximations of the residual processes might be very complicate for the non-stationary case. Hence we only consider a direct approach.

We have the following theorem.

THEOREM 7.4.1.

Under the following conditions

1) $\{x_1(t)$ and $\{x_2(t)\}$ satisfy the conditions of definition 6.3.1.

2) Each of $\{x_1(t)\}$, $\{x_2(t)\}$, $\{x_3(t)\}$, $\{w_1(t)\}$ and $\{w_2(t)\}$ has an spectral representation w.r.t. the same family of function.

3) $A_t(\lambda)$ is "slowly" varying over time, i.e.,

$$A_{t_1}(\lambda)\approx A_{t_2}(\lambda) \qquad \text{when } t_1\neq t_2.$$

then we have the following conclusion:

$$f_{t,\varepsilon_1}(\lambda)\approx f_{t,x_1}(\lambda)-f_{t,x_1x_3}(\lambda)f^{-1}_{t,x_3x_3}(\lambda)f_{t,x_3x_1}(\lambda).$$

PROOF

We know that $x_1(t)=w_1(t)+\varepsilon_1(t)$. By the assumption

$$x_1(t)=\int A_t(\lambda)e^{i\lambda t}dZ_{x_1}(\lambda)$$
$$x_3(t)=\int A_t(\lambda)e^{i\lambda t}dZ_{x_3}(\lambda)$$
$$w_1(t)=\int A_t(\lambda)e^{i\lambda t}dZ_{w_1}(\lambda)$$
$$\varepsilon_3(t)=\int A_t(\lambda)e^{i\lambda t}dZ_{v_1}(\lambda)$$

Then we have

$$\int A_t(\lambda)e^{i\lambda t}dZ_{x_1}(\lambda)$$
$$=\Sigma b_{1t}(u)\int A_{t-u}(\lambda)e^{i\lambda(t-u)}dZ_{x_3}(\lambda)$$
$$+\int A_t(\lambda)e^{i\lambda t}dZ_{v_1}(\lambda).$$
$$=\int A_t(\lambda)e^{i\lambda t}[\Sigma b_{1t}(u)e^{-i\lambda u}dZ_{x_3}(\lambda)+dZ_{v_1}(\lambda)],$$

as $A_t(\lambda)\simeq A_{t-u}(\lambda)$, for any u. Then

$$dZ_{x_1}(\lambda)\simeq D_t(\lambda)dZ_{x_3}(\lambda)+dZ_{v_1}(\lambda),$$

where $D_t(\lambda)=\Sigma b_{1t}(u)e^{-i\lambda u}$. We know that $dZ_{x_3}(\lambda)$ and $dZ_{v_1}(\lambda)$ are orthogonal, thus

$$f_{t,x_1,t}(\lambda)\simeq D_t(\lambda)f_{t,x_3x_3}(\lambda)D_t(\lambda)^c+f_{t,\varepsilon_1}(\lambda),$$

i.e.,

$$f_{t,\varepsilon_1}(\lambda)\simeq f_{t,x_1}(\lambda)-f_{t,x_1x_3}(\lambda)f^{-1}_{x_{t,3}x_3}(\lambda)f_{t,x_3x_1}(\lambda).$$

Under similar conditions, we also have

$$f_{t,\varepsilon_2}(\lambda)\simeq f_{t,x_2}(\lambda)-f_{t,x_2x_3}(\lambda)f^{-1}_{x_{t,3}x_3}(\lambda)f_{t,x_3x_2}(\lambda)$$

and

$$f_{t,\varepsilon_1\varepsilon_2}(\lambda)\simeq f_{t,x_1x_2}(\lambda)-f_{t,x_1x_3}(\lambda)f^{-1}_{t,x_3x_3}(\lambda)f_{t,x_2x_3}(\lambda).$$

Based on Theorem 7.4.1, we can obtain the estimate of the evolutionary spectral densities for ε_1 and ε_2, i.e., $f^{\#}_{t,\varepsilon_1}(\lambda)$, $f^{\#}_{t,\varepsilon_2}(\lambda)$and $f^{\#}_{t,\varepsilon_1\varepsilon_2}(\lambda)$. Then following the same procedure described in Section 7.4 (see (7.4.1) and (7.4.2)), we can obtain $\tau^{\#}_{t,x_1x_2.x_3}(\lambda)$ or $\tau^{\#}_{t,12.3}$, the estimate of partial group delay.

8.0 SUMMARY AND DISCUSSION

In this chapter, we will briefly summarize this dissertation.

This dissertation proposes procedures for estimating partial group delay. The partial group delay between two stochastic processes adjusted for the third stochastic process is a spectral parameter. However, no estimating procedure has been proposed for partial group delay.

In this dissertation two procedures are proposed. The estimates of both procedures are weakly consistent. Also the asymptotic distributions of both estimates are derived.

This dissertation also extends the result of the time lag relationship between two stochastic processes (Deaton and Foutz, 1980) to the case of more than two stochastic processes. Partial group delay plays an important role in that result.

This dissertation extends the concept of partial group delay to non-stationary stochastic processes. The corresponding estimating procedure is investigated. However the sample properties of the proposed estimate need to be found.

The results of this dissertation may be extended to multivariate processes. For example, let X(t), Y(t) and Z(t) be p-dimensional stochastic time series, the partial group delay between $X(t)$ and $Y(t)$ adjusted for $Z(t)$ could be considered. For this a general definition of group delay between two multivariate stochastic processes can be established. Based on this, by using the approaches of multivariate analysis the corresponding result for the partial group delay and its estimation may be obtained.

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