A STUDY OF ANISOTROPIC AND VISCOELASTIC

DUCTILE FRACTURE

by

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## LIST OF ILLUSTRATIONS

FIGURE PAGE

## NOTATION

| | |
|---|---|
| $A_1, B_1$ | constants given by Equation (3.3.36) |
| $A(\vec{x},t)$, $B(\vec{x},t)$ | given vector valued functions |
| $B_{ij\kappa\ell}$, $C_{ij\kappa\ell}$ | elastic constants |
| $C^e$ | solution on boundary for an elastic problem |
| $E$ | Young's modulus for isotropic material |
| $E_i$ | Young's modulus along $x_i$ - axis |
| $G$, $\mu$ | elastic shearing modulus |
| $G_{ij}$ | shearing modulus in $x_i$ - $x_j$ plane |
| $G_1$, $G_2$ | relaxation functions in shear and dilatation |
| $G_{ij\kappa\ell}$ | generalized relaxation modulus |
| Im | imaginary part |
| $H(t)$ | Heaviside step function |
| $J_1$, $J_2$ | creep function in shear and dilatation |
| $J_{ij\kappa\ell}$ | generalized creep compliance |
| $K$, $\kappa$ | bulk modulus |
| $K_1(t)$ | parameter given by Equation (4.11) |
| $P$, $P'$, $Q$, $Q'$ | differential or integral operators |
| $P_{ij\kappa\ell}$, $Q_{ij\kappa\ell}$ | generalized differential or integral operator |
| Re | real part |
| $S$ | surface boundary |
| $S_1$, $S_2$ | sub-regions of surface boundary |
| $T$ | applied tensile stress |

| | |
|---|---|
| $V$ | velocity of crack |
| $Y$ | yield stress |
| $a$ | $s + \ell$ |
| $a_{ij}$ | coefficients of deformation |
| $e_{ij}$ | strain deviator tensor |
| $f_i$ | body forces per unit volume |
| $i$ | complex number $\sqrt{-1}$ |
| $k(\mu,\kappa)$, $k(C_{ij\kappa\ell})$, $k(B_{ij\kappa\ell})$ | parameter containing only elastic constants |
| $\ell$ | crack half-length |
| $m_i$ | roots of quartic equation |
| $p$ | Laplace transform variable |
| $r$ | radial measure |
| $s_{ij}$ | stress deviator tensor |
| $s$ | plastic zone length |
| $t$ | time |
| $u$, $v$ | displacements in $x$, $y$ directions |
| $u_i$ | displacement in $x_i$ direction |
| $w$, $w_1$, $w_2$ | complex transformation functions for $z$, $z_1$, $z_2$ |
| $x$, $y$ | cartesian coordinates |
| $z$ | cartesian coordinate or complex variable $x+iy$ or $\eta+iy$ |
| $z_i$ | complex variable $x+i\beta_i y$ or $\eta+i\beta_i y$ |
| $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ | parameters reflecting anisotropy and velocity |

| | |
|---|---|
| $\delta_{ij}$ | Kronecker delta |
| $\varepsilon_{ij}$ | infinitesimal strain tensor |
| $\zeta$ | transformed complex variable |
| $\eta$ | viscosity coefficient or variable of substitution, $\eta = x - Vt$ |
| $\theta$ | angular coordinate |
| $\theta_2$ | $\cos^{-1} \ell/a$ |
| $\nu$, $\nu_{ij}$ | Poisson's ratio |
| $\rho$ | density |
| $\sigma_{ij}$ | stress tensor |
| $\sigma_2$ | $e^{i\theta_2}$ |
| $\tau$ | dummy variable |
| $\phi(z_1)$, $\psi(z_2)$ | analytic functions in x,y plane |
| $\phi'(z_1)$, $\psi'(z_2)$ | derivative with respect to $z_1$, $z_2$ |
| $\overline{\phi(z_1)}$, $\overline{\psi(z_2)}$ | conjugate of analytic functions |
| $\chi$ | stress function |
| $\chi_i$ | body force per unit mass |
| $\sum_{o}^{s}$ | summation over index from 0 to s |
| $\Phi(\zeta)$, $\Psi(\zeta)$ | analytic functions in conformal plane |
| $\mathcal{L}$ | Laplace transform |
| $\mathcal{L}^{-1}$ | inverse Laplace transform |
| $\sigma_{ij},_j$ | $\frac{\partial \sigma_{ij}}{\partial x_j}$ - summation from j= 1,.....N |

| | |
|---|---|
| [ 1968 ] | preceeded by a name - a reference by the author published in 1968 |

# 1. INTRODUCTION

## 1.1 Viscoelasticity

Viscoelastic stress analysis, as usually performed, is based upon Alfrey's correspondence principle [1948]. Essentially, in this technique, Laplace transforms are taken of the governing viscoelastic equations in order to reduce the viscoelastic problem to an associated elastic problem. If the solution to the associated elastic problem is known and can be found, then the viscoelastic solution may be found by replacing elastic constants by viscoelastic operators and inverting the transform. This approach can only be applied directly when the body shape and boundary do not vary and the boundary conditions do not change in type with time, for otherwise the transform of a prescribed quantity, which is an integral from zero to infinity in time, cannot be evaluated to determine the corresponding quantity for the associated elastic problem. Application to ablating bodies with time dependent boundaries, and crack propagation problems in which boundaries have different values prescribed varying with time call for a more general approach which has been developed for particular groups of problems. Radok [1957] has developed a technique called functional equations and Graham [1968] has developed a more general correspondence principal to deal with a certain group of the above mentioned problems.

Similar relations, as those between an isotropic viscoelastic and isotropic elastic solution, exist also in the case of anisotropic viscoelastic bodies and anisotropic elastic bodies. Some significant contributions have been reported by Biot [1954] and Hilton [1961] in the derivation of the stress-strain relations for viscoelastic materials. Some practical problems also have been solved by Lee [1955, 1956, 1957, 1960, 1964] with many of his co-workers, Radok [1957] as well as Bienick, Spillers, and Freudental [1962] and by Moghe and Hsiao [1965].

## 1.2 Fracture

When the stress in the region of the tip of a crack is high enough the material ahead of the crack will fail or fracture in a manner determined by both the stress field and the nature of the material.

If one considers an elastic body containing cracks which are of zero opening in the unloaded state, and if the elastic body is subjected to a system of loads either on the crack surfaces or at other points of the body, then classical fracture mechanics predicts a stress singularity at the crack tip of the usual inverse square root type. This singularity from a physical point of view is of course not realistic, and some mechanism must occur in the material to eliminate it. One possible material mechanism is the development of a plastic zone near the crack tip.

Elastic-plastic problems of isotropic materials with cracks are very difficult to treat analytically, and the complete solution of the problem has been found so far only for anti-plane (longitudinal) shearing stress. Dugdale [1960] proposed a solution for the problem of tension perpendicular to a crack in an infinite plate, using the assumption that the plastic regions are confined to lie along the crack line and the tensile stress in this region has a constant value corresponding to the yield stress of the material. Agreement between his proposed solution and experimental results for mild steel was quite good.

Bilby [1963] discussed the same problem for plane and antiplane shearing stress using the theory of dislocations and the same assumption about plastic yield regions and obtained similar results. Using Dugdale's results, Goodier and Field [1963] were able to evaluate the plastic energy dissipation by the methods of elastic perfectly-plastic continuum mechanics.

Yoffe [1951] has calculated the stress field about a straight crack moving through an elastic medium. The stresses were shown to depend on the velocity and reduced to Inglis' [1913] solution when the velocity was zero. She was also able to find a limiting speed before branching of the crack would occur.

Craggs [1960] has solved the problem in which a semi-infinite crack in an infinite medium is extended by finite forces. The conclusion is drawn that the force required to maintain a steady rate of extension of the crack decreases as the rate increases. It is also observed that various criteria which may be assumed for crack division lead to limiting velocities of propagation of a single crack.

Cotterell [1964] showed that the dynamic elastic-wave equations can be integrated simply in the neighborhood of the tip of a moving crack. A simple approximation to the exact solution is given. From examination of the dynamic stress field, arguments are presented to explain the increase in fracture toughness and fracture surface roughness with crack velocity.

Baker [1962] has solved the transient problem in which a semi-infinite crack suddenly appears and grows at constant velocity in a stretched elastic body. The problem is solved by transform methods including the Weiner-Hopf and Cagniard techniques.

Kanninen [1968] has solved a steady state dynamic problem for a Dugdale crack propagating at a constant speed using the Sneddon-Radok [1956] formulation of the dynamic plane elasticity equations. The dynamic stress distribution and the Tresca yield condition are used to determine the limiting ductile crack propagation speed. It is shown that the limiting brittle crack speed of Yoffe and Craggs are satisfied as well.

Goodier and Field [1963] examined a slowly extending internal Dugdale slit crack under tension and a rapidly extending semi-infinite Dugdale crack under traveling wedge pressures.

## 1.3 Viscoelastic Fracture

Graham [1966] has solved for the distribution of stress and displacement in an infinite linear viscoelastic body which contains an extending plane crack. Both the symmetrically loaded two dimensional and the axisymmetrically loaded three dimensional cases were solved. The solution was obtained utilizing an extended correspondence principle which he had previously developed. This extended correspondence principle will be discussed in detail in Chapter 2.

Wnuk and Knauss [1968] have solved for a penny-shaped crack in a viscoelastic body assuming the crack is surrounded by a Dugdale type thin plastic zone. They have also assumed that the yield point is a function of time, as suggested by Crochet [1966]. They also used Graham's extended correspondence principle.

Williams [1965] has looked at viscoelastic fracture using an energy formulation. Assuming the viscoelastic criterion for fracture has the same character for all geometries as for the spherical cavity which he studied, the appropriate extension of the Griffith initiation criterion is believed established.

## 1.4 Anisotropic Fracture

The majority of past theoretical and experimental effort was directed toward isotropic materials. Since the formulation of this fracture theory is based on continuum mechanics where material differences are not considered, it is expected that fracture mechanics should be equally applicable to anisotropic materials.

Sih, Paris, and Irwin [1965] have derived the general equations for crack tip stress fields in anisotropic bodies making use of a complex variable approach. The stress-intensity factors are defined and are evaluated directly from stress functions. Some individual boundary value problem solutions are given in closed form and discussed with reference to their companion solutions for isotropic bodies. It was found that an elastic stress singularity of the order $r^{-1/2}$ is always present at the crack tip in a body with anisotropy of the rectilinear type.

Ang and Williams [1961] using a formulation in integral equations presented a solution for the combined extension and bending for the case of an infinite orthotropic flat plate containing a finite crack. Qualitatively no major difference in behavior due to orthotropy was found, although certain quantitative features were noted, mainly as a function of the characteristic rigidity ratio $(Ex/Ey)^{1/2}$. The $r^{-1/2}$ singularity was again present as in the isotropic theory.

Chapkis and Williams [1958] have investigated stress singularities for a sharp-notched polarly orthotropic plate. Characteristic equations for all the usual boundary conditions are derived, and the qualitative effect of orthotropy upon the character of the stress is determined for the particular case of a cracked plate under free-free extension.

Gotoh [1967] has discussed problems of bonded dissimilar anisotropic plates with cracks along the bond.

Gonzalez [1968] has found the stresses and displacements associated with a Dugdale crack in an orthotropic plate, and has found a limit on the degree of orthotropy for yield zones to progress along the crack line.

## 1.5 Anisotropic Viscoelastic Fracture

Anisotropic materials play an important role in modern technology. Missile and aircraft designers, geophysicists and in general people in the natural sciences all must deal with a variety of anisotropic problems. It is now possible to produce artificial anisotropy as in corregated plates and in reinforcing of certain structures. The importance of anisotropy is clearly seen.

In the solution of many engineering problems the assumption of isotropy is usually made because it greatly simplifies the problem. Thus solutions for isotropic bodies have been thoroughly studied. In today's sophisticated world we must take into account anisotropy to get a truer picture of material behavior.

The increasing use of polymers and plastics call for extension of the theory of stress analysis to bodies which exhibit more complex stress strain relations, in particular those which involve time derivatives of both the stress and strain. The group of this type of materials which provides the simplest analysis is the linearly viscoelastic materials which are governed by linear relations between stress and strain and their time derivatives.

Recently considerable effort has been expended to investigating the brittle fracture of polymers and the isotropic ductile fracture of metals. Very little research, however, has been conducted on the ductile fracture of polymers and anisotropic effects on the ductile fracture of metals.

A natural extension of the previously cited work is to investigate fracture for anisotropic and viscoelastic materials. Concerning ourselves mainly with ductile fracture, where a Dugdale type plastic zone is assumed, we shall find static and dynamic solutions for anisotropic materials and static anisotropic solutions for viscoelastic materials.

# 2. VISCOELASTIC THEORY

## 2.1 Governing Equations

For quasi-static problems in which inertia forces and thermal effects are negligible, the constitutive equations for linear isotropic viscoelastic materials are given by Lee [1955, 1956] in operator form as,

$$Ps_{ij} = Qe_{ij} \tag{2.1.1}$$

$$P'\sigma_{ii} = Q'\varepsilon_{ii} \tag{2.1.2}$$

where P, Q, P', and Q' may be differential or integral operators.

Defining P, Q, P', and Q' as differential operators they are of the form,

$$P = \sum_{o}^{m} a_n D^n \tag{2.1.3}$$

$$Q = \sum_{o}^{r} b_n D^n \tag{2.1.4}$$

$$P' = \sum_{o}^{s} c_n D^n \tag{2.1.5}$$

$$Q' = \sum_{o}^{t} d_n D^n \tag{2.1.6}$$

and

$$D^n = \frac{\partial^n}{\partial t^n} \tag{2.1.7}$$

The coefficients $a_n$, $b_n$, $c_n$, $d_n$ and the numbers m, r, s, t are in general different. $\sigma_{ij}$ and $\varepsilon_{ij}$ are the stress and infinitesimal strain tensors respectively. $s_{ij}$ and $e_{ij}$ are respectively their deviators defined in the usual way,

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{\kappa\kappa}\,\delta_{ij} \; *$$

$$e_{ij} = \sigma_{ij} - \frac{1}{3}\varepsilon_{\kappa\kappa}\,\delta_{ij} \qquad (2.1.8)$$

As an example of the use of (2.1.1) and (2.1.2) consider a material which behaves as a delayed elastic Kelvin material in shear and as a Maxwellian fluid in dilitation,

$$\begin{aligned} P &= 1 \\ Q &= G + \eta\frac{\partial}{\partial t} \\ P' &= \eta/E\frac{\partial}{\partial t} + 1 \\ Q' &= \eta\frac{\partial}{\partial t} \end{aligned} \qquad (2.1.9)$$

where G is the elastic shearing modulus, E is the modulus of elasticity, and $\eta$ in a viscosity coefficient.

For a purely elastic material we would have,

$$\begin{aligned} P &= 1 \\ Q &= 2G \\ P' &= 1 \\ Q' &= 3K \end{aligned} \qquad (2.1.10)$$

where K is the bulk modulus.

Defining P, Q, P', Q' to be integral operators (2.1.1) and (2.1.2) take the form,

---

*

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$s_{ij}(\vec{x},t) = G_1(\vec{x},t)e_{ij}(\vec{x},o) + \int_o^t G_1(t-\tau)\frac{\partial}{\partial\tau} e_{ij}(\vec{x},\tau)d\tau$$
$$\sigma_{\kappa\kappa}(\vec{x},t) = G_2(\vec{x},t)\varepsilon_{\kappa\kappa}(\vec{x},o) + \int_o^t G_2(t-\tau)\frac{\partial}{\partial\tau} \varepsilon_{\kappa\kappa}(\vec{x},\tau)d\tau \tag{2.1.11}$$

where $G_1$ and $G_2$ are the relaxation functions in shear and dilatation respectively. Equations (2.1.11) may be inverted and written as,

$$e_{ij}(\vec{x},t) = J_1(\vec{x},t)s_{ij}(\vec{x},o) + \int_o^t J_1(t-\tau)\frac{\partial}{\partial\tau} s_{ij}(\vec{x},\tau)d\tau$$
$$\varepsilon_{\kappa\kappa}(\vec{x},t) = J_2(\vec{x},t)\sigma_{\kappa\kappa}(\vec{x},o) + \int_o^t J_2(t-\tau)\frac{\partial}{\partial\tau} \sigma_{\kappa\kappa}(\vec{x},\tau)d\tau \tag{2.1.12}$$

where $J_1$ and $J_2$ are the creep functions in shear and isotropic compression respectively.

We now consider the solutions to problems of bodies subjected to prescribed body forces $f_i(\vec{x},t)$ per unit volume, tractions $T_i(\vec{x},t)$ on the surface S or displacements on the surface $u_i(\vec{x},t)$, or any combination of these (see Fig. 2.1).

A complete solution consists of the stress $\sigma_{ij}(\vec{x},t)$ and displacements $u_i(\vec{x},t)$ specified throughout the body V for all times. This solution must satisfy the constitutive relations (2.1.1,2), the equilibrium equations,

$$\sigma_{ij},_j = f_i(\vec{x},t) \tag{2.1.13}$$

and the strain displacement relations,

$$\varepsilon_{ij} = \frac{1}{2}(u_i,_j + u_j,_i) \tag{2.1.14}$$

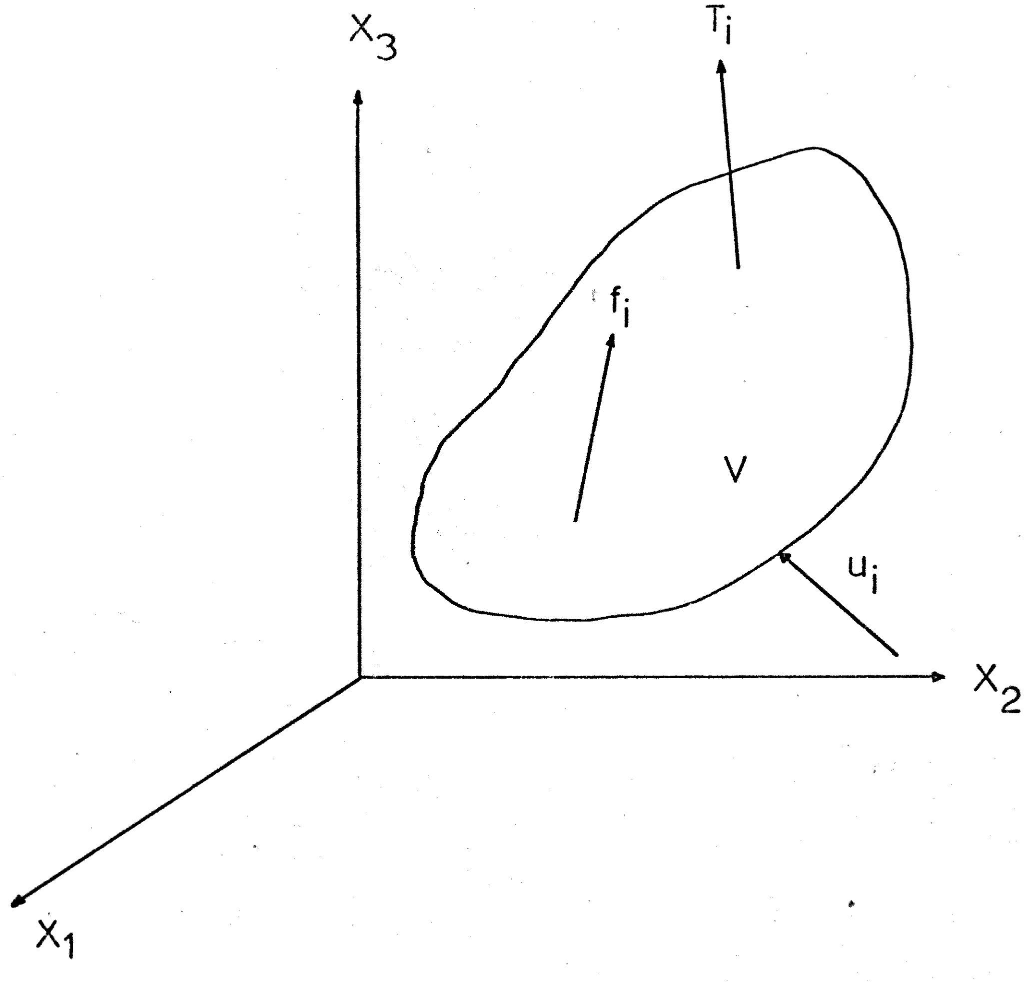



Figure 2.1
GENERAL BOUNDARY VALUE PROBLEM OF QUASI-STATIC STRESS ANALYSIS

The displacement must be compatible with the prescribed surface displacements, and the stresses with the prescribed surface tractions according to,

$$T_i = \sigma_{ij} n_j \text{ on } S, \tag{2.1.15}$$

where $n_i$ is the outward normal.

## 2.2 General Correspondence

The classical method of solving viscoelastic boundary value problems is to apply an integral transform (with respect to time) to the time-dependent field equations and boundary conditions. The transformed equations then have the same form as the field equations of elasticity theory and if a solution to these, which is compatible with the transformed boundary conditions, can be found, then the solution to the original problem is reduced to transform inversion. This method of solving viscoelastic stress analysis problems is referred to as Alfrey's [1948] correspondence principle.

Applying the Laplace transform to the field equations (2.1.1,2) or (2.1.11) and (2.1.13,14) and boundary conditions (2.1.15) we have,

$$\bar{\sigma}_{ij},_j(\vec{x},p) = \bar{f}_i(p) \tag{2.2.1}$$

$$2\ \bar{\varepsilon}_{ij}(\vec{x},p) = \bar{u}_i,_j(\vec{x},p) + \bar{u}_j,_i(\vec{x},p) \tag{2.2.2}$$

and

$$\bar{s}_{ij}(\vec{x},p) = p\bar{G}_1(p)\bar{e}_{ij}(\vec{x},p) \tag{2.2.3}$$

$$\bar{\sigma}_{\kappa\kappa}(\vec{x},p) = p\bar{G}_2(p)\bar{\varepsilon}_{\kappa\kappa}(\vec{x},p) \tag{2.2.4}$$

or

$$\bar{P}(p)\bar{s}_{ij} = \bar{Q}(p)\bar{e}_{ij} \tag{2.2.3a}$$

$$\bar{P}'(p)\bar{\sigma}_{\kappa\kappa} = \bar{Q}'(p)\bar{\varepsilon}_{\kappa\kappa} \tag{2.2.4a}$$

and

$$\bar{T}_i(\vec{x},p) = \bar{\sigma}_{ij}(\vec{x},p)n_j \tag{2.2.5}$$

(2.2.1-5) represent a stress analysis problem for an elastic body of the same shape as the viscoelastic body with elastic constants a function of the parameter p. When the stresses $\bar{\sigma}_{ij}(\vec{x},p)$ have been determined, inversion of the Laplace transform gives the desired stresses $\sigma_{ij}(\vec{x},t)$ for the viscoelastic problem, and similarly for the other variables. Therefore solutions in the theory of elasticity may be used to solve viscoelastic problems.

Although the body shape for the associated elastic problem is the same, the prescribed distribution of surface tractions, displacements, and body forces may be quite different since the Laplace transform $\bar{\phi}(\vec{x},p)$ will in general have an entirely different space distribution than $\phi(\vec{x},t)$. Thus the associated elastic problem is not in general simply related to the viscoelastic problem. There is however a common type of problem, we shall call proportional loading, in which the space and time dependence of prescribed quantities separates out with a common time dependence. Thus we have,

$$T_i(\vec{x},t) = T_i'(\vec{x})\ f_1(t)$$

$$f_i(\vec{x},t) = f_i'(\vec{x})\ f_1(t) \tag{2.2.6}$$

$$u_i(\vec{x},t) = u_i'(\vec{x})\ f_1(t)$$

In this case the Laplace transform simply changes $f_1(t)$ to $\bar{f}_1(p)$ leaving the space dependence unchanged. Since $\bar{f}_1(p)$ then merely appears as a multiplying factor, the associated elastic problem contains the same spatial distribution of prescribed quantities as the viscoelastic problem. This represents a considerable simplification since the associated elastic problem can be solved when the values $T_i'$, $f_i'$, $u_i'$, are prescribed and are independent of p. Groups of proportional loading and prescribed displacements each with different time dependence can be considered separately and superposed since we are considering only a linear theory.

Lee [1955] has presented a simple example of the above correspondence principle. Consider a point force P(t) acting normally at a fixed point on the surface of a semi-infinite viscoelastic body (see Fig. 2.2).

The point of action is the origin or cylindrical coordinates $(r,\theta,z)$ with the body occupying the space $z \geq o$. Since only one point force is a case of proportional loading, the associated elastic problem is that of a semi-infinite elastic body with a normal point load of magnitude of $\bar{P}(p)$.

The solution to this elastic problem is given by Timoshenko with the $\sigma_r$ stress given by,

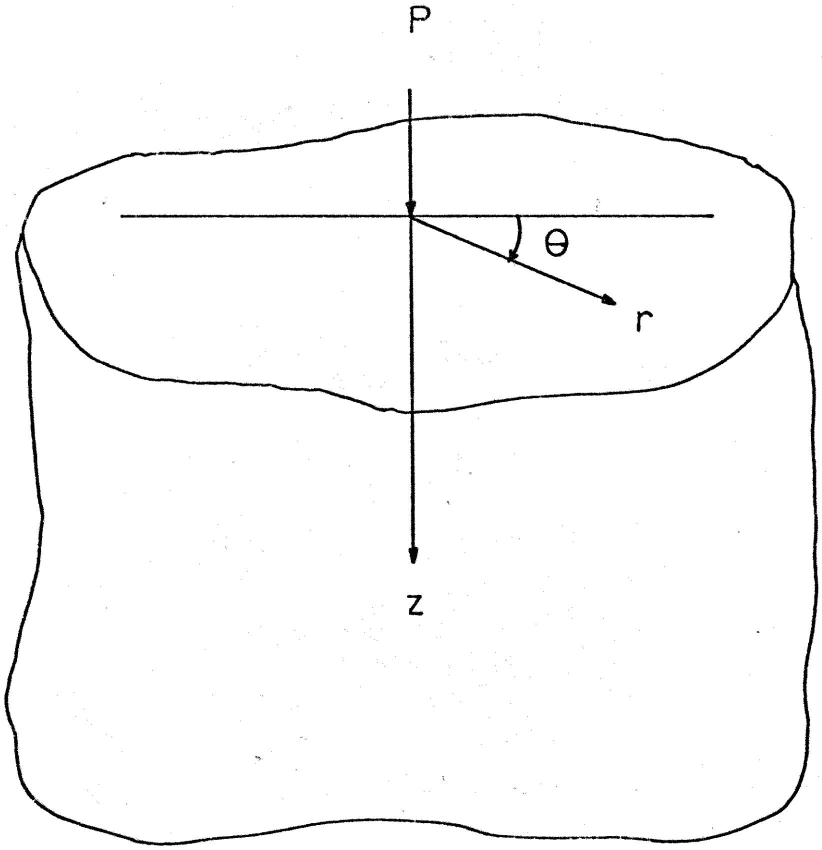



Figure 2.2

VISCOELASTIC BOUSSINESQ PROBLEM

$$\sigma_r = \frac{P}{2\pi}\left[(1-2\nu)\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2 + z^2)^{-1/2}\right] -3r^2 z(r^2 + z^2)^{-5/2}\right] \quad (2.2.7)$$

or

$$\sigma_r = \frac{P}{2\pi}\left[\frac{3G}{3K+G}\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2 + z^2)^{-1/2}\right] -3r^2 z(r^2 + z^2)^{-5/2}\right] \quad (2.2.8)$$

From (2.1.10) we see,

$$\begin{aligned} \overline{Q}(p)/\overline{P}(p) &= 2G \\ \overline{Q}'(p)/\overline{P}'(p) &= 3K \end{aligned} \quad (2.2.9)$$

Taking the Laplace transform of (2.2.7) and using (2.2.9) we have (assuming zero initial conditions),

$$\sigma_r = \frac{\overline{P}}{2\pi}\left[\frac{3Q(p)/P(p)}{[2\overline{Q}'(p)/\overline{P}'(p)+\overline{Q}(p)/\overline{P}(p)]}\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2 + z^2)^{-1/2}\right] -3r^2 z(r^2 + z^2)^{-5/2}\right] \quad (2.2.10)$$

For a prescribed material, with known operators P, Q, P', Q' the inverse of (2.2.10) gives the radial stress in the viscoelastic problem. Assuming Kelvin behavior in shear and elastic behavior in dilatation equations (2.1.1,2) become,

$$\begin{aligned} s_{ij} &= (G + \eta \frac{\partial}{\partial t})\, e_{ij} \\ \sigma_{\kappa\kappa} &= E\, \varepsilon_{\kappa\kappa} \end{aligned} \quad (2.2.11)$$

Also assuming $P = P_o H(t)$* and using (2.2.11) and (2.2.10) we have,

$$\bar{\sigma}_r = \frac{P_o}{2\pi p}\left[\frac{3(\eta p+G)}{2E+\eta p+G}\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2+z^2)^{-1/2}\right] - 3r^2 z(r^2+z^2)^{-5/2}\right] \quad (2.2.12)$$

Inverting we have for the radial stress

$$\sigma_r = \frac{P_o}{2\pi}\left[\frac{3}{2E+G}\left[G + 2E\exp\left[-(2E+G)t/\eta\right]\right]\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2+z^2)^{-1/2}\right]\right.$$

$$\left. -3r^2 z(r^2+z^2)^{-5/2}\right] \qquad t > o \quad (2.2.13)$$

## 2.3 Extended Correspondence

The correspondence principle is clearly applicable whenever the type of boundary condition prescribed is the same at all points of the boundary. For mixed boundary value problems (i.e. problems for which different field quantities are prescribed over separate parts of the boundary) the method is still applicable provided the regions over which different types of boundary conditions are given do not vary with time. There remain those viscoelastic mixed boundary value problems where the regions, over which different types of boundary conditions are given, do vary with time. Particular examples are indentation and crack propagation problems. For problems of this type there will be points of the boundary at which only partial histories of some field quantities will be prescribed. When this is the case the transforms of these quantities are not directly obtainable and the classical correspondence principle is not applicable.

---

* $H(t) = \begin{cases} 1 & t > o \\ 0 & t < o \end{cases}$

An extension to the classical correspondence principle is clearly needed in order to take case of moving boundaries. Radok [1957] used an approach called functional equations and Graham [1968] developed an extended correspondence principle to deal with mixed boundary value problems.

### 2.3.1 Functional Equations

Radok [1957] has suggested that a broader range of problems can be treated, which fall outside the scope of the transform method, by taking a one-parameter family of solutions of the elastic problem in the time parameter, t, with the same boundary conditions as the viscoelastic problem, and by replacing the elastic constants by appropriate viscoelastic operators in the expressions for stress components. These may then comprise tractable mathematical equations for evaluating the stress components for the viscoelastic body.

For the purpose of comparison we will look at the same problem which was solved by Lee. Recalling (2.2.10) we have,

$$\sigma_r = \frac{P(t)}{2\pi}\left[\frac{3Q/P}{[2Q'/P'+Q/P]}\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2+z^2)^{-1/2}\right] - 3r^2z(r^2+z^2)^{-5/2}\right] \tag{2.3.1}$$

For the same material considered by Lee,

$$\begin{aligned} P &= 1 \\ Q &= G + \eta\frac{\partial}{\partial t} \\ P' &= 1 \\ Q' &= E \end{aligned} \tag{2.3.2}$$

substituting (2.2.18) in (2.2.17) we have,

$$(2E + G + \eta \frac{\partial}{\partial t}) \sigma_r = \frac{1}{2\pi} \left[ 3(\eta \frac{\partial}{\partial t} + G) P(t) \left[\frac{1}{r^2} - \frac{z}{r^2} (r^2 + z^2)^{-1/2}\right] - (2E + G + \eta \frac{\partial}{\partial t}) P(t) 3r^2 z(r^2 + z^2)^{-5/2} \right] \quad (2.3.3)$$

Now for $P(t) = P_o H(t)$ we have a differential equation for $\sigma_r$. This is a first order equation and therefore one initial condition for $\sigma_r$ is needed. There is a discontinuity at the origin due to $P(t) = P_o H(t)$. We must integrate across this discontinuity to get the initial condition for $\sigma_r$. Doing this we have,

$$\sigma_r(r,z,o) = \frac{P_o}{2\pi} \left[ 3 \left[\frac{1}{r^2} - \frac{z}{r^2} (r^2 + z^2)^{-1/2}\right] - 3r^2 z(r^2 + z^2)^{-5/2} \right] \quad (2.3.4)$$

Solving (2.3.3) using (2.3.4) we have,

$$\sigma_r = \frac{P_o}{2\pi} \left[ \frac{3}{2E+G} \left[G + 2E \exp\left[-(2E + G)t/\eta\right]\right] \left[\frac{1}{r^2} - \frac{z}{r^2} (r^2 + z^2)^{-1/2}\right] - 3r^2 z(r^2 + z^2)^{-5/2} \right] \quad (2.3.5)$$

which is the same result arrived at by Lee.

This procedure is open to some question since the manipulation of the elastic constants in the determination of the elastic solution could involve procedures which are not valid for operators, or which lead to the introduction of new functions in the operator case.

### 2.3.2 Graham's Method

Graham [1968] has derived conditions, in terms of a one parameter family of static elastic solutions meeting mixed boundary values, which are sufficient to ensure that the viscoelastic problem determined by the mixed boundary values may be reduced to one to which the correspondence principle may be applied.

Denoting $u_n$ and $u_s$ ($\sigma_n$ and $\sigma_s$) the vector components of the displacement vector (traction vector) normal and tangential to the surface S respectively. Thus $u_n$, $u_s$, $\sigma_n$, $\sigma_s$ are vector valued functions of both $\vec{x}$ and t which are defined for all $(\vec{x},t)$ on S. Consider a matrix which denotes all possible boundary conditions as follows,

$$\begin{vmatrix} \sigma_s & \sigma_s & \sigma_n & \sigma_n & u_s & u_s & u_n & u_n \\ u_n & \sigma_n & u_s & \sigma_s & \sigma_n & u_n & \sigma_s & u_s \\ \sigma_n & u_n & \sigma_s & u_s & u_n & \sigma_n & u_s & \sigma_s \end{vmatrix}$$

Now suppose that a, b, c, are the elements of any column of the matrix. Boundary conditions for a viscoelastic problem might take the form,

$$\begin{aligned} a(\vec{x},t) &= A(\vec{x},t) & \quad \vec{x} \text{ on } S \\ b(\vec{x},t) &= B(\vec{x},t) & \quad \vec{x} \text{ on } S \end{aligned} \tag{2.3.6}$$

where A and B are vector valued functions which are given for all nonnegative times at each point $\vec{x}$ on S. Therefore if a is the normal stress $\sigma_n$, the value of the normal stress at the boundary would be A.

Applying the Laplace transform to (2.3.6) we have,

$$\bar{a}(\vec{x},p) = \bar{A}(\vec{x},p) \qquad \vec{x} \text{ on } S$$
$$\bar{b}(\vec{x},p) = \bar{B}(\vec{x},p) \qquad \vec{x} \text{ on } S \qquad (2.3.7)$$

Thus, if the boundary conditions are of the type (2.3.6) then the viscoelastic problem may be solved by the usual correspondence principle.

Consider now the boundary value problem governed by the boundary conditions,

$$a(\vec{x},t) = A(\vec{x},t) \qquad \vec{x} \text{ on } S \qquad \text{a)}$$
$$b(\vec{x},t) = B(\vec{x},t) \qquad \vec{x} \text{ on } S_1(t) \qquad \text{b)} \qquad (2.3.8)$$
$$c(\vec{x},t) = 0 \qquad \vec{x} \text{ on } S_2(t) \qquad \text{c)}$$

where A and B are vector valued functions. $S_1$ and $S_2$ are subregions of S so that $S_1(t)$ and $S_2(t)$ comprise the entire region S. When $S_1$ and $S_2$ vary with time it is easy to see that there will be points of $\vec{x}$ of S for which neither $b(\vec{x},t)$ nor $c(\vec{x},t)$ are known for all nonnegative times. In this case the correspondence principle is not applicable as a method of solution since the required Laplace transforms of the boundary conditions are not obtainable.

Consider, however, the one parameter family of static elastic boundary value problems with the strain-displacement relations, the equilibrium equations and constitutive relations given by,

$$s_{ij}(\vec{x},t) = 2\mu\, e_{ij}(\vec{x},t) \quad *$$
$$\sigma_{\kappa\kappa}(\vec{x},t) = 3\kappa\, \varepsilon_{\kappa\kappa}(\vec{x},t) \qquad (2.1.10a)$$

* For simplicity later, G & K have been replaced by μ & κ respectively.

Suppose that the solutions to these problems, which we denote by $[u_i^e(\vec{x},t),\ \varepsilon_{ij}^e(\vec{x},t),\ \sigma_{ij}^e(\vec{x},t)]$ are such that,

$$b^e(\vec{x},t) = B^e(\vec{x},t) \qquad \vec{x} \text{ on } S \qquad \text{a)}$$

$$c^e(\vec{x},t) = k(\mu,\kappa)\, C^e(\vec{x},t) \quad \vec{x} \text{ on } S \qquad \text{b)} \qquad (2.3.9)$$

where $B^e$ and $C^e$ are independent of elastic constants and k is a function of these constants only, and the subscripts e refer to the elastic solution.

Since the boundary conditions (2.3.8) must be satisfied we have,

$$B^e(\vec{x},t) = B(\vec{x},t) \qquad \vec{x} \text{ on } S_1(t)$$

$$C^e(\vec{x},t) = o \qquad \vec{x} \text{ on } S_2(t) \qquad (2.3.10)$$

Now (2.3.10) together with (2.3.9) means that for the elastic solution we must have,

$$b^e(\vec{x},t) = B^e(\vec{x},t) \qquad \vec{x} \text{ on } S_2(t)$$

$$c^e(\vec{x},t) = k(\mu,\kappa)C^e(\vec{x},t) \qquad \vec{x} \text{ on } S_1(t) \qquad (2.3.11)$$

Consider now the viscoelastic boundary value problem governed by the field equations together with boundary conditions (2.3.8a) and boundary conditions,

$$b(\vec{x},t) = B^e(\vec{x},t) \qquad \vec{x} \text{ on } S \qquad (2.3.12)$$

Since both A and $B^e$ are given for all nonnegative time t at each point $\vec{x}$ of S we may apply the Laplace transform to A and $B^e$ and find,

$$\bar{a}(\vec{x},p) = \bar{A}(\vec{x},p) \qquad \vec{x} \text{ on } S$$
$$\bar{b}(\vec{x},p) = \bar{B}^e(\vec{x},p) \qquad \vec{x} \text{ on } S \tag{2.3.13}$$

Comparing (2.3.13) and (2.2.1-4) with (2.3.13), (2.2.3,4) and (2.1.10), we see that for the problem now under consideration,

$$[\bar{u}_i(\vec{x},p),\ \bar{\varepsilon}_{ij}(\vec{x},p),\ \bar{\sigma}_{ij}(\vec{x},p)] = [\bar{u}_i^{\,e}(\vec{x},p),\ \bar{\varepsilon}_{ij}^{\,e}(\vec{x},p),\ \bar{\sigma}_{ij}^{\,e}(\vec{x},p)];$$

$$\mu = \frac{p}{2}\,\bar{G}_1(p),\ \kappa = \frac{p}{3}\,\bar{G}_2(p) \tag{2.3.14}$$

Now taking the Laplace transform of (2.3.9b) we have,

$$\bar{c}^e(\vec{x},p) = k(\mu,\kappa)\ \bar{C}^e(\vec{x},p) \qquad \vec{x} \text{ on } S \tag{2.3.15}$$

which by (2.3.14) implies, that for the viscoelastic displacement-stress field meeting the boundary conditions (2.3.8a) and (2.3.12),

$$\bar{c}(\vec{x},p) = k\ \left(\frac{p}{2}\,\bar{G}_1(p),\ \frac{p}{3}\,\bar{G}_2(p)\right)\ \bar{C}^e(\vec{x},p) \qquad \vec{x} \text{ on } S \tag{2.3.16}$$

Inverting the above we have,

$$c(\vec{x},t) = K(t)C^e(\vec{x},o) + \int_o^t K(t-\tau)\ \frac{\partial}{\partial\tau}\,[C^e(\vec{x},\tau)]d\tau \qquad \vec{x} \text{ on } S \tag{2.3.17}$$

where,

$$K(t) = \mathcal{L}^{-1}\left[\frac{1}{p}\,k\ \left(\frac{p}{2}\,G_1(p),\ \frac{p}{3}\,G_2(p)\right)\right] \tag{2.3.18}$$

Now consider the case when $S_1(t)$ is monotonically increasing. Substituting (2.3.10b) into (2.3.17) it is found that in these circumstances $C(\vec{x},t)$

as defined through (2.3.17) satisfies,

$$c(\vec{x},t) = 0 \qquad \vec{x} \text{ on } S_2(t) \tag{2.3.19}$$

Thus provided $S_1(t)$ is monotonically increasing with time and the requirements of (2.3.11) are met we have by virtue of (2.3.10a) and (2.3.19) that the Laplace transform of the required viscoelastic field meeting the boundary conditions (2.3.8) is given by (2.3.14). When these conditions are met it is found that, for the viscoelastic problem determined by the conditions (2.3.8), $b(\vec{x},t)$, $\vec{x}$ on $S_2(t)$ has the same values as those computed from the static elastic analysis, while $c(\vec{x},t)$, $\vec{x}$ on $S_1(t)$ is generated by (2.3.17).

We will now give an example which is solved by Graham [1968] to illustrate the use of the extended correspondence principle. Suppose that, in terms of circular cylindrical coordinates $(\rho,\theta,z)$, the region R is the half space $z \geq 0$ with boundary S given by the plane $z = 0$. We consider the axisymmetric problem governed by the following boundary conditions:

$$\begin{aligned}
&\sigma_{\rho z}(\rho,0,t) = \sigma_{\theta z}(\rho,0,t) = 0, && \rho \geq 0 && \text{(a)} \\
&u_z(\rho,0,t) = D(t) - \beta(\rho), && 0 \leq \rho \leq a(t) && \text{(b)} \\
&\sigma_{zz}(\rho,0,t) = 0 \quad , && \rho > a(t) && \text{(c)}
\end{aligned} \tag{2.3.20}$$

where the field quantities are independent of $\theta$. These boundary conditions have the same form as (2.3.8), corresponding to the special circumstances in which the quantities a, b, c are chosen from the first column of the matrix of section 2.3.2. The region $S_1(t)$ is now that part of the plane $z = 0$ for which $0 \leq \rho \leq a(t)$, while $S_2(t)$ is the part of $z = 0$ for which $\rho > a(t)$. In accordence with the conditions of the extended correspondence principle given in the previous section we will restrict ourselves to the particular circumstances that $S_1(t)$ is monotone increasing with time so that $a(t)$ is a monotone increasing function of time. We assume that the function $\beta(\rho)$ appearing in (2.3.20b) is at least once continuously differentiable and that its derivative is never negative. The boundary conditions (2.3.20) then correspond to the physical circumstances of an axisymmetric punch of curved profile being pressed against the surface of a visoelastic half space in such a way that the radius of the circular area of contact is monotone increasing with time.

If we can find a solution to the system of equations (2.1.13), (2.1.14) and (2.1.10a) which satisfies the conditions (2.3.20) then the corresponding viscoelastic solution meeting (2.1.11,13,14) and (2.3.20) will be given through (2.3.14) provided the requirements of (2.3.9) are satisfied. In this instance the solution to the elastic problem governed by the field equations (2.1.13,14,10a) and the boundary conditions (2.3.20) is given by Sneddon [1965], therefore,

$$\sigma_{zz}(\rho,o,t) = \frac{2\mu(\mu + 3\kappa)}{(3\kappa + 4\mu)\rho} \frac{d}{d\rho} \int_{\rho}^{a(t)} \frac{y\, g(y,t)\, dy}{\left[y^2 - \rho^2\right]^{1/2}},$$

$$0 \leq \rho \leq a(t) \qquad (2.3.21)$$

where

$$g(y,t) = \frac{2}{\pi}\left[D(t) - y\int_0^y \frac{d\beta/d\rho}{\left[y^2-\rho^2\right]^{1/2}}\,d\rho\right] \tag{2.3.22}$$

Further we have that outside the area of contact the normal displacement of the surface of the half space is given by,

$$u_z(\rho,0,t) = \int_0^{a(t)} \frac{g(y,t)\,dy}{\left[\rho^2-y^2\right]^{1/2}}\,, \qquad \rho > a(t) \tag{2.3.23}$$

Since the elastic constants are absent from (2.3.23) and appear in a separate factor in (2.3.21) the requirements of (2.3.9) are satisfied and the extension of the correspondence principle obtained in the previous section is applicable. Thus we find that for the viscoelastic boundary value problem determined by (2.3.11,13,14) and (2.3.20) the normal displacement outside the contact area is given by (2.3.23), where g is given by (2.3.22) and since the punch is assumed to have a curved profile, D is related to a(t) through the equation,

$$D(t) = a(t)\int_0^{a(t)} \frac{d\beta/d\rho}{\left[a^2(t)-\rho^2\right]^{1/2}}\,d\rho\,, \tag{2.3.24}$$

provided only that a(t) remains a monotonic increasing function of time. By combining (2.3.17) and (2.3.18) with (2.3.20c) and (2.3.21) we find that,

$$\sigma_{zz}(\rho,0,t) = K(t)\mathcal{E}(\rho,0) + \int_0^t K(t-\tau)\,\frac{\partial}{\partial\tau}\,[\mathcal{E}(\rho,\tau)]d\tau \tag{2.3.25}$$

where

load is small compared with stress propagation speeds, inertia may be neglected and the problem treated as quasi-static. If, however the velocity of the load is no longer small compared with stress propagation speeds, inertia may no longer be neglected. The problem then becomes a dynamic problem. We will now consider the solution of viscoelastic problems of both the quasi-static and dynamic types.

### 2.4.1 Quasi-Static

For the quasi-static elastic moving load problem where the load is moving along the x-axis, the results are the same as for the load fixed, but with the coordinates carried along with the load. The moving load solution is thus the solution for the fixed load with x replaced by x-Vt (where V is the velocity of the load).

For a viscoelastic material with a moving load, a solution might be obtained by taking the Laplace transform of the moving surface tractions, but an alternative method is described which may be more convenient. Instead of taking the transform of the prescribed boundary conditions and body forces to determine the associated elastic problem, and then solving this, the prescribed viscoelastic problem could first be solved assuming it to be elastic (with the same boundary conditions as the viscoelastic problem). At each instant the quasi-static elastic problem is solved simply by keeping the current tractions held constant. This is of course not true for a viscoelastic problem in which the history of loading has an influence. Thus the solution of the elastic problem for the series of instantaneous distributions of surface tractions gives the sequence of stress values at each point and the transform of these values

$$\mathcal{C}(\rho,t) = \frac{H(a(t)-\rho)}{\rho} \frac{d}{d\rho} \int_{\rho}^{a(t)} \frac{y\, g(y,t)}{\left[y^2-\rho^2\right]^{1/2}}\, dy \tag{2.3.26}$$

and where K is given by

$$K(t) = \mathcal{L}^{-1}\left[\frac{\overline{G}_1(p)[(\overline{G}_1(p) + 2\overline{G}_2(p)]}{2[2\overline{G_1(p)} + \overline{G_2}(p)]}\right] \tag{2.3.27}$$

The total pressure acting on the punch is given by

$$P(t) = -2\pi \int_{0}^{a(t)} \rho\, \sigma_{zz}(\rho,t,0)d\rho \tag{2.3.28}$$

Substituting from (2.3.25) and (2.3.26) into (2.3.28) we find that,

$$P(t) = K(t)\mathcal{P}(0) + \int_{0}^{t} K(t-\tau) \frac{\partial}{\partial\tau} [\mathcal{P}(\tau)]d\tau \tag{2.3.29}$$

where,

$$\mathcal{P}(t) = 2\pi \int_{0}^{a(t)} g(y,t)dy \tag{2.3.30}$$

If we take $\beta(\rho) = \rho^2/2R$, where R is a constant, the solution given here reduces to that which was originally given by Lee and Radok [1960] for the problem of a viscoelastic half space against whose surface is pressed a rigid paraboloid of revolution. Their solution was derived at by the method of functional equations.

## 2.4 Moving Loads

Distributed loads moving over the surface of a deformable material are of frequent occurrence in engineering. When the velocity of the gives the required solution of the associated elastic problem. The elastic constants are functions of the transform parameter p associated with the transformed stress-strain relations, but since they are not time dependent this adds no complication to the determination of the transform of the stresses. On inverting to obtain the stresses for the viscoelastic problem, the fact that these elastic constants are functions of p modifies the inversion.

### 2.4.2 Dynamic

We will now consider problems where the inertia terms are no longer negligible. We assumed in the previous section for moving loads, if the velocity of the load was small compared to stress propagation speeds, then inertia effects might be neglected. For large velocities of moving loads (with respect to stress propagation speeds) inertial terms must be included. The governing differential equations are taken in the form,

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i},_{j} + u_{j},_{i}\right) \tag{2.4.1}$$

$$\sigma_{ij},_{j} + \rho X_{i} = \rho\, \ddot{u}_{i} \tag{2.4.2}$$

$$\sigma_{ij} = G_{ijk\ell}(\vec{x},t)\varepsilon_{k\ell}(x,o) + \int_{o}^{t} G_{ijk\ell}(\vec{x},t-\tau)\,\frac{\partial\varepsilon_{k\ell}}{\partial\tau}(\vec{x},\tau)d\tau \tag{2.4.3}$$

It is seen that time derivatives occur (2.4.2) and therefore the correspondence principle in the form stated for quasi-static problems is no longer applicable. Applying the Laplace transform to the above equations we have,

$$\bar{\varepsilon}_{ij} = \frac{1}{2}\left(\bar{u}_{i},_{j} + \bar{u}_{j},_{i}\right) \tag{2.4.4}$$

$$\bar{\sigma}_{ij},_{j} + \rho\bar{X}_{i} = \rho p^{2}\bar{u}_{i} \tag{2.4.5}$$

$$\bar{\sigma}_{ij} = p\bar{G}_{ij\kappa\ell}\bar{\varepsilon}_{\kappa\ell} \tag{2.4.6}$$

The governing equations for an elastic solid differ from the above only in that the constitutive equations are replaced by,

$$\sigma_{ij} = C_{ij\kappa\ell}\varepsilon_{\kappa\ell} \tag{2.4.7}$$

Equations (2.4.6) differs from equations (2.4.7) only in that $p\bar{G}_{ij\kappa\ell}$ are replaced by $C_{ij\kappa\ell}$. Therefore a correspondence principle for dynamic problems can be stated.

> "The solution of a dynamic problem for a viscoelastic material can be obtained from the solution of the corresponding problem for an elastic solid by applying the Laplace transform to the elastic solution, replacing the elastic constants by the corresponding viscoelastic modulli (or compliances) and finally inverting the transform." Brull [1953], Berry [1958]

## 2.5 Anisotropy in Materials

### 2.5.1 Elastic Materials

Anisotropic materials play an important role in modern technology. Missle and aircraft designers, specialists in mining problems, solid state physicists, manufacturers of certain parts and materials, and in general, people engaged in the material sciences, all must deal with a variety of anisotropic problems.

In order to obtain the relations between the components of stress and the components of deformation in an elastic body, it is necessary to choose some model which reflects the elastic properties. In what follows, we shall always assume that the components of strain are linear functions of the components of stress. Assuming the absence of stress components in the undeformed state, the most general form of the linear relation between stresses and strains can be written as,

$$\varepsilon_{ij} = B_{ij\kappa\ell}\sigma_{\kappa\ell} \qquad (i,j,\kappa,\ell = 1,2,3) \qquad (2.5.1)$$

where $B_{ij\kappa\ell}$ are constants. Equations (2.5.1) are called the generalized Hooke's law for an elastic solid. There appear to be 81 constants in (2.5.1), but for the case when $\sigma_{ij}$ and $\varepsilon_{ij}$ are both symmetric we can show that there are at most 36 independent elastic constants. We can therefore re-index (2.5.1) so that it will read,

$$\varepsilon_{\alpha} = a_{\alpha\beta}\sigma_{\beta} \qquad (\alpha,\beta = 1,2,3,4,5,6) \qquad (2.5.2)$$

where in (2.5.2) we have used,

$$\begin{array}{llll} \sigma_1 = \sigma_{11}, & \sigma_4 = \sigma_{23} = \sigma_{32}, & \varepsilon_1 = \varepsilon_{11}, & \varepsilon_4 = \varepsilon_{23} = \varepsilon_{32} \\ \sigma_2 = \sigma_{22}, & \sigma_5 = \sigma_{31} = \sigma_{13}, & \varepsilon_2 = \varepsilon_{22}, & \varepsilon_5 = \varepsilon_{31} = \varepsilon_{13} \\ \sigma_3 = \sigma_{33}, & \sigma_6 = \sigma_{12} = \sigma_{21}, & \varepsilon_3 = \varepsilon_{33}, & \varepsilon_6 = \varepsilon_{12} = \varepsilon_{21} \end{array} \qquad (2.5.3)$$

and

$$a_{11} = B_{1111}$$

$$a_{12} = B_{1122}$$

$$a_{14} = 2B_{1123} = 2B_{1132}$$

$$a_{44} = 2B_{2323} = 2B_{2332} = 2B_{3223} = 2B_{3232}$$

$$a_{45} = 2B_{2331} = 2B_{2313} = 2B_{3231} = 2B_{3213}$$

etc. (2.5.4)

and also we have used the symmetry of the stress and strain tensors, i.e.,

$$B_{ij\kappa\ell} = B_{ji\kappa\ell} = B_{ij\ell\kappa} \tag{2.5.5}$$

By solving the above equations for the stress components, we obtain an equivalent form for the equations of the generalized Hooke's law,

$$\sigma_{ij} = C_{ij\kappa\ell}\varepsilon_{\kappa\ell}$$

and $$\sigma_\alpha = A_{\alpha\beta}\varepsilon_\beta \tag{2.5.6}$$

According to Bekhterev, [1926], the constants $a_{\alpha\beta}$ are called the coefficients of deformation and the constants $A_{\alpha\beta}$ are called the moduli of elasticity. The moduli of elasticity can be uniquely expressed in terms of the coefficients of deformation when the determinants of the sixth order, which are composed of the constants $a_{\alpha\beta}$ and $A_{\alpha\beta}$ set down in order, differ from zero. If an elastic potential exists, the number of elastic constants in the most general case of anisotropy is reduced to 21. Such an elastic potential exists when the variation of the body under deformation occurs isothermally or adiabatically. We shall assume that the variations for deformation occur isothermally; that is the temperature of each element remains constant. Therefore with an elastic potential assumed to exist we find,

$$B_{ij\kappa\ell} = B_{\kappa\ell ij} \tag{2.5.7}$$

$$C_{ij\kappa\ell} = C_{\kappa\ell ij}$$

$$a_{\alpha\beta} = a_{\beta\alpha}$$

$$A_{\alpha\beta} = A_{\beta\alpha}$$

and therefore the constants are reduced to 21.

In the most general case of an anisotropic body the 21 independent elastic constants are interpreted as,

$$B_{1111} = a_{11} = \frac{1}{E_1}\ , \quad B_{2222} = a_{22} = \frac{1}{E_2}\ , \quad B_{3333} = a_{33} = \frac{1}{E_3}$$

$$B_{1122} = a_{12} = -\frac{\nu_{21}}{E_1} = -\frac{\nu_{12}}{E_2}\ , \quad B_{2233} = a_{23} = -\frac{\nu_{32}}{E_2} = -\frac{\nu_{23}}{E_3}$$

$$B_{1133} = a_{13} = -\frac{\nu_{13}}{E_3} = -\frac{\nu_{31}}{E_1}$$

$$B_{2323} = a_{44} = \frac{1}{G_{23}}\ , \quad B_{1313} = a_{55} = \frac{1}{G_{13}}\ , \quad B_{1212} = a_{66} = \frac{1}{G_{12}}$$

$$B_{1312} = a_{56} = \frac{\mu_{1231}}{G_{13}} = \frac{\mu_{3112}}{G_{12}}$$

$$B_{2312} = a_{46} = \frac{\mu_{1223}}{G_{23}} = \frac{\mu_{2312}}{G_{12}}$$

$$B_{2313} = a_{45} = \frac{\mu_{3112}}{G_{23}} = \frac{\mu_{2332}}{G_{13}}$$

$$B_{1123} = a_{14} = \frac{\eta_{23,1}}{E_1} = \frac{\eta_{1,23}}{G_{23}} \tag{2.5.8}$$

$$B_{2213} = a_{25} = \frac{\eta_{31,2}}{E_2} = \frac{\eta_{2,31}}{G_{13}}$$

$$B_{3312} = a_{36} = \frac{\eta_{12,3}}{E_3} = \frac{\eta_{3,12}}{G_{12}}$$

$$B_{2223} = a_{24} = \frac{\eta_{23,2}}{E_2} = \frac{\eta_{2,23}}{G_{23}}$$

$$B_{3313} = a_{36} = \frac{\eta_{31,3}}{E_3} = \frac{\eta_{3,31}}{G_{13}}$$

$$B_{1112} = 4_{16} = \frac{\eta_{12,1}}{E_1} = \frac{\eta_{1,12}}{G_{12}}$$

$$B_{3323} = a_{34} = \frac{\eta_{23,3}}{E_3} = \frac{\eta_{3,23}}{G_{23}}$$

$$B_{1113} = a_{15} = \frac{\eta_{31,1}}{E_1} = \frac{\eta_{1,31}}{G_{13}}$$

$$B_{2212} = a_{26} = \frac{\eta_{12,2}}{E_2} = \frac{\eta_{2,12}}{G_{12}} \qquad (2.5.8)$$

$E_1$, $E_2$, $E_3$ are the Young's moduli (for tension-compression) with respect to the directions $x_1$, $x_2$, $x_3$ ; $G_{23,}$ $G_{13}$, $G_{12}$ are the shear moduli for planes which are perpendicular to the coordinate axis $x_1$, $x_2$, $x_3$ respectively; $\nu_{21}$, $\nu_{32}$, $\nu_{13}$, $\nu_{12}$, $\nu_{23}$, $\nu_{31}$ are the Poisson coefficients which characterize the transverse compression for tension in the direction of the axis of the coordinate (thus, $\nu_{21}$ is a coefficient which characterizes the decrease in the $x_1$ direction for tension in the $x_2$ direction). The constants $\mu_{3123}$, $\mu_{1223}$,..., $\mu_{2312}$ are called "the coefficients of Chentsev." These constants characterize the shear in the planes which are parallel to the coordinates and which induce tangential stresses parallel to the other coordinate planes. Thus, for example $\mu_{3123}$ characterizes the shear in the plane parallel to the $x_2$-$x_3$ plane which induces the stress $\sigma_{13}$ and so forth. Finally $\eta_{23,1}, \eta_{31,1}, \ldots, \eta_{12,3}$ are called the coefficients of mutual influence of the first kind and $\eta_{1,23}, \eta_{2,23}, \ldots, \eta_{3,12}$ are called the coefficients of mutual influence of the second kind; the first characterize the stretching in the directions parallel to the axes which are induced by the tangential stresses; the second are shears in the planes parallel to the coordinates under the influence of normal stresses. In the following we will present the stress-strain equations for an elastic body under various degrees of anisotropy. The most general case of anisotropy has already been given above.

One plane of elastic symmetry

A plane of symmetry is a plane so that any properties that are symmetric relative to this plane are equivalent with respect to their material properties. Directing the $x_3$ axis normal to the plane of elastic symmetry, we obtain for the Hooke's law,

$$\varepsilon_{11} = a_{11}\sigma_{11} + a_{12}\sigma_{22} + a_{13}\sigma_{33} + a_{16}\sigma_{12}$$

$$\varepsilon_{22} = a_{12}\sigma_{11} + a_{22}\sigma_{22} + a_{23}\sigma_{33} + a_{26}\sigma_{12}$$

$$\varepsilon_{33} = a_{13}\sigma_{11} + a_{23}\sigma_{22} + a_{33}\sigma_{33} + a_{36}\sigma_{12}$$

$$\varepsilon_{23} = a_{44}\sigma_{23} + a_{45}\sigma_{13}$$

$$\varepsilon_{13} = a_{45}\sigma_{23} + a_{55}\sigma_{13}$$

$$\varepsilon_{12} = a_{16}\sigma_{11} + a_{26}\sigma_{22} + a_{36}\sigma_{33} + a_{66}\sigma_{12} \tag{2.5.9}$$

where the number of independent elastic constants are now 13.

Three planes of elastic symmetry

Assuming that three orthogenal planes of elastic symmetry pass through each point of a body and by directing the axes of the coordinates perpendicular to these planes, we obtain the following for the Hooke's law,

$$\begin{aligned}
\varepsilon_{11} &= a_{11}\sigma_{11} + a_{12}\sigma_{22} + a_{13}\sigma_{33} \\
\varepsilon_{22} &= a_{12}\sigma_{11} + a_{22}\sigma_{22} + a_{23}\sigma_{33} \\
\varepsilon_{33} &= a_{13}\sigma_{11} + a_{23}\sigma_{22} + a_{33}\sigma_{33} \\
\varepsilon_{23} &= a_{44}\sigma_{23} \\
\varepsilon_{13} &= a_{55}\sigma_{13} \\
\varepsilon_{12} &= a_{66}\sigma_{12}
\end{aligned} \tag{2.5.10}$$

where the number of independent elastic constants are now 9.

A plane of isotropy

A plane of isotropy is one in which all in-plane directions are mechanically equivalent. Assuming that the curvilinear coordinate $x_3$ is always perpendicular to this plane, the generalized Hooke's laws are,

$$\varepsilon_{11} = a_{11}\sigma_{11} + a_{12}\sigma_{22} + a_{13}\sigma_{33}\,, \quad \varepsilon_{12} = a_{66}\sigma_{12}$$

$$\varepsilon_{22} = a_{21}\sigma_{11} + a_{22}\sigma_{22} + a_{23}\sigma_{33}\,, \quad \varepsilon_{13} = a_{55}\sigma_{13}$$

$$\varepsilon_{33} = a_{31}\sigma_{11} + a_{32}\sigma_{22} + a_{33}\sigma_{33} , \quad \varepsilon_{23} = a_{44}\sigma_{23} \tag{2.5.11}$$

where

$$a_{13} = a_{23}$$

$$a_{66} = 2(a_{11} - a_{12})$$

$$a_{55} = a_{44}$$

where the number of independent elastic constants are now 6.

Complete symmetry - an isotropic body

Some bodies are equivalent in all directions, any plane being a plane of elastic symmetry. By denoting the Young's modulus, the shear modulus, and the Poisson ratio by E, G, $\nu$ respectively (identical in all directions), the Hooke's law is,

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{E}\,[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] , \quad \varepsilon_{12} = \frac{1}{G}\,\sigma_{12} \\ \varepsilon_{22} &= \frac{1}{E}\,[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})] , \quad \varepsilon_{13} = \frac{1}{G}\,\sigma_{13} \\ \varepsilon_{33} &= \frac{1}{E}\,[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})] , \quad \varepsilon_{23} = \frac{1}{G}\,\sigma_{23} \end{aligned} \tag{2.5.12}$$

where the number of independent elastic constants are now 2.

### 2.5.2 Viscoelastic Materials

The stress-strain relations in the classical theory of linear infinitesimal isotropic viscoelasticity have been expressed in section 2.1 in operator form, where the operators are either differential or integral.

Generalizing the stress-strain relations to the anisotropic case we have for the differential operator,

$$P_{ij\kappa\ell}\sigma_{\kappa\ell} = Q_{ij\kappa\ell}\varepsilon_{\kappa\ell} \tag{2.5.13}$$

where

$$\begin{aligned} P_{ij\kappa\ell} &= (\alpha_o)_{ij\kappa\ell} + (\alpha_1)_{ij\kappa\ell}\frac{\partial}{\partial t} + (\alpha_2)_{ij\kappa\ell}\frac{\partial^2}{\partial t^2} + \cdots \\ Q_{ij\kappa\ell} &= (\beta_o)_{ij\kappa\ell} + (\beta_1)_{ij\kappa\ell}\frac{\partial}{\partial t} + (\beta_2)_{ij\kappa\ell}\frac{\partial^2}{\partial t^2} + \cdots \end{aligned} \tag{2.5.14}$$

and for the integral operator,

$$\sigma_{ij} = G_{ij\kappa\ell}(\vec{x},t)\varepsilon_{\kappa\ell}(\vec{x},o) + \int_o^t G_{ij\kappa\ell}(\vec{x},t-\tau)\frac{\partial\varepsilon_{\kappa\ell}}{\partial\tau}(\vec{x},\tau)d\tau \tag{2.5.15}$$

or

$$\varepsilon_{ij} = J_{ij\kappa\ell}(\vec{x},t)\sigma_{\kappa\ell}(\vec{x},o) + \int_o^t J_{ij\kappa\ell}(\vec{x},t-\tau)\frac{\partial\sigma_{\kappa\ell}}{\partial\tau}(\vec{x},\tau)d\tau \tag{2.5.16}$$

where $G_{ij\kappa\ell}$ is the relaxation function or relaxation modulus, and $J_{ij\kappa\ell}$ is the creep function or creep compliance. The symmetry of the stress and strain tensors require that,

$$G_{ij\kappa\ell} = G_{ji\kappa\ell} = G_{ij\ell\kappa} = G_{\kappa\ell ij} \tag{2.5.17}$$

$$J_{ij\kappa\ell} = J_{ji\kappa\ell} = J_{ij\ell\kappa} = J_{\kappa\ell ij}$$

Through the use of the convolution integral, equations (2.5.15) and (2.5.16) become in the transform space,

$$\sigma_{ij}(\vec{x},p) = p\overline{G}_{ij\kappa\ell}(\vec{x},p)\varepsilon_{\kappa\ell}(\vec{x},p) \tag{2.5.18}$$

$$\varepsilon_{ij}(\vec{x},p) = p\overline{J}_{ij\kappa\ell}(\vec{x},p)\sigma_{\kappa\ell}(\vec{x},p) \tag{2.5.19}$$

For a purely elastic anisotropic material we would have,

$$\varepsilon_{ij} = B_{ij\kappa\ell}\sigma_{\kappa\ell} \tag{2.5.20}$$

or

$$\sigma_{ij} = C_{ij\kappa\ell}\varepsilon_{\kappa\ell} \tag{2.5.21}$$

and therefore (2.3.14) must be of the form,

$$[\overline{u}_i, \overline{\varepsilon}_{ij}, \overline{\sigma}_{ij}] = [\overline{u}_i^{\,e}, \overline{\varepsilon}_{ij}^{\,e}, \overline{\sigma}_{ij}^{\,e}];\ C_{ij\kappa\ell} = p\overline{G}_{ij\kappa\ell} \tag{2.5.22}$$

or

$$[\overline{u}_i, \overline{\varepsilon}_{ij}, \overline{\sigma}_{ij}] = [\overline{u}_i^{\,e}, \overline{\varepsilon}_{ij}^{\,e}, \overline{\sigma}_{ij}^{\,e}];\ B_{ij\kappa\ell} = p\overline{J}_{ij\kappa\ell} \tag{2.5.23}$$

Thus we see that the extension of the various correspondence principles to the anisotropic case is straight forward. The same regard must be given to the boundary conditions as in the isotropic case and the essential difference when working with an anisotropic problem lies only in the substitution of

$$p\overline{G}_{ij\kappa\ell} \longrightarrow C_{ij\kappa\ell} \tag{2.5.24}$$

or

$$p\bar{J}_{ij\kappa\ell} \rightarrow B_{ij\kappa\ell} \tag{2.5.25}$$

instead of

$$p\bar{G}_1 \rightarrow 2\mu$$
$$p\bar{G}_2 \rightarrow 3\kappa \tag{2.5.26}$$

The material coefficients which are retained for a viscoelastic body correspond to those which are retained in the elastic solution. Thus if the coefficient $C_{2213}$ vanishes for a particular case of symmetry, then $B_{2213}$, $G_{2213}$, $J_{2213}$, $P_{2213}$ and $Q_{2213}$ will vanish. In other words, all the comments made on elastic anisotropy given in the preceding section may also be made for viscoelastic anisotropy.

## 2.6 Inversion of the Laplace Transform

We have seen from the various correspondence principles previously stated that the usual method of obtaining a viscoelastic solution is by inverting an associated elastic solution from the transform plane. While the transformed solution might be easily obtained, the real time dependent response might be quite complicated depending on the nature of the transform solution. We shall now explore the inversion of the Laplace transform.

Let F(t) be a function of t (time) specified for t > o. Then the Laplace transform of F(t), denoted by $\bar{F}(p)$ is defined by,

$$\mathcal{L}\left[F(t)\right] = \bar{F}(p) = \int_o^\infty e^{-pt}\, F(t)dt \tag{2.6.1}$$

The Laplace transform of F(t) is said to exist if the integral (2.6.1) converges for some value of p, otherwise it does not exist.

If the Laplace transform of a function F(t) is $\bar{F}(p)$, i.e. if $\mathcal{L}(F(t)) = \bar{F}(p)$, then F(t) is called an inverse Laplace transform of $\bar{F}(p)$ and we write $F(t) = \mathcal{L}^{-1}(\bar{F}(p))$ where $\mathcal{L}^{-1}$ is called the inverse Laplace transform operator. Thus $\mathcal{L}^{-1}(\bar{F}(p))$ is given by,

$$F(t) = \mathcal{L}^{-1}(\bar{F}(p)) = \frac{1}{2\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}\, \bar{F}(p)dp, \quad t > o \tag{2.6.2}$$

and F(t) = o for t < 0. This result is called the complex inversion formula and is also known as Bromwich's integral formula. The result provides a direct means for obtaining the inverse Laplace transform of a given function $\bar{F}(p)$.

The integration in (2.6.2) is performed along a line $p = \gamma$ in the complex plane where $p = x + iy$. The real number $\gamma$ is chosen so that $p = \gamma$ lies to the right of all the singularities (poles, branch points or essential singularities) but is otherwise arbitrary.

<u>Methods of Finding Inverse Laplace Transform</u>

1. Partial Fractions Method -

Any rational function $\bar{P}(p)/\bar{Q}(p)$ where $\bar{P}(p)$ and $\bar{Q}(p)$ are polynomials, with the degree of $\bar{P}(p)$ less than that of $\bar{Q}(p)$, can be written as the sum of rational functions (called partial fractions) having the form $\frac{A}{(ap+b)^r}$, $\frac{Ap+B}{(ap^2+bp+c)}$, where r = 1,2,3,.... By finding the inverse Laplace transform of each of the partial fractions, we can find $\mathcal{L}^{-1}(\bar{P}(p)/\bar{Q}(p))$.

2. Use of Tables -

There has been an extensive literature of tables for the inversion of Laplace transforms published which can be used if the Laplace transform can be put in an appropriate form.

3. The Complex Inversion Formula -

The Bromwich integral formula (see equation (2.6.2)) supplies a powerful direct method for finding inverse Laplace transforms.

4. Approximate Methods -

When determining a function $F(t)$ from its Laplace transform $\overline{F}(p)$ one might apply one of the formal methods previously mentioned. When, however, the associated elastic solution is not available in analytical form suitable for formal inversion, numerical values of the transformed viscoelastic solution must be obtained which correspond to certain discrete values of the transform parameter. If the time variation of the viscoelastic moduli of the particular material of interest is known, it is possible to express the equivalent moduli of the material in terms of the Laplace transform of the viscoelastic moduli. Since the equivalent moduli and the transformed boundary conditions can be evaluated for particular values of the transform parameter, the associated elastic solution can be evaluated for particular values of the transform parameter. The transformed viscoelastic solution then can be found for real values of the transform parameter. If a method of inverting the solution for discrete values of the transform parameter is available, the viscoelastic solution to the problem may be obtained. For further information on these methods one is referred to Bellman [1966], Cost [1964], Kaplan [1962], Papoulis [1957], Schapery [1962].

Two other approximate methods for the inversion of the Laplace transform are inversion by series and by continued fractions Akin [1968].

# 3. ANISOTROPIC FRACTURE

## 3.1 The Isotropic Dugdale Model

Fracture may be defined as the phenomenon of structural failure by catastropic crack propagation at average stresses well below yield strength. Why a crack propagates and under what conditions it is most likely to occur are the two most important questions engineers working in fracture mechanics have been trying to solve since Coulomb, in 1773, expressed the view that fracture of a solid would occur if the maximum shear strain at some point surpassed a critical value characterizing the mechanical strength of the material.

Most of the modern day crack propagation theories are based on the work of A. A. Griffith [1920]. In his classic paper he concluded that an existing crack would propagate in a cataclysmic fashion, if the available elastic strain energy exceeded the increase in surface energy of the crack. There have been many different reactions to this concept since the time of its conception.

It wasn't until the late 1940's that Orowan [1952] and Irwin [1948] showed independently that the Griffith type energy balance must be between the strain energy stored in the specimen and the surface energy plus the work done in plastic deformation. For ductile materials the work done against surface tension is generally not significant in comparison with the work done against plastic deformation.

Dugdale [1960] while investigating the yielding of steel sheets containing slits, observed the yielded region was shaped as a thin extension of the crack. The three hypotheses that he formulated were;

1. The material in the yielded zone is under a uniform tensile yield stress Y. (see Figure 3.1)
2. The thickness of the yielded zone is so small that the elastic region outside may be regarded as bounded internally by a flattened ellipse of length $2(\ell + s)$, where $\ell$ is the half length of the crack and s the half length of the plastic zone. (see Figure 3.2)
3. The length s is such that there is no stress singularity at the ends of the flattened ellipse.

With these hypotheses and noting the problem of a straight cut loaded over part of its edge had been examined by Muskhelishvili [1953], Dugdale was able to get a relation between the extent of plastic yielding and the external load. Muskhelishvili's stress functions were found to assume a simple form when account was taken of the symmetry of the problem. Introducing variables $\alpha$ and $\theta_2$ defined by,

$$x = a \cos h\alpha \qquad (3.1.1)$$

$$\ell = a \cos \theta_2 \qquad (3.1.2)$$

the stress $\sigma_y$ acting at points on $y = o$ was determined in the form of a series in ascending powers of $\alpha$ having a loading term,

$$\sigma_y = -\frac{2Y\theta_2}{\pi\alpha} \qquad (3.1.3)$$

The analogous expression for stress due to the external loading was,

$$\sigma_y = \frac{T}{\alpha} \qquad (3.1.4)$$

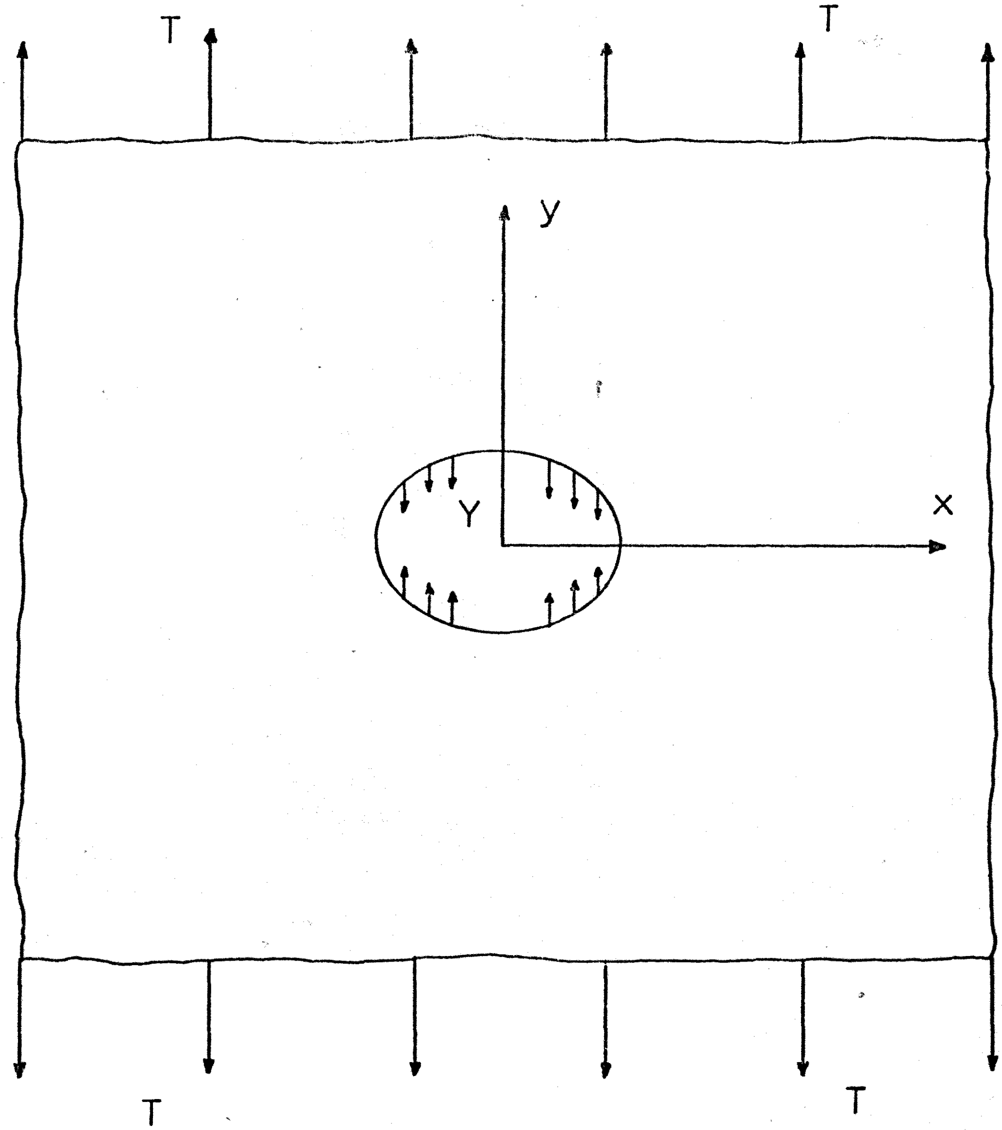



Figure 3.1

PLASTIC ZONE REPLACED BY YIELD STRESS

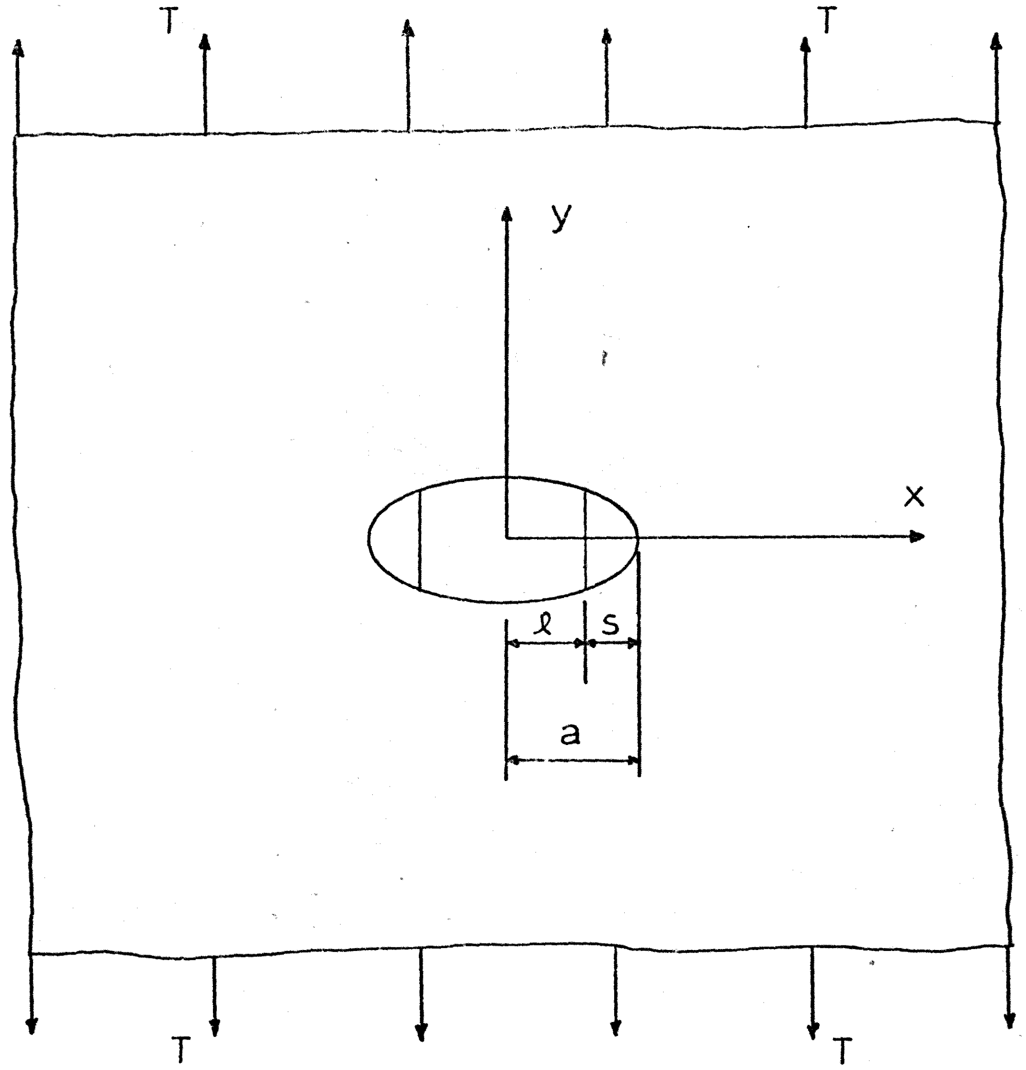



Figure 3.2

WEDGE SHAPED PLASTIC ZONE

When these stresses were superposed, use was made of the third hypothesis that the stress at the point $\alpha = o$ (i.e. $x = a$) should not be infinite, so the coefficient of $1/\alpha$ must vanish. Therefore,

$$T - \frac{2Y\theta_2}{\pi} = o \qquad (3.1.5)$$

or

$$\theta_2 = \frac{\pi}{2}\frac{T}{Y} \qquad (3.1.6)$$

This readily leads to the relation with T/Y very small,

$$\frac{s}{\ell} = 1.23(T/Y)^2 \qquad (3.1.7)$$

Using the results of Dugdale, Goodier and Field [1963] were able to evaluate the plastic energy dissipation by the methods of elastic perfectly-plastic continuum mechanics. A slowly extending internal slit crack under tension and a rapidly extending semi-infinite crack under traveling wedge pressures were examined by them.

Gonzalez [1968] incorporated orthotropic material properties with the Dugdale Model and obtained a lower bound on T/Y ratios for which the Dugdale Model is applicable.

## 3.2 The Anisotropic Dugdale Model

A natural extension of earlier orthotropic work is to extend Dugdale's technique to a general anisotropic material.

We will now find the stresses and displacements about a crack in an anisotropic infinite plate loaded in tension at infinity in a direction perpendicular to the line of the crack. A Dugdale model is assumed, and stress functions for the problem are obtained by use of Muskhelishvili's complex variable methods.

Beginning with usual plane stress assumptions $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = o$, the stress-strain relations for an anisotropic medium under plane stress conditions,

$$\begin{aligned}
\varepsilon_{xx} &= a_{11}\sigma_{xx} + a_{12}\sigma_{yy} + a_{16}\sigma_{xy} \\
\varepsilon_{yy} &= a_{12}\sigma_{xx} + a_{22}\sigma_{yy} + a_{26}\sigma_{xy} \\
\varepsilon_{xy} &= a_{16}\sigma_{xx} + a_{26}\sigma_{yy} + a_{66}\sigma_{xy}
\end{aligned} \tag{3.2.1}$$

where the $a_{ij}$ are the coefficients of deformation as previously defined in Chapter 2. The equilibrium and compatibility equations are independent of mechanical properties and are assuming no body forces,

$$\begin{aligned}
\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} &= o \\
\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} &= o
\end{aligned} \tag{3.2.2}$$

$$\frac{\partial^2\varepsilon_{yy}}{\partial x^2} + \frac{\partial^2\varepsilon_{xx}}{\partial y^2} = \frac{\partial^2\varepsilon_{xy}}{\partial x\partial y} \tag{3.2.3}$$

A stress function can be chosen in the usual form,

$$\sigma_{xx} = \frac{\partial^2\chi}{\partial y^2}$$

$$\sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2}$$

$$\sigma_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} \qquad (3.2.4)$$

Combining equations (3.1.8-11) we get,

$$a_{22}\frac{\partial^4 \chi}{\partial x^4} - 2a_{26}\frac{\partial^4 \chi}{\partial x^3 \partial y} + (2a_{12} + a_{66})\frac{\partial^4 \chi}{\partial x^2 \partial y^2} - 2a_{16}\frac{\partial^4 \chi}{\partial x \partial y^3} + a_{11}\frac{\partial^4 \chi}{\partial y^4} = o \qquad (3.2.5)$$

The general solution of (3.2.5) may be assumed to be of the form,

$$\chi = f(x + my) \qquad (3.2.6)$$

so that (3.2.5) becomes,

$$a_{22} - 2a_{26}m + (2a_{12} + a_{66})m^2 - 2a_{16}m^3 + a_{11}m^4 = o \qquad (3.2.7)$$

Thus the general solution of (3.2.5) depends on the roots of (3.2.7).

For the case of unequal roots the solution will be of the form,

$$\chi(x,y) = F_1(x+m_1y) + F_2(x+m_2y) + F_3(x+m_3y) + F_4(x+m_4y) \qquad (3.2.8)$$

Lekhnitskii [1937] has shown that the roots of (3.2.7) are not real. Let us assume therefore,

$$m_1 = \alpha_1 + i\beta_1 \qquad m_2 = \alpha_2 + i\beta_2$$
$$m_3 = \alpha_1 - i\beta_1 \qquad m_4 = \alpha_2 - i\beta_2 \qquad (3.2.9)$$

where $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ are real constants and it is always possible to arrange $\beta_1$, $\beta_z > o$. Now for,

$$\begin{aligned} &\beta_1, \beta_2 > o \\ &\beta_1 \neq \beta_2 \end{aligned} \tag{3.2.10}$$

and letting

$$\begin{aligned} z_1 &= x + m_1 y = x + \alpha_1 y + i\beta_1 y \\ z_2 &= x + m_2 y = x + \alpha_2 y + i\beta_2 y \end{aligned} \tag{3.2.11}$$

(3.2.8) can be written as,

$$\chi(x,y) = F_1(z_1) + F_2(z_2) + \overline{F_1(z_1)} + \overline{F_2(z_2)} \tag{3.2.12}$$

where $F_1(z_1)$ and $F_2(z_2)$ are analytic functions and $\overline{F_1(z_1)}$ and $\overline{F_2(z_2)}$ are respectively their conjugates. Now let,

$$\frac{dF_1}{dz_1} = \phi(z_1) , \qquad \frac{dF_2}{dz_2} = \psi(z_2) \tag{3.2.13}$$

and from (3.2.13) we see that,

$$\frac{d\overline{F}_1}{d\overline{z}_1} = \overline{\phi(z_1)} , \qquad \frac{d\overline{F}_2}{d\overline{z}_2} = \overline{\psi(z_2)} \tag{3.2.14}$$

Inserting (3.2.12) into (3.2.4) and using (3.2.13) and (3.2.14) we obtain,

$$\sigma_{xx} = 2\text{Re}\ [m_1^{\ 2}\ \phi'(z_1) + m_2^{\ 2}\ \psi'(z_2)] \tag{3.2.15}$$

$$\sigma_{yy} = 2\text{Re}\ [\phi'(z_1) + \psi'(z_2)]$$

$$\sigma_{xy} = -\ 2\text{Re}\ [m_1\phi'(z_1) + m_2\psi'(z_2)] \qquad (3.2.15)$$

The strain-displacement relations are independent of the mechanical properties and they are,

$$\frac{\partial u}{\partial x} = \varepsilon_{xx} = a_{11}\sigma_{xx} + a_{12}\sigma_{yy} + a_{16}\sigma_{xy}$$

$$\frac{\partial v}{\partial y} = \varepsilon_{yy} = a_{12}\sigma_{xx} + a_{22}\sigma_{yy} + a_{26}\sigma_{xy} \qquad (3.2.16)$$

Substituting (3.2.15) into (3.2.16) and integrating yields,

$$u = 2\text{Re}\ [p_1\phi(z_1) + p_2\psi(z_2)] - \gamma_o y + \alpha_o$$

$$v = 2\text{Re}\ [q_1\phi(z_1) + q_2\psi(z_2)] + \gamma_o x + \beta_o \qquad (3.2.17)$$

where,

$$p_1 = a_{11}m_1^2 + a_{12} - a_{16}m_1\ , \qquad p_2 = a_{11}m_2^2 + a_{12} - a_{16}m_2\ ,$$

$$q_1 = \frac{a_{12}m_1^2 + a_{22} - a_{26}m_1}{m_1}\ , \qquad q_2 = \frac{a_{12}m_2^2 + a_{22} - a_{26}m_2}{m_2}\ , \qquad (3.2.18)$$

The terms $-\ \gamma_o y + \alpha_o$ and $\gamma_o x + \beta_o$ are the expressions for rigid displacements of the entire body and can be disregarded when investigating elastic equilibrium.

In addition to (3.2.5), $\chi$ must also satisfy certain boundary conditions on the contour of the investigated area and these depend on the given problem. The projections $X_n$, $Y_n$ on the coordinate axes of the external forces acting on the contour L of the area S are supposed given. Therefore the stress boundary conditions would be,

$$\begin{aligned} X_n &= \sigma_{xx} \cos (n,x) + \sigma_{xy} \cos (n,y) \\ Y_n &= \sigma_{xy} \cos (n,x) + \sigma_{yy} \cos (n,y) \end{aligned} \tag{3.2.19}$$

where,

$$\cos (n,x) = \frac{dy}{d\lambda} \qquad\qquad \cos (n,y) = - \frac{dx}{d\lambda} \tag{3.2.20}$$

therefore,

$$\begin{aligned} X_n &= \frac{\partial^2 X}{\partial y^2} \frac{dy}{d\lambda} + \frac{\partial^2 X}{\partial x \partial y} \frac{dx}{d\lambda} = \frac{d}{d\lambda} \left[\frac{\partial X}{\partial y}\right] \\ Y_n &= - \frac{\partial^2 X}{\partial x \partial y} \frac{dy}{d\lambda} - \frac{\partial^2 X}{\partial x^2} \frac{dx}{d\lambda} = - \frac{d}{d\lambda} \left[\frac{\partial X}{\partial x}\right] \end{aligned} \tag{3.2.21}$$

and

$$\begin{aligned} \frac{\partial X}{\partial x} &= - \int_o^\lambda Y_n d\lambda + C_1 \\ \frac{\partial X}{\partial y} &= \int_o^\lambda X_n d\lambda + C_2 \end{aligned} \tag{3.2.22}$$

where $\lambda$ is an arc measured from an arbitrary point of the contour L of the area S. Using (3.2.12) and (3.2.15) the boundary conditions for $\phi(z_1)$ and $\psi(z_2)$ become,

$$2\text{Re}\,[\phi(z_1) + \psi(z_2)] = -\int_0^{\lambda} Y_n ds + C_1 = f_1$$

$$2\text{Re}\,[m_1\phi(z_1) + m_2\psi(z_2)] = \int_0^{\lambda} X_n ds + C_2 = f_2 \qquad (3.2.23)$$

Thus the solution of problems where external forces are given is reduced to the determination of the two functions $\phi(z_1)$ and $\psi(z_2)$, and the application of the boundary conditions given by (3.2.23).

Using Dugdale's hypothesis the analytic functions for the Dugdale crack, Figure 3.1, may be determined by superposition of the three states shown in Figure 3.3. These analytic functions may be found from the functions given by Muskhelishvili and adapted to the anisotropic case by Savin [1961] for an infinite sheet containing an elliptical hole which is loaded by uniform normal forces acting over a portion of its surface.

Superposing the two stress states in Figure 3.4, taking the limit as the ellipse flattens to a crack and adding the uniform tension at infinity, yields the analytic functions for the anisotropic Dugdale crack. The problem then reduces to the following: Given an infinitely large anisotropic elastic plate containing an elliptical hole with forces acting on the contour of the hole given, what are the analytic functions for this particular geometry?

To get the functions for states 2 and 3 we must therefore first have the function for the case of an elliptical hole part of which is subjected to normal pressure as in Figure 3.5. It is assumed that a uniform pressure p is applied only to the portion AB of the hole contour and that no external forces are applied to the remaining part BCA.

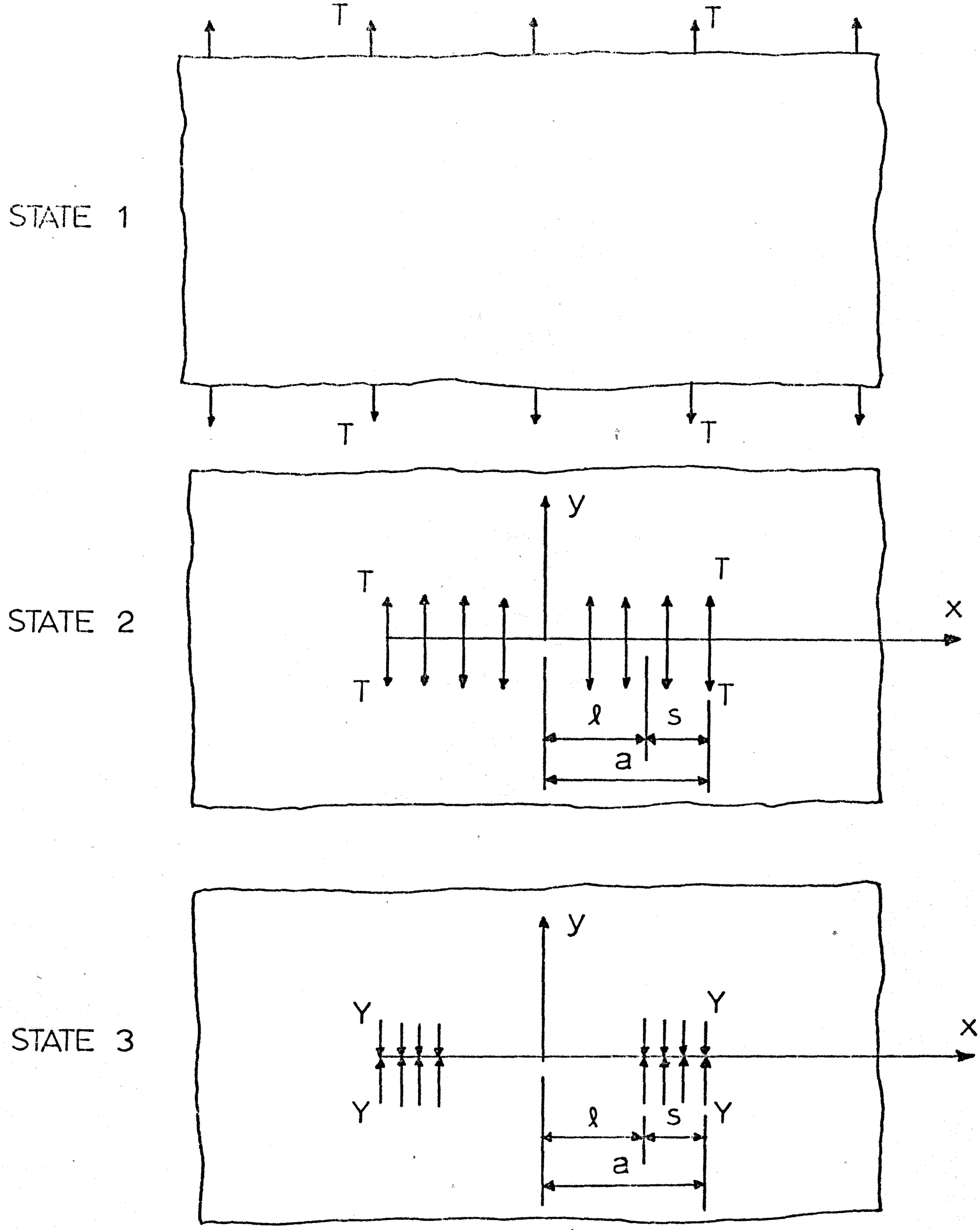
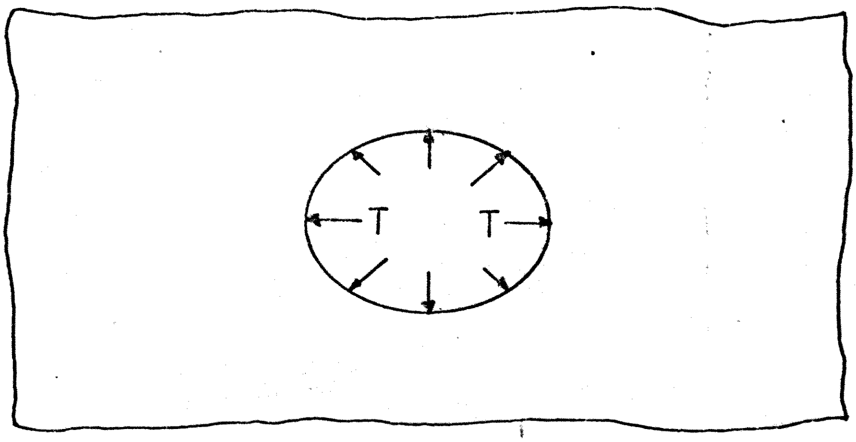



Figure 3.3
COMPONENT STRESS STATES FOR DUGDALE CRACK

STATE 2




STATE 3

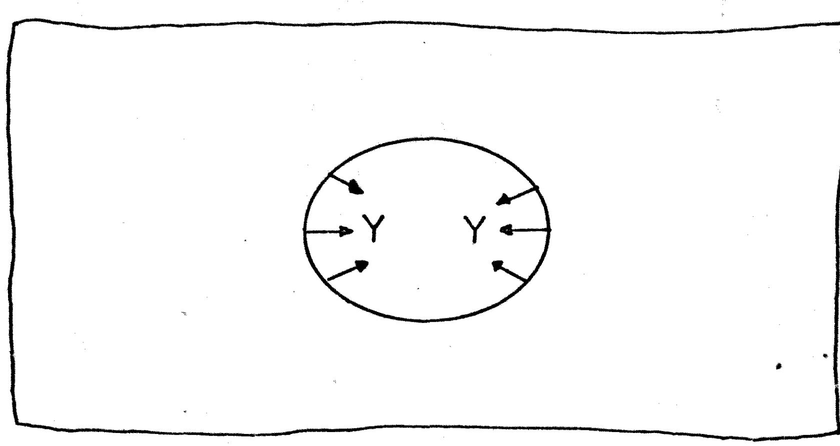



Figure 3.4

SUPERPOSITION FOR COMPONENT STRESS STATES

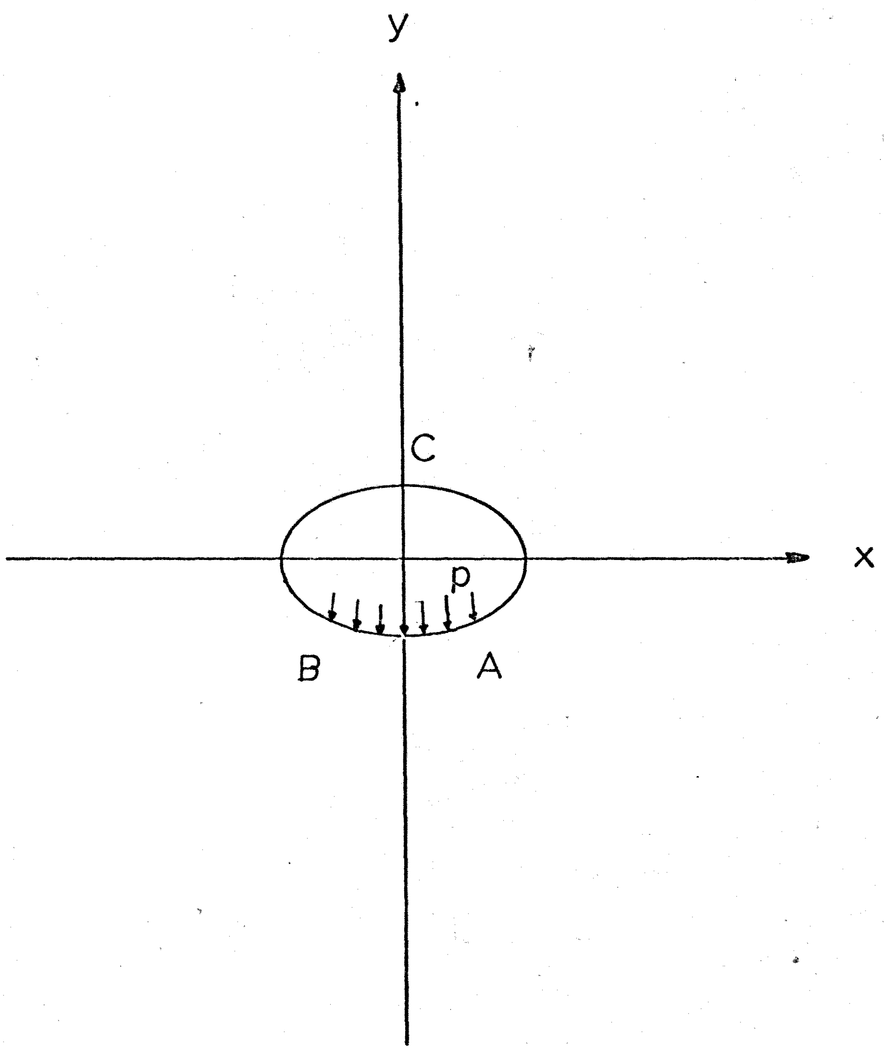



Figure 3.5

LOADING ALONG ELLIPTICAL HOLE

The coordinate axes x and y are chosen in the directions of the axes of the ellipse and the semi-axes are a and b, where for a crack along the x-axis of length 2a, b = 0.

In addition to the given plane $z = x + iy$, the planes $z_1$ and $z_2$ obtained from this plane by an affine transformation (3.2.11) will also be investigated.

The ellipse in the z plane is transformed into ellipses in the $z_1$ and $z_2$ planes. The areas outside these ellipses will be denoted by S, $S^1$, $S^2$ respectively and the functions which give a conformal representation of the areas on the inside of the unit circle are to be determined.

Now,

$$z = w(\zeta) = \frac{a - b}{2} \zeta + \frac{a + b}{2} \frac{1}{\zeta} \tag{3.2.24}$$

gives the conformal representation of the area S on the inside of the unit circle, where the coordinates of the contour points of the ellipse of area S are,

$$x = a \cos \theta \ , \qquad\qquad y = - b \sin \theta \tag{3.2.25}$$

but,

$$z_1 = x + m_1 y = a \cos \theta - m_1 b \sin \theta \ , \tag{3.2.26}$$

therefore,

$$z_1 = w_1(\zeta) = \frac{a + im_1 b}{2} \zeta + \frac{a - im_1 b}{2} \frac{1}{\zeta} \tag{3.2.27}$$

and similarly,

$$z_2 = w_2(\zeta) = \frac{a + im_2 b}{2} \zeta + \frac{a - im_2 b}{2} \zeta \tag{3.2.28}$$

Now our two functions $\phi(z_1)$ and $\psi(z_2)$ can be written in the conformal plane as,

$$\begin{aligned} \phi(z_1) &= \phi(w_1(\zeta)) = \Phi(\zeta) \\ \psi(z_2) &= \psi(w_2(\zeta)) = \Psi(\zeta) \end{aligned} \tag{3.2.29}$$

The derivatives of these functions are,

$$\begin{aligned} \phi'(z_1) &= \frac{\Phi'(\zeta)}{w_1'(\zeta)} \\ \psi'(z_2) &= \frac{\Phi'(\zeta)}{w_2'(\zeta)} \end{aligned} \tag{3.2.30}$$

Equation (3.2.15) becomes,

$$\begin{aligned} \sigma_{xx} &= 2\mathrm{Re}\left[m_1^2 \frac{\Phi'(\zeta)}{w_1'(\zeta)} + m_2^2 \frac{\Psi'(\zeta)}{w_2'(\zeta)}\right] \\ \sigma_{yy} &= 2\mathrm{Re}\left[\frac{\Phi'(\zeta)}{w_1'(\zeta)} + m_2 \frac{\Psi'(\zeta)}{w_2'(\zeta)}\right] \\ \sigma_{xy} &= -2\mathrm{Re}\left[m_1 \frac{\Phi'(\zeta)}{w_1'(\zeta)} + m_2 \frac{\Psi'(\zeta)}{w_2'(\zeta)}\right] \end{aligned} \tag{3.2.31}$$

and (3.2.17) becomes,

$$u = 2\mathrm{Re}\,[p_1\Phi(\zeta) + p_2\Psi(\zeta)]$$

$$v = 2\text{Re}\ [q_1\Phi(\zeta) + q_2\Psi(\zeta)] \qquad (3.2.32)$$

The points A and B in the conformal plane are $\sigma_1$ and $\sigma_2$ where $\sigma_1 = e^{i\theta_1}$ and $\sigma_2 = e^{i\theta_2}$, Figure 3.6.

Assuming b = 0, the stress functions are,

$$\Phi(\zeta) = \left[- (A' + iA'') - \frac{im_2pa(t - r)}{4\pi(m_1 - m_2)}\right] \ln \zeta$$

$$+ \frac{ipam_2}{4\pi(m_1-m_2)} \left[(\zeta + \frac{1}{\zeta} - r) \ln (\sigma_1 - \zeta) - (\zeta + \frac{1}{\zeta} - t) \ln (\sigma_2 - \zeta) + \frac{1}{\zeta} \ln \frac{\sigma_2}{\sigma_1}\right]$$

$$\Psi(\zeta) = - \left[(B' + iB'') - \frac{im_1pa(t - r)}{4\pi(m_1 - m_2)}\right] \ln \zeta \qquad (3.2.33)$$

$$- \frac{ipam_1}{4\pi(m_1-m_2)} \left[(\zeta + \frac{1}{\zeta} - r) \ln (\sigma_1 - \zeta) - (\zeta + \frac{1}{\zeta} - t) \ln (\sigma_2 - \zeta) + \frac{1}{\zeta} \ln \frac{\sigma_2}{\sigma_1}\right]$$

where,

$$r = \sigma_1 + \frac{1}{\sigma_1}$$

$$(3.2.34)$$

$$t = \sigma_2 + \frac{1}{\sigma_2}$$

A'. A'', B', B'' are constants which have a factor of (t - r).

For state 2 (see Figure 3.7) the stress functions reduce with the entire contour loaded and therefore with $\sigma_1 = \sigma_2$, $r = t$, $\ln \frac{\sigma_2}{\sigma_1} = 2\pi i$, $p = T$ to,

$$\Phi_2(\zeta) = \frac{iTam_2}{4\pi(m_1-m_2)}\ \frac{2\pi i}{\zeta} = - \frac{Tam_2}{2(m_1-m_2)\zeta}$$

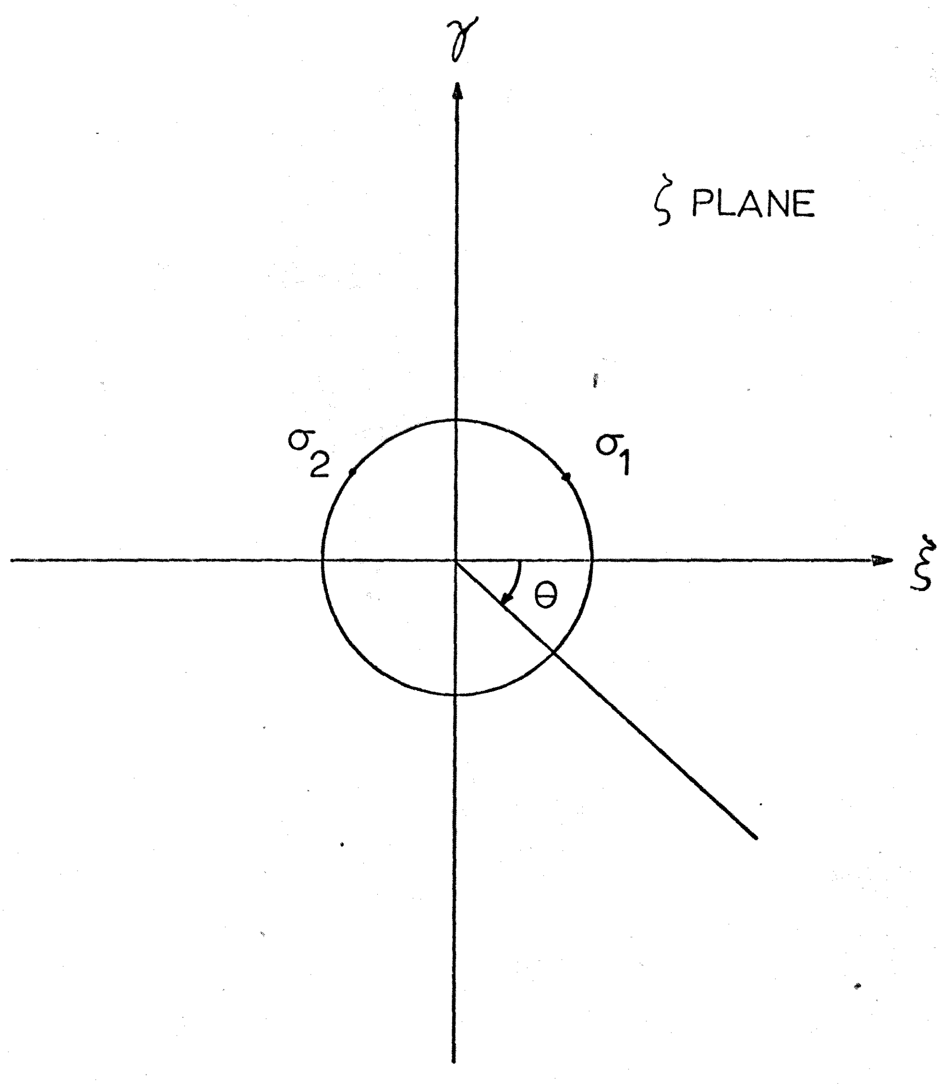



Figure 3.6

CONFORMAL PLANE

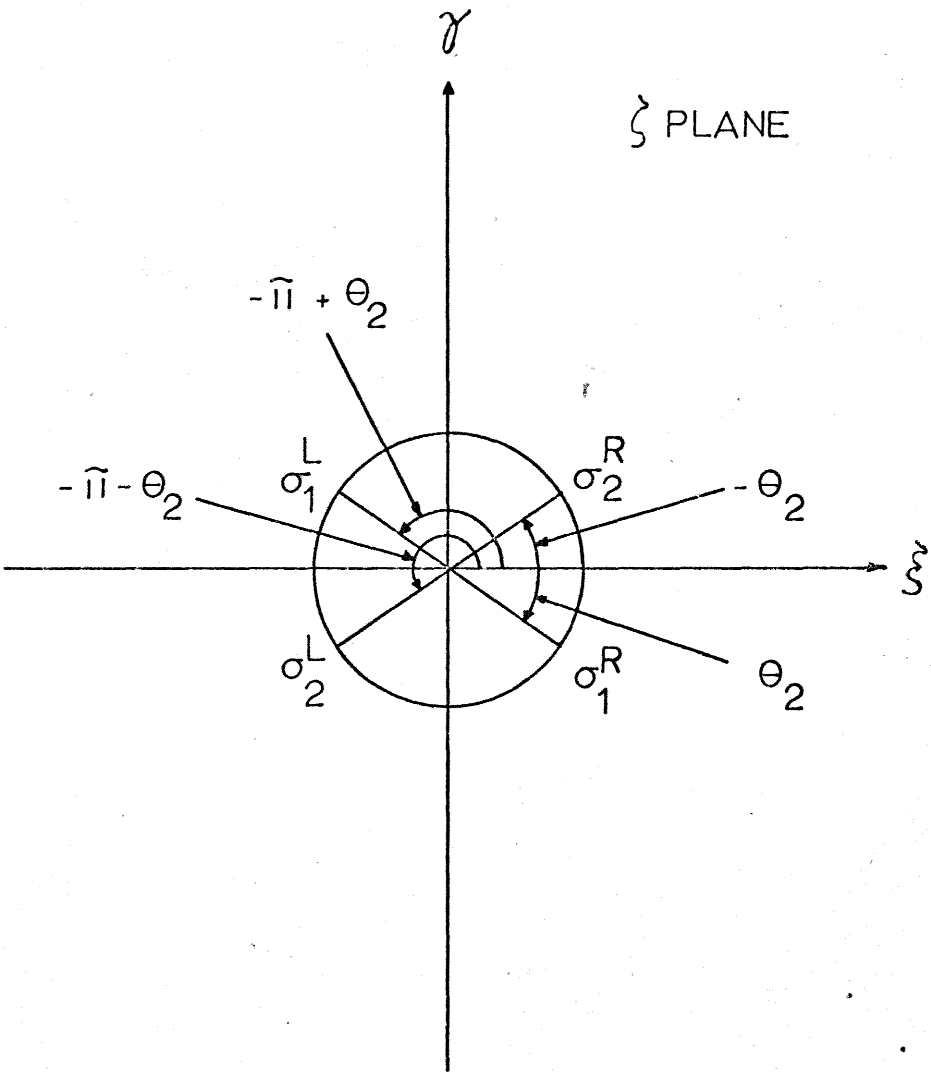



Figure 3.7

LOADING IN CONFORMAL PLANE

$$\Psi_2(\zeta) = -\frac{iTam_1}{4\pi(m_1-m_2)} \frac{2\pi i}{\zeta} = \frac{Tam_1}{2(m_1-m_2)\zeta} \qquad (3.2.35)$$

For state 3 with the contour partially loaded (see Figure 3.7) and noting,

$$\begin{aligned}
&\sigma_2{}^R = e^{-i\theta_2} = \sigma_2 && r^R = \sigma_1{}^R + \frac{1}{\sigma_1{}^R} = \bar{\sigma}_2 + \frac{1}{\bar{\sigma}_2} = \bar{t} = \frac{2\ell}{a} \\
&\sigma_1{}^R = e^{i\theta_2} = \bar{\sigma}_2 && t^R = \sigma_2{}^R + \frac{1}{\sigma_2{}^R} = 2\cos\theta_2 = \frac{2\ell}{a} = t \\
&\sigma_1{}^L = e^{-i(\pi-\theta_2)} = -\bar{\sigma}_2 && r^L = \sigma_1{}^L + \frac{1}{\sigma_1{}^L} = -\bar{t} \qquad (3.2.36) \\
&\sigma_2{}^L = e^{-i(\pi+\theta_2)} = -\sigma_2 && t^L = \sigma_2{}^L + \frac{1}{\sigma_2{}^L} = -t \\
&p = -Y && \theta_2 = \cos^{-1}(\ell/a)
\end{aligned}$$

the stress functions are,

$$\Phi_3(\zeta) = -\frac{Yam_2}{4\pi i(m_1-m_2)}\left[-\frac{4i\theta_2}{\zeta} - (\zeta + \frac{1}{\zeta})\ln\frac{(\sigma_2-\zeta)(\sigma_2+\zeta)}{(\bar{\sigma}_2-\zeta)(\bar{\sigma}_2+\zeta)}\right.$$

$$\left. -t\ln\frac{(\bar{\sigma}_2-\zeta)(\sigma_2+\zeta)}{(\sigma_2-\zeta)(\bar{\sigma}_2+\zeta)}\right]$$

$$\Psi_3(\zeta) = \frac{Yam_1}{4\pi i(m_1-m_2)}\left[-\frac{4i\theta_2}{\zeta} - (\zeta + \frac{1}{\zeta})\ln\frac{(\sigma_2-\zeta)(\sigma_2+\zeta)}{(\bar{\sigma}_2-\zeta)(\bar{\sigma}_2+\zeta)}\right.$$

$$-t\ \ln \frac{(\bar{\sigma}_2-\zeta)(\sigma_2+\zeta)}{(\sigma_2-\zeta)(\bar{\sigma}_2+\zeta)}\Bigg] \tag{3.2.37}$$

where in the above the positive direction of $\theta$ is changed from clockwise to counter-clockwise*.

Superposing states 2 and 3 we get for the Dugdale model exclusive of the uniform tension at infinity,

$$\Phi(\zeta) = \left[\frac{Yam_2\theta_2}{\pi(m_1-m_2)} - \frac{Tam_2}{2(m_1-m_2)}\right]\frac{1}{\zeta} + \frac{Yam_2}{4\pi i(m_1-m_2)}\Bigg[(\zeta+\frac{1}{\zeta})\ \ln\frac{(\sigma_2-\zeta)(\sigma_2+\zeta)}{(\sigma_2-\zeta)(\sigma_2+\zeta)}$$

$$+\frac{2\ell}{a}\ \ln\frac{(\bar{\sigma}_2-\zeta)(\sigma_2+\zeta)}{(\sigma_2-\zeta)(\bar{\sigma}_2+\zeta)}\Bigg]$$

$$\Psi(\zeta) = \left[-\frac{Yam_1\theta_2}{\pi(m_1-m_2)} + \frac{Tam_1}{2(m_1-m_2)}\right]\frac{1}{\zeta} - \frac{Yam_1}{4\pi i(m_1-m_2)}\Bigg[(\zeta+\frac{1}{\zeta})\ \ln\frac{(\sigma_2-\zeta)(\sigma_2+\zeta)}{(\bar{\sigma}_2-\zeta)(\bar{\sigma}_2+\zeta)}$$

$$+\frac{2\ell}{a}\ \ln\frac{(\bar{\sigma}_2-\zeta)(\sigma_2+\zeta)}{(\sigma_2-\zeta)(\bar{\sigma}_2+\zeta)}\Bigg] \tag{3.2.38}$$

The Dugdale's finiteness condition requires that,

$$\frac{Yam_2\theta_2}{\pi(m_1-m_2)} - \frac{Tam_2}{2(m_1-m_2)} = -\frac{Yam_1\theta_2}{\pi(m_1-m_2)} + \frac{Tam_1}{2(m_1-m_2)} = 0 \tag{3.2.39}$$

or

$$\boxed{\theta_2 = \frac{\pi T}{2Y}} \tag{3.2.40}$$

*This is done in order to check the results with Goodier and Field.

Therefore the stress functions reduce to, using (3.2.40),

$$\Phi(\zeta) = \frac{Yam_2}{4\pi i(m_1-m_2)}\left[(\zeta + \frac{1}{\zeta}) \ln \frac{(\sigma_2-\zeta)(\sigma_2+\zeta)}{(\bar{\sigma}_2-\zeta)(\bar{\sigma}_2+\zeta)} + \frac{2\ell}{a} \ln \frac{(\bar{\sigma}_2-\zeta)(\sigma_2+\zeta)}{(\sigma_2-\zeta)(\bar{\sigma}_2+\zeta)}\right]$$

$$\Psi(\zeta) = -\frac{Yam_1}{4\pi i(m_1-m_2)}\left[(\zeta + \frac{1}{\zeta}) \ln \frac{(\sigma_2-\zeta)(\sigma_2+\zeta)}{(\bar{\sigma}_2-\zeta)(\bar{\sigma}_2+\zeta)} + \frac{2\ell}{a} \ln \frac{(\bar{\sigma}_2-\zeta)(\sigma_2+\zeta)}{(\sigma_2-\zeta)(\bar{\sigma}_2+\zeta)}\right] \quad (3.2.41)$$

For the stresses along the line of the crack, letting $\zeta = \xi$ (i.e. $y = 0$) we have,

$$\sigma_{xx} = \frac{2Y}{\pi} (\beta_1\beta_2 - \alpha_1\alpha_2) \tan^{-1} \frac{\sin 2\theta_2}{\xi^2 - \cos 2\theta_2}$$

$$\sigma_{yy} = \frac{2Y}{\pi} \tan^{-1} \frac{\sin 2\theta_2}{\xi^2 - \cos 2\theta_2} + T \quad (3.2.42)$$

Taking $\xi = 1$ (the tip of the plastic zone) we have, $\xi + \frac{1}{\xi} = \frac{x}{a} = 1$

$$\sigma_{yy} = Y$$

$$\sigma_{xx} = (\beta_1\beta_2 - \alpha_1\alpha_2)(Y - T) \quad (3.2.43)$$

at this point as well as the rest of the x-axis, $\sigma_{xy}$ vanishes by symmetry.

Displacements at the surface of the crack and the elastic plastic interface on which $\zeta = e^{i\theta}$ and $\sigma_2 = e^{i\theta_2}$ are derived by substituting (3.2.38) into (3.2.17)

$$u = \frac{2Ya\theta_2}{\pi} [a_{11}(\alpha_1\alpha_2 - \beta_1\beta_2) - a_{12}] \cos \theta$$

$$v = -\frac{Ya}{\pi(\alpha_1^2+\beta_1^2)(\alpha_2^2+\beta_2^2)} \quad Re\Bigg[\Bigg(a_{22}[\beta_1\beta_2(\beta_1+\beta_2) + \alpha_1^2\beta_2 + \alpha_2^2\beta_1]$$

$$+ i\Big[a_{22}[\alpha_1\alpha_2(\alpha_1+\alpha_2) + (\alpha_1^2\beta_2^2 + \beta_1^2\alpha_2) - a_{26}[(\alpha_1^2+\beta_1^2)(\alpha_2^2 \tag{3.2.44}$$

$$+ \beta_2^2)]\Big]\Bigg)\Big[2i\theta_2 \cos\theta + 2\cos\theta \ln\sin(\theta+\theta_2) + \cos\theta_2 \ln(\sin\theta_2$$

$$- \sin\theta)^2 - \frac{\cos\theta_2 + \cos\theta}{2} \ln(\cos^2\theta_2 - \cos^2\theta)^2\Big]\Bigg]$$

Now we note for the orthotropic case that

$$\alpha_1 = \alpha_2 = 0 \tag{3.2.45}$$

and

$$a_{11} = \frac{1}{E_x} \qquad a_{16} = a_{26} = 0$$

$$a_{22} = \frac{1}{E_y} \qquad a_{66} = \frac{1}{G_{xy}} \tag{3.2.46}$$

$$a_{12} = -\frac{\nu_x}{E_x} = -\frac{\nu_y}{E_y}$$

therefore (3.2.5) reduces to

$$\frac{\partial^4\chi}{\partial x^4} + 2A\frac{\partial^2\chi}{\partial x^2 dy^2} + B\frac{\partial^4\chi}{\partial y^4} = 0 \tag{3.2.47}$$

where

$$A = \frac{E_y}{2G_{xy}} - \nu_y \qquad B = \frac{E_y}{E_x} = \frac{\nu_y}{\nu_x} \tag{3.2.48}$$

also (3.2.42) and (3.2.44) reduce to

$$\sigma_{xx} = \frac{2Y}{\pi} \sqrt{\frac{Ex}{Ey}} \tan^{-1} \frac{\sin 2\theta_2}{\xi^2 - \cos 2\theta_2}$$

$$\sigma_{yy} = \frac{2Y}{\pi} \tan^{-1} \left[\frac{\sin 2\theta_2}{\xi^2 - \cos 2\theta_2}\right] + T \tag{3.2.49}$$

and

$$u = \frac{2Ya(\nu_x - \beta_1\beta_2)}{\pi E_x} \theta_2 \cos\theta \tag{3.2.50}$$

$$v = \frac{Ya(\beta_1+\beta_2)}{2Ey\beta_1\beta_2} \left[\cos\theta \ln\left[\frac{\sin(\theta-\theta_2)}{\sin(\theta+\theta_2)}\right]^2 + \cos\theta_2 \ln\left[\frac{\sin\theta + \sin\theta_2}{\sin\theta - \sin\theta_2}\right]^2\right]$$

Equations (3.2.47,49.50) are seen to be the same relations which had been previously arrived at by Gonzalez [1968] for the orthotropic case.

Equations (3.2.49) are plotted in Figures (3.8,9) for some T/Y.

## 3.3 The Dynamic Orthotropic Dugdale Model

In the solution of the dynamic Dugdale model use is made of Radok's [1956] application of complex variable methods to solve the dynamic equations of plane anisotropic elasticity. For the solution a stress function must be introduced that satisfies a fourth order partial differential equation which reflects both the dynamic and anisotropic influences. By restricting the solution to steady motion, this stress function can be replaced by two analytic functions of two different complex variables, as was done in the static-anisotropic case (see Section 3.1). The stress and the displacement components are then expressed in terms of these functions. The solution to dynamic-anisotropic problems then is the

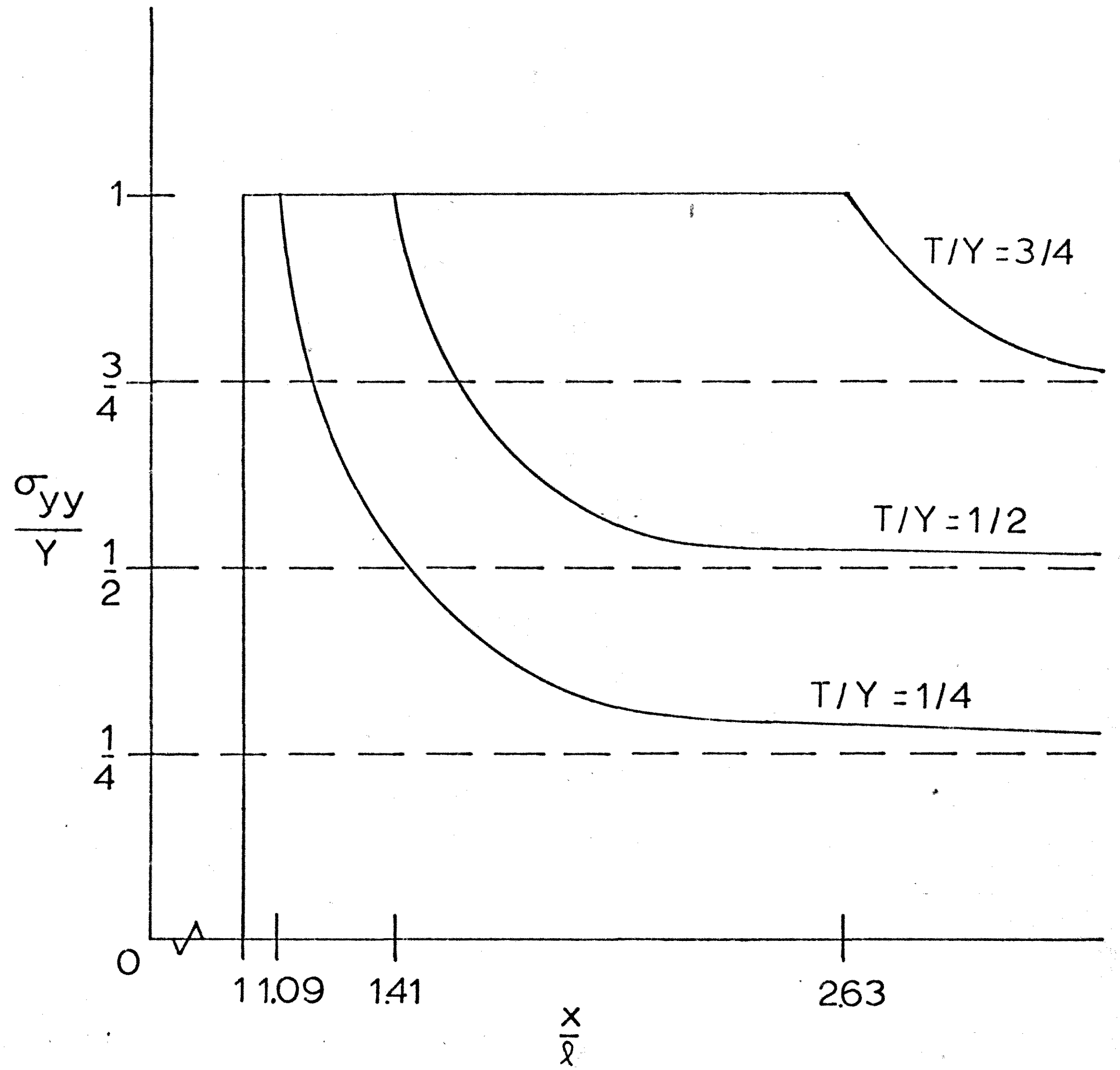



Figure 3.8

$\sigma_{yy}$ STRESS ALONG x - AXIS FOR ANISOTROPIC STATIC DUGDALE MODEL

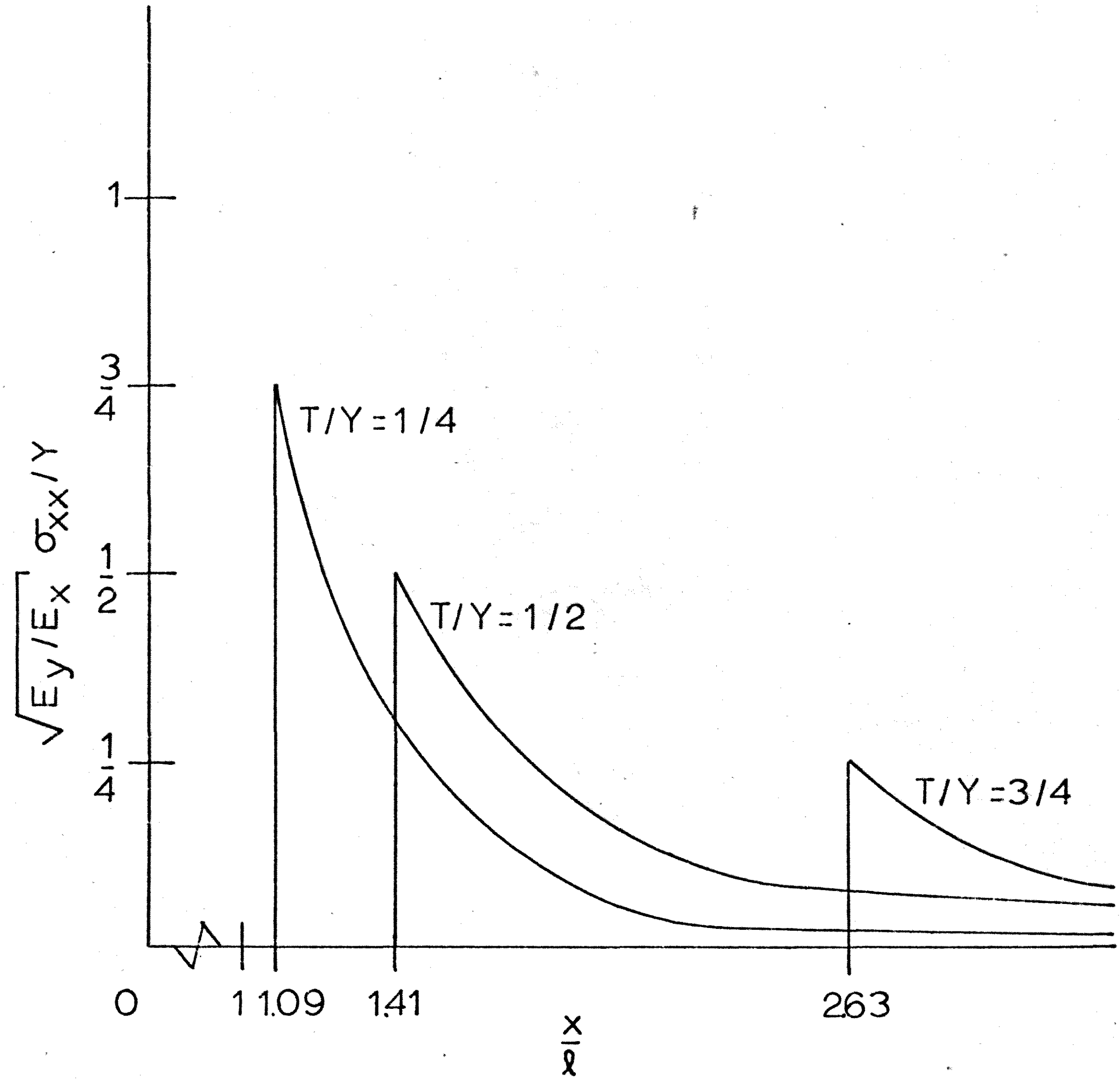



Figure 3.9

$\sigma_{xx}$ STRESS ALONG x-AXIS FOR ORTHOTROPIC STATIC DUGDALE MODEL determination of the two functions, which satisfy the equations and the boundary conditions. The Radok equations can be derived for the anisotropic case by considering the two equations of motion and the three Hooke's law which are applicable in the plane stress case.

In the dynamic anisotropic case for plane stress conditions the equations of motion are,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2} \tag{3.3.1}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2} \tag{3.3.2}$$

and the Hooke's law are the same as in the static case (3.2.1)

$$\varepsilon_{xx} = a_{11}\sigma_{xx} + a_{12}\sigma_{yy} + a_{16}\sigma_{xy} = \frac{\partial u}{\partial x} \tag{3.3.3}$$

$$\varepsilon_{yy} = a_{12}\sigma_{xx} + a_{22}\sigma_{yy} + a_{26}\sigma_{xy} = \frac{\partial v}{\partial y} \tag{3.3.4}$$

$$\varepsilon_{xy} = a_{16}\sigma_{xx} + a_{26}\sigma_{yy} + a_{66}\sigma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{3.3.5}$$

Differentiating (3.3.1) with respect to x, (3.3.2) with respect to y then adding and subtracting we get,

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + 2\,\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = \rho \frac{\partial^2}{\partial t^2} (\varepsilon_x + \varepsilon_y) \tag{3.3.6}$$

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} = \rho \frac{\partial^2}{\partial t^2} (\varepsilon_x - \varepsilon_y) \tag{3.3.7}$$

Differentiating (3.3.5) with respect to x and y we have,

$$\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = a_{16} \frac{\partial^2 \sigma_{xx}}{\partial x \partial y} + a_{26} \frac{\partial^2 \sigma_{yy}}{\partial x \partial y} + a_{66} \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} \tag{3.3.8}$$

Solving for $\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}$ in (3.3.8) and substituting in (3.3.6) we have,

$$\begin{aligned} &\left[\frac{\partial^2}{\partial x^2} - \frac{2a_{16}}{a_{66}} \frac{\partial^2}{\partial x \partial y}\right] \sigma_{xx} + \left[\frac{\partial^2}{\partial y^2} - \frac{2a_{26}}{a_{66}} \frac{\partial^2}{\partial x \partial y}\right] \sigma_{yy} \\ &= \left[\rho \frac{\partial^2}{\partial t^2} - \frac{2}{a_{66}} \frac{\partial^2}{\partial y^2}\right] \varepsilon_{xx} + \left[\rho \frac{\partial^2}{\partial t^2} - \frac{2}{a_{66}} \frac{\partial^2}{\partial x^2}\right] \varepsilon_{yy} \end{aligned} \tag{3.3.9}$$

Substituting (3.3.3,4.5) into (3.3.7) we have,

$$\begin{aligned} \left[\frac{\partial^2}{\partial x^2} - \rho(a_{11} - a_{12}) \frac{\partial^2}{\partial t^2}\right] \sigma_{xx} &= \left[\frac{\partial^2}{\partial y^2} - \rho(a_{22} - a_{12}) \frac{\partial^2}{\partial t^2}\right] \sigma_{yy} \\ &+ \left[\rho \frac{\partial^2}{\partial t^2} (a_{16} - a_{26})\right] \sigma_{xy} \end{aligned} \tag{3.3.10}$$

Now for the special case of orthotropy we have, $a_{16} = a_{26} = o$. (3.3.10) can now be satisfied by introducing a stress function of the form,

$$\begin{aligned} \sigma_{xx} &= \left[\frac{\partial^2}{\partial y^2} - \rho(a_{22} - a_{12}) \frac{\partial^2}{\partial t^2}\right] \chi \\ \sigma_{yy} &= \left[\frac{\partial^2}{\partial x^2} - \rho(a_{11} - a_{12}) \frac{\partial^2}{\partial t^2}\right] \chi \end{aligned} \tag{3.3.11}$$

(3.3.9) can be written as,

$$\left[\left(1+\frac{2a_{12}}{a_{66}}\right)\frac{\partial^2}{\partial x^2}+\frac{2a_{11}}{a_{66}}\frac{\partial^2}{\partial y^2}-\rho(a_{12}+a_{11})\frac{\partial^2}{\partial t^2}\right]\sigma_{xx}$$

$$+\left[\left(1+\frac{2a_{12}}{a_{66}}\right)\frac{\partial^2}{\partial y^2}+\frac{2a_{22}}{a_{66}}\frac{\partial^2}{\partial x^2}-\rho(a_{12}+a_{22})\frac{\partial^2}{\partial t^2}\right]\sigma_{yy}=o \tag{3.3.12}$$

For a crack moving along the x-axis at constant velocity V we introduce,

$$\eta = x - Vt \tag{3.2.13}$$

which therefore indicates that,

$$\frac{\partial^2}{\partial x^2}=\frac{\partial^2}{\partial \eta^2}$$

$$\frac{\partial^4}{\partial x^4}=\frac{\partial^4}{\partial \eta^4}$$

$$\frac{\partial^2}{\partial t^2}=V^2\frac{\partial^2}{\partial \eta^2} \tag{3.3.14}$$

$$\frac{\partial^4}{\partial t^4}=V^4\frac{\partial^4}{\partial \eta^4}$$

For an orthotropic material we have from Chapter 2,

$$a_{11}=\frac{1}{E_x}$$

$$a_{12}=-\frac{\nu_{yx}}{E_x}=-\frac{\nu_{xy}}{E_y} \tag{3.3.15}$$

$$a_{22} = \frac{1}{E_y}$$

$$a_{66} = \frac{1}{G_{xy}} \tag{3.3.15}$$

Substituting (3.3.11,14,15) into (3.3.12) we have,

$$\frac{\partial^4 \chi}{\partial \eta^4} + 2A \frac{\partial^4 \chi}{\partial \eta^2 \partial y^2} + B \frac{\partial^4 \chi}{\partial y^4} = o \tag{3.3.16}$$

which is of the same form as (3.2.5) where,

$$2A = \frac{(E_x E_y - 2\nu_{yx} E_y G_{xy}) - \rho V^2 (E_y + (1 - \nu_{xy}\nu_{yx})\, G_{xy})}{E_x G_{xy} - \rho V^2 (E_x + (1 - \nu_{xy}\nu_{yx})\, G_{xy}) + \rho^2 V^4 (1 - \nu_{xy}\nu_{yx})} \tag{3.3.17}$$

$$B = \frac{E_y G_{xy}}{E_x G_{xy} - \rho V^2 (E_x + (1 - \nu_{xy}\nu_{yx})\, G_{xy} + \rho^2 V^4\ (1 - \nu_{xy}\nu_{yx})} \tag{3.3.18}$$

The general solution of (3.3.16) depends on the roots of the characteristic equation as before (see Eq. (3.2.7)),

$$Bm^4 + 2Am^2 + 1 = o \tag{3.3.19}$$

The roots are,

$$m_1 = i\sqrt{\frac{A - C}{B}}$$

$$m_2 = i\sqrt{\frac{A + C}{B}} \tag{3.3.20}$$

$$m_3 = -i\sqrt{\frac{A-C}{B}}$$

$$m_4 = -i\sqrt{\frac{A+C}{B}} \qquad (3.3.20)$$

where,

$$C = \sqrt{A^2 - B}$$

$$A > C \qquad (3.3.21)$$

Again as in (3.2.8), for the case of unequal roots the solution of (3.3.16) is,

$$\chi(\eta,y) = F_1(z_1) + F_2(z_2) + \overline{F_1(z_1)} + \overline{F_2(z_2)} \qquad (3.3.25)$$

where $F_1$ and $F_2$ are analytic functions and $\overline{F}_1$and $\overline{F}_2$ are respectively their conjugates.

$$z_1 = \eta + i\beta_1 y$$

$$z_2 = \eta + i\beta_2 y \qquad (3.3.26)$$

and,

$$\phi(z_1) = \frac{dF_1}{dz_1} \qquad \psi(z_2) = \frac{dF_2}{dz_2} \qquad (3.3.27)$$

$$\overline{\phi(z_1)} = \frac{d\overline{F}_1}{d\overline{z}_1} \qquad \overline{\psi(z_2)} = \frac{d\overline{F}_2}{d\overline{z}_2} \qquad (3.3.28)$$

Using (3.3.14), (3.3.11) becomes,

$$\sigma_{yy} = [1 - \rho V^2(a_{11} - a_{12})] \frac{\partial^2\chi}{\partial\eta^2} \tag{3.3.29}$$

and using (3.3.25,27,28) we have,

$$\sigma_{yy} = 2\,[1 - \rho V^2(a_{11} - a_{12})]\ \mathrm{Re}\ [\phi'(z_1) + \psi'(z_2)] \tag{3.3.30}$$

Similarly,

$$\sigma_{xx} = -\,2\mathrm{Re}\ [(\beta_1{}^2 + \rho V^2(a_{22} - a_{12}))\phi'(z_1) + (\beta_2{}^2 + \rho V^2(a_{22} - a_{12}))\psi(z_2)] \tag{3.3.31}$$

Now,

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta} \tag{3.3.32}$$

using (3.3.32) and (3.3.3) and integrating we have,

$$u = 2\mathrm{Re}\ \Big[ \Big(a_{12}(1 - \rho V^2(a_{11} - a_{12})) - a_{11}(\beta_1{}^2 + \rho V^2(a_{22} - a_{12}))\Big)\ \phi(z_1) + \Big(a_{12}(1 - \rho V^2(a_{11} - a_{12})) - a_{11}(\beta_2{}^2 + \rho V^2(a_{22} - a_{12}))\Big)\ \psi(z_2)\Big] \tag{3.3.33}$$

and similarly for v we have,

$$v = -\,2\mathrm{Rei}\ \left[\frac{[a_{22}(1 - \rho V^2(a_{11} - a_{12})) - a_{12}(\beta_1{}^2 + \rho V^2(a_{22} - a_{12}))]}{\beta_1}\ \phi(z_1)\right.$$

$$+ \frac{[a_{22}(1 - \rho V^2(a_{11} - a_{12})) - a_{12}(\beta_2{}^2 + \rho V^2(a_{22} - a_{12}))]}{\beta_2} \psi(z_2) \Bigg] \quad (3.3.34)$$

Using (3.3.5) we have,

$$\sigma_{xy} = 2\text{Rei}\ [A_1\phi'(z_1) + B_1\psi'(z_2)] \quad (3.3.35)$$

where,

$$a_{66}A_1 = 2\beta_1 a_{12} - \frac{a_{22}}{\beta_1} + (a_{11}a_{22} - a_{12}{}^2)\ \rho V^2 (\frac{1}{\beta_1} - \beta_1) - \beta_1{}^3 a_{11}$$

$$a_{66}B_1 = 2\beta_2 a_{12} - \frac{a_{22}}{\beta_2} + (a_{11}a_{22} - a_{12}{}^2)\ \rho V^2 (\frac{1}{\beta_2} - \beta_2) - \beta_2^3 a_{11} \quad (3.3.36)$$

Thus (3.3.30,31,33,34,35) represent the stresses and displacements for the general problem of a crack loaded along its boundary moving along the x-axis in an orthotropic material.

To get the solution for a dynamic Dugdale model we will superpose to the solution of the dynamic Griffith crack (i.e. crack with no load on its boundary and a uniform tension, T, at infinity) a solution for a crack moving along the x-axis which is partially loaded symmetrically along its contour with a load Y. The solutions for these problems are derived by finding the stress functions which satisfy the boundary conditions and the governing equations.

For the moving Griffith crack, Figure 3.10 the boundary conditions are,

$$\sigma_{yy} = o \qquad |\eta| < a \quad (3.3.37)$$

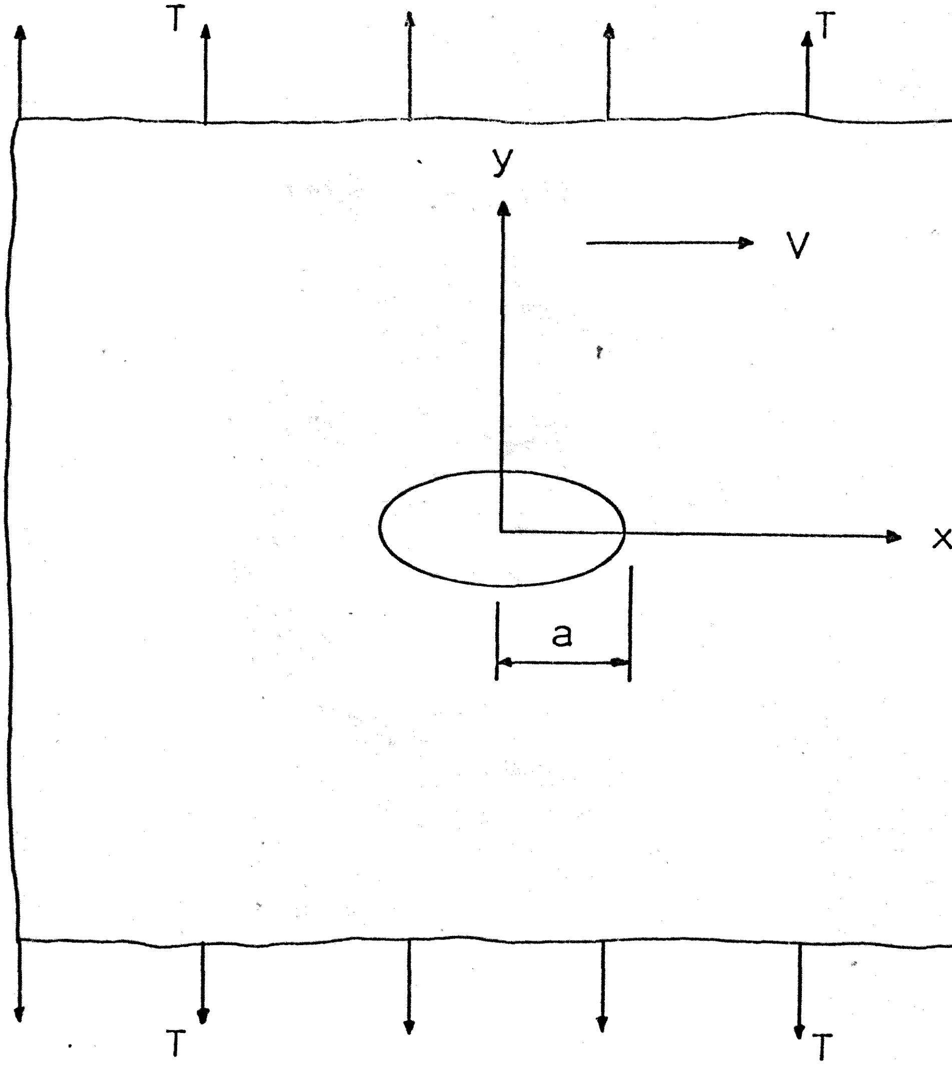



Figure 3.10

MOVING GRIFFITH CRACK

$$\sigma_{xy} = o \qquad |\eta| < a$$

$$\sigma_{yy}^{(\infty)} = T\ , \quad \sigma_{xx}^{(\infty)} = o\ , \quad \sigma_{xy}^{(\infty)} = o \tag{3.3.37}$$

From (3.2.30) therefore the boundary conditions take the form,

$$\text{Re}\ [\phi'(z_1) + \psi'(z_2)] = o \qquad |\eta| < a \tag{3.3.38}$$

$$2\text{Rei}\ [A_1\phi'(z_1) + B_1\psi'(z_2)] = o \ |\eta| < a \tag{3.3.39}$$

To find the stresses and displacements we must find $\phi(z_1)$ and $\psi(z_2)$ so that we have the required stresses at infinity and satisfy the boundary conditions (3.3.38,39).

The uniform loading at infinity will be caused by linear terms in $\phi(z_1)$ and $\psi(z_2)$, therefore $\phi(z_1)$ and $\psi(z_2)$ must be of the form,

$$\begin{aligned} \phi_1(z_1) &= (A_2 + iB_2)\ z_1 + \phi_o(z_1) \\ \psi_1(z_2) &= (A_3 + iB_3)z_2 + \psi_o(z_2) \end{aligned} \tag{3.3.40}$$

where as $z_1$ and $z_2$ go to infinity $\phi_o(z_1)$ and $\psi_o(z_2)$ go to zero. Using (3.3.30 and 31,32) and (3.3.40) we find,

$$\sigma_{yy}^{(\infty)} = 2[1 - \rho V^2(a_{11} - a_{12})]\ [A_2 + A_3]$$

$$\begin{aligned} \sigma_{xx}^{(\infty)} = -\ 2[ & \left(\beta_1^{\ 2} + \rho V^2(a_{22} - a_{12})\right) A_2 \\ & + \left(\beta_2^{\ 2} + \rho V^2(a_{22} - a_{12})\right) A_3] \end{aligned} \tag{3.3.41}$$

$$a_{66}\sigma_{xy}^{(\infty)} = - A_1B_2 - B_1B_3 \qquad (3.3.41)$$

From the boundary conditions at infinity we find,

$$B_2 = B_3 = o$$

$$A_2 = \frac{T}{2[1 - \rho V^2(a_{11} - a_{12})]} \left[\frac{\beta_2{}^2 + \rho V^2(a_{22} - a_{12})}{(\beta_2{}^2 - \beta_1{}^2)}\right]$$

$$A_3 = \left[\frac{-\left[\beta_1{}^2 + \rho V^2(a_{22} - a_{12})\right]}{2(\beta_2{}^2-\beta_1{}^2)(1-\rho V^2(a_{11}-a_{12}))}\right] T \qquad (3.3.42)$$

Substituting (3.3.40) into the boundary conditions (3.2.38,39) we have,

$$\mathrm{Re}\left[\phi'_o(z_1) + \psi'_o(z_2)\right] = - (A_2 + A_3)$$

$$? \rightarrow \mathrm{Rei}\ [A_1\phi'_o(z_1) + B_1\psi'_o(z_2)] = o \qquad (3.3.43)$$

Mapping the crack into a unit circle in the conformal plane where,

$$z_{1,2} = \frac{a}{2}\left(\zeta + \frac{1}{\zeta}\right)$$

$$a\zeta = z_{1,2} - (z_{1,2}{}^2 - a^2)^{1/2} \qquad (3.3.44)$$

(3.3.43) becomes,

$$\phi'_o(\zeta) + \psi'_o(\zeta) = - a(A_2 + A_3)$$

$$i \quad A_1\phi'_o(\zeta) + B_1\psi'_o(\zeta) = o \tag{3.3.45}$$

Solving for $\phi'_o(\zeta)$ and $\psi'_o(\zeta)$ from (3.3.45) and integrating and going back to the real plane we have,

$$\begin{aligned}
\phi_o(z_1) &= \frac{T}{2[1 - \rho V^2(a_{11} - a_{12})]} \left[\frac{B_1}{A_1 - B_1}\right] \left[z_1 - (z_1{}^2 - a^2)^{1/2}\right] \\
\psi_o(z_2) &= -\frac{T}{2[1 - \rho V^2(a_{11} - a_{12})]} \left[\frac{A_1}{A_1 - B_1}\right] \left[z_2 - (z_2{}^2 - a^2)^{1/2}\right]
\end{aligned} \tag{3.3.46}$$

and therefore substituting (3.3.42) and (3.3.46) into (3.3.40) we have,

$$\begin{aligned}
\phi_1(z_1) &= \frac{T}{2[1 - \rho V^2(a_{11} - a_{12})]} \left[\left[\frac{\beta_2{}^2 + \rho V^2(a_{22} - a_{12})}{(\beta_2{}^2 - \beta_1{}^2)}\right] z_1 \right. \\
&\quad \left. + \left[\frac{B_1}{A_1 - B_1}\right] \left[z_1 - (z_1{}^2 - a^2)^{1/2}\right]\right] \\
\psi_1(z_2) &= -\frac{T}{2[1 - \rho V^2(a_{11} - a_{12})]} \left[\left[\frac{\beta_1{}^2 + \rho V^2(a_{22} - a_{12})}{(\beta_2{}^2 - \beta_1{}^2)}\right] z_2 \right. \\
&\quad \left. + \left[\frac{A_1}{A_1 - B_1}\right] \left[z_2 - (z_2{}^2 - a^2)^{1/2}\right]\right]
\end{aligned} \tag{3.3.47}$$

From (3.3.30) therefore,

$$\sigma_{yy} = \frac{T}{A_1 - B_1} \operatorname{Re}\left[\frac{-z_1B_1}{(z_1{}^2 - a^2)^{1/2}} + \frac{z_2A_1}{(z_2{}^2 - a^2)^{1/2}}\right] \tag{3.3.48}$$

and from (3.3.31) therefore,

$$\sigma_{xx} = - \frac{T}{[1 - \rho V^2(a_{11} - a_{12})]} \text{Re}\left[ \frac{\beta_1{}^2 B_1 - \beta_2{}^2 A_1}{A_1 - B_1} - \rho V^2 (a_{22} - a_{12}) \right.$$

$$- \frac{B_1}{A_1 - B_1} \left[\beta_1{}^2 + \rho V^2(a_{22} - a_{12})\right] \frac{z_1}{(z_1{}^2 - a^2)^{1/2}} \tag{3.3.49}$$

$$\left. + \frac{A_1}{A_1 - B_1} \left[ \beta_2{}^2 + \rho V^2(a_{22} - a_{12})\right] \frac{z_2}{(z_2{}^2 - a^2)^{1/2}} \right]$$

and from (3.3.35)

$$\sigma_{xy} = \frac{T}{[1 - \rho V^2(a_{11} - a_{12})]} \text{Rei}\left[ \frac{1}{(\beta_2{}^2 - \beta_1{}^2)} \left[A_1\beta_2{}^2 - B_1\beta_1{}^2 \right.\right.$$

$$\left.\left. + (A_1B_1)\ \rho V^2(a_{22} - a_{12})\right] + \frac{A_1B_1}{A_1 - B_1} \left[ \frac{-z_1}{(z_1{}^2 + a^2)^{1/2}} + \frac{z_2}{(z_2{}^2 - a^2)^{1/2}} \right] \right]$$

(3.3.50)

Along the real axis where $z_1 = z_2 = \eta$ we would have,

$$\sigma_{yy} = \begin{cases} o & |\eta| < a \\ \dfrac{T\eta}{(\eta^2 - a^2)^{1/2}} & |\eta| < a \end{cases}$$

$$\sigma_{xy} = o \qquad \text{for all } \eta$$

$$\sigma_{xx} = \frac{1}{[1 - \rho V^2(a_{11} - a_{12})]} \left[ \frac{-\beta_2{}^2 A_1 + \beta_1{}^2 B_1}{A_1 - B_1} - \rho V^2(a_{22} - a_{12}) \right]$$

$$\begin{cases} -T & |\eta| < a \\ -\left[1 - \dfrac{\eta}{(\eta^2 - a^2)^{1/2}}\right] T & |\eta| > a \end{cases} \tag{3.3.51}$$

Now to get the Dugdale model we must superpose on the Griffith solution another solution to give the proper boundary conditions. The boundary conditions for this second problem are of the form, see Figure 3.11.

$$\sigma_{yy} = \begin{cases} o & |\eta| < \ell \\ Y & \ell < |\eta| < a \end{cases}$$

$$\sigma_{xy} = o \qquad |\eta| < a$$

$$R\sigma_{xx},\ R\sigma_{yy},\ R\sigma_{xy} \longrightarrow o \tag{3.3.52}$$

$$\text{as } \sqrt{\eta^2 + y^2} \longrightarrow \infty$$

Following a procedure set forth by Sneddon [1961], two analytic functions, each containing an arbitrary complex constant, can be devised. The Radok equations can be used to express the boundary stresses in terms of these constants. The constants are then evaluated from the boundary conditions. The two analytic functions to be found must have the same functional form on the real axis. Consequently, these must have the form,

$$\phi'_2(z_1) = Y\,(C_1 + iD_1)\, f\,(z_1) \tag{3.3.53}$$

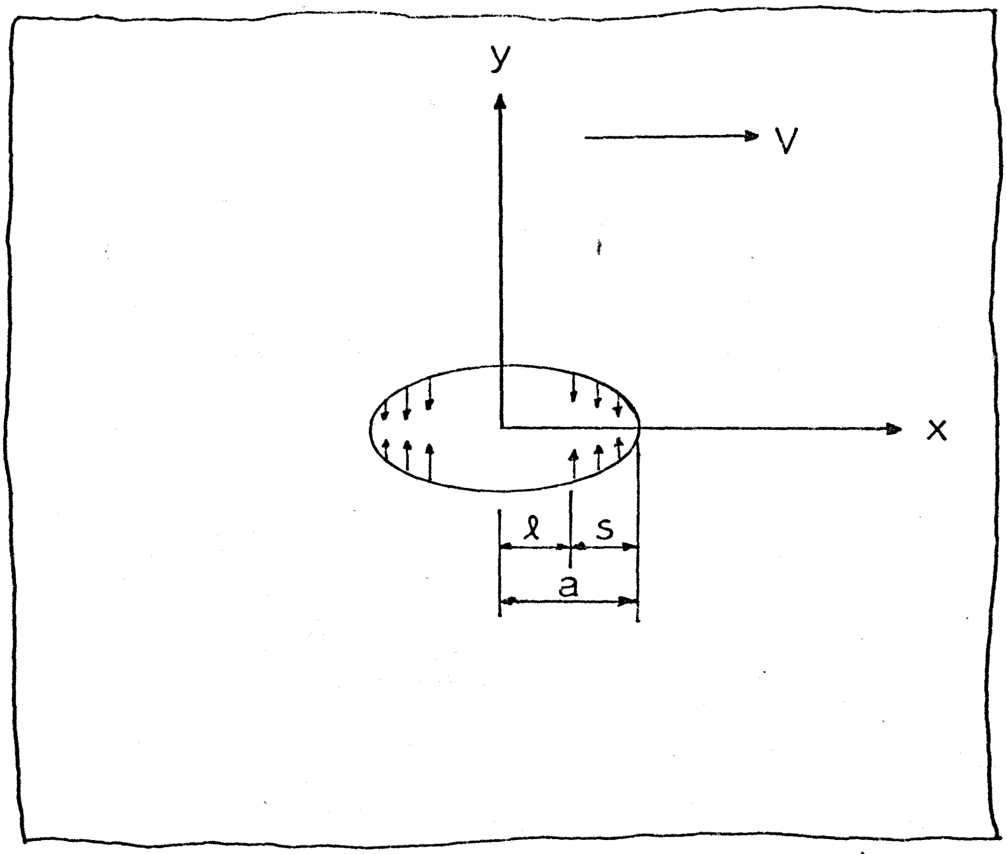



Figure 3.11

MOVING CRACK LOADED WITH YIELD STRESS

$$\psi'_2(z_2) = Y\ (C_2 + iD_2)\ f(z_2) \tag{3.3.54}$$

where $C_1$, $C_2$, $D_1$, $D_2$ are real constants and $f(z_1)$ and $f(z_2)$ are analytic functions.

Now for $y = 0$ we have $z_1 = z_2 = \eta$ and therefore,

$$\sigma_{yy}(\eta,o) = 2[1 - \rho V^2(a_{11} - a_{12})]\ Y\Big[(C_1 + C_2)\ \mathrm{Re}\ f(\eta)$$

$$- (D_1 + D_2)\ \mathrm{Im}\ f(\eta)\Big] \tag{3.3.55}$$

$$\sigma_{xy} = -\ 2Y\ \Big[(A_1C_1 + B_1C_2)\ \mathrm{Im}\ f(\eta) + (A_1D_1 + B_1D_2)\ \mathrm{Re}\ f(\eta)\Big]$$

From (3.3.52) and from (3.3.56) we have,

$$\begin{aligned} A_1C_1 + B_1C_2 &= o \\ A_1D_1 + B_1D_2 &= o \end{aligned} \tag{3.3.57}$$

Now guided by the isotropic static solution we take, Kanninen [1968],

$$f(z) = \log \frac{z\sqrt{a^2-\ell^2} - i\ell\sqrt{z^2-a^2}}{z\sqrt{a^2-\ell^2} + i\ell\sqrt{z^2-a^2}} + i\pi - \frac{i\alpha z}{\sqrt{z^2-a^2}} \tag{3.3.58}$$

For $R >> a$ it can be shown,

$$f(z) \rightarrow \log \frac{\sqrt{a^2-\ell^2} - i\ell}{\sqrt{a^2-\ell^2} + i\ell} + i\pi - \frac{i\alpha z}{\sqrt{z^2-a^2}} + O(R^{-2})$$

$$= i\left[\pi - 2\tan^{-1}\frac{\ell}{\sqrt{a^2-\ell^2}} - \alpha\right] + O(R^{-2}) \tag{3.3.59}$$

The boundary conditions at infinity can therefore be satisfied by taking,

$$\alpha = \pi - 2\tan^{-1}\frac{\ell}{\sqrt{a^2-\ell^2}} = 2\cos^{-1}\ell/a \tag{3.3.60}$$

or

$$\cos(\alpha/2) = \ell/a \tag{3.3.61}$$

Using (3.3.58) with (3.3.55) we have,

$$\sigma_{yy} = 2[1 - \rho V^2(a_{11} - a_{12})]\, Y \begin{cases} (C_1+C_2) \log \dfrac{\ell\sqrt{a^2-\eta^2} + \eta\sqrt{a^2-\ell^2}}{\ell\sqrt{a^2-\eta^2} - \eta\sqrt{a^2-\ell^2}} - \dfrac{\alpha\eta}{\sqrt{a^2-\eta^2}} & |\eta| < \ell \\[2ex] (C_1+C_2) \log \dfrac{\eta\sqrt{a^2-\ell^2} + \ell\sqrt{a^2-\eta^2}}{\eta\sqrt{a^2-\ell^2} - \ell\sqrt{a^2-\eta^2}} - \pi(D_1 + D_2) & \ell < |\eta| < a \end{cases} \tag{3.3.62}$$

Satisfying the boundary conditions (3.3.52) for $\sigma_{yy}$ we must have,

$$\begin{gathered} C_1 + C_2 = 0 \\ -\pi(D_1 + D_2)\, 2[1 - \rho V^2(a_{11} - a_{12})] = 1 \end{gathered} \tag{3.3.63}$$

Solving (3.3.57) and (3.3.63) for $C_1$, $C_2$, $D_1$, $D_2$ we have,

$$C_1 = C_2 = o$$

$$D_2 = -\frac{A_1}{(A_1 - B_1)2\pi[1 - \rho V^2(a_{11} - a_{12})]} \tag{3.3.64}$$

$$D_1 = \frac{B_1}{(A_1 - B_1)2\pi[1 - \rho V^2(a_{11} - a_{12})]}$$

where $A_1$, $B_1$ are defined by (3.3.36).

Therefore (3.3.53) and (3.3.54) can now be written as,

$$\phi'_2(z_1) = \frac{iYB_1}{2\pi(A_1 - B_1)[1 - \rho V^2(a_{11} - a_{12})]} \; f(z_1)$$

$$\psi'_2(z_2) = -\frac{iYA_1}{2\pi(A_1 - B_1)[1 - \rho V^2(a_{11} - a_{12})]} \; f(z_2) \tag{3.3.65}$$

Now combining the Griffith crack solution (3.3.47) with the solution of the problem just solved we will have the solution to the Dugdale - dynamic model. Combining (3.3.65) with (3.3.47) (really the derivative), we have,

$$\phi'(z_1) = \frac{T}{2[1 - \rho V^2(a_{11}-a_{12})]}\left[\left[\frac{\beta_2^{\,2}+\rho V^2(a_{22}-a_{12})}{(\beta_2^{\,2} - \beta_1^{\,2})}\right] + \frac{B_1}{A_1 - B_1}\right]$$

$$+\frac{iY\beta_1}{2\pi(A_1-B_1)[1-\rho V^2(a_{11}-a_{12})]}\log\frac{z_1\sqrt{a^2-\ell^2}-i\ell\sqrt{z_1^2-a^2}}{z_1\sqrt{a^2-\ell^2}+i\ell\sqrt{z_1^2-a^2}}$$

$$-\frac{YB_1}{2(A_1-B_1)[1-\rho V^2(a_{11}-a_{12})]}+\left[\frac{YB_1\alpha}{2\pi(A_1-B_1)[1-\rho V^2(a_{11}-a_{12})]}\right.$$

$$\left.-\frac{B_1T}{(A_1-B_1)2[1-\rho V^2(a_{11}-a_{12})]}\right]\frac{z_1}{(z_1^2-a^2)^{1/2}}$$

$$\psi'(z_2)=-\frac{T}{2[1-\rho V^2(a_{11}-a_{12})]}\left[\left[\frac{\beta_1^2+\rho V^2(a_{22}-a_{12})}{(\beta_2^2-\beta_1^2)}\right]+\frac{A_1}{A_1-B_1}\right]$$

$$-\frac{iYA_1}{2\pi(A_1-B_1)[1-\rho V^2(a_{11}-a_{12})]}\log\frac{z_2\sqrt{a^2-\ell^2}-i\ell\sqrt{z_2^2-a^2}}{z_2\sqrt{a^2-\ell^2}+i\ell\sqrt{z_2^2-a^2}}$$

$$+\frac{YA_1}{2(A_1-B_1)[1-\rho V^2(a_{11}-a_{12})]}+\left[-\frac{YA_1\alpha}{2\pi(A_1-B_1)[1-\rho V^2(a_{11}-a_{12})]}\right.$$

$$\left.+\frac{TA_1}{2(A_1-B_1)[1-\rho V^2(a_{11}-a_{12})]}\right]\frac{z_2}{(z_2^2-a^2)^{1/2}} \qquad (3.3.66)$$

The final term in each of the equations (3.3.66) is singular at $z = a$. According to the Dugdale hypothesis these singularities cannot exist and therefore they must vanish. We therefore have,

$$\alpha = \frac{\pi T}{Y} \qquad (3.3.67)$$

or substituting in (3.3.61) we have,

$$\cos \frac{\pi T}{2Y} = \ell/a \tag{3.3.68}$$

which is the same relation we arrived at in both the static isotropic case and the static anisotropic case.

Equation (3.3.66) reduces to,

$$\phi'(z_1) = \frac{T}{2[1-\rho V^2(a_{11}-a_{12})]}\left[\frac{\beta_2^{\,2}+\rho V^2(a_{22}-a_{12})}{(\beta_2^{\,2}-\beta_1^{\,2})} + \frac{B_1}{A_1 - B_1}\right]$$

$$+ \frac{YB_1}{2\pi(A_1-B_1)[1-\rho V^2(a_{11}-a_{12})]}\left[-\pi + i\log\frac{z_1\sqrt{a^2-\ell^2} - i\ell\sqrt{z_1^{\,2}-a^2}}{z_1\sqrt{a^2-\ell^2} + i\ell\sqrt{z_1^{\,2}-a^2}}\right]$$

$$\psi'(z_2) = -\frac{T}{2[1-\rho V^2(a_{11}-a_{12})]}\left[\frac{\beta_1^{\,2}+\rho V^2(a_{22}-a_{12})}{(\beta_2^{\,2}-\beta_1^{\,2})} + \frac{A_1}{A_1 - B_1}\right] \tag{3.3.69}$$

$$- \frac{YA_1}{2\pi(A_1-B_1)[1-\rho V^2(a_{11}-a_{12})]}\left[-\pi + i\log\frac{z_2\sqrt{a^2-\ell^2} - i\ell\sqrt{z_2^{\,2}-a^2}}{z_2\sqrt{a^2-\ell^2} + i\ell\sqrt{z_2^{\,2}-a^2}}\right].$$

Substituting (3.3.69) into (3.3.30) and simplifying we have,

$$\sigma_{yy} = \frac{2Y}{\pi}\tan^{-1}\frac{\eta}{\ell}\sqrt{\frac{a^2-\ell^2}{\eta^2-a^2}}, \qquad |\eta| > a \tag{3.3.70}$$

and substituting it into (3.3.31) we have,

$$\sigma_{xx} = \mathrm{Re}\left[\left[\frac{B_1(\beta_1^{\,2}+\rho V^2(a_{22}-a_{12})) - A_1(\beta_2^{\,2}+\rho V^2(a_{22}-a_{12}))}{[1-\rho V^2(a_{11}-a_{12})]\ (A_1-B_1)}\right]\right.$$

$$\left.\left[-\frac{Y}{\pi}\left[-\pi + i\log\frac{\eta\sqrt{a^2-\ell^2}-i\ell\sqrt{\eta^2-a^2}}{\eta\sqrt{a^2-\ell^2}+i\ell\sqrt{\eta^2-a^2}}\right] - T\right]\right] \tag{3.3.71}$$

$$= \left[\frac{(B_1\beta_1^{\,2}-A_1\beta_2^{\,2})}{[1-\rho V^2(a_{11}-a_{12})]\ (A_1-B_1)} - \frac{\rho V^2(a_{22}-a_{12})}{[1-\rho V^2(a_{11}-a_{12})]}\right][\sigma_{yy} - T]$$

For the isotropic dynamic case we would have,

$$\begin{aligned} a_{11} &= a_{22} = \frac{1}{E} \\ a_{12} &= -\frac{\nu}{E} \\ a_{66} &= \frac{1}{G} \\ \beta_1^{\,2} &= 1 - \frac{\rho V^2(1-\nu)}{2G} \\ \beta_2^{\,2} &= 1 - \frac{\rho V^2}{G} \end{aligned} \tag{3.3.72}$$

Equations (3.3.70) and (3.3.71) become with some manipulation

$$\sigma_{yy} = \frac{2Y}{\pi}\tan^{-1}\frac{\eta}{\ell}\sqrt{\frac{a^2-\ell^2}{\eta^2-a^2}}\,, \qquad |\eta| > a \tag{3.3.73}$$

$$\sigma_{xx} = \left[ \frac{2(\beta_1^{\ 2}-\beta_2^{\ 2})(1+\beta_2^{\ 2})}{4\beta_1\beta_2 - (1+\beta_2^{\ 2})^2} - 1 \right] [\sigma_{yy} - T] \tag{3.3.74}$$

which are the same as the results obtained by Kanninen [1968].

The coefficient of (3.3.74) is shown plotted in figure 3.12 versus various crack speeds for $\nu = 0.3$. It can be seen that since the coefficient becomes infinite when $V = C_R$, the velocity of Rayleigh surface waves, $\sigma_{xx}$ becomes infinite as V approaches $C_R$ (see Kanninen [1968]).

For the orthotropic static case we would have,

$$V = 0 \, , \tag{3.3.75}$$

$\beta_1$ and $\beta_2$ would be the same as given by Gonzalez [1968]. Equations (3.3.70) and (3.3.71) become with manipulation,

$$\sigma_{yy} = \frac{2Y}{\pi} \tan^{-1} \left[ \frac{\sin \frac{\pi T}{Y}}{\xi^2 - \cos \frac{\pi T}{Y}} \right] \tag{3.3.76}$$

$$\sigma_{xx} = \frac{2Y}{\pi} \beta_1\beta_2 \tan^{-1} \left[ \frac{\sin \frac{\pi T}{Y}}{\xi^2 - \cos \frac{\pi T}{Y}} \right] \tag{3.3.77}$$

where

$$\xi + \frac{1}{\xi} = \frac{2x}{a} \tag{3.3.78}$$

which are the same as the results obtained by Gonzalez [1968].

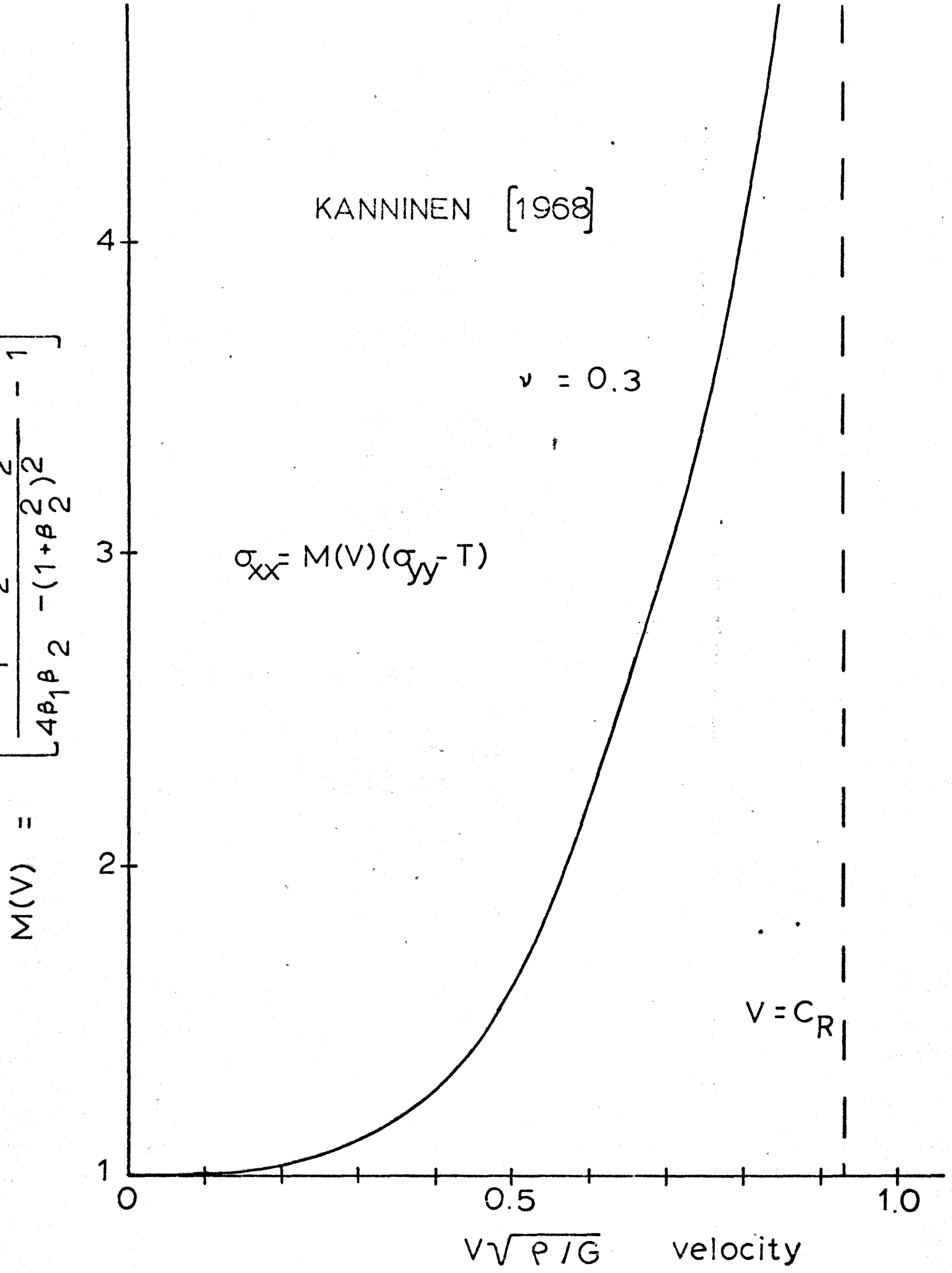



Figure 3.12
COEFFICIENT FOR MOVING ISOTROPIC DUGDALE CRACK

The coefficient of equation (3.3.71) for the orthotropic dynamic case is shown in Figure 3.13. It should be noted that when the velocity of the crack is zero the coefficient simply reduces to that for the static orthotropic case. Also when the quantity,

$$A_1 - B_1 = o \tag{3.3.79}$$

it may be seen that the coefficient becomes infinite and therefore the $\sigma_{xx}$ stress becomes infinite. Equation (3.3.79) may also be written as

$$2a_{12} + \frac{a_{22}}{\beta_1\beta_2} - (a_{11}a_{22} - a_{12}^{\ 2})\,\rho V^2 \left[\frac{1}{\beta_1\beta_2} + 1\right]$$

$$- a_{11}\,(\beta_1^{\ 2} + \beta_1\beta_2 + \beta_2^{\ 2}) = o \tag{3.3.80}$$

For given $a_{ij}$, equation (3.3.80) may be solved for V and this V = $V_\infty$, would be the ultimate speed a crack moving across an orthotropic plate may have.

## 3.4 Yield Criteria

Up to now the material in the plastic zone has been assumed to be perfectly plastic and in our solution we have assumed that the zone could be removed and replaced by a constant stress on the boundary of the zone. We have made no mention of what this constant stress is other than it is related in an arbitrary way to a yield criterion. The relation between the yield criterion and the material properties of the plate will now be investigated.

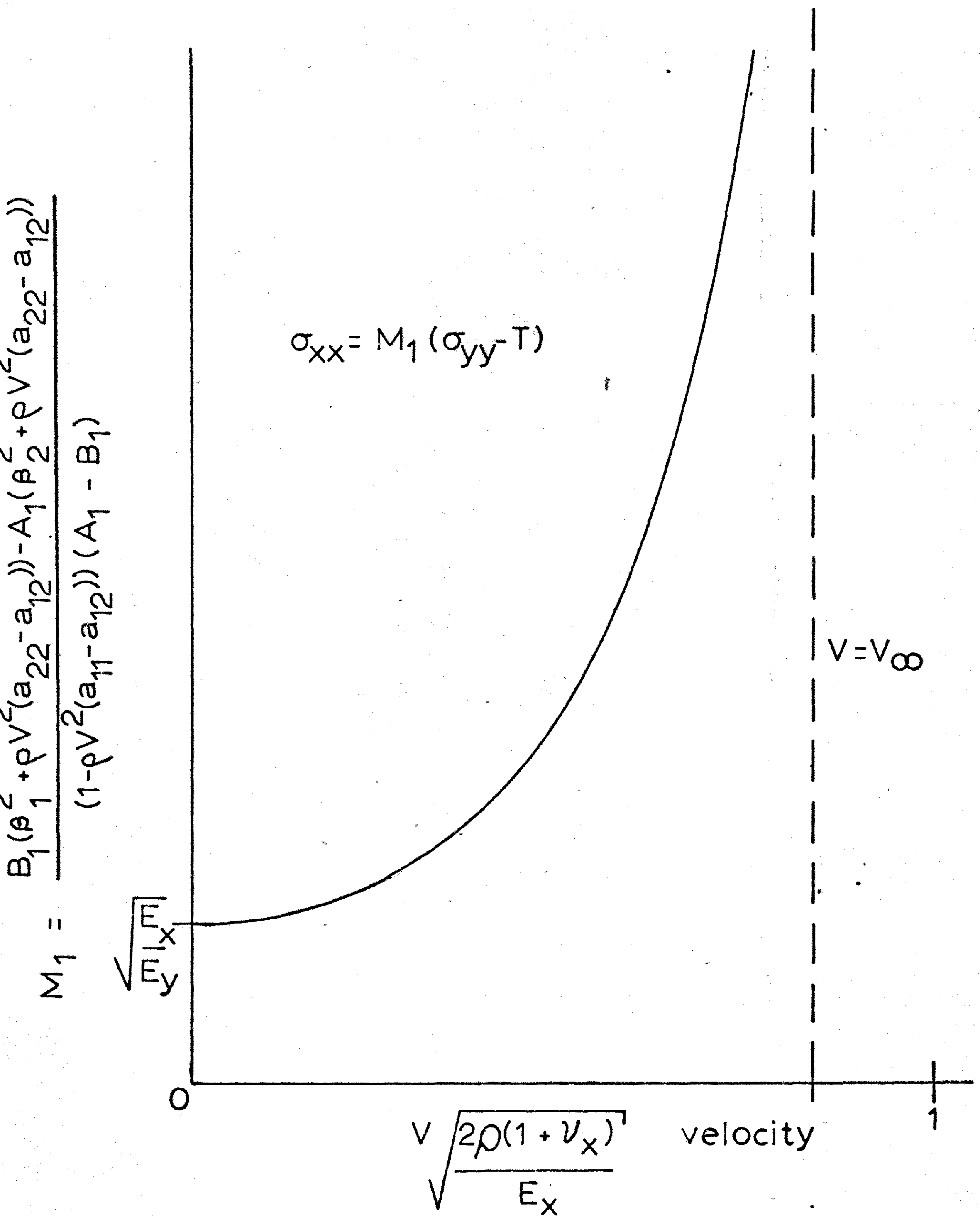



Figure 3.13

COEFFICIENT FOR MOVING ORTHOTROPIC DUGDALE CRACK

The conditions for deciding which combination of multi-axial stresses will cause yielding are called yield criteria. Numerous criteria have been proposed for the yielding of solids going as far back as Coulomb in 1773. Many of these were originally suggested as criterion for failure of brittle materials and were later adopted as yield criteria for ductile materials. One of the inadequacies of these theories is that they are not applicable for materials with different tensile strengths in various directions.

The yield criterion of Von Mises has been shown to be in excellent agreement with experiment for many isotropic ductile materials, for example copper, nickel, aluminum, iron, cold-worked mild steel, medium carbon and alloy steels. For an isotropic material the Von Mises criterion is,

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_{yp}^2 \qquad (3.4.1)$$

where $\sigma_{yp}$ is the yield stress in simple tension.

Equation (3.4.1) can be modified to provide for materials which have different simple tensile yield strengths in different directions as follows, Marin [1957],

$$[K_a(\sigma_1-\sigma_2)]^2 + [K_b(\sigma_2-\sigma_3)]^2 + (K_c(\sigma_3-\sigma_1)]^2 = 2\sigma^2 \qquad (3.4.2)$$

where $K_a$, $K_b$, and $K_c$ are constants to be determined.

For two-dimensional stresses, $\sigma_3$ = o and (3.4.2) becomes,

$$a\sigma_1^2 + b^2\sigma_1\sigma_2 + c^2\sigma_2^2 = 1 \qquad (3.4.3)$$

where,

$$a^2 = (K_a^{\ 2}+K_c^{\ 2})/2\sigma^2 \ , \quad b^2 = -\ K_a^{\ 2}/\sigma^2 \ , \quad c^2 = (K_a^{\ 2}+K_b^{\ 2})/2\sigma^2 \qquad (3.4.4)$$

the values of a, b, c can be obtained from three simple strength properties of the material. For the case when the values a, b, c are found from the simple tensile yield strengths in the $\sigma_1$ and $\sigma_2$ directions and from the simple torsional yield strength equation (3.4.3) becomes,

$$\frac{\sigma_1^{\ 2}}{T_1^{\ 2}} + \left[\frac{1}{T_1^{\ 2}} + \frac{1}{T_2^{\ 2}} - \frac{1}{T_3^{\ 2}}\right] \sigma_1\sigma_2 + \frac{\sigma_2^{\ 2}}{T_2^{\ 2}} = 1 \qquad (3.4.5)$$

where now $\sigma_1$, $\sigma_2$ are the principle stresses, $T_1$, $T_2$ are the yield strengths in these directions and $T_3$ is the torsional yield strength.

As a load is applied to a structure, the stresses at any point in the material increase numerically and remain proportional to one another so that they can be expressed in terms of one of them.

Considering the yield criterion along the line of the crack we have for the static case,

$$\sigma_{xx} = \frac{2Y(\beta_1\beta_2-\alpha_1\alpha_2)}{\pi} \tan^{-1} \frac{\sin 2\theta_2}{\xi^2 - \cos 2\theta_2} \qquad (3.4.6)$$

and

$$\sigma_{yy} = \frac{2Y}{\pi} \tan^{-1}\left[\frac{\sin 2\theta_2}{\xi^2 - \cos 2\theta_2}\right] + T \qquad (3.4.7)$$

and for the dynamic orthotropic case,

$$\sigma_{xx} = \mathrm{Re}\left[\frac{\beta_1(\beta_1{}^2 + \rho V^2(a_{22}-a_{12})) - A_1(\beta_2{}^2 + \rho V^2(a_{22}-a_{12}))}{(1-\rho V^2(a_{11}-a_{12})]\ (A_1-B_1)}\right.$$

$$\left.\left[-\frac{Y}{\pi}\left[-\pi + i\log\frac{\eta\ a^2-\ell^2 - i\ell\ \eta^2-a^2}{\eta\ a^2-\ell^2 + i\ell\ \eta^2-a^2}\right] - T\right]\right] \tag{3.4.8}$$

and

$$\sigma_{yy} = \frac{2Y}{\pi}\ \tan^{-1}\frac{\eta}{\ell}\sqrt{\frac{a^2-\ell^2}{\eta^2-a^2}} \tag{3.4.9}$$

The relation between the stresses $\sigma_{xx}$ and $\sigma_{yy}$ is therefore in the static case,

$$\sigma_{xx} = (\beta_1\beta_2 - \alpha_1\alpha_2)(\sigma_{yy} - T) \tag{3.4.10}$$

and in the dynamic case is,

$$\sigma_{xx} = \left[\frac{(B_1\beta_1{}^2 - A_1\beta_2{}^2)}{[1-\rho V^2(a_{11}-a_{12})](A_1-B_1)} - \frac{\rho V^2(a_{22}-a_{12})}{[1-\rho V^2(a_{11}-a_{12})]}\right][\sigma_{yy} - T] \tag{3.4.11}$$

Substituting these relations into equation (3.4.5) we find in the static case,

$$\sigma_{yy} = -\frac{d_1 + \sqrt{d_1{}^2 - 4e_1{}^2}}{2} \tag{3.4.12}$$

and in the dynamic case,

$$\sigma_{yy} = -\frac{d_2 + \sqrt{d_2{}^2 - 4e_2{}^2}}{2} \tag{3.4.13}$$

where $d_1$, $d_2$, $e_1$, $e_2$ are constants evaluated by the substitution.

If we therefore set,

$$Y_{STATIC} = -\frac{d_1 + \sqrt{d_1^2 - 4e_1^2}}{2} \qquad a)$$

$$Y_{DYNAMIC} = -\frac{d_2 + \sqrt{d_2^2 - 4e_2^2}}{2} \qquad b) \qquad (3.4.14)$$

the stress distribution will follow a Von Mises yield criterion for plane stress conditions.

Because the coefficients of $(\sigma_{yy} - T)$ in equations (3.4.10,11) can vary depending on the state of anisotropy and also on the velocity of propagation, the Von Mises conditions continue to be satisfied as long as $\sigma_{xx} < \sigma_{yy}$. Yoffe [1951] supposes that the crack will always propagate in the direction normal to the maximum tensile stress. If $\sigma_{xx}$ becomes greater than $\sigma_{yy}$, $\sigma_{xx}$ becomes the maximum principal stress. The material then tends to yield in a direction normal to the crack line Kanninen [1968]. Hence the value of the coefficients of equations (3.4.10,11) at which $\sigma_{xx} = \sigma_{yy}$ must be the greatest value for which the present analysis is valid. Then from equations (3.4.10,11) this greatest value is determined by,

$$(\beta_1\beta_2 - \alpha_1\alpha_2)(Y_S - T) = Y_S \qquad (3.4.15)$$

in the static case and,

$$\left[\frac{(\beta_1^2 B_1 - A_1\beta_2^2)}{[1-\rho V^2(a_{11}-a_{12})](A_1-B_1)} - \frac{\rho V^2(a_{22}-a_{12})}{[1-\rho V^2(a_{11}-a_{12})]}\right](Y_D - T) = Y_D \qquad (3.4.16)$$

in the dynamic case.

Therefore our analysis is valid only if,

$$(\beta_1\beta_2 - \alpha_1\alpha_2) < \frac{Y_S}{Y_S-T} \tag{3.4.17}$$

or

$$\frac{T}{Y_S} > \frac{\beta_1\beta_2 - \alpha_1\alpha_2 - 1}{\beta_1\beta_2 - \alpha_1\alpha_2} \tag{3.4.18}$$

in the static case and,

$$\frac{(\beta_1{}^2B_1 - A_1\beta_2{}^2)}{(1-\rho V^2(a_{11}-a_{12})](A_1-B_1)} - \frac{\rho V^2(a_{22}-a_{12})}{[1-\rho V^2(a_{11}-a_{12})]} < \frac{Y_D}{Y_D-T} \tag{3.4.19}$$

or

$$\frac{T}{Y_D} > 1 - \cfrac{1}{\cfrac{(\beta_1{}^2B_1 - A_1\beta_2{}^2)}{[1-\rho V^2(a_{11}-a_{12})](A_1-B_1)} - \cfrac{\rho V^2(a_{22}-a_{12})}{[1-\rho V^2(a_{11}-a_{12})]}} \tag{3.4.20}$$

in the dynamic case.

It should be noted that in both the static and dynamic case at the point where $\sigma_{xx} = \sigma_{yy}$ the stress is generally uniform over a wide area in front of the crack and if this is the case and the maximum normal stress propagation theory is used the crack is likely to propagate in any direction.

For the orthotropic-static case we have $\alpha_1 = \alpha_2 = o$ therefore the limit on our analysis for the orthotropic static case would be,

$$\beta_1\beta_2 = \sqrt{\frac{E_x}{E_y}} < \frac{1}{1-T/Y_S} . \tag{3.4.21}$$

For a given material, $Y_S$ would be given by equation (3.4.14a). For increasing values of T, therefore, yielding will take place along the line of the crack as long as equation (3.4.21) is satisfied. The limiting value of $\beta_1\beta_2$ is shown in Figure 3.14.

For the isotropic dynamic case the limit on our analysis would be,

$$\frac{2(\beta_1^{\,2} - \beta_2^{\,2})(1 + \beta_2^{\,2})}{4\beta_1\beta_2 - (1 + \beta_2^{\,2})^2} - 1 < \frac{1}{1 - \frac{T}{Y_D}} \tag{3.4.22}$$

For a given material, $Y_D$ would be given by equation (3.4.14b) and $\beta_1$, $\beta_2$ are given by (3.3.72). For given T, $Y_D$, $\rho$, G, $\nu$ therefore a value of V can be found for which the analysis is valid if the velocity of the crack is below this value.

This must therefore be the greatest speed at which a ductile crack can propagate. A plot of $V_{MAX}$ versus $T/Y_D$ for various values of $\nu$ is shown in Figure (3.15) (Kanninen [1968]).

For the orthotropic-dynamic problem the effect of both the anisotropy and the velocity of the crack need to be considered in arriving at a limit of the analysis. If we consider a material whose orthotropy is such that equation (3.4.21) is satisfied we can find limiting Dugdale crack speeds for values of $T/Y_D$. This is shown in Figure 3.16, where it is noted that at a value of $T/Y_D = 1$ the velocity is that which was calculated from equation (3.3.80).

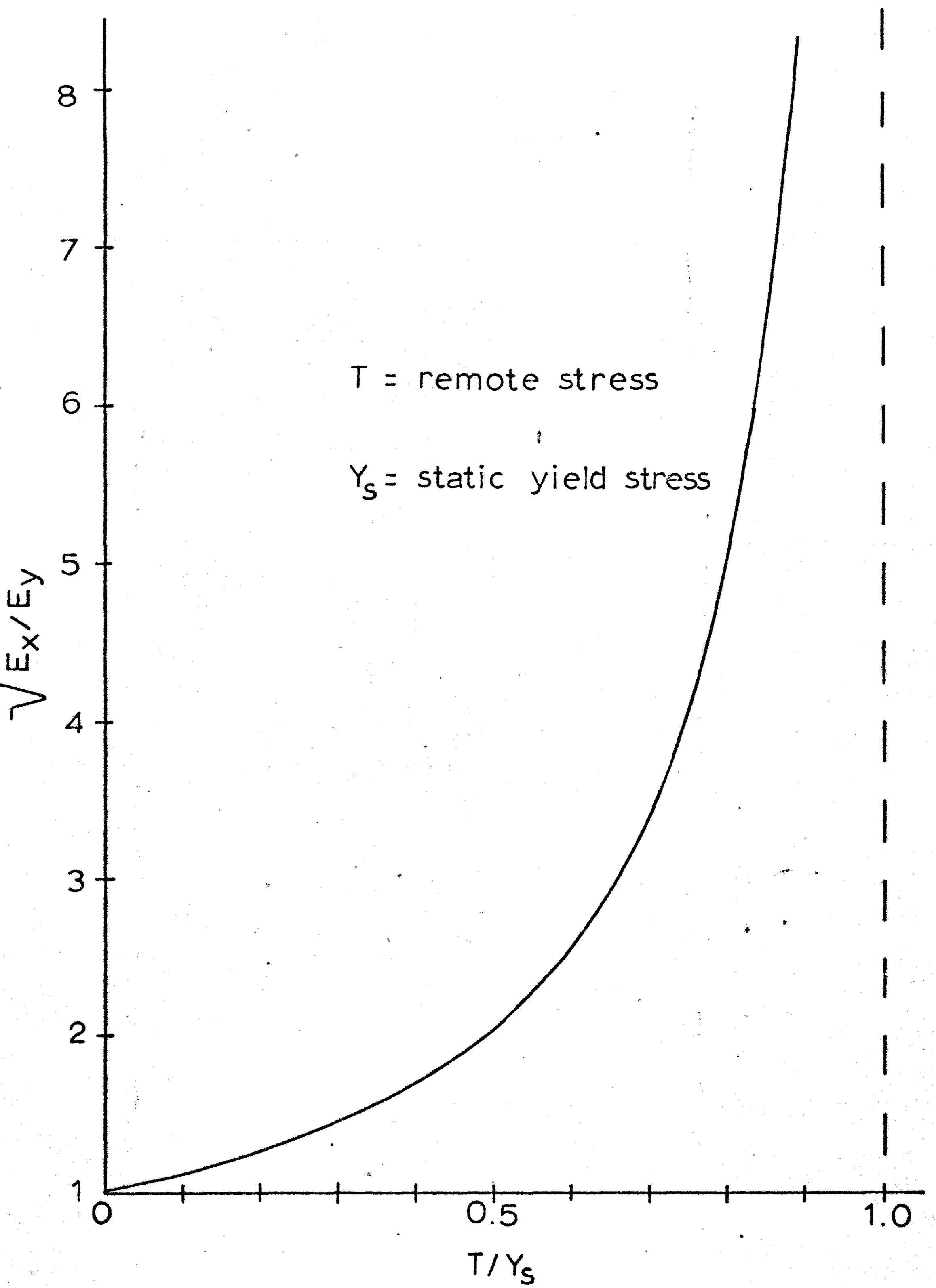



Figure 3.14
LIMIT ON ORTHOTROPY FOR ORTHOTROPIC DUGDALE CRACK

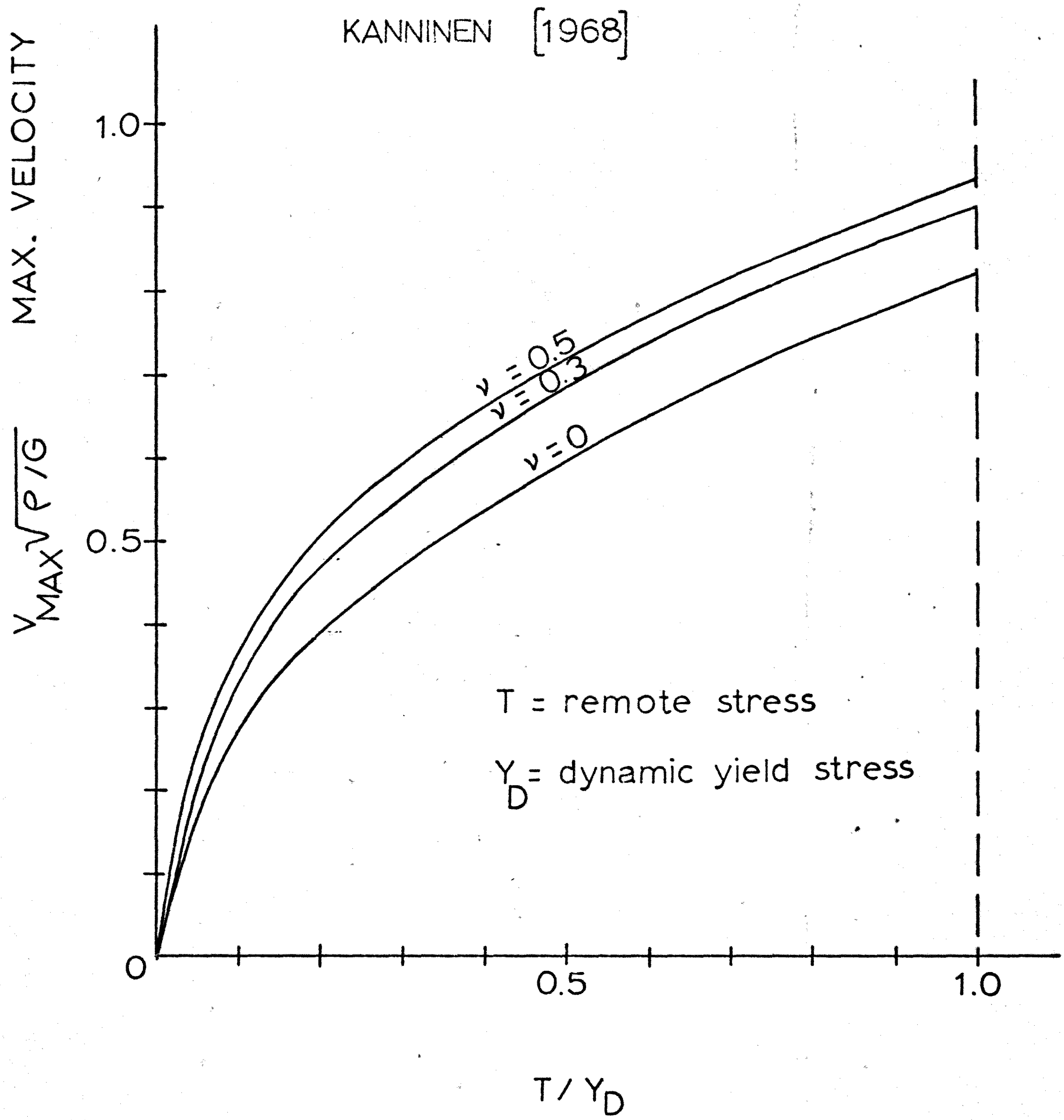



Figure 3.15

LIMITING DUGDALE CRACK SPEED
FOR ISOTROPIC MATERIAL

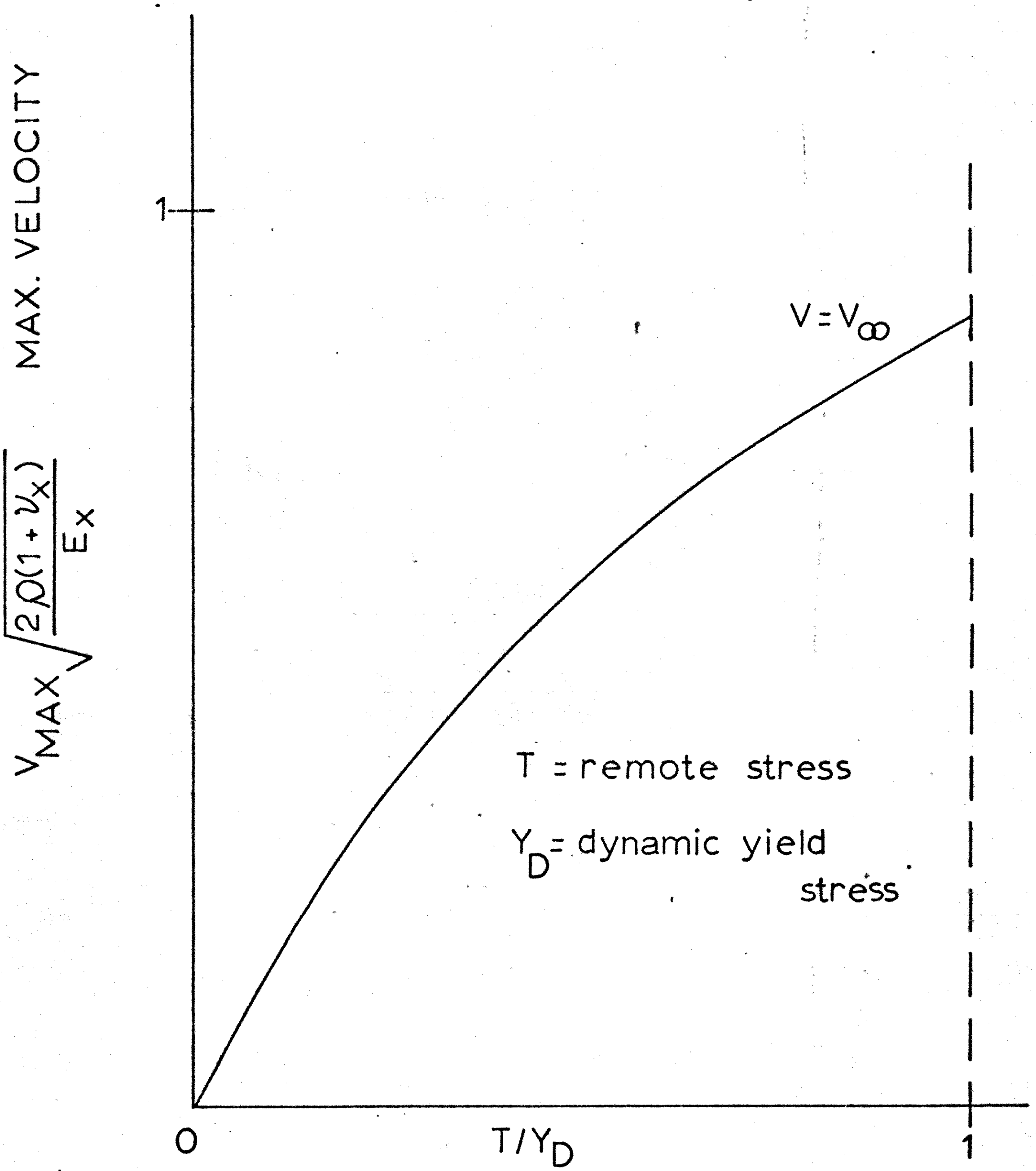



Figure 3.16

LIMITING DUGDALE CRACK SPEED FOR ORTHOTROPIC MATERIAL

## 3.5 General Anisotropic - Isotropic Relations

At this point it seems proper to look at the solutions for the static and dynamic cases which we have found and see the form of these solutions. Looking at these various solutions (i.e. equations (3.2.42), (3.2.44), (3.3.70) and (3.3.71)) we see that they all may be written as,

$$f_A = \alpha f_I \tag{3.5.1}$$

where $f_A$ is an anisotropic function, $f_I$ is the corresponding isotropic function and $\alpha$ is a coefficient which contains elastic constants. Thus an anisotropic function may be obtained from its corresponding isotropic function if the coefficient $\alpha$ for the particular problem is known. This is illustrated in Figure (3.17) where an anisotropic function is $f_A$ is plotted versus an anisotropic coefficient, $\alpha$ for constant values of the isotropic function, $f_I$.

We also see that a similar conclusion may be arrived at in the dynamic case, that is, a dynamic solution or function may be derived from the corresponding static solution or function by multiplying by some appropriate coefficient.

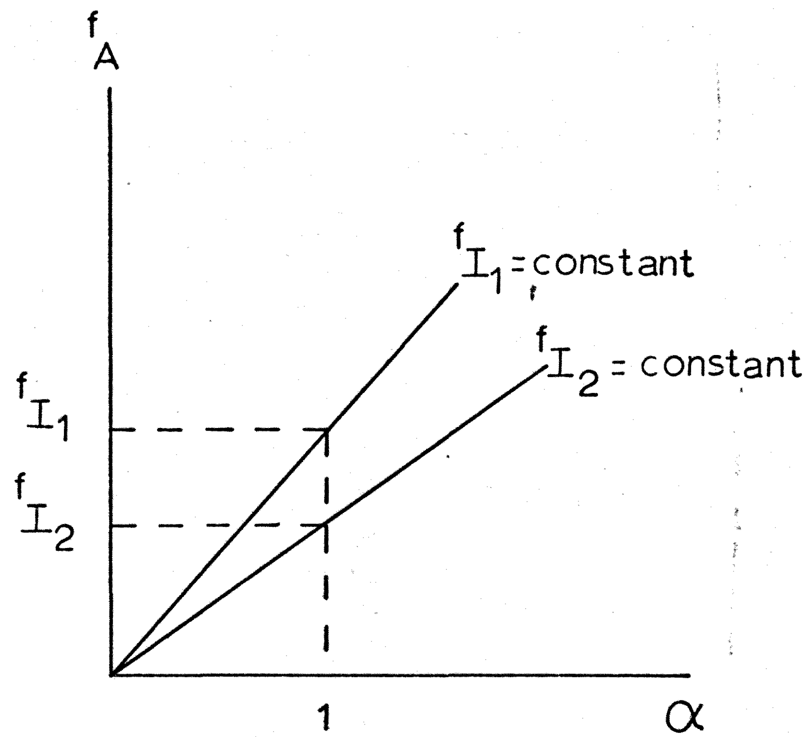



Figure 3.17

GENERAL ANISOTROPIC - ISOTROPIC RELATION

## 4. ANISOTROPIC VISCOELASTIC FRACTURE

In the following, the theory for anisotropic viscoelastic fracture will be applied to certain problems under known loading conditions and with known moduli. The governing equations for anisotropic viscoelastic fracture are more complex than the more familiar isotropic-elastic case and solutions can become extremely complex. Since the emphasis of this paper is to develop a comprehensive theory, the following solutions serve only as guidelines for future investigators who wish to explore a particular anisotropic material. The problems picked are of the simplest type and are not intended to reflect any particular material behavior.

The solution of the Dugdale problem for elastic-anisotropic materials has been found in section 3.2. It is noted in this solution that the size of the plastic zone is given as a function of the external load as is shown by,

$$\frac{\ell}{a} = \cos \left[ \frac{\pi T}{2Y} \right] \tag{4.1}$$

where 2a is the crack length plus plastic zone. It can be seen that if T is applied over a period of time the plastic zone would grow as a function of time (i.e., when T increases a must increase and since the length of the crack remains the same the plastic zone must increase).

The viscoelastic solution cannot be obtained from the elastic solution by the usual correspondence techniques since we are considering a mixed boundary value problem where the regions over which the different types of boundary conditions which are prescribed are time dependent, and therefore this does not allow the evaluation of the required integral transforms with respect to time. The viscoelastic solution has to be obtained from an extension of the correspondence principle if, that is, the problem fits the restriction set forth by these extensions.

For the important case when all the time-dependent quantities are monotonically increasing with time, we will now show that the Dugdale model does indeed meet the conditions (2.3.11).

We first set the Dugdale problem up as the superposition of two separate boundary value problems as in Figure 4.1. (Omitting the constant stress $\sigma_{yy} = T$ at infinity). The boundary conditions for state 1 are,

$$\begin{aligned} &\sigma_{xy}(x,0) = 0 & & \\ &\sigma_{yy}(x,0) = -T & &|x| < a(t) \\ &v(x,0) = 0 & &|x| > a(t) \end{aligned} \tag{4.2}$$

for state 2 they are,

$$\begin{aligned} &\sigma_{xy}(x,0) = 0 & & \\ &\sigma_{yy}(x,0) = Y\,H(|x| - \ell)^{*} & &|x| < a(t) \\ &v(x,0) = 0 & &|x| > a(t) \end{aligned} \tag{4.3}$$

---

* $H(|x| - \ell) = 1 \quad |x| > \ell$

$\qquad\qquad\quad 0 \quad |x| < \ell$

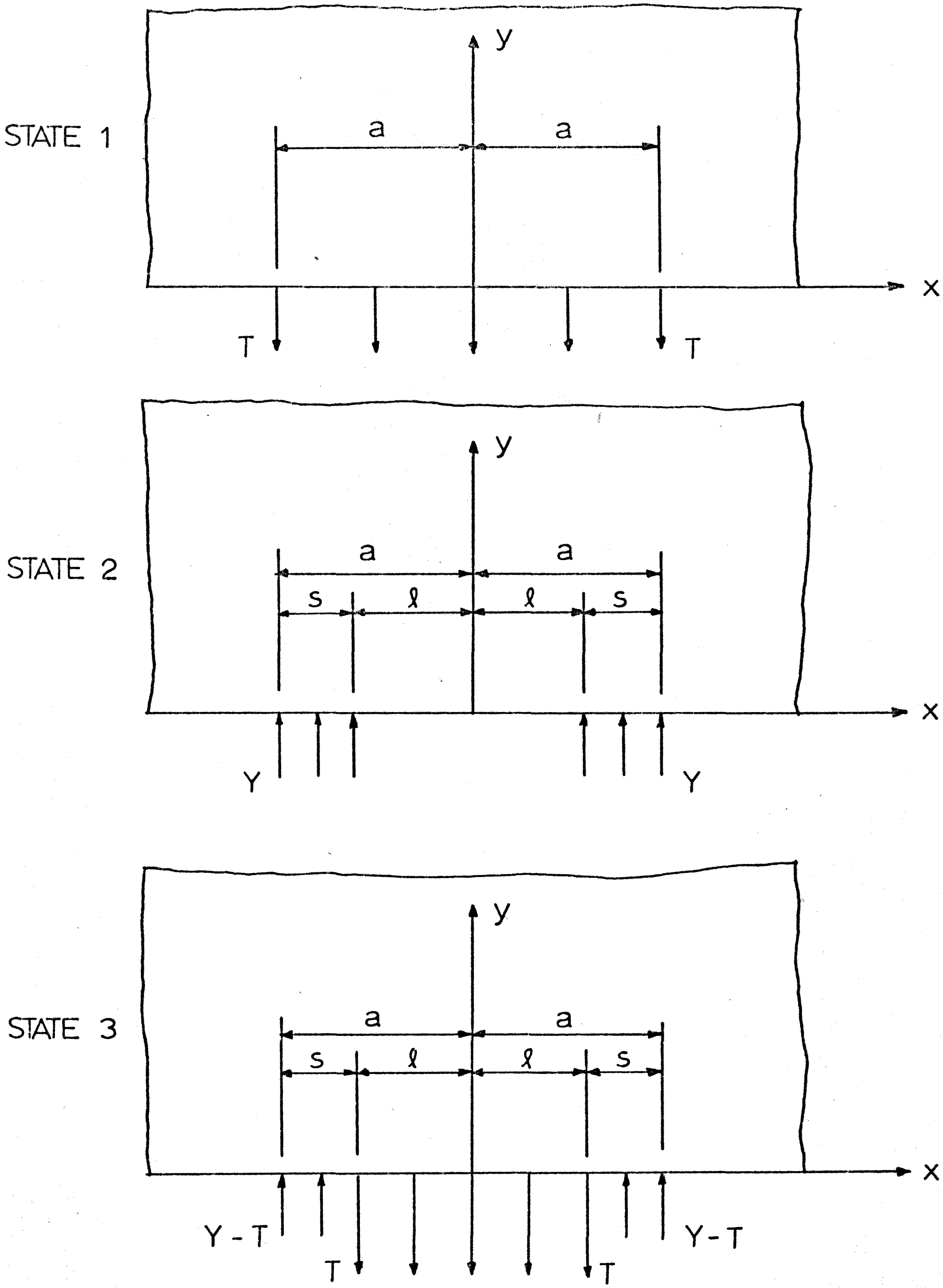



Figure 4.1
BOUNDARY CONDITIONS FOR DUGDALE CRACK

and for the superposition of states 1 and 2, i.e. state 3 they are. (including the constant stress at infinity),

$$\sigma_{xy}(x,0) = 0$$

$$\sigma_{yy}(x,0) = Y\, H(|x| - \ell) \qquad |x| < a(t) \tag{4.4}$$

$$v(x,0) = 0 \qquad |x| > a(t)$$

We compare (4.4) with (2.3.8) and we see that they are of the same form. Thus with reference to Graham's method we see that a solution is valid only if in the solution to the Dugdale problem we have (in accordance with (2.3.11),)

$$\begin{aligned} \sigma_{yy}(x,0) &= f(x,t)^{*} \qquad && |x| > a(t) \\ v(x,0) &= k(B_{ij\kappa\ell})\, g(x,t) \qquad && |x| < a(t) \end{aligned} \tag{4.5}$$

Specifying apriori that the crack is monotonically increasing in length with time, we see from (3.2.42) and (3.2.44) that Graham's requirements indeed are met and thus a solution by extended correspondence is possible.

It should be mentioned that in both the static and dynamic cases the $\sigma_{yy}$ stress along the x-axis is independent of elastic constants and therefore in the viscoelastic case the $\sigma_{yy}$ stress would be the same as in the elastic case.

---

* $f(x,t)$ and $g(x,t)$ are functions of the geometry and time only, $k(B_{ij\kappa\ell})$ is a function of the material properties only.

In the next sections, we will solve for the static $\sigma_{xx}$ stress in an orthotropic material for several cases of viscoelastic material properties. As previously stated solutions of this nature are simple but they do illustrate the procedures which must be used in the solution of more difficult problems.

From (3.2.42) we see that the $\sigma_{xx}$ stress is,

$$\sigma_{xx} = \frac{2Y}{\pi} (\beta_1\beta_2 - \alpha_1\alpha_2) \tan^{-1}\left( \frac{\sin 2\theta_2}{\xi^2 - \cos 2\theta_2} \right) \tag{4.6}$$

and for the orthotropic case we have (i.e., $\alpha_1 = \alpha_2 = 0$)

$$\sigma_{xx} = \frac{2Y}{\pi} \beta_1\beta_2 \tan^{-1} \left( \frac{\sin 2\ _2}{\xi^2 - \cos 2\theta_2} \right) \tag{4.7}$$

Substituting in for $\beta_1$, $\beta_2$ for the orthotropic static case we have,

$$\sigma_{xx} = \frac{2Y}{\pi} \sqrt{\frac{E_x}{E_y}} \tan^{-1} \left( \frac{\sin 2\theta_2}{\xi^2 - \cos 2\theta_2} \right) = k_1 \frac{2Y}{\pi} \tan^{-1} \left( \frac{\sin 2\theta_2}{\xi^2 - \cos 2\theta_2} \right) \tag{4.8}$$

now taking the Laplace transform of (4.8) according to (2.3.15) we have,

$$\bar{\sigma}_{xx} = k_1 \quad \pounds \left[ \frac{2Y}{\pi} \tan^{-1}\left( \frac{\sin 2\theta_2}{\xi^2 - \cos 2\theta_2} \right) \right] \tag{4.9}$$

$$= k_1 \bar{S}_x$$

where $k_1$ may be considered either $k_1(C_{ij\kappa\ell})$ or $k_1(B_{ij\kappa\ell})$. In accordance with (2.3.17) the viscoelastic $\sigma_{xx}$ stress is therefore,

$$\sigma_{xx}(x,t) = K_1(t)S_x(x,0) + \int_o^t K_1(t-\tau) \frac{\partial S_x(x,\tau)}{\partial \tau} d\tau \tag{4.10}$$

where

$$K_1(t) = \mathcal{L}^{-1} [\frac{1}{p} k_1(p\, \bar{G}_{ij\kappa\ell})] \quad \text{if} \quad k_1 = k_1(C_{ij\kappa\ell}) \tag{4.11}$$

or

$$K_1(t) = \mathcal{L}^{-1} [\frac{1}{p} k_1(p\, \bar{J}_{ij\kappa\ell})] \quad \text{if} \quad k_1 = k_1(b_{ij\kappa\ell}), \tag{4.12}$$

and

$$S_x(x,t) = \frac{2Y}{\pi} \tan^{-1} \left( \frac{\sin \frac{\pi T}{Y}}{\xi^2 - \cos \frac{\pi T}{Y}} \right) \tag{4.13}$$

For the case when the external stress is held constant, equation (4.10) can be written as,

$$\sigma_{xx}(x,t) = K_1(t)\, S_x(x) \tag{4.14}$$

Thus we see that the time dependence and the spatial dependence of the $\sigma_{xx}$ can be separated for the case of constant external load. The spatial dependence of $\sigma_{xx}$ is given in Figure 3.9, while the time dependence, $K_1(t)$, depends on the particular viscoelastic properties chosen for the material.

For the specific problem we are solving we have,

$$k_1 = \beta_1\beta_2 = \sqrt{\frac{E_x}{E_y}} \tag{4.15}$$

and

$$\sqrt{\frac{E_x}{E_y}} = \sqrt{\frac{B_{2222}}{B_{1111}}} \tag{4.16}$$

and therefore

$$k_1(p\bar{J}_{ij\kappa\ell}) = \sqrt{\frac{\bar{J}_{2222}}{\bar{J}_{1111}}} \tag{4.17}$$

thus

$$K_1(t) = \mathcal{L}^{-1}\left[\frac{1}{p}\sqrt{\frac{\bar{J}_{2222}}{\bar{J}_{1111}}}\right] \tag{4.18}$$

From (2.5.18,19) the following useful inverse matrix relation may be derived,

$$\frac{1}{p^2}\left|\bar{J}_{ij\kappa\ell}(\vec{x},p)\right| = \left|\bar{G}_{ij\kappa\ell}(\vec{x},p)\right|^{-1} \tag{4.19}$$

therefore for the case of orthotropic plane stress conditions equation (4.18) may be written as,

$$K_1(t) = \mathcal{L}^{-1}\left[\frac{1}{p}\sqrt{\frac{\bar{J}_{2222}}{\bar{J}_{1111}}}\right] = \mathcal{L}^{-1}\left[\frac{1}{p}\sqrt{\frac{\bar{G}_{1111}}{\bar{G}_{2222}}}\right] \tag{4.20}$$

It should be pointed out that equation (4.20) would not be true for the general anisotropic case.

It only remains to choose the proper relaxation moduli or creep compliance to solve for $\sigma_{xx}$.

In the following examples both Maxwell and Kelvin behavior will be assumed.

Maxwell behavior is represented by a spring and dashpot in series, where the modulus of the spring is E and the viscosity of the dashpot is $\eta$. The relaxation modulus for a Maxwell material is,

$$G = Ee^{-\lambda t} \tag{4.21}$$

where

$$\lambda = E/\eta \quad . \tag{4.22}$$

Kelvin behavior is represented by a spring and dashpot in parallel, where the modulus of the spring is E and the viscosity of the dashpot is $\eta$. The creep compliance for a Kelvin material is,

$$J = \frac{1}{E}\,(1 - e^{-\lambda t}) \tag{4.23}$$

where

$$\lambda = E/\eta \quad . \tag{4.24}$$

For our first example, let us consider a material which behaves as a Maxwell material in the direction of the crack and as an elastic material normal to the crack. Therefore

$$G_{1111} = E_1 e^{-\lambda_1 t} \quad ; \quad \bar{G}_{1111} = \frac{E_1}{(p + \lambda_1)} \tag{4.25}$$

$$G_{2222} = E_2 \quad ; \quad \bar{G}_{2222} = \frac{E_2}{P}$$

Equation (4.20) may therefore be written as,

$$K_1(t) = \sqrt{\frac{E_1}{E_2}} \quad \pounds^{-1} \left[ \sqrt{\frac{1}{p(p + \lambda_1)}} \right] \tag{4.26}$$

therefore

$$K_1(t) = \sqrt{\frac{E_1}{E_2}} \quad e^{-\frac{\lambda_1 t}{2}} \quad I_o \left( \frac{\lambda_1 t}{2} \right) \tag{4.27}$$

---

* $I_\nu(t)$ is the modified Bessel function of the first kind, defined by:

$$I_\nu(t) = \sum^{\infty} \frac{(1/2t)^{\nu+2m}}{m! \, \Gamma(\nu + m + 1)}$$

and $\Gamma$ is the gamma function defined by:

$$\Gamma(z) = \int_o^\infty e^{-t} \, t^{z-1} \, dt$$

and equation (4.14) becomes

$$\sigma_{xx}(x,t) = \sqrt{\frac{E_1}{E_2}}\; e^{-\frac{\lambda_1 t}{2}} \quad I_o\left(\frac{\lambda_1 t}{2}\right) S_x(x) \tag{4.28}$$

Equation (4.27) is plotted versus time in Figure 4.2 and equation (4.28) is plotted for T/Y = 1/4, in Figure 4.3.

It can be seen from Figures 4.2 and 4.3 that the $\sigma_{xx}$ stress begins at the elastic value and decays to zero as the time goes to infinity. This means that the stress in the x-direction relaxes as time goes on.

We now consider a material which behaves elastically in the direction of the crack and as a Maxwell material normal to the crack. Therefore,

$$\begin{aligned} G_{1111} &= E_1 \quad ; \quad \bar{G}_{1111} = \frac{E_1}{p} \\ G_{2222} &= E_2\, e^{-\lambda_2 t} \quad ; \quad \bar{G}_{2222} = \frac{E_2}{(p + \lambda_2)} \end{aligned} \tag{4.29}$$

Equation (4.20) may therefore be written as,

$$K_1(t) = \sqrt{\frac{E_1}{E_2}}\;\; \pounds^{-1}\left[\sqrt{\frac{p + \lambda_2}{p^3}}\right] \tag{4.30}$$

therefore,

$$K_1(t) = \sqrt{\frac{E_1}{E_2}}\;\left[\frac{\lambda_2 t}{2}\; e^{-\frac{\lambda_2 t}{2}} \left(I_o\left(\frac{\lambda_2 t}{2}\right) + I_1\left(\frac{\lambda_2 t}{2}\right)\right) + \frac{e^{\frac{\lambda_2 t}{2}}\; I_o\left(\frac{\lambda_2 t}{2}\right)}{2}\right] \tag{4.31}$$

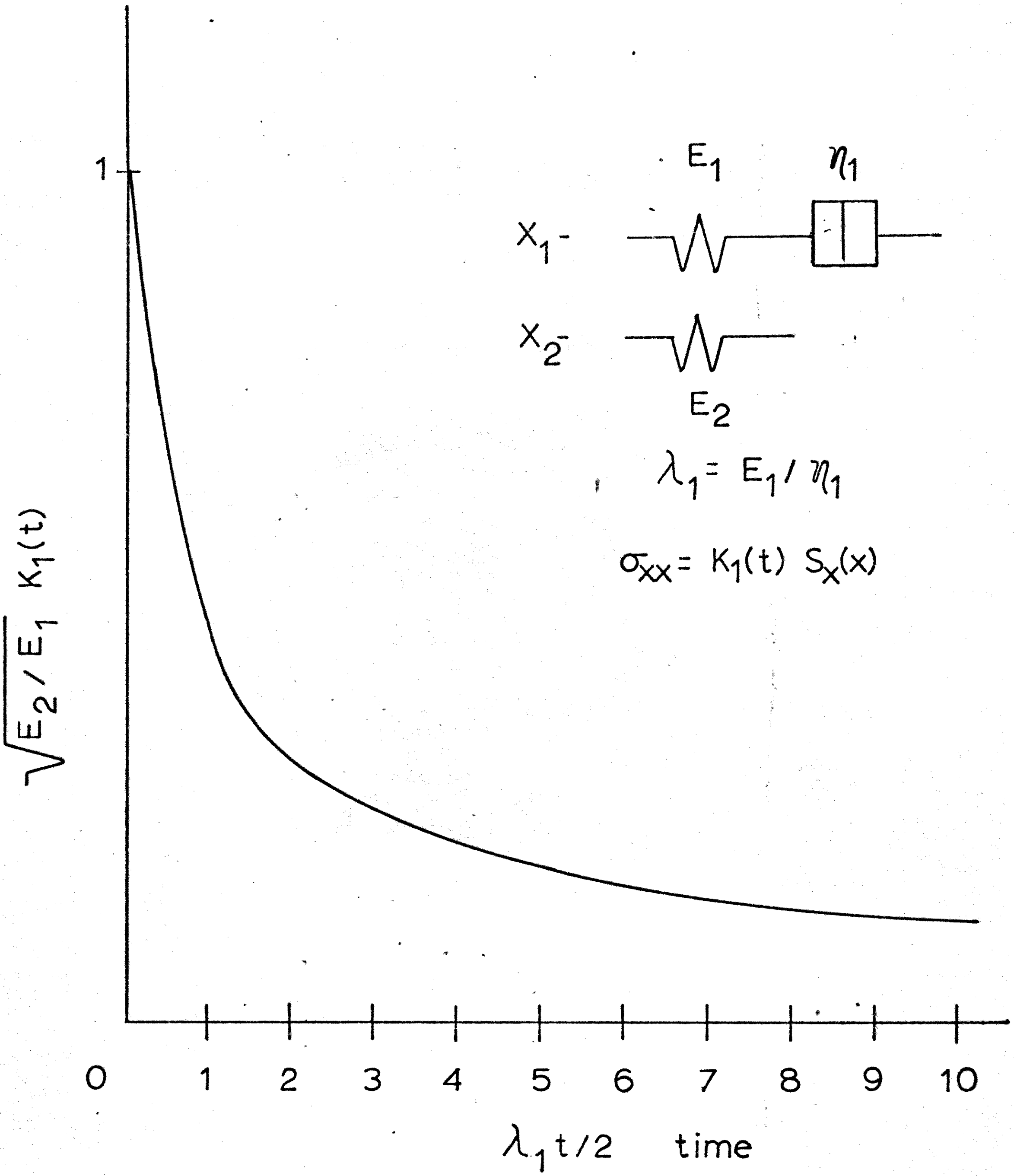



Figure 4.2

$K_1(t)$ FOR MATERIAL MAWELLEAN ALONG CRACK AND ELASTIC NORMAL TO THE CRACK

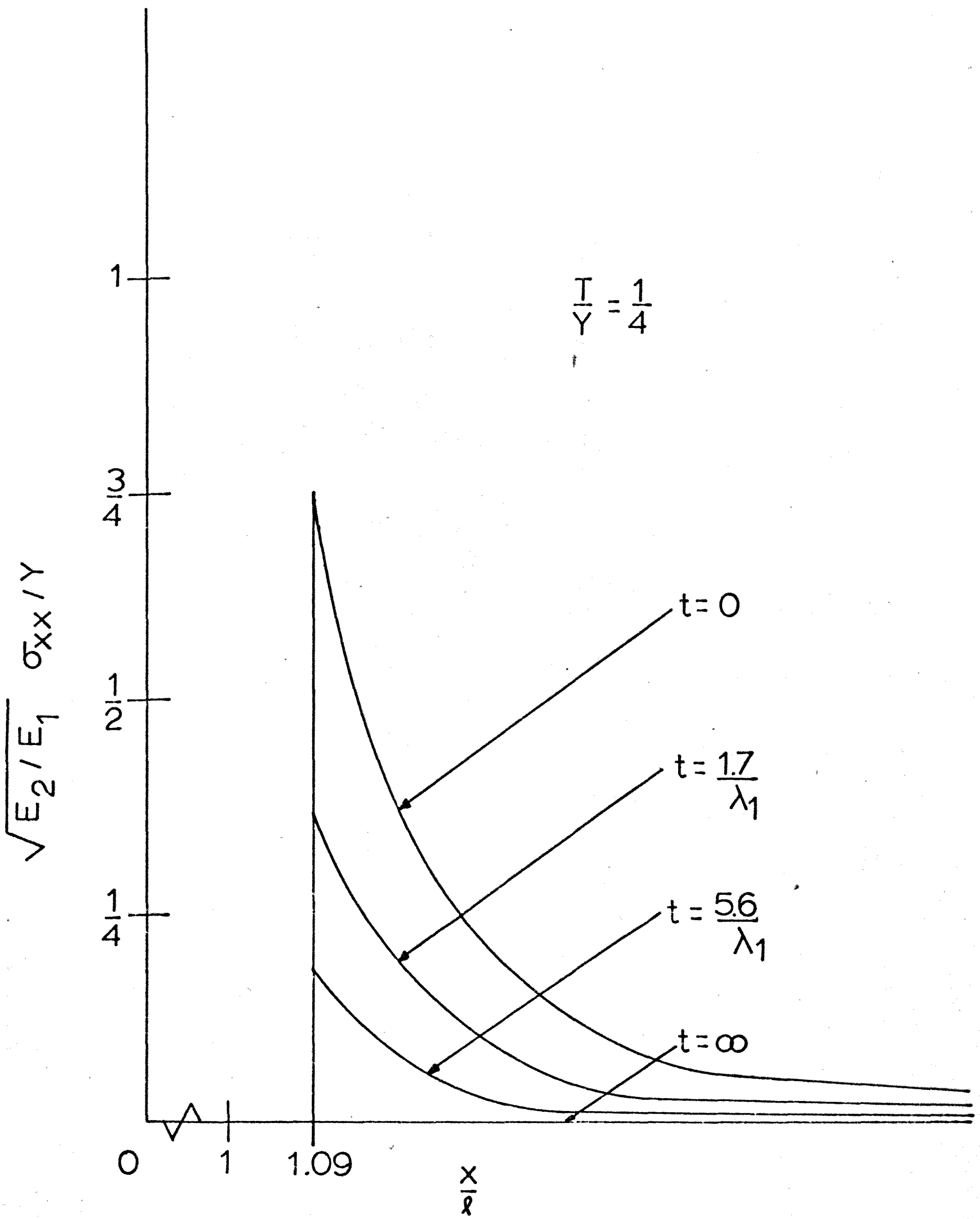



Figure 4.3

TIME VARIATION OF $\sigma_{xx}$, MAXWELL MATERIAL ALONG CRACK and equation (4.14) becomes

$$\sigma_{xx}(x,t) = \sqrt{\frac{E_1}{E_2}}\ \left[\frac{\lambda_2 t}{2}\ e^{-\frac{\lambda_2 t}{2}}\ \left(I_o\left(\frac{\lambda_2 t}{2}\right) + I_1\left(\frac{\lambda_2 t}{2}\right)\right) + \frac{e^{-\frac{\lambda_2 t}{2}}\ I_o\left(\frac{\lambda_2 t}{2}\right)}{2}\right]\ S_x(x) \tag{4.32}$$

Equation (4.31) is plotted versus time in Figure 4.4 and equation (4.32) is plotted for T/Y = 1/4 in Figure 4.5.

From Figures 4.4 and 4.5 we see that the $\sigma_{xx}$ stress begins at a value one half of its elastic value. At a time of $t = 2.6/\lambda_2$ the $\sigma_{xx}$ stress reaches its elastic value while for time approaching infinity the $\sigma_{xx}$ stress would continue to increase to a value of infinity.

Now consider a material which behaves as a Kelvin solid in the direction of the crack and as an elastic material normal to the crack. Therefore

$$\begin{aligned} J_{1111} &= \frac{1}{E_1}\ (1 - e^{-\lambda_1 t}) \quad ; \quad \bar{J}_{1111} = \frac{1}{\eta_1}\ \frac{1}{p(p + \lambda_1)} \\ J_{2222} &= \frac{1}{E_2} \qquad\qquad\qquad ; \quad \bar{J}_{2222} = \frac{1}{pE_2} \end{aligned} \tag{4.33}$$

Equation (4.20) may therefore be written as,

$$K_1(t) = \sqrt{\frac{\eta_1}{E_2}}\qquad \mathcal{L}^{-1}\left[\frac{1}{p}\ \sqrt{p + \lambda_1}\ \right] \tag{4.34}$$

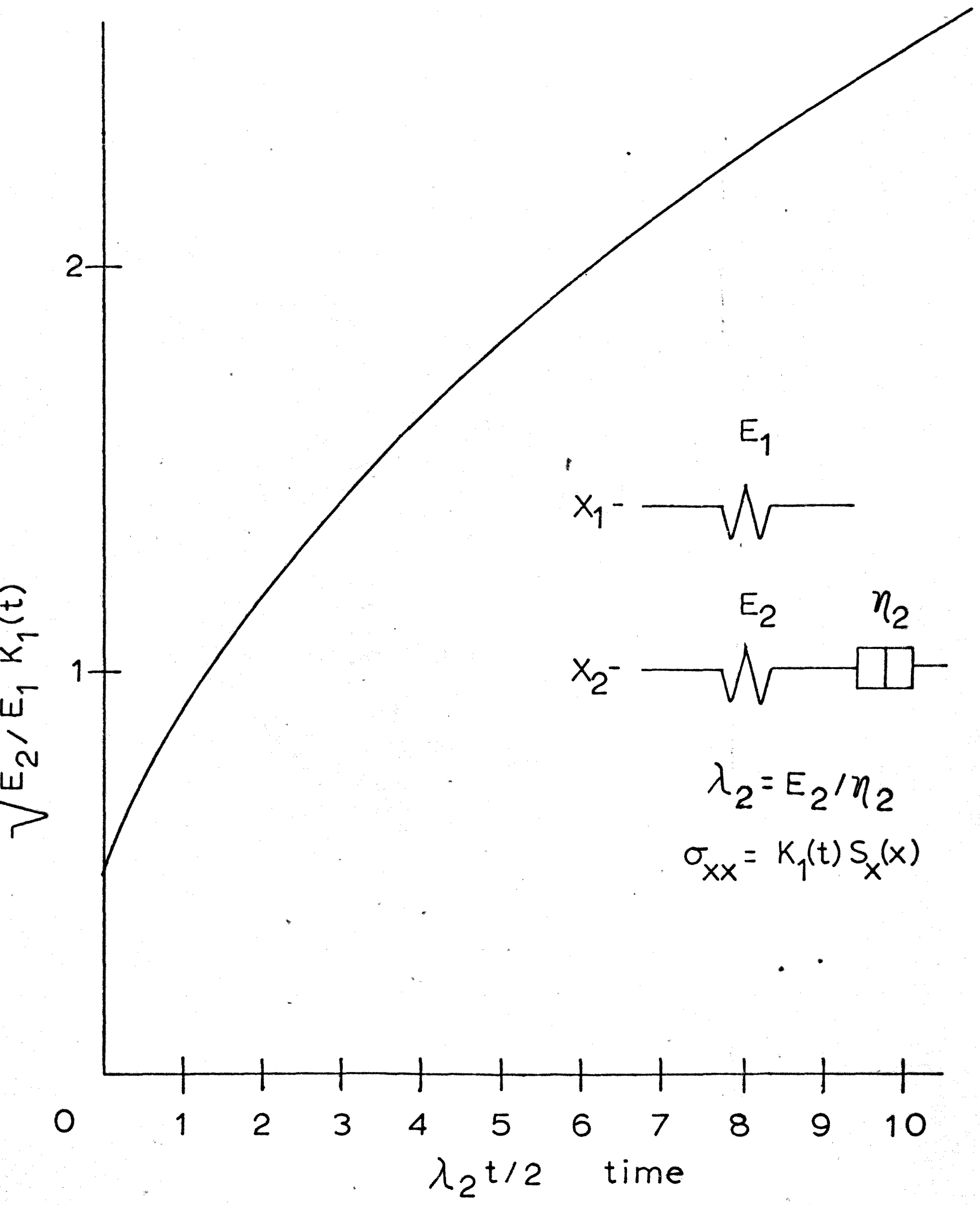



Figure 4.4

$K_1(t)$ FOR MATERIAL ELASTIC ALONG CRACK

AND MAWELLEAN NORMAL TO THE CRACK

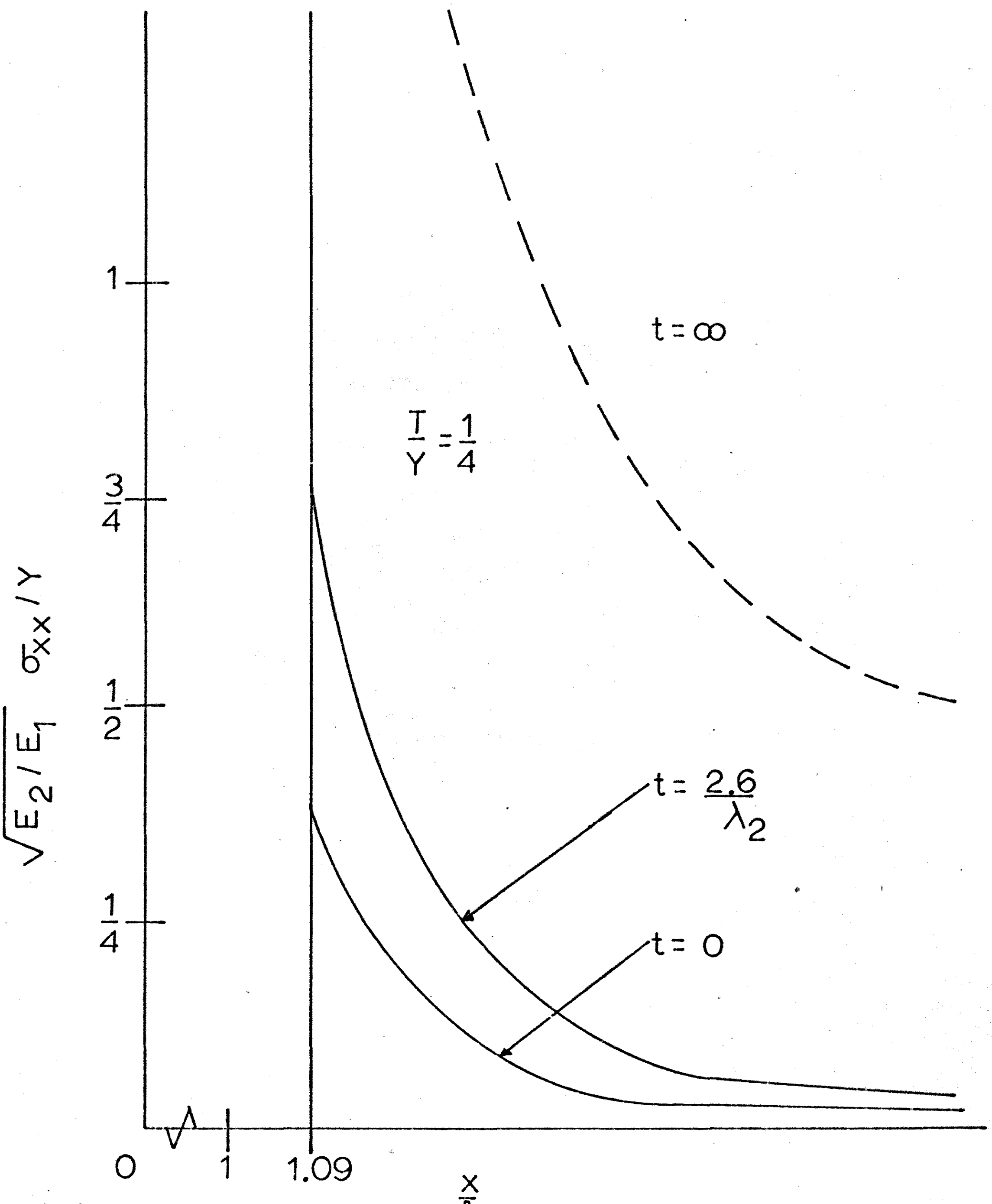



Figure 4.5

TIME VARIATION OF $\sigma_{xx}$, MAXWELL MATERIAL NORMAL TO CRACK therefore,

$$K_1(t) = \sqrt{\frac{E_1}{E_2}} \left[ \frac{e^{-\lambda_1 t}}{\sqrt{\pi}\ \sqrt{\lambda_1 t}} + \text{erf}\ \sqrt{\lambda_1 t} \right] \; * \tag{4.35}$$

and equation (4.14) becomes,

$$\sigma_{xx}(x,t) = \sqrt{\frac{E_1}{E_2}} \left[ \frac{e^{-\lambda_1 t}}{\sqrt{\pi}\ \sqrt{\lambda_1 t}} + \text{erf}\ \sqrt{\lambda_1 t} \right] S_x(x) \tag{4.36}$$

Equation (4.35) is plotted versus time in Figure 4.6 and equation (4.36) is plotted for T/Y = 1/4 in Figure 4.7.

From Figures 4.6 and 4.7 we can see that at $t = 0$ the stress along the x-axis would be infinite. This would seem to indicate that for a plastic zone to exist there would need to be infinite stresses. The material would of course fracture under these conditions. We may therefore conclude that for a material exhibiting the behavior we have assumed, no plastic zone would form before fracture.

We now consider a material which behaves elastically in the direction of the crack and as a Kelvin solid normal to the crack. Therefore,

$$J_{1111} = \frac{1}{E_1} \qquad ; \qquad \bar{J}_{1111} = \frac{1}{pE_1}$$

---

* erf is the error function defined by:

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\, du .$$

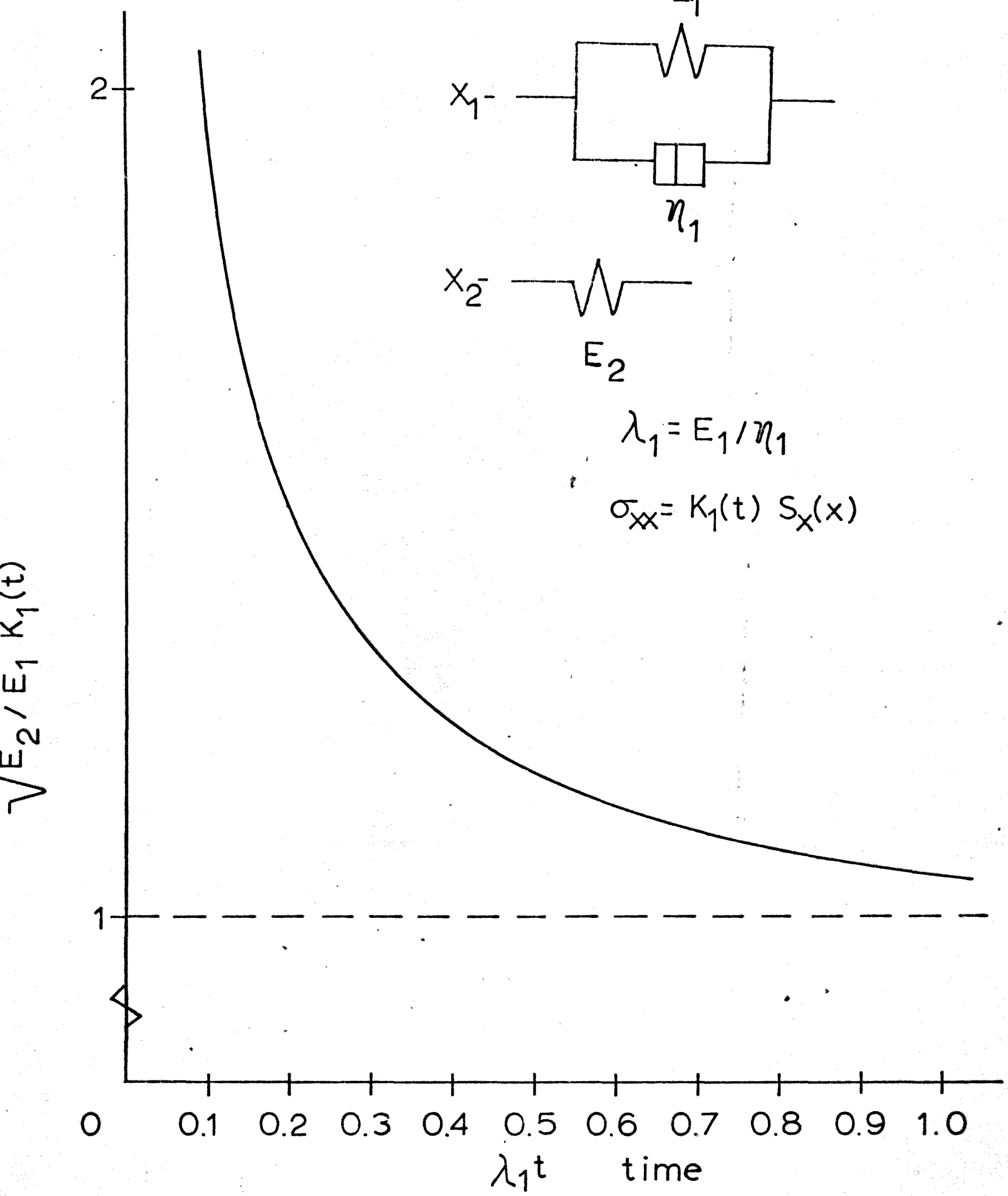



Figure 4.6

$K_1(t)$ FOR MATERIAL KELVIN ALONG CRACK

AND ELASTIC NORMAL TO THE CRACK

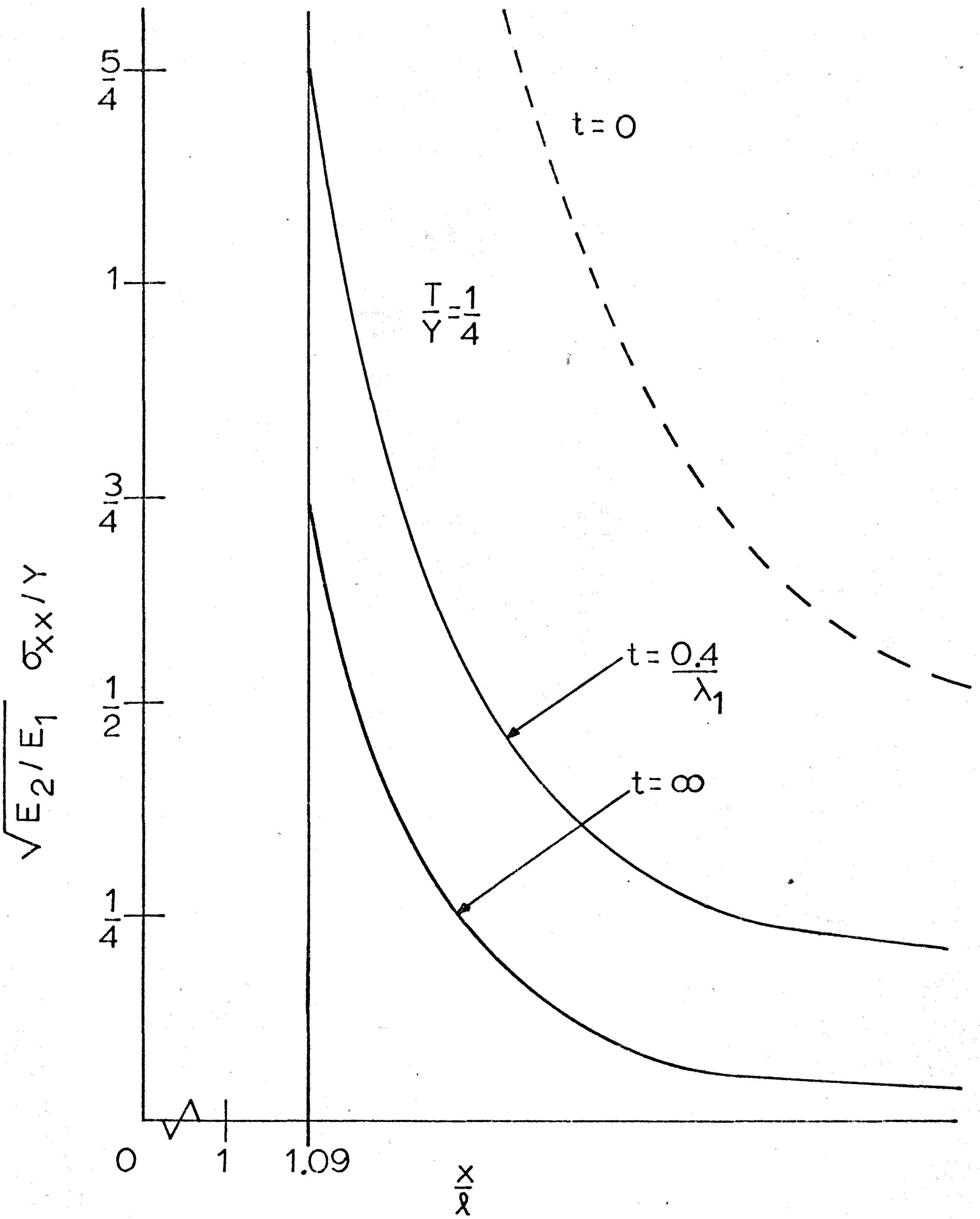



Figure 4.7

TIME VARIATION OF $\sigma_{xx}$, KELVIN MATERIAL ALONG CRACK

$$J_{2222} = \frac{1}{E_2}(1-e^{-\lambda_2 t}) \quad ; \quad \bar{J}_{2222} = \frac{1}{\eta_2} \quad \frac{1}{p(p+\lambda_2)} \tag{4.37}$$

Equation (4.20) can therefore be written as,

$$K_1(t) = \sqrt{\frac{E_1}{\eta_2}} \quad \mathcal{L}^{-1}\left[\frac{1}{p}\sqrt{\frac{1}{p+\lambda_2}}\right] \tag{4.38}$$

therefore,

$$K_1(t) = \sqrt{\frac{E_1}{E_2}} \quad \text{erf} \quad \sqrt{\lambda_2 t} \tag{4.39}$$

and equation (4.14) becomes,

$$\sigma_{xx}(x,t) = \sqrt{\frac{E_1}{E_2}} \quad \text{erf} \quad \sqrt{\lambda_2 t} \quad S_x(x) \tag{4.40}$$

Equation (4.39) is plotted versus time in Figure 4.8 and equation (4.40) is plotted for T/Y = 1/4 in Figure 4.9.

From Figures 4.8 and 4.9 we see that the $\sigma_{xx}$ stress exhibits no instanteous elastic respondence. As time proceeds we see that $\sigma_{xx}$ stress in the limit approaches the elastic value.

From these four examples of elasticity in one direction and visco-elasticity in the other the following conclusions may be made:

1. If the viscoelasticity is in the direction normal to the crack line the relative behavior of the $\sigma_{xx}$ stress is the same as the relative behavior of the model assumed. For example for the Kelvin model in a

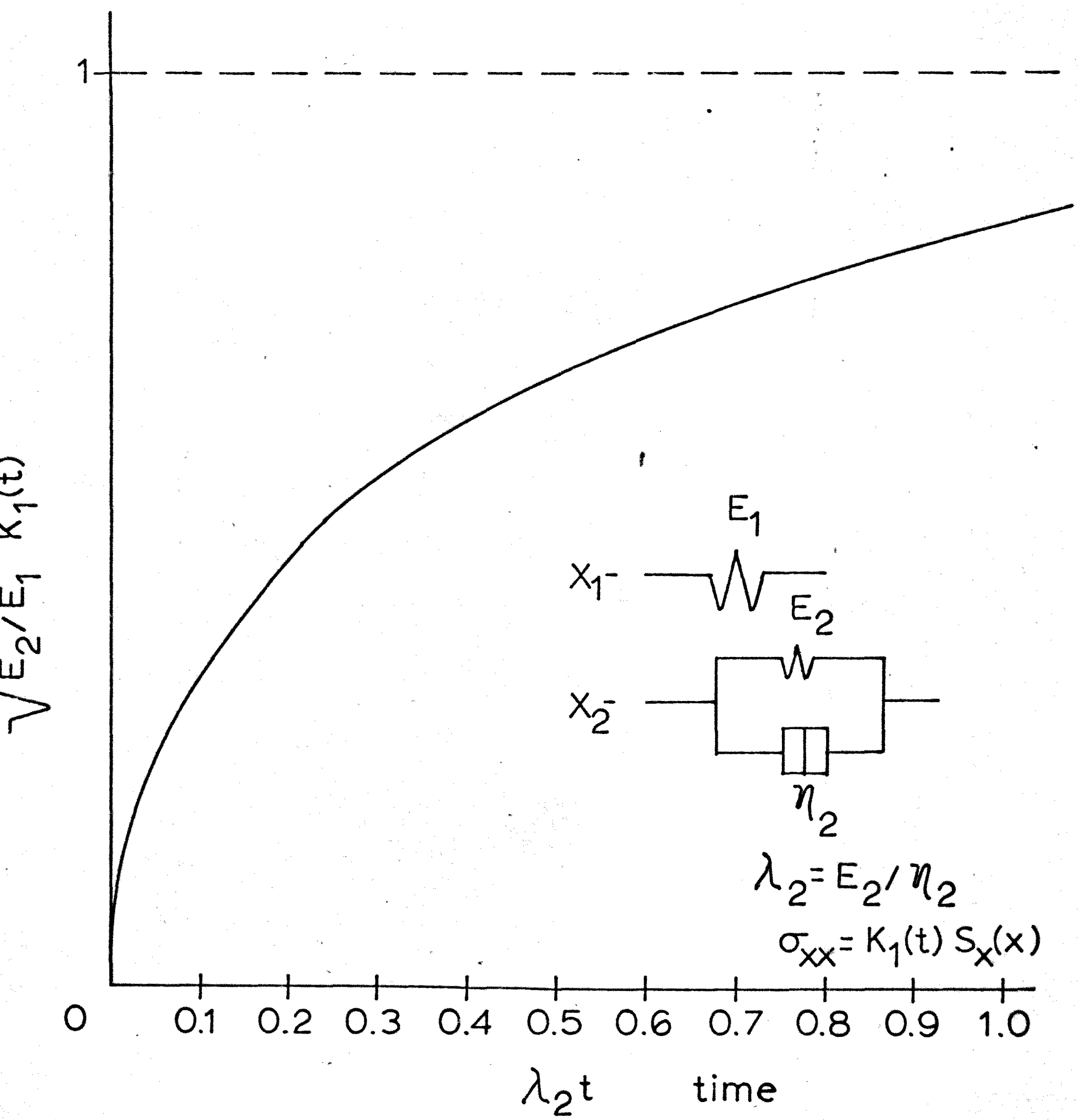



Figure 4.8

$K_1(t)$ FOR MATERIAL ELASTIC ALONG CRACK AND KELVIN NORMAL TO THE CRACK

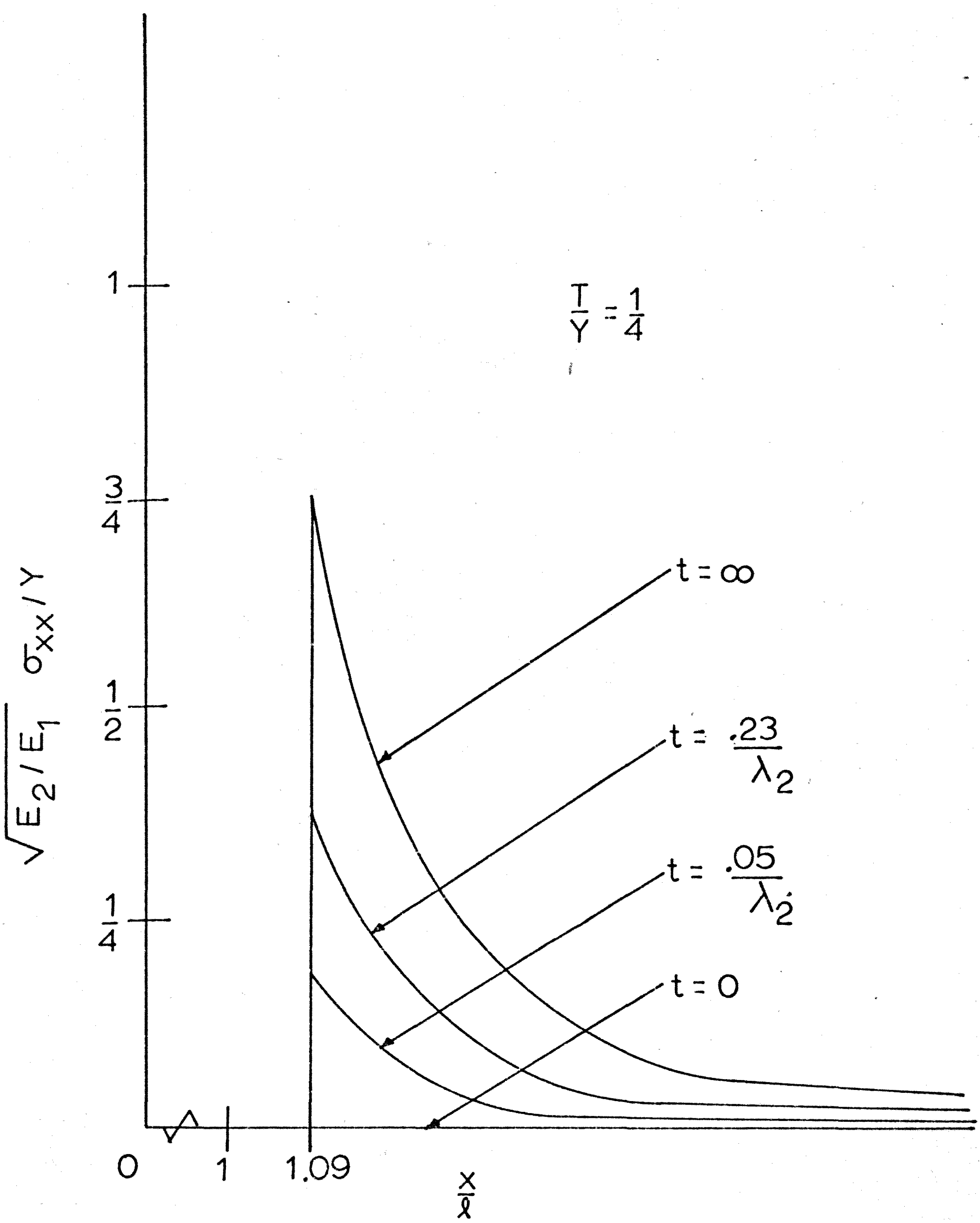



Figure 4.9

TIME VARIATION OF $\sigma_{xx}$, KELVIN MATERIAL NORMAL TO CRACK creep test there would be no initial elastic response but there would be a limiting value and for the Maxwell model there would be an initial elastic response but there would be no limiting value.

2. If the viscoelasticity in the direction of the crack line is characterized by fluid type behavior (Maxwell) the $\sigma_{xx}$ stress decays from its elastic value to zero. If however the viscoelasticity is characterized by solid type behavior (Kelvin) the $\sigma_{xx}$ stress decays from infinity to its elastic value.

The preceding examples seem well suited for composite materials, since they have such different behavior in different directions.

For a homogeneous anisotropic material one might expect to have similar behavior with time in different directions. The behavior may not be as simple as proposed here, i.e. a simple multiple, but might be of a more complex nature, such as Maxwell or Kelvin response in every direction, with different values for E and $\eta$.

For our final example, let us choose,

$$J_{1111} = E(t) \quad ; \quad \bar{J}_{1111} = \bar{E}(p) \tag{4.41}$$

$$J_{2222} = \gamma^2 E(t) \qquad \bar{J}_{2222} = \gamma^2 \bar{E}(p)$$

where E(t) is a characteristic creep compliance and $\gamma^2$ is a constant. It is seen therefore that the creep compliance in one direction is simply a multiple of the creep compliance in the other. Equation (4.20) becomes

$$K_1(t) = \mathcal{L}^{-1}\left[\frac{\gamma}{p}\right] \tag{4.42}$$

therefore

$$K_1(t) = \gamma H(t) \tag{4.43}$$

and equation (4.14) becomes

$$\sigma_{xx}(x,t) = \gamma\, S_x(x)\, H(t) \tag{4.44}$$

Equation (4.43) is shown plotted in Figure 4.10 and equation (4.44) is plotted for T/Y = 1/4 in Figure 4.11.

From Figures 4.10 and 4.11 we see that the $\sigma_{xx}$ stress is a constant times its elastic value. This means that the $\sigma_{xx}$ stress will not change with time. This is not surprising since for the purely isotropic elastic case the $\sigma_{xx}$ stress does not contain any elastic constants and therefore the $\sigma_{xx}$ stress for a isotropic viscoelastic material would be constant. This may be seen in Figures 4.10 and 4.11 and equation (4.41) for $\gamma^2 = 1$.

It should be pointed out, however, that for a more complex behavior (i.e. Kelvin or Maxwell in every direction) the $\sigma_{xx}$ stress would no longer remain constant with time since $K_1(t)$ would no longer be a constant but would be some definite function of time.

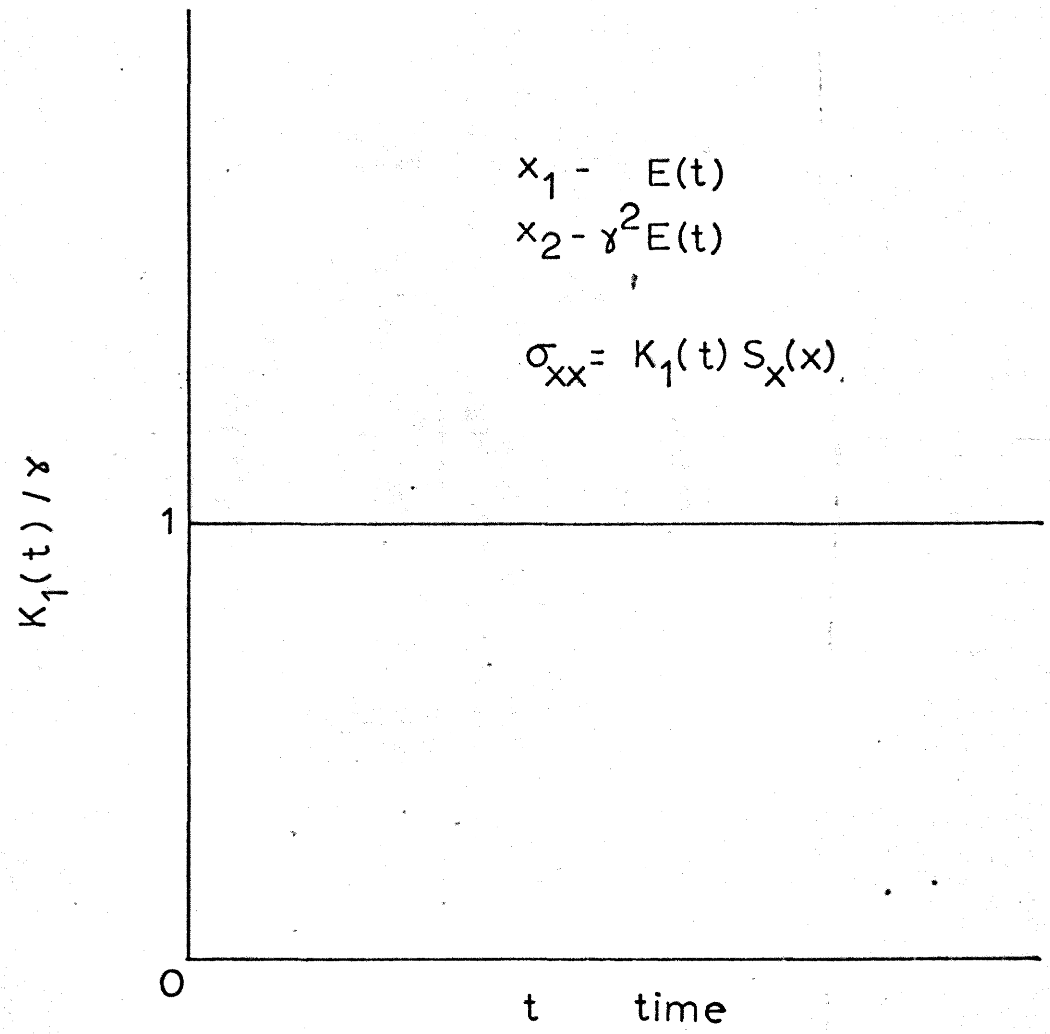



Figure 4.10

$K_1(t)$ FOR MATERIAL SIMILAR IN BOTH DIRECTIONS

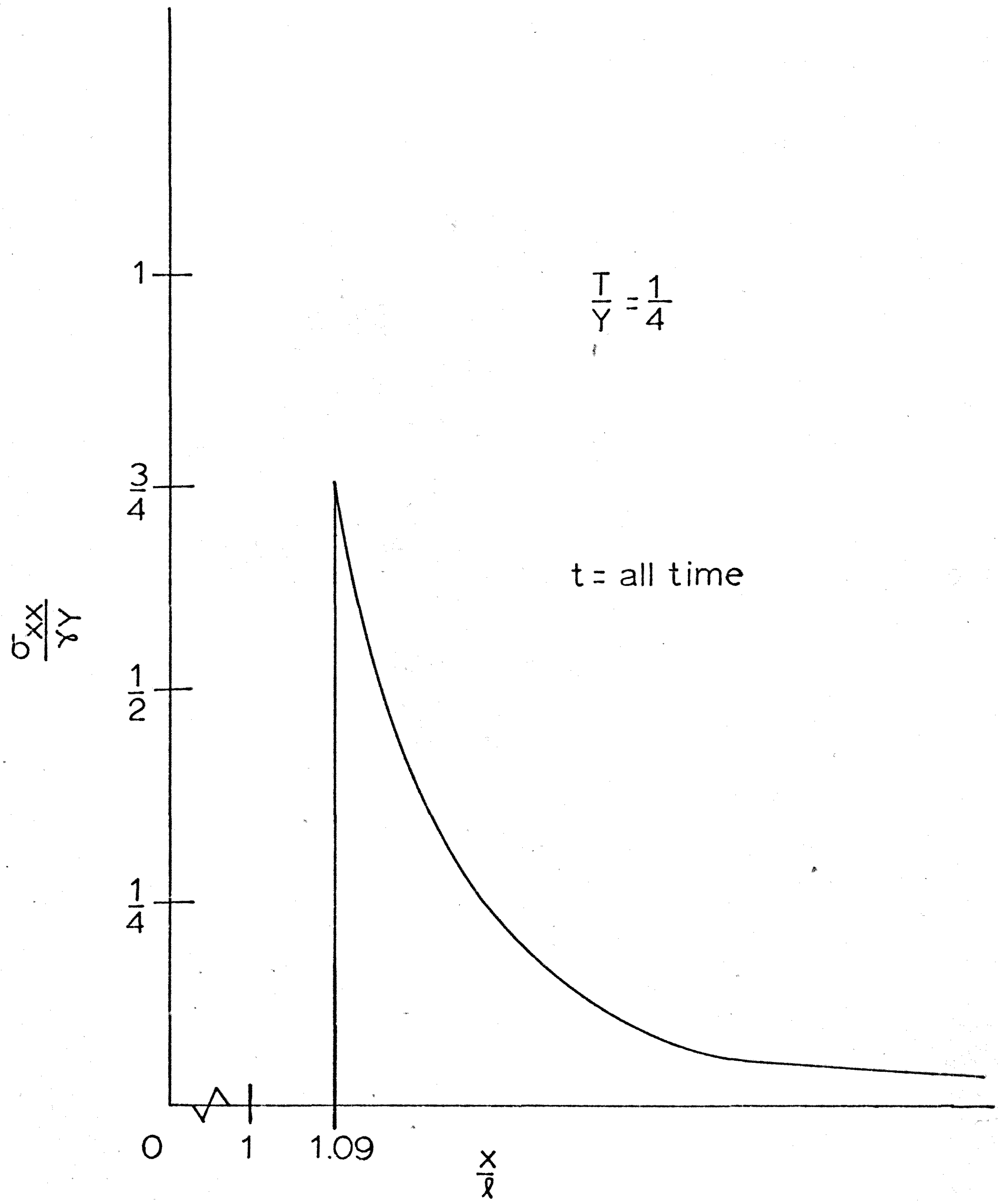



Figure 4.11

TIME VARIATION OF $\sigma_{xx}$, MATERIAL WITH SIMILAR BEHAVIOR

We can conclude then that the only reason the $\sigma_{xx}$ stress varies for an anisotropic problem is due to the anisotropy itself. If the anisotropy were not present, as has been previously stated, the viscoelastic and elastic stress distribution would be the same. This would not be true, however, for other quantities such as displacements, which even in the isotropic case contain elastic constants.

A general analysis as well as individual solutions have shown that for the case of an orthotropic viscoelastic material the $\sigma_{xx}$ stress distribution under constant external load varies with time. Hence, the $\sigma_{xx}$ stress cannot be equated to the solution for the isotropic viscoelastic or elastic $\sigma_{xx}$ stress.

It may be seen from the various solutions obtained that the variation of the $\sigma_{xx}$ stress for the whole time interval $0 < t < \infty$ depends also upon the individual material constants (i.e. $E_i$, $\eta_i$, $\lambda_i$).

Finally it should be mentioned that although the analysis is confined to the case of constant external load, the solution for $\sigma_{xx}$ can also be obtained for any arbitrary external load T, which is an increasing function of time which varies slowly so that the quasi-static nature of the problem is preserved.

## 5. DISCUSSION

The introduction of the complex variable approach into the study of anisotropic and dynamic elasticity has considerably widened the scope for solving specific problems in these fields. In both cases the stress functions have to satisfy a generalized biharmonic equation, where the stress functions can be represented in terms of two analytic functions of two different complex variables and the stress and displacement components may be found in terms of these functions. In this way boundary value problems can be reduced to problems of complex function theory. The anisotropic and dynamic solutions are thus seen to be completely analogous, and similar relations in the solutions of problems may thus be expected. This indeed has been shown to be the case in this paper.

A brief statement on the validity of choosing a constant length crack for the orthotropic dynamic problem is in order. In the orthotropic dynamic solution we have assumed that the crack moving across the plate remains of constant length. This is of course unrealistic. As Yoffe does however, we assume that the stress distribution at one end of the crack does not influence the stress distribution at the other end. We could of course insure this by specifying a semi-infinite crack, but it is preferable to retain the Yoffe crack so we may compare the solution with the static-anisotropic crack.

For the static anisotropic and the orthotropic - dynamic cases the following results are obtained:

(1) The plastic zone is given by the same relation as in the isotropic-static case,

$$\frac{\ell}{a} = \cos\left[\frac{\pi T}{2Y}\right]$$

(2) Any anisotropic function or corresponding dynamic function (ie. $\sigma_{xx}$; $\sigma_{yy}$; u; v; ....) may be obtained from the corresponding isotropic or static function by simply multiplying the isotropic or static function by a suitable coefficient.

(3) A limit on yielding along the line of the crack, and therefore a limit on the anisotropy and the velocity are derived, see section 3.4.

(4) It has been shown in the paper that all the results for the static anisotropic and orthotropic dynamic cases can be reduced to the static isotropic case.

It was mentioned in the introduction that work in the field of viscoelastic stress analysis has been stimulated by the growing use of components made of polymers which may exhibit viscoelasticity in their final state. Since the stress distribution at a given time may be a function of the entire history of loading, it may be necessary to calculate the complete history of the varying stress distribution to assess the adequacy of a design.

To this end we have solved for the complete stress history of the Dugdale model. The use of Graham's extension of the correspondence principle has been shown to be applicable for our particular problem.

The correspondence principle is a convenient means for the solution of anisotropic viscoelastic problems. In a number of cases the solutions of elastic problems may be employed in a simple manner, whereas for more complicated problems approximate inversion methods may have to be used.

In the paper we have made mention of the generalized creep compliance $J_{ijk\ell}$ and the generalized relaxation modulus $G_{ijk\ell}$ . These would of course have to be determined experimentally before any analytic solution may be found. Thus, for example, the $J_{1111}$ and $J_{2222}$ may be determined by creep test in the $x_1$ and $x_2$ directions, where a constant stress is applied in the $x_i$ direction and the variation with time of the strain in the $x_i$ direction is recorded. More complex creep and relaxation functions may also be found experimentally. For example, $J_{1112} = J_{1121} = J_{1211} = J_{2111}$, may be found by applying a constant pure shear stress $\sigma_{12}$ and measuring the resultant strain along the $x_1$ axis as a function of time. Having these time variations we can try curve fitting with the use of viscoