OPTIMAL LARGE ANGLE SPACECRAFT ROTATIONAL MANEUVERS

by

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CHAPTER I

INTRODUCTION

1.1 STATEMENT OF PROBLEM

Central to the success of many current and planned space missions is the knowledge and control of the orientation or attitude state of the space vehicle.

Once it is possible to reliably determine and control the vehicle's attitude state, a number of important functions onboard the vehicle can be carried out; e.g., solar arrays can be kept pointing toward the sun to generate electrical power, and antennas can be directed so that commands and information can be transmitted and received.

Clearly the success in carrying out these functions critically depends on the following: (1) knowledge of the attitude state, (2) active control, and (3) an adequate dynamical model of the vehicle; in order for the control system to generate appropriate actuator commands to achieve mission objectives.

The difficulties encountered in carrying out any of the items mentioned above depends on whether the vehicle can be represented as a rigid or a distributed parameter system, and whether the equations of motion, while the vehicle is reorienting, can be linearized or not.

Furthermore, the complexity of the problem depends on whether the time history of the attitude state is constrained or is determined as part of some solution process. Clearly when the freedom exists for determining the attitude time history the scope of the problem has been broadened enough so as to allow optimal processes to be considered. It is to this end that the focus of this thesis has been directed.

The importance of this class of problems to the aerospace community can begin to be appreciated when one considers the recent upsurge in the number of publications on various aspects of this problem. A partial list of these articles would include [1]-[6] , [9]-[12] , [14] , [16] , [17] , [19] , [22] , [23] , [26] , [28]-[32] , and [35]-[43] . Many of these articles deal with rigid and flexible body degrees of freedom, but limit the maneuvers to single axis special cases or place some restrictions in order that a linearization of the system dynamics and kinematics is permitted.

The treatment of optimal large angle reorientational maneuvers, per se, has only recently received attention; e.g., [9] , [12] , [23] , [30] , [31] , [42] , and [43] . It is the objective of this thesis to more fully develop the ideas contained in the articles by Junkins and Turner ([23] , [42] , and [43]), where arbitrary rotational maneuvers are considered.

1.2 MOTIVATION AND OBJECTIVES

The motivation for this study is based on the desire to address the problems associated with both rigid and flexible vehicles during large angle reorientational maneuvers.

The principal objectives of this research on optimal large angle reorientational maneuvers are:

(1) Provide complete dynamical formulations, valid for large angle reorientations of a generic model for a space vehicle, where both rigid and flexible body dynamical models are considered;

(2) Derive the necessary conditions from Pontryagin's principle for all dynamical models considered, such that during a large angle reorientation maneuver some integral measure of the system performance is minimized; and

(3) Provide numerical methods for solving the two-point-boundary-value problems derived in (2).

1.3 SCOPE OF THE DISSERTATION

The structure of this dissertation is split into three distinct parts; based principally on a criteria that measures the difficulty of each particular problem. The three distinct parts of the dissertation are: (1) The rigid body problem for single axis and three-dimensional maneuvers; (2) The flexible body case for single axis maneuvers; and (3) The flexible body case for three-dimensional maneuvers. The three parts are interconnected, not only because all are attitude control problems, but because the numerical solution process developed in part (1) provides the insight to tackle the problem contained in part (2), and likewise the solution process in part (3) requires a hybrid combination of the methods developed in parts (1) and (2). A brief summary of each chapter now follows.

Chapter II treats the rigid body problem. Treating in detail the single axis special case. Then the three-dimensional problem is presented and an iterative boundary condition relaxation process is developed for the solution of the problem. The relaxation process belongs to a general class of methods for solving nonlinear equations. In the literature several closely related methods are known as Davidenko's method, imbedding, continuation, and homotopy chain methods (see [8] , [15] , and [45]). Common to all of these methods is the idea that a problem can be made to depend, in a continuous way, on an artificially introduced parameter. If a solution is known for one value of the parameter, then the real problem of interest can be solved by analytically continuing the known solution, as a function of the parameter; until the parameter assumes the value which restores the original problem.

Chapter III develops the single axis flexible body problem. The equations of motion are obtained from Hamilton's principle using the assumed modes method for the flexural degrees of freedom. The optimal control problem is formulated, and the resulting two-point-boundary-value problem is solved using a differential equation relaxation process. Several example maneuvers are presented, where different dynamical models are treated.

Chapter IV extends the formulation presented in Chapter III in order to treat three-dimensional maneuvers of flexible spacecraft. The optimal control problem is formulated and an iterative hybrid relaxation process is presented; which incorporates the boundary condition relaxation of Chapter II and the differential relaxation of Chapter III. A previously unknown problem is identified, concerning high geometrical symmetry in the dynamical model, which must be circumvented before the methods described herein can be applied.

Chapter V summaries the results developed and recommends areas for future study.

CHAPTER II

RIGID BODY OPTIMAL CONTROL

2.1 SINGLE AXIS PROBLEM

The problem considered here is the simplest case , reorientation of a rigid space vehicle which is restricted to a maneuver about one of its principal lines. The boundary conditions for orientation (θ) and angular rate ($\dot{\theta}$) are specified both initially and finally. The objective is then to generate the torque history $U(t)$ which takes the vehicle from $(\theta(t_o),\dot{\theta}(t_o))$ to $(\theta(t_f),\dot{\theta}(t_f))$ in the fixed time interval (t_o,t_f) while minimizing the integral of $\frac{1}{2}U^2$ during the maneuver.

The principal value of this problem lies in the fact that closed form solutions are obtained for the state and co-state variables. Once obtained, these solutions provide the non-iterative initial conditions required in the imbedding or relaxation procedure, described in Section 2.2.4, for the solution of the general three-dimensional rigid body problem.

One of the key features of this formulation is that Euler parameters (see Appendix D) are used for the orientation kinematics which are a once redundant set of variables. However, the price paid for carrying this redundant set (i.e., the difficulty in visualizing the motion and the implicit nonlinear constraint) is well rewarded when the nonsingular necessary conditions for the three-dimensional version of this same problem are obtained.

The results in this chapter are presented in two parts: First, single axis optimal rotational maneuvers are treated; and second, general three-dimensional optimal rotational maneuvers are considered. A brief summary of each of the sections of this chapter now follows.

In Section 2.1.1 the equation of motion for a single axis rotational maneuver is derived. Next, in Section 2.1.2 the state space form of the equation of motion is established. Leading to Section 2.1.3, where Pontryagin's principle is used to obtain the state and co-state differential equation necessary conditions for the optimal solution. Closed form solutions are obtained in Section 2.1.4 for the necessary conditions found in Section 2.1.3. Three numerical examples are presented in Section 2.1.5 and the results for the single axis rotational maneuvers are discussed in Section 2.1.6. As shown in Section 2.1.5 when the time interval for the maneuver is fixed, that somewhat surprising optimal maneuvers can result. In many optimal control problems, fixing the final is somewhat artificial and thus free final time maneuvers are of interest. The modifications of the methods to determine free final time maneuvers are rather minor and their treatment is given in Section 2.1.6 for single axis maneuvers.

For three dimensional rigid body optimal rotational maneuvers, the equations of motion are obtained in Section 2.2.1. In Section 2.2.2 the state space form for the differential equations is defined. The optimal control problem is treated in Section 2.2.3

and Pontryagin's principle is used to obtain the necessary conditions for the solution. A boundary condition relaxation process is presented in Section 2.2.4 for iteratively solving the necessary conditions obtained in Section 2.2.3. Numerical results are presented in Section 2.2.5 and the results for the three-dimensional rigid body problem are discussed in Section 2.2.6.

2.1.1 EQUATIONS OF MOTION

At the most fundamental level the dynamics of a rigid body is governed by the relation that the external torque is equal to the inertial time rate of change of the angular momentum. Though the most general form of Euler's equations are not required for the applications of this section. In anticipation of later needs, the general results are derived for later reference and reduced to special forms for the results of this section.

The angular momentum $\underline{H}$ is defined:

$$\underline{H} = \int_B \underline{r} \times \frac{d\underline{r}}{dt}\bigg|_N \, dm \qquad (2.1.1)$$

where $\frac{d}{dt}(\ )\Big|_N$ is the time derivative relative to the inertial frame N; dm is the generic mass element; and $\underline{r}$ locates the generic mass element dm in the body relative to the vehicle's center of mass.

On taking the inertial time derivative of (2.1.1), one obtains

$$\left.\frac{d\underline{H}}{dt}\right|_N = \underline{L}_{ex} \tag{2.1.2}$$

where L_{ex} is the external torque; and

$$\left.\frac{d\underline{H}}{dt}\right|_N = \left.\frac{d\underline{H}}{dt}\right|_B + \underline{\omega} \times \underline{H} \tag{2.1.3}$$

where $\left.\frac{d}{dt}(\ )\right|_B$ is the time derivative relative to the body frame,

and $\underline{\omega} = \omega_1\hat{\underline{b}}_1 + \omega_2\hat{\underline{b}}_2 + \omega_3\hat{\underline{b}}_3$ is the angular velocity of the body frame $(\hat{b}_1\ \ \hat{b}_2\ \ \hat{b}_3)^T$ relative to the inertial frame $(\hat{n}_1\ \ \hat{n}_2\ \ \hat{n}_3)^T$.

Substituting (2.1.3) into (2.1.2), then (2.1.2) becomes

$$\underline{L}_{ex} = \left.\frac{d\underline{H}}{dt}\right|_B + \underline{\omega} \times \underline{H} \tag{2.1.4}$$

Next, introducing (2.1.3) into (2.1.1); thereby, replacing the term $\left.\frac{d\underline{r}}{dt}\right|_N$ in terms of its body fixed component form, (2.1.1) becomes

$$\underline{H} = \int_B \underline{r} \times \left(\left.\frac{d\underline{r}}{dt}\right|_B + \underline{\omega} \times \underline{r}\right)\, dm \tag{2.1.5}$$

On observing that

$$\left.\frac{d\underline{r}}{dt}\right|_B \equiv 0 \tag{2.1.6}$$

then (2.1.5) reduces to

$$\underline{H} = \underline{\underline{J}}\ \underline{\omega} \tag{2.1.7}$$

where $$\underline{\underline{J}} = \int_B \left[(\underline{r}\cdot\underline{r})\ \underline{\underline{I}} - \underline{r}\ \underline{r}^T\right] dm \tag{2.1.8}$$

and $\underline{\underline{J}}$ denotes the moment of inertia tensor.

Introducing (2.1.7) into (2.1.3) and noting that the moment of inertia tensor $\underline{\underline{J}}$ is constant in the body frame, leads to

$$\underline{L}_{ex} = \underline{\underline{J}}\, \dot{\underline{\omega}} + \underline{\omega} \times (\underline{\underline{J}}\, \underline{\omega})^{*} \tag{2.1.9}$$

Alternatively if the vector cross product is written in matrix form, (2.1.9) becomes

$$\underline{L}_{ex} = \underline{\underline{J}}\, \dot{\underline{\omega}} + \tilde{\underline{\underline{\omega}}}\, \underline{\underline{J}}\, \underline{\omega} \tag{2.1.10}$$

where

$$\tilde{\underline{\underline{\omega}}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{2.1.11}$$

Equation (2.1.10) represents Euler's rotational equations of motion in their most general form for a rigid body. To reduce the number of terms in (2.1.10) to the greatest extent possible, one requires that the body fixed axis are chosen to lie along the principal axis of the body; thus reducing the moment of inertia tensor $\underline{\underline{J}}$ to the diagonal form

$$\underline{\underline{J}} = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} \tag{2.1.12}$$

Solving (2.1.10) for the angular acceleration $\dot{\underline{\omega}}$, in preparing to write the final form of the equation of motion, leads to

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* Use has been made of the following equation for the angular velocity,

$$\left.\frac{d\underline{\omega}}{dt}\right|_B = \left.\frac{d\underline{\omega}}{dt}\right|_N + (-\underline{\omega}) \times \underline{\omega} = \left.\frac{d\underline{\omega}}{dt}\right|_N$$

$$\dot{\underline{\omega}} = - \underline{\underline{J}}^{-1} \tilde{\underline{\underline{\omega}}} \, \underline{\underline{J}} \, \underline{\omega} + \underline{\underline{J}}^{-1} \underline{L}_{ex} \tag{2.1.13}$$

Finally introducing (2.1.12) into (2.1.13), one obtains

$$\begin{Bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{Bmatrix} = - \begin{bmatrix} J_{11}^{-1} & 0 & 0 \\ 0 & J_{22}^{-1} & 0 \\ 0 & 0 & J_{33}^{-1} \end{bmatrix} \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} + \begin{bmatrix} J_{11}^{-1} & 0 & 0 \\ 0 & J_{22}^{-1} & 0 \\ 0 & 0 & J_{33}^{-1} \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} \tag{2.1.14}$$

which can be reduced upon carrying out the implied algebra to

$$\left.\begin{aligned} \dot{\omega}_1 &= - \left(\frac{J_{33} - J_{22}}{J_{11}} \right) \omega_2 \, \omega_3 + \frac{1}{J_{11}} L_1 \\ \dot{\omega}_2 &= - \left(\frac{J_{11} - J_{33}}{J_{22}} \right) \omega_1 \, \omega_3 + \frac{1}{J_{22}} L_2 \\ \dot{\omega}_3 &= - \left(\frac{J_{22} - J_{11}}{J_{33}} \right) \omega_1 \, \omega_2 + \frac{1}{J_{33}} L_3 \end{aligned}\right\} \tag{2.1.15}$$

Equations (2.1.15) represent the principal axis form of Euler's rotational equations of motion, and the desired structural form for the treatment of the rigid body optimal control problem.

It can be easily seen that for any single axis attitude maneuver about the i-th axis, that (2.1.15) reduces to

$$\dot{\omega}_i = J_i^{-1} L_i \quad ; \quad J_i = J_{ii} \tag{2.1.16}$$

and

$$\left.\begin{aligned} \omega_j &= 0 \\ L_j &= 0 \end{aligned}\right\} \text{for } j \neq i \qquad (2.1.17)$$

Equation (2.1.16) governs the dynamics of the single axis problem. To complete the proper statement of the problem the kinematic Euler parameter differential equations (see Appendix D) are now reduced to their special form for an i-th axis rotational maneuver.

The general Euler parameter differential equations are (see Appendix D)

$$\left.\begin{aligned} \dot{\beta}_0 &= \frac{1}{2}\,(-\omega_1\beta_1 - \omega_2\beta_2 - \omega_3\beta_3) \\ \dot{\beta}_1 &= \frac{1}{2}\,(\omega_1\beta_0 - \omega_2\beta_3 + \omega_3\beta_2) \\ \dot{\beta}_2 &= \frac{1}{2}\,(\omega_1\beta_3 + \omega_2\beta_0 - \omega_3\beta_1) \\ \dot{\beta}_3 &= \frac{1}{2}\,(-\omega_1\beta_2 + \omega_2\beta_1 + \omega_3\beta_0) \end{aligned}\right\} \qquad (2.1.18)$$

which reduce for a rotation about the i-th axis, to

$$\left.\begin{aligned} \dot{\beta}_0 &= -\frac{1}{2}\,\omega_i\beta_i \\ \dot{\beta}_i &= \frac{1}{2}\,\omega_i\beta_0 \\ \beta_j &= \dot{\beta}_j = 0 \quad ; \quad \text{for } i \neq j \end{aligned}\right\} \qquad (2.1.19)$$

In summary, the equation of motion for generally asymmetric rigid body about its i-th principal line is given by (2.1.17) and the corresponding Euler parameter kinematical equations are governed by (2.1.19).

2.1.2 STATE SPACE FORMULATION

To consider the rotational dynamics of a rigid vehicle about its i-th principal line the natural choice for the state variables are the two Euler parameters (β_0, β_i) and the angular velocity (ω_i); the state differential equations are thus (2.1.19) together with Euler's rotational equation of motion (2.1.16), as listed below:

$$\dot{\omega}_i = J_i^{-1} U_i \tag{2.1.20}$$

where $U_i = L_i$ and U_i is the unknown control torque to be determined, and

$$\left.\begin{aligned} \dot{\beta}_0 &= -\frac{1}{2}\,\omega_i \beta_i \\ \dot{\beta}_i &= \frac{1}{2}\,\omega_i \beta_0 \end{aligned}\right\} \tag{2.1.21}$$

2.1.3 OPTIMAL CONTROL PROBLEM

In this section, as well as in all other optimal control problem discussions of this thesis, the term "optimal control" is to be taken to mean that some integral measure of the system performance has been extremized.* For the case of rigid vehicles, in this thesis, this integral measure is to be taken as:

$$J = \frac{1}{2}\int_{t_o}^{t_f} U_i^2\, dt \tag{2.1.22}$$

which is a minimum torque criteria**, where the limits t_o and t_f are taken as being fixed.

The problem is to determine a solution to (2.1.20) and

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* When an integral is extremized the conditions for the first variation of the integral to vanish are determined [25,Ch.4].

** J is referred to as the performance index or performance measure, where (2.1.22) is the i-th axis specialization of (2.2.7).

(2.1.21), satisfying the prescribed initial and final orientation and angular velocity given by:

$$\left.\begin{aligned} \beta_r(t_o) &= \beta_{ro} \\ \beta_r(t_f) &= \beta_{rf} \\ \omega_i(t_o) &= \omega_{io} \\ \omega_i(t_f) &= \omega_{if} \end{aligned}\right\} \text{for } r=0,i \qquad (2.1.23)$$

where the Euler parameter boundary conditions must be consistent with the constraint $\beta_o^2 + \beta_i^2 = 1$ (see (D.2)), so that only two degrees of freedom exist.

Hence, the objective for the optimal control problem is to find the torque history $U_i(t)$ generating an optimal solution of (2.1.20) and (2.1.21), satisfying the boundary conditions of (2.1.23), which extremizes the performance index (2.1.22).*

The Hamiltonian associated with minimizing (2.1.21) along the trajectories of (2.1.19) and (2.1.20) is given by

$$H = \frac{1}{2} U_i^2 + \lambda_i U_i J_i^{-1} + \gamma_o(-\frac{1}{2}\omega_i\beta_i) + \gamma_i(\frac{1}{2}\omega_i\beta_o) \qquad (2.1.24)$$

where $\overset{\lambda_i}{\cancel{\gamma_i}}$ is the co-state variable associated with the angular velocity ω_i , and γ_o , γ_i are co-state variables associated with the Euler parameters β_o , β_i .

Pontryagin's principle requires as necessary conditions that λ_i , γ_o , and γ_i satisfy co-state differential equations derivable from

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* The torque history U(t) is restricted to be piecewise continuous.

$$\left.\begin{aligned}\dot{\gamma}_o &= -\frac{\partial H}{\partial \beta_o} = -\frac{1}{2}\gamma_i\omega_i \\ \dot{\gamma}_i &= -\frac{\partial H}{\partial \beta_i} = \frac{1}{2}\gamma_o\omega_i \\ \dot{\lambda}_i &= -\frac{\partial H}{\partial \omega_i} = \frac{1}{2}(\gamma_o\beta_i - \gamma_i\beta_o)\end{aligned}\right\} \tag{2.1.25}$$

In addition to (2.1.25) Pontryagin's principle requires that the gradient of the Hamiltonian with respect to the control torque U_i, vanish everywhere along the optimal solution, which leads to

$$\frac{\partial H}{\partial U_i} = 0 = U_i + \lambda_i J_i^{-1} \tag{2.1.26}$$

so that the control torque U_i is determined as a function of the angular velocity co-state variable λ_i as

$$U_i = -\lambda_i J_i^{-1} \tag{2.1.27}$$

The state and co-state differential equations forming the boundary value problem are (2.1.19) and (2.1.20) after using (2.1.27) together with (2.1.25), which follow as

(a) state equations:

$$\dot{\beta}_o = -\frac{1}{2}\omega_i\beta_i \tag{2.1.28a}$$

$$\dot{\beta}_i = \frac{1}{2}\omega_i\beta_o \tag{2.1.28b}$$

$$\dot{\omega}_i = -\lambda_i J_i^{-2} \tag{2.1.28c}$$

(b) co-state equations:

$$\dot{\gamma}_o = -\frac{1}{2}\omega_i\gamma_i \tag{2.1.29a}$$

$$\dot{\gamma}_i = \frac{1}{2}\omega_i\gamma_o \tag{2.1.29b}$$

$$\dot{\lambda}_i = \frac{1}{2}\,(\gamma_o\beta_i - \gamma_i\beta_o) \qquad (2.1.29c)$$

Inspection of (2.1.28a), (2.1.28b), (2.1.29a), and (2.1.29b) reveals that these differential equations have the same form. Since it is well known that the Euler parameters satisfy a constraint (see Appendix D) of the form $\beta_o^2 + \beta_i^2 = 1$. It is clear that Euler parameter co-state variables satisfy a constraint of the form $\gamma_o^2 + \gamma_i^2 = B^2$; where B^2 is some unknown constant.

2.1.4 METHOD OF SOLUTION FOR THE TWO-POINT-BOUNDARY-VALUE PROBLEM

In this section a detailed account is given for the solution of the state and co-state variables with particular emphasis being focused on the angular velocity co-state variable λ_i . Since it has been determined that an ambiguity exists surrounding the norm of the Euler parameter co-state variables γ_o and γ_i which creeps into the solution for λ_i . The resolution of this ambiguity motivated the development of the boundary condition relaxation process of Section 2.2.4 .

To place things in their proper prospective, it should be pointed out that although the complete solution is obtained for this problem, the real interest in this section lies in the fact that the initial co-states are determined. Since it is these initial co-states that are required as the initial guesses for the differential correction algorithm mentioned above.

To start with the solutions for (2.1.28a), (2.1.28b), (2.1.29a), and (2.1.29b) can be easily obtained by using the complex variables Γ and Λ , defined by

$$\Gamma(t) = \beta_o(t) + i\ \beta_i(t) \tag{2.1.30a}$$

$$\Lambda(t) = \gamma_o(t) + i\ \gamma_i(t) \tag{2.1.30b}$$

where

$$i^2 = -1 \tag{2.1.31}$$

Using (2.1.30a) and (2.1.30b) the differential equations of (2.1.28a), (2.1.28b), (2.1.29a), and (2.1.29b) become

$$\dot{\Gamma}(t) = \frac{1}{2}\ i\ \omega_i(t)\ \Gamma(t) \tag{2.1.32}$$

and

$$\dot{\Lambda}(t) = \frac{1}{2}\ i\ \omega_i(t)\ \Lambda(t) \tag{2.1.33}$$

Integrating the equations above the solutions follow as

$$\Gamma(t) = A\ \exp.\left[\frac{1}{2}\ i\ (\int_{t_o}^{t} \omega_i(\tau)\ d\tau + \phi_{io})\right] \tag{2.1.34}$$

and

$$\Lambda(t) = B\ \exp.\left[\frac{1}{2}\ i\ (\int_{t_o}^{t} \omega_i(\tau)\ d\tau + \theta_{io})\right] \tag{2.1.35}$$

Taking the real and imaginary parts of (2.1.34) and (2.1.35) yields the following solutions for $\beta_o(t)$, $\beta_i(t)$, $\gamma_o(t)$, and $\gamma_i(t)$:

$$\beta_o(t) = A\ \cos\ \left[\frac{1}{2} \int_{t_o}^{t} \omega_i(\tau)\ d\tau + \frac{1}{2}\ \phi_{io}\right] \tag{2.1.36a}$$

$$\beta_i(t) = A\ \sin\ \left[\frac{1}{2} \int_{t_o}^{t} \omega_i(\tau)\ d\tau + \frac{1}{2}\ \phi_{io}\right] \tag{2.1.36b}$$

$$\gamma_o(t) = B\ \cos\ \left[\frac{1}{2} \int_{t_o}^{t} \omega_i(\tau)\ d\tau + \frac{1}{2}\ \theta_{io}\right] \tag{2.1.37a}$$

$$\gamma_i(t) = B \sin \left[\frac{1}{2} \int_{t_o}^{t} \omega_i(\tau)\ d\tau + \frac{1}{2}\theta_{io}\right] \tag{2.1.37b}$$

To complete the solutions in (2.1.36) and (2.1.37) the four constants A , B , ϕ_{io} , and θ_{io} must be specified. In (2.1.36) the phase angle ϕ_{io} is just the initial orientation angle. Thus, eliminating one of the unknowns, recalling that the Euler parameter of (2.1.36) must satisfy the constraint $\beta_o^2(t) + \beta_i^2(t) = 1$ leads immediately to the conclusion that $A \equiv 1$; leaving only B and θ_{io} to be determined. Unfortunately, a bit more effort is required in order to determine B and θ_{io} . The first step in determining B comes by substituting (2.1.36a), (2.1.36b), (2.1.37a), and (2.1.37b) into (2.1.29c), yielding

$$\dot{\lambda}_i(t) = \frac{1}{2}\left[B \cos \{\frac{1}{2}\int_{t_o}^{t} \omega_i(\tau)\ d\tau + \frac{1}{2}\theta_{io}\} \sin \{\frac{1}{2}\int_{t_o}^{t} \omega_i(\tau)\ d\tau + \frac{1}{2}\phi_{io}\}\right.$$
$$\left. - B \sin \{\frac{1}{2}\int_{t_o}^{t} \omega_i(\tau)\ d\tau + \frac{1}{2}\theta_{io}\} \cos \{\frac{1}{2}\int_{t_o}^{t} \omega_i(\tau)\ d\tau + \frac{1}{2}\phi_{io}\}\right] \tag{2.1.38}$$

which reduces to

$$\dot{\lambda}_i(t) = \frac{1}{2} B \sin (\frac{1}{2}\phi_{io} - \frac{1}{2}\theta_{io}) = \text{constant} \tag{2.1.39}$$

From which one concludes that an infinity of choices exist for θ_{io} and B , yielding the same constant for $\dot{\lambda}_i$; thus, arbitrarily choosing θ_{io} as

$$\theta_{io} = \phi_{io} - \pi \tag{2.1.40}$$

so that $\sin (\frac{1}{2}\phi_{io} - \frac{1}{2}\theta_{io}) = \sin (\frac{\pi}{2}) = 1$, such that the constant

B is minimized*, and (2.1.39) becomes

$$\dot{\lambda}_i(t) = \frac{1}{2} B = \text{constant}^{**} \tag{2.1.41}$$

Equation (2.1.41) integrates immediately to

$$\lambda_i(t) = \frac{1}{2} B (t - t_o) + \lambda_{io} \tag{2.1.42}$$

where $\lambda_{io} = \lambda_i(t_o)$ is the constant of integration.

Substitution of (2.1.42) into (2.1.28c) and integrating yields the angular velocity solution as:

$$\omega_i(t) = \dot{\phi}_{io} - \frac{\lambda_{io}}{J_i^2} (t - t_o) - \frac{B}{4 J_i^2} (t - t_o)^2 \tag{2.1.43}$$

where $\dot{\phi}_{io} = \omega_i(t_o)$ is the constant of integration.

Substituting (2.1.43) into the following equation

$$\phi_i(t) = \phi_{io} + \int_{t_o}^{t} \omega_i(\tau)\, d\tau \tag{2.1.44}$$

and integrating, yields the solution for the angle of rotation as the cubic polynomial

$$\phi_i(t) = \phi_{io} + \dot{\phi}_{io} (t - t_o) + \frac{1}{2} \ddot{\phi}_{io} (t - t_o)^2 + \frac{1}{6} \dddot{\phi}_{io} (t - t_o)^3 \tag{2.1.45}$$

where the integration constants B and λ_{io} have the interpretations

$$B = - 2 J_i^2 \dddot{\phi}_{io} \tag{2.1.46a}$$

---

* It is this criterion that the Euler parameter co-states norm be minimized that leads to the particular differential correction algorithm of Section 2.2.4.

** Referring to (2.1.29), one observes that B = 0 corresponds to either zero torque (a coast) or a constant torque maneuver. This statement generalizes for the three-dimensional case.

$$\lambda_{io} = - J_i^2 \ddot{\phi}_{io} \tag{2.1.46b}$$

Since (2.1.36) and (2.1.43) must satisfy the terminal boundary conditions (2.1.23), equation (2.1.45) must therefore satisfy the conditions $\phi_i(t_f) = \phi_{if}$ and $\dot{\phi}_i(t_f) = \dot{\phi}_{if}$; imposing these conditions on (2.1.45), leads to

$$\ddot{\phi}_{io} = 6 \frac{(\phi_{if} - \phi_{io})}{(t_f - t_o)^2} - 2 \frac{(2 \dot{\phi}_{io} + \dot{\phi}_{if})}{(t_f - t_o)} \tag{2.1.47a}$$

$$\dddot{\phi}_{io} = - 12 \frac{(\phi_{if} - \phi_{io})}{(t_f - t_o)^3} + 6 \frac{(\dot{\phi}_{io} + \dot{\phi}_{if})}{(t_f - t_o)^2} \tag{2.1.47b}$$

Substitution of (2.1.47) into (2.1.46) determines the initial co-state constants of integration in terms of the non-trivial initial and final boundary conditions. The Euler parameter co-states initial conditions are obtained from (2.1.37) by evaluating at $t = t_o$, and making use of (2.1.40), leading to

$$\gamma_o(t_o) = B \cos (\frac{1}{2} \phi_{io} - \frac{1}{2} \pi) = B \sin \frac{1}{2} \phi_{io} = B \beta_i(t_o) \tag{2.1.48a}$$

$$\gamma_i(t_o) = B \sin (\frac{1}{2} \phi_{io} - \frac{1}{2} \pi) = - B \cos \frac{1}{2} \phi_{io} = - B \beta_o(t_o) \tag{2.1.48b}$$

With (2.1.48) all the initial conditions for the problem have been determined; hence, to obtain the optimal solution, the four variables ϕ_{io} , $\dot{\phi}_{io}$, ϕ_{if} , and $\dot{\phi}_{if}$ must be assigned whatever their arbitrary values might be, and then the solution follows as

$$\left.\begin{aligned}
&\beta_o(t) = \cos\frac{1}{2}\phi \quad ; \ \beta_i(t) = \sin\frac{1}{2}\phi \quad ; \ \beta_j(t) = 0 \text{ for } i\neq j\\
&\gamma_o(t) = -\,2\,J_i^2\,\dddot{\phi}_{io}\,\sin\frac{1}{2}\phi_i \ ; \ \gamma_i(t) = 2\,J_i^2\,\dddot{\phi}_{io}\,\cos\frac{1}{2}\phi_i \ ;\\
&\qquad\quad \gamma_j(t) = 0 \text{ for } i\neq j\\
&\phi_i(t) = \phi_{io} + \dot{\phi}_{io}\,(t - t_o) + \frac{1}{2}\ddot{\phi}_{io}\,(t - t_o)^2\\
&\qquad\qquad\qquad + \frac{1}{6}\dddot{\phi}_{io}\,(t - t_o)^3\\
&\omega_i(t) = \dot{\phi}_i(t) \ ; \ \omega_j(t) = 0 \text{ for } i\neq j\\
&\ddot{\phi}_{io} = 6\,\frac{(\phi_{if} - \phi_{io})}{(t_f - t_o)^2} - 2\,\frac{(2\,\dot{\phi}_{io} + \dot{\phi}_{if})}{(t_f - t_o)}\\
&\dddot{\phi}_{io} = -\,12\,\frac{(\phi_{if} - \phi_{io})}{(t_f - t_o)^3} + 6\,\frac{(\dot{\phi}_{io} + \dot{\phi}_{if})}{(t_f - t_o)^2}\\
&\lambda_i(t) = -\,J_i^2\,\ddot{\phi}_{io} - J_i^2\,\dddot{\phi}_{io}\,(t - t_o) \ ; \ \lambda_j(t) = 0 \text{ for } i\neq j\\
&U_i(t) = -\,\frac{\lambda_i(t)}{J_i} = J_i\left[\ddot{\phi}_{io} + \dddot{\phi}_{io}\,(t - t_o)\right] = J_i\,\dot{\omega}_i \ ;\\
&\qquad\quad U_j(t) = 0 \text{ for } i\neq j
\end{aligned}\right\} \quad (2.1.49)$$

In summary the initial co-states are collected in Table 2.1.

2.1.5 NUMERICAL RESULTS

The three numerical examples previously mentioned are summarized below,

- Case-I is a "rest-to-rest maneuver corresponding to a reorientation about the $\hat{\underline{b}}_1$ principal axis, with zero initial and final angular velocity. Table 2.2 summarizes the initial and final boundary

Table 2.1: Initial Co-States for Three Principal Axis Solutions.

For Set-i
$\gamma_o(t_o) = -2\, J_i^2\, \dddot{\phi}_{io}\, \beta_i(t_o)$
$\gamma_i(t_o) = -2\, J_i^2\, \dddot{\phi}_{io}\, \beta_o(t_o)$
$\gamma_j(t_o) = 0 \quad \text{for } j \neq i$
$\lambda_i(t_o) = -\, J_i^2\, \ddot{\phi}_{io}$
$\lambda_j(t_o) = 0 \quad \text{for } j \neq i$

ϕ_{io}, $\dot{\phi}_{io}$, ϕ_{if}, $\dot{\phi}_{if}$ are given.

$\ddot{\phi}_{io}$, $\dddot{\phi}_{io}$ are found from (2.1.49).

Table 2.2: Assumed Data for Case-I, Case-II, and Case-III.

		Case-I Maneuver		Case-II Maneuver		Case-III Maneuver	
		$I_1 = 1.0$, $I_2 = 0.8$, $I_3 = 0.5$ $(kg.m^2)$		$I_1 = 1.00$, $I_2 = 0.83$, $I_3 = 0.92$ $(10^6\ kg.m^2)$		$I_1 = 1.00$, $I_2 = 0.83$, $I_3 = 0.92$ $(10^6\ kg.m^2)$	
		Initial State $t_o=0$	Final State $t_f=60$ sec	Initial State $t_o=0$	Final State $t_f=60$ sec	Initial State $t_o=0$	Final State $t_f=100$ sec
θ_1	rad	0	$\pi/2$	0	0	0	0
θ_2	rad	0	0	0	$\pi/2$	0	$\pi/2$
θ_3	rad	0	0	0	0	0	0
β_o		1	0.707	1	0.707	1	0.707
β_1		0	0.707	0	0	0	0
β_2		0	0	0	0.707	0	0.707
β_3		0	0	0	0	0	0
ω_1	rad/sec	0	0	0	0	0	0
ω_2	rad/sec	0	0	0	0.047	0	0.5
ω_3	rad/sec	0	0	0	0	0	0

conditions. Recognizing that Case-I corresponds to the Set-1 family, the initial co-states can be determined to be

$$\left.\begin{aligned} \underline{\gamma}^T(t_o) &= \{0\ ,\ -37.699\ ,\ 0\ ,\ 0\} \\ \underline{\lambda}^T(t_o) &= \{-9.425\ ,\ 0\ ,\ 0\} \end{aligned}\right\} \qquad (2.1.50)$$

The control and state variables are sketched in Figures 2.1, (a)-(d).

- Case-II is a "spin-up" maneuver corresponding to a reorientation about the $\hat{\underline{b}}_3$ principal axis, with zero initial angular velocity and a final angular velocity of 4.712 rad/sec (this case represents a transition case to be discussed in Section 2.1.6). Table 2.2 summarizes the remaining boundary conditions. Recognizing that Case-II corresponds to the Set-2 family, the initial co-states can be determined to be

$$\left.\begin{aligned} \underline{\gamma}^T(t_o) &= \{0\ ,\ 0\ ,\ -1.309\text{x}10^7\ ,\ 0\} \\ \underline{\lambda}^T(t_o) &= \{0\ ,\ 0\ ,\ 0\} \end{aligned}\right\} \qquad (2.1.51)$$

The control and state variables are sketched in Figures 2.2, (a)-(d).

- Case-III is a "spin-up" maneuver corresponding to reorientation about the $\hat{\underline{b}}_2$ principal axis, with zero initial angular velocity and a final angular velocity of 0.5 rad/sec. Table 2.2 summarizes the remaining boundary conditions. Recognizing that Case-III corresponds to the Set-2 family, the initial co-states can be determined to be

$$\left.\begin{aligned} \gamma^T(t_o) &= \{0\ ,\ 0\ ,\ -3.905\text{x}10^8\ ,\ 0\} \\ \lambda^T(t_o) &= \{0\ ,\ 6.290\text{x}10^8\ ,\ 0\} \end{aligned}\right\} \qquad (2.1.52)$$

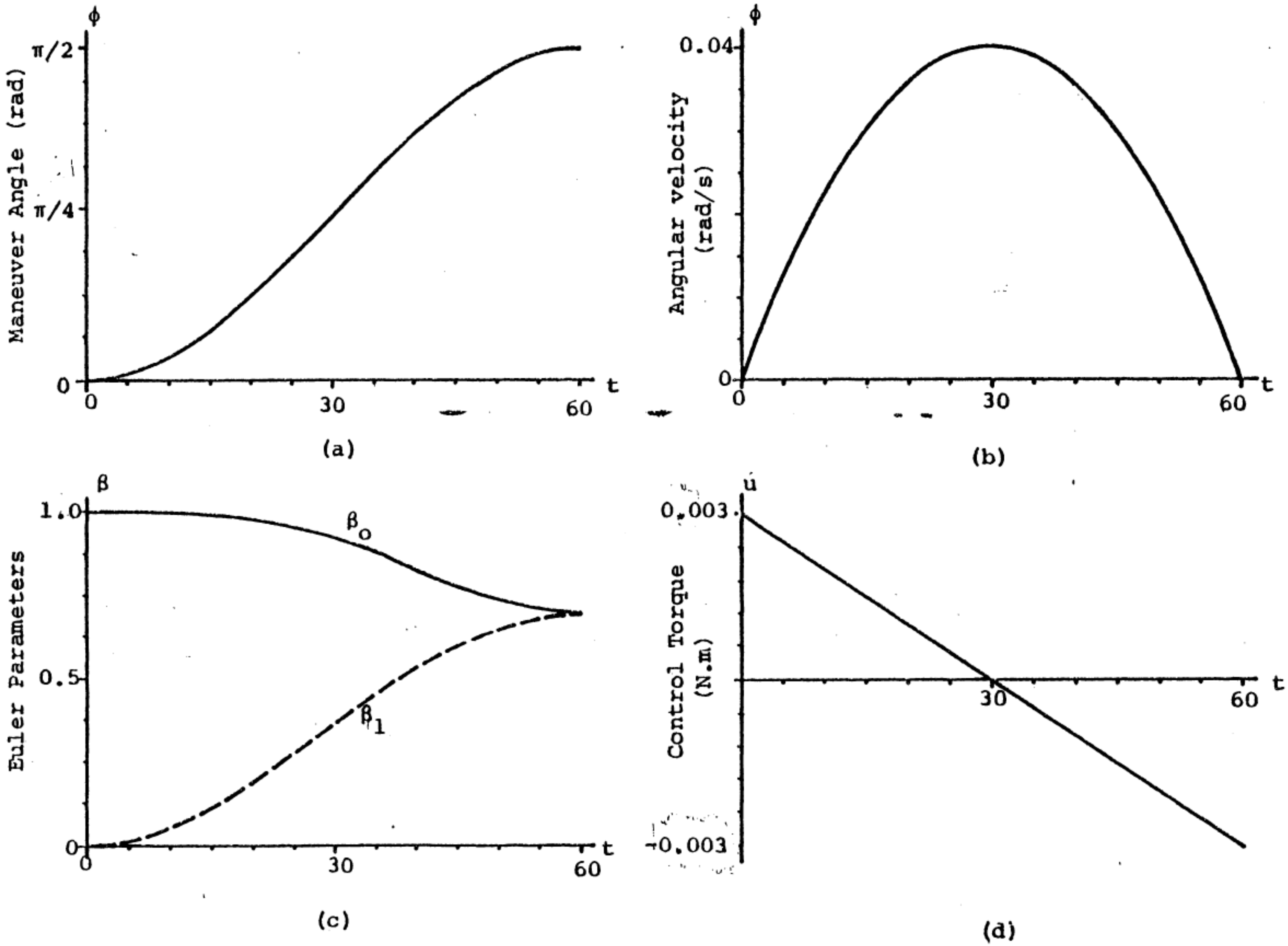


Figure 2.1: Case-I - Rigid Body "Rest-to-Rest" Maneuver

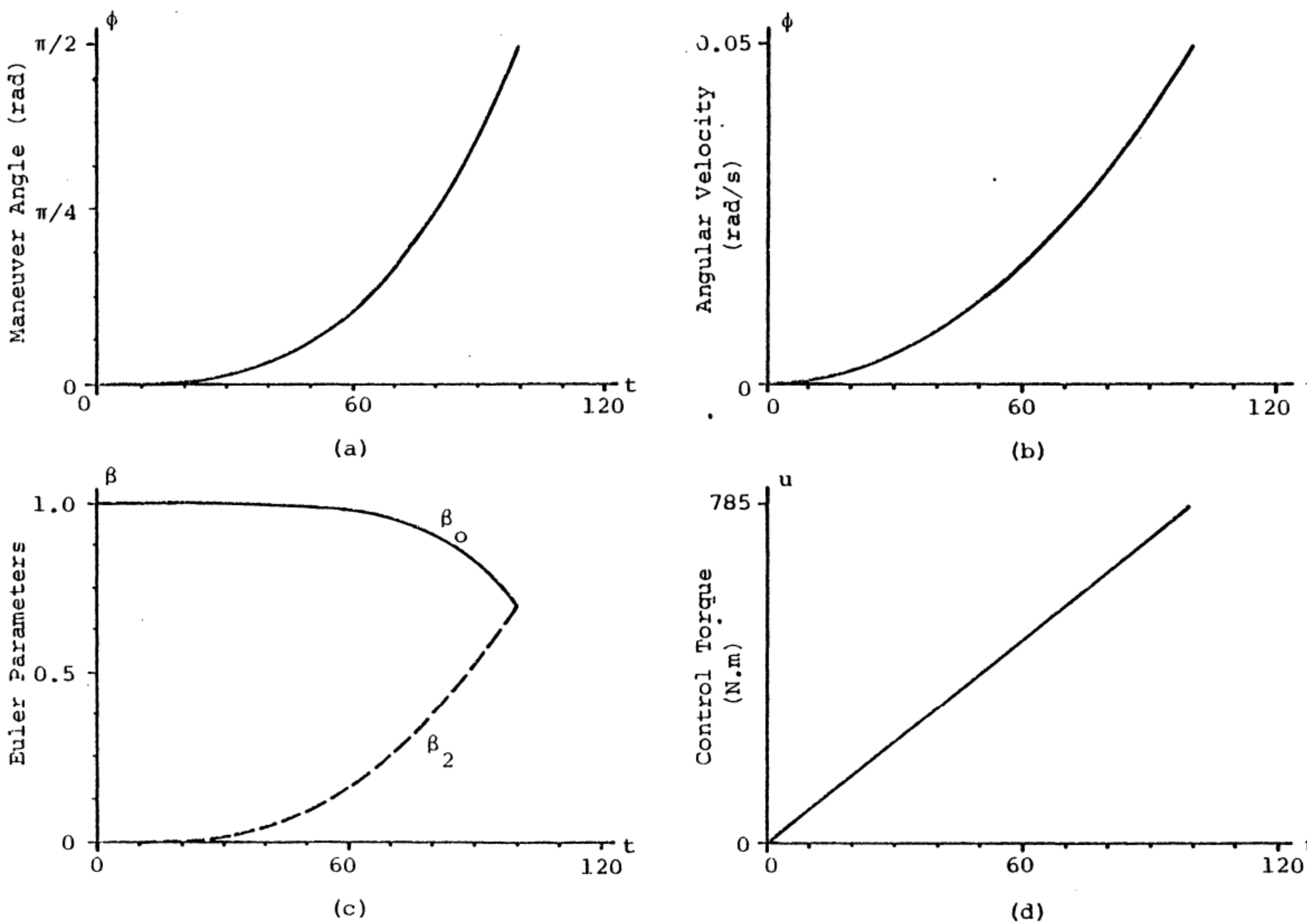


Figure 2.2: Case-II - Rigid Body "Spin-Up" Maneuver.

The control and state variables are sketched in Figures 2.3, (a)-(d).

2.1.6 DISCUSSION

In the numerical examples of the previous section two general types of maneuvers are considered. The Case-I maneuver is the near trivial case which displays for this linear problem the anti-symmetric symmetry of the control torque $U_1(t)$, the angular velocity $\dot{\phi}_1(t)$, and angle $\phi_1(t)$ about the mid point time for the maneuver, t = 30 seconds. In this simple example where the results for $U_1(t)$, $\dot{\phi}_1(t)$, and $\phi_1(t)$ are easily motivated on physical grounds one finds the results difficult to relate to for the Euler parameters. As mentioned earlier it is this difficulty in visualizing the physical coordinate $\phi_1(t)$'s time history, in terms of the Euler parameters that represents one minor limitation which must be endured through their use. Of course, computationally, the Euler parameters are smoothly varying functions, free of geometric singularities, and thus ideally suited for the three-dimensional problem.

In the Case-II and Case-III maneuvers attention is drawn to a somewhat counter intuitive result for which Case-II represents the transition case. In Case-II the vehicle always rotates in the direction of the desired terminal boundary conditions, and in Case-III the vehicle must rotate first in the opposite direction and then reversing its angular velocity to finally arrive at the desired terminal boundary conditions.

This vehicle counter rotation phenomena arises from the

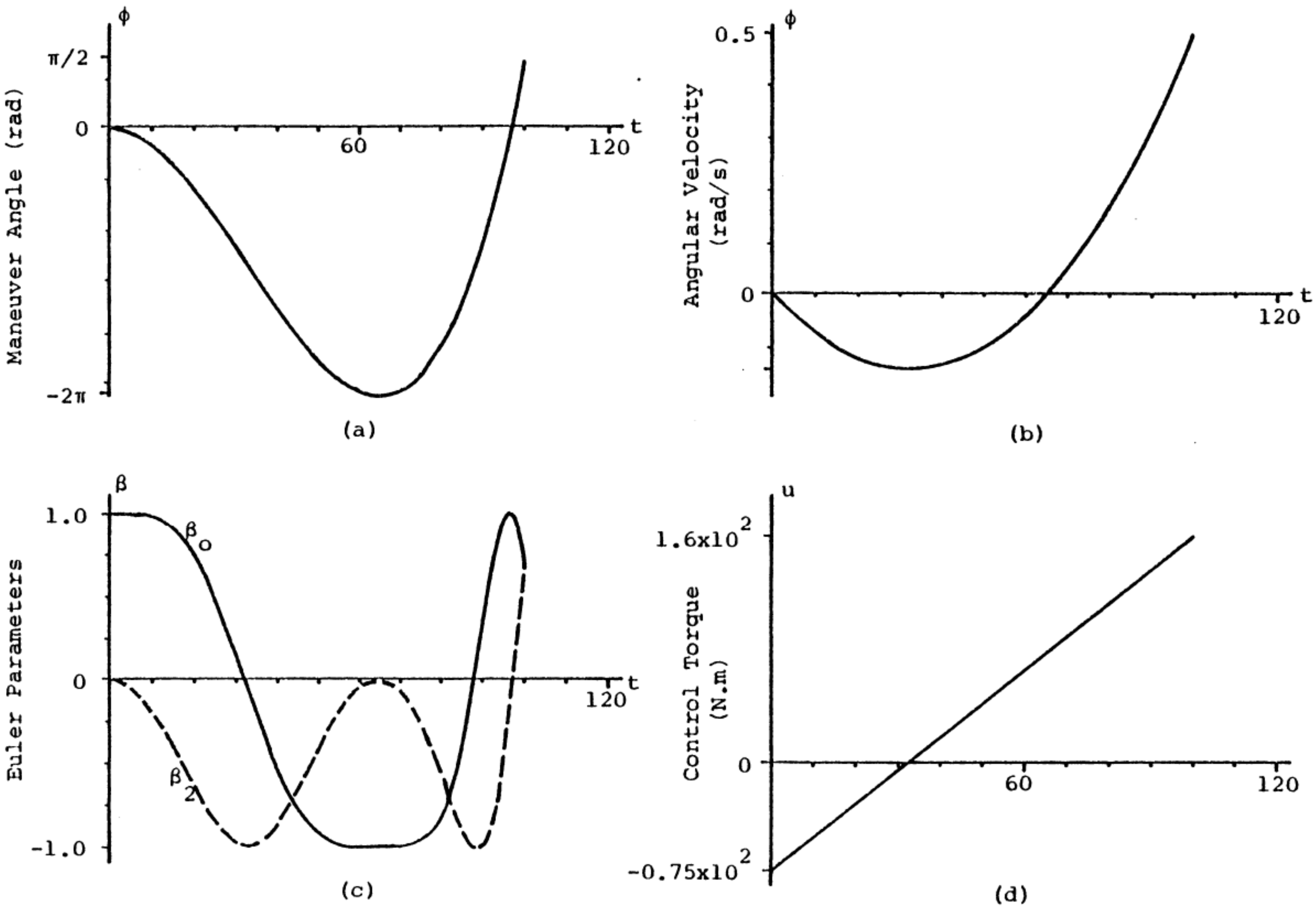


Figure 2.3: Case-III - Rigid Body "Spin-Up" Maneuver

somewhat artificial specification of time interval for the maneuver. As shown below, when the final time t_f is allowed to go free this reversal phenomenon disappears. Functionally the condition defining the transition case of Case-II has the form

$$f(\phi_{io}, \phi_{if}, \dot{\phi}_{io}, \dot{\phi}_{if}, (t_f - t_o)) = 0 \qquad (2.1.53)$$

To obtain the explicit form for f, the quadratic equation for $\dot{\phi}_i(t)$, given by

$$\dot{\phi}_i(t) = \dot{\phi}_{io} + \ddot{\phi}_{io}\,(t - t_o) + \frac{1}{2}\dddot{\phi}_{io}\,(t - t_o)^2 \qquad (2.1.54)$$

is solved for the maneuver time $(t - t_o)$ such that $\dot{\phi}_i(t) \equiv 0$ *.

On caryying out the necessary steps, one obtains

$$(t - t_o) = \frac{-\ddot{\phi}_{io} \pm \sqrt{(\ddot{\phi}_{io})^2 - 2\,\dot{\phi}_{io}\,\dddot{\phi}_{io}}}{\dddot{\phi}_{io}} \qquad (2.1.55)$$

Since $\ddot{\phi}_{io}$ and $\dddot{\phi}_{io}$ are function of the final time, the additional requirement that a single maneuver time $(t_f - t_o)$ exists is imposed on (2.1.55), yielding the explicit form for f:

$$f(\phi_{io}, \phi_{if}, \dot{\phi}_{io}, \dot{\phi}_{if}, (t_f - t_o)) =$$

$$(t_f - t_o) + \frac{3\,(\phi_{io} - \phi_{if})(\dot{\phi}_{io} + \dot{\phi}_{if} \pm \sqrt{\dot{\phi}_{io}\,\dot{\phi}_{if}})}{\dot{\phi}_{io}^2 + \dot{\phi}_{io}\,\dot{\phi}_{if} + \dot{\phi}_{if}^2} = 0 \qquad (2.1.56)$$

On introducing the Case-II boundary conditions into (2.1.56) verifies that this maneuver represents required transition case; i.e., $\dot{\phi}_{if}$ is given by

---

* The condition being imposed represents the requirement that the time history for $\phi_{io}(t)$ have a single extremum on its curve.

$$\dot{\phi}_{if} = \frac{3\pi}{200} \text{ rad/sec} = 4.712 \times 10^{-2} \text{ rad/sec} \tag{2.1.57}$$

If the tranversality condition given by

$$H(\beta_o(t),\ \beta_i(t),\ \omega_i(t),\ \gamma_o(t),\ \gamma_i(t),\ \lambda_i(t),\ (t - t_o))\Big|_{t=t_f} = 0 \tag{2.1.58}$$

is solved for the final time t_f, in lieu of fixing t_f a priori, the solution yields precisely the condition given by (2.1.56). From (2.1.56) the condition for vehicle counter rotation is given by

$$f(\phi_{io},\ \phi_{if},\ \dot{\phi}_{io},\ \dot{\phi}_{if},\ (t_f - t_o)) > 0 \tag{2.1.59}$$

In the plot of Figure 2.2(a) it can be seen that the principal angle $\phi_i(t)$, which start at $\phi_i(t_o) = 0$ radians, increases strictly monotonically until it arrives at $\phi_i(t_f) = \frac{\pi}{2}$ radians at $t_f = 100$ seconds. Of course the time history of this maneuver agrees well with what intuition would expect, and it is only when more than one solution for t exists in (2.1.55) that somewhat counter intuitive results can occur. The boundary conditions of Case-III represents one such case where the condition of (2.1.55) is not satisfied.

In conclusion, the key feature about the single axis case is that the initial co-states are determined in closed form. Hence, with the complete solution for the single axis problem now in hand, interest can now be shifted to concentrate upon the natural generalization of the formulation above to the three-dimensional optimal control attitude maneuver problem.

2.2 THREE-DIMENSIONAL PROBLEM

The problem to be considered here is concerned with the reorientation of a rigid vehicle for which no restrictions have been imposed which limit the maneuvers to single axis special cases. Therefore, the emphasis in this section shall be focused upon obtaining the necessary conditions for an arbitrary reorientation/ detumble of a generally asymmetric rigid vehicle; as well as presenting an iterative relaxation process for the solution of the resulting nonlinear differential equations.

2.2.1 EQUATIONS OF MOTION

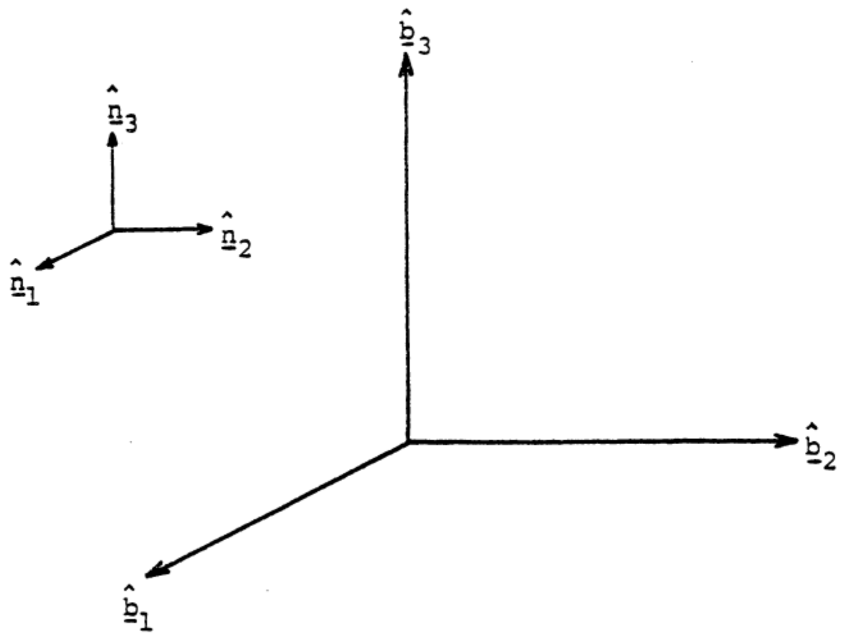
To proceed, in this section, the orientation kinematics are first introduced. Referring to Figure 2.4(a), it can be seen that two coordinate frames have been defined. Where $\{\hat{n}\}$ is an orthonormal set of unit vectors arbitrarily oriented, but fixed in inertial space; and $\{\hat{b}\}$ is an orthonormal set of unit vectors set along the vehicle's principal axis. The relative orientation of the vehicle to the inertial axis is defined by

$$\{\hat{b}\} = [C]\{\hat{n}\} \tag{2.2.1}$$

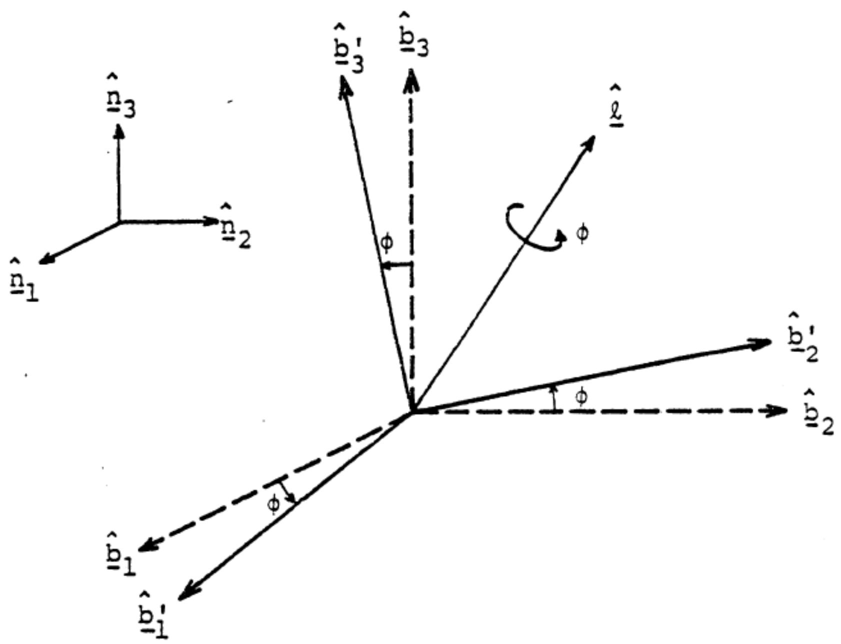
where $[C]$ is the direction cosine matrix.

In lieu of any three angle description of orientation the Euler parameters are adopted (see Appendix D), and are defined as

$$\left.\begin{aligned} &\beta_o = \cos\frac{1}{2}\phi\ ,\ \beta_i = \ell_i \sin\frac{1}{2}\phi\ ;\ \text{for } i=1,2,3 \\ \text{and}\quad &\underline{\beta} \triangleq \{\beta_o\ \ \beta_1\ \ \beta_2\ \ \beta_3\}^T \end{aligned}\right\} \tag{2.2.2}$$



(a) before principal axis rotation



(b) after principal axis rotation

Figure 2.4: Inertial and Body Fixed Coordinate Frames Before and After Rotation of the Body.

where $\hat{\underline{\ell}} = \ell_1\hat{\underline{n}}_1 + \ell_2\hat{\underline{n}}_2 + \ell_3\hat{\underline{n}}_3 \equiv \ell_1\hat{\underline{b}}'_1 + \ell_2\hat{\underline{b}}'_2 + \ell_3\hat{\underline{b}}'_3$ is the principal vector, and ϕ is the principal angle.

The existence of $\hat{\underline{\ell}}$ and ϕ corresponding to arbitrary admissible values for the elements of $[C]$ is guaranteed by Euler's Principal Rotation Theorem [45] which states that a completely general reorientation of a rigid body can be accomplished by rigidly rotating through ϕ about $\hat{\underline{\ell}}$ (see Figure 2.4(b)).

In terms of the Euler parameters of (2.2.2) the direction cosine matrix $[C]$ becomes

$$[C] = \begin{bmatrix} (\beta_o^2+\beta_1^2-\beta_2^2-\beta_3^2) & 2(\beta_1\beta_2+\beta_o\beta_3) & 2(\beta_1\beta_3-\beta_o\beta_2) \\ 2(\beta_1\beta_2-\beta_o\beta_3) & (\beta_o^2-\beta_1^2+\beta_2^2-\beta_3^2) & 2(\beta_2\beta_3+\beta_o\beta_1) \\ 2(\beta_1\beta_3+\beta_o\beta_2) & 2(\beta_2\beta_3-\beta_o\beta_1) & (\beta_o^2-\beta_1^2-\beta_2^2+\beta_3^2) \end{bmatrix} \qquad (2.2.3)$$

By the method of Appendix D, if it is desired, the Euler parameters can be related to the Euler angles.

Recalling (2.1.18) the differential equations for the Euler parameters can be cast in either of the forms

$$\dot{\underline{\beta}}(t) = \underline{\underline{\Omega}}\ \underline{\beta} = \underline{\underline{\beta}}\ \underline{\omega} \qquad (2.2.4)$$

where

$$\underline{\underline{\Omega}} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}, \quad \underline{\underline{\beta}} = \frac{1}{2}\begin{bmatrix} -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_o & -\beta_3 & \beta_2 \\ \beta_3 & \beta_o & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_o \end{bmatrix}; \quad (\dot{\ }) = \frac{d(\ )}{dt}$$

Recalling (2.1.15) Euler's rotational equations of motion can be written as

$$\underline{\dot{\omega}} = \underline{f}(\underline{\omega}) + \underline{\underline{D}}\ \underline{U} \qquad (2.2.5)$$

where $f = (- \mathcal{I}_1\omega_2\omega_3\ ,\ - \mathcal{I}_2\omega_1\omega_3\ ,\ - \mathcal{I}_3\omega_1\omega_2)^T$;

$$\underline{\underline{D}} = \begin{bmatrix} J_{11}^{-1} & 0 & 0 \\ 0 & J_{22}^{-1} & 0 \\ 0 & 0 & J_{33}^{-1} \end{bmatrix} ;$$

I_1 , I_2 , and I_3 are the vehicle principal inertial; U_1 , U_2 , and U_3 are the $\hat{\underline{b}}_i$ components of the control torque; and

$$\mathcal{I}_1 = \frac{(I_3 - I_2)}{I_1}\ ,\ \mathcal{I}_2 = \frac{(I_1 - I_3)}{I_2}\ ,\ \text{and } \mathcal{I}_3 = \frac{(I_2 - I_1)}{I_3}\ .$$

2.2.2 STATE SPACE FORMULATION

For this problem the natural choice for the state variable are the Euler parameters $\underline{\beta}(t)$ and the angular velocity $\underline{\omega}(t)$. For which the state differential equation are thus (2.2.4) and (2.2.5), as listed below:

$$\underline{\dot{\beta}} = \underline{\underline{\Omega}}\ \underline{\beta}$$

and

$$\underline{\dot{\omega}} = \underline{f}(\underline{\omega}) + \underline{\underline{D}}\ \underline{U}$$

2.2.3 OPTIMAL CONTROL PROBLEM

The current attitude control problem is to seek a solution to (2.2.4) and (2.2.5) satisfying the prescribed initial and final orientation and angular velocity given by:

$$\left.\begin{aligned} \beta_i(t_o) &= \beta_{io} \\ \omega_j(t_o) &= \omega_{jo} \\ \beta_i(t_f) &= \beta_{if} \\ \omega_j(t_f) &= \omega_{jf} \end{aligned}\right\} \text{ for } i=1,2,3,4 \text{ and } j=1,2,3 \tag{2.2.6}$$

where the Euler parameters must be consistent with the constraint $\sum_{i=0}^{3} \beta_i^2 = 1$ so that only 12 degrees of freedom exist in the 14 prescribed conditions above.

The optimal control problem is to seek the torque history $U_i(t)$ generating an optimal solution of (2.2.4) and (2.2.5), satisfying the boundary conditions of (2.2.6), which minimizes the performance index

$$J = \frac{1}{2} \int_{t_o}^{t_f} \underline{U}^T \underline{U}\, dt \;^{*} \tag{2.2.7}$$

The Hamiltonian function associated with minimizing (2.2.7) along trajectories of (2.2.4) and (2.2.5) is given by

$$H = \frac{1}{2} \underline{U}^T \underline{U} + \underline{\gamma}^T \underline{\underline{\beta}}\, \underline{\omega} + \underline{\lambda}^T \{\underline{f}(\underline{\omega}) + \underline{\underline{D}}\, \underline{U}\} \tag{2.2.8}$$

where γ_i and λ_j are co-state variables associated with β_i and ω_j . Pontryagin's principle requires as necessary conditions for $\underline{U}$ to be an optimal control that the γ's and the λ's satisfy co-state differential equations derivable from

---

* Attention is restricted to a piecewise continuous torque history $U_i(t)$.

$$\dot{\underline{\gamma}} = - \left(\frac{\partial H}{\partial \underline{\beta}}\right)^T = \underline{\underline{\Omega}}\ \underline{\gamma} \; ^{*} \tag{2.2.9a}$$

and

$$\dot{\underline{\lambda}} = - \left(\frac{\partial H}{\partial \underline{\omega}}\right)^T = - \left(\frac{\partial \underline{f}}{\partial \underline{\omega}}\right)^T \underline{\lambda} - \underline{\underline{\beta}}^T \underline{\gamma} \tag{2.2.9b}$$

In addition, Pontryagin's principle requires as a necessary condition for an arbitrary unbounded admissible control that

$$\frac{\partial H}{\partial \underline{U}} = 0 \; ^{**} \tag{2.2.10}$$

On introducing (2.2.8) into (2.2.10), one finds

$$0 = \underline{U} + \underline{\underline{D}}\ \underline{\lambda} \; ^{\dagger} \tag{2.2.11}$$

From which the optimal control is determined as a function of the angular velocity co-state variables as

$$\underline{U} = - \underline{\underline{D}}\ \underline{\lambda} \tag{2.2.12}$$

Introducing (2.2.12) into (2.2.5) yields the state and co-state differential equations defining the two-point-boundary-value problem as

(a) state equations:

$$\dot{\underline{\beta}} = \underline{\underline{\Omega}}\ \underline{\beta} \tag{2.2.13a}$$

$$\dot{\underline{\omega}} = \underline{f}(\underline{\omega}) - \underline{\underline{D}}\ \underline{\underline{D}}\ \underline{\lambda} \tag{2.2.13b}$$

---

* Use has been made of the fact that $\underline{\underline{\Omega}}^T = - \underline{\underline{\Omega}}$.

** Use has been made of the fact that $\underline{\underline{D}} = \underline{\underline{D}}^T$.

Observing that $\frac{\partial^2 H}{\partial \underline{U}^2} \equiv [I]$ leads to the conclusion that the equations being obtained are in fact the sufficient conditions for J to be locally minimized.

(b) co-state equations:

$$\dot{\underline{\gamma}} = \underline{\underline{\Omega}}\ \underline{\gamma} \tag{2.2.14a}$$

$$\dot{\underline{\lambda}} = -\left(\frac{\partial \underline{f}}{\partial \underline{\omega}}\right)^T \underline{\lambda} - \underline{\underline{\beta}}^T\ \underline{\gamma} \tag{2.2.14b}$$

Inspection of (2.2.13a) and (2.2.14a) reveals that the Euler parameter state and co-state variables satisfy skew-symmetric matrix differential equation of the same form. Hence, referring to Appendix D where it is shown that the parameter co-state norm is also preserved.

Thus, with the state (2.2.13) and the co-state (2.2.14) equations together with the boundary conditions of (2.2.6) this completes the specification of the optimal control two-point-boundary-value problem. Where $\underline{\beta}$ and $\underline{\omega}$ are known both initially and finally, and where all boundary conditions for $\underline{\gamma}$ and $\underline{\lambda}$ are unknown.

In the next section a method is presented for solving for the unknown co-state variables $\underline{\gamma}$ and $\underline{\lambda}$.

2.2.4 TERMINAL BOUNDARY CONDITION RELAXATION PROCESS FOR THE SOLUTION OF THE TWO-POINT-BOUNDARY-VALUE PROBLEM

The problem one faces when attempting a solution to the equations derived in the previous section is that the equations are nonlinear. Which is a fact that precludes the application of most analytical methods; e.g., superposition and transform methods. A consequence of nonlinearity is that some numerical method must be employed which permits the extraction of solutions from these equations in an efficient and reliable way.

Many approximate methods have been proposed and developed (e.g., quasilinearization [6], [13], [27], dynamic programming/ invariant imbedding [6], [13], [27], parameterization/parameter optimization [18], [20], [21], and function space gradient methods [24]) which to varying degrees, depending on the skills and preferences of the user, address the solution of problems of the type being considered here. In lieu of any of these methods the remainder of this section will be devoted to the development of an iterative process for the solution of the problem; which make efficient use of the information obtained from the closed form solution for the single axis special case of Section 2.1.4.

The relaxation process being considered here occurs in the space of the terminal boundary conditions; thus, providing solutions to a sequence of neighboring boundary value problems. The sequence of boundary conditions is such that the first set results in an analytical solution and the last set contains the true boundary conditions for the real problem of interest. The motivation for this imbedding approach comes from a (natural) desire to somehow make use of the closed form single axis solutions of Section 2.1.4. On examining (2.2.13) and (2.2.14), one can see that any deviation from a single axis special case set of boundary conditions has the consequence that the describing state and co-state equations are nonlinear*. Thereby, invalidating the results obtained for the

* Assume for the moment that the participation of the nonlinearity is infinitesimal; i.e., that the boundary conditions are only infinitesimally different from a single axis special case.

single axis special case. But, if one were to introduce a differential correction algorithm into the solution process, where the strength of the nonlinearity is such as to allow a local linearization of the current unknown initial co-state, then one could iteratively correct the initial guesses for the unknown initial co-state until satisfactory convergence has been achieved.*

Having obtained the initial co-states for this slightly nonlinear problem one could proceed to alter, in an arbitrary way, the boundary conditions of this slightly nonlinear problem.** Subject only to the requirement that a local linearization about the initial co-state again be valid. Permitting an iterative refinement of the initial co-state in order to satisfy the terminal constraints of this new problem. Continuing in this manner it is possible to reach any set of terminal boundary conditions for the three-dimensional problem by the marching process outlined above. The process begins by choosing to solve one of the three-single axis special cases; thus, providing the initial guesses for the differential correction algorithm described in this section.

The process described above is a form of imbedding where the linear single axis problem is fixed in or imbedded in a larger family of problems. All of which can be reached using a single parameter (the relaxation parameter) to control how quickly one

---

* Satisfactory convergence is measured by the degree to which terminal boundary conditions are satisfied for a given set of initial co-states.

** Clearly if this process is to be of any value the manner in which the boundary conditions are to be altered must be well defined.

marches in the direction of the boundary conditions of the problem one desires to solve.

As is mentioned several times, in the paragraphs above, the boundary conditions for any one of the intermediate or relaxation states have to be altered in some way. To make this boundary condition altering process specific, a sequence of boundary condition vectors is introduced; and the n-th boundary condition vector is denoted by

$$\underline{X}_n = \big[\theta_{1n}(t_o) \quad \theta_{2n}(t_o) \quad \theta_{3n}(t_o) \quad \omega_{1n}(t_o) \quad \omega_{2n}(t_o) \quad \omega_{3n}(t_o)$$
$$\theta_{1n}(t_f) \quad \theta_{2n}(t_f) \quad \theta_{3n}(t_f) \quad \omega_{1n}(t_f) \quad \omega_{2n}(t_f) \quad \omega_{3n}(t_f)\big]^T ;$$
$$\text{for } n=0,1,.....,N \qquad (2.2.15)$$

where $\underline{X}_N = \underline{X}_{true}$ is the true desired boundary conditions, and

$\underline{X}_o = \underline{X}_{start}$ is a set of boundary conditions for which the initial co-state variable can be determined exactly (e.g., the single axis solutions of Section 2.1.4).

Introducing the relaxation parameter α such that

$$\{0 \leqslant \alpha_1 < \alpha_2 < \alpha_3 < \,..... < \alpha_N = 1\}$$

the boundary conditions for the n-th step in the relaxation process are obtained from the linear sweep equation

$$\underline{X}_n = \underline{X}_o + \alpha_n \, (\underline{X}_N - \underline{X}_o) \qquad (2.2.16)$$

To see what (2.2.16) accomplishes, observe that $\alpha_n = 0$ yields $\underline{X}_o$; and $\alpha_n = 1$ produces $\underline{X}_N$, with all other admissible values for α_n yielding a linear combination of the boundary conditions of $\underline{X}_o$ and $\underline{X}_N$. Hence, once the algorithm is started, each stage of the relaxation process solves a problem closer (in terms of the boundary conditions) to the real problem of interest.

Since X_n is defined in terms of $\theta_{in}(t_o)$ and $\theta_{in}(t_f)$ in (2.2.15) the corresponding relaxation Euler parameters must be determined from these values of θ_{in} ; a compact algorithm is given in Appendix D. Now, with the boundary condition relaxation process defined by (2.2.16), the next obvious question is "how to change α_n". Based on several numerical experiments it is determined that the following sequence of α_n values provided adequate convergence:

$$\left.\begin{aligned} \alpha_1 &= 0.001 \\ \alpha_n &= \frac{n}{N} \quad \text{for } n=2,\ldots\ldots,N \end{aligned}\right\} \tag{2.2.17}$$

where N typically is taken to be 4 or 5 .
The fact that such a simple scheme has worked may only be the result of having examined a limited number of problems. Clearly though, there is nothing that precludes the possibility of allowing the process which determines α_n to become adaptive; where the incremental change in α could be based on some heurestically defined measure of the differential correction algorithms difficulty in obtaining converged initial co-states.

In (2.2.17) the reason for setting $\alpha_1 = 0.001$ in the first stage in the relaxation process is to insure that only small changes occur in the boundary conditions of (2.2.16). Then once this first small step is taken it has been found that relatively large increments in α_n still permit reliable convergence for initial co-states in the subsequent relaxation states (since linear or higher order extrapolations are possible).

After completing the first relaxation state starting estimates for the initial co-states are obtained from the first order Taylor expansion

$$\tilde{\underline{P}}_n = \underline{P}_{n-1} + (\alpha_n - \alpha_{n-1})\left\{\frac{d\underline{P}}{d\alpha}\Big|_{n-1}\right\} \tag{2.2.18}$$

where

$$\underline{P}_n = \left[\gamma_{on}(t_o) \quad \gamma_{1n}(t_o) \quad \gamma_{2n}(t_o) \quad \gamma_{3n}(t_o) \quad \lambda_{1n}(t_o) \quad \lambda_{2n}(t_o) \quad \lambda_{3n}(t_o)\right]^T \tag{2.2.19}$$

and the finite difference derivative of the co-state vector with respect to the relaxation parameter α is given by

$$\left\{\frac{d\underline{P}}{d\alpha}\Big|_{n-1}\right\} = \left\{\begin{array}{ll} \left(\dfrac{1}{\alpha_{n-1} - \alpha_{n-2}}\right)\{\underline{P}_{n-1} - \underline{P}_{n-2}\} & \text{for } n > 1 \\ \{0\} & \end{array}\right\} \tag{2.2.20}$$

where the $\underline{P}_n$ vectors are the converged co-state vectors (2.2.19) resulting from previous applications of the algorithm of Figure 2.5 .

An accelerated quadratic estimate for the initial co-state vector (2.2.18) is obtained from

$$\hat{\underline{P}}_n = \tilde{\underline{P}}_n + \underline{\Delta P}_{n-1} + (\alpha_n - \alpha_{n-1})\left\{\frac{d\underline{\Delta P}}{d\alpha}\Big|_{n-1}\right\} \quad \text{for } n > 2 \tag{2.2.21}$$

where $\underline{\Delta P}_n$ represents the actual error in the linearly predicted co-states in the n-th relaxation state.

The derivative of the linear co-state prediction error with respect to the relaxation parameter α is approximated via finite differences as

$$\left\{\frac{d\underline{\Delta P}}{d\alpha}\bigg|_{n-1}\right\} = \left\{\begin{array}{ll} \left(\frac{1}{\alpha_{n-1} - \alpha_{n-2}}\right)\{\underline{\Delta P}_{n-1} - \underline{\Delta P}_{n-2}\} & \text{for } n > 3 \\ \{0\} \quad \text{for } n \leq 2 & \end{array}\right\} \tag{2.2.22}$$

Thus, each increment of the state boundary conditions (2.2.16) is supported by a linear or better extrapolation of the co-state initial conditions via (2.2.21), followed by a differential correction refinement using the algorithm of Figure 2.5 . The approximations clearly become exact as the α increments approach zero.

To complete the solution process the differential correction algorithm is presented next. As is pointed out in Section 2.1.4 the Euler parameter co-state norm is bounded below for any of the three single axis special case solutions. Based on this knowledge, the unique feature of the differential correction algorithm is the minimization of the Euler parameter co-state norm; when the Euler parameter norm equation is given by

$$\gamma_o^2(t_o) + \gamma_1^2(t_o) + \gamma_2^2(t_o) + \gamma_3^2(t_o) = B^2 = \text{constant} \tag{2.2.23}$$

where B^2 is unknown.

Hence, the differential correction strategy is to determine the initial co-state initial conditions $\underline{\gamma}(t_o)$ and $\underline{\lambda}(t_o)$ which satisfy $\underline{\beta}(t_f) = \underline{\beta}_f$ and $\underline{\omega}(t_f) = \underline{\omega}_f$, subject to the criterion that $B^2 = \sum_{j=0}^{3} \gamma_j^2(t_o)$ be minimized.

Since the Euler parameters must satisfy the constraint

Given: $\beta_i(t_o)$, $\beta_i(t_f)$, i=0,1,2,3

$\omega_j(t_o)$, $\omega_j(t_f)$, j=1,2,3

Approximate $\hat{\underline{\gamma}}(t_o), \hat{\underline{\lambda}}(t_o)$

(A) →

Solve the differential (2.2.13) and (2.2.14)

to determine $\hat{\beta}_i = \beta_i(\hat{\underline{\gamma}}, \hat{\underline{\lambda}}, t_f)$ i=1,2,3

$\hat{\omega}_i = \omega_i(\hat{\underline{\gamma}}, \hat{\underline{\lambda}}, t_f)$

and determine (using method of Appendix B) the partial derivatives (2.2.28)

↓

Calculate the residual vectors

$(\underline{\beta}_f' - \hat{\underline{\beta}}')$ & $(\underline{\omega}_f - \hat{\underline{\omega}})$

Stop if sufficiently small

↓

Calculate $\tilde{A}$ from (2.2.31)
Calculate $\underline{\Delta\gamma}$ from (2.2.37)
Calculate $\underline{\Delta\lambda}$ from (2.2.35)

↓

Apply corrections

$\hat{\underline{\gamma}}(t_o) = \hat{\underline{\gamma}}(t_o) + \underline{\Delta\gamma}$

$\hat{\underline{\lambda}}(t_o) = \hat{\underline{\lambda}}(t_o) + \underline{\Delta\lambda}$

↓

(A)

Figure 2.5: Differential Correction Algorithm for the Rigid Body Problem.

$\sum_{j=0}^{3} \beta_j^2(t) = 1$ at $t = t_f$, it is necessary to formally constraint only three of Euler parameters $\underline{\beta}(t_f) = \underline{\beta}_f$ to their prescribed terminal values; the remaining $\underline{\beta}_{if}$ then automatically satisfy the constraint (by virtue of the norm preserving character of the Euler parameter's matrix differential equation). These considerations motivate the following successive approximation strategy to solve the two-point-boundary-value problem: Given the starting estimates for the co-state vectors

$$\underline{\hat{\gamma}}(t_o) = [\hat{\gamma}_o(t_o) \quad \hat{\gamma}_1(t_o) \quad \hat{\gamma}_2(t_o) \quad \hat{\gamma}_3(t_o)]^T \tag{2.2.24a}$$

$$\underline{\hat{\lambda}}(t_o) = [\hat{\lambda}_1(t_o) \quad \hat{\lambda}_2(t_o) \quad \hat{\lambda}_3(t_o)]^T \tag{2.2.24b}$$

one seeks the correction vectors $\Delta\underline{\gamma}$ and $\Delta\underline{\lambda}$ which minimize

$$\begin{aligned} B^2 &= [\underline{\hat{\gamma}}(t_o) + \Delta\underline{\gamma}]^T [\underline{\hat{\gamma}}(t_o) + \Delta\underline{\gamma}] \\ &= \underline{\hat{\gamma}}^T(t_o)\, \underline{\hat{\gamma}}(t_o) + 2\, \Delta\underline{\gamma}^T\, \underline{\gamma}(t_o) + \Delta\underline{\gamma}^T\, \Delta\underline{\gamma} \end{aligned} \tag{2.2.25}$$

subject to the terminal constraints

$$\underline{\beta}_f' - \underline{\beta}'\ [\underline{\hat{\gamma}}(t_o) + \Delta\underline{\gamma}\ ,\ \underline{\hat{\lambda}}(t_o) + \Delta\underline{\lambda}\ ,\ t_f] = 0 \tag{2.2.26a}$$

$$\underline{\omega}_f - \underline{\omega}\ [\underline{\hat{\gamma}}(t_o) + \Delta\underline{\gamma}\ ,\ \underline{\hat{\lambda}}(t_o) + \Delta\underline{\lambda}\ ,\ t_f] = 0 \tag{2.2.26b}$$

where

$$\underline{\beta}'(t) = [\beta_1(t) \quad \beta_2(t) \quad \beta_3(t)]^T$$

$$\underline{\omega}(t) = [\omega_1(t) \quad \omega_2(t) \quad \omega_3(t)]^T$$

and

$$\underline{\beta}_f' = \underline{\beta}'(t_f)\ ;\ \underline{\omega}_f = \underline{\omega}(t_f)$$

Letting the notations $\underline{\hat{\beta}}' = \underline{\beta}'[\underline{a},\underline{b},t]$ and $\underline{\hat{\omega}} = \underline{\omega}[\underline{a},\underline{b},t]$ denote solutions

(2.2.13) and (2.2.14) with the initial co-states $\underline{\lambda}(t_o) = \underline{b}$ and $\underline{\gamma}(t_o) = \underline{a}$. One obtains on linearizing (2.2.26)

$$\underline{\beta}_f' - \hat{\underline{\beta}}' - \underline{\underline{A}}_{\beta\gamma}\,\underline{\Delta\gamma} - \underline{\underline{A}}_{\beta\lambda}\,\underline{\Delta\lambda} = \underline{0} \tag{2.2.27a}$$

$$\underline{\omega}_f - \hat{\underline{\omega}} - \underline{\underline{A}}_{\omega\gamma}\,\underline{\Delta\gamma} - \underline{\underline{A}}_{\omega\lambda}\,\underline{\Delta\lambda} = 0 \tag{2.2.27b}$$

where

$$\underline{\underline{A}}_{\beta\gamma} = \left|\frac{\partial \underline{\beta}'^{T}|\underline{\gamma}(t_o),\underline{\lambda}(t_o),t|}{\partial\{\underline{\gamma}(t_o)\}}\bigg|_{\underline{a},\underline{b},t_f}\right|^{T} = \left|\frac{\partial \underline{\beta}_f'^{T}}{\partial \underline{\gamma}_o}\right|^{T} \tag{2.2.28a}$$

$$\underline{\underline{A}}_{\beta\lambda} = \left[\frac{\partial \underline{\beta}'^{T}[\underline{\gamma}(t_o),\underline{\lambda}(t_o),t]}{\partial\{\underline{\lambda}(t_o)\}}\bigg|_{\underline{a},\underline{b},t_f}\right]^{T} = \left[\frac{\partial \underline{\beta}_f'^{T}}{\partial \underline{\lambda}_o}\right]^{T} \tag{2.2.28b}$$

$$\underline{\underline{A}}_{\omega\gamma} = \left[\frac{\partial \underline{\omega}^{T}[\underline{\gamma}(t_o),\underline{\lambda}(t_o),t]}{\partial\{\underline{\gamma}(t_o)\}}\bigg|_{\underline{a},\underline{b},t_f}\right]^{T} = \left[\frac{\partial \underline{\omega}_f^{T}}{\partial \underline{\gamma}_o}\right]^{T} \tag{2.2.28c}$$

$$\underline{\underline{A}}_{\omega\lambda} = \left[\frac{\partial \underline{\omega}^{T}[\underline{\gamma}(t_o),\underline{\lambda}(t_o),t]}{\partial\{\underline{\lambda}(t_o)\}}\bigg|_{\underline{a},\underline{b},t_f}\right]^{T} = \left[\frac{\partial \underline{\omega}_f^{T}}{\partial \underline{\lambda}_o}\right]^{T} \tag{2.2.28d}$$

Calculation of the derivatives of (2.2.28) is a separate issue dealt with in Appendix B.

On observing that (2.2.27a) and (2.2.27b) provide the coupling relationships between $\underline{\Delta\gamma}$ and $\underline{\Delta\lambda}$, and recalling that (2.2.25) does not contain an explicit dependence upon $\underline{\Delta\lambda}$, it is then convenient to reduce (2.2.27) to a single constraint as a function of $\underline{\Delta\gamma}$; in order to carry out the desired minimization. Since $\underline{\underline{A}}_{\omega\lambda}$ is a square matrix and presumed to be nonsingular, one can determine $\underline{\Delta\lambda}$ from (2.2.27b) as a function of $\underline{\Delta\gamma}$, which leads to

$$\underline{\Delta\lambda} = \underline{\underline{A}}_{\omega\lambda}^{-1}\,(\underline{\omega}_f - \hat{\underline{\omega}}) - \underline{\underline{A}}_{\omega\lambda}^{-1}\,\underline{\underline{A}}_{\omega\gamma}\,\underline{\Delta\gamma} \tag{2.2.29}$$

Substitution of (2.2.29) into (2.2.27a) then yields the single constraining relationship, replacing (2.2.27), depending only on

$\underline{\Delta\gamma}$ as

$$(\underline{\beta}_f' - \hat{\underline{\beta}}') - \underline{\underline{A}}_{\beta\lambda}\, \underline{\underline{A}}_{\omega\lambda}\, (\underline{\omega}_f - \hat{\underline{\omega}}) - \tilde{\underline{\underline{A}}}\, \underline{\Delta\gamma} = \underline{0} \tag{2.2.30}$$

where

$$\tilde{\underline{\underline{A}}} = \underline{\underline{A}}_{\beta\gamma} - \underline{\underline{A}}_{\beta\lambda}\, \underline{\underline{A}}_{\omega\lambda}^{-1}\, \underline{\underline{A}}_{\omega\gamma} \tag{2.2.31}$$

Using the Lagrange multiplier rule to minimize (2.2.25) subject to the constraint (2.2.30), the augmented objective function $\hat{\Phi}$ is introduced

$$\hat{\Phi} = \hat{\underline{\gamma}}^T(t_o)\, \underline{\gamma}(t_o) + 2\, \hat{\underline{\gamma}}^T(t_o)\, \underline{\Delta\gamma} + \underline{\Delta\gamma}^T\, \underline{\Delta\gamma} + \underline{\Lambda}^T \Big[(\underline{\beta}_f' - \hat{\underline{\beta}}') - \underline{\underline{A}}_{\beta\lambda}\, \underline{\underline{A}}_{\omega\lambda}^{-1}\, (\underline{\omega}_f - \hat{\underline{\omega}}) - \tilde{\underline{\underline{A}}}\, \underline{\Delta\gamma} \Big] \tag{2.2.32}$$

where $\underline{\Lambda}$ is a 3 by 1 vector of (as yet unknown) Lagrange multipliers. The necessary conditions for $\hat{\Phi}$ to possess a minimum follow as

$$\frac{\partial \hat{\Phi}}{\partial (\underline{\Delta\gamma})} = \underline{0} = 2\, \hat{\underline{\gamma}}(t_o) + 2\, \underline{\Delta\gamma} - \tilde{\underline{\underline{A}}}^T\, \underline{\Lambda}\ ^{*} \tag{2.2.33}$$

and

$$\frac{\partial \hat{\Phi}}{\partial (\underline{\Lambda})} = \underline{0} = (\underline{\beta}_f' - \hat{\underline{\beta}}') - \underline{\underline{A}}_{\beta\lambda} \cdot \underline{\underline{A}}_{\omega\lambda}^{-1}\, (\underline{\omega}_f - \hat{\underline{\omega}}) - \tilde{\underline{\underline{A}}}\, \underline{\Delta\gamma} \tag{2.2.34}$$

Solving (2.2.33) for the optimum correction $\underline{\Delta\gamma}$ as a function of $\underline{\Lambda}$, one obtains

$$\underline{\Delta\gamma} = \frac{1}{2}\, \tilde{\underline{\underline{A}}}^T\, \underline{\Lambda} - \hat{\underline{\gamma}}(t_o) \tag{2.2.35}$$

Introducing (2.2.35) into (2.2.34) yields the solution for $\underline{\Lambda}$:

$$\frac{1}{2}\, \underline{\Lambda} = (\tilde{\underline{\underline{A}}}\, \tilde{\underline{\underline{A}}}^T)^{-1} \left[(\underline{\beta}_f' - \hat{\underline{\beta}}') - \underline{\underline{A}}_{\beta\lambda}\, \underline{\underline{A}}_{\omega\lambda}^{-1}\, (\underline{\omega}_f - \hat{\underline{\omega}}) + \tilde{\underline{\underline{A}}} \right] \tag{2.2.36}$$

---

* Taking the second partial derivative of Φ with respect to $\underline{\Delta\gamma}$, leads to $\left[\frac{\partial}{\partial (\underline{\Delta\gamma})} \left(\frac{\partial \hat{\Phi}}{\partial (\underline{\Delta\gamma})} \right)^T \right] = I$, which is positive definite; hence the conditions of (2.2.33) and (2.2.34) are in fact the sufficient conditions for Φ to be locally minimized.

Finally, substitution of (2.2.36) into (2.2.35) yields the solution for $\underline{\Delta\gamma}$:

$$\underline{\Delta\gamma} = \tilde{\underline{\underline{A}}}^T \left(\tilde{\underline{\underline{A}}}\,\tilde{\underline{\underline{A}}}^T\right)^{-1} \left[(\underline{\beta}_f' - \hat{\underline{\beta}}') - \underline{\underline{A}}_{\beta\lambda}\,\underline{\underline{A}}_{\omega\lambda}^{-1}\,(\underline{\omega}_f - \hat{\underline{\omega}}) + \tilde{\underline{\underline{A}}}\,\hat{\underline{\gamma}}(t_o)\right] - \hat{\underline{\gamma}}(t_o) \tag{2.2.37}$$

From (2.2.37) the solution for $\underline{\Delta\lambda}$ then follows immediately from (2.2.29).

With (2.2.37) this completes the derivation of all equations required in the iterative relaxation process diagrammed in Figure 2.5. The only significant assumption in route to the algorithm is the local linearization of (2.2.26) to obtain (2.2.27).

In summary, the relaxation process is started by using the closed form solution for one of the single axis special cases. Next, one then employs (2.2.16) to alter the chosen single axis boundary condition such that the linearization of (2.2.26) is valid. Equations (2.2.37) and (2.2.29) then provide the means to iteratively refine the initial co-state extrapolations of (2.2.23) until the terminal boundary conditions of (2.2.16) have been adequately satisfied. The process above continues until n = N in (2.2.17), at which time the desired solution has been obtained.

2.2.5 NUMERICAL RESULTS

In this section the results are presented for a rather general "detumble" maneuver of an asymmetric rigid vehicle which are referred to as Case-IV ; the bodies principal moment of inertias are given by

$I_1 = 1.0\times10^6 \text{ kg.m}^2$, $I_2 = 0.833\times10^6 \text{ kg.m}^2$, and $I_3 = 0.917\times10^6 \text{ kg.m}^2$

and the terminal boundary condition vectors corresponding to $\underline{X}_o$ and $\underline{X}_n$ are

$$\left.\begin{aligned} &\underline{X}_o = \left[0,0,0,0.01,0,0,\tfrac{5\pi}{2},0,0,0,0,0\right]^T \\ \text{and} \quad & \\ &\underline{X}_N = \left[0,0,0,0.01,0.005,0.001,\tfrac{5\pi}{2},\tfrac{\pi}{3},\tfrac{\pi}{4},0,0,0\right]^T \end{aligned}\right\} \qquad (2.2.38)$$

where the single axis solution about the $\hat{\underline{b}}_1$ axis has been chosen to start the relaxation process since it possesses the greatest initial angular rate. This is not the only acceptable choice. In fact, for this particular maneuver, identical results have been obtained by choosing rotations about the $\hat{\underline{b}}_2$ and $\hat{\underline{b}}_3$ axis as the starting solutions (this is reassuring, since one has no a priori guarantee that different starting solutions are not leading to different local extremum trajectories for the control torque histories of $\underline{U}(t)$).*

The relaxation boundary conditions generated by (2.2.16) for Case-IV are summarized in Table 2.3 . In addition, Table 2.3 summarizes the converged initial co-state determined by the differential correction algorithm of Figure 2.5 . For each of the five relaxation stages, an average of four differential corrections are required. The rigorous partial derivatives of the state transition matrix are computed and used for the first differential correction for each relaxation state, with approximate partial

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* The fact that the same solution was obtained represents at least a necessary condition for the solution obtained to be the global minimum for J.

Table 2.3: Boundary Condition Relaxation for Case-IV

n		0	1	2	3	4	5
α_n		0	0.001	0.25	0.50	0.75	1.00
Tolerance*		-	10^{-2}	10^{-2}	10^{-2}	10^{-2}	10^{-5}
No. of Iterations		-	1	4	8	6	4
Initial State	$\theta_1(t_o)$	0	0	0	0	0	0
	$\theta_2(t_o)$	0	0	0	0	0	0
	$\theta_3(t_o)$	0	0	0	0	0	0
	$\omega_1(t_o)$	0.01000 rad/s	0.01000	0.01000	0.01000	0.01000	0.01000
	$\omega_2(t_o)$	0	0.00005	0.00125	0.00250	0.00375	0.00500
	$\omega_3(t_o)$	0	0.00001	0.00025	0.00050	0.00075	0.00100

* Maximum relative error in ω's (normalized by 0.001), and β's (normalized by 1).

Table 2.3 (continued): Boundary Condition Relaxation for Case-IV

n		0	1	2	3	4	5
Final State	$\theta_1(t_f)$	7.85397	7.85397	7.85397	7.85397	7.85397	7.85397 ($\equiv 5\pi/2$)
	$\theta_2(t_f)$	0	0.01048	0.260829	0.52249	0.78471	1.04731 ($\equiv \pi/3$)
	$\theta_3(t_f)$	0	0.07857	0.19620	0.39248	0.58909	0.78546 ($\equiv \pi/4$)
	$\omega_1(t_f)$	0	0	0	0	0	0
	$\omega_2(t_f)$	0	0	0	0	0	0
	$\omega_3(t_f)$	0	0	0	0	0	0
Converged Co-State ($\times 10^9$)	$\gamma_o(t_o)$	-4.3124	-4.31237	-4.22029	-4.01835	-3.81454	-3.64945
	$\gamma_1(t_o)$	0	-0.01576	-0.37601	-0.67991	-0.91423	-1.10589
	$\gamma_2(t_o)$	0	-0.03034	-0.73721	-1.34988	-1.80119	-2.13870
	$\gamma_3(t_o)$	0	0	0	0	0	0
	$\lambda_1(t_o)$	-0.17649	-0.17649	-0.17685	-0.17546	-0.17082	-0.16634
	$\lambda_2(t_o)$	0	-0.00063	-0.01501	-0.02674	-0.03496	-0.04117
	$\lambda_3(t_o)$	0	-0.00039	-0.01101	-0.02672	-0.04411	-0.05838

derivatives being generated for successive differential corrections by the method of Appendix C.

The control and state vectors are sketched in Figures 2.6, (a)-(c).

2.2.6 DISCUSSION

Considering the rather general nature of the Case-IV example and relative ease with which the solution is obtained, leads one to believe that the relaxation method of this chapter provides a practical tool, capable of handling this class of optimal control problems. Of course, conspicuous by its absence here is any mathematical proof of convergence or guarantee that different sequencies of relaxation parameters will not find bifurcation points in the solution process leading to different optimal torque histories for $\underline{U}(t)$. The answers to the concerns mentioned above are not available and are not considered here. Instead, the thrust of this development is concerned with developing and testing methods for obtaining solutions to certain classes of nonlinear problems of interest to the aerospace industry. It seems reasonable to anticipate further theoretical and numerical study of these more subtle issues.

If one reflects for a moment, it should be clear that the relaxation process just presented, can in fact be used by many of the other methods used to solve complicated, nonlinear, two-point-boundary-value problems; where one should seek to imbed the

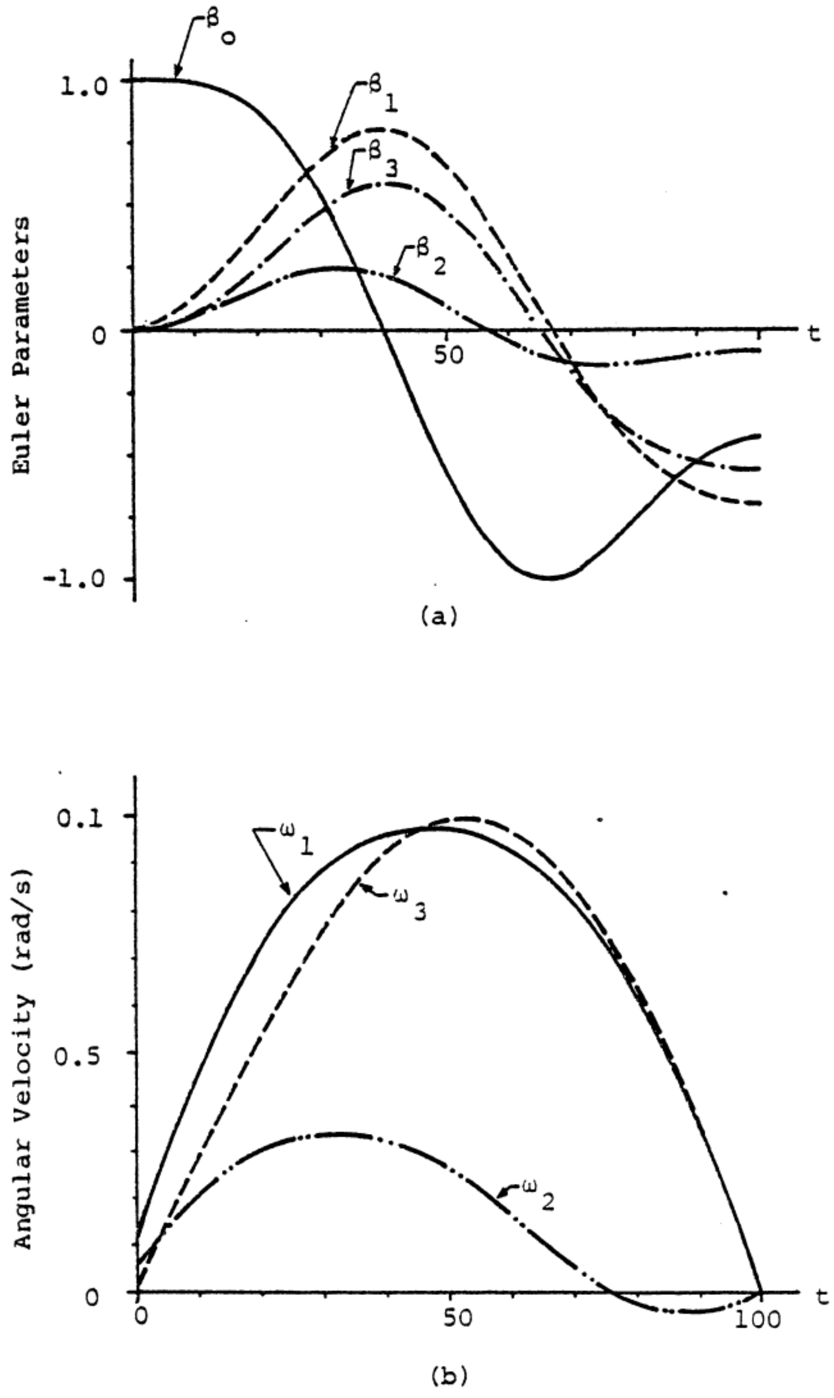

Figure 2.6: Case-IV - Three-Dimensional Rigid Body Maneuver.

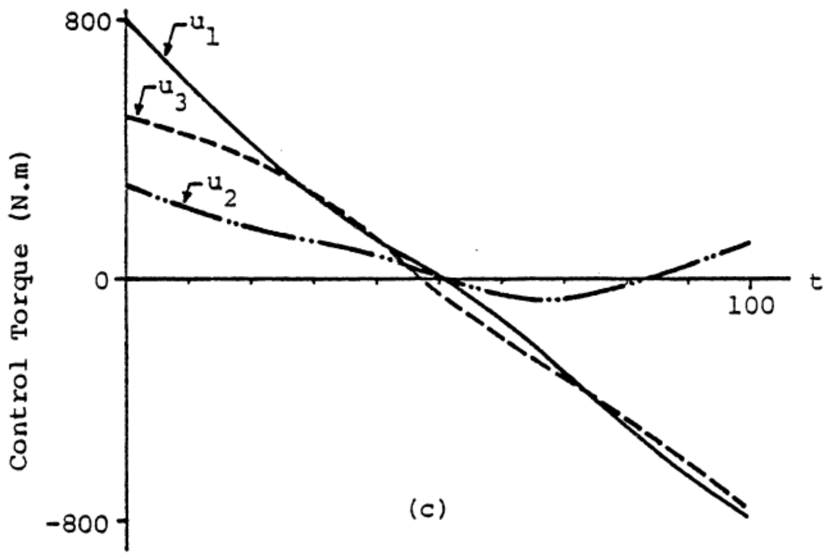

Figure 2.6 (continued): Case-IV - Three Dimensional Rigid Body Maneuver.

particular problem in a family problems, in which one member of the family has an analytical solution. The principal merit of this approach lies in the fact that the solution to some difficult problems can be reduced to seeking special case solutions; which once obtained, can be refined by the relaxation process until the desired problem of interest has been solved. If no convenient special case solutions result by simply altering the problem boundary conditions, then the methods of Chapter III provides an alternative relaxation process designed to handle such cases.

This concludes the treatment of the rigid body optimal attitude control problem. In the next chapter the simplest generalization of the rigid body problem is considered.

CHAPTER III

SINGLE AXIS FLEXIBLE BODY OPTIMAL MANEUVERS

This chapter is concerned with the attitude and configuration control of large flexible space vehicles, restricted to maneuvers about one of the principal axis of the undeformed vehicles. In particular, this chapter is concerned with obtaining, from Pontryagin's principle, the necessary conditions for optimal large angle single axis reorientations of these vehicles. The flexural deformations of the elastic bodies are modeled by the "assumed modes" method and only small (linear) flexural displacements are considered. Furthermore, the control system for the vehicle is assumed to be capable of generating an arbitrary continuous control torque which is to be applied on a rigid part of the structure.

The equations of motion and their state space forms are obtained in Sections 3.1 and 3.2 . The formulation of the necessary conditions are presented in two parts: The first part (Section 3.3.1), deals with linearized version of this problem, and the second part (Section 3.3.2), deals with nonlinear version. A relaxation process is presented in Section 3.4 , for the solution of the nonlinear problem which makes efficient use of the closed form solution algorithm for the initial co-states in the linearized problem. Several examples are treated in Section 3.5 which consist of rest-to-rest and spin-up maneuvers with various numbers of modes being used in the dynamical model. Finally, in Section 3.6 the numerical results of Section 3.5 are discussed.

3.1 EQUATIONS OF MOTION

The motion of this class of vehicles considered is described by a system of hybrid coordinates, using a combination of discrete coordinates (for translations and rotations of rigid bodies) and distributed or modal coordinates (for the deformation of elastic bodies) [28].

A specific model is considered in this chapter (see Figure 3.1 - (b)), which consists of a rigid hub with four identical elastic appendages attached symmetrically about the central hub. Since only single axis maneuvers are being considered, the flexible members are restricted to anti-symmetric deformational modes (see Figure 3.1 - (b)) in the plane normal to the axis of rotation.

The procedure used herein for analyzing hybrid systems is a discretization, whereby the displacements of the continuous elastic members are replaced by a finite series; in which prescribed (admissible) space-dependent functions are multiplied by time-dependent (to be determined) modal amplitudes and summed, to model the instantaneous deflections of the structure [34].

For the body being considered, the equations of motion can be obtained from Hamilton's extended principle [33],

$$\int_{t_1}^{t_2} (\delta L + \delta W)\, dt = 0 \qquad (3.1.1)$$

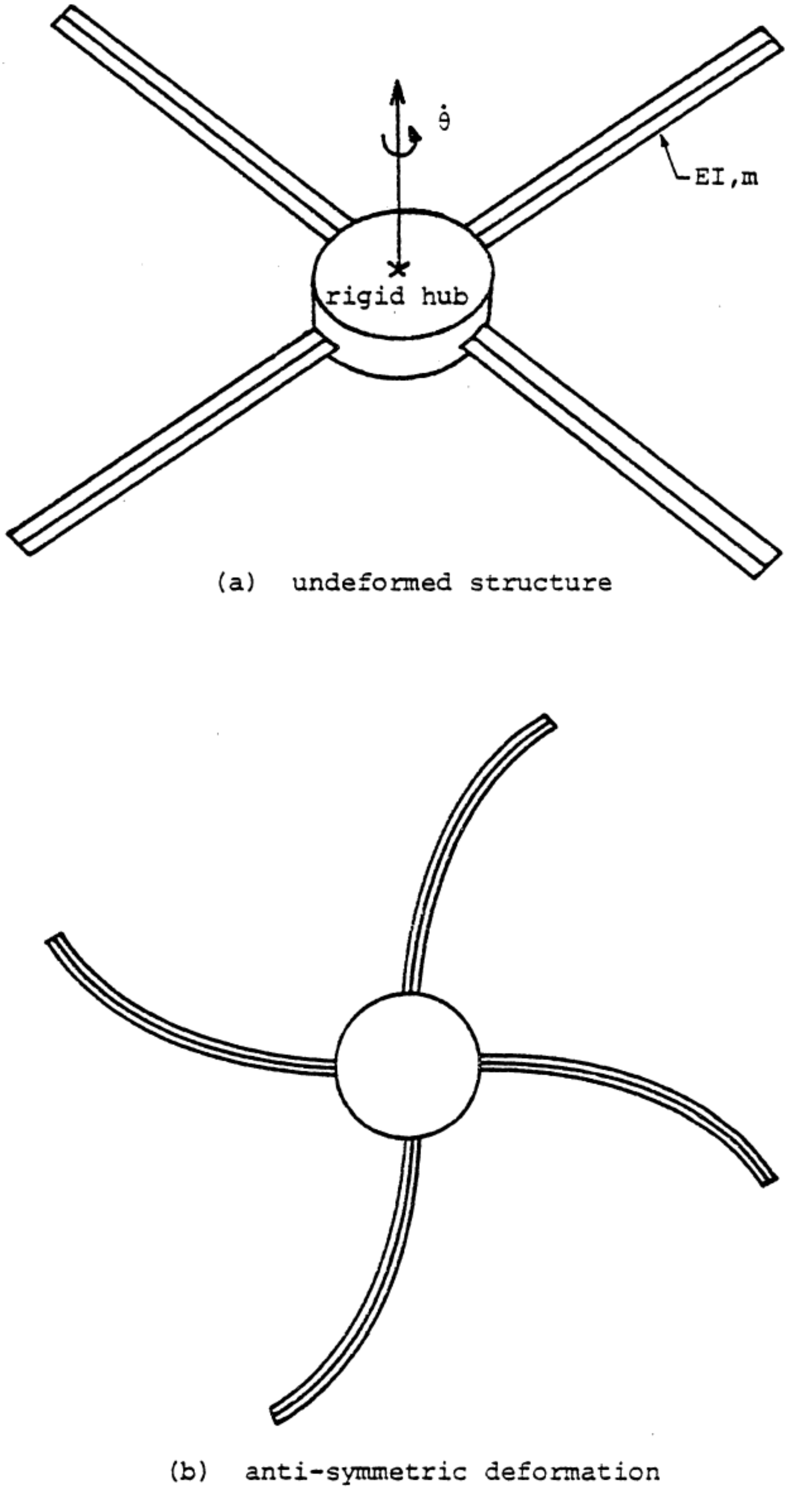

Figure 3.1: Undeformed and Deformed Structure.

which is subject to the boundary conditions

$$\delta\theta = \delta U = 0 \quad , \text{ at } t_1 \ , \ t_2 \tag{3.1.2}$$

where $L = T - V$ is the system Lagrangian, δW represents the virtual work, δr represents a virtual rotation, and δU represents a virtual elastic displacements.

The general expression for the kinetic energy of a continuum is given by

$$T = \frac{1}{2} \int_B \left.\frac{d\underline{r}}{dt}\right|_N \cdot \left.\frac{d\underline{r}}{dt}\right|_N dm \tag{3.1.3}$$

where $\left.\frac{d(\ )}{dt}\right|_N$ is the time derivative relative to the inertial frame N , dm is the generic mass element, and $\underline{r}$ locates the generic mass element dm in the continuum relative to the origin of the inertial frame.

Referring to (A.8), (3.1.3) becomes

$$T = \frac{1}{2} \int_B \left.\frac{d\underline{u}}{dt}\right|_B \cdot \left.\frac{d\underline{u}}{dt}\right|_B dm + \underline{\omega} \cdot \int_B \underline{\rho} \times \left.\frac{d\underline{\rho}}{dt}\right|_B dm + \frac{1}{2}\, \underline{\omega} \cdot \int_B \underline{\rho} \times (\underline{\omega} \times \underline{\rho})\ dm \tag{3.1.4}$$

where $\underline{\rho} = \underline{r} + \underline{u}$ locates the generic mass element dm in the deformed body relative to the bodies mass center* , $\underline{r}$ locates the generic mass element dm in the undeformed body relative to the body's mass center, and $\underline{u}$ represents the vector of elastic displacements.

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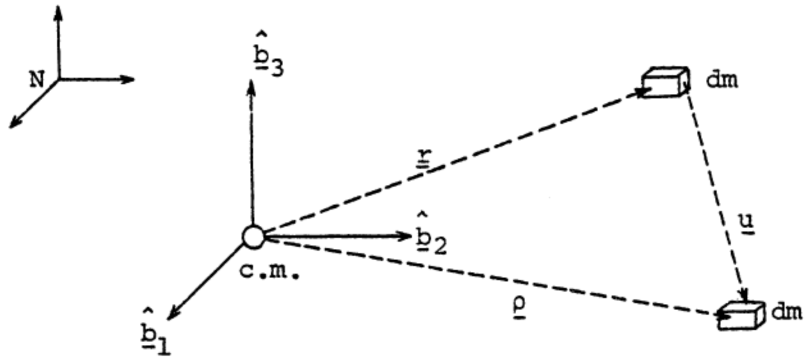
* For the single axis and anti-symmetric case being considered the body's mass center does not move in inertial space.



Treating the four identical elastic appendages as one dimensional elastic domains and specializing (3.1.4) for a single axis maneuver, leads to

$$T = \frac{1}{2} J_{hub}\, \dot{\theta}^2 + 2 \int_A \left[(x^2 + u^2)\, \dot{\theta}^2 + \dot{u}^2 + 2\, x\, \dot{\theta}\, \dot{u} \right] dm \qquad (3.1.5)$$

where A denotes that the integration is to be taken over one of the elastic appendages*, and J_{hub} denotes the moment of inertia of the rigid hub.

In order to take into account the shortening of the appendages, due to deformation, an arc-length constraint is imposed to first order in the integration over the appendage, where the approximate arc-length equation is given by

$$L = \int_A ds = \int_0^x \sqrt{dx^2 + du^2} \triangleq \int_0^x \left[1 + \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 \right] dx \qquad (3.1.6)$$

so that $L = x + \delta x$

where $\delta x \equiv \int_0^x \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 dx$ is the shortening of the appendage due to

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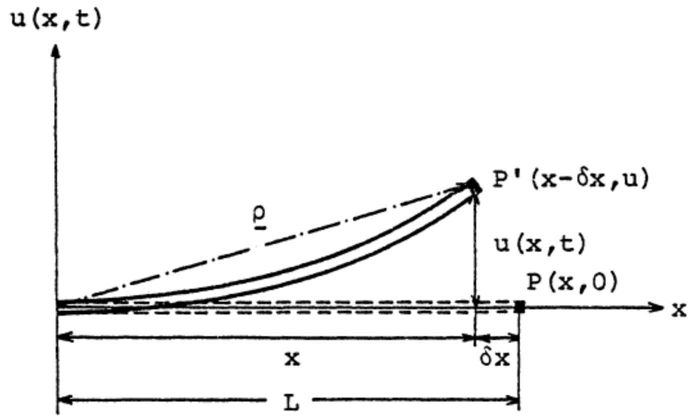
* In the integral over "A" , the finite radius of the appendage to the hub is ignored in comparison to its length.

elastic deformation, and L is the undeformed appendage length.



Taking (3.1.6) into account, the integral in (3.1.5) becomes

$$\int_A \left[(x^2 + u^2)\ \dot{\theta}^2 + \dot{u}^2 + 2\ x\ \dot{\theta}\ \dot{u}\right]\ dm \triangleq$$

$$\int_0^L \left[(x^2 - 2\ x\ \delta x + u^2)\ \dot{\theta}^2 + \dot{u}^2 + 2\ x\ \dot{\theta}\ \dot{u}\right]\ dm\ ^* \tag{3.1.7}$$

In (3.1.7), using integration by parts, the integrand containing δx can be written as

$$\int_0^L 2\ x\ \delta x\ dx = \mathcal{U}\ \mathcal{V}\Big|_0^L - \int_0^L \mathcal{V}\ d\mathcal{U} = \int_0^L p^2\ dx \tag{3.1.8}$$

where $\mathcal{U} = \delta x = \int_0^x \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$; $\mathcal{V} = x^2$; $d\mathcal{U} = \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2$;

$d\mathcal{V} = 2\ x\ dx$, and $p^2 = \frac{1}{2}\ (L^2 - x^2)\left(\frac{\partial u}{\partial x}\right)^2$.

Introducing (3.1.8) into (3.1.7) the final expression for the kinetic energy becomes

---

* Only first order terms in δx have been retained.

$$T = \frac{1}{2}\dot{\theta}^2 \left[I + 4\int_A (u^2 - p^2)\ dm\right] + 2\int_A \dot{u}^2\ dm + 4\ \dot{\theta}\int_A x\ \dot{u}\ dm\Big] \tag{3.1.9}$$

where I denotes the moment of inertia of the body about the spin axis in the undeformed state.

Since only the elastic potential energy is considered in this problem the expression for the potential becomes

$$V = 2\int_A EI\left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx \tag{3.1.10}$$

The virtual work in this example just consists of the control torque generated to bring about the maneuver and hence δW has the form

$$\delta W = U_T \cdot \delta\theta \tag{3.1.11}$$

where U_T represents the control torque.

Before applying Hamilton's extended principle the elastic displacements are first expressed, by the assumed modes method, as the following series

$$U(x,t) = \sum_{k=1}^{n} \eta_k(t)\ \phi_k(x) \tag{3.1.12}$$

where $\eta_k(t)$ represents the time varying modal amplitude, $\phi_k(x)$ represents an admissible assumed mode shape, and n denotes the number of modes to be used in the approximation.

On introducing (3.1.12) into (3.1.9) and (3.1.10) the system Lagrangian becomes

$$L = \frac{I}{2}\dot{\theta}^2 + \frac{1}{2}\dot{\underline{\eta}}^T \underline{\underline{M}}_{\eta\eta}\ \dot{\underline{\eta}} + \dot{\theta}\ \underline{M}_{\dot{\theta}\eta}\ \dot{\underline{\eta}} + \frac{1}{2}\underline{\eta}^T\left[\dot{\theta}^2\ (\underline{\underline{M}}_{\eta\eta} - \underline{\underline{M}}_p) - \underline{\underline{K}}_{\eta\eta}\right]\underline{\eta} \tag{3.1.13}$$

where $\underline{\eta} = \left[\eta_1 \quad \eta_2 \,\ldots\ldots\, \eta_n\right]^T$

$$\left[\underline{\underline{M}}_{\eta\eta}\right]_{k\ell} = 4 \int_A \phi_k \, \phi_\ell \, dm$$

$$\left[\underline{\underline{M}}_p\right]_{k\ell} = 2 \int_A (L^2 - x^2) \, \phi_k' \, \phi_\ell' \, dm \; *$$

$$\left[\underline{\underline{K}}_{\eta\eta}\right]_{k\ell} = 4 \int_A EI \, \phi_k'' \, \phi_\ell'' \, dx$$

$$\left[\underline{M}_{\dot{\theta}\eta}\right]_k = 4 \int_A x \, \phi_k \, dm$$

Subsituting (3.1.11) and (3.1.13) into (3.1.1) yields Lagrange's equations

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= U_T \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{\eta}}}\right) - \frac{\partial L}{\partial \underline{\eta}} &= \underline{0} \end{aligned} \qquad (3.1.14)$$

From which the equations of motion follow as

$$\ddot{\theta} = \frac{1}{(I + \underline{\eta}^T \, \underline{\underline{M}}_{\eta\eta} \, \underline{\eta})} \left[U_T - \underline{M}_{\dot{\theta}\eta}^T \, \ddot{\underline{\eta}} - 2\, \dot{\theta} \, \dot{\underline{\eta}}^T \, \underline{\underline{M}}_{\eta\eta} \, \underline{\eta}\right] \qquad (3.1.16)$$

$$\underline{\underline{M}}_{\eta\eta} \, \ddot{\eta} + \left[\underline{\underline{K}} + \dot{\theta}^2 \, \underline{\underline{\mathcal{M}}}\right] \underline{\eta} = - \ddot{\theta} \, \underline{\underline{M}}_{\dot{\theta}\eta} \qquad (3.1.17)$$

where $\underline{\underline{\mathcal{M}}} = \underline{\underline{M}}_p - \underline{\underline{M}}_{\eta\eta}$

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* $(\;)' \equiv \frac{\partial (\;)}{\partial x}$; $(\;)'' \equiv \frac{\partial^2 (\;)}{\partial x^2}$

3.2 STATE SPACE FORMULATION

Since only small (linear) flexural deformations have been assumed, the quadratic deflection terms in (3.1.16) and (3.1.17) can be deleted. Further, since the linearized problem is to be treated first, the $\dot{\theta}^2$ term can be ignored and (3.1.16) and (3.1.17) can be cast in the usual linear matrix form

$$\underline{\underline{M}}\,\ddot{\underline{\zeta}} + \underline{\underline{K}}\,\underline{\zeta} = U_T \begin{bmatrix} 1 & \underline{0}^T \end{bmatrix}^T \tag{3.2.1}$$

where

$$\underline{\zeta} = \begin{Bmatrix} \theta \\ \underline{\eta} \end{Bmatrix} \;, \quad \underline{\underline{M}} = \begin{bmatrix} I & \underline{M}_{\dot{\theta}\eta}^T \\ \underline{M}_{\dot{\theta}\eta} & \underline{\underline{M}}_{\eta\eta} \end{bmatrix} \;, \quad \underline{\underline{K}} = \begin{bmatrix} 0 & \underline{0}^T \\ \underline{0} & \underline{\underline{K}}_{\eta\eta} \end{bmatrix}$$

$\underline{\underline{M}} = \underline{\underline{M}}^T$ is a positive definite matrix, and $\underline{\underline{K}} = \underline{\underline{K}}^T$ is a positive semidefinite matrix (the null space corresponds to the rigid body mode).

Equation (3.2.1) represents the linearized coupled matrix equation of motion for the vehicle being considered. Since the coefficient mass and stiffness matrices are constant, it is then convenient to introduce a linear coordinate transformation which completely uncouple the equations of motion. As a consequence the optimal control problem is solved in modal space.

In order to uncouple the equation in (3.2.1) the following generalized eigenvalue problem must be solved:

$$\lambda_r^2\,\underline{\underline{M}}\,\underline{\xi}_r = \underline{\underline{K}}\,\underline{\xi}_r \tag{3.2.2}$$

where λ_r^2 represents the eigenvalue and $\underline{\xi}_r$ is the eigenvector, to obtain the modal matrix $\underline{\underline{E}} = \left[\underline{\xi}_1 \quad \underline{\xi}_2 \; \cdots\cdots \; \underline{\xi}_n\right]$ subject to the normalization

$$\underline{\underline{E}}^T \, \underline{\underline{M}} \, \underline{\underline{E}} = \underline{\underline{I}} \tag{3.2.3}$$

which leads to

$$\underline{\underline{E}}^T \, \underline{\underline{K}} \, \underline{\underline{E}} = \underline{\underline{\Lambda}}^2 = \text{diag.}\left[\lambda_1^2 \quad \lambda_2^2 \; \cdots\cdots \; \lambda_n^2\right] \tag{3.2.4}$$

Introducing the coordinate transformation

$$\underline{\zeta} = \underline{\underline{E}} \, \underline{t} \tag{3.2.5}$$

into (3.2.1) and premultiplying the resulting equation by $\underline{\underline{E}}^T$, while taking note of (3.2.3) and (3.2.4), leads to the uncoupled matrix equation

$$\ddot{\underline{t}} + \underline{\underline{\Lambda}}^2 \, \underline{t} = U_T \, \underline{V} \tag{3.2.6}$$

where $\underline{V} = \underline{\underline{E}}^T \left[1 \quad \underline{0}^T\right]^T = \left[E_{11} \quad E_{12} \; \cdots\cdots \; E_{1N}\right]^T$ and $N = n + 1$.

Defining the state variable subsets

$$\begin{aligned} \underline{s}_1 &= \underline{t} \\ \underline{s}_2 &= \dot{\underline{t}} \end{aligned} \tag{3.2.7}$$

leads to the first order differential equations

$$\begin{aligned} \dot{\underline{s}}_1 &= \underline{s}_2 \\ \dot{\underline{s}}_2 &= - \underline{\underline{\Lambda}}^2 \, \underline{s} + U_T \, \underline{V} \end{aligned} \tag{3.2.8}$$

Then letting $\underline{s} = \left|\underline{s}_1^T \quad \underline{s}_2^T\right|^T$ the state space form for the linearized differential equations become

$$\dot{\underline{s}} = \underline{\underline{A}} \, \underline{s} + \underline{\underline{B}} \, U_T \tag{3.2.9}$$

where $\underline{\underline{A}} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{I}} \\ -\underline{\underline{\Lambda}}^2 & \underline{\underline{0}} \end{bmatrix}$, $\underline{\underline{B}} = \begin{Bmatrix} \underline{0}' \\ \underline{V} \end{Bmatrix}$, $\underline{0}' = \begin{Bmatrix} 0 \\ \underline{0} \end{Bmatrix}$

For future reference the state equation for the nonlinear problem is summarized next. Only key equations are given since the derivation closely parallels that of (3.2.9). Proceeding from (3.1.16) and (3.1.17), and retaining the θ^2 nonlinearity, the transformed matrix equation of motion (see (3.2.6)) for the nonlinear problem is found to be

$$\ddot{\underline{t}} + \underline{\underline{\Lambda}}^2 \underline{t} = U_T \underline{V} - \alpha \, (\underline{V}^T \dot{\underline{t}})^2 \underline{\underline{L}} \, \underline{t} \; * \tag{3.2.10}$$

where $\dot{\theta} = \underline{V}^T \dot{\underline{t}}$

$$\underline{\underline{L}} = \underline{\underline{E}}^T \begin{bmatrix} 0 & \underline{0}^T \\ \underline{0} & \underline{\underline{M}} \end{bmatrix} \underline{\underline{E}}$$

Taking the same state variable definitions as in (3.2.6) leads to the nonlinear state equation

$$\dot{\underline{s}} = \underline{\underline{A}}(\underline{s}) \, \underline{s} + \underline{B} \, U_T \tag{3.2.11}$$

where $\underline{\underline{A}}(\underline{s}) = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{I}} \\ \underline{\underline{A}}_{21}(\underline{s}) & \underline{\underline{0}} \end{bmatrix}$ and $\underline{\underline{A}}_{21}(\underline{s}) = -\underline{\underline{\Lambda}}^2 - \alpha \, (\underline{V}^T \underline{s}_2)^2 \, \underline{\underline{L}}$.

3.3 OPTIMAL CONTROL PROBLEM

The problem considered here is the optimal rotational motions of a flexible space vehicle restricted to a single axis maneuver. The necessary conditions are derived from Pontryagin's

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* α is a relaxation parameter which has been artificially introduced into the problem in order to control the participation of the nonlinear term in a sequence of optimal control problems ($\alpha = 0$, linear problem (3.2.6) and $\alpha = 1$, the true nonlinear problem).

principle for the linear version of the problem in Section 3.3.1 and the nonlinear version in Section 3.3.2 . The same performance index J , is used for the linear and nonlinear problem. As in Chapter II the performance index is taken as a minimum torque criteria with the additional requirement that the modal space amplitudes and their rates be penalized during the entire maneuver.

In the spirit of attempting to retain the fidelity of the dynamic model (in the simplest manner possible) a conventional quadratic performance index is chosen which minimizes the sum of the weighted quadratic products of modal amplitudes, modal amplitude rates, and the applied control torque:

$$J = \frac{1}{2} \int_0^t \left[U_T \, W_{uu} \, U_T + \underline{s}^T \, \underline{\underline{W}}_{ss} \, \underline{s} \right] dt \tag{3.3.1}$$

where W_{uu} is a positive scalar weight on the control torque U_T and

$\underline{\underline{W}}_{ss}$ is a positive semidefinite weight matrix for the state variable $\underline{s}$.

One consequence of this choice for the performance index J is that the necessary conditions for the optimal maneuver simultaneously treats the problem of attitude control and structural (or configuration) control for the vehicle. Furthermore, it is assumed that the vehicle is to be in a quiescent state (with respect to elastic deformations) at the final time; therefore, at the final time t_f the modal amplitudes and their rates must be suppressed (i.e., there are to be no elastic deformations taking place with respect to an equilibrium configuration fixed relative to the rigid part of the body).

3.3.1 LINEAR PROBLEM

The problem is to seek a solution of (3.2.9), satisfying the prescribed terminal states given by:

$$\underline{\zeta}_0 = \begin{bmatrix} \theta_0 & \underline{\eta}^T(0) \end{bmatrix}^T \quad , \quad \dot{\underline{\zeta}}_0 = \begin{bmatrix} \dot{\theta}_0 & \dot{\underline{\eta}}^T(0) \end{bmatrix}^T \tag{3.3.2a}$$

and

$$\underline{\zeta}_f = \begin{bmatrix} \theta_f & \underline{\eta}^T(t_f) \end{bmatrix}^T \quad , \quad \dot{\underline{\zeta}}_f = \begin{bmatrix} \dot{\theta}_f & \dot{\underline{\eta}}^T(t_f) \end{bmatrix}^T \tag{3.3.2b}$$

where the requirement that $\underline{\eta}(t_f) = \dot{\underline{\eta}}(t_f) = \underline{0}$ is imposed on the right hand side of (3.3.2b).

The optimal control formulation seeks to determine the torque history $U_T(t)$ generating a solution of (3.2.9), subject to two criteria: First, the solution must satisfy the boundary conditions of (3.3.2); and Second, the solution must minimize the performance index (3.3.1) (as in the earlier problems of Chapter II , attention is restricted to a piecewise continuous torque history for U_T).

The Hamiltonian functional associated with minimizing (3.3.1) along trajectories of (3.2.9) is given by

$$H = \frac{1}{2} (U_T W_{uu} U_T + \underline{s}^T \underline{\underline{W}}_{ss} \underline{s}) + \underline{\Lambda}^T (\underline{\underline{A}} \underline{s} + \underline{B} U_T) \tag{3.3.3}$$

where the $\underline{\Lambda}$'s* are co-state variables associated with state variable $\underline{s}$.

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* $\underline{\Lambda} = \begin{bmatrix} \underline{\lambda}_1^T & \underline{\lambda}_2^T \end{bmatrix}^T$ where $\underline{\lambda}_1$ is the co-state variable corresponding to the transformed physical space angle (θ) and amplitudes ($\underline{\eta}$), and $\underline{\lambda}_2$ is the co-state variable corresponding to the transformed physical space angular rate ($\dot{\theta}$) and amplitude rates ($\dot{\underline{\eta}}$).

Pontryagin's principle requires as necessary conditions that the $\underline{\Lambda}$'s satisfy a co-state differential equation derivable from

$$\dot{\underline{\Lambda}} = - \frac{\partial H}{\partial s} = - \underline{W}_{ss} \, \underline{s} - \underline{\underline{A}}^T \, \underline{\Lambda} \tag{3.3.4}$$

In addition, Pontryagin's principle requires as a necessary condition for an arbitrary smooth unbounded admissible control that

$$\frac{\partial H}{\partial U_T} = \underline{0} = W_{uu} \, U_T + \underline{B}^T \, \underline{\Lambda} \tag{3.3.5}$$

from which the optimal torque U_T is determined as a function of the co-state variable as

$$U_T = - W_{uu}^{-1} \, \underline{B}^T \, \underline{\Lambda} \tag{3.3.6}$$

Introducing (3.3.6) into (3.2.9) yields the state and co-state differential equations defining the two-point-boundary-value problem as

(a) state equations:

$$\dot{\underline{s}} = \underline{\underline{A}} \, \underline{s} - \underline{B} \, W_{uu} \, \underline{B}^T \, \underline{\Lambda} \quad ; \quad \underline{s}(t_o) = \underline{s}_o \quad ; \quad \underline{s}(t_f) = \underline{s}_f \tag{3.3.7a}$$

(b) co-state equations:

$$\dot{\underline{\Lambda}} = - \underline{\underline{W}}_{ss} \, \underline{s} - \underline{\underline{A}}^T \, \underline{\Lambda} \tag{3.3.7b}$$

where (3.3.7) defines the necessary condition for the linearized problem. As expected, the co-state vector's initial and terminal conditions are unknown a priori, thus leading to a two-point-boundary-value problem.

3.3.2 NONLINEAR PROBLEM

Recalling the state equation for the nonlinear problem (3.2.11), and defining the optimal control problem exactly as is done for the linear case, leads to the following state and co-state necessary conditions for the nonlinear problem:

(a) state equation:

$$\dot{\underline{s}} = \underline{\underline{A}}(\underline{s})\ \underline{s} - \underline{B}\ W_{uu}^{-1}\ \underline{B}^{T}\ \underline{\Lambda} \tag{3.3.8a}$$

(b) co-state equation:

$$\dot{\underline{\Lambda}} = -\ \underline{\underline{W}}_{ss}\ \underline{s} - \underline{\underline{C}}(\underline{s})\ \underline{\Lambda} \tag{3.3.8b}$$

where

$$\underline{\underline{C}}(\underline{s}) = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{A}}_{21}^{T}(\underline{s}) \\ \underline{\underline{I}} & -2\alpha(\underline{V}^{T}\underline{s}_{2})\underline{V}\underline{s}_{1}^{T}\underline{\underline{L}}^{T} \end{bmatrix}$$

3.4 DIFFERENTIAL EQUATION RELAXATION PROCESS FOR THE SOLUTION OF THE TWO-POINT-BOUNDARY-VALUE PROBLEM

As the title of this section suggests, the relaxation process being described alters the structure of the differential equations at each stage of the relaxation process. To focus the ideas for altering the structure of the differential equations, one can assume the following general expression for a vector-matrix differential equation

$$\dot{\underline{X}}(t) = \underline{\underline{A}}(t)\ \underline{X}(t) + \underline{B}(\underline{X}(t),t) \tag{3.4.1}$$

where t represents the independent variable time, $\underline{X}(t)$ represents some vector function to be integrated, $\underline{\underline{A}}(t)$ represents a linear operator which acts on $\underline{X}(t)$, and $\underline{B}$ denotes the generally nonlinear combinations of $\underline{X}(t)$.*

By artificially introducing the relaxation parameter α, into (3.4.1) one obtains

$$\begin{aligned}\dot{\underline{X}}(t;\alpha) &= \underline{F}(\underline{X}(t;\alpha),t;\alpha)\\ &= \underline{\underline{A}}(t)\ \underline{X}(t;\alpha) + \alpha\ \underline{B}(X(t;\alpha),t) \qquad (3.4.2)\end{aligned}$$

where α is usually introduced as a multiplier times the nonlinear terms in $\underline{F}$.

On setting $\alpha = 0$ in (3.4.2) one recovers the set of linear differential equations

$$\begin{aligned}\dot{\underline{X}}(t;0) &= \underline{F}(\underline{X}(t;0),t;0)\\ &= \underline{\underline{A}}(t)\ \underline{X}(t) \qquad (3.4.3)\end{aligned}$$

and setting $\alpha = 1$ in (3.4.2) one obtains the nonlinear equations of (3.4.1); i.e.,

$$\begin{aligned}\dot{\underline{X}}(t;1) &= \underline{F}(\underline{X}(t;1),t;1)\\ &= \underline{\underline{A}}(t)\ \underline{X}(t) + \underline{B}(\underline{X}(t),t) \qquad (3.4.4)\end{aligned}$$

Since (3.4.4) can be obtained from (3.4.3) in a continuous way by varying α from 0 to 1, one can see that the solution of (3.4.3) is imbedded in the space containing the solution for (3.4.2). Hence, the idea is to start with the solution of (3.4.3) and then march via the relaxation process until the solution for (3.4.4) is obtained. The motivation for considering this approach comes about from the

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* The problem being treated here is a prototype state and co-state two-point-boundary-value problem, where initial conditions are known for half the state vector $\underline{X}(t)$, and the remaining half of the initial conditions are to be determined; subject to satisfying the known boundary conditions of $\underline{X}(t)$ at the final time.

observation that the single axis flexible body state and co-state equations are a nonlinear perturbation of the corresponding linear problem.

For the system of linear equations in (3.4.3) many methods of solution exist. Once having obtained a solution one can then increase α in (3.4.2) such that a local linearization about the initial co-state boundary conditions is valid. Then one can introduce a differential correction algorithm to iteratively refine the initial co-state approximations subject to the satisfaction of the problem terminal boundary conditions. Continuing in this fashion, by increasing α , this process permits the known solution of (3.4.3) to be marched in the direction of the real problem of interest. The number of intermediate α values required is problem dependent, but typically less than 5 have been required.

In Chapter II the relaxation process at each stage α_n solves a problem whose boundary conditions are closer to the real problem of interest; whereas in this chapter the relaxation process at each stage α_n solves the problem closer to the real problem of interest, in terms of the structure of the differential equations.

Before developing the differential correction algorithm for the relaxation process, the initial co-states for the linear problem are obtained first. Referring to (3.3.2) and (3.3.7), one has the standard form for two-point-boundary-value problem, where the terminal conditions for $\underline{s}$ are known initially and finally, but where all boundary conditions for $\underline{\Lambda}$ are unknown. Since the equations of (3.3.7) are linear one can obtain a state transition matrix solution for $\underline{s}$ and $\underline{\Lambda}$ as follows: First, defining the merged state vector

$$\underline{X} = \left[\underline{s}^T \quad \underline{\Lambda}^T\right]^T \qquad (3.4.5)$$

then (3.3.7) becomes

$$\dot{\underline{X}} = \underline{\underline{\Omega}}\ \underline{X} \qquad (3.4.6)$$

where $\underline{\underline{\Omega}} = \begin{bmatrix} \underline{\underline{A}} & -\underline{B}\underline{W}_{uu}^{-1}\underline{B}^T \\ -\underline{\underline{W}}_{ss} & -\underline{\underline{A}}^T \end{bmatrix}$ is a constant coefficient matrix.

Since $\underline{\underline{\Omega}}$ is constant, it is well known that (3.4.6) possesses the solution:

$$\underline{X}(t) = \underline{\underline{\Phi}}(t,0)\ \underline{X}(0) \qquad (3.4.7)$$

where $\dot{\underline{\underline{\Phi}}}(t,0) = \underline{\underline{\Omega}}\ \underline{\underline{\Phi}}(t,0)$; $\underline{\underline{\Phi}}(0,0) = \underline{\underline{I}}^*$, and $\underline{\underline{\Phi}}$ is the 4N by 4N state transition matrix.

Expanding (3.4.7), one obtains

$$\begin{Bmatrix} \underline{s}(t_f) \\ \underline{\Lambda}(t_f) \end{Bmatrix} = \begin{bmatrix} \underline{\underline{\Phi}}_{ss} & \underline{\underline{\Phi}}_{s\Lambda} \\ \underline{\underline{\Phi}}_{\Lambda s} & \underline{\underline{\Phi}}_{\Lambda\Lambda} \end{bmatrix} \begin{Bmatrix} \underline{s}(0) \\ \underline{\Lambda}(0) \end{Bmatrix} \qquad (3.4.9)$$

and upon carrying out the partitioned matrix multiplication, one finds for $\underline{s}(t_f)$

$$\underline{s}(t_f) = \underline{\underline{\Phi}}_{ss}\ \underline{s}(0) + \underline{\underline{\Phi}}_{s\Lambda}\ \underline{\Lambda}(0) \qquad (3.4.10)$$

which can be solved for $\underline{\Lambda}(0)$ as

$$\underline{\Lambda}(0) = \underline{\underline{\Phi}}_{s\Lambda}^{-1}\left[\underline{s}(t_f) - \underline{\underline{\Phi}}_{ss}\ \underline{s}(0)\right] \qquad (3.4.11)$$

where $\underline{s}(0)$ and $\underline{s}(t_f)$ are given by

$$\underline{s}(0) = \left[\{\underline{\underline{E}}^T\ \underline{\underline{M}}\ \underline{\zeta}(0)\}^T \quad \{\underline{\underline{E}}^T\ \underline{\underline{M}}\ \dot{\underline{\zeta}}(0)\}^T\right]^T \qquad (3.4.12a)$$

---

* A closed form solution is presented for $\underline{\underline{\Phi}}$ in Appendix B when a linear eigenvalue analysis can determine the right and left eigenvectors of $\underline{\underline{\Omega}}$.

$$\underline{s}(t_f) = \left[\{\underline{\underline{E}}^T \underline{\underline{M}}\ \underline{\zeta}(t_f)\}^T \quad \{\underline{\underline{E}}^T \underline{\underline{M}}\ \underline{\zeta}(t_f)\}^T\right]^T \tag{3.4.12b}$$

Thus, with (3.4.11) and (3.4.7) solution for the linear problem is now complete.

The fact that the solution for the initial co-states in (3.4.11) can be obtained in a closed form algorithm provides the principal motivation for introducing the differential equation relaxation process. This process iteratively refines the estimates for the initial co-states in the marching process that begins with the linear problem ($\alpha = 0$), and ends with the nonlinear problem ($\alpha = 1$). These considerations thus motivate the following successive approximation strategy to solve the two-point-boundary-value problem.

Letting the approximate initial co-states be denoted by

$$\hat{\underline{\Lambda}}(t_o) = \left[\hat{\underline{\lambda}}_1^T(t_o) \quad \hat{\underline{\lambda}}_2^T(t_o)\right]^T \ , \ \hat{\underline{\lambda}}_i = \left[\hat{\lambda}_{i1} \quad \hat{\lambda}_{i2} \ \ldots\ldots \ \hat{\lambda}_{iN}\right]^T \ ; \ i=1,2 \tag{3.4.13}$$

the differential correction strategy is to seek the correction vector $\underline{\Delta\Lambda}$ subject to the terminal constraint

$$\underline{s}_f - \underline{s}\ (\hat{\underline{\Lambda}}(0)+\underline{\Delta\Lambda}, t_f) = 0 \ ^* \tag{3.4.14}$$

where $\underline{\Delta\Lambda} = \left[\underline{\Delta\lambda}_1^T \ , \ \underline{\Delta\lambda}_2^T\right]$

On linearizing (3.4.14), one obtains

$$\underline{s}_f - \hat{\underline{s}} - \underline{\underline{A}}_{s\Lambda}\ \underline{\Delta\Lambda} = \underline{0} \tag{3.4.15}$$

where $\hat{\underline{s}}$ denotes the solution of (3.3.7) for some specific choice of

---

* The terminal constraint satisfaction is enforced in the transformed $\underline{s}$ space.

$\underline{\Lambda}(0)$, and

$$\underline{\underline{A}}_{s\Lambda} = \left[\frac{\partial \underline{s}^T}{\partial \underline{\Lambda}(0)} \bigg|_f \right]^T = \underline{\underline{\Phi}}_{s\Lambda} \tag{3.4.16}$$

$\underline{\underline{\Phi}}_{s\Lambda}$ denotes the block of the state transition matrix for the variables $\underline{s}$ and $\underline{\Lambda}$.

Now defining

$$\underline{\Delta s}_f = \underline{s}_f - \hat{\underline{s}} \tag{3.4.17}$$

and introducing (3.4.17) into (3.4.15) leads to

$$\underline{\Delta s}_f = \underline{\underline{\Phi}}_{s\Lambda} \, \underline{\Delta\Lambda} \tag{3.4.18}$$

where the solution for $\underline{\Delta\Lambda}$ can be obtained from (3.4.18) using a variety of methods; e.g., one could explicitly compute $\underline{\underline{\Phi}}_{s\Lambda}^{-1}$ or one could use a linear equation solver for $\underline{\Delta\Lambda}$.*

For large order systems the second solution path becomes the most attractive method since algorithms for linear equation solvers are more numerically stable than matrix inversion algorithms.

The calculation of the state transition matrix partial derivatives is a separate issue, dealt with in Appendix B.

Equation (3.4.18) completes the derivation of all results required in the differential equation relaxation process diagrammed in Figure 3.2 . The only significant assumption enroute to the algorithm is the local linearization of (3.4.14) to obtain (3.4.15). Based on several numerical experiments, the following sequence of

---

* Several schemes exist for the solution of $\underline{\Delta\Lambda}$ which depend upon partitioning $\underline{\underline{\Phi}}_{s\Lambda}$ (see equation (40) of [42]; in fact this equation is implemented in the existing computer code).

Given: $\underline{s}(0)$, $\underline{s}(t_f)$

Approximate $\underline{\Lambda}(0)$ (3.4.11)

A

Increment α

B

Solve the differential equations of (3.3.11) to determine

$$\hat{\underline{s}} = \underline{s}(\hat{\underline{\Lambda}}(0), t_f)$$

and determine (using method of Appendix B) the partial derivatives of (3.4.15)

Calculate the residual vector $\Delta\underline{s}_f$

If small and $\alpha = 1$ $\longrightarrow$ End

If small and $\alpha \neq 1$ $\longrightarrow$ A

If large, continue

Calculate $\Delta\underline{\Lambda}$ from (3.4.18)

Apply corrections

$$\hat{\underline{\Lambda}}(0) = \hat{\underline{\Lambda}}(0) + \Delta\underline{\Lambda}$$

B

Figure 3.2: Differential Correction Algorithm for the Single Axis Problem.

relaxation parameters (α_n)

$$\left.\begin{aligned} \alpha_1 &= 0.001 \\ \alpha_n &= \frac{n}{N}\ ; \ n=2,\ldots\ldots,N \end{aligned}\right\} \qquad (3.4.19)$$

have provided reliable convergence; where N was taken to be 4 or 5 . In fact, the discussion following (2.1.17) remains valid for this new relaxation process, regarding the possibility of having the incremental change in α_n determined by some dynamic process. These relaxation processes differ in that one deals with equations and the other deals with boundary conditions.

After completing the solution for the first relaxation stage, starting estimates for the initial co-states are obtained from the first order Taylor expansion

$$\hat{\underline{P}}_n = \underline{P}_{n-1} + (\alpha_n - \alpha_{n-1})\left(\frac{d\underline{P}}{d\alpha}\bigg|_{n-1}\right) \qquad (3.4.20)$$

where

$$\underline{P}_n = \underline{\Lambda}_n(t_o) \qquad (3.4.21)$$

and the finite difference derivative of the co-state vector with respect to the relaxation parameter α is given by

$$\left(\frac{d\underline{P}}{d\alpha}\bigg|_{n-1}\right) = \left\{\begin{array}{ll} \left(\dfrac{1}{\alpha_{n-1} - \alpha_{n-2}}\right)\{\underline{P}_{n-1} - \underline{P}_{n-2}\}\ , & \text{for } n > 1 \\ \{0\} & ,\ \text{for } n = 0 \end{array}\right\} \qquad (3.4.22)$$

where the $\underline{P}_n$ vectors are the converged co-state vectors (3.4.21) resulting from previous application of the algorithm of Figure 3.2 .

Thus, each incremental change in the structure of the governing differential equation is supported by a linear extrapolation of the co-state initial conditions via (3.4.20), followed by a differential correction refinement using the algorithm of Figure 3.2 .

In summary, the relaxation process is started using the initial co-states obtained from (3.4.11). Then choosing the relaxation parameter α_n to alter the structure of the governing differential equations such that a local linearization (3.4.14) is valid. Then the solution of (3.4.18) provides the means to iteratively refine the initial co-state extrapolation of (3.4.20) until the terminal boundary conditions of (3.4.12) are adequately satisfied. The process above continues until $n = N$ in (3.4.19), at which time the real problem of interest has been solved.

3.5 NUMERICAL RESULTS

In this section several numerical examples are given for rest-to-rest and spin-up maneuvers assuming various dynamical models for the flexible spacecraft. For all maneuvers considered the following system parameters are assumed: I is the inertia of the undeformed structure about the spin axis, which is equal to 7,000 $kg.m^2$; ρ is the mass per unit length of the four identical elastic appendages, which is equal to $0.4x10^{-3}$ kg/m; L is the length of each undeformed cantilevered appendage, which is equal to 150 m; EI is the flexural rigidity of the elastic appendages, which is equal to 1,500 $kg.m/s^2$; and the vehicle configuration is that of Figure 3.1, (a).

In the assumed modes expansion of the elastic displacements (3.1.12) the following admissible assumed mode shapes for $\phi_r(x)$ are adopted

$$\phi_r(x) = \left(\frac{x}{L}\right)^{r+1} \quad , \; r=1,2,\ldots\ldots,\infty \tag{3.5.1}$$

which satisfy the geometric boundary conditions of a clamped-free appendage. The upper limit for r in (3.5.1) in all cases considered, has been set to $r_{max} = 4$.

Introducing (3.5.1) into (3.1.13) and carrying out the integration over the appendage length (ignoring the hub radius in comparison to the appendage length), leads to

$$\left[\underline{\underline{M}}_{\eta\eta}\right]_{k\ell} = \frac{4\ \rho\ L}{k + \ell + 3} \tag{3.5.2a}$$

$$\left[\underline{\underline{M}}_{p}\right]_{k\ell} = 4\ \rho\ L\ \frac{(k + 1)(\ell + 1)}{(k + \ell + 1)(k + \ell + 3)} \tag{3.5.2b}$$

$$\left[\underline{\underline{K}}_{\eta\eta}\right]_{k\ell} = \frac{4\ EI}{L^3}\ \frac{k\ \ell\ (k + 1)(\ell + 1)}{k + \ell - 1} \tag{3.5.2c}$$

$$\left[\underline{\underline{M}}\right]_{k\ell} = \frac{4\ \rho\ L\ k\ \ell}{(k + \ell + 1)(k + \ell + 3)} \tag{3.5.2d}$$

$$\left[\underline{M}_{\dot{\theta}\eta}\right]_{k} = \frac{4\ \rho\ L^2}{k + 3} \tag{3.5.2e}$$

With reference to Table 3.1 , seven models for the system dynamics are considered with N = 0 , 1 , 2 , and 4 assumed modes. For the flexible cases both the linear ($\alpha = 0$) and nonlinear ($\alpha = 1$) solutions are presented. For the nonlinear cases, the two-point-boundary-value problem is solved by relaxing α from zero to unity with a finite number of intermediate values (the largest number of relaxation steps is 5 , required for Case-3N). For each α value,

Table 3.1: Description of Test Case Maneuvers.

Case #	Qualitative Description	# of Modes (N)	θ_o (rad)	$\dot{\theta}_o$ (rad/sec)	θ_f (rad)	$\dot{\theta}_f$ (rad/sec)	W_{uu}	W_{ss}
1	Rigid Appendages Rest-to-Rest Maneuver $t_f = 14.221$ sec	0	0	0	0.1	0	1.0	$[0]$
2L	Linear Kinematics Rest-to-Rest Maneuver $t_f = 2\pi/\omega_1 = 14.221$ sec	1	0	0	0.1	0	1.0	$[I]$
2N	Nonlinear Kinematics Rest-to-Rest Maneuver $t_f = 2\pi/\omega_1 = 14.221$ sec	1	0	0	0.1	0	1.0	$[I]$
3L	Linear Kinematics Spin-Up Maneuver $t_f = 60$ sec	2	0	0	2π	0.5	1.0	$[I]$
3N	Nonlinear Kinematics Spin-Up Maneuver $t_f = 60$ sec	2	0	0	2π	0.5	1.0	$[I]$
4L	Linear Kinematics Rest-to-Rest Maneuver $t_f = 60$ sec	4	0	0	π	0	1.0	$[I]$
4N	Nonlinear Kinematics Rest-to-Rest Maneuver $t_f = 60$ sec	4	0	0	π	0	1.0	$[I]$

co-state extrapolations are obtained via (3.4.16) from converged neighboring solutions; following this approach, the algorithm of Figure 3.2 reliably converged requiring an average of three differential corrections.

3.6 DISCUSSION

In this section the numerical examples of Section 3.5 are discussed qualitatively for the various maneuvers presented in Table 3.1 .

Note for Case-2L , that the torque required (Figure 3.4 - (c)) to carry out the rigid rotation and arrest the terminal amplitude and amplitude rate is anti-symmetric with respect to the time axis about the midpoint time for the maneuver. Further, the control torque of Figure 3.4 - (c) oscilates about the straight line torque of Figure 3.3 - (c), for the rigid case. The anti-symmetric torque history is a direct consequence of the fact that the specified final time is equal to the period of the first mode. For the maneuvers shown in Figure 3.5 to Figure 3.7, several modes are controlled where the natural frequencies of the modes are not commensurable with themselves or the final time; the result being that the control torque is generally asymmetric. Also of significance is the fact that the linear (Case-2L) and the nonlinear (Case-2N) solutions are identical to plotting accuracy with only slight variations occuring in the fourth significant digit of the state variables.

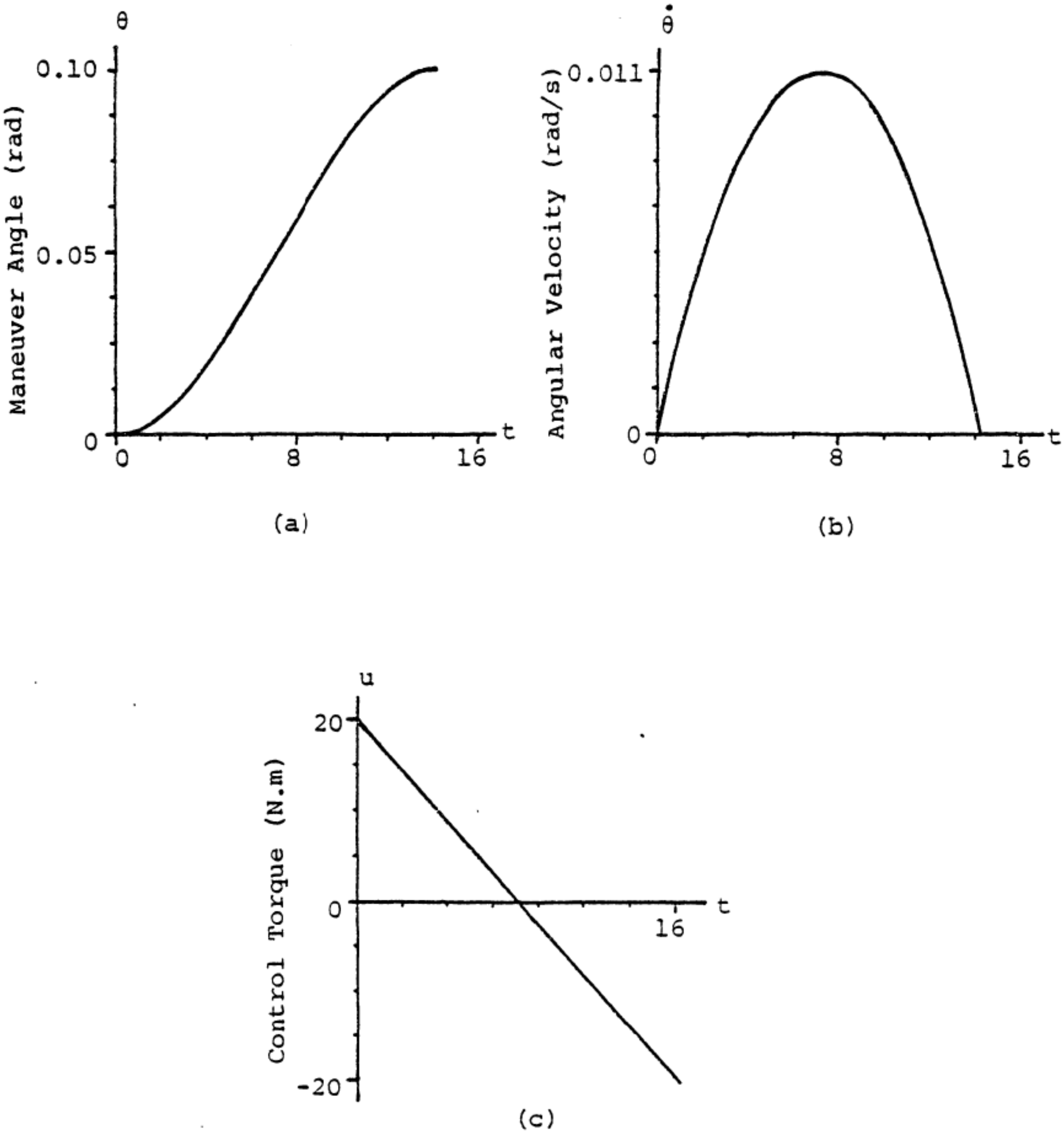

Figure 3.3: Case-I - Rigid Body Solution.

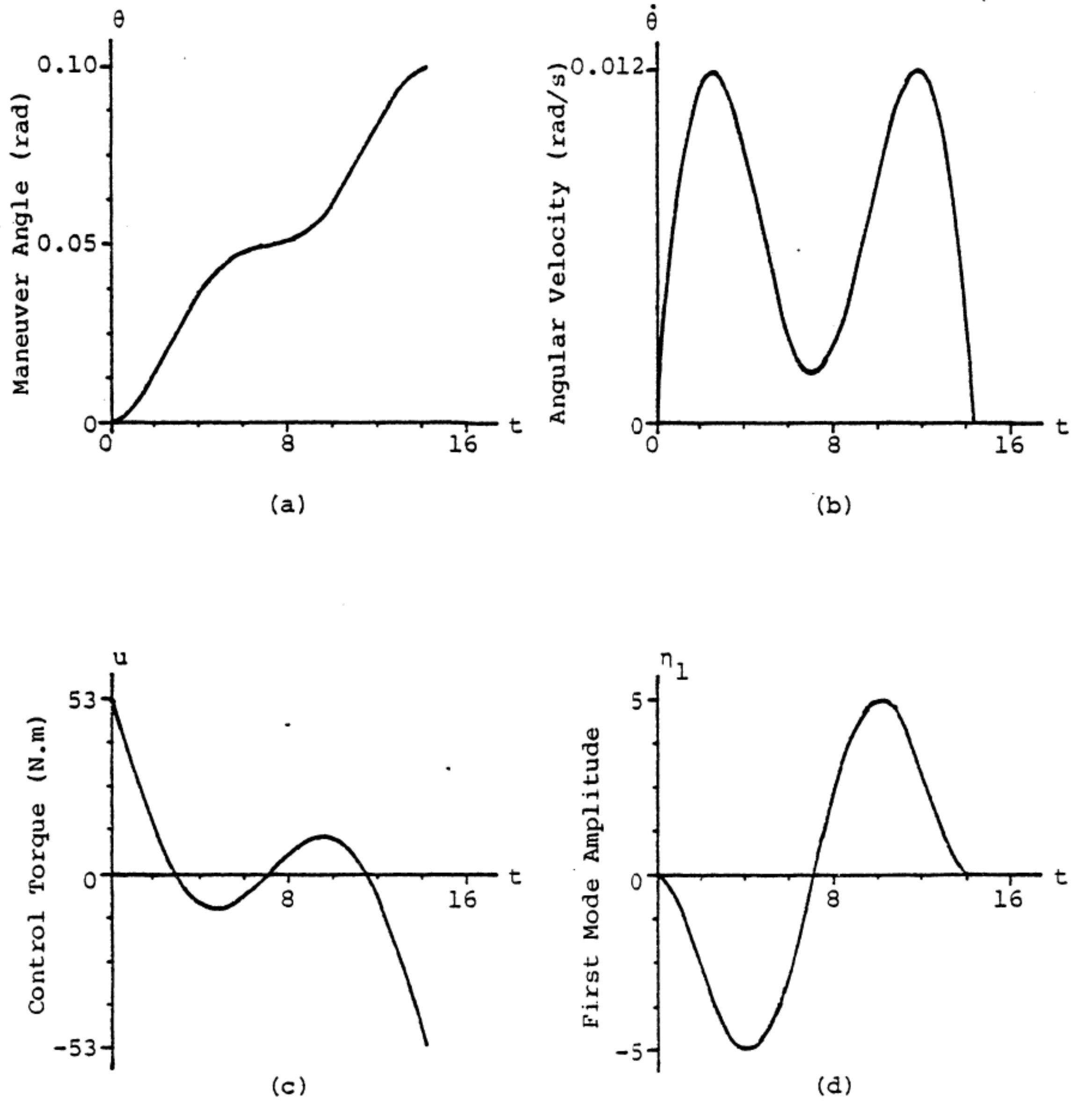

Figure 3.4: Cases-2L and -2N - Flexible Appendage Rest-to-Rest Maneuver $t_f = 2\pi/\omega_1 = 14.221$ seconds.

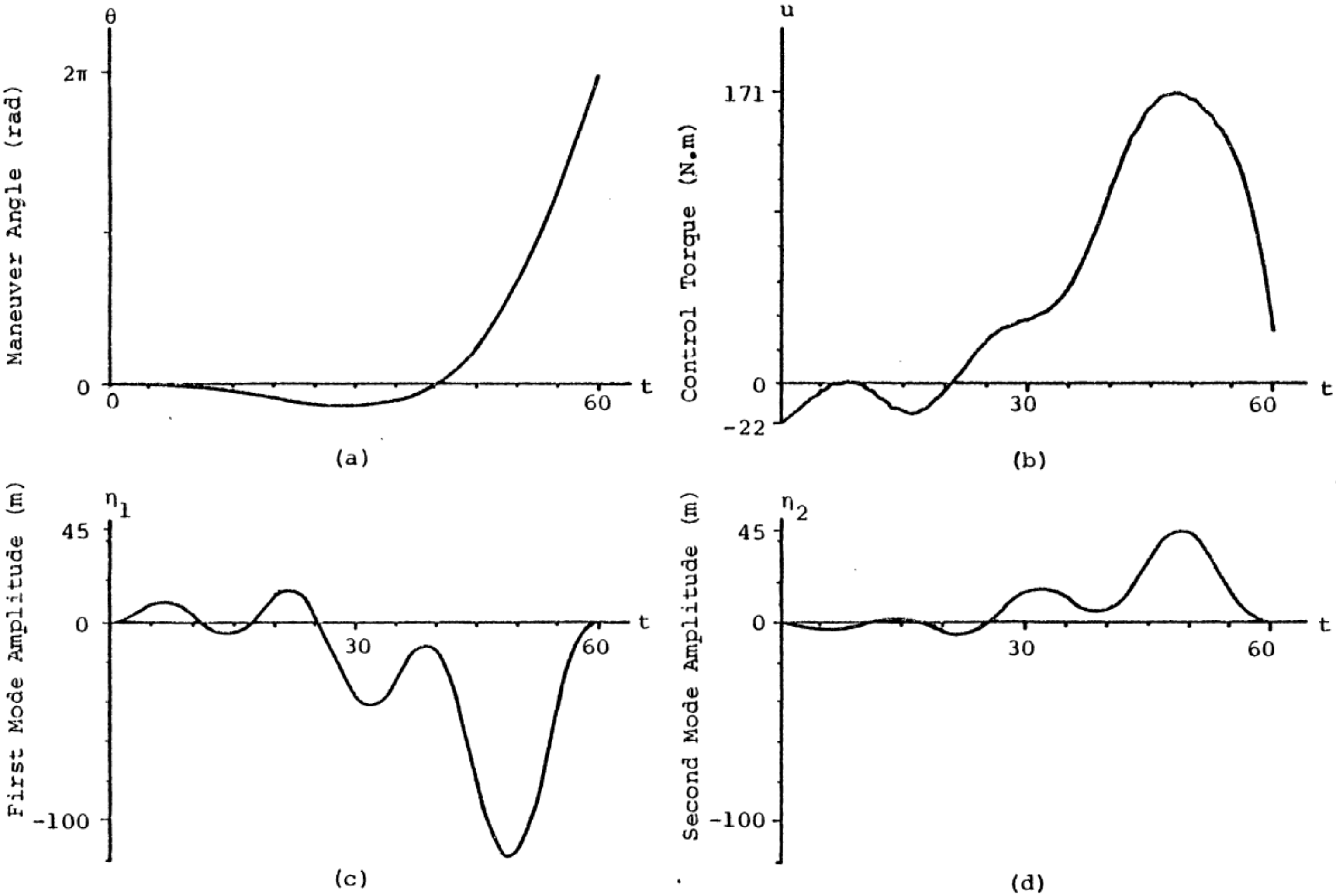

Figure 3.5: Case-3L - Linear Spin-Up Maneuver.

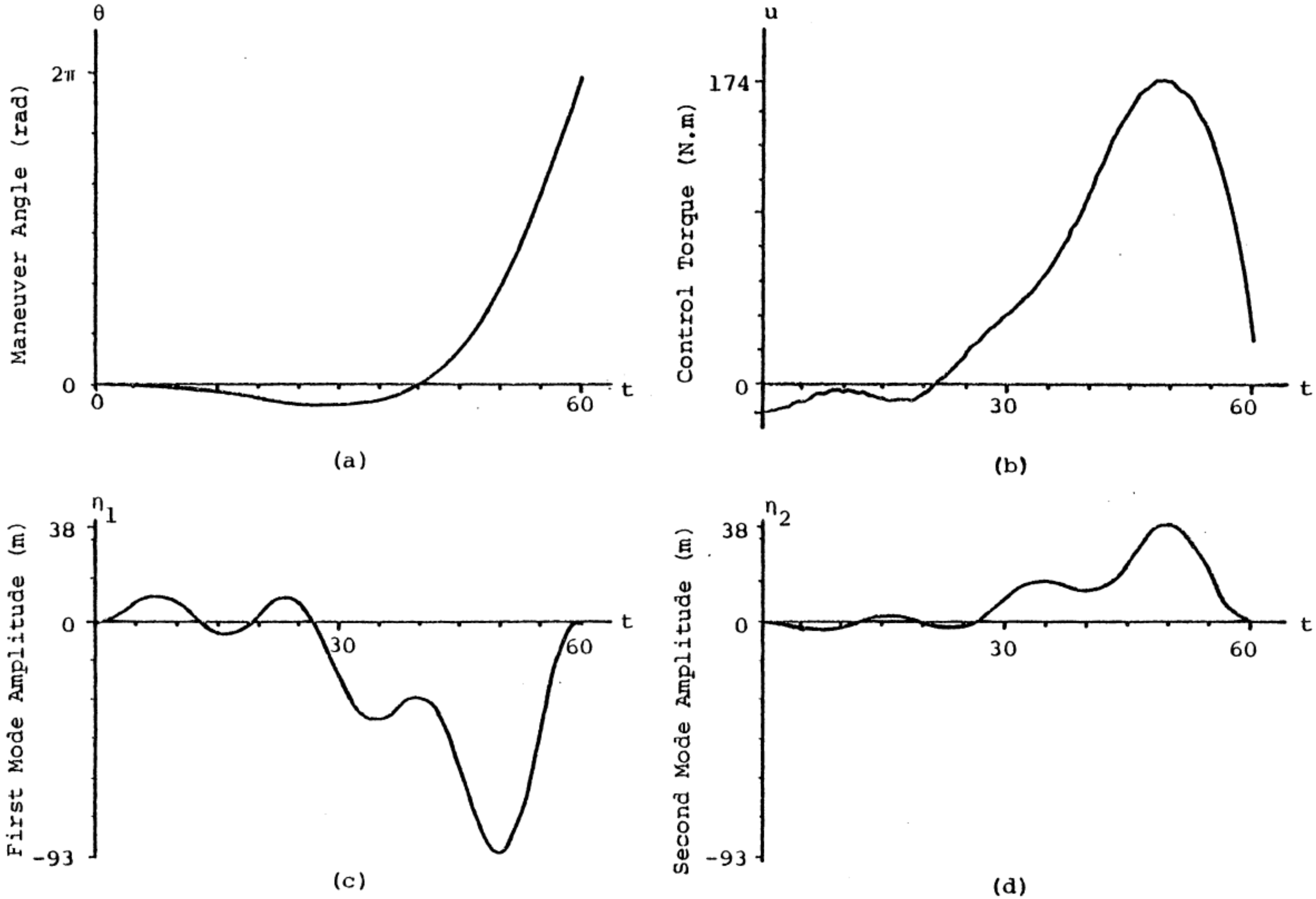

Figure 3.6: Case-3N - Nonlinear Spin-Up Maneuver.

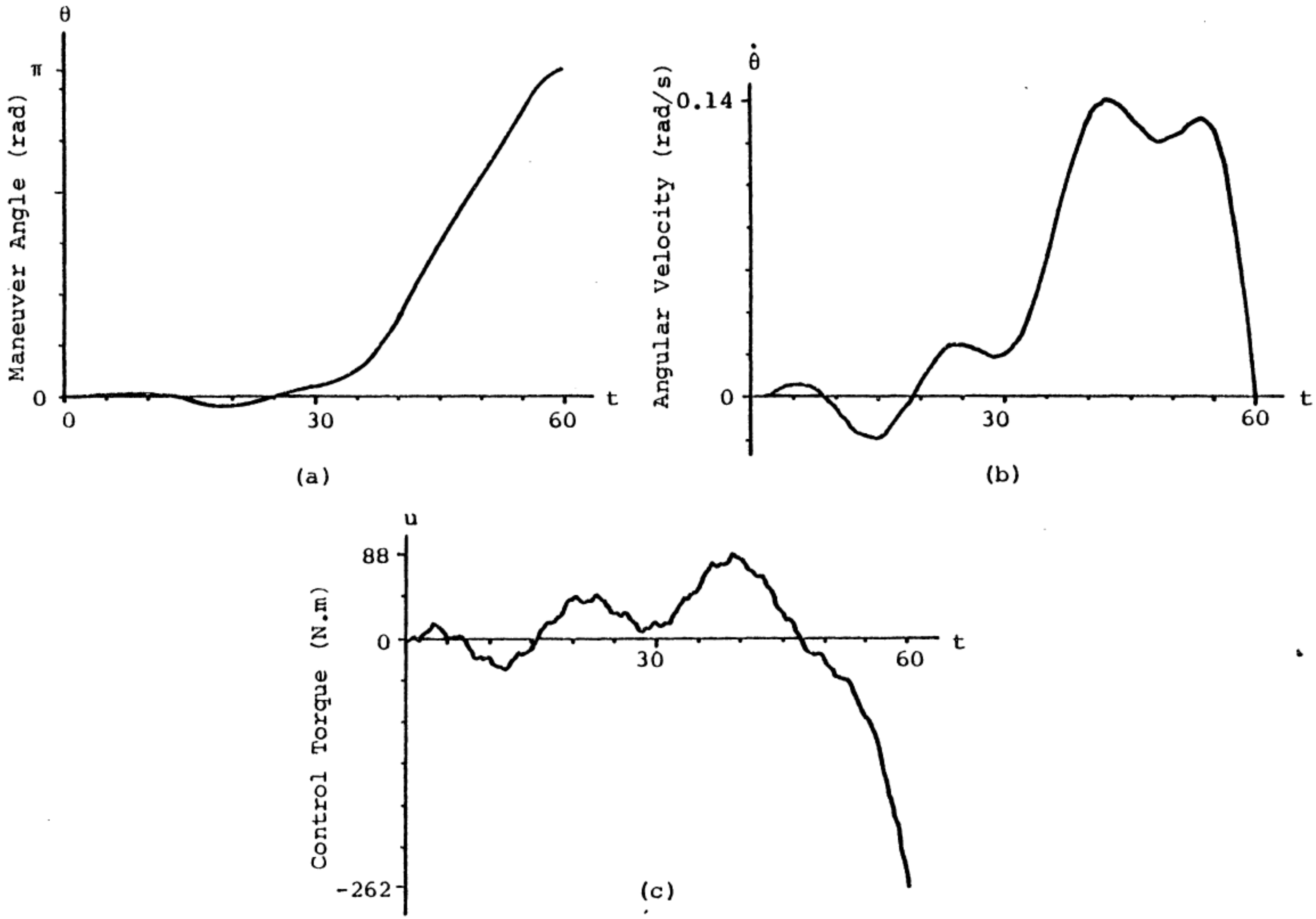

Figure 3.7: Cases-4L and -4N - Rest-to-Rest Maneuver Controlling and Arresting Four Modes.

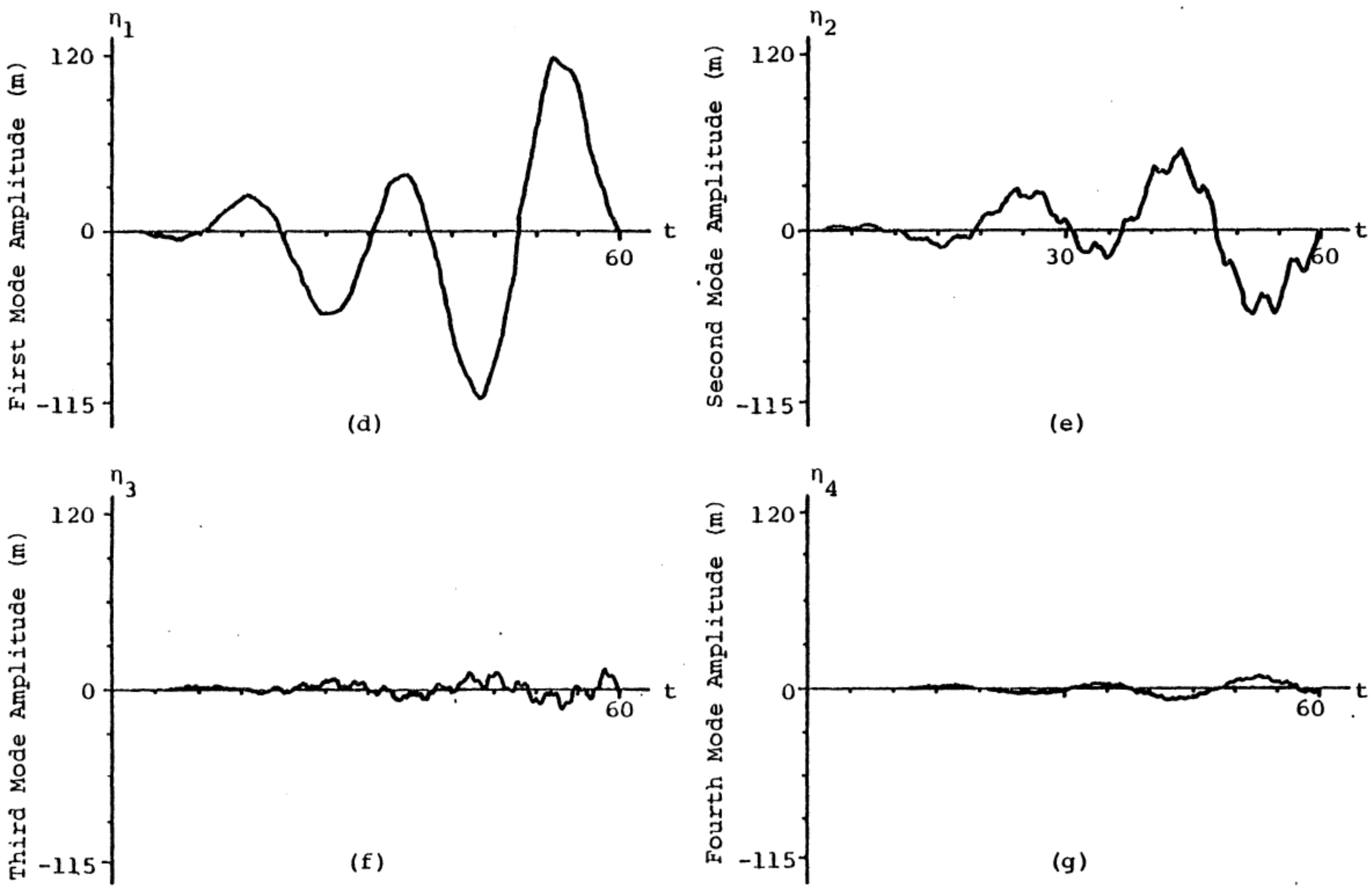

Figure 3.7 (continued): Cases-4L and -4N - Rest-to-Rest Maneuver Controlling and Arresting Four Modes.

For all maneuvers shown the modal amplitude rates are not displayed, though in all cases they had terminal values of 10^{-7} m/s or smaller.

Cases-3L and -3N deal with a spin-up maneuver, where the torque and principle angle undergo the reversal phenomena discussed in Section 2.1.6 for the rigid body. In this case it can be seen that the linear (Case-3L) and the nonlinear (Case-3N) solutions depict significant differences, which can be observed in both the shape and scale changes for control torque and modal amplitudes. The nonlinear $\dot{\theta}^2$ term serves to reduce the amplitude of vibration, which is intuitively consistent with the centrifugal stiffening effect.

In Cases 4L and 4N (Figure 3.7) demonstrates the capacity of the formulation to simultaneously control and arrest several (4) modes and accomplish a large rotational maneuver (180°). As in the Cases-2L and -2N the linear and nonlinear solutions of Cases-4L and -4N typically agreed to three digits in the state variables. Since only Cases-3L and -3N processed high angular rates, this leads one to conclude that, in applications, "slow" maneuvers of the configuration of Figure 3.1 may require only linear coupling between the structural and rotational degrees of freedom.

In each of the nonlinear examples above, within each relaxation stage, an average of three differential corrections have been required. The rigorous partial derivatives of the state transition matrix are computed and used for the first differential correction in each relaxation state, with approximate partial derivatives being generated for the successive differential corrections by the method in Appendix C .

It is of interest to point out in (3.5.2d) that if the arc-length constraint of (3.1.6) is not imposed, that the resulting differential equations for the modal amplitudes are unstable for high rotation rates. This conclusion follows from the observation that the matrix M consists of all negative elements; which for high angular rates leads to the possibility that the matrix $K_{\eta\eta} + \dot{\theta}^2 M$ could possess negative elements, leading to a destabilizing effect for the modes involved. Physically, one expects that increasing the rotation rates leads to a stiffer system, due to centrifugal stiffening effects. For stiff near-linear systems this effect is not of great importance; but, when large angular rates are involved or the appendage flexibility is significant, an order of magnitude analysis dictates that this effect must be taken into account.

Based on the success of the relaxation process in extracting solutions for the nonlinear examples of Table 3.1, it is anticipated that the differential equation imbedding/analytic continuation process of this chapter will find wide applications as a tool for solving a broad class of optimal control problems.

This concludes the treatment of the optimal single axis flexible body large angle attitude control problem. In Chapter IV the necessary conditions are determined for the nonlinear three-dimensional flexible body optimal attitude control problem, and a relaxation process proposed for the solution of the necessary conditions which employs simultaneously the relaxation processes presented in Chapters II and III .

In a recent paper by Breakwell [9], prepared simultaneously to and independently of [42], considers the linear case ($\alpha = 0$) and obtains essentially identical analytical results and analogous numerical results * for that special case. In addition to the control formulation, Breakwell reports that laboratory experiments have been conducted which verify the linear control formulation.

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* It is important to note in these formulations that long maneuver times encounter numerical difficulties due to the co-states growing to large values. This weak instability of the adjoint system, while not serious in the present calculations, may be a limitation in other applications.

CHAPTER IV

THREE-DIMENSIONAL FLEXIBLE BODY OPTIMAL CONTROL

This chapter is concerned with the attitude and configuration control of large flexible spacecraft during large angle rotational maneuvers of these vehicles. The necessary conditions for these maneuvers to be optimal are obtained from Pontryagin's principle. The flexible structure is assumed to be linearly elastic; hence, the structure is assumed to undergo only small deformations, while the rigid body motion can be completely arbitrary. The flexural deformations are modeled by the "assumed modes" method.

The methods presented in this chapter assume that an arbitrary continuous torque is applied on a rigid part of the structure, for simultaneously reorienting the vehicle and controlling the elastic deformations during the maneuver. At the end of the maneuver, as part of the problem's boundary conditions, the vibratory motion of the elastic appendages is required to be suppressed.

The equations of motion are obtained in Section 4.1 . In Section 4.2 the state space form for the equations of motion is established. The optimal control problem is defined and the necessary conditions are obtained from Pontryagin's principle in Section 4.3 , where only the nonlinear problem is given treatment. A relaxation method is proposed in Section 4.4 , which combines the terminal boundary condition method of Section 2.4 and the differential equation method of Section 3.4 . In Section 4.4 , a discussion is presented which identifies the stumbling block to the solution of this problem.

4.1 EQUATIONS OF MOTION

The motion of the vehicle being modeled is described by a system of hybrid coordinates, using a combination of discrete coordinates (for translations and rotations of rigid bodies) and distributed or modal coordinates (for the deformations of elastic bodies) [28].

The specific model considered in this chapter (see Figure 4.1) consists of a rigid hub with four identical elastic appendages, attached symmetrically about the central hub. Both in- and out-of-plane deformations of the flexible members are considered.

As in Chapter III, the procedure used for analyzing these hybrid systems is a discretization, whereby the displacements of the continuous elastic members are replaced by a finite series; in which prescribed (admissible) space-dependent functions are multiplied by time-dependent (to be determined) generalized coordinates and summed, to model the instantaneous deflections of the structure [34].

The equations of motion are obtained using Newton's law for the vehicle's angular velocity and the Lagrangian formulation for the generalized coordinates representing the vehicle's elastic degrees of freedom.

Using the notation of Appendix A the angular momentum of the vehicle about its instantaneous center of mass can be written as

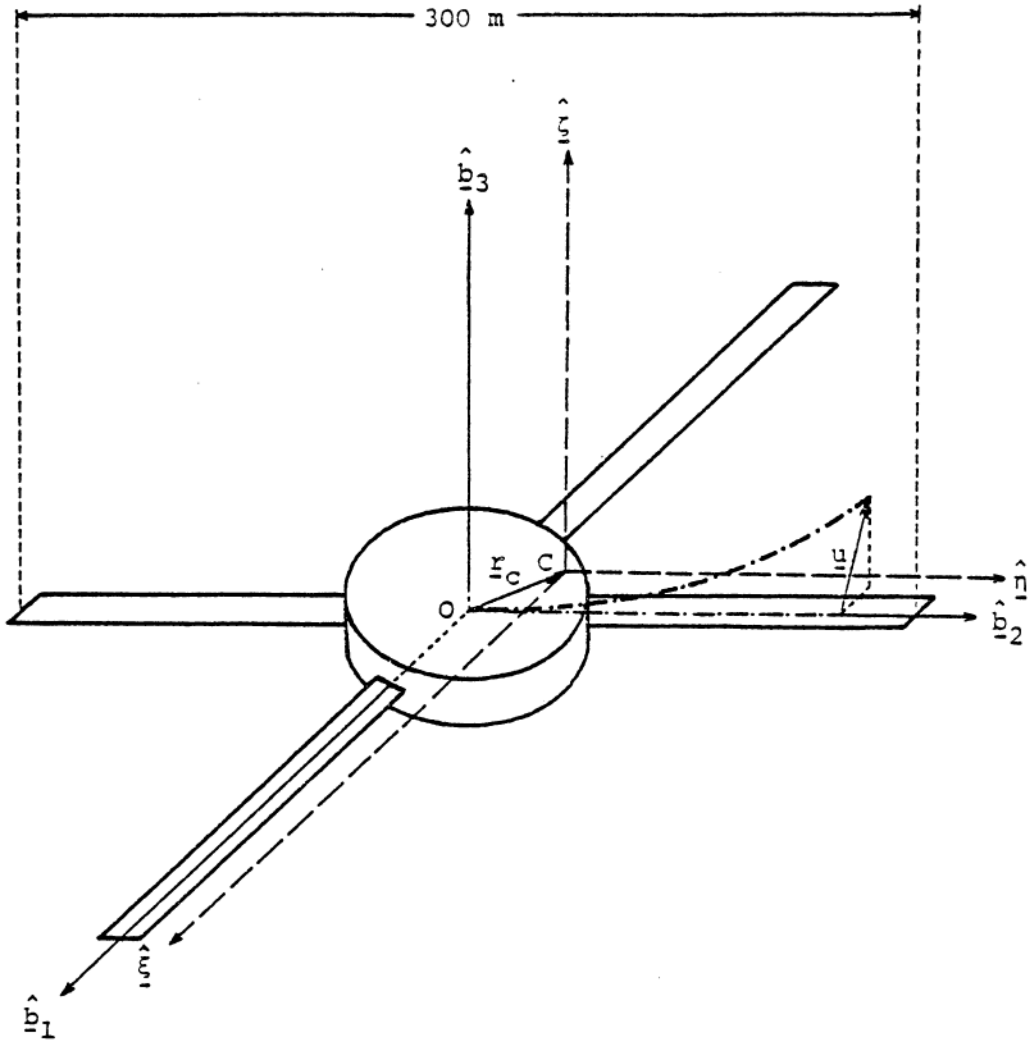

Figure 4.1: Satellite Configuration.

$$\underline{H} = \int_B \underline{\rho} \times \left.\frac{d\underline{\rho}}{dt}\right|_N dm \qquad (4.1.1)$$

where $\underline{\rho}$ locates the generic mass element dm in the vehicle, relative to the vehicle's instantaneous mass center, $\left.\frac{d\underline{\rho}}{dt}\right|_N$ denotes the time rate of change of $\underline{\rho}$ in the inertial frame N , and B denotes that the integral is to be carried out over the entire vehicle.

On making use of the vector identity

$$\left.\frac{d\underline{\rho}}{dt}\right|_N = \left.\frac{d\underline{\rho}}{dt}\right|_B + \underline{\omega} \times \underline{\rho} \qquad (4.1.2)$$

where B denotes the body frame and $\underline{\omega}$ denotes the angular velocity of the body frame with respect to the inertial frame N .

Equation (4.1.1) becomes

$$\underline{H} = \underline{H}_E + \underline{\underline{J}}\,\underline{\omega} \qquad (4.1.3)$$

where $\underline{H}_E = \int_B \underline{\rho} \times \left.\frac{d\underline{\rho}}{dt}\right|_B dm$ is the elastic angular momentum;

$\underline{\underline{J}} = \int_B \left[(\underline{\rho} \cdot \underline{\rho})\, \underline{\underline{E}} - \underline{\rho}\, \underline{\rho}^T\right] dm$ is the moment of inertia tensor for the vehicle about its instantaneous mass center, and $\underline{\underline{E}}$ is a 3 by 3 identity matrix.

Taking the inertial time derivative of the $\underline{H}$ and using the vector identity of (4.1.2) yields

$$\underline{L}_{ex} = \left.\frac{d\underline{H}}{dt}\right|_B + \underline{\omega} \times \underline{H} \qquad (4.1.4)$$

where $\underline{L}_{ex}$ is the external torque (Newton's law).

Substituting the right hand side of (4.1.3) into (4.1.4), one obtains

$$\underline{L}_{ex} = \dot{\underline{H}}_E + \dot{\underline{\underline{J}}}\ \underline{\omega} + \underline{\underline{J}}\ \dot{\underline{\omega}} + \underline{\omega} \times (\underline{H}_E + \underline{\underline{J}}\ \underline{\omega}) \tag{4.1.5}$$

where $(\dot{\ }) = \left.\frac{d(\ )}{dt}\right|_B$.

The differential equation for the vehicle's angular velocity can be obtained from (4.1.5), by solving for $\dot{\underline{\omega}}$, yielding

$$\dot{\underline{\omega}} = \underline{\underline{J}}^{-1} \left[\underline{L}_{ex} - \dot{\underline{\underline{J}}}\ \underline{\omega} - \dot{\underline{H}}_E - \underline{\omega} \times (\underline{H}_E + \underline{\underline{J}}\ \underline{\omega})\right] \tag{4.1.6}$$

With (4.1.6) the equation of motion for the generalized coordinates representing the elastic degrees of freedom is obtained next.

Referring to (A.9) the system kinetic energy is given by

$$T = \frac{1}{2}\, m \left.\frac{d\underline{R}_c}{dt}\right|_N \cdot \left.\frac{d\underline{R}_c}{dt}\right|_N + \frac{1}{2} \sum_{j=0}^{n} \int_{D_j} \left.\frac{d\underline{u}_{jc}}{dt}\right|_{D_j} \cdot \left.\frac{d\underline{u}_{jc}}{dt}\right|_{D_j} dm_j$$

$$+ \underline{\omega} \cdot \underline{H}_E + \frac{1}{2}\, \underline{\omega} \cdot \underline{\underline{J}} \cdot \underline{\omega} \tag{4.1.7}$$

where $\underline{R}_c$ locates the vehicles instantaneous mass center in inertia space; $\underline{u}_{jc} = \underline{u}_j - \underline{r}_c$, $\underline{u}_j$ denotes an elastic deflection in the j-th domain, $\underline{r}_c$ denotes the position vector to the vehicle's instantaneous mass center in the deformed vehicle from the vehicle's undeformed mass center position; m is the total vehicle's mass, n = 4 (four elastic appendages), and $\underline{\omega}$, $\underline{H}_E$, and $\underline{\underline{J}}$ have been previously defined.

The potential energy * for the vehicle is given by (see (A.10))

* It is assumed that only elastic potential energy is considered.

$$V = \frac{1}{2} \sum_{j=0}^{n} \int_{D_j} EI_j \left[\left(\frac{\partial^2 u_{2j}}{\partial x^2} \right)^2 + \left(\frac{\partial^2 u_{3j}}{\partial x^2} \right)^2 \right] dx \qquad (4.1.8)$$

where u_{2j} and u_{3j} are the in- and out-of-plane bending deflections for the j-th domain (for rigid part of the vehicle, $j = 0$, $u_{20} = u_{30} = 0$).

To obtain the equations of motion for the elastic deflections, the elastic deflections are first expressed, by the assumed modes method, as the following series:

$$u_{2j} = \sum_{k=1}^{m} \phi_{jk}^2(x)\ \eta_{jk}^2(t)\ ^* = \underline{\phi}_{2j}^T\ \underline{\eta}_{2j} \qquad (4.1.9a)$$

$$u_{3j} = \sum_{k=1}^{m} \phi_{jk}^3(x)\ \eta_{jk}^3(t) = \underline{\phi}_{3j}^T\ \underline{\eta}_{3j} \qquad (4.1.9b)$$

where $\underline{\phi}_{pj}^T = \left[\phi_{j1}^p \quad \phi_{j2}^p \ \ldots\ldots\ \phi_{jm}^p\right]^T$, $p=2,3$;

$\underline{\eta}_{pj}^T = \left[\eta_{j1}^p \quad \eta_{j2}^p \ \ldots\ldots\ \eta_{jm}^p\right]^T$, $p=2,3$;

$\phi_{kj}^2(x)$ and $\eta_{kj}^2(t)$ represent the in-plane admissible assumed deflection shapes and generalized coordinates for the j-th domain; $\phi_{kj}^3(x)$ and $\eta_{kj}^3(t)$ represent the out-of-plane admissible assumed deflection shapes and generalized coordinates for the j-th domain. (It has been assumed that $\underline{\phi} = \underline{\phi}_{2j} = \underline{\phi}_{3j}$ for all calculations).

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* m is the number of terms in the expansion; in theory, m is infinite, and in practice, m is truncated at some reasonable level which represents a workable compromise between model fidelity and computer budget.

On introducing (4.1.9a) and (4.1.9b) into (4.1.7) and (4.1.8) the system Lagrangian * becomes

$$L = \omega_1 \left[\underline{\eta}_3^T \underline{\underline{H}}_{11} \dot{\underline{\eta}}_2 + (\underline{H}_1^T + \underline{\eta}_2^T \underline{\underline{H}}_{12})\, \dot{\underline{\eta}}_3\right] + \omega_2 \left[\underline{\eta}_3^T \underline{\underline{H}}_{21} \dot{\underline{\eta}}_2 + (\underline{H}_2^T + \underline{\eta}_2^T \underline{\underline{H}}_{22})\, \dot{\underline{\eta}}_3\right]$$

$$+ \omega_3 \left[(\underline{H}_3^T + \underline{\eta}_2^T \underline{\underline{H}}_{31})\, \dot{\underline{\eta}}_2\right] + \frac{1}{2}\, \dot{\underline{\eta}}_2^T \underline{\underline{M}}_{22} \dot{\underline{\eta}}_2 + \frac{1}{2}\, \dot{\underline{\eta}}_3^T \underline{\underline{M}}_{33} \dot{\underline{\eta}}_3 + \frac{1}{2}\, \underline{\omega}^T \underline{\underline{J}} \underline{\omega}$$

$$- \underline{\eta}_2^T \underline{\underline{K}}_{22} \underline{\eta}_2 - \underline{\eta}_3^T \underline{\underline{K}}_{33} \underline{\eta}_3 \tag{4.1.10}$$

The terms contained in (4.1.10) are defined in the following equations.

The in- and out-of-plane deflection amplitudes $\underline{\eta}_2$ and $\underline{\eta}_3$ in (4.1.10) follow as

$$\left.\begin{aligned} \underline{\eta}_2 &= \left[\underline{\eta}_{21}^T \quad \underline{\eta}_{22}^T \quad \underline{\eta}_{23}^T \quad \underline{\eta}_{24}^T\right] \\ \underline{\eta}_3 &= \left[\underline{\eta}_{31}^T \quad \underline{\eta}_{32}^T \quad \underline{\eta}_{33}^T \quad \underline{\eta}_{34}^T\right] \end{aligned}\right\} \tag{4.1.11}$$

The mass matrices $\underline{\underline{M}}_{22}$ and $\underline{\underline{M}}_{33}$ in (4.1.10) are defined by

$$\left.\begin{aligned} \underline{\underline{M}}_{22} &= \hat{\underline{\underline{M}}}_{22} - m\left[\underline{\underline{R}}_{11} + \underline{\underline{R}}_{22}\right] \\ \underline{\underline{M}}_{33} &= \hat{\underline{\underline{M}}}_{33} - m\left[\underline{\underline{R}}_{33}\right] \end{aligned}\right\} \tag{4.1.12}$$

where $\underline{\underline{M}}_{22}$ and $\underline{\underline{M}}_{33}$ are the generalized mass matrices for the in- and out-of-plane generalized coordinates, respectively.

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* The first term in (4.1.7) has been dropped because the interest in this problem is strictly in the rotational motion of the vehicle. All second order terms have been retained in (4.1.10), under the assumption that $\underline{\omega}^T = (\omega_1 \quad \omega_2 \quad \omega_3)$ can be arbitrarily large.

The matrices $\hat{\underline{\underline{M}}}_{22}$ and $\hat{\underline{\underline{M}}}_{33}$ in (4.1.12) are given by

$$\hat{\underline{\underline{M}}}_{22} = \hat{\underline{\underline{M}}}_{33} = \begin{bmatrix} \hat{\underline{\underline{M}}} & 0 & 0 & 0 \\ & \hat{\underline{\underline{M}}} & 0 & 0 \\ & & \hat{\underline{\underline{M}}} & 0 \\ \text{sym.}^{*} & & & \hat{\underline{\underline{M}}} \end{bmatrix} \tag{4.1.13}$$

where $\hat{\underline{\underline{M}}} = \int_D \underline{\phi}\ \underline{\phi}^T\ dm$. **

The matrices $\underline{\underline{R}}_{11}$, $\underline{\underline{R}}_{22}$, and $\underline{\underline{R}}_{33}$ in (4.1.12), result from the motion of the center of mass in the deformed vehicle; these matrices are defined in (4.1.26). The stiffness matrices $\underline{\underline{K}}_{22}$ and $\underline{\underline{K}}_{33}$ in (4.1.10) follow as

$$\underline{\underline{K}}_{22} = \underline{\underline{K}}_{33} = \begin{bmatrix} \hat{\underline{\underline{K}}} & 0 & 0 & 0 \\ & \hat{\underline{\underline{K}}} & 0 & 0 \\ & & \hat{\underline{\underline{K}}} & 0 \\ \text{sym.} & & & \hat{\underline{\underline{K}}} \end{bmatrix} \tag{4.1.15}$$

where $\underline{\underline{K}}_{22}$ and $\underline{\underline{K}}_{33}$ are the generalized stiffness matrices for the in- and out-of-plane generalized coordinates, respectively; and

$$\hat{\underline{\underline{K}}} = \int_D EI\ \underline{\phi}''\ (\underline{\phi}'')^T\ dx\ ^{\dagger} \tag{4.1.16}$$

The terms in (4.1.10) arising from the elastic angular momentum $\underline{H}_E$ follow from the second order expansion for $\underline{H}_E$:

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* The abbreviation "sym." stands for "symmetric".

** "D" denotes that the integral is over one of the elastic appendages.

† $(\ )'' = \dfrac{d^2(\ )}{dx^2}$

$$\underline{H}_E = \int_B \underline{r} \times \dot{\underline{u}}\, dm + \int_B \underline{u} \times \dot{\underline{u}}\, dm - m\, \underline{r}_c \times \dot{\underline{r}}_c = \underline{\underline{H}} \begin{Bmatrix} \dot{\underline{\eta}}_2 \\ \cdots \\ \dot{\underline{\eta}}_3 \end{Bmatrix} \qquad (4.1.17)$$

where

$$\underline{\underline{H}} = \begin{bmatrix} \underline{\eta}_3^T \underline{\underline{H}}_{11} & \underline{H}_1^T + \underline{\eta}_2^T \underline{\underline{H}}_{12} \\ \underline{\eta}_3^T \underline{\underline{H}}_{21} & \underline{H}_2^T + \underline{\eta}_2^T \underline{\underline{H}}_{22} \\ \underline{H}_3^T + \underline{\eta}_2 \underline{\underline{H}}_{31} & \underline{0}^T \end{bmatrix} \qquad (4.1.18)$$

The vectors $\underline{H}_1$, $\underline{H}_2$, and $\underline{H}_3$ in (4.1.18) follow as

$$\left.\begin{aligned} \underline{H}_1^T &= \begin{bmatrix} -s_1 \underline{h}_3^T & -s_2 \underline{h}_3^T & -s_3 \underline{h}_3^T & -s_4 \underline{h}_3^T \end{bmatrix} * \\ \underline{H}_2^T &= \begin{bmatrix} c_1 \underline{h}_3^T & c_2 \underline{h}_3^T & c_3 \underline{h}_3^T & c_4 \underline{h}_3^T \end{bmatrix} \\ \underline{H}_3^T &= \begin{bmatrix} \underline{h}_2^T & \underline{h}_2^T & \underline{h}_2^T & \underline{h}_2^T \end{bmatrix} \end{aligned}\right\} \qquad (4.1.19)$$

where

$$\left.\begin{aligned} \underline{h}_2 &= \int_D x\, \underline{\phi}\, dm \\ \underline{h}_3 &= - \int_D x\, \underline{\phi}\, dm \end{aligned}\right\} \qquad (4.1.20)$$

The matrices $\underline{\underline{H}}_{11}$, $\underline{\underline{H}}_{12}$, $\underline{\underline{H}}_{21}$, $\underline{\underline{H}}_{22}$, and $\underline{\underline{H}}_{31}$ in (4.1.18) are defined by

---

* $s = \sin$, $c = \cos$, $\begin{bmatrix} s_1 & s_2 & s_3 & s_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}$, $\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}$; c_j and s_j are direction cosines locating the four appendages in the vehicle relative to a frame fixed in the undeformed vehicle.

$$\left.\begin{aligned}
\underline{\underline{H}}_{11} &= - \underline{\underline{M}}_c + m\, \underline{\underline{R}}_{23} \\
\underline{\underline{H}}_{12} &= - \underline{\underline{M}}_c - m\, \underline{\underline{R}}_{23}^T \\
\underline{\underline{H}}_{21} &= - \underline{\underline{M}}_s - m\, \underline{\underline{R}}_{13}^T \\
\underline{\underline{H}}_{22} &= \underline{\underline{M}}_s + m\, \underline{\underline{R}}_{13} \\
\underline{\underline{H}}_{31} &= m\, (\underline{\underline{R}}_{12}^T - \underline{\underline{R}}_{12})
\end{aligned}\right\} \qquad (4.1.21)$$

The matrices $\underline{\underline{M}}_c$ and $\underline{\underline{M}}_s$ in (4.1.21) can be shown to be

$$\underline{\underline{M}}_c = \begin{bmatrix} c_1\hat{\underline{\underline{M}}} & 0 & 0 & 0 \\ & c_2\hat{\underline{\underline{M}}} & 0 & 0 \\ & & c_3\hat{\underline{\underline{M}}} & 0 \\ \text{sym.} & & & c_4\hat{\underline{\underline{M}}} \end{bmatrix} \qquad (4.1.22a)$$

$$\underline{\underline{M}}_s = \begin{bmatrix} s_1\hat{\underline{\underline{M}}} & 0 & 0 & 0 \\ & s_2\hat{\underline{\underline{M}}} & 0 & 0 \\ & & s_3\hat{\underline{\underline{M}}} & 0 \\ \text{sym.} & & & s_4\hat{\underline{\underline{M}}} \end{bmatrix} \qquad (4.1.22b)$$

where $\hat{\underline{\underline{M}}}$ is defined in (4.1.14).

The vector locating the instantaneous center of mass of the vehicle, relative to the vehicle's undeformed center of mass, is defined by

$$\underline{r}_c = \frac{1}{m} \int_B \underline{u}\, dm = \left[\underline{R}_1^T \underline{\eta}_2 \quad \underline{R}_2^T \underline{\eta}_2 \quad \underline{R}_3^T \underline{\eta}_3\right] \qquad (4.1.23)$$

where B denotes that the integral is taken over the entire vehicle.

The vectors $\underline{R}_1$, $\underline{R}_2$, and $\underline{R}_3$ in (4.1.23) follow as

$$\left.\begin{aligned} \underline{R}_1^T &= \frac{1}{m}\left[-s_1\underline{r}^T \quad -s_2\underline{r}^T \quad -s_3\underline{r}^T \quad -s_4\underline{r}^T\right] \\ \underline{R}_2^T &= \frac{1}{m}\left[c_1\underline{r}^T \quad c_2\underline{r}^T \quad c_3\underline{r}^T \quad c_4\underline{r}^T\right] \\ \underline{R}_3^T &= \frac{1}{m}\left[\underline{r}^T \quad \underline{r}^T \quad \underline{r}^T \quad \underline{r}^T\right] \end{aligned}\right\} \tag{4.1.24}$$

where the vector $\underline{r}$ in (4.1.24) is defined by

$$\underline{r} = \int_D \underline{\phi}\ dm \tag{4.1.25}$$

The matrices $\underline{\underline{R}}_{11}$, $\underline{\underline{R}}_{22}$, $\underline{\underline{R}}_{33}$, $\underline{\underline{R}}_{12}$, $\underline{\underline{R}}_{13}$, and $\underline{\underline{R}}_{23}$ in (4.1.12) and (4.1.21) are obtained from the relation

$$\underline{\underline{R}}_{ij} = \underline{R}_i\ \underline{R}_j^T \tag{4.1.26}$$

where i,j=1,2,3 and $\underline{R}_i$ is defined by (4.1.24).

Referring to (4.1.3) and carrying the integral over the vehicle, the moment of inertia tensor for the vehicle about its instantaneous center of mass is given by

$$\underline{\underline{J}} = \underline{\underline{I}}_o + \sum_{j=1}^{4} \underline{\underline{L}}_j^T\ \underline{\underline{I}}_{uu}^j\ \underline{\underline{L}}_j + \sum_{j=1}^{4} \underline{\underline{L}}_j^T\ \underline{\underline{I}}_{ur}^j\ \underline{\underline{L}}_j - m\left[(\underline{r}_c \cdot \underline{r}_c)\ \underline{\underline{E}} - \underline{r}_c\ \underline{r}_c^T\right] \tag{4.1.27}$$

where

$$\underline{\underline{J}} = \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ & J_{yy} & J_{yz} \\ \text{sym.} & & J_{zz} \end{bmatrix} \tag{4.1.28}$$

$\underline{\underline{L}}_j$ is the direction cosine matrix relating the j-th appendage coordinate frame to the undeformed body frame, and $\underline{\underline{E}}$ has been previously defined.

For the undeformed vehicle, the moment of inertia tensor $\underline{\underline{I}}_o$ in (4.1.27) is given by

$$\underline{\underline{I}}_o = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \tag{4.1.29a}$$

where I_{xx} , I_{yy} , and I_{zz} are the vehicle's principal moments of inertia.

The elastic moment of inertia tensor $\underline{\underline{I}}^j_{uu}$ in (4.1.27), for the j-th appendage follows as

$$\underline{\underline{I}}^j_{uu} = \int_{D_j} \left[(\underline{u} \cdot \underline{u}) \, \underline{\underline{E}} - \underline{u} \, \underline{u}^T \right] dm_j \tag{4.1.29b}$$

where D_j denotes that the integral is taken over the j-th appendage. The cross coupling moment of inertia tensor $\underline{\underline{I}}^j_{ur}$ in (4.1.27) for the j-th appendage is defined by

$$\underline{\underline{I}}^j_{ur} = \int_{D_j} \left[2 \, (\underline{u} \cdot \underline{r}) \, \underline{\underline{E}} - \underline{u} \, \underline{r}^T - \underline{r} \, \underline{u}^T \right] dm_j \tag{4.1.29c}$$

To carry out the integrals in (4.1.29), two definitions * are required: First, the elastic appendages are one-dimensional elastic domains; and Second, only transverse flexural deflections are considered. Mathematically, these two requirements assume the form

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* These definitions are made to simplify the analysis involved with treating the vehicle shown in Figure 4.2.

Given: $\underline{\zeta}_0$, $\underline{\zeta}_f$

Choose the principal axis to start
the relaxation process

↓

Solve the linear single axis flexible
body problem (3.4.10)

↓

Map the modal space initial co-states to
three-dimensional physical space initial co-states
(using Appendix H and (4.4.6))

A →

Increment α

B →

Solve the differential equations of (4.3.8)
and (4.3.9) to determine

$$\hat{\underline{\zeta}}' = \underline{\zeta}'\{\underline{\Lambda}(0), t_f\}$$

and determine (by the method in Appendix B)
the partial derivative of (4.4.9)

↓

Calculate the residual vector $\Delta\underline{\zeta}_f'$

If small and $\alpha = 1$ → End

If small and $\alpha \neq 1$ → A

If large, continue

↓

Calculate $\underline{\Delta\Lambda}$ from (4.4.11)

↓

Apply corrections $\underline{\Lambda}(0) = \underline{\Lambda} + \underline{\Delta\Lambda}$

↓

B

Figure 4.2: Differential Correction Algorithm for the
Three-Dimensional Flexible Body Problem.

$$\left.\begin{aligned} \underline{r}^T &= (x\ ,\ 0\ ,\ 0) \\ \underline{u} &= (0\ ,\ u_2\ ,\ u_3) \end{aligned}\right\} \qquad (4.1.30)$$

The integrals are carried out assuming the j-th coordinate frame is oriented such that the x-axis lies along the j-th appendage in the undeformed vehicle.

On carrying out the integrals in (4.1.29b) and (4.1.29c), using (4.1.30), the elements of the moment of inertia tensor follow as

$$\left.\begin{aligned} J_{xx} &= I_{xx} + \underline{\eta}_2^T \underline{\underline{J}}_{22xx} \underline{\eta}_2 + \underline{\eta}_3^T \underline{\underline{J}}_{33xx} \underline{\eta}_3 \\ J_{xy} &= \underline{J}_{2xy}^T \underline{\eta}_2 + \underline{\eta}_2^T \underline{\underline{J}}_{12xy} \underline{\eta}_2 \\ J_{xz} &= \underline{J}_{3xz}^T \underline{\eta}_3 + \underline{\eta}_2^T \underline{\underline{J}}_{13xz} \underline{\eta}_3 \\ J_{yy} &= I_{yy} + \underline{\eta}_2^T \underline{\underline{J}}_{22yy} \underline{\eta}_2 + \underline{\eta}_3^T \underline{\underline{J}}_{33yy} \underline{\eta}_3 \\ J_{yz} &= \underline{J}_{3yz}^T \underline{\eta}_3 + \underline{\eta}_2^T \underline{\underline{J}}_{23yz} \underline{\eta}_3 \\ J_{zz} &= I_{zz} + \underline{\eta}_2^T \underline{\underline{J}}_{22zz} \underline{\eta}_2 + \underline{\eta}_3^T \underline{\underline{J}}_{33zz} \underline{\eta}_3 \end{aligned}\right\} \qquad (4.1.31)$$

In (4.1.31) the elastic inertia vectors and matrices are defined by

$$\begin{aligned} \underline{\underline{J}}_{22xx} &= \underline{\underline{M}}_{22xx} - m\,\underline{\underline{R}}_{22} \\ \underline{\underline{J}}_{33xx} &= \underline{\underline{M}}_{33xx} - m\,\underline{\underline{R}}_{33} \\ \underline{J}_{2xy} &= \underline{M}_{2xy} \\ \underline{J}_{3xz} &= \underline{M}_{3xz} \\ \underline{\underline{J}}_{12xy} &= m\,\underline{\underline{R}}_{12} \end{aligned}$$

$$\left.\begin{aligned}
\underline{\underline{J}}_{13xz} &= m\,\underline{\underline{R}}_{13} \\
\underline{\underline{J}}_{22yy} &= \underline{\underline{M}}_{22yy} - m\,\underline{\underline{R}}_{11} \\
\underline{\underline{J}}_{33yy} &= \underline{\underline{M}}_{33yy} - m\,\underline{\underline{R}}_{33} \\
\underline{J}_{3yz} &= \underline{M}_{3yz} \\
\underline{\underline{J}}_{23yz} &= \underline{\underline{M}}_{23yz} + m\,\underline{\underline{R}}_{23} \\
\underline{\underline{J}}_{22zz} &= \underline{\underline{M}}_{22zz} - m\,(\underline{\underline{R}}_{11} + \underline{\underline{R}}_{22}) \\
\underline{\underline{J}}_{33zz} &= \underline{\underline{M}}_{33zz}
\end{aligned}\right\} \qquad (4.1.32)$$

The mass vectors and matrices in (4.1.32) are given by

$$\underline{\underline{M}}_{22xx} = \begin{bmatrix} \hat{\underline{\underline{M}}} & 0 & 0 & 0 \\ & \hat{\underline{\underline{M}}}' & 0 & 0 \\ & & \hat{\underline{\underline{M}}} & 0 \\ \text{sym.} & & & \hat{\underline{\underline{M}}}' \end{bmatrix} \qquad (4.1.33a)$$

$$\underline{\underline{M}}_{33xx} = \begin{bmatrix} \hat{\underline{\underline{M}}} & 0 & 0 & 0 \\ & \hat{\underline{\underline{M}}}+\hat{\underline{\underline{M}}}' & 0 & 0 \\ & & \hat{\underline{\underline{M}}} & 0 \\ \text{sym.} & & & \hat{\underline{\underline{M}}}+\hat{\underline{\underline{M}}}' \end{bmatrix} \qquad (4.1.33b)$$

$$\underline{M}^T_{2xy} = \begin{bmatrix} \underline{h}^T_3 & -\underline{h}^T_3 & \underline{h}^T_3 & -\underline{h}^T_3 \end{bmatrix} \qquad (4.1.33c)$$

$$\underline{M}^T_{3xz} = \begin{bmatrix} \underline{h}^T_3 & -\underline{h}^T_3 & -\underline{h}^T_3 & \underline{h}^T_3 \end{bmatrix} \qquad (4.1.33d)$$

$$\underline{\underline{M}}_{22yy} = \begin{bmatrix} \hat{\underline{\underline{M}}}' & 0 & 0 & 0 \\ & \hat{\underline{\underline{M}}} & 0 & 0 \\ & & \hat{\underline{\underline{M}}}' & 0 \\ \text{sym.} & & & \hat{\underline{\underline{M}}} \end{bmatrix} \qquad (4.1.33e)$$

$$\underline{\underline{M}}_{33yy} = \begin{bmatrix} \hat{\underline{\underline{M}}}+\hat{\underline{\underline{M}}}' & 0 & 0 & 0 \\ & \hat{\underline{\underline{M}}} & 0 & 0 \\ & & \hat{\underline{\underline{M}}}+\hat{\underline{\underline{M}}}' & 0 \\ \text{sym.} & & & \hat{\underline{\underline{M}}} \end{bmatrix} \qquad (4.1.33f)$$

$$\underline{M}^T_{3yz} = \begin{bmatrix} \underline{0}^T & \underline{h}^T_3 & \underline{0}^T & -\underline{h}^T_3 \end{bmatrix} \qquad (4.1.33g)$$

$$\underline{\underline{M}}_{23yz} = \begin{bmatrix} \underline{\underline{M}} & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & -\underline{\underline{M}} & 0 \\ \text{sym.} & & & 0 \end{bmatrix} \qquad (4.1.33h)$$

$$\underline{\underline{M}}_{22zz} = \begin{bmatrix} \hat{\underline{\underline{M}}}+\hat{\underline{\underline{M}}}' & 0 & 0 & 0 \\ & \hat{\underline{\underline{M}}}+\hat{\underline{\underline{M}}}' & 0 & 0 \\ & & \hat{\underline{\underline{M}}}+\hat{\underline{\underline{M}}}' & 0 \\ \text{sym.} & & & \hat{\underline{\underline{M}}}+\hat{\underline{\underline{M}}}' \end{bmatrix} \qquad (4.1.33i)$$

$$\underline{\underline{M}}_{33zz} = \begin{bmatrix} \hat{\underline{\underline{M}}} & 0 & 0 & 0 \\ & \hat{\underline{\underline{M}}} & 0 & 0 \\ & & \hat{\underline{\underline{M}}} & 0 \\ \text{sym.} & & & \hat{\underline{\underline{M}}} \end{bmatrix} \qquad (4.1.33j)$$

where $\hat{\underline{\underline{M}}}$ is defined by (4.1.14); $\hat{\underline{\underline{M}}}'$ is given by

$$\hat{\underline{\underline{M}}}' = - \int_D \left(\frac{L^2 - x^2}{2}\right) \underline{\phi}' \; (\underline{\phi}')^T \, dm \tag{4.1.34}$$

and L is the length of an appendage.

Assuming no external forces or moments are acting directly on the elastic appendages, and using (4.1.10), Lagrange's equations for the in- and out-of-plane modes take the form

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{\eta}}_2}\right) - \frac{\partial L}{\partial \underline{\eta}_2} &= \underline{0} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{\eta}}_3}\right) - \frac{\partial L}{\partial \underline{\eta}_3} &= \underline{0} \end{aligned} \tag{4.1.35}$$

From which the equations of motion for the elastic degrees of freedom follow as

$$\underline{\underline{M}}_{22}\ddot{\underline{\eta}} + \underline{\underline{K}}_{22}\underline{\eta} + \left[\underline{\underline{H}}_{11}^T\underline{\eta}_3 \quad \underline{\underline{H}}_{21}^T\underline{\eta}_3 \quad \underline{H}_3 + \underline{\underline{H}}_{31}^T\underline{\eta}_2\right] \dot{\underline{\omega}}$$

$$= \left[(\underline{\underline{H}}_{12} - \underline{\underline{H}}_{11}^T)\dot{\underline{\eta}}_3 \quad (\underline{\underline{H}}_{22} - \underline{\underline{H}}_{21}^T)\dot{\underline{\eta}}_3 \quad (\underline{\underline{H}}_{31} - \underline{\underline{H}}_{31}^T)\dot{\underline{\eta}}_2\right] \underline{\omega} + \frac{1}{2}\underline{\omega}^T \left[\frac{\partial \underline{\underline{J}}}{\partial \underline{\eta}_2}\right] \underline{\omega} \tag{4.1.36}$$

$$\underline{\underline{M}}_{33}\ddot{\underline{\eta}}_3 + \underline{\underline{K}}_{33}\underline{\eta}_3 + \left[(\underline{H}_1 + \underline{\underline{H}}_{12}^T\underline{\eta}_2) \quad (\underline{H}_2 + \underline{\underline{H}}_{22}^T\underline{\eta}_2) \quad (\underline{0})\right] \dot{\underline{\omega}}$$

$$= \left[(\underline{\underline{H}}_{11} - \underline{\underline{H}}_{12}^T)\dot{\underline{\eta}}_2 \quad (\underline{\underline{H}}_{21} - \underline{\underline{H}}_{22}^T)\dot{\underline{\eta}}_2 \quad (\underline{0})\right] \underline{\omega} + \frac{1}{2}\underline{\omega}^T \left[\frac{\partial \underline{\underline{J}}}{\partial \underline{\eta}_3}\right] \underline{\omega} \tag{4.1.37}$$

On combining (4.1.6), (4.1.36), and (4.1.37) the coupled nonlinear equations of motion for the vehicle can be cast in the matrix form *

$$\underline{\underline{M}} \begin{Bmatrix} \dot{\underline{\omega}} \\ \cdots \\ \ddot{\underline{\eta}} \end{Bmatrix} + \underline{\underline{K}} \begin{Bmatrix} \underline{0} \\ \cdots \\ \underline{\eta} \end{Bmatrix} = \underline{\underline{I}}\, \underline{U}_T + \underline{L} \tag{4.1.38}$$

where $\underline{\eta}^T = \begin{bmatrix} \underline{\eta}_2^T & \underline{\eta}_3^T \end{bmatrix}$ is the vector of in- and out-of-plane bending deflections, and $\underline{U}_T$ is the control torque ** which drives the vehicle.

The time varying mass matrix $\underline{\underline{M}}$ in (4.1.38) is given by

$$\underline{\underline{M}} = \left[\begin{array}{ccc|cc} J_{xx} & J_{xy} & J_{xz} & \underline{0}^T & \underline{H}_1^T \\ J_{yx} & J_{yy} & J_{yz} & \underline{0}^T & \underline{H}_2^T \\ J_{zx} & J_{zy} & J_{zz} & \underline{H}_3^T & \underline{0}^T \\ \hline \underline{\underline{H}}_{11}^T\underline{\eta}_3 & \underline{\underline{H}}_{21}^T\underline{\eta}_3 & \underline{H}_3+\underline{\underline{H}}_{31}^T\underline{\eta}_2 & \underline{\underline{M}}_{22} & \underline{\underline{0}} \\ \underline{H}_1+\underline{\underline{H}}_{12}^T\underline{\eta}_2 & \underline{H}_2+\underline{\underline{H}}_{22}^T\underline{\eta}_2 & \underline{0} & \underline{\underline{0}} & \underline{\underline{M}}_{33} \end{array}\right]^{\dagger}. \tag{4.1.39}$$

The stiffness matrix $\underline{\underline{K}}$ in (4.1.38) is given by

$$\underline{\underline{K}} = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & \underline{0}^T & \underline{0}^T \\ 0 & 0 & 0 & \underline{0}^T & \underline{0}^T \\ 0 & 0 & 0 & \underline{0}^T & \underline{0}^T \\ \hline \underline{0} & \underline{0} & \underline{0} & \underline{\underline{K}}_{22} & \underline{\underline{0}} \\ \underline{0} & \underline{0} & \underline{0} & \underline{\underline{0}} & \underline{\underline{K}}_{33} \end{array}\right] \tag{4.1.40}$$

---

* Only first order terms are retained in the elastic deflections.

† $\underline{0}$ is a null vector of dimension p by 1 where p is the same dimension as $\underline{\eta}$, and $\underline{\underline{0}}$ is a null matrix of dimension p by p.

** $\underline{U}_T \equiv \underline{L}_{ex}$, where $\underline{L}_{ex}$ is defined by (4.1.4)

The control matrix $\underline{\underline{\Pi}}$ in (4.1.38) follows as

$$\underline{\underline{\Pi}} = \left\{ \begin{array}{c} \underline{\underline{E}} \\ \cdots\cdots \\ \underline{\underline{0}}' \end{array} \right\} \tag{4.1.41}$$

where $\underline{\underline{E}}$ has been previously defined and $\underline{\underline{0}}'$ is a null matrix of dimension 2p by 3 .

The vector $\underline{L}$ in (4.1.38) containing nonlinear terms is given by

$$\underline{L}^T = \left[\underline{L}_1^T \quad \underline{L}_2^T \right] \text{*} \tag{4.1.42}$$

where

$$\underline{L}_1 = - \dot{\underline{\underline{J}}}\, \underline{\omega} - \tilde{\underline{\underline{\omega}}}\, (\underline{\underline{J}}\, \underline{\omega} + \underline{H}_E) \tag{4.1.43a}$$

$$\underline{L}_2 = \left\{ \begin{array}{c} \left[(\underline{\underline{H}}_{12} - \underline{\underline{H}}_{11}^T)\dot{\underline{\eta}}_3 \quad (\underline{\underline{H}}_{22} - \underline{\underline{H}}_{21}^T)\dot{\underline{\eta}}_3 \quad (\underline{\underline{H}}_{31} - \underline{\underline{H}}_{31}^T)\dot{\underline{\eta}}_2 \right] \underline{\omega} \\ + \frac{1}{2}\, \underline{\omega}^T \left[\frac{\partial \underline{\underline{J}}}{\partial \underline{\eta}_2} \right] \underline{\omega} \\ \text{-------------------------} \\ \left[(\underline{\underline{H}}_{11} - \underline{\underline{H}}_{12}^T)\underline{\eta}_2 \quad (\underline{\underline{H}}_{21} - \underline{\underline{H}}_{22}^T)\underline{\eta}_2 \quad \underline{0} \right] \underline{\omega} \\ + \frac{1}{2}\, \underline{\omega}^T \left[\frac{\partial \underline{\underline{J}}}{\partial \underline{\eta}_3} \right] \underline{\omega} \end{array} \right\} \tag{4.1.43b}$$

and

$$\tilde{\underline{\underline{\omega}}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{4.1.43c}$$

As in Chapter II , in lieu of any three parameter description of

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* The $\underline{L}_1$ is a vector of dimension 3 by 1 , and $\underline{L}_2$ is a vector of dimension 2p by 1 .

orientation, the Euler parameters (see Appendix D) are selected; in order to provide a singularity free description of the vehicle's orientation as a function of time. The differential equation for the Euler parameters follows as (see (2.2.4))

$$\dot{\underline{\beta}} = \underline{\underline{\Omega}}\ \underline{\beta} = \underline{\underline{\beta}}\ \underline{\omega} \tag{4.1.44}$$

where $\underline{\underline{\Omega}}$ and $\underline{\underline{\beta}}$ are defined following (2.2.4).
Equations (4.1.38) and (4.1.44), on integration with respect to time, provide a complete description of the vehicle's angular velocity and orientation as a function of time.

4.2 STATE SPACE FORMULATION

In preparing to establish the state space representation of (4.1.38), it is first convenient to premultiply (4.1.38) by the inverse mass matrix * $\underline{\underline{M}}^{-1}$, given by

$$\underline{\underline{M}}^{-1} = \begin{bmatrix} \underline{\underline{m}}_{11} & \underline{\underline{m}}_{12} \\ \underline{\underline{m}}_{21} & \underline{\underline{m}}_{22} \end{bmatrix} \tag{4.2.1}$$

where the partitions of $\underline{\underline{M}}^{-1}$ have the following dimensions: $\underline{\underline{m}}_{11}$ is 3 by 3 ; $\underline{\underline{m}}_{12}$ is 3 by 2p ; $\underline{\underline{m}}_{21}$ is 2p by 3 ; and $\underline{\underline{m}}_{22}$ is 2p by 2p .

In addition, the stiffness matrix $\underline{\underline{K}}$ is partitioned as follows:

---

* Since $\underline{\underline{M}}$ is time varying a linear eigenvalue analysis cannot be used efficiently to uncouple the equations of motion in (4.1.38).

$$\underline{\underline{K}} = \begin{bmatrix} \underline{\underline{0}} & (\underline{\underline{0}}')^T \\ \underline{\underline{0}}' & \underline{\underline{K}}_{22} \end{bmatrix} \tag{4.2.2}$$

where

$$\underline{\underline{K}}_{22} = \begin{bmatrix} \underline{\underline{K}}_{22} & \hat{\underline{\underline{0}}} \\ \hat{\underline{\underline{0}}} & \underline{\underline{K}}_{33} \end{bmatrix} * \tag{4.2.3}$$

On premultiplying (4.1.38) by (4.2.1), one obtains

$$\left\{\begin{array}{c} \dot{\underline{\omega}} \\ \hdashline \ddot{\underline{\eta}} \end{array}\right\} + \underline{\underline{M}}^{-1} \underline{\underline{K}} \left\{\begin{array}{c} \underline{0} \\ \hdashline \underline{\eta} \end{array}\right\} = \underline{\underline{M}}^{-1} \underline{\underline{\Pi}} \, \underline{U}_T + \underline{\underline{M}}^{-1} \underline{L} \tag{4.2.4}$$

After carrying out the partitioned matrix multiplication in (4.2.4), according to (4.2.1) and (4.2.2), (4.2.4) reduces to

$$\dot{\underline{\omega}} = \underline{\underline{m}}_{11} (\underline{U}_T + \underline{L}_1) + \underline{\underline{m}}_{12} (\underline{L}_2 - \underline{\underline{K}}_{22} \underline{\eta}) \tag{4.2.5}$$

$$\ddot{\underline{\eta}} = \underline{\underline{m}}_{22} (\underline{U}_T + \underline{L}_1) + \underline{\underline{m}}_{22} (\underline{L}_2 - \underline{\underline{K}}_{22} \underline{\eta}) \tag{4.2.6}$$

With the equations above, the state space form of (4.1.44), (4.2.5), and (4.2.6) is established as follows: Defining the state variable subsets as

$$\left.\begin{array}{l} \underline{s}_1 = \underline{\omega} \\ \underline{s}_2 = \underline{\eta} \\ \underline{s}_3 = \dot{\underline{\eta}} \\ \underline{s}_4 = \underline{\beta} \end{array}\right\} \tag{4.2.7}$$

leads to the first order differential equations

---

* $\hat{\underline{\underline{0}}}$ is a null matrix of dimension p by p .

$$
\left.
\begin{aligned}
\dot{\underline{s}}_1 &= \underline{\underline{m}}_{11} \, (\underline{U}_T + \underline{L}_1) + \underline{\underline{m}}_{12} \, (\underline{L}_2 - \underline{\underline{K}}_{22} \, \underline{s}_2) \\
\dot{\underline{s}}_2 &= \underline{s}_3 \\
\dot{\underline{s}}_3 &= \underline{\underline{m}}_{21} \, (\underline{U}_T + \underline{L}_1) + \underline{\underline{m}}_{22} \, (\underline{L}_2 - \underline{\underline{K}}_{22} \, \underline{s}_2) \\
\dot{\underline{s}}_4 &= \underline{\underline{\Omega}} \, \underline{s}_4 \quad \text{or} \quad \underline{\underline{\beta}} \, \underline{s}_1
\end{aligned}
\right\} \qquad (4.2.8)
$$

where $\underline{\underline{\Omega}}$ and $\underline{\underline{\beta}}$ are defined by (2.2.4).

Equation (4.2.8) represents the state space form for the equations of motion.

4.3 OPTIMAL CONTROL PROBLEM

The problem considered here is the optimal rotational motions of a flexible space vehicle, where general motions for the vehicle are considered. The necessary conditions for the maneuvers to be optimal are derived from Pontryagin's principle. A conventional quadratic performance index is chosen which minimizes the sum of weighted products of torque, elastic deflections, and deflection rates:

$$
J = \frac{1}{2} \int_0^t \left[\underline{U}_T^T \, \underline{\underline{W}}_{uu} \, \underline{U}_T + \underline{s}_2^T \, \underline{\underline{W}}_{22} \, \underline{s}_2 + \underline{s}_3^T \, \underline{\underline{W}}_{33} \, \underline{s}_3 \right] dt \qquad (4.3.1)
$$

where $\underline{\underline{W}}_{uu}$ is a 3 by 3 positive definite weight matrix for the control torque $\underline{U}_T$; $\underline{\underline{W}}_{22}$ is a p by p positive semidefinite weight matrix for $\underline{s}_2$; and $\underline{\underline{W}}_{33}$ is a p by p positive semidefinite weight matrix for $\underline{s}_3$.

One consequence of selecting this particular form for the performance index J , is that the necessary conditions for the optimal maneuver simultaneously address the problems of attitude and structural control for the vehicle. In addition to (4.3.1), as part of the complete specification of the problem, the elastic deflections and deflection rates are required to be suppressed at the end of the maneuver, with respect to a rigid body frame fixed in the undeformed vehicle.

The problem is to seek a solution of (4.2.8), satisfying the prescribed terminal states given by:

$$\underline{\zeta}_0 = \left[\theta_1(0) \quad \theta_2(0) \quad \theta_3(0) \quad \underline{\omega}^T(0) \quad \underline{\eta}^T(0) \quad \dot{\underline{\eta}}^T(0)\right]^T \tag{4.3.2a}$$

$$\underline{\zeta}_f = \left[\theta_1(t_f) \quad \theta_2(t_f) \quad \theta_3(t_f) \quad \underline{\omega}^T(t_f) \quad \underline{\eta}^T(t_f) \quad \dot{\underline{\eta}}^T(t_f)\right]^T \tag{4.3.2b}$$

where θ_1 , θ_2 , and θ_3 * denote an Euler angle rotation sequence (1-2-3) specifying the attitude state of the vehicle, $\underline{\omega}$ denotes the vehicle's angular velocity, $\underline{\eta}$ denotes the in- and out-of-plane deformation coordinates, $\underline{\eta}$ denotes the time derivative of $\underline{\eta}$, and $\underline{\eta}(t_f) = \underline{\eta}(t_f) = 0$ as required by the problem boundary conditions.

The optimal control formulation has as its objective the torque history $\underline{U}_T(t)$ generating a solution of (4.2.8) which (a) satisfies the boundary conditions of (4.3.2), and (b) minimizes the performance index (4.3.1). Only a piecewise continuous torque history for $\underline{U}_T$ is considered.

---

* Before any calculations are performed the angles θ_1 , θ_2 , and θ_3 are converted into Euler parameters using (D.9).

The Hamiltonian functional associated with minimizing (4.3.1) along optimal trajectories of (4.1.38) is given by

$$H = \frac{1}{2}\left(\underline{U}_T^T \underline{\underline{W}}_{uu} \underline{U}_T + \underline{s}_2^T \underline{\underline{W}}_{22} \underline{s}_2 + \underline{s}_3^T \underline{\underline{W}}_{33} \underline{s}_3\right) + \underline{\lambda}_1^T \{\underline{\underline{m}}_{11}(\underline{U}_T + \underline{L}_1) + \underline{\underline{m}}_{12}(\underline{L}_2 - \underline{\underline{K}}_{22}\underline{s}_2)\}$$
$$+ \underline{\lambda}_2^T \underline{s}_3 + \underline{\lambda}_3^T \{\underline{\underline{m}}_{21}(\underline{U}_T + \underline{L}_1) + \underline{\underline{m}}_{22}(\underline{L}_2 - \underline{\underline{K}}_{22}\underline{s}_2)\}$$
$$+ \underline{\lambda}_4^T (\underline{\underline{\Omega}}\underline{s}_4 \text{ or } \underline{\underline{\beta}}\underline{s}_1) \tag{4.3.3}$$

where $\underline{\lambda}_1$, $\underline{\lambda}_2$, $\underline{\lambda}_3$, and $\underline{\lambda}_4$ are co-state variables associated with the state variables $\underline{s}_1$, $\underline{s}_2$, $\underline{s}_3$, and $\underline{s}_4$, respectively. Pontryagin's principle requires as necessary conditions that $\underline{\lambda}_1$, $\underline{\lambda}_2$, $\underline{\lambda}_3$, and $\underline{\lambda}_4$ satisfy co-state differential equations derivable from

$$\dot{\underline{\lambda}}_i = -\frac{\partial H}{\partial \underline{s}_i} \quad (i=1,\ldots,4) \tag{4.3.4}$$

On carrying out the implied partial differentiation in (4.3.4), one finds

$$\dot{\underline{\lambda}}_1 = -\underline{\underline{\beta}}^T \underline{\lambda}_4 - \underline{\lambda}_1^T \left(\underline{\underline{m}}_{11}\left[\frac{\partial \underline{L}_1}{\partial \underline{s}_1}\right] + \underline{\underline{m}}_{12}\left[\frac{\partial \underline{L}_2}{\partial \underline{s}_1}\right]\right) - \underline{\lambda}_3^T \left(\underline{\underline{m}}_{21}\left[\frac{\partial \underline{L}_1}{\partial \underline{s}_1}\right] + \underline{\underline{m}}_{22}\left[\frac{\partial \underline{L}_2}{\partial \underline{s}_1}\right]\right) \tag{4.3.5a}$$

$$\dot{\underline{\lambda}}_2 = -\underline{\underline{W}}_{22}\underline{s}_2 + \underline{\underline{K}}_{22}\left(\underline{\underline{m}}_{21}\underline{\lambda}_1 + \underline{\underline{m}}_{22}\underline{\lambda}_3\right)$$
$$- \underline{\lambda}_1^T \left(\left[\frac{\partial \underline{\underline{m}}_{11}}{\partial \underline{s}_2}\right](\underline{U}_T + \underline{L}_1) + \left[\frac{\partial \underline{\underline{m}}_{12}}{\partial \underline{s}_2}\right](\underline{L}_2 - \underline{\underline{K}}_{22}\underline{s}_2) + \underline{\underline{m}}_{11}\left[\frac{\partial \underline{L}_1}{\partial \underline{s}_2}\right] + \underline{\underline{m}}_{12}\left[\frac{\partial \underline{L}_2}{\partial \underline{s}_2}\right]\right)$$
$$- \underline{\lambda}_3^T \left(\left[\frac{\partial \underline{\underline{m}}_{21}}{\partial \underline{s}_2}\right](\underline{U}_T + \underline{L}_1) + \left[\frac{\partial \underline{\underline{m}}_{22}}{\partial \underline{s}_2}\right](\underline{L}_2 - \underline{\underline{K}}_{22}\underline{s}_2) + \underline{\underline{m}}_{21}\left[\frac{\partial \underline{L}_1}{\partial \underline{s}_2}\right] + \underline{\underline{m}}_{22}\left[\frac{\partial \underline{L}_2}{\partial \underline{s}_2}\right]\right) \tag{4.3.5b}$$

$$\dot{\underline{\lambda}}_3 = - \underline{\underline{W}}_{33}\underline{s}_3 - \underline{\lambda}_2 - (\underline{\lambda}_1^T \underline{\underline{m}}_{11} + \underline{\lambda}_3^T \underline{\underline{m}}_{21}) \left[\frac{\partial \underline{L}_1}{\partial \underline{s}_3}\right] - (\underline{\lambda}_1^T \underline{\underline{m}}_{12} + \underline{\lambda}_3^T \underline{\underline{m}}_{22}) \left[\frac{\partial \underline{L}_2}{\partial \underline{s}_3}\right] \qquad (4.3.5c)$$

$$\dot{\underline{\lambda}}_4 = \underline{\underline{\Omega}}\ \underline{\lambda}_4 \qquad (4.3.5d)$$

In addition, Pontryagin's principle requires as a necessary condition for an arbitrary smooth unbounded admissible control that

$$\frac{\partial H}{\partial \underline{U}_T} = 0 = \underline{\underline{W}}_{uu}\ \underline{U}_T + \underline{\underline{m}}_{11}\ \underline{\lambda}_1 + \underline{\underline{m}}_{21}^T\ \underline{\lambda}_3 \qquad (4.3.6)$$

from which the optimal torque vector $\underline{U}_T$ is determined as a function of the co-state variables $\underline{\lambda}_1$ and $\underline{\lambda}_3$ as

$$\underline{U}_T = - \underline{\underline{W}}_{uu}^{-1}\ (\underline{\underline{m}}_{11}\ \underline{\lambda}_1 + \underline{\underline{m}}_{21}^T\ \underline{\lambda}_3) \qquad (4.3.7)$$

Introducing (4.3.7) into (4.2.8) and (4.3.5) yields the state and co-state differential equations defining the two-point-boundary-value problem as

(a) state equations:

$$\underline{s}_1 = \underline{\underline{m}}_{11}\left[\underline{L}_1 - \underline{\underline{W}}_{uu}^{-1}(\underline{\underline{m}}_{11}\underline{\lambda}_1 + \underline{\underline{m}}_{21}^T\underline{\lambda}_3)\right] + \underline{\underline{m}}_{12}(\underline{L}_2 - \underline{\underline{K}}_{22}\underline{s}_2) \qquad (4.3.8a)$$

$$\underline{s}_3 = \underline{s}_3 \qquad (4.3.8b)$$

$$\underline{s}_4 = \underline{\underline{m}}_{21}\left[\underline{L}_1 - \underline{\underline{W}}_{uu}^{-1}(\underline{\underline{m}}_{11}\underline{\lambda}_1 + \underline{\underline{m}}_{21}^T\underline{\lambda}_3)\right] + \underline{\underline{m}}_{22}(\underline{L}_2 - \underline{\underline{K}}_{22}\underline{s}_2) \qquad (4.3.8c)$$

$$\underline{s}_4 = \underline{\underline{\Omega}}\ \underline{s}_4 \quad \text{or} \quad \underline{\underline{\beta}}\ \underline{s}_1 \qquad (4.3.8d)$$

(b) co-state equations:

$$\dot{\underline{\lambda}}_1 = -\underline{\underline{\beta}}^T\underline{\lambda}_4 - \underline{\underline{\lambda}}_1^T\left(\underline{\underline{m}}_{11}\left[\frac{\partial \underline{L}_1}{\partial \underline{s}_1}\right] + \underline{\underline{m}}_{12}\left[\frac{\partial \underline{L}_2}{\partial \underline{s}_1}\right]\right) - \underline{\underline{\lambda}}_3^t\left(\underline{\underline{m}}_{21}\left[\frac{\partial \underline{L}_1}{\partial \underline{s}_1}\right] + \underline{\underline{m}}_{22}\left[\frac{\partial \underline{L}_2}{\partial \underline{s}_1}\right]\right) \tag{4.3.9a}$$

$$\begin{aligned}
\dot{\underline{\lambda}}_2 = & -\underline{\underline{W}}_{22}\underline{s}_2 + \underline{\underline{K}}_{22}(\underline{\underline{m}}_{21}\underline{\lambda}_1 + \underline{\underline{m}}_{22}\underline{\lambda}_3) \\
& - \underline{\lambda}_1^T\left(\left[\frac{\partial \underline{\underline{m}}_{11}}{\partial \underline{s}_2}\right]\left[\underline{L}_1 - \underline{\underline{W}}_{uu}^{-1}(\underline{\underline{m}}_{11}\underline{\lambda}_1 + \underline{\underline{m}}_{21}^T\underline{\lambda}_3)\right] + \left[\frac{\partial \underline{\underline{m}}_{12}}{\partial \underline{s}_2}\right](\underline{L}_2 - \underline{\underline{K}}_{22}\underline{s}_2)\right. \\
& \qquad \left. + \underline{\underline{m}}_{11}\left[\frac{\partial \underline{L}_1}{\partial \underline{s}_2}\right] + \underline{\underline{m}}_{12}\left[\frac{\partial \underline{L}_2}{\partial \underline{s}_2}\right]\right) \\
& - \underline{\lambda}_3^T\left(\left[\frac{\partial \underline{\underline{m}}_{21}}{\partial \underline{s}_2}\right]\left[\underline{L}_1 - \underline{\underline{W}}_{uu}^{-1}(\underline{\underline{m}}_{11}\underline{\lambda}_1 + \underline{\underline{m}}_{21}^T\underline{\lambda}_3)\right] + \left[\frac{\partial \underline{\underline{m}}_{22}}{\partial \underline{s}_2}\right](\underline{L}_2 - \underline{\underline{K}}_{22}\underline{s}_2)\right. \\
& \qquad \left. + \underline{\underline{m}}_{21}\left[\frac{\partial \underline{L}_1}{\partial s_2}\right] + \underline{\underline{m}}_{22}\left[\frac{\partial \underline{L}_2}{\partial s_2}\right]\right)
\end{aligned} \tag{4.3.9b}$$

$$\dot{\underline{\lambda}}_3 = -\underline{\underline{W}}_{33}\underline{s}_3 - \underline{\lambda}_2 - (\underline{\lambda}_1^T\underline{\underline{m}}_{11} + \underline{\lambda}_3^T\underline{\underline{m}}_{21})\left[\frac{\partial \underline{L}_1}{\partial \underline{s}_3}\right] - (\underline{\lambda}_1^T\underline{\underline{m}}_{12} + \underline{\lambda}_3^T\underline{\underline{m}}_{22})\left[\frac{\partial \underline{L}_2}{\partial \underline{s}_3}\right] \tag{4.3.9c}$$

$$\dot{\underline{\lambda}}_4 = \underline{\underline{\Omega}}\,\underline{\lambda}_4 \tag{4.3.9d}$$

As usual, the co-state vector's initial and terminal conditions are unknown a priori, leading to a two-point-boundary-value problem.

4.4 HYBRID RELAXATION PROCESS FOR THE SOLUTION OF THE TWO-POINT-BOUNDARY-VALUE PROBLEM

The hybrid relaxation described in this section combines the boundary condition relaxation process of Chapter II with the differential equation relaxation process of Chapter III . The procedure is this: First, a decision is made to start the relaxation process about one of the vehicle's principle axis * ; then, a sequence of boundary condition vectors is introduced where the n-th boundary condition vector is denoted by

$$\underline{X}_n = \Big[\theta_{1n}(t_0) \quad \theta_{2n}(t_0) \quad \theta_{3n}(t_0) \quad \underline{\omega}_n^T(t_0) \quad \underline{\eta}_n^T(t_0) \quad \dot{\underline{\eta}}_n^T(t_0)$$

$$\theta_{1n}(t_f) \quad \theta_{2n}(t_f) \quad \theta_{3n}(t_f) \quad \underline{\omega}_n^T(t_f) \quad \underline{\eta}_n^T(t_f) \quad \dot{\underline{\eta}}_n^T(t_f)\Big]^T ;$$

$$n=0,1,\ldots,N \qquad (4.4.1)$$

where $\underline{X}_N = \underline{X}_{true}$ is the true desired boundary conditions, and

$\underline{X}_0 = \underline{X}_{start}$ is a set of single axis boundary condition for which a closed form solution exists.

Before any calculations are performed, the Euler angles θ_{1n} , θ_{2n} , and θ_{3n} are converted into Euler parameters using (D.9). If the Euler parameters were introduced directly into (4.4.1), and then relaxed, the implicit constraint on the sum square of the Euler parameters would be violated; and the Euler parameters would no longer represent a valid set of attitude variables. Hence, by relaxing on three angles the difficulties associated with relaxing the Euler parameters can be avoided.

Introducing the relaxation parameter α such that

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* As in Chapter II , this decision is somewhat arbitrary; one successful criteria for selection has been based on choosing the axis with the largest angular velocity, either initially or finally.

$$0 \leqslant \alpha_n \leqslant 1 \ , \ \{0 \leqslant \alpha_1 < \alpha_2 < \ldots\ldots < \alpha_N = 1\} \ ,$$

the boundary conditions for the n-th step in the relaxation process are obtained from the linear sweep equation

$$\underline{X}_n = \underline{X}_0 + \alpha_n \ (\underline{X}_N - \underline{X}_0) \tag{4.4.2}$$

As in (4.4.2), $\alpha_n = 0$ produces $\underline{X}_0$ and $\alpha_n = 1$ produces $\underline{X}_N$; thus, as n increases from 0 to N the relaxation process solves a problem closer (in terms of the boundary conditions) to the real problem of interest.

In the second part of the hybrid process, the relaxation parameter is imbedded in the equation of motion (4.1.38); where α_n multiplies the time varying terms in the mass matrix $\underline{\underline{M}}$, as well as multiplying the vector $\underline{L}$, as follows:

$$\underline{\underline{M}}(\alpha_n) \begin{Bmatrix} \underline{\dot{\omega}} \\ \cdots\cdots \\ \underline{\ddot{\eta}} \end{Bmatrix} + \underline{\underline{K}} \begin{Bmatrix} \underline{0} \\ \cdots\cdots \\ \underline{\eta} \end{Bmatrix} = \underline{\underline{\Pi}} \ \underline{U}_T + \alpha_n \ \underline{L} \tag{4.4.3}$$

The mass matrix $\underline{\underline{M}}(\alpha_n)$ in (4.4.3) has the following structure

$$\underline{\underline{M}}(\alpha_n) = \begin{bmatrix} J_{xx}(\alpha_n) & J_{xy}(\alpha_n) & J_{xz}(\alpha_n) & \underline{0}^T & \underline{H}_1^T \\ J_{yx}(\alpha_n) & J_{yy}(\alpha_n) & J_{yz}(\alpha_n) & \underline{0}^T & \underline{H}_2^T \\ J_{zx}(\alpha_n) & J_{zy}(\alpha_n) & J_{zz}(\alpha_n) & \underline{H}_3^T & \underline{0}^T \\ \alpha_n \underline{\underline{H}}_{11}^T \eta_3 & \alpha_n \underline{\underline{H}}_{21}^T \eta_3 & \underline{H}_3 + \alpha_n \underline{\underline{H}}_{31}^T \eta_2 & \underline{\underline{M}}_{22} & \underline{\underline{0}} \\ \underline{H}_1 + \alpha_n \underline{\underline{H}}_{12}^T \eta_2 & \underline{H}_2 + \alpha_n \underline{\underline{H}}_{22}^T \eta_2 & \underline{0} & \underline{\underline{0}} & \underline{\underline{M}}_{33} \end{bmatrix} \tag{4.4.4}$$

where the elements of the moment of inertia tensor $\underline{\underline{J}}$ become

$$\left.\begin{aligned}
J_{xx}(\alpha_n) &= I_{xx} + \alpha_n\,(\underline{\eta}_2^T\underline{\underline{J}}_{22xx}\underline{\eta}_2 + \underline{\eta}_3^T\underline{\underline{J}}_{33xx}\underline{\eta}_3)\\
J_{xy}(\alpha_n) &= \qquad\;\;\alpha_n\,(\underline{J}_{2xy}^T\underline{\eta}_2 + \underline{\eta}_2^T\underline{\underline{J}}_{12xy}\underline{\eta}_2)\\
J_{xz}(\alpha_n) &= \qquad\;\;\alpha_n\,(\underline{J}_{3xz}^T\underline{\eta}_3 + \underline{\eta}_2^T\underline{\underline{J}}_{13xz}\underline{\eta}_3)\\
J_{yy}(\alpha_n) &= I_{yy} + \alpha_n\,(\underline{\eta}_2^T\underline{\underline{J}}_{22yy}\underline{\eta}_2 + \underline{\eta}_3^T\underline{\underline{J}}_{33yy}\underline{\eta}_3)\\
J_{yz}(\alpha_n) &= \qquad\;\;\alpha_n\,(\underline{J}_{3yz}^T\underline{\eta}_3 + \underline{\eta}_2^T\underline{\underline{J}}_{23yz}\underline{\eta}_3)\\
J_{zz}(\alpha_n) &= I_{zz} + \alpha_n\,(\underline{\eta}_2^T\underline{\underline{J}}_{22zz}\underline{\eta}_2 + \underline{\eta}_3^T\underline{\underline{J}}_{33zz}\underline{\eta}_3)
\end{aligned}\right\rbrace \qquad (4.4.5)$$

The steps following (4.1.38) leading to the state and co-state remain the same; Only now all equations are explicit functions of α_n . In (4.4.3) when $\alpha_n = 0$, the equation of motion possesses the constant coefficient matrices $\underline{\underline{M}}$, $\underline{\underline{K}}$, and $\underline{\underline{\Pi}}$; on the other hand when $\alpha_n = 1$, the full time varying character of the equations of motion is restored. By setting $\alpha_n = 0$, and using (4.4.2) and (4.4.3), the equations of motion and the problem boundary conditions reduce to the linear single axis flexible body case treated in Section 3.3.1. The fact that the linear single axis case possesses a closed form solution (combined with the ability of the relaxation parameter to reduce the general equations of motion and the problem boundary conditions to that special case), provides the principle motivation for introducing the hybrid relaxation process.

Hence, as α_n is increased from 0 to 1 , the hybrid relaxation process analytically continues the closed form solution of Section 3.3.1 , through a sequence of intermediate problem; defined by boundary conditions and the structure of the equation of motion.

In order to set up the proper boundary conditions for the three-dimensional problem the single axis results of Section 3.3.4 must be converted into an equivalent three-dimensional form. The equations to accomplish this transformation are found in Appendix H , where weighting matrices and co-state variables are related in physical and modal space representations of the equation of motion and optimal control problem formulations. Since the results of Chapter III do not use Euler parameters, the co-state variables for the Euler parameters must be determined by some other means. The simplest way to determine the proper form for the Euler parameter co-states, is to formulate the optimal control problem in Chapter III using Euler parameters. The result of carrying out this formulation is that the Euler parameter co-states for the three-dimensional problem are found to be

$$\left.\begin{aligned} \lambda_{4,1}(t_0) &= 2\,\dot{\lambda}_{\omega i}(t_0)\ \beta_i(t_0) \\ \lambda_{4,i+1}(t_0) &= -\,2\,\dot{\lambda}_{\omega i}(t_0)\ \beta_0(t_0) \\ \lambda_{4,j+1}(t_0) &= 0\ , \quad \text{for } i \neq j\ ,\ j=1,2,3 \end{aligned}\right\} \qquad (4.4.6)$$

where $\dot{\lambda}_{\omega i}(t_0)$ denotes the differential equation for the physical space co-state angular velocity evaluated at $t = t_0$, and "i" denotes that the single axis problem of Chapter III is solved about the i-th axis of the three-dimensional problem.

To obtain $\dot{\lambda}_{\omega i}(t_0)$, (3.3.7b) is mapped to physical coordinates by the methods of Appendix H , and then $\dot{\lambda}_{\omega i}(t_0)$ is taken as the first element of $\dot{\underline{\lambda}}_2$, which is defined in (3.3.3).

These considerations thus motivate the following successive approximation strategy to solve the two-point-boundary-value problem.

Letting the approximate co-states be denoted by

$$\underline{\Lambda}(0) = \left[\underline{\lambda}_1(t_0) \quad \underline{\lambda}_2(t_0) \quad \underline{\lambda}_3(t_0) \quad \underline{\lambda}_4(t_0)\right] \qquad (4.4.7)$$

the differential correction strategy is to seek the correction vector $\Delta\underline{\Lambda}$ subject to the terminal constraint

$$\underline{\zeta}_f' - \underline{\zeta}'\left[\underline{\Lambda}(0)+\Delta\underline{\Lambda}, t_f\right] = 0 \qquad (4.4.8)$$

where $\underline{\zeta}_f'$ denotes that the three angles $\theta_1(t_f)$, $\theta_2(t_f)$, and $\theta_3(t_f)$ in (4.3.2b) have been replaced by $\beta_1(t_f)$, $\beta_2(t_f)$, $\beta_3(t_f)$ * ,

and $\Delta\underline{\Lambda} = \left[\Delta\underline{\lambda}_1^T \quad \Delta\underline{\lambda}_2^T \quad \Delta\underline{\lambda}_3^T \quad \Delta\underline{\lambda}_4^T\right]^T$.

On linearizing (4.4.8), one obtains

$$\underline{\zeta}_f' - \hat{\underline{\zeta}}' - \underline{\underline{A}}_{s\Lambda}\, \Delta\underline{\Lambda} = \underline{0} \qquad (4.4.9)$$

where $\hat{\underline{\zeta}}'$ denotes the solution of (4.3.8) and (4.3.9) for some specific choice of $\underline{\Lambda}(0)$; and

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* $\beta_0(t_f)$ is not considered because the Euler parameters are a once redundant set of attitude variables; in fact, any three of the Euler parameters can be chosen and the procedure is not altered.

$$\underline{\underline{A}}_{s\Lambda} = \left[\frac{\partial(\underline{\zeta}')^T}{\partial \underline{\Lambda}(0)} \bigg|_{t_f} \right]^T = \underline{\underline{\phi}}_{\zeta'\Lambda}{}^{*} \tag{4.4.10}$$

$\underline{\underline{\phi}}_{\zeta'\Lambda}$ denotes the block of the state transition matrix for the variables $\underline{\zeta}'$ and $\underline{\Lambda}$.

The calculation of the state transition matrix partial derivatives is a separate issue, dealt with in Appendix B. Now defining

$$\underline{\Delta\zeta}'_f = \underline{\zeta}'_f - \hat{\underline{\zeta}}' \tag{4.4.11}$$

and introducing (4.4.11) into (4.4.9), one obtains

$$\underline{\Delta\zeta}'_f = \underline{\underline{\phi}}_{\zeta'\Lambda}\, \underline{\Delta\Lambda} \tag{4.4.12}$$

In the normal scheme of things (4.4.12) is inverted for the differential corrections $\underline{\Delta\Lambda}$, as follows:

$$\underline{\Delta\Lambda} = \underline{\underline{\phi}}^{\dagger}_{\zeta'\Lambda}\, \underline{\Delta\zeta}_f \tag{4.4.13}$$

where $\underline{\underline{\phi}}^{\dagger}_{\zeta'\Lambda}$ denotes the generalized inverse of $\underline{\underline{\phi}}_{\zeta'\Lambda}$.

A generalized inverse is required because $\underline{\underline{\phi}}_{\zeta'\Lambda}$ is rectangular, since one of the Euler parameters is not constrained. As currently formulated (4.4.13) represents the stumbling block to the solution of the three-dimensional flexible body problem; there are two reasons for this: (1) currently available generalized matrix

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* $\underline{\underline{\phi}}_{\zeta'\Lambda}$ is not a square matrix, because only the final three Euler parameters are being constrained to satisfy their prescribed terminal values.

inverse algorithms * have failed to produce convergent solutions in the differential correction algorithm of Figure 4.2 (apparently due to the row and column rank deficiency in $\underset{=}{\phi}_{\zeta',\Lambda}$), and (2) matrix partitioning strategies have failed since it is found that the row and column rank of $\underset{=}{\phi}_{\zeta',\Lambda}$ is time varying ** . In Section 4.5 a more complete discussion is provided detailing the various methods that have been applied in the attempt to obtain a workable solution to (4.4.12).

If it were possible to invert (4.4.13) for the corrections $\underline{\Delta\Lambda}$ the solution process would proceed exactly analogous to the solution of Chapters II and III , as diagrammed in Figure 4.2 , and the initial co-state extrapolations would be determined from an appropriately dimensioned form of (3.4.9).

In all simulations run the admissible assumed deflection shapes are taken to be

$$\underline{\phi} = \left[\phi_1 \quad \phi_2 \; \ldots\ldots \; \phi_p\right] , \; \phi_p = x^{p+1} \qquad (4.4.14)$$

4.5 DISCUSSION

As pointed out in Section 4.4 the inversion of (4.4.13)

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* For example, singular value matrix decomposition routines.

** Without a priori knowledge of the rank of the linear system, partitioning strategies are defeated at the onset, since they depend on the knowledge of the rank of the linear system.

represents the major obstacle to the solution of the presently formulated three-dimensional flexible body optimal control problem. Many strategies have been tried in an effort to invert (4.4.13); e.g., the weighted minimum norm algorithm, the weighted least square algorithm, combined weighted minimum norm and least square algorithms, matrix partitioning methods, gradient optimization methods, and singular value matrix decomposition* routines.

The most promising of these methods is the singular value matrix decomposition method; yet, even so, this technique has not been found to yield solutions which have lead to convergence in the algorithm shown in Figure 4.2.

Two observations are appropriate: First, the difficulties associated with inverting (4.4.13) appear to be more fundamental than just the failure of one or more of the methods applied; and Second, new methods may be required in order to invert row and column deficient systems like the one encountered in (4.4.13)

In regard to the first observation, the fundamental source of difficulty in inverting (4.4.13) arises from two facts: (1) the vehicle possesses high geometrical symmetry (e.g., four identical elastic appendages located symmetrically about a rigid central hub)**, and (2) the motion considered is near the pure spin anti-

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* The singular value decomposition of a matrix is used to obtain a generalized inverse, in the sense that some rank deficient systems can be solved, when ordinarily the minimum norm or least square operator would fail to exist [44].

** It can be anticipated that any vehicle being modeled, which possesses high geometrical symmetry can be expected to experience similar difficulties.

symmetric case which can be fully described by a subset of the generalized coordinates.

The result of this high symmetry is that several rows and columns of $\underset{=}{\phi}_{\zeta'\Lambda}$ are repeated or nearly repeated depending upon initial condition variations; thus, leading to the row and column rank deficiency of this matrix. To compound the problem, as the mass center of the vehicle moves and the elastic deflection coordinates interact, the row and column rank $\underset{=}{\phi}_{\zeta'\Lambda}$ changes as a function of time. Without a priori knowledge of the instantaneous rank of the system, there seems little reason to anticipate a quick resolution of the difficulties associated with inverting (4.4.13).

In Chapter III the same symmetry problem exists, but because only anti-symmetric elastic deflections are permitted, the problem can be treated with only one u(x,t) deflection coordinate; thereby eliminating what is believed to be the root cause of the rank deficiency.

Further work is required before the above difficulties can be properly resolved. One simple alternative strategy for this problem is to abandon the quest for the general solution to the problem, and instead, solve three separate single axis maneuvers to accomplish the desired reorientation of the vehicle. In this way, the solution of Chapter III provides a method to accomplish the maneuver in a sub-optimal fashion. From a practical view point this last strategy represents at least a workable solution for the problem, and in view of the execution time for the problem in Chapter III is 30 to 1 (computer time versus real time) compared with 180 to 1 for the formulation of this chapter, it may prove to be the most attractive.

Of course, there still remains the possibility that methods may be developed for inverting (4.4.13); only further study and computational tests can answer this possibility.

This concludes the treatment of the three-dimensional problem.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

5.1 SUMMARY AND CONCLUSIONS

During the course of this research several significant problems of interest to the aerospace community have been addressed and solved. These include:

(1) Optimal large angle rotational maneuvers for rigid vehicles undergoing very general attitude maneuvers; and

(2) Optimal large angle single axis rotational maneuvers of flexible vehicles, restricted to antisymmetric deformation modes in the plane normal to the rotation axis.

In the case of optimal general large angle rotational maneuvers of flexible vehicles, a solution has not been obtained, though a previously unrecognized stumbling block has been identified.

Common to all of the problems receiving treatment, in this thesis, is their inherent nonlinearity. For the rigid body problem (1) a gyroscopic nonlinearity occurs in Euler's equation, where for three-dimensional maneuvers the angular velocities interact quadratically. In the single axis flexible body case (2) the nonlinearity comes into the problem from two sources: First, from kinematic terms, and Second, from structural terms arising from the moment of inertia matrix being time varying. Finally, in the three-dimensional flexible body problem the nonlinearity arises from both the angular velocities and the in- and out-of-plane deflections coupling in the general motion.

The strategy that has been developed for dealing with the nonlinear problems in this thesis is the following:

(1) Obtain a closed form solution for a special case of each of the problems by either altering the problem's boundary conditions or altering the structure of the governing differential equations; and

(2) Introducing a well defined process, whereby the solution obtained in (1) can be analytically continued (in this thesis by discrete processes) until the real problem of interest has been solved.

Three analytic continuation methods have been applied to the problems of this thesis. These methods include:

(a) Boundary condition relaxation processes (see Chapter II);

(b) Differential equation relaxation processes (see Chapter III); and

(c) Hybrid relaxation processes combining (a) and (b) above (see Chapter IV).

These analytic continuation methods have made it possible to obtain optimal torque histories for systems of nonlinear differential equations, satisfying the rigorous necessary conditions derived from Pontryagin's principle, for optimal spacecraft rotational maneuvers. Many other methods have been proposed for solving the nonlinear two-point-boundary-value problem receiving treatment in this thesis; e.g., shooting methods, invariant imbedding, parameterization/parameter optimization, and function space gradient methods. From the point of view of someone examining the available literature on solution methods for two-point-boundary-value problems, the methods used in this thesis should be viewed as providing a practical, systematic procedure that can be used either independently of or in conjunction with the older methods. The unique feature and strength of the continuation methods resides in the fact that a sequence of increasingly more difficult problems is solved starting from a known solution, where the increased participation of the nonlinearity in each intermediate problem can be controlled. The ability to control the change in the participation of the nonlinearity in the sequence of problems is the cardinal virtue of this method.

The principal results of this research are:

(1) A general nonsingular optimal maneuver formulation for rigid body optimal large angle rotational maneuvers; treating all kinematic and dynamical nonlinearities, and considering general orientation and angular velocity boundary conditions. The maneuver times are treated as fixed, and the performance index consists of the integral of the sum square of the control torque;

(2) Development of a boundary condition relaxation/analytic continuation method for the solution of the general rigid body problem;

(3) A general method for determining optimal single axis large angle rotational maneuvers for a class of flexible space vehicles; treating all kinematic and structural nonlinearities. The maneuver times are treated as fixed, and the performance

index consists of the integral of the weighted square of the control torque and transformed angle, angular rate, assumed modes, and mode rates;

(4) Development of a differential equation relaxation/analytic continuation method for the solution of the single axis flexible body case;

(5) Formulation of the general solution for three-dimensional maneuvers of flexible space vehicles. The maneuvers times are treated as fixed, and the performance index consists of the weighted sum square of the control torques and weighted sum square of the modal amplitudes and amplitude rates;

(6) Formulation of a hybrid relaxation/analytic continuation method that combines the boundary condition method of (2) with the differential equation method of (4); and

(7) A heretofore unidentified stumbling block has been identified, which must be circumvented before the formulation and method of (5) and (6) can be applied. In particular, this stumbling block has been found to be high symmetry in the dynamical model for the vehicle; which leads to a rank deficiency in the process which differentially corrects estimates for the initial co-states.

To place things in proper perspective, the methods developed in this thesis treat the problem of large angle rapid slewing maneuvers; yet, these methods are inefficient computationally if the desired maneuver is a small angle correction about some desired nominal attitude state. Hence, the complete attitude control problem must be viewed as two separate problems: First, the nonlinear large angle and large rate problem treated in this thesis, and Second, the small angle and small rate problem treated, e.g., by Meirovitch and Öz (see [35] , [36] , and [39]). Combining these methods provides the means to rapidly reorient a vehicle and then to keep the vehicle in the vicinity of the desired attitude state.

5.2 RECOMMENDATIONS

The results of this research provide the insight to move on to tackle more interesting and challenging variations of the problems solved herein. Many important issues remain to be addressed before the gap separating the theoretical results obtained and practical implementation onboard a space vehicle can be closed. Several of the more important issues recommended for further study are:

(1) Finding new ways of easing the computational burden of carrying out the iterative solution of systems of nonlinear differential equations (for the present time the solution processes would overly tax the onboard computational capability of current satellites and indeed, the computations for high degree of freedom simulations are challenging for ground-based computation);

(2) Giving consideration to the effects of the unmodeled degrees of freedom in the truncated dynamical models assumed for flexible spacecraft;

(3) Answering such questions as how stable and controllable the

physical systems are;

(4) Exploring how insensitive or robust the control systems are to parameter uncertainties or system changes (e.g., use of expendable materials, thermoelastic effects, onboard machinery, structural configuration shifts due to antenna motion or remaining expendable fluid supplies, and aerodynamic and other environmental effects);

(5) Examining what the effects of allowing greater freedom in the optimal control problem would be (e.g., allowing the final time to be free, modeling real control systems, formulating feedback control loops to track the open loop optimal torque slewing profiles). Furthermore, one could consider treating various alternative forms for the performance index (e.g., minimum energy, figure control, thermal control, and pointing control); multiple performance index optimal control problems for simultaneously extremizing integral measures of torque, energy, time, and structural configuration changes; looking into what suboptimal control laws might be useful, allowing some elements of the state to be free variables (e.g., allowing arbitrary phasing on reaction wheels, number of revolutions for the vehicle, etc.), distributed control, and nondeterministic stochastic optimal control; and

(6) Methods need to be developed to overcome the rank deficiency that is found to plague the solution process for the three-dimensional flexible body problem (where specifically the row and column rank of the state transition matrix, required in the

relaxation process is found to be time varying for vehicles possessing geometrical symmetry in the elastic degrees of freedom). One reasonable approach to this problem consists of reducing the three-dimensional results of Chapter IV to an equivalent single axis special case. Then by considering anti-symmetric flexural deformation cases for the appendages, perhaps insight can be gained which will lead to the solution of the three-dimensional problem.

In addition to the items above there are many areas of interest to be explored concerning computational strategies for solving the resulting control problem i.e., new relaxation/ continuation methods can be considered or combined and used with older methods; new ways can be developed for propagating the sensitivity matrices required in the differential correction processes.

From the discussion above it is clear that there are many fruitful, interesting, and challenging avenues work of remaining to receive treatment in the area spacecraft rotational maneuvers. The items listed in (1)-(6) provide a partial list of the areas still requiring treatment and these are recommended for future work. This brings the presentation of the results of this research to conclusion.

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APPENDIX A

DERIVATION OF THE KINETIC AND POTENTIAL ENERGY EQUATIONS

The results obtained in this appendix are valid for the general motion of an elastically deforming continuum. Expressions for both the kinetic and potential energy are obtained. Referring to Figure A.1 the vehicle being described consists of (n+1) domains, denoted by D_i , where i=0 refers to the rigid part of the vehicle; and i=1,.....,n refers to the remaining parts of the vehicle possessing elastic properties. The vectors and coordinate frames depicted in Figure A.1 have the following definitions:

$N_{n_1 n_2 n_3}$ is the Newtonian frame* ;

$O_{b_1 b_2 b_3}$ is the body fixed frame located at the vehicle's undeformed mass center O ;

$O'_{\zeta\xi\eta}$ is the body fixed frame, parallel to $O_{b_1 b_2 b_3}$, located at the deformed vehicle's mass center O' ;

$O_{\zeta_i \xi_i \eta_i}$ is the local frame locating the undeformed domain D_i with respect to $O'_{\zeta\xi\eta}$;

L_i is the constant direction cosine matrix relating $O'_{\zeta_i \xi_i \eta_i}$ to $O'_{\zeta\xi\eta}$;

$\underline{R}_{co}$ locates the undeformed vehicles mass center O relative to $N_{n_1 n_2 n_3}$;

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* The symbol A_{abc} denotes a dextral orthogonal set of unit vectors $\hat{\underline{a}}$, $\hat{\underline{b}}$, and $\hat{\underline{c}}$ with origin set at A .

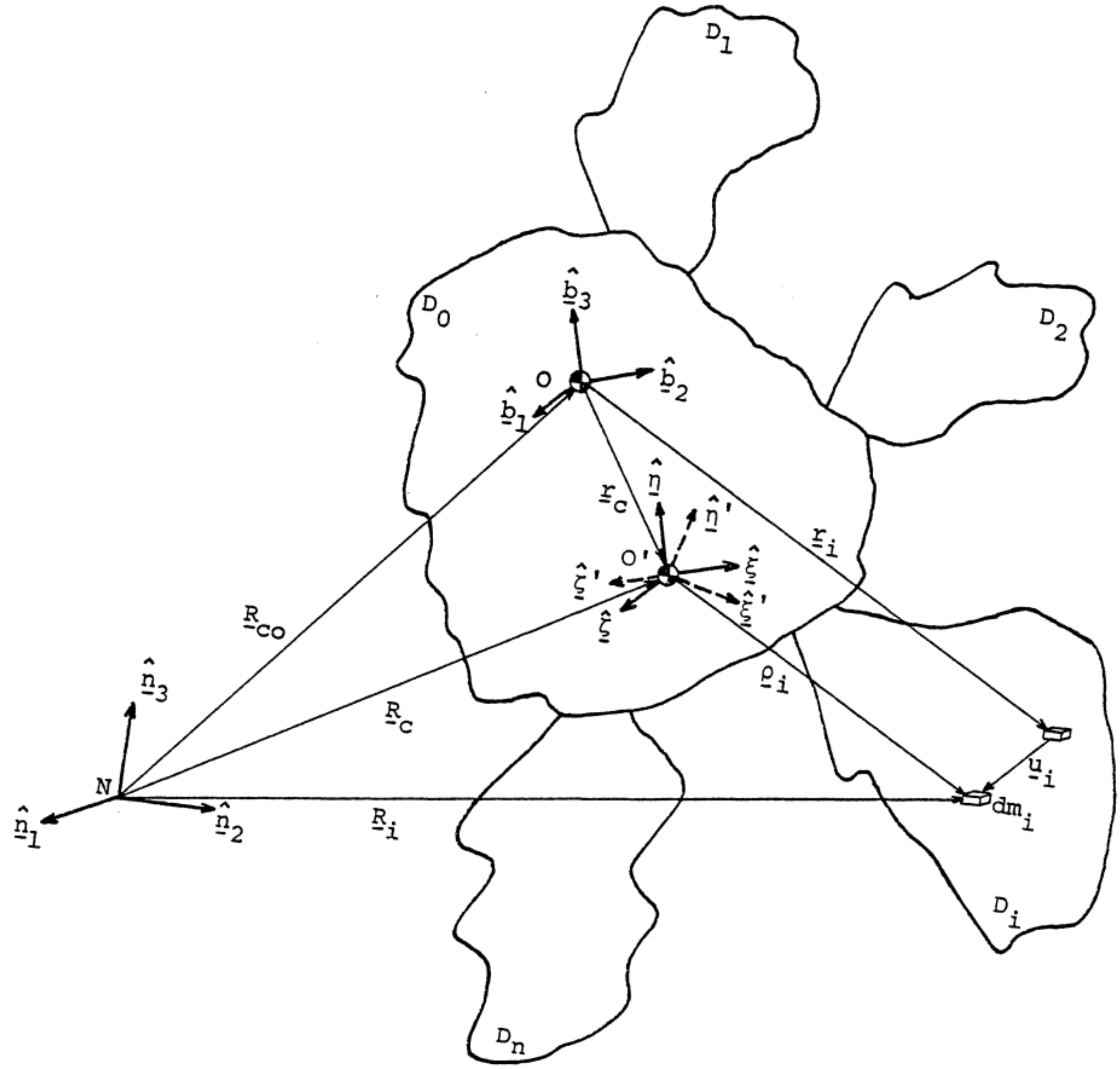

Figure A.1: Elastically Deforming Vehicle Containing n Elastic Domains.

$\underline{R}_c$ locates the deformed vehicles mass center O' relative to $N_{n_1 n_2 n_3}$;

$\underline{r}_c$ locates the deformed mass center O' with respect to the undeformed mass center O , such that $\underline{R}_c \equiv \underline{R}_{co} + \underline{r}_o$;

$\underline{dm}_i$ denotes a generic mass element in D_i ;

$\underline{r}_i$ locates dm_i in the undeformed vehicle;

$\underline{u}_i$ locates dm_i in the deformed vehicle;

$\underline{\rho}_i$ locates dm_i with respect to $O'_{\zeta_i \xi_i \eta_i}$, such that $\underline{\rho}_i \equiv \underline{r}_i + \underline{u}_{ic}$ where $\underline{u}_{ic} = \underline{u}_i - \underline{r}_c$;

$\underline{R}_i$ locates dm_i with respect to $N_{n_1 n_2 n_3}$ and represents the absolute position of dm_i at any time;

${}^{N}\underline{\omega}^{B}$ denotes the angular velocity of $O_{b_1 b_2 b_3}$ with respect to $N_{n_1 n_2 n_3}$; and

${}^{N}\underline{\omega}^{D_i}$ denotes the angular velocity of $O'_{\zeta_i \xi_i \eta_i}$ with respect to $N_{n_1 n_2 n_3}$, such that ${}^{N}\underline{\omega}^{B} = {}^{N}\underline{\omega}^{D_i} = \underline{\omega}$.

Since all integrations for the kinetic energy are carried out relative to the deformed vehicle's mass center, it is necessary to determine the vector $\underline{r}_c$. The vector $\underline{r}_c$ can be determined by integrating the vector locating dm_i with respect to $O_{b_1 b_2 b_3}$ over the entire vehicle, as follows:

$$\underline{r}_c = \frac{1}{m} \sum_{j=0}^{n} \int_{D_j} (\underline{r}_j + \underline{u}_j)\, dm_j \qquad \text{(A.1)}$$

where $m = \sum_{j=0}^{n} \int_{D_j} dm_j$

Equation (A.1) reduces to

$$\underline{r}_c = \frac{1}{m} \sum_{j=0}^{n} \int_{D_j} \underline{u}_j \, dm_j \tag{A.2}$$

where $\frac{1}{m} \sum_{j=0}^{n} \int_{D_j} \underline{r}_j \, dm_j = 0$ by definition of the center of mass.

The vector denoting the absolute position of dm_j in inertial space is given by

$$\underline{R}_j = \underline{R}_c + \underline{\rho}_j \tag{A.3}$$

In order to obtain the time derivative of (A.3) in N , the following vector identity is used

$$\left.\frac{d}{dt}\right|_N = \left.\frac{d}{dt}\right|_N + {}^N\underline{\omega}^R \times \tag{A.4}$$

where N and R denote the coordinate frames in which the differentiations are carried out.

Introducing (A.3) into (A.4), leads to

$$\left.\frac{d\underline{R}_j}{dt}\right|_N = \left.\frac{d\underline{R}_c}{dt}\right|_N + \left.\frac{d\underline{\rho}_j}{dt}\right|_N$$

$$= \left.\frac{d\underline{R}_c}{dt}\right|_N + \left.\frac{d\underline{\rho}_j}{dt}\right|_{D_j} + \underline{\omega} \times \underline{\rho}_j \tag{A.5}$$

Recalling that the kinetic energy for the entire vehicle is defined by

$$T = \frac{1}{2} \sum_{j=0}^{n} \int_{D_j} \left.\frac{d\underline{R}_j}{dt}\right|_N \cdot \left.\frac{d\underline{R}_j}{dt}\right|_N dm_j \tag{A.6}$$

and substituting (A.5) into (A.6), one obtains

$$T = \frac{1}{2} \sum_{j=0}^{n} \int_{D_j} \left(\left.\frac{d\underline{R}_c}{dt}\right|_N + \left.\frac{d\underline{\rho}_j}{dt}\right|_{D_j} + \underline{\omega}\times\underline{\rho}_j\right) \cdot \left(\left.\frac{d\underline{R}_c}{dt}\right|_N + \left.\frac{d\underline{\rho}_j}{dt}\right|_{D_j} + \underline{\omega}\times\underline{\rho}_j\right) dm_j \tag{A.7}$$

On carrying out the indicated vector dot products in (A.7) and then integrating the resulting equation over the mass distribution, one finds

$$T = \frac{1}{2} m \left.\frac{d\underline{R}_c}{dt}\right|_N \cdot \left.\frac{d\underline{R}_c}{dt}\right|_N + \frac{1}{2} \sum_{j=0}^{n} \int_{D_j} \left.\frac{d\underline{u}_{jc}}{dt}\right|_{D_j} \cdot \left.\frac{d\underline{u}_{jc}}{dt}\right|_{D_j} dm_j$$

$$+ \underline{\omega} \cdot \sum_{j=0}^{n} \int_{D_j} \underline{\rho}_j \times \left.\frac{d\underline{\rho}_j}{dt}\right|_{D_j} dm_j$$

$$+ \frac{1}{2} \underline{\omega}^T \cdot \sum_{j=0}^{n} \int_{D_j} \underline{\rho}_j \times (\underline{\omega} \times \underline{\rho}_j)\, dm_j \cdot \underline{\omega} \qquad \text{(A.8)}$$

where several integrals in (A.7) have vanished by virtue of the definition of the center of mass.

Equation (A.8) can be further simplified, yielding

$$T = \frac{1}{2} m \left.\frac{d\underline{R}_c}{dt}\right|_N \cdot \left.\frac{d\underline{R}_c}{dt}\right|_N + \frac{1}{2} \sum_{j=0}^{n} \int_{D_j} \left.\frac{d\underline{u}_{jc}}{dt}\right|_{D_j} \cdot \left.\frac{d\underline{u}_{jc}}{dt}\right|_{D_j} dm_j$$

$$+ \underline{\omega} \cdot \underline{H}_E + \frac{1}{2} \underline{\omega} \cdot \underline{\underline{J}} \cdot \underline{\omega} \qquad \text{(A.9)}$$

where $\underline{H}_E = \sum_{j=0}^{n} \underline{\underline{L}}_j^T \int_{D_j} \underline{\rho}_j \times \left.\frac{d\underline{\rho}_j}{dt}\right|_{D_j} dm_j$, is the elastic angular momentum;

$\underline{\underline{J}} = \sum_{j=0}^{n} \underline{\underline{L}}_j^T \underline{\underline{I}}_j \underline{\underline{L}}_j$, is the moment of inertia tensor for the vehicle;

$\underline{\underline{I}}_j = \int_{D_j} \left[(\underline{\rho}_j \cdot \underline{\rho}_j)\, \underline{\underline{U}} - \underline{\rho}_j\, \underline{\rho}_j^T \right] dm_j$, is the moment of inertia tensor for the j-th domain;

$\underline{\underline{L}}_j$ is the direction cosine matrix relating the j-th domain to the body frame; and $\underline{\underline{U}}$ is a 3 by 3 identity matrix such that

$\underline{\underline{U}}\, \underline{\underline{A}} = \underline{\underline{A}}\, \underline{\underline{U}} = \underline{\underline{A}}$.

Equation (A.9) represents the desired form for the kinetic energy.

In the problems being considered, the potential energy is restricted to the elastic potential energy. In particular, the axial deformations and torsion are ignored, so that only the transverse deformations are considered.

Subject to the restrictions above the potential energy takes the form

$$V = \frac{1}{2} \sum_{j=0}^{n} \int_{D_j} EI_j \left(\left[\frac{d^2 u_{2j}}{dx^2} \right]^2 + \left[\frac{d^2 u_{3j}}{dx^2} \right]^2 \right) dx \qquad (A.10)$$

where EI_j is the flexural rigidity; u_{2j} and u_{3j} are the allowed in- and out-of-plane deflections for the j-th domain.

With (A.9) and (A.10) this completes the derivation of the kinetic and potential energies.

APPENDIX B

STATE TRANSITION MATRIX INTEGRATION

The integration of the state transition matrix is split into two parts, based upon the following criteria:

(A) The differential equation describing the state is integrable only by numerical means; and

(B) The differential equation describing the state has the special form $\dot{\underline{S}}(t) = \underline{\underline{F}}\ \underline{S}$, where $\underline{\underline{F}}$ is an arbitrary real constant coefficient matrix possessing a complete set of right and left eigenvectors.

For part (A), in order to determine the necessary partial derivatives, the composite state vector S * is defined by:

$$\underline{S}(t) = \left[\underline{S}_1^T(t) \quad \underline{S}_2^T(t) \quad \underline{\lambda}_1^T(t) \quad \underline{\lambda}_2^T(t)\right]^T \tag{B.1}$$

where $\underline{S}_1(t) = \left[S_{11}^T(t) \quad S_{12}^T(t) \quad \ldots\ldots \quad S_{1n}^T(t)\right]^T \equiv$ state;

$$\underline{S}_2(t) = \left[S_{21}(t) \quad S_{22}(t) \quad \ldots\ldots \quad S_{2n}(t)\right]^T = \dot{\underline{S}}_1(t);$$

$$\underline{\lambda}_1(t) = \left[\lambda_{11}(t) \quad \lambda_{12}(t) \quad \ldots\ldots \quad \lambda_{1n}(t)\right]^T \equiv \text{co-state};$$

$$\underline{\lambda}_2(t) = \left[\lambda_{21}(t) \quad \lambda_{22}(t) \quad \ldots\ldots \quad \lambda_{2n}(t)\right]^T = \dot{\underline{\lambda}}_1(t); \quad \text{and}$$

n denotes the dimension of the state.

The differential equation describing (B.1) is assumed to be given by:

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* The system described in (B.1) is assumed to be governed by second order differential equations.

$$\dot{\underline{S}}(t) = \underline{F}\{\underline{S}(t),t\} \tag{B.2}$$

where (B.2) has the formal solution

$$\underline{S}(t) = \underline{S}(t_o) + \int_{t_o}^{t} \underline{F}\{\underline{S}(\tau),\tau\}\, d\tau \tag{B.3}$$

Taking the partial derivative of (B.3) with respect to $\underline{S}(t_o)$, leads to

$$\left[\frac{\partial \underline{S}(t)}{\partial \underline{S}(t_o)}\right] = [I] + \int_{t_o}^{t} \left[\frac{\partial \underline{F}\{\underline{S}(\tau),\tau\}}{\partial \underline{S}(\tau)}\right]\left[\frac{\partial \underline{S}(\tau)}{\partial \underline{S}(t_o)}\right] d\tau \tag{B.4}$$

where $[I]$ is a 4n by 4n identity matrix.

On examining (B.4) one is lead to the conclusion that the initial conditions for $\left[\frac{\partial \underline{S}(t)}{\partial \underline{S}(t_o)}\right]$ are given by

$$\left[\left.\frac{\partial \underline{S}(t)}{\partial \underline{S}(t_o)}\right|_{t=t_o}\right] = [I] \tag{B.5}$$

Finally, taking the time derivative of (B.4), one obtains

$$\dot{\underline{\underline{\phi}}}(t,t_o) = \underline{\underline{F}}\{\underline{S}(t),t\}\ \underline{\underline{\phi}}(t,t_o) \tag{B.6}$$

$$\text{where } \underline{\underline{\phi}}(t,t_o) = \left[\frac{\partial \underline{S}(t)}{\partial \underline{S}(t_o)}\right] \tag{B.7}$$

is the state transition matrix; and

$$\underline{\underline{F}}\{\underline{S}(t),t\} = \left[\frac{\partial \underline{F}\{\underline{S}(t),t\}}{\partial \underline{S}(t)}\right] \tag{B.8}$$

Hence, the solution process is the following: Given $\underline{S}(t_o)$ one integrates simultaneously (B.2) and (B.6) subject to (B.5) from t_o to t_f ; where at $t = t_f$ the required partial derivatives are obtained from the appropriate blocks of $\underline{\underline{\phi}}(t_o,t_f)$.

The method described in part (A) has the advantage that it works in all situations (barring numerical integration instabilities), and the disadvantage that is very expensive to carry out. For those situations where the conditions described in part (B) are valid, the following eigenvalue-eigenvector analysis yields a computationally superior algorithm.

Assuming that (B.2) takes the form

$$\dot{\underline{S}}(t) = \underline{\underline{F}}\ \underline{S}(t) \tag{B.9}$$

where $\underline{\underline{F}}$ is an arbitrary real constant coefficient matrix.
The first step in the solution process is to solve the following eigenvalue problems:

$$\underline{\underline{F}}\ \underline{\underline{R}} = \underline{\underline{R}}\ \underline{\underline{\Lambda}} \tag{B.10}$$

and

$$\underline{\underline{F}}^{T}\ \underline{\underline{L}} = \underline{\underline{L}}\ \underline{\underline{\Lambda}} \tag{B.11}$$ *

where $\underline{\underline{R}}$ denotes the matrix of the right eigenvectors of $\underline{\underline{F}}$; $\underline{\underline{L}}$ denotes the matrix of the left eigenvectors of $\underline{\underline{F}}$; and $\underline{\underline{\Lambda}}$ denotes the diagonal matrix of eigenvalues of $\underline{\underline{F}}$.**

The eigenvector matrices $\underline{\underline{R}}$ and $\underline{\underline{L}}$ satisfy a biorthogonality relationship which is established as follows: Given the i-th eigenvector of $\underline{\underline{R}}$ and the j-th eigenvector of $\underline{\underline{L}}$, and introducing these results into (B.10) and (B.11), one finds

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* Equations (B.10) and (B.11) have the same eigenvalues since the following relationship holds

$$\det.\left[\underline{\underline{F}} - \lambda\ \underline{\underline{I}}\right] = \det.\left[\underline{\underline{F}} - \lambda\ \underline{\underline{I}}\right]^{T} = \det.\left[\underline{\underline{F}}^{T} - \lambda\ \underline{\underline{I}}\right] \text{ , Q.E.D.}$$

** In general $\underline{\underline{\Lambda}}$ could possess a Jordan Block Structure, though for the applications considered here it possesses a strictly diagonal form ($\underline{\underline{R}}$, $\underline{\underline{L}}$, $\underline{\underline{\Lambda}}$ are in general complex).

$$\underline{\underline{F}}\ \underline{r}_i = \Lambda_i\ \underline{r}_i \tag{B.12a}$$

and

$$\underline{\underline{F}}^T\ \underline{\ell}_j = \Lambda_j\ \underline{\ell}_j \tag{B.12b}$$

Premultiplying (B.12a) by $\underline{\ell}_j^T$ and (B.12b) by $\underline{r}_i^T$ yields

$$\underline{\ell}_j^T\ \underline{\underline{F}}\ \underline{r}_i = \Lambda_i\ \underline{\ell}_j^T\ \underline{r}_i \tag{B.13a}$$

and

$$\underline{r}_i^T\ \underline{\underline{F}}^T\ \underline{\ell}_j = \Lambda_i\ \underline{r}_i^T\ \underline{\ell}_j \tag{B.13b}$$

Taking the transpose of (B.13b) and subtracting this result from (B.13a), one obtains

$$(\Lambda_i - \Lambda_j)\ \underline{\ell}_j^T\ \underline{r}_i = 0 \tag{B.14}$$

For the case that $i \neq j$, it is clear that

$$\underline{\ell}_j^T\ \underline{r}_i = 0 \tag{B.15}$$

which establishes the desired biorthogonality condition. From (B.14) it is clear that for $i = j$ that $\underline{r}_i$ and $\underline{\ell}_j$ can be normalized by taking

$$\underline{r}_i^T\ \underline{\ell}_j = 1 \tag{B.16}$$

So that in matrix form the (normalized) biorthogonality condition becomes

$$\underline{\underline{L}}^T\ \underline{\underline{R}} = \underline{\underline{I}} \tag{B.17}$$

One consequence of (B.17) is that an explicit equation for the matrix of eigenvalues of $\underline{\underline{F}}$ can be obtained as follows: Rewriting (B.13a) for all eigenvectors and eigenvalues as

$$\underline{\underline{F}}\ \underline{\underline{R}} = \underline{\underline{R}}\ \underline{\underline{\Lambda}} \tag{B.18}$$

then premultiplying (B.18) by $\underline{\underline{L}}^T$ and taking note of (B.17), one obtains

$$\underline{\underline{L}}^T\ \underline{\underline{F}}\ \underline{\underline{R}} = \underline{\underline{\Lambda}} \tag{B.19}$$

With the results of (B.17) and (B.19) the solution for the state transition matrix proceeds as follows: Introducing the coordinate transformation

$$\underline{S}(t) = \underline{\underline{R}}\ \underline{T}(t) \tag{B.20}$$

into (B.9), yields

$$\underline{\underline{R}}\ \dot{\underline{T}}(t) = \underline{\underline{F}}\ \underline{\underline{R}}\ \underline{T}(t) \tag{B.21}$$

On premultiplying (B.21) by $\underline{\underline{L}}^T$ and recalling (B.17) and (B.19), (B.21) becomes

$$\dot{\underline{T}}(t) = \underline{\underline{L}}^T\ \underline{\underline{F}}\ \underline{\underline{R}}\ \underline{T}(t) = \underline{\underline{\Lambda}}\ \underline{T}(t) \tag{B.22}$$

Since it is assumed that $\underline{\underline{\Lambda}}$ is diagonal, (B.22) possesses the solution

$$\underline{T}(t) = e^{[\underline{\underline{\Lambda}}t]}\ \underline{T}(t_o) \tag{B.23}$$

where

$$e^{[\underline{\underline{\Lambda}}t]} = \begin{bmatrix} e^{\Lambda_1 t} & 0 & \cdots\cdots & 0 \\ 0 & e^{\Lambda_2 t} & \cdots\cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots\cdots & e^{\Lambda_n t} \end{bmatrix} \tag{B.24}$$

is the exponential matrix, and Λ_i is the i-th eigenvalue of $\underline{\underline{\Lambda}}$.

Equation (B.24) represents the state transition matrix in terms of the transformation variable $\underline{T}(t)$. To obtain the desired results in terms of physical variables $\underline{S}(t)$, one first premultiplies (B.20) by $\underline{\underline{L}}^T$ and recalling (B.17), leads to

$$\underline{T}(t) = \underline{\underline{L}}^T \underline{S}(t) \tag{B.25}$$

Next, introducing (B.25) into both sides of (B.23), one obtains

$$\underline{\underline{L}}^T \underline{S}(t) = e^{[\underline{\underline{\Lambda}} t]} \underline{\underline{L}}^T \underline{S}(t_o) \tag{B.26}$$

Finally, premultiplying (B.26) by $\underline{\underline{R}}$ and recalling (B.17), one obtains

$$\underline{S}(t) = \underline{\underline{R}} \, e^{[\underline{\underline{\Lambda}} t]} \underline{\underline{L}}^T \underline{S}(t_o) \tag{B.27}$$

where the desired result for the state transition matrix is given by

$$\underline{\underline{\phi}}(t,t_o) = \underline{\underline{R}} \, e^{[\underline{\underline{\Lambda}} t]} \underline{\underline{L}}^T \tag{B.28}$$

This concludes the section on obtaining the state transition matrix.

APPENDIX C

DERIVATIVE UPDATE ALGORITHM

The algorithm presented in this section deals with an approximate method for extrapolating, between iterations, the state transition matrices required in the iterative solution of the two-point-boundary-value problem.

Two observations motivate the development of this algorithm: (1) it is expensive to integrate the n^2 additional equations required in the state transition matrix integration (where n is the dimension of the state equation), and (2) since the differential correction algorithm operates successfully only in a linear regime, the state transition matrix can be likewise extrapolated by a similar linear assumption. With these ideas in mind the formal development of the algorithm now begins.

Since the problem being addressed is a two-point-boundary-value problem, a natural criteria for measuring the acceptability of a current set of parameter estimates is the vanishing of some set of terminal constraints; e.g.,

$$\underline{\psi}(\underline{P}) = \left[\psi_1(\underline{P}) , \psi_2(\underline{P}) , \ldots\ldots , \psi_m(\underline{P})\right]^T \tag{C.1}$$

where

$$\underline{P} = \left[P_1 , P_2 , \ldots\ldots , P_n\right]^T \tag{C.2}$$

and $\psi_j(\underline{P})$ denotes the j-th terminal constraint, and P_j denotes the j-th current parameter estimate (when $\underline{P}_D = \underline{P}_{desired}$ then

$\underline{\psi}(\underline{P}_D) = [0 , 0 , \ldots\ldots , 0]^T$).

Letting the superscript "k" denotes the k-th set of parameter estimates, (C.1) can be expanded in the Taylor series:

$$\underline{\psi}^{(k+1)} = \underline{\psi}^{(k)} + \left[\frac{\partial\underline{\psi}}{\partial\underline{P}}\bigg|_{\underline{P}^{(k)}}\right] \underline{\Delta P}^{(k)} + \ldots\ldots + \text{H.O.T.}^{*} \qquad \text{(C.3)}$$

where $\underline{P}^{(k)} = \left[P_1^{(k)} , P_2^{(k)} , \ldots\ldots , P_n^{(k)}\right]^T$; $\underline{\Delta P}^{(k)} = \left[\Delta P_1^{(k)} , \Delta P_2^{(k)} , \ldots\ldots , \Delta P_n^{(k)}\right]^T$ is the parameter correction vector; $\underline{P}^{(k+1)} = \underline{P}^{(k)} + \underline{\Delta P}^{(k)}$; and $\left[\frac{\partial\underline{\psi}}{\partial\underline{P}}\bigg|_{\underline{P}^{(k)}}\right]$ denotes the matrix of partial derivatives ** relating the sensitivity of the terminal constraints to the k-th parameter estimates.

Now defining

$$\underline{\delta\psi}^{(k)} = \underline{\psi}^{(k+1)} - \underline{\psi}^{(k)} \qquad \text{(C.4)}$$

then (C.3) becomes

$$\underline{\delta\psi}^{(k)} = \left[\frac{\partial\underline{\psi}}{\partial\underline{P}}\bigg|_{\underline{P}^{(k)}}\right] \underline{\Delta P}^{(k)} + \ldots\ldots + \text{H.O.T.} \qquad \text{(C.5)}$$

If the effect of the higher order terms is lumped into a derivative correction matrix, (C.5) becomes

$$\underline{\delta\psi}^{(k)} = \left[\frac{\partial\underline{\psi}}{\partial\underline{P}}\bigg|_{\underline{P}^{(k)}} + \underline{\underline{C}}^{(k)}\right] \underline{\Delta P}^{(k)} \qquad \text{(C.6)}$$

Equation (C.6) states that the linear predictions in the parameters are to agree with the actual changes in the terminal constraints.

At this point in the iteration process $\underline{P}^{(k+1)}$ and $\underline{\psi}^{(k+1)}$

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* The abbreviation "H.O.T." stands for "Higher Order Terms".

** This partial derivative matrix is in fact the state transition matrix, or a sub-matrix thereof.

are known; but, the partial derivative matrix $\left[\left.\frac{\partial \underline{\psi}}{\partial \underline{P}}\right|_{\underline{P}^{(k+1)}}\right]$ needed for differential correction is unknown.

The assumption made in the derivative update algorithm is that the partial derivative matrix $\left[\left.\frac{\partial \underline{\psi}}{\partial \underline{P}}\right|_{\underline{P}^{(k+1)}}\right]$ can be approximated by

$$\left[\left.\frac{\partial \underline{\psi}}{\partial \underline{P}}\right|_{\underline{P}^{(k+1)}}\right] \triangleq \left[\left.\frac{\partial \underline{\psi}}{\partial \underline{P}}\right|_{\underline{P}^{(k)}} + \underline{\underline{C}}^{(k)}\right] \tag{C.7}$$

where $\underline{\underline{C}}^{(k)}$ has yet to be determined.

One method for determining $\underline{\underline{C}}^{(k)}$ from (C.6) is the following:

$$\text{Minimize} \; : \; \phi = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(C_{ij}^{(k)}\right)^2 \; * \tag{C.8}$$

$$\text{Subject to:} \; \underline{\delta\psi}^{(k)} - \left[\left.\frac{\partial \underline{\psi}}{\partial \underline{P}}\right|_{\underline{P}^{(k)}}\right] \underline{\Delta P}^{(k)} - \underline{\underline{C}}^{(k)} \, \underline{\Delta P}^{(k)} = \underline{0} \tag{C.9}$$

Using the Lagrange multiplier rule to adjoin the constraint (C.9) to (C.8) to form the augmented performance index, one obtains

$$\hat{\phi} = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(C_{ij}^{(k)}\right)^2 + \underline{\Lambda}^T \left(\underline{\delta\psi}^{(k)} - \left[\left.\frac{\partial \underline{\psi}}{\partial \underline{P}}\right|_{\underline{P}^{(k)}}\right] \underline{\Delta P}^{(k)} - \underline{\underline{C}}^{(k)} \, \underline{\Delta P}^{(k)}\right) \tag{C.10}$$

where $\underline{\Lambda} = \left[\Lambda_1 \, , \Lambda_2 \, , \, \ldots\ldots \, , \Lambda_n\right]^T$ is a vector of as yet undetermined multipliers.

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* This choice for ϕ is arbitrary but consistent with assumption that correction matrix elements C_{ij} are "small" departures from the elements of $\left[\left.\frac{\partial \underline{\psi}}{\partial \underline{P}}\right|_{\underline{P}^{(k)}}\right]_{ij}$.

The necessary conditions for (C.10) to possess a minimum are

$$\nabla_{C_{ij}} \hat{\phi} = 2\, C_{ij}^{(k)} - \Lambda_i \, \Delta P_j^{(k)} = 0 \quad , \quad \forall_{ij}\ ^{*} \qquad (C.11)$$

and

$$\nabla_{\underline{\Lambda}} \phi = \underline{\delta\psi}^{(k)} - \left[\frac{\partial \underline{\psi}}{\partial \underline{P}}\bigg|_{\underline{P}^{(k)}}\right] \underline{\Delta P}^{(k)} - \underline{\underline{C}}^{(k)} \, \underline{\Delta P}^{(k)} = 0 \qquad (C.12)$$

Solving (C.11) for $C_{ij}^{(k)}$ yields

$$C_{ij}^{(k)} = \frac{1}{2} \Lambda_i \, \Delta P_j^{(k)}$$

$$= \frac{1}{2} \left[\underline{\Lambda} \left[\underline{\Delta P}^{(k)}\right]^T\right]_{ij} \qquad (C.13)$$

or

$$\underline{\underline{C}}^{(k)} = \frac{1}{2} \underline{\Lambda} \left[\underline{\Delta P}^{(k)}\right]^T \qquad (C.14)$$

Introducing (C.14) into (C.12) yields the solution for the Lagrange multipliers

$$\underline{\Lambda} = \frac{2\left(\underline{\delta\psi}^{(k)} - \left[\frac{\partial \underline{\psi}}{\partial \underline{P}}\bigg|_{\underline{P}^{(k)}}\right] \underline{\Delta P}^{(k)}\right)}{\left[\underline{\Delta P}^{(k)}\right]^T \underline{\Delta P}^{(k)}} \qquad (C.15)$$

Finally, substituting (C.15) into (C.14) yields the desired correction matrix $\underline{\underline{C}}^{(k)}$ as

$$\underline{\underline{C}}^{(k)} = \frac{\left(\underline{\delta\psi}^{(k)} - \left[\frac{\partial \underline{\psi}}{\partial \underline{P}}\bigg|_{\underline{P}^{(k)}}\right] \underline{\Delta P}^{(k)}\right)\left(\underline{\Delta P}^{(k)}\right)^T}{\left[\underline{\Delta P}^{(k)}\right]^T \underline{\Delta P}^{(k)}} \qquad (C.16)$$

On introducing (C.16) into (C.7) the desired approximate partial

---

* $\forall_{ij}$ means for all i and j .

derivative matrix is obtained. The derivative update algorithm can be summarized as in Figure C.1 .

The loop returning to the k = 0 block is necessary because there are times, when after several differential corrections, that the approximate matrix $\left[\frac{\partial \underline{\psi}}{\partial \underline{P}}\bigg|_{\underline{P}^{(k+1)}}\right]$ no longer adequately represents the true partial derivative matrix. The cure for this difficulty is to substract out the corrections leading to an increase in the norm of the terminal constraints, and then to integrate rigorously the partial derivative matrix again; in order to provide good starting iteratives for the subsequent partial derivative matrix extrapolations.

The specification of the matrix $\underline{\underline{C}}^{(k)}$ in (C.7) amounts to a multi-dimensional generalization of the "secant" or "reguli-falsi" derivative approximation process treated in [20].

This concludes the section on the derivative updating algorithm.

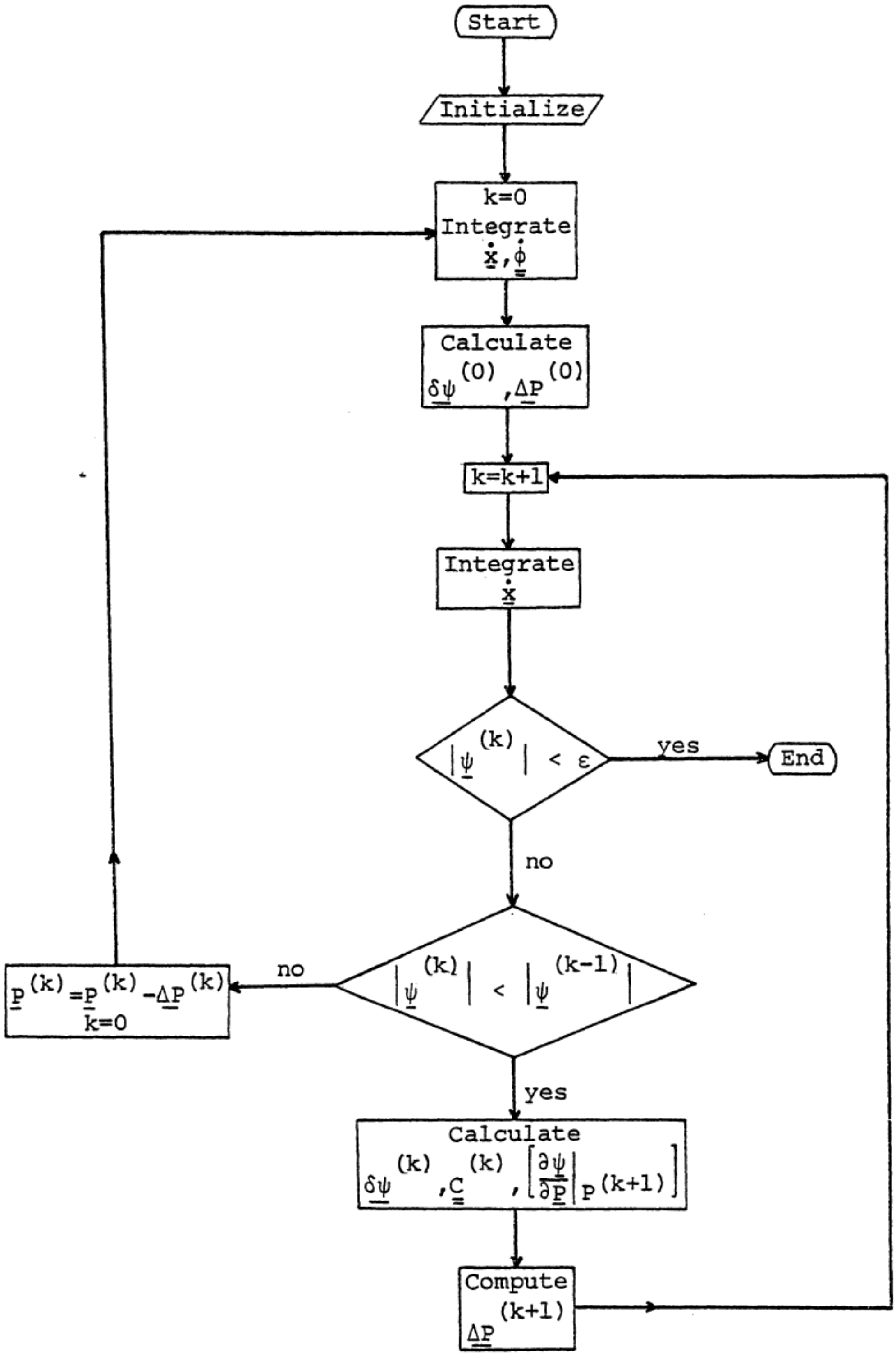

Figure C.1: Flow Diagram for the Derivative Update Algorithm.

APPENDIX D

EULER PARAMETERS

The Euler parameters consist of a four parameter set of attitude variables which are used as orientation coordinates for rigid body motion. These parameters are important for several reasons: (1) They eliminate the geometrical singularity present in every three parameter description of orientation; (2) They have derivatives which are linearly related (without approximation) to angular velocity; and (3) They are known to yield numerically superior (with respect to both accuracy and speed) algorithms for angular motion integration, if large angular motions are involved.

The Euler parameters (β_0 , β_1 , β_2 , β_3) may be interpreted geometrically in terms of Euler's theorem which states: An arbitrary reorientation of rigid body* can be accomplished by a single rotation of the body about the principal line $\hat{\underline{\ell}}$ through the angle ϕ (see Figure D.1).

In terms of $\hat{\underline{\ell}}$ and ϕ the Euler parameters are expressed as

$$\left.\begin{aligned} \beta_0 &= \cos \frac{1}{2} \phi \\ \beta_i &= \ell_i \sin \frac{1}{2} \phi \quad , \; i=1,2,3 \end{aligned}\right\} \qquad (D.1)$$

From (D.1) it is clear that the Euler parameters satisfy the constraint

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* For flexible body dynamics the term rigid body is replaced by "coordinate frame".

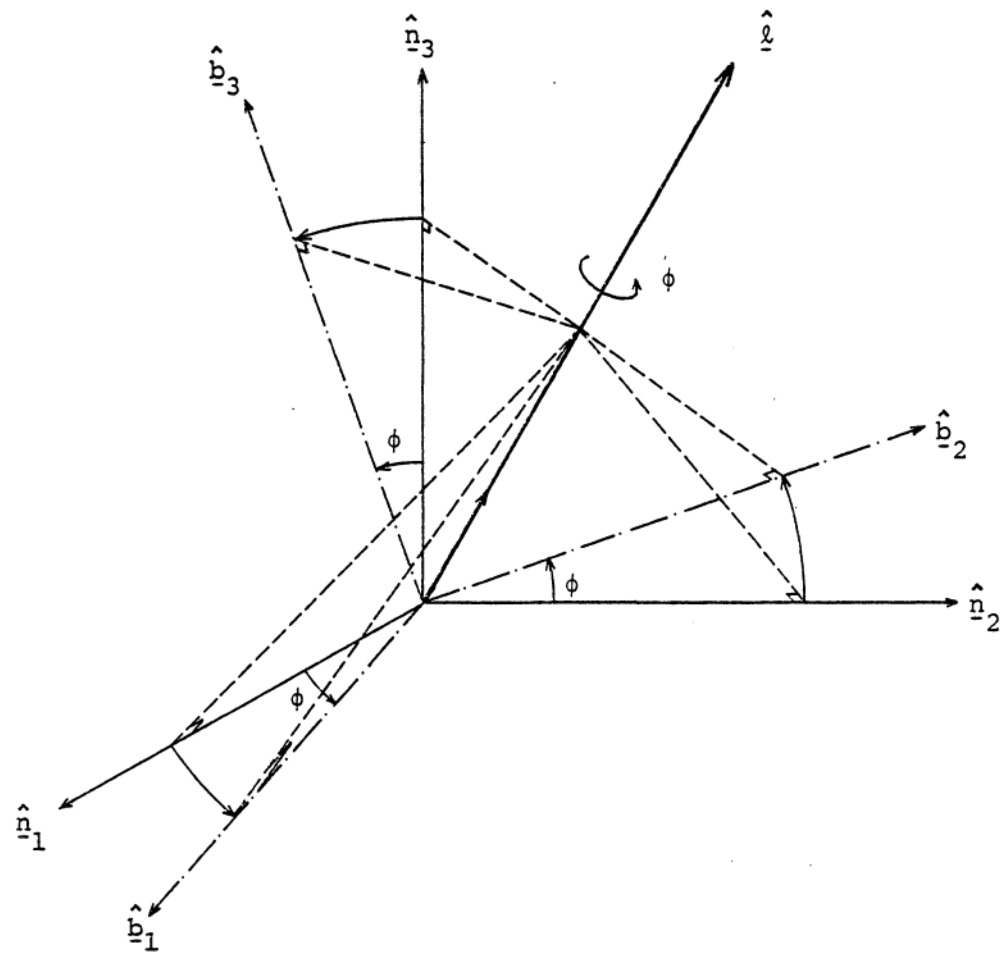

Figure D.1: Principal Axis Rotation about the Principal Line $\hat{\underline{\ell}}$ through the Angle ϕ .

$$\sum_{i=0}^{3} \beta_i^2 = 1 \tag{D.2}$$

and hence are a once redundant set of variables. These variables have the interesting topological interpretation that every physical angular motion traces out two paths on a four-dimensional sphere (these paths are reflections).

The direction cosine matrix $[C]$ relating the orientation of the body frame $\{\hat{\underline{b}}\}$ and the inertial frame $\{\hat{\underline{n}}\}$ is given by

$$\{\hat{\underline{b}}\} = [C]\{\hat{\underline{n}}\} \tag{D.3}$$

where $[C]$ is parameterized in terms of the Euler parameters as

$$\left[C(\beta_0,\beta_1,\beta_2,\beta_3)\right] = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2(\beta_1\beta_2+\beta_0\beta_3) & 2(\beta_1\beta_3-\beta_0\beta_2) \\ 2(\beta_1\beta_2-\beta_0\beta_3) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2(\beta_2\beta_3+\beta_0\beta_1) \\ 2(\beta_1\beta_3+\beta_0\beta_2) & 2(\beta_2\beta_3-\beta_0\beta_1) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix}$$

(D.4)

The procedure used to obtain (D.4) can be found in [46] .

The matrix differential equation relating the time derivatives of Euler parameters to the body frame $\{\hat{\underline{b}}\}$ components of angular velocity $\{\omega\}$, is given by

$$\begin{Bmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}^{*} \qquad \text{(D.5)}$$

The equations of (D.5) are directly obtained on introducing (D.4) into the following fundamental kinematic differential equation for direction cosines

$$[\dot{C}][C]^T = -[\tilde{\omega}] \qquad \text{(D.6)}$$

where

$$[\tilde{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \qquad \text{(D.7)}$$

Since motion in terms of Euler parameters is difficult to visualize, it is convenient to have at hand the relationships which carry any set of Euler angles into the corresponding set of Euler parameters, and vice-versa. In particular for the 3-1-3 set of Euler angles (ϕ , θ , ψ) the transformation is given by [46]:

$$\beta_0 = \cos\frac{1}{2}\theta \cos\frac{1}{2}(\phi+\psi) \qquad \text{(D.8a)}$$

$$\beta_1 = \sin\frac{1}{2}\theta \cos\frac{1}{2}(\phi-\psi) \qquad \text{(D.8b)}$$

$$\beta_2 = \sin\frac{1}{2}\theta \sin\frac{1}{2}(\phi-\psi) \qquad \text{(D.8c)}$$

$$\beta_3 = \cos\frac{1}{2}\theta \sin\frac{1}{2}(\phi+\psi) \qquad \text{(D.8d)}$$

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* Notice, in place of the usual transcendental coefficient matrix, for any set of three Euler angles, the coefficient matrix is orthogonal and contains the β's linearly.

Morton [23] has recently developed a general transformation analogous to (D.8) which is valid for all twelve rotation sequences; these results are summarized in Table D.1. In Table D.1 the following rotational conventions are adopted:

(1) θ_1 is the rotation angle about axis $\hat{\underline{n}}_\alpha$;

(2) θ_2 is the rotation angle about the once rotated axis $\hat{\underline{n}}_\beta$;

(3) θ_3 is the rotation angle about the twice rotated axis $\hat{\underline{n}}_\gamma$; and

let α-β-γ denote the Euler angle rotation sequence (e.g., α-β-γ $\Longrightarrow$ 3-1-3 in (D.8)).

Morton's universal matrix equation is

$$\begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} = \left[R_{\alpha-\beta-\gamma}\right] \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{D.9}$$

where

$$\left[R_{\alpha-\beta-\gamma}\right] = \left[\cos\tfrac{1}{2}\theta_3\ \underline{\underline{R}}_0 + \sin\tfrac{1}{2}\theta_3\ \underline{\underline{R}}_\gamma\right]\left[\cos\tfrac{1}{2}\theta_2\ \underline{\underline{R}}_0 + \sin\tfrac{1}{2}\theta_2\ \underline{\underline{R}}_\beta\right]$$
$$\left[\cos\tfrac{1}{2}\theta_1\ \underline{\underline{R}}_0 + \sin\tfrac{1}{2}\theta_1\ \underline{\underline{R}}_\alpha\right] \tag{D.10}$$

and

$$\underline{\underline{R}}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{\underline{R}}_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{D.11a,b}$$

Table D.1: Transformation from the Twelve Sets of Euler Angles to Euler Parameters.*

α-β-γ	β_0	β_1	β_2	β_3
1-2-1	$c\,\frac{\theta_2}{2}\,c\left(\frac{\theta_1+\theta_3}{2}\right)$	$c\,\frac{\theta_2}{2}\,s\left(\frac{\theta_1+\theta_3}{2}\right)$	$s\,\frac{\theta_2}{2}\,c\left(\frac{\theta_1-\theta_3}{2}\right)$	$s\,\frac{\theta_2}{2}\,s\left(\frac{\theta_1-\theta_3}{2}\right)$
2-3-2	$c\,\frac{\theta_2}{2}\,c\left(\frac{\theta_1+\theta_3}{2}\right)$	$s\,\frac{\theta_2}{2}\,s\left(\frac{\theta_1-\theta_3}{2}\right)$	$c\,\frac{\theta_2}{2}\,s\left(\frac{\theta_1+\theta_3}{2}\right)$	$s\,\frac{\theta_2}{2}\,c\left(\frac{\theta_1-\theta_3}{2}\right)$
3-1-3	$c\,\frac{\theta_2}{2}\,c\left(\frac{\theta_1+\theta_3}{2}\right)$	$s\,\frac{\theta_2}{2}\,c\left(\frac{\theta_1-\theta_3}{2}\right)$	$s\,\frac{\theta_2}{2}\,s\left(\frac{\theta_1-\theta_3}{2}\right)$	$c\,\frac{\theta_2}{2}\,s\left(\frac{\theta_1+\theta_3}{2}\right)$
1-3-1	$c\,\frac{\theta_2}{2}\,c\left(\frac{\theta_3+\theta_1}{2}\right)$	$c\,\frac{\theta_2}{2}\,s\left(\frac{\theta_3+\theta_1}{2}\right)$	$s\,\frac{\theta_2}{2}\,s\left(\frac{\theta_3-\theta_1}{2}\right)$	$s\,\frac{\theta_2}{2}\,c\left(\frac{\theta_3-\theta_1}{2}\right)$
2-1-2	$c\,\frac{\theta_2}{2}\,c\left(\frac{\theta_3+\theta_1}{2}\right)$	$s\,\frac{\theta_2}{2}\,c\left(\frac{\theta_3-\theta_1}{2}\right)$	$c\,\frac{\theta_2}{2}\,s\left(\frac{\theta_3+\theta_1}{2}\right)$	$s\,\frac{\theta_2}{2}\,s\left(\frac{\theta_3-\theta_1}{2}\right)$
3-2-3	$c\,\frac{\theta_2}{2}\,c\left(\frac{\theta_3+\theta_1}{2}\right)$	$s\,\frac{\theta_2}{2}\,s\left(\frac{\theta_3-\theta_1}{2}\right)$	$s\,\frac{\theta_2}{2}\,c\left(\frac{\theta_3-\theta_1}{2}\right)$	$c\,\frac{\theta_2}{2}\,s\left(\frac{\theta_3+\theta_1}{2}\right)$
1-2-3	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2} - s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2} + c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2} - s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2} + s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$

* $c \equiv \cos$, $s \equiv \sin$.

Table D.1 (continued): Transformation from the Twelve Sets of Euler Angles to Euler Parameters. *

α-β-γ	β_0	β_1	β_2	β_3
2-3-1	$c\frac{\theta_1}{2}\,c\frac{\theta_2}{2}\,c\frac{\theta_3}{2} - s\frac{\theta_1}{2}\,s\frac{\theta_2}{2}\,s\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\,c\frac{\theta_2}{2}\,s\frac{\theta_3}{2} + s\frac{\theta_1}{2}\,s\frac{\theta_2}{2}\,c\frac{\theta_3}{2}$	$s\frac{\theta_1}{2}\,c\frac{\theta_2}{2}\,c\frac{\theta_3}{2} + c\frac{\theta_1}{2}\,s\frac{\theta_2}{2}\,s\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\,s\frac{\theta_2}{2}\,c\frac{\theta_3}{2} - s\frac{\theta_1}{2}\,c\frac{\theta_2}{2}\,s\frac{\theta_3}{2}$
3-1-2	$c\frac{\theta_1}{2}\,c\frac{\theta_2}{2}\,c\frac{\theta_3}{2} - s\frac{\theta_1}{2}\,s\frac{\theta_2}{2}\,s\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\,s\frac{\theta_2}{2}\,c\frac{\theta_3}{2} - s\frac{\theta_1}{2}\,c\frac{\theta_2}{2}\,s\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\,c\frac{\theta_2}{2}\,s\frac{\theta_3}{2} + s\frac{\theta_1}{2}\,s\frac{\theta_2}{2}\,c\frac{\theta_3}{2}$	$s\frac{\theta_1}{2}\,c\frac{\theta_2}{2}\,c\frac{\theta_3}{2} + c\frac{\theta_1}{2}\,s\frac{\theta_2}{2}\,s\frac{\theta_3}{2}$
1-3-2	$c\frac{\theta_1}{2}\,c\frac{\theta_2}{2}\,c\frac{\theta_3}{2} + s\frac{\theta_1}{2}\,s\frac{\theta_2}{2}\,s\frac{\theta_3}{2}$	$s\frac{\theta_1}{2}\,c\frac{\theta_2}{2}\,c\frac{\theta_3}{2} - c\frac{\theta_1}{2}\,s\frac{\theta_2}{2}\,s\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\,c\frac{\theta_2}{2}\,s\frac{\theta_3}{2} - s\frac{\theta_1}{2}\,s\frac{\theta_2}{2}\,c\frac{\theta_3}{2}$	$c\frac{\theta_1}{2}\,s\frac{\theta_2}{2}\,c\frac{\theta_3}{2} + s\frac{\theta_1}{2}\,c\frac{\theta_2}{2}\,s\frac{\theta_3}{2}$

* $c \equiv \cos$, $s \equiv \sin$.

Table D.1 (continued): Transformation from the Twelve Sets of Euler Angles to Euler Parameters. *

α-β-γ	β_0	β_1	β_2	β_3
2-1-3	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2} + s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2} + s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2} - c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2} - s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$
3-2-1	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2} + s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2} - s\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2}$	$c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2} + s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$	$s\,\frac{\theta_1}{2}\,c\,\frac{\theta_2}{2}\,c\,\frac{\theta_3}{2} - c\,\frac{\theta_1}{2}\,s\,\frac{\theta_2}{2}\,s\,\frac{\theta_3}{2}$

* $c \equiv \cos$, $s \equiv \sin$.

$$\underset{=}{R}_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad ; \quad \underset{=}{R}_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(D.11c,d)}$$

With (D.9), the Euler angles θ_1 , θ_2 , and θ_3 can be transformed into the corresponding Euler parameters β_0 , β_1 , β_2 , and β_3 . When the inverse of this transformation is desired, it is first necessary to parameterize the direction cosine matrix in terms of the desired Euler angle rotation sequence α-β-γ ; from which the Euler angles can be recovered in terms of inverse trigonometric functions as functions of the direction cosine matrix. Table D.2 summarizes these inverse transformations for all twelve Euler angle rotation sequences. Since the direction cosine matrix can be parameterized in terms of the Euler parameters, then any Euler angle rotation sequence can be recovered using the appropriate entry of Table D.2 .

This concludes the section on presenting the expressions relating the Euler parameters to Euler angles and angular rates for the attitude control problem.

Table D.2: Transformation from the Direction Cosines to the Twelve Sets of Euler Angles.

α-β-γ	θ_1	θ_2	θ_3	Geometric Singularity
1-2-1	$\tan^{-1}(C_{12}/-C_{13})$	$\cos^{-1}(C_{11})$	$\tan^{-1}(C_{21}/C_{31})$	$\theta_2 = 0$
2-3-2	$\tan^{-1}(C_{23}/-C_{21})$	$\cos^{-1}(C_{22})$	$\tan^{-1}(C_{32}/C_{12})$	$\theta_2 = 0$
3-1-3	$\tan^{-1}(C_{31}/-C_{32})$	$\cos^{-1}(C_{33})$	$\tan^{-1}(C_{13}/C_{23})$	$\theta_2 = 0$
1-3-1	$\tan^{-1}(C_{13}/C_{12})$	$\cos^{-1}(C_{11})$	$\tan^{-1}(C_{31}/-C_{21})$	$\theta_2 = 0$
2-1-2	$\tan^{-1}(C_{21}/C_{23})$	$\cos^{-1}(C_{22})$	$\tan^{-1}(C_{12}/-C_{32})$	$\theta_2 = 0$
3-2-3	$\tan^{-1}(C_{32}/C_{31})$	$\cos^{-1}(C_{33})$	$\tan^{-1}(C_{23}/-C_{13})$	$\theta_2 = 0$
1-2-3	$\tan^{-1}(-C_{32}/C_{33})$	$\sin^{-1}(C_{31})$	$\tan^{-1}(-C_{21}/C_{11})$	$\theta_2 = \pm\,\pi/2$
2-3-1	$\tan^{-1}(-C_{13}/C_{11})$	$\sin^{-1}(C_{12})$	$\tan^{-1}(-C_{32}/C_{22})$	$\theta_2 = \pm\,\pi/2$
3-1-2	$\tan^{-1}(-C_{21}/C_{22})$	$\sin^{-1}(C_{23})$	$\tan^{-1}(-C_{13}/C_{33})$	$\theta_2 = \pm\,\pi/2$
1-3-2	$\tan^{-1}(C_{23}/C_{22})$	$\sin^{-1}(-C_{21})$	$\tan^{-1}(C_{31}/C_{11})$	$\theta_2 = \pm\,\pi/2$
2-1-3	$\tan^{-1}(C_{31}/C_{33})$	$\sin^{-1}(-C_{32})$	$\tan^{-1}(C_{12}/C_{22})$	$\theta_2 = \pm\,\pi/2$
3-2-1	$\tan^{-1}(C_{12}/C_{11})$	$\sin^{-1}(-C_{13})$	$\tan^{-1}(C_{23}/C_{33})$	$\theta_2 = \pm\,\pi/2$

where $[C_{ij}]$ = function of $(\beta_0,\beta_1,\beta_2,\beta_3)$ = equation (D.4).

APPENDIX E

PARTIAL DERIVATIVES FOR THE RIGID BODY FORMULATION

Given the state space form for the state and co-state differential equations for the rigid body as (see (B.2)):

$$\dot{\underline{s}} = \underline{F}(\underline{s},t) \tag{E.1}$$

where $\underline{s}(t) = \left[\beta_0(t)\ldots\beta_3(t)\ \omega_1(t)\ldots\omega_3(t)\ \gamma_0(t)\ldots\gamma_3(t)\ \lambda_1(t)\ldots\lambda_3(t)\right]$

and $\underline{F}$ is defined by (2.2.13) and (2.2.14).

The partial derivative of (E.1) with respect to the current state follow as

$$\frac{\partial\dot{\underline{s}}(t)}{\partial\underline{s}(t)} = \frac{\partial\underline{F}(\underline{s}(t),t)}{\partial\underline{s}(t)} = \underline{\underline{F}} \tag{E.2}$$

In partitioned form (E.2) becomes

$$\underline{\underline{F}} = \begin{array}{cc} \begin{array}{cccc} a & b & a & b \end{array} & \\ \begin{bmatrix} \underline{\underline{F}}_{11} & \underline{\underline{F}}_{12} & 0 & 0 \\ 0 & \underline{\underline{F}}_{22} & 0 & \underline{\underline{F}}_{24} \\ 0 & \underline{\underline{F}}_{32} & \underline{\underline{F}}_{33} & 0 \\ \underline{\underline{F}}_{41} & \underline{\underline{F}}_{42} & \underline{\underline{F}}_{43} & \underline{\underline{F}}_{44} \end{bmatrix} & \begin{array}{c} a \\ b \\ a \\ b \end{array} \end{array} \tag{E.3}$$

where a and b denote the row and column dimensions for the partitions of (E.3); with $a = 4$ and $b = 3$.

The non-zero partitions of $\underline{\underline{F}}$ are given by

$$\underline{\underline{F}}_{11} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{E.4}$$

$$\underline{\underline{F}}_{12} = \frac{1}{2}\begin{bmatrix} -\beta_1 & -\beta_2 & \beta_3 \\ \beta_0 & -\beta_2 & \beta_2 \\ \beta_3 & \beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \tag{E.5}$$

$$\underline{\underline{F}}_{22} = \begin{bmatrix} 0 & -I_1\omega_3 & -I_1\omega_2 \\ -I_2\omega_3 & 0 & -I_2\omega_1 \\ -I_3\omega_2 & -I_3\omega_1 & 0 \end{bmatrix} \tag{E.6}$$

$$\underline{\underline{F}}_{24} = \begin{bmatrix} -I_1^{-2} & 0 & 0 \\ 0 & -I_2^{-2} & 0 \\ 0 & 0 & -I_3^{-2} \end{bmatrix} \tag{E.7}$$

$$\underline{\underline{F}}_{32} = \frac{1}{2}\begin{bmatrix} -\gamma_1 & -\gamma_2 & -\gamma_3 \\ \gamma_0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & \gamma_0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix} \tag{E.8}$$

$$\underline{\underline{F}}_{33} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{E.9}$$

$$\underline{\underline{F}}_{41} = \frac{1}{2}\begin{bmatrix} -\gamma_1 & \gamma_0 & \gamma_3 & -\gamma_2 \\ -\gamma_2 & -\gamma_3 & \gamma_0 & \gamma_1 \\ -\gamma_3 & \gamma_2 & -\gamma_1 & \gamma_0 \end{bmatrix} \tag{E.10}$$

$$\underline{\underline{F}}_{42} = \begin{bmatrix} 0 & \mathcal{I}_3\gamma_3 & \mathcal{I}_2\gamma_2 \\ \mathcal{I}_2\gamma_3 & 0 & \mathcal{I}_1\gamma_1 \\ \mathcal{I}_2\gamma_2 & \mathcal{I}_1\gamma_1 & 0 \end{bmatrix} \tag{E.11}$$

$$\underline{\underline{F}}_{43} = \frac{1}{2}\begin{bmatrix} \beta_1 & -\beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & -\beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & -\beta_0 \end{bmatrix} \tag{E.12}$$

$$\underline{\underline{F}}_{44} = \begin{bmatrix} 0 & \mathcal{I}_2\omega_3 & \mathcal{I}_3\omega_2 \\ \mathcal{I}_1\omega_3 & 0 & \mathcal{I}_3\omega_1 \\ \mathcal{I}_1\omega_2 & \mathcal{I}_2\omega_1 \cdot & 0 \end{bmatrix} \tag{E.13}$$

where I_1 , I_2 , and I_3 represent the principal axis moments of inertia for the vehicle, and $\mathcal{I}_1 = (I_3 - I_2)/I_1$, $\mathcal{I}_2 = (I_1 - I_3)/I_2$, and $\mathcal{I}_3 = (I_2 - I_1)/I_3$.

With (E.13) this completes the specification of all partials necessary in order to integrate the state transition matrix $\underline{\underline{\phi}}(t,t_o)$ (see (B.6)).

Upon integrating the state transition matrix the submatrices $\underline{\underline{A}}_{\beta\gamma}$, $\underline{\underline{A}}_{\beta\lambda}$, $\underline{\underline{A}}_{\omega\gamma}$, and $\underline{\underline{A}}_{\omega\lambda}$ (see (2.2.28a-d)) are obtained from the state transition matrix

$$\underline{\underline{\Phi}}(t_f,t_o) = \left[\begin{array}{c:c:c:c} \underline{\underline{\phi}}_{11} & \underline{\underline{\phi}}_{12} & \underline{\underline{\phi}}_{13} & \underline{\underline{\phi}}_{14} \\ \hdashline \underline{\underline{\phi}}_{21} & \underline{\underline{\phi}}_{22} & \underline{\underline{\phi}}_{23} & \underline{\underline{\phi}}_{24} \\ \hdashline \underline{\underline{\phi}}_{31} & \underline{\underline{\phi}}_{32} & \underline{\underline{\phi}}_{33} & \underline{\underline{\phi}}_{34} \\ \hdashline \underline{\underline{\phi}}_{41} & \underline{\underline{\phi}}_{42} & \underline{\underline{\phi}}_{43} & \underline{\underline{\phi}}_{44} \end{array}\right] \qquad \text{(E.14)}$$

where the partitions $\underline{\underline{\Phi}}_{ij}$ have the same dimensions as $\underline{\underline{F}}_{ij}$ in (E.3). The submatrices $\underline{\underline{A}}_{\beta\gamma}$, $\underline{\underline{A}}_{\beta\lambda}$, $\underline{\underline{A}}_{\omega\gamma}$, and $\underline{\underline{A}}_{\omega\lambda}$ are obtained from (E.14) as follows:

$$\underset{(3x4)}{\underline{\underline{A}}_{\beta\gamma}} \equiv \text{last 3 rows of } \underset{(4x4)}{\underline{\underline{\Phi}}_{13}}(t_f,t);$$

$$\underset{(3x3)}{\underline{\underline{A}}_{\beta\lambda}} \equiv \text{last 3 rows of } \underset{(4x3)}{\underline{\underline{\Phi}}_{14}}(t_f,t);$$

$$\underset{(3x4)}{\underline{\underline{A}}_{\omega\gamma}} = \underset{(3x4)}{\underline{\underline{\Phi}}_{23}}(t_f,t_o); \quad \text{and}$$

$$\underset{(3x3)}{\underline{\underline{A}}_{\omega\lambda}} = \underset{(3x3)}{\underline{\underline{\Phi}}_{24}}(t_f,t_o) .$$

APPENDIX F

PARTIAL DERIVATIVES FOR THE SINGLE AXIS FLEXIBLE BODY PROBLEM

Given the state and co-state differential equations in the index form (see (3.3.8)):

$$\dot{s}_{1i} = s_{2i} \tag{F.1}$$

$$\dot{s}_{2i} = \alpha\ (v_q\ s_{2q})^2\ L_{ij}\ s_{ij} - \Lambda_i\ s_{1i} - \frac{\lambda_{2k}\ v_k\ v_i}{W_{uu}} \tag{F.2}$$

$$\dot{\lambda}_{1i} = -\ W_{11ij}\ s_{1j} - \alpha\ (v_q\ s_{2q})^2\ L_{ji}\ s_{2j} + \Lambda_i\ \lambda_{2i} \tag{F.3}$$

$$\dot{\lambda}_{2i} = -\ W_{22ij}\ s_{2j} - \lambda_{1i} - 2\ \alpha\ v_q\ s_{2q}\ \lambda_{2p}\ L_{pr}\ s_{1r}\ v_i \tag{F.4}$$

where W_{11} is the weight matrix for the $\underline{s}_1$ variables; W_{22} is the weight matrix for the $\underline{s}_2$ variables; W_{uu} is the scalar weight for the control u ; Λ_i is the i-th tranformation eigenvalue (see (3.2.2)); α is the relaxation parameter; $\underline{v} = [E_{11}\ ,\ E_{12}\ ,\ \ldots\ldots\ ,\ E_{1n}]^T$ (see (3.2.6)), and E_{ij} is the i-j-th element of the normalized transformation eigenvector matrix (see (3.2.3) and (3.2.4)).

The partial derivatives of (F.1) - (F.4) are now obtained with respect to the variables $\underline{s}_1$, $\underline{s}_2$, $\underline{\lambda}_1$, and $\underline{\lambda}_3$ yielding the partials necessary for the state transition matrix integration of Appendix B.

For $\dot{\underline{s}}_1$, the partials are

$$\frac{\partial s_{1i}}{\partial s_{1k}} = 0 \tag{F.5a}$$

$$\frac{\partial \dot{s}_{1i}}{\partial s_{2k}} = \delta_{ik} \quad * \tag{F.5b}$$

$$\frac{\partial \dot{s}_{1i}}{\partial \lambda_{1k}} = 0 \tag{F.5c}$$

$$\frac{\partial \dot{s}_{1i}}{\partial \lambda_{2k}} = 0 \tag{F.5d}$$

For $\dot{\underline{s}}_2$, the partials are

$$\frac{\partial \dot{s}_{2i}}{\partial s_{1k}} = \alpha \, (v_q \, s_{2q})^2 \, L_{ik} - \Lambda_i \, \delta_{ik} \tag{F.6a}$$

$$\frac{\partial \dot{s}_{2i}}{\partial s_{2k}} = 2 \, \alpha \, (v_q \, s_{2q}) \, v_k \, L_{ij} \, s_{1j} \tag{F.6b}$$

$$\frac{\partial \dot{s}_{2i}}{\partial \lambda_{1k}} = 0 \tag{F.6c}$$

$$\frac{\partial \dot{s}_{2i}}{\partial \lambda_{2k}} = - \frac{v_i \, v_k}{W_{uu}} \tag{F.6d}$$

For $\dot{\underline{\lambda}}_1$, the partials are

$$\frac{\partial \lambda_{1i}}{\partial s_{1k}} = - W_{11ik} \tag{F.7a}$$

$$\frac{\partial \lambda_{1i}}{\partial s_{2k}} = - 2 \, \alpha \, (v_q \, s_{2q}) \, v_k \, L_{qi} \, \lambda_{2q} \tag{F.7b}$$

$$\frac{\partial \lambda_{1i}}{\partial \lambda_{1k}} = 0 \tag{F.7c}$$

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* $\frac{\partial s_{1i}}{\partial s_{2k}} = \delta_{ik}$

$$\frac{\partial \dot{\lambda}_{1i}}{\partial \lambda_{2k}} = - \alpha \; (v_q \; s_{2q})^2 \; L_{ki} + \Lambda_i \; \delta_{ik} \tag{F.7d}$$

For $\dot{\underline{\lambda}}_2$, the partials are

$$\frac{\partial \dot{\lambda}_{2i}}{\partial s_{1k}} = - 2 \; \alpha \; (v_q \; s_{2q}) \; L_{kp} \; \lambda_{2p} \tag{F.8a}$$

$$\frac{\partial \dot{\lambda}_{2i}}{\partial s_{2k}} = - W_{22ik} - 2 \; \alpha \; \lambda_{2p} \; L_{pr} \; s_{1r} \; v_i \; v_k \tag{F.8b}$$

$$\frac{\partial \dot{\lambda}_{2i}}{\partial \lambda_{1k}} = - \delta_{ik} \tag{F.8c}$$

$$\frac{\partial \dot{\lambda}_{2i}}{\partial \lambda_{2k}} = - 2 \; \alpha \; v_q \; s_{2q} \; L_{kr} \; s_{1r} \; v_i \tag{F.8d}$$

With (F.8) this completes all equations necessary to integrate the state transition matrix (B.6).

APPENDIX G

THE PARTIAL DERIVATIVES FOR THE THREE-DIMENSIONAL FLEXIBLE BODY FORMULATION

The partial derivatives of the state and co-state equations are listed in back substitutions form, in order to reduce unncessary repetition of many of the various terms. To begin with the equations describing the right hand sides of the angular velocity and in- and out-of-plane modal equations are listed below, to be followed by their respective partial derivatives.

$$\underline{L}_1 = -\dot{\underline{\underline{J}}}\,\underline{\omega} - \tilde{\underline{\underline{\omega}}}\,(\underline{\underline{J}}\,\underline{\omega} + \underline{H}_E) \tag{G.1}$$

$$\underline{L}_2 = \underline{\underline{L}}_2\,\underline{s}_2 + \underline{\underline{L}}_3\,\underline{s}_3 + \underline{\ell}_2 \tag{G.2}$$

where

$$\underline{\underline{L}}_2 = \left[\begin{array}{c|c} \begin{array}{c}\omega_1^2\underline{\underline{J}}_{22xx}+\omega_2^2\underline{\underline{J}}_{22yy}+\omega_3^2\underline{\underline{J}}_{22zz}\\ +\omega_1\omega_2(\underline{\underline{J}}_{12xy}+\underline{\underline{J}}_{12xy}^T)\end{array} & \omega_1\omega_3\underline{\underline{J}}_{13xz}+\omega_2\omega_3\underline{\underline{J}}_{23yz} \\ \hline \omega_1\omega_3\underline{\underline{J}}_{13xz}^T+\omega_2\omega_3\underline{\underline{J}}_{23yz}^T & \omega_1^2\underline{\underline{J}}_{33xx}+\omega_2^2\underline{\underline{J}}_{33yy}+\omega_3^2\underline{\underline{J}}_{33zz} \end{array}\right]$$

$$\underline{\underline{L}}_3 = \left[\begin{array}{c|c} \omega_3\underline{\underline{H}}_{3131} & \omega_1\underline{\underline{H}}_{1211}+\omega_2\underline{\underline{H}}_{2221} \\ \hline -(\omega_1\underline{\underline{H}}_{1211}+\underline{\underline{H}}_{2221})^T & \underline{\underline{0}} \end{array}\right]^*$$

and

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* $\underline{\underline{H}}_{1211} = \underline{\underline{H}}_{12} - \underline{\underline{H}}_{11}^T$; $\underline{\underline{H}}_{2221} = \underline{\underline{H}}_{22} - \underline{\underline{H}}_{21}^T$; and $\underline{\underline{H}}_{3131} = \underline{\underline{H}}_{31} - \underline{\underline{H}}_{31}^T$.

$$\underline{\ell}_2 = \begin{Bmatrix} \omega_1 \omega_2 \underline{J}_{2xy} \\ \omega_1 \omega_3 \underline{J}_{3xz} + \omega_2 \omega_3 \underline{J}_{3yz} \end{Bmatrix}$$

Equation (G.1) in index form becomes

$$L_{1i} = - \dot{J}_{ij} \, \omega_j - \tilde{\omega}_{ij} \, (J_{jp} \, \omega_p + H_{Ej}) \tag{G.3}$$

The partial derivatives of (G.3) follow as

$$\frac{\partial L_{1i}}{\partial s_{1k}} = - \dot{J}_{ik} - \tilde{\omega}_{ij,k} \, (J_{jp} \, \omega_p + H_{Ej}) - \tilde{\omega}_{ij} \, J_{jk} \tag{G.3a}$$

$$\frac{\partial L_{1i}}{\partial s_{2k}} = - \tilde{\omega}_{ij} \, \frac{\partial J_{jp}}{\partial s_{2k}} \, \omega_p \tag{G.3b}$$

$$\frac{\partial L_{1i}}{\partial s_{3k}} = - \frac{\partial J_{ij}}{\partial s_{3k}} \, \omega_j - \tilde{\omega}_{ij} \, \frac{\partial H_{Ej}}{\partial s_{3k}} \tag{G.3c}$$

$$\frac{\partial^2 L_{1i}}{\partial s_{1k} \, \partial s_{1q}} = - \tilde{\omega}_{ij,k} \, J_{jq} - \tilde{\omega}_{ij,q} \, J_{ik} \tag{G.3d}$$

$$\frac{\partial^2 L_{1i}}{\partial s_{1k} \, \partial s_{2q}} = - \tilde{\omega}_{ij,k} \, \frac{\partial J_{jp}}{\partial s_{2q}} \, \omega_p - \tilde{\omega}_{ij} \, \frac{\partial J_{jk}}{\partial s_{2q}} \tag{G.3e}$$

$$\frac{\partial^2 L_{1i}}{\partial s_{1k} \, \partial s_{3q}} = - \frac{\partial J_{ik}}{\partial s_{3k}} - \tilde{\omega}_{ij,k} \, \frac{\partial H_{Ej}}{\partial s_{3q}} \tag{G.3f}$$

where

$$\tilde{\omega}_{ij,k} \equiv \begin{bmatrix} 0 & -\delta_{3k} & \delta_{2k} \\ \delta_{3k} & 0 & -\delta_{1k} \\ -\delta_{2k} & \delta_{1k} & 0 \end{bmatrix}$$

and δ_{ik} is the Kronecker delta symbol.

All other partial derivatives of $\underline{L}_1$ vanish identically.

Equation (G.2) possesses partial derivatives with a structure that is simple enough that index form is not an advantage, hence partial derivatives are presented in the form

$$\frac{\partial \underline{L}_2}{\partial s_{1k}} = \frac{\partial \underline{\underline{L}}_2}{\partial s_{1k}}\, \underline{s}_2 + \frac{\partial \underline{\underline{L}}_3}{\partial s_{1k}}\, \underline{s}_3 + \frac{\partial \underline{\ell}_2}{\partial s_{1k}} \tag{G.4}$$

where

(a) for k = 1 :

$$\left.\begin{aligned}
\frac{\partial \underline{\underline{L}}_2}{\partial s_{1k}} &= \left[\begin{array}{c|c} 2\omega_1 \underline{\underline{J}}_{22xx} + \omega_2 (\underline{\underline{J}}_{12xy} + \underline{\underline{J}}^T_{12xy}) & \omega_3 \underline{\underline{J}}_{13xz} \\ \hline \omega_3 \underline{\underline{J}}^T_{13xz} & 2\omega_1 \underline{\underline{J}}_{33xx} \end{array}\right] \\
\frac{\partial \underline{\underline{L}}_3}{\partial s_{1k}} &= \left[\begin{array}{c|c} \underline{\underline{0}} & \underline{\underline{H}}_{1211} \\ \hline -\underline{\underline{H}}^T_{1211} & \underline{\underline{0}} \end{array}\right] \\
\frac{\partial \underline{\ell}_2}{\partial s_{1k}} &= \left\{\begin{array}{c} \omega_2 \underline{J}_{2xy} \\ \hline \omega_3 \underline{J}_{3xz} \end{array}\right\}
\end{aligned}\right\} \tag{G.4a}$$

(b) for k = 2 :

$$\left.\begin{aligned}
\frac{\partial \underline{\underline{L}}_2}{\partial s_{1k}} &= \left[\begin{array}{c|c} 2\omega_2 \underline{\underline{J}}_{22yy} + \omega_1 (\underline{\underline{J}}_{12xy} + \underline{\underline{J}}^T_{12xy}) & \omega_3 \underline{\underline{J}}_{23yz} \\ \hline \omega_3 \underline{\underline{J}}^T_{23yz} & 2\omega_2 \underline{\underline{J}}_{33yy} \end{array}\right] \\
\frac{\partial \underline{\underline{L}}_3}{\partial s_{1k}} &= \left[\begin{array}{c|c} \underline{\underline{0}} & \underline{\underline{H}}_{2221} \\ \hline -\underline{\underline{H}}^T_{2221} & \underline{\underline{0}} \end{array}\right] \\
\frac{\partial \underline{\ell}_2}{\partial s_{1k}} &= \left\{\begin{array}{c} \omega_1 \underline{J}_{2xy} \\ \hline \omega_3 \underline{J}_{3yz} \end{array}\right\}
\end{aligned}\right\} \tag{G.4b}$$

(c) for k = 3

$$\frac{\partial \underline{\underline{L}}_2}{\partial s_{1k}} = \left[\begin{array}{c|c} 2\omega_3 \underline{\underline{J}}_{22zz} & \omega_1 \underline{\underline{J}}_{13xz} + \omega_2 \underline{\underline{J}}_{23yz} \\ \hline \omega_1 \underline{\underline{J}}_{13xz} + \omega_3 \underline{\underline{J}}^T_{23yz} & 2\omega_3 \underline{\underline{J}}_{33zz} \end{array}\right]$$

$$\left.\begin{aligned} \frac{\partial \underline{\underline{L}}_3}{\partial s_{1k}} &= \left[\begin{array}{c:c} \underline{\underline{H}}_{3131} & \underline{\underline{0}} \\ \hdashline \underline{\underline{0}} & \underline{\underline{0}} \end{array}\right] \\ \frac{\partial \underline{\ell}_2}{\partial s_{1k}} &= \left\{\begin{array}{c} \underline{0} \\ \hdashline \omega_1 \underline{J}_{3xz} + \omega_2 \underline{J}_{3yz} \end{array}\right\} \end{aligned}\right\} \qquad \text{(G.4c)}$$

The second order partials of (G.2) with respect to $\underline{s}_1$ follow as

$$\frac{\partial^2 \underline{L}_2}{\partial s_{1i}\, \partial s_{1j}} = \frac{\partial^2 \underline{\underline{L}}_2}{\partial s_{1i}\, \partial s_{1j}}\, \underline{s}_2 + \frac{\partial^2 \underline{\ell}_2}{\partial s_{1i}\, \partial s_{1j}} \qquad \text{(G.5)}$$

where

(a) for i = 1 and j = 1 :

$$\frac{\partial^2 \underline{\underline{L}}_2}{\partial s_{1i}\, \partial s_{1j}} = \left[\begin{array}{c:c} 2\underline{\underline{J}}_{22xx} & \underline{\underline{0}} \\ \hdashline \underline{\underline{0}} & 2\underline{\underline{J}}_{33xx} \end{array}\right] ; \quad \frac{\partial^2 \underline{\ell}_2}{\partial s_{1i}\, \partial s_{1j}} = \left\{\begin{array}{c} \underline{0} \\ \hdashline \underline{0} \end{array}\right\} \qquad \text{(G.5a)}$$

(b) for i = 1 and j = 2 :

$$\frac{\partial^2 \underline{\underline{L}}_2}{\partial s_{1i}\, \partial s_{1j}} = \left[\begin{array}{c:c} \underline{\underline{J}}_{12xy} + \underline{\underline{J}}^T_{12xy} & \underline{\underline{0}} \\ \hdashline \underline{\underline{0}} & \underline{\underline{0}} \end{array}\right] ; \quad \frac{\partial^2 \underline{\ell}_2}{\partial s_{1i}\, \partial s_{1j}} = \left\{\begin{array}{c} \underline{J}_{2xy} \\ \hdashline \underline{0} \end{array}\right\} \qquad \text{(G.5b)}$$

(c) for i = 1 and j = 3 :

$$\frac{\partial^2 \underline{\underline{L}}_2}{\partial s_{1i}\, \partial s_{1j}} = \left[\begin{array}{c:c} \underline{\underline{0}} & \underline{\underline{J}}_{13xz} \\ \hdashline \underline{\underline{J}}^T_{13xz} & \underline{\underline{0}} \end{array}\right] ; \quad \frac{\partial^2 \underline{\ell}_2}{\partial s_{1i}\, \partial s_{1j}} = \left\{\begin{array}{c} \underline{0} \\ \hdashline \underline{J}_{3xz} \end{array}\right\} \qquad \text{(G.5c)}$$

(d) for i = 2 and j = 2 :

$$\frac{\partial^2 \underline{\underline{L}}_2}{\partial s_{1i}\,\partial s_{1j}} = \left[\begin{array}{c:c} 2\underline{\underline{J}}_{22yy} & \underline{\underline{0}} \\ \hdashline \underline{\underline{0}} & 2\underline{\underline{J}}_{33yy} \end{array}\right] \; ; \; \frac{\partial^2 \underline{\ell}_2}{\partial s_{1i}\,\partial s_{1j}} = \left\{\begin{array}{c} \underline{0} \\ \hdashline \underline{0} \end{array}\right\} \tag{G.5d}$$

(e) for i = 2 and j = 3 :

$$\frac{\partial^2 \underline{\underline{L}}_2}{\partial s_{1i}\,\partial s_{1j}} = \left[\begin{array}{c:c} \underline{\underline{0}} & \underline{\underline{J}}_{23yz} \\ \hdashline \underline{\underline{J}}^T_{23yz} & \underline{\underline{0}} \end{array}\right] \; ; \; \frac{\partial^2 \underline{\ell}_2}{\partial s_{1i}\,\partial s_{1j}} = \left\{\begin{array}{c} \underline{0} \\ \hdashline \underline{J}_{3yz} \end{array}\right\} \tag{G.5e}$$

(f) for i = 3 and j = 3 :

$$\frac{\partial^2 \underline{\underline{L}}_2}{\partial s_{1i}\,\partial s_{1j}} = \left[\begin{array}{c:c} 2\underline{\underline{J}}_{22zz} & \underline{\underline{0}} \\ \hdashline \underline{\underline{0}} & 2\underline{\underline{J}}_{33zz} \end{array}\right] \; ; \; \frac{\partial^2 \underline{\ell}_2}{\partial s_{1i}\,\partial s_{1j}} = \left\{\begin{array}{c} \underline{0} \\ \hdashline \underline{0} \end{array}\right\} \tag{G.5f}$$

The first and second order partials of (G.2) with respect to $\underline{s}_2$ follow as (in index form)

$$\left.\begin{aligned} &\frac{\partial L_{2j}}{\partial s_{2k}} = L_{2jk} \qquad (j,k=1,\ldots\ldots,n) \\ &\text{and} \\ &\frac{\partial^2 L_{2j}}{\partial s_{1r}\,\partial s_{2k}} = \frac{\partial L_{2jk}}{\partial s_{1r}} \quad (r=1,2,3;j,k=1,\ldots\ldots,n) \end{aligned}\right\} \tag{G.6}$$

The first and second order partials of (G.2) with respect to $\underline{s}_3$ follow as (in index form)

$$\left.\begin{aligned} &\frac{\partial L_{2j}}{\partial s_{3k}} = L_{3jk} \qquad (j,k=1,\ldots\ldots,n) \\ &\frac{\partial^2 L_{2j}}{\partial s_{1r}\,\partial s_{3k}} = \frac{\partial L_{3jk}}{\partial s_{1r}} \quad (r=1,2,3;\ j,k=1,\ldots\ldots,n) \end{aligned}\right\} \tag{G.7}$$

All other partial derivatives of (G.2) vanish.

The elements of the moment of inertia tensor are listed below

$$\underline{\underline{J}} = \begin{bmatrix} A+\frac{2}{3}m_{AP}\ell^2 & \underline{J}^T_{2xy}\underline{\eta}_2 & \underline{J}^T_{3xz}\underline{\eta}_3 \\ & B+\frac{2}{3}m_{AP}\ell^2 & \underline{J}^T_{3yz}\underline{\eta}_3 \\ \text{sym.} & & C+\frac{4}{3}m_{AP}\ell^2 \end{bmatrix} * \qquad \text{(G.8)}$$

where m_{AP} and ℓ denote the mass and length of one of the appendages, respectfully; and A, B, and C denote the principal moment of inertia for the rigid central hub.

The partial derivatives of (G.8) with respect to $\underline{s}_2$ follow as

$$\frac{\partial \underline{\underline{J}}}{\partial s_{2k}} = \begin{bmatrix} 0 & J_{2xyk} & 0 \\ & 0 & 0 \\ \text{sym.} & & 0 \end{bmatrix} \quad (k=1,\ldots\ldots,\tfrac{n}{2}) \qquad \text{(G.9a)}$$

and

$$\frac{\partial \underline{\underline{J}}}{\partial s_{2k}} = \begin{bmatrix} 0 & 0 & J_{3yzk} \\ & 0 & J_{3yzk} \\ \text{sym.} & 0 & \end{bmatrix} \quad (k=\tfrac{n}{2}+1,\ldots\ldots,n) \qquad \text{(G.9b)}$$

All other partials of $\underline{\underline{J}}$ vanish identically.

The partials of $\dot{\underline{\underline{J}}}$ with respect to $\underline{s}_3$ follow as

$$\frac{\partial \dot{\underline{\underline{J}}}}{\partial s_{3k}} = \begin{bmatrix} 0 & J_{2xyk} & 0 \\ & 0 & 0 \\ \text{sym.} & & 0 \end{bmatrix} \quad (k=1,\ldots\ldots,\tfrac{n}{2}) \qquad \text{(G.10a)}$$

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* Only first order terms are retained.

and

$$\frac{\partial \underline{\underline{J}}}{\partial s_{3k}} = \begin{bmatrix} 0 & 0 & J_{3xzk} \\ & 0 & J_{3yzk} \\ \text{sym.} & & 0 \end{bmatrix} \quad (k=\tfrac{n}{2}+1,\ldots\ldots,n) \tag{G.10b}$$

All other partials of $\dot{\underline{\underline{J}}}$ vanish identically.

On comparing (G.9) and (G.10) it is clear that

$$\frac{\partial \underline{\underline{J}}}{\partial s_2} \equiv \frac{\partial \dot{\underline{\underline{J}}}}{\partial s_3} \tag{G.11}$$

which is a convenient observation since it reduces computer storage of identical partial derivatives.

The elastic angular momentum can be written as

$$H_E = \begin{Bmatrix} \underline{\eta}_3^T \underline{\underline{H}}_{11} \dot{\underline{\eta}}_2 + (\underline{H}_1^T + \underline{\eta}_2^T \underline{\underline{H}}_{12}) \dot{\underline{\eta}}_3 \\ \underline{\eta}_3^T \underline{\underline{H}}_{21} \dot{\underline{\eta}}_2 + (\underline{H}_2^T + \underline{\eta}_2^T \underline{\underline{H}}_{22}) \dot{\underline{\eta}}_3 \\ (\underline{H}_3^T + \underline{\eta}_2 \underline{\underline{H}}_{31}) \dot{\underline{\eta}}_2 \end{Bmatrix} \tag{G.12}$$

The partial derivatives of (G.12) follow as (using index form for the right hand side of the equations):

$$\frac{\partial \underline{H}_E}{\partial \eta_{2k}} = \begin{Bmatrix} H_{12kj} \dot{\eta}_{3j} \\ H_{22kj} \dot{\eta}_{3j} \\ H_{31kj} \dot{\eta}_{2j} \end{Bmatrix} \tag{G.13a}$$

$$\frac{\partial \underline{H}_E}{\partial \eta_{3k}} = \begin{Bmatrix} H_{11kj} \dot{\eta}_{2j} \\ H_{21kj} \dot{\eta}_{2j} \\ 0 \end{Bmatrix} \tag{G.13b}$$

$$\frac{\partial \underline{H}_E}{\partial \dot{\eta}_{2k}} = \left\{ \begin{array}{c} H_{11jk}\eta_{3j} \\ H_{21jk}\eta_{3j} \\ H_{3k} + H_{31jk}\eta_{2j} \end{array} \right\} \qquad \text{(G.13c)}$$

$$\frac{\partial \underline{H}_E}{\partial \dot{\eta}_{3k}} = \left\{ \begin{array}{c} H_{1k} + H_{12jk}\eta_{2j} \\ H_{2k} + H_{22jk}\eta_{2j} \\ 0 \end{array} \right\} \qquad \text{(G.13d)}$$

and

$$\frac{\partial \underline{H}_E}{\partial s_{2k}} = \left\{ \begin{array}{ll} \partial H_E/\partial \eta_{2k}\ , & \text{for } k=1,\ldots\ldots,\frac{n}{2} \\ & \\ \partial H_E/\partial \eta_{3k}\ , & \text{for } k=\frac{n}{2}+1,\ldots\ldots,n \end{array} \right\} \qquad \text{(G.14a)}$$

$$\frac{\partial \underline{H}_E}{\partial s_{3k}} = \left\{ \begin{array}{ll} \partial H_E/\partial \dot{\eta}_{2k}\ , & \text{for } k=1,\ldots\ldots,\frac{n}{2} \\ & \\ \partial H_E/\partial \dot{\eta}_{3k}\ , & \text{for } k=\frac{n}{2}+1,\ldots\ldots,n \end{array} \right\} \qquad \text{(G.14b)}$$

The control torque can be written as

$$U_{Ti} = - W^{-1}_{uuij} \left(m_{11jk}\lambda_{1k} + m_{21pj}\lambda_{3p} \right) \qquad \text{(G.15)}$$

where $i,j,k=1,2,3$ and $r,p=1,\ldots\ldots,n$.

The partial derivative

$$\frac{\partial U_{Ti}}{\partial s_{2r}} = - W^{-1}_{uuij} \left(\frac{\partial m_{11jk}}{\partial s_{2r}} \lambda_{1k} + \frac{\partial m_{21pj}}{\partial s_{2r}} \lambda_{3p} \right) \qquad \text{(G.16a)}$$

$$\frac{\partial U_{Ti}}{\partial \lambda_{1\ell}} = - W^{-1}_{uuij}\, m_{11j\ell} \qquad \text{(G.16b)}$$

$$\frac{\partial U_{Ti}}{\partial \lambda_{3r}} = - W^{-1}_{uuij} \; m_{21rj} \tag{G.16c}$$

The partitions of the inverse mass matrix can be written as

$$\begin{bmatrix} \underline{\underline{m}}_{11} & \underline{\underline{m}}_{12} \\ \underline{\underline{m}}_{21} & \underline{\underline{m}}_{22} \end{bmatrix} = \begin{bmatrix} \underline{\underline{J}} & \underline{\underline{H}}_1 \\ \underline{\underline{H}}_2 & \underline{\underline{M}} \end{bmatrix}^{-1} \tag{G.17}$$

where

$$\underline{\underline{m}}_{11} = (\underline{\underline{J}} - \underline{\underline{H}}_1 \; \underline{\underline{M}}^{-1} \; \underline{\underline{H}}_2)^{-1} \tag{G.17a}$$

$$\underline{\underline{m}}_{12} = - \underline{\underline{m}}_{11} \; \underline{\underline{H}}_1 \; \underline{\underline{M}}^{-1} \tag{G.17b}$$

$$\underline{\underline{m}}_{21} = - \underline{\underline{M}}^{-1} \; \underline{\underline{H}}_2 \; \underline{\underline{m}}_{11} \tag{G.17c}$$

$$\underline{\underline{m}}_{22} = \underline{\underline{M}}^{-1} - \underline{\underline{M}}^{-1} \; \underline{\underline{H}}_2 \; \underline{\underline{m}}_{12}$$

The matrices $\underline{\underline{H}}_1$ and $\underline{\underline{H}}_2$ in (G.17) are defined in (G.20) and (G.21), respectively.

The first order partials of (G.17) follow as

$$\frac{\partial \underline{\underline{m}}_{11}}{\partial s_{2k}} = - \underline{\underline{m}}_{11} \left(\frac{\partial \underline{\underline{J}}}{\partial s_{2k}} - \underline{\underline{H}}_1 \; \underline{\underline{M}}^{-1} \; \frac{\partial \underline{\underline{H}}_2}{\partial s_{2k}} \right) \underline{\underline{m}}_{11} \; {}^{*} \tag{G.18a}$$

$$\frac{\partial \underline{\underline{m}}_{12}}{\partial s_{2k}} = - \frac{\partial \underline{\underline{m}}_{11}}{\partial s_{2k}} \; \underline{\underline{H}}_1 \; \underline{\underline{M}}^{-1} \tag{G.18b}$$

$$\frac{\partial \underline{\underline{m}}_{21}}{\partial s_{2k}} = - \underline{\underline{M}}^{-1} \left(\frac{\partial \underline{\underline{H}}_2}{\partial s_{2k}} \; \underline{\underline{m}}_{11} + \underline{\underline{H}}_2 \; \frac{\partial \underline{\underline{m}}_{11}}{\partial s_{2k}} \right) \tag{G.18c}$$

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* If $\underline{\underline{A}} = \underline{\underline{B}}^{-1}$, then $\underline{\underline{A}} \; \underline{\underline{B}}^{-1} = \underline{\underline{I}}$, differentiating this last result yields $\frac{d\underline{\underline{A}}}{dp} \underline{\underline{B}}^{-1} + \underline{\underline{A}} \frac{d\underline{\underline{B}}^{-1}}{dp} = 0$ or $\frac{d\underline{\underline{B}}^{-1}}{dp} = - \underline{\underline{B}} \frac{d\underline{\underline{A}}}{dp} \underline{\underline{B}}$.

$$\frac{\partial \underline{\underline{m}}_{22}}{\partial s_{2k}} = - \underline{\underline{M}}^{-1} \left(\frac{\partial \underline{\underline{H}}_2}{\partial s_{2k}} \underline{\underline{m}}_{12} + \underline{\underline{H}}_2 \frac{\partial \underline{\underline{m}}_{12}}{\partial s_{2k}} \right) \tag{G.18d}$$

The second order partials of (G.17) follow as

$$\frac{\partial^2 \underline{\underline{m}}_{11}}{\partial s_{2k} \partial s_{2p}} = - \frac{\partial \underline{\underline{m}}_{11}}{\partial s_{2p}} \left(\frac{\partial \underline{\underline{J}}}{\partial s_{2k}} - \underline{\underline{H}}_1 \underline{\underline{M}}^{-1} \frac{\partial \underline{\underline{H}}_2}{\partial s_{2k}} \right) \underline{\underline{m}}_{11} - \underline{\underline{m}}_{11} \left(\frac{\partial \underline{\underline{J}}}{\partial s_{2k}} - \underline{\underline{H}}_1 \underline{\underline{M}}^{-1} \frac{\partial \underline{\underline{H}}_2}{\partial s_{2k}} \right) \frac{\partial \underline{\underline{m}}_{11}}{\partial s_{2k}} \tag{G.19a}$$

$$\frac{\partial^2 \underline{\underline{m}}_{12}}{\partial s_{2k} \partial s_{2p}} = - \frac{\partial^2 \underline{\underline{m}}_{11}}{\partial s_{2k} \partial s_{2p}} \underline{\underline{H}}_1 \underline{\underline{M}}^{-1} \tag{G.19b}$$

$$\frac{\partial^2 \underline{\underline{m}}_{21}}{\partial s_{2k} \partial s_{2p}} = - \underline{\underline{M}}^{-1} \left(\frac{\partial \underline{\underline{H}}_2}{\partial s_{2k}} \frac{\partial \underline{\underline{m}}_{11}}{\partial s_{2p}} + \frac{\partial \underline{\underline{H}}_2}{\partial s_{2p}} \frac{\partial \underline{\underline{m}}_{11}}{\partial s_{2k}} + \underline{\underline{H}}_2 \frac{\partial^2 \underline{\underline{m}}_{11}}{\partial s_{2k} \partial s_{2p}} \right) \tag{G.19c}$$

$$\frac{\partial^2 \underline{\underline{m}}_{22}}{\partial s_{2k} \partial s_{2p}} = - \underline{\underline{M}}^{-1} \left(\frac{\partial \underline{\underline{H}}_2}{\partial s_{2k}} \frac{\partial \underline{\underline{m}}_{12}}{\partial s_{2p}} + \frac{\partial \underline{\underline{H}}_2}{\partial s_{2p}} \frac{\partial \underline{\underline{m}}_{12}}{\partial s_{2k}} + \underline{\underline{H}}_2 \frac{\partial^2 \underline{\underline{m}}_{12}}{\partial s_{2k} \partial s_{2p}} \right) \tag{G.19d}$$

where

$$\underline{\underline{H}}_1 = \begin{bmatrix} \underline{0}^T & \underline{H}_1^T \\ \underline{0}^T & \underline{H}_2^T \\ \underline{H}_3^T & \underline{0}^T \end{bmatrix} \tag{G.20}$$

$$\underline{\underline{H}}_2 = \begin{bmatrix} \underline{\underline{H}}_{11}^T \underline{\eta}_3 & \underline{\underline{H}}_{21}^T \underline{\eta}_3 & \underline{H}_3 + \underline{\underline{H}}_{31}^T \underline{\eta}_2 \\ \underline{H}_1 + \underline{\underline{H}}_{12}^T \underline{\eta}_2 & \underline{H}_3 + \underline{\underline{H}}_{22}^T \underline{\eta}_2 & 0 \end{bmatrix} \tag{G.21}$$

and

$$\frac{\partial \underline{\underline{H}}_1}{\partial s_{2k}} \equiv 0 \tag{G.22}$$

$$\frac{\partial \underline{\underline{H}}_2}{\partial s_{2k}} = \left\{ \begin{array}{ll} \begin{bmatrix} 0 & 0 & \underline{\underline{H}}_{31}^T \dfrac{\partial \underline{\eta}_2}{\partial s_{2k}} \\ \underline{\underline{H}}_{12}^T \dfrac{\partial \underline{\eta}_2}{\partial s_{2k}} & \underline{\underline{H}}_{22}^T \dfrac{\partial \underline{\eta}_2}{\partial s_{2k}} & 0 \end{bmatrix} & ;\ k=1,\ldots,\frac{n}{2} \\ \begin{bmatrix} \underline{\underline{H}}_{11}^T \dfrac{\partial \underline{\eta}_3}{\partial s_{2k}} & \underline{\underline{H}}_{21}^T \dfrac{\partial \underline{\eta}_3}{\partial s_{2k}} & 0 \\ 0 & 0 & 0 \end{bmatrix} & ;\ k=\frac{n}{2}+1,\ldots,n \end{array} \right\} \tag{G.23}$$

where

$$\frac{\partial \underline{\eta}_2}{\partial s_{2k}} = \Big[\, 0 \quad 0 \ldots 0 \quad \underset{\substack{\downarrow \\ \text{k-th}}}{1} \quad 0 \ldots 0 \quad 0 \,\Big] \tag{G.24a}$$

$$\frac{\partial \underline{\eta}_3}{\partial s_{2k}} = \Big[\, 0 \quad 0 \ldots 0 \quad \underset{\substack{\downarrow \\ \text{k-th}}}{1} \quad 0 \ldots 0 \quad 0 \,\Big] \tag{G.24b}$$

In vector matrix form the necessary conditions for state and co-state equations are given by

$$\dot{\underline{s}}_1 = \underline{\underline{m}}_{11} (\underline{U}_T + \underline{L}_1) + \underline{\underline{m}}_{12} (\underline{L}_2 - \underline{\underline{K}}_{22}\, \underline{s}_2) \tag{G.25a}$$

$$\dot{\underline{s}}_2 = \underline{s}_3 \tag{G.25b}$$

$$\dot{\underline{s}}_3 = \underline{\underline{m}}_{21} (\underline{U}_T + \underline{L}_1) + \underline{\underline{m}}_{22} (\underline{L}_2 - \underline{\underline{K}}_{22}\, \underline{s}_2) \tag{G.25c}$$

$$\dot{\underline{s}}_4 = \underline{\underline{\Omega}}\, \underline{s}_4 = \underline{\underline{\beta}}\, \underline{s}_1 \tag{G.25d}$$

$$\dot{\underline{\lambda}}_1 = - \underline{\underline{\beta}}^T \underline{\lambda}_4 - \underline{\lambda}_1^T \left(\underline{\underline{m}}_{11} \left[\frac{\partial \underline{L}_1}{\partial \underline{s}_1}\right] + \underline{\underline{m}}_{12} \left[\frac{\partial \underline{L}_2}{\partial \underline{s}_1}\right]\right) - \underline{\lambda}_3^T \left(\underline{\underline{m}}_{21} \left[\frac{\partial \underline{L}_1}{\partial \underline{s}_1}\right] + \underline{\underline{m}}_{22} \left[\frac{\partial \underline{L}_2}{\partial \underline{s}_1}\right]\right) \tag{G.25e}$$

$$\dot{\underline{\lambda}}_2 = - \underline{\underline{W}}_{22}\, \underline{s}_2 + \underline{\underline{K}}_{22} (\underline{\underline{m}}_{21}\, \underline{\lambda}_1 + \underline{\underline{m}}_{22}\, \underline{\lambda}_3) - \underline{\lambda}_1^T \left\{ \left[\frac{\partial \underline{\underline{m}}_{11}}{\partial \underline{s}_2}\right] (\underline{U}_T + \underline{L}_1) + \left[\frac{\partial \underline{\underline{m}}_{12}}{\partial \underline{s}_2}\right] (\underline{L}_2 - \underline{\underline{K}}_{22} \underline{s}_2) + \underline{\underline{m}}_{11} \left[\frac{\partial \underline{L}_1}{\partial \underline{s}_2}\right] + \underline{\underline{m}}_{12} \left[\frac{\partial \underline{L}_2}{\partial \underline{s}_2}\right] \right\}$$

$$- \underline{\lambda}_3^T \left\{ \left[\frac{\partial \underline{\underline{m}}_{21}}{\partial \underline{s}_2} \right] (\underline{U}_T + \underline{L}_1) + \left[\frac{\partial \underline{\underline{m}}_{22}}{\partial \underline{s}_2} \right] (\underline{L}_2 - \underline{\underline{K}}_{22} \underline{s}_2) + \underline{\underline{m}}_{21} \left[\frac{\partial \underline{L}_1}{\partial \underline{s}_2} \right] + \underline{\underline{m}}_{22} \left[\frac{\partial \underline{L}_2}{\partial \underline{s}_2} \right] \right\}$$

(G.25f)

$$\dot{\underline{\lambda}}_3 = - \underline{\underline{W}}_{33} \underline{s}_3 - \underline{\lambda}_2 - (\underline{\lambda}_1^T \underline{\underline{m}}_{11} + \underline{\lambda}_3^T \underline{\underline{m}}_{21}) \left[\frac{\partial \underline{L}_1}{\partial \underline{s}_3} \right]$$

$$- (\underline{\lambda}_1^T \underline{\underline{m}}_{12} + \underline{\lambda}_3^T \underline{\underline{m}}_{22}) \left[\frac{\partial \underline{L}_2}{\partial \underline{s}_3} \right] \tag{G.25g}$$

$$\dot{\underline{\lambda}}_4 = \underline{\underline{\Omega}} \, \underline{\lambda}_4 \tag{G.25h}$$

where $\underline{U}_T = - \underline{\underline{W}}_{uu}^{-1} (\underline{\underline{m}}_{11} \underline{\lambda}_1 + \underline{\underline{m}}_{21}^T \underline{\lambda}_3)$ (G.25i)

The procedure used in obtaining the required partial derivative in (G.25) is the following: First, the equations are written in index form (where the Einstein summation convention is assumed), and Second, the partial derivatives are presented in index form. Equation (G.25a) in index form becomes

$$\dot{s}_{1i} = m_{11ij} (U_{Tj} + L_{1j}) + m_{12iq} (L_{2q} - K_{22qp} s_{2p}) \tag{G.26}$$

where i,j=1,2,3 and q,p=1,2,...,n . *

The partial derivatives of (G.26) follow as

$$\frac{\partial \dot{s}_{1i}}{\partial s_{1k}} = m_{11ij} \frac{\partial L_{1j}}{\partial s_{1k}} + m_{12iq} \frac{\partial L_{2q}}{\partial s_{1k}} \tag{G.26a}$$

$$\frac{\partial \dot{s}_{1i}}{\partial s_{2k}} = \frac{\partial m_{11ij}}{\partial s_{2k}} (U_{Tj} + L_{1j}) + m_{11ij} \left(\frac{\partial U_{Tj}}{\partial s_{2k}} + \frac{\partial L_{1j}}{\partial s_{2k}} \right)$$

---

* n has the dimension of the combine number of degrees of freedom for the in- and out-of-plane modes and their corresponding rates.

$$+ \frac{\partial m_{12iq}}{\partial s_{2k}} (L_{2q} - K_{22qp} s_{2p}) + m_{12iq} \left(\frac{\partial L_{2q}}{\partial s_{2k}} - K_{22qk} \right) \; *$$

(G.26b)

$$\frac{\partial \dot{s}_{1i}}{\partial s_{3k}} = m_{11ij} \frac{\partial L_{ij}}{\partial s_{3k}} \tag{G.26c}$$

$$\frac{\partial \dot{s}_{1i}}{\partial s_{4k}} = 0 \tag{G.26d}$$

$$\frac{\partial \dot{s}_{1i}}{\partial \lambda_{1k}} = m_{11ij} \frac{\partial U_{Tj}}{\partial \lambda_{1k}} \tag{G.26e}$$

$$\frac{\partial \dot{s}_{1i}}{\partial \lambda_{2k}} = 0 \tag{G.26f}$$

$$\frac{\partial \dot{s}_{1k}}{\partial \lambda_{3k}} = m_{11ij} \frac{\partial U_{Tj}}{\partial \lambda_{3k}} \tag{G.26g}$$

$$\frac{\partial \dot{s}_{1k}}{\partial \lambda_{4k}} = 0 \tag{G.26h}$$

Equation (G.25b) in index form becomes

$$\dot{s}_{2i} = s_{3i} \tag{G.27}$$

where $i = 1,\ldots\ldots,n$.

The partial derivatives of (G.27) follow as

$$\frac{\partial \dot{s}_{2i}}{\partial s_{1k}} = 0 \tag{G.27a}$$

---

* The identity $K_{22qk} = K_{22qp} \dfrac{\partial s_{2p}}{\partial s_{2k}} = K_{22qp}\, \delta_{pk}$ has been used, where δ_{ij} is the Kronecker delta symbol.

$$\frac{\partial \dot{s}_{2i}}{\partial s_{2k}} = 0 \qquad \text{(G.27b)}$$

$$\frac{\partial \dot{s}_{2i}}{\partial s_{3k}} = \delta_{ik} \qquad \text{(G.27c)}$$

$$\frac{\partial \dot{s}_{2i}}{\partial s_{4k}} = 0 \qquad \text{(G.27d)}$$

$$\frac{\partial \dot{s}_{2i}}{\partial \lambda_{1k}} = 0 \qquad \text{(G.27e)}$$

$$\frac{\partial \dot{s}_{2i}}{\partial \lambda_{2k}} = 0 \qquad \text{(G.27f)}$$

$$\frac{\partial \dot{s}_{2i}}{\partial \lambda_{3k}} = 0 \qquad \text{(G.27g)}$$

$$\frac{\partial \dot{s}_{2i}}{\partial \lambda_{4k}} = 0 \qquad \text{(G.27h)}$$

Equation (G.25c) in index form becomes

$$\dot{s}_{3i} = m_{21ij}\,(U_{Tj} + L_{1j}) + m_{22iq}\,(L_{2q} - K_{22qp}\,s_{2p}) \qquad \text{(G.28)}$$

where $j=1,2,3$ and $i,q,p=1,\ldots\ldots,n$.

The partial derivatives of (G.4) follow as

$$\frac{\partial \dot{s}_{3i}}{\partial s_{1k}} = m_{21ij}\,\frac{\partial L_{1j}}{\partial s_{1k}} + m_{22iq}\,\frac{\partial L_{2q}}{\partial s_{1k}} \qquad \text{(G.28a)}$$

$$\frac{\partial \dot{s}_{3i}}{\partial s_{2k}} = \frac{\partial m_{21ij}}{\partial s_{2k}}\,(U_{Tj}+L_{1j}) + m_{21ij}\left(\frac{\partial U_{Tj}}{\partial s_{2k}} + \frac{\partial L_{1j}}{\partial s_{2k}}\right)$$
$$+ \frac{\partial m_{22iq}}{\partial s_{2k}}(L_{2q}-K_{22qp}s_{2p}) + m_{22iq}\left(\frac{\partial L_{2q}}{\partial s_{2k}} - K_{22qk}\right) \qquad \text{(G.28b)}$$

$$\frac{\partial \dot{s}_{3i}}{\partial s_{3k}} = m_{21ij} \frac{\partial L_{1j}}{\partial s_{3k}} + m_{22iq} \frac{\partial L_{2q}}{\partial s_{3k}} \tag{G.28c}$$

$$\frac{\partial \dot{s}_{3i}}{\partial s_{4k}} = 0 \tag{G.28d}$$

$$\frac{\partial \dot{s}_{3i}}{\partial \lambda_{1k}} = m_{21ij} \frac{\partial U_{Tj}}{\partial \lambda_{1k}} \tag{G.28e}$$

$$\frac{\partial \dot{s}_{3i}}{\partial \lambda_{2k}} = 0 \tag{G.28f}$$

$$\frac{\partial \dot{s}_{3i}}{\partial \lambda_{3k}} = m_{21ij} \frac{\partial U_{Tj}}{\partial \lambda_{3k}} \tag{G.28g}$$

$$\frac{\partial \dot{s}_{3i}}{\partial \lambda_{4k}} = 0 \tag{G.28h}$$

Equation (G.25d) in index form becomes

$$\dot{s}_{4i} = \Omega_{ij}\, s_{4j} = \beta_{iq}\, s_{1q} \tag{G.29}$$

where $i,j=1,\ldots\ldots,4$ and $q=1,\ldots\ldots,3$.

The partial derivatives of (G.5) follow as

$$\frac{\partial \dot{s}_{4i}}{\partial s_{1k}} = \beta_{ik} \tag{G.29a}$$

$$\frac{\partial \dot{s}_{4i}}{\partial s_{2k}} = 0 \tag{G.29b}$$

$$\frac{\partial \dot{s}_{4i}}{\partial s_{3k}} = 0 \tag{G.29c}$$

$$\frac{\partial \dot{s}_{4i}}{\partial s_{4k}} = \Omega_{ik} \tag{G.29d}$$

$$\frac{\partial \dot{s}_{4i}}{\partial \lambda_{1k}} = 0 \tag{G.29e}$$

$$\frac{\partial \dot{s}_{4i}}{\partial \lambda_{2k}} = 0 \tag{G.29f}$$

$$\frac{\partial \dot{s}_{4i}}{\partial \lambda_{3k}} = 0 \tag{G.29g}$$

$$\frac{\partial \dot{s}_{4i}}{\partial \lambda_{4k}} = 0 \tag{G.29h}$$

Equation (G.25e) in index form becomes

$$\dot{\lambda}_{1i} = -\beta_{ri}\lambda_{4r} - \lambda_{1h}\left(m_{11hj}\frac{\partial L_{1j}}{\partial s_{1i}} + m_{12hp}\frac{\partial L_{2p}}{\partial s_{1i}}\right) - \lambda_{3q}\left(m_{21qj}\frac{\partial L_{1j}}{\partial s_{1i}} + m_{22qp}\frac{\partial L_{2p}}{\partial s_{1i}}\right) \tag{G.30}$$

where i,h,j=1,2,3 ; r=1,.....,4 ; and p,q=1,.....,n .

The partial derivatives of (G.30) follow as

$$\frac{\partial \dot{\lambda}_{1i}}{\partial s_{1k}} = -\lambda_{1h}\left(m_{11hj}\frac{\partial^2 L_{1j}}{\partial s_{1i}\,\partial s_{1k}} + m_{12hp}\frac{\partial^2 L_{2p}}{\partial s_{1i}\,\partial s_{1k}}\right) - \lambda_{3q}\left(m_{21qj}\frac{\partial^2 L_{1j}}{\partial s_{1i}\,\partial s_{2k}} + m_{22qp}\frac{\partial^2 L_{2p}}{\partial s_{1i}\,\partial s_{1k}}\right) \tag{G.30a}$$

$$\frac{\partial \dot{\lambda}_{1i}}{\partial s_{2k}} = -\lambda_{1h}\left(\frac{\partial m_{11hj}}{\partial s_{2k}}\frac{\partial L_{1j}}{\partial s_{1i}} + m_{11hj}\frac{\partial^2 L_{1j}}{\partial s_{2k}\,\partial s_{1i}} + \frac{\partial m_{12hp}}{\partial s_{2k}}\frac{\partial L_{2p}}{\partial s_{1i}} + m_{12hp}\frac{\partial^2 L_{2p}}{\partial s_{2k}\,\partial s_{1i}}\right)$$

$$- \lambda_{3q}\left(\frac{\partial m_{21qj}}{\partial s_{2k}}\frac{\partial L_{1j}}{\partial s_{1i}} + m_{21qj}\frac{\partial^2 L_{1j}}{\partial s_{2k}\,\partial s_{1i}}\right.$$

$$+ \frac{\partial m_{22qp}}{\partial s_{2k}} \frac{\partial L_{2p}}{\partial s_{1i}} + m_{22qp} \frac{\partial^2 L_{2p}}{\partial s_{2k}\, \partial s_{1i}}\Bigg) \tag{G.30b}$$

$$\frac{\partial \dot{\lambda}_{1i}}{\partial s_{3k}} = - \lambda_{1h} \left(m_{11hj} \frac{\partial^2 L_{1j}}{\partial s_{3k}\, \partial s_{1i}} + m_{12hp} \frac{\partial^2 L_{2p}}{\partial s_{1i}\, \partial s_{3k}} \right)$$

$$- \lambda_{3q} \left(m_{21qj} \frac{\partial^2 L_{1j}}{\partial s_{3k}\, \partial s_{1i}} + m_{22qp} \frac{\partial^2 L_{2p}}{\partial s_{1i}\, \partial s_{3k}} \right) \tag{G.30c}$$

$$\frac{\partial \dot{\lambda}_{1i}}{\partial s_{4k}} = - \frac{\partial \beta_{pi}}{\partial s_{4k}} \lambda_{4p} \tag{G.30d}$$

$$\frac{\partial \dot{\lambda}_{1i}}{\partial \lambda_{1k}} = - m_{11kj} \frac{\partial L_{1j}}{\partial s_{1i}} - m_{12kp} \frac{\partial L_{2p}}{\partial s_{1i}} \tag{G.30e}$$

$$\frac{\partial \dot{\lambda}_{1i}}{\partial \lambda_{2k}} = 0 \tag{G.30f}$$

$$\frac{\partial \dot{\lambda}_{1i}}{\partial \lambda_{3k}} = - m_{21kj} \frac{\partial L_{1j}}{\partial s_{1i}} - m_{22kp} \frac{\partial L_{2p}}{\partial s_{1i}} \tag{G.30g}$$

$$\frac{\partial \dot{\lambda}_{1i}}{\partial \lambda_{4k}} = - \beta_{ki} \tag{G.30h}$$

Equation (G.25f) in index form becomes

$$\dot{\lambda}_{2i} = - W_{22ij}\, s_{2j} + K_{22ij} \left(m_{21jp}\, \lambda_{1p} + m_{22jq}\, \lambda_{3q} \right)$$

$$- \lambda_{1r} \left(\frac{\partial m_{11rp}}{\partial s_{2i}} (U_{Tp} + L_{1p}) + \frac{\partial m_{12rq}}{\partial s_{2i}} (L_{2q} - K_{22qt}\, s_{2t}) \right.$$

$$\left. + m_{11rp} \frac{\partial L_{1p}}{\partial s_{2i}} + m_{12rq} \frac{\partial L_{2q}}{\partial s_{2i}} \right)$$

$$- \lambda_{3\ell} \left(\frac{\partial m_{21\ell p}}{\partial s_{2i}} (U_{Tp} + L_{1p}) + \frac{\partial m_{22\ell q}}{\partial s_{2i}} (L_{2q} - K_{22qt}\, s_{2t}) \right.$$

$$\left. + m_{21\ell p} \frac{\partial L_{1p}}{\partial s_{2i}} + m_{22\ell q} \frac{\partial L_{2q}}{\partial s_{2i}} \right) \tag{G.31}$$

where p,r=1,2,3 and i,j,q,t,ℓ=1,.....,n .

The partial derivatives of (G.31) follow as

$$\frac{\partial\dot{\lambda}_{2i}}{\partial s_{1k}} = -\lambda_{1r}\left(\frac{\partial m_{11rp}}{\partial s_{2i}}\frac{\partial L_{1p}}{\partial s_{1k}} + \frac{\partial m_{12rq}}{\partial s_{2i}}\frac{\partial L_{2q}}{\partial s_{1k}} + m_{11rp}\frac{\partial^2 L_{1p}}{\partial s_{2i}\,\partial s_{1k}} + m_{12rq}\frac{\partial^2 L_{2q}}{\partial s_{2i}\,\partial s_{1k}}\right)$$
$$- \lambda_{3\ell}\left(\frac{\partial m_{21\ell p}}{\partial s_{2i}}\frac{\partial L_{1p}}{\partial s_{1k}} + \frac{\partial m_{22\ell q}}{\partial s_{2i}}\frac{\partial L_{2q}}{\partial s_{1k}} + m_{21\ell p}\frac{\partial^2 L_{1p}}{\partial s_{2i}\,\partial s_{1k}} + m_{22\ell q}\frac{\partial^2 L_{2q}}{\partial s_{2i}\,\partial s_{1k}}\right) \tag{G.31a}$$

$$\frac{\partial\dot{\lambda}_{2i}}{\partial s_{2k}} = -W_{22ik} - \lambda_{1r}\left\{\frac{\partial^2 m_{11rp}}{\partial s_{2i}\,\partial s_{2k}}(U_{Tp} + L_{1p}) + \frac{\partial m_{11rp}}{\partial s_{2i}}\left(\frac{\partial U_{Tp}}{\partial s_{2k}} + \frac{\partial L_{1p}}{\partial s_{2k}}\right) + \frac{\partial m_{11rp}}{\partial s_{2k}}\frac{\partial L_{1q}}{\partial s_{2i}} + \frac{\partial m_{12rq}}{\partial s_{2i}}\left(\frac{\partial L_{2q}}{\partial s_{2k}} - K_{22qk}\right) + \frac{\partial m_{12rq}}{\partial s_{2k}}\left(\frac{\partial L_{2q}}{\partial s_{2i}} - K_{22qi}\right)\right\}$$
$$- \lambda_{3\ell}\left\{\frac{\partial^2 m_{21\ell p}}{\partial s_{2i}\,\partial s_{2k}}(U_{Tp}+L_{1p}) + \frac{\partial^2 m_{22\ell q}}{\partial s_{2i}\,\partial s_{2k}}(L_{2q}-K_{22qt}s_{2t}) + \frac{\partial m_{21\ell p}}{\partial s_{2i}}\left(\frac{\partial U_{Tp}}{\partial s_{2k}} + \frac{\partial L_{1p}}{\partial s_{2i}}\right) + \frac{\partial m_{21\ell p}}{\partial s_{2k}}\frac{\partial L_{1p}}{\partial s_{2i}} + \frac{\partial m_{22\ell p}}{\partial s_{2i}}\left(\frac{\partial L_{2p}}{\partial s_{2k}} - K_{22\ell k}\right) + \frac{\partial m_{22\ell p}}{\partial s_{2k}}\left(\frac{\partial L_{2p}}{\partial s_{2i}} - K_{22pi}\right)\right\} \tag{G.31b}$$

$$\frac{\partial\dot{\lambda}_{2i}}{\partial s_{3k}} = -\lambda_{1r}\left(\frac{\partial m_{11rp}}{\partial s_{2i}}\frac{\partial L_{1p}}{\partial s_{3k}} + \frac{\partial m_{rq}}{\partial s_{2i}}\frac{\partial L_{2q}}{\partial s_{3k}}\right)$$

$$- \lambda_{3\ell} \left(\frac{\partial m_{21\ell p}}{\partial s_{2i}} \frac{\partial L_{1p}}{\partial s_{3k}} + \frac{\partial m_{22\ell q}}{\partial s_{2i}} \frac{\partial L_{2q}}{\partial s_{3k}} \right) \tag{G.31c}$$

$$\frac{\partial \dot{\lambda}_{2i}}{\partial s_{4k}} = 0 \tag{G.31d}$$

$$\frac{\partial \dot{\lambda}_{2i}}{\partial \lambda_{1k}} = - \left\{ \frac{\partial m_{11kp}}{\partial s_{2i}} (U_{Tp} + L_{1p}) + m_{11kp} \frac{\partial L_{1p}}{\partial s_{2i}} + \frac{\partial m_{12kq}}{\partial s_{2i}} (L_{2q} - K_{22qt} s_{2t}) + m_{12kq} \left(\frac{\partial L_{2q}}{\partial s_{2i}} - K_{22qi} \right) \right\}$$
$$- \lambda_{1r} \frac{\partial m_{11rp}}{\partial s_{2i}} \frac{\partial U_{Tp}}{\partial \lambda_{1k}} - \lambda_{3\ell} \frac{\partial m_{21\ell p}}{\partial s_{2i}} \frac{\partial U_{Tp}}{\partial \lambda_{1k}} \tag{G.31e}$$

$$\frac{\partial \dot{\lambda}_{2i}}{\partial \lambda_{2k}} = 0 \tag{G.31f}$$

$$\frac{\partial \dot{\lambda}_{2i}}{\partial \lambda_{3k}} = - \left\{ \frac{\partial m_{21kp}}{\partial s_{2i}} (U_{Tp} + L_{1p}) + m_{21kp} \frac{\partial L_{1p}}{\partial s_{2i}} + \frac{\partial m_{22kq}}{\partial s_{2i}} (L_{2q} - K_{22qt} s_{2t}) + m_{22kq} \left(\frac{\partial L_{2q}}{\partial s_{2i}} - K_{22qi} \right) \right\}$$
$$- \lambda_{1r} \frac{\partial m_{11rp}}{\partial s_{2i}} \frac{\partial U_{Tp}}{\partial \lambda_{3k}} - \lambda_{3\ell} \frac{\partial m_{21\ell p}}{\partial s_{2i}} \frac{\partial U_{Tp}}{\partial \lambda_{3k}} \tag{G.31g}$$

$$\frac{\partial \dot{\lambda}_{2i}}{\partial \lambda_{4k}} = 0 \tag{G.31h}$$

In index form (G.25h) becomes

$$\dot{\lambda}_{3i} = - W_{33iq} s_{3q} - \lambda_{2i} - (\lambda_{1p} m_{11pj} + \lambda_{3q} m_{21qj}) \frac{\partial L_{1j}}{\partial s_{3i}}$$
$$- (\lambda_{1p} m_{12pq} + \lambda_{3\ell} m_{22\ell q}) \frac{\partial L_{2q}}{\partial s_{3i}} \tag{G.32}$$

where $p,j=1,2,3$ and $i,q,\ell=1,\ldots\ldots,n$.

The partial derivatives of (G.32) follow as

$$\frac{\partial \dot{\lambda}_{3i}}{\partial s_{1k}} = - (\lambda_{1p}\, m_{11pj} + \lambda_{3q}\, m_{21qj}) \frac{\partial^2 L_{1j}}{\partial s_{3i}\, \partial s_{1j}}$$

$$- (\lambda_{1p}\, m_{12pq} + \lambda_{3\ell}\, m_{22\ell q}) \frac{\partial^2 L_{2q}}{\partial s_{3i}\, \partial s_{1k}} \qquad \text{(G.32a)}$$

$$\frac{\partial \dot{\lambda}_{3i}}{\partial s_{2k}} = - \left(\lambda_{1p} \frac{\partial m_{11pj}}{\partial s_{2k}} + \lambda_{3q} \frac{\partial m_{21qj}}{\partial s_{2k}} \right) \frac{\partial L_{1j}}{\partial s_{3i}} \qquad \text{(G.32b)}$$

$$\frac{\partial \dot{\lambda}_{3i}}{\partial s_{3k}} = - W_{33ik} \qquad \text{(G.32c)}$$

$$\frac{\partial \dot{\lambda}_{3i}}{\partial s_{4k}} = 0 \qquad \text{(G.32d)}$$

$$\frac{\partial \dot{\lambda}_{3i}}{\partial \lambda_{1k}} = - m_{11kj} \frac{\partial L_{1j}}{\partial s_{3i}} \qquad \text{(G.32e)}$$

$$\frac{\partial \dot{\lambda}_{3i}}{\partial \lambda_{2k}} = - \delta_{ik} \qquad \text{(G.32f)}$$

$$\frac{\partial \dot{\lambda}_{3i}}{\partial \lambda_{3k}} = - m_{21kj} \frac{\partial L_{1j}}{\partial s_{3i}} \qquad \text{(G.32g)}$$

$$\frac{\partial \dot{\lambda}_{3i}}{\partial \lambda_{4k}} = 0 \qquad \text{(G.32h)}$$

In index form (G.25h) becomes

$$\dot{\lambda}_{4i} = \Omega_{iq}\, \lambda_{4q} \qquad \text{(G.33)}$$

where $i,q=1,.....,4$.

The partial derivatives of (G.33) follow as

$$\frac{\partial \dot{\lambda}_{4i}}{\partial s_{1k}} = \frac{\partial \Omega_{iq}}{\partial s_{1k}} \lambda_{4q} \qquad \text{(G.33a)}$$

$$\frac{\partial \dot{\lambda}_{4i}}{\partial s_{2k}} = 0 \tag{G.33b}$$

$$\frac{\partial \dot{\lambda}_{4i}}{\partial s_{3k}} = 0 \tag{G.33c}$$

$$\frac{\partial \dot{\lambda}_{4i}}{\partial s_{4k}} = 0 \tag{G.33d}$$

$$\frac{\partial \dot{\lambda}_{4i}}{\partial \lambda_{1k}} = 0 \tag{G.33e}$$

$$\frac{\partial \dot{\lambda}_{4i}}{\partial \lambda_{2k}} = 0 \tag{G.33f}$$

$$\frac{\partial \dot{\lambda}_{4i}}{\partial \lambda_{3k}} = 0 \tag{G.33g}$$

$$\frac{\partial \dot{\lambda}_{4i}}{\partial \lambda_{4k}} = \Omega_{ik} \tag{G.33h}$$

This concludes the section on obtaining the partial derivatives necessary for integrating the state transition matrix (B.6).

APPENDIX H

CORRESPONDENCE BETWEEN PHYSICAL SPACE AND MODAL SPACE

NECESSARY CONDITIONS

In order to guarantee that an optimal solution obtained in physical space corresponds to the solution in modal space, several identities must be established. These identities are determined by considering the following constant coefficient vector matrix differential equation

$$\underline{\underline{M}}\,\ddot{\underline{x}} + \underline{\underline{K}}\,\underline{x} = \underline{\underline{B}}\,\underline{U} \tag{H.1}$$

where $\underline{x}$ is a q by 1 state vector; $\underline{U}$ is a p by 1 control vector;

$\underline{\underline{M}}$ is a symmetric q by q positive definite mass matrix; $\underline{\underline{K}}$ is a symmetric q by q positive semidefinite stiffness matrix; and

$\underline{\underline{B}}$ is an arbitrary q by p control matrix.

The first step is to obtain the modal space form of (H.1). To do this, the following eigenvalue - eigenvector problem is solved

$$\lambda_r^2\,\underline{\underline{M}}\,\underline{\xi}_r = \underline{\underline{K}}\,\underline{\xi}_r \tag{H.2}$$

for the eigenvalues λ_r^2 and the eigenvectors $\underline{\xi}_r^2$, where the eigenvectors are normalized by

$$\underline{\underline{E}}^T\,\underline{\underline{M}}\,\underline{\underline{E}} = \underline{\underline{I}} \tag{H.3}$$

leading to

$$\underline{\underline{E}}^T\,\underline{\underline{K}}\,\underline{\underline{E}} = \underline{\underline{\Omega}} \tag{H.4}$$

where $\underline{\underline{E}} = \left[\underline{\xi}_1 \quad \underline{\xi}_2 \;\ldots\ldots\; \underline{\xi}_q\right]$ is the matrix of eigenvectors, and

$\underline{\underline{\Omega}} = \text{diag.}\left[\lambda_1 \quad \lambda_2 \;\ldots\ldots\; \lambda_q\right]$.

Introducing the tranformation

$$\underline{x} = \underline{\underline{E}}\ \underline{t} \tag{H.5}$$

into (H.2) and premultiplying the resulting equation by $\underline{\underline{E}}^T$, leads to

$$\ddot{\underline{t}} + \underline{\underline{\Omega}}\ \underline{t} = \underline{\underline{E}}^T\ \underline{\underline{B}}\ \underline{U} \tag{H.6}$$

where (H.6) represents modal space representation of (H.1). Next, if (H.1) is premultiplied by $\underline{\underline{M}}^{-1}$ the following equation of motion for physical variables $\underline{x}$ is obtained

$$\ddot{\underline{x}} + \underline{\underline{M}}^{-1}\ \underline{\underline{K}}\ \underline{x} = \underline{\underline{M}}^{-1}\ \underline{\underline{B}}\ \underline{U} \tag{H.7}$$

In preparing to define the optimal control problem for (H.6) and (H.7), these equations are cast into state space form. Introducing the state space variable subsets

$$\begin{aligned} \underline{s}_1 &= \underline{x} \\ \underline{s}_2 &= \dot{\underline{x}} \end{aligned} \tag{H.8}$$

and

$$\begin{aligned} \underline{\delta}_1 &= \underline{t} \\ \underline{\delta}_2 &= \dot{\underline{t}} \end{aligned} \tag{H.9}$$

leads to the first order differential equations

$$\begin{aligned} \dot{\underline{s}}_1 &= \underline{s}_2 \\ \dot{\underline{s}}_2 &= \underline{\underline{M}}^{-1}\ \underline{\underline{B}}\ \underline{U} - \underline{\underline{M}}^{-1}\ \underline{\underline{K}}\ \underline{s}_1 \end{aligned} \tag{H.10}$$

and

$$\begin{aligned} \dot{\underline{\delta}}_1 &= \underline{\delta}_2 \\ \dot{\underline{\delta}}_2 &= \underline{\underline{E}}^T\ \underline{\underline{B}}\ \underline{U} - \underline{\underline{\Omega}}\ \underline{\delta}_1 \end{aligned} \tag{H.11}$$

For the optimal control problem the performance indices for (H.10) and (H.11) are defined as

$$J_x = \frac{1}{2}\int_{t_o}^{t} (\underline{U}^T \underline{\underline{W}}_{uu} \underline{U} + \underline{s}_1^T \underline{\underline{W}}_{11} \underline{s}_1 + \underline{s}_2^T \underline{\underline{W}}_{22} \underline{s}_2)\ dt \tag{H.12}$$

and

$$J_t = \frac{1}{2}\int_{t_o}^{t} (\underline{U}^T \underline{\underline{\mathcal{W}}}_{uu} \underline{U} + \underline{\delta}_1^T \underline{\underline{\mathcal{W}}}_{11} \underline{\delta}_1 + \underline{\delta}_2^T \underline{\underline{\mathcal{W}}}_{22} \underline{\delta}_2)\ dt \tag{H.13}$$

For the physical space variables, $\underline{\underline{W}}_{uu}$ is the weight matrix for $\underline{U}$, $\underline{\underline{W}}_{11}$ is the weight matrix for $\underline{s}_1$, and $\underline{\underline{W}}_{22}$ is the weight matrix for $\underline{s}_2$. For the modal space variables, $\underline{\underline{\mathcal{W}}}_{uu}$ is the weight matrix for $\underline{U}$, $\underline{\underline{\mathcal{W}}}_{11}$ is the weight matrix for $\underline{\delta}_1$, and $\underline{\underline{\mathcal{W}}}_{22}$ is the weight matrix for $\underline{\delta}_2$.

The Hamiltonian functionals associated with (H.12) and (H.13) follow as

$$\mathcal{H}_x = \frac{1}{2} (\underline{U}^T \underline{\underline{W}}_{uu} \underline{U} + \underline{s}_1^T \underline{\underline{W}}_{11} \underline{s}_1 + \underline{s}_2^T \underline{\underline{W}}_{22} \underline{s}_2) + \underline{\lambda}_1^T \underline{s}_2 + \underline{\lambda}_2^T (\underline{\underline{M}}^{-1} \underline{\underline{B}} \underline{U} - \underline{\underline{M}}^{-1} \underline{\underline{K}} \underline{s}_1) \tag{H.14}$$

and

$$\mathcal{H}_t = \frac{1}{2} (\underline{U}^T \underline{\underline{\mathcal{W}}}_{uu} \underline{U} + \underline{\delta}_1^T \underline{\underline{\mathcal{W}}}_{11} \underline{\delta}_1 + \underline{\delta}_2^T \underline{\underline{\mathcal{W}}}_{22} \underline{\delta}_2) + \underline{\Lambda}_1^T \underline{\delta}_2 + \underline{\Lambda}_2^T (\underline{\underline{E}}^T \underline{\underline{B}} \underline{U} - \underline{\underline{\Omega}} \underline{\delta}_1) \tag{H.15}$$

where $\underline{\lambda}_1$ and $\underline{\lambda}_2$ denote co-state variables for the physical space variables $\underline{s}_1$ and $\underline{s}_2$; and $\underline{\Lambda}_1$ and $\underline{\Lambda}_2$ denote co-state variables for the modal space variables $\underline{\delta}_1$ and $\underline{\delta}_2$.

In order to guarantee that the optimal solution found in physical space corresponds to the optimal solution in modal space, two requirements are imposed: The first requirement is

$$J_x = J_t \tag{H.16}$$

This condition can be enforce by equating the quadratic terms in

(H.12) and (H.13) as

$$\underset{\sim}{U}^T \underset{\approx}{W}_{uu} \underset{\sim}{U} = \underset{\sim}{U}^T \underset{\approx}{\mathcal{W}}_{uu} \underset{\sim}{U} \qquad \text{(H.17a)}$$

$$\underset{\sim}{s}_1^T \underset{\approx}{W}_{11} \underset{\sim}{s}_1 = \underset{\sim}{\delta}_1 \underset{\approx}{\mathcal{W}}_{11} \underset{\sim}{\delta}_1 \qquad \text{(H.17b)}$$

$$\underset{\sim}{s}_2^T \underset{\approx}{W}_{22} \underset{\sim}{s}_2 = \underset{\sim}{\delta}_2 \underset{\approx}{\mathcal{W}}_{22} \underset{\sim}{\delta}_2 \qquad \text{(H.17c)}$$

In (H.17a) the correspondence is obtained by requiring

$$\underset{\approx}{\mathcal{W}}_{uu} = \underset{\approx}{W}_{uu} \qquad \text{(H.18)}$$

In (H.17b) and (H.17c) the $\underset{\sim}{s}_1$ and $\underset{\sim}{s}_2$ variables are first transformed to $\underset{\sim}{\delta}_1$ and $\underset{\sim}{\delta}_2$, via the transformation

$$\begin{aligned} \underset{\sim}{s}_1 &= \underset{\approx}{E} \underset{\sim}{\delta}_1 \\ \underset{\sim}{s}_2 &= \underset{\approx}{E} \underset{\sim}{\delta}_2 \end{aligned} \qquad \text{(H.19)}$$

On introducing (H.19) into (H.17b) and (H.17c) the following identities are established

$$\underset{\approx}{\mathcal{W}}_{11} = \underset{\approx}{E}^T \underset{\approx}{W}_{11} \underset{\approx}{E} \qquad \text{(H.20)}$$

$$\underset{\approx}{\mathcal{W}}_{22} = \underset{\approx}{E}^T \underset{\approx}{W}_{22} \underset{\approx}{E} \qquad \text{(H.21)}$$

The second requirement is that the following terms in (H.14) and (H.15) are equated

$$\underset{\sim}{\lambda}_1^T \underset{\sim}{s}_2 \equiv \underset{\sim}{\Lambda}_1^T \underset{\sim}{\delta}_2 \qquad \text{(H.22a)}$$

$$\underset{\sim}{\lambda}_2^T (\underset{\approx}{M}^{-1} \underset{\approx}{B} \underset{\sim}{U} - \underset{\approx}{M}^{-1} \underset{\approx}{K} \underset{\sim}{s}_1) = \underset{\sim}{\Lambda}_2^T (\underset{\approx}{E}^T \underset{\approx}{B} \underset{\sim}{U} - \underset{\approx}{\Omega} \underset{\sim}{\delta}_1) \qquad \text{(H.22b)}$$

Introducing the transformations of (H.19) into (H.22), leads to the identifications

$$\underline{\lambda}_1 = \underline{\underline{M}}\ \underline{\underline{E}}\ \underline{\Lambda}_1$$
$$\underline{\lambda}_2 = \underline{\underline{M}}\ \underline{\underline{E}}\ \underline{\Lambda}_2 \quad * \qquad \text{(H.23)}$$

With (H.18) - (H.23) the relationships necessary for determining consistent weighting matrices and the transformations for mapping co-state variables between physical and modal space representations of the equations of motion, have been determined.

* This result follows immediately on introducing the expression $\underline{\underline{\Omega}} = \underline{\underline{E}}^T\ \underline{\underline{K}}\ \underline{\underline{E}}$ into (H.22b).

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OPTIMAL LARGE ANGLE SPACECRAFT ROTATIONAL MANEUVERS

by

James Daniel Turner

(ABSTRACT)

Pontryagin's principle is applied to several significant problems associated with optimal large angle spacecraft rotational maneuvers. Both rigid and flexible body dynamical models for these vehicles are considered. Three relaxation/analytic continuation methods are developed for iteratively solving the two-point-boundary-value problem which results in the treatment of these problems. The solutions obtained are required to rigorously satisfy the necessary conditions derived from Pontryagin's principle. These methods include: (1) boundary condition relaxation processes; (2) differential equation relaxation processes; and (3) hybrid relaxation processes, combining (1) and (2) above. In the literature these relaxation processes are closely related to a number of methods for solving nonlinear equations, known as Davidenko's method, imbedding, and homotopy chain methods.

For rigid vehicles a general nonsingular optimal maneuver formulation is obtained, treating all kinematic and dynamical nonlinearities, for general orientation and angular velocity boundary conditions.

For flexible vehicles restricted to single axis maneuvers and anti-symmetric elastic deflection modes, a general optimal

maneuver formulation is obtained; treating all kinematic, dynamical, and first order structural nonlinearities.

In the case of general motion for a flexible vehicle a general formulation is provided, though a solution is not obtained; due to a previously unidentified and as yet unresolved computational difficulty associated symmetry in the dynamical model for the spacecraft.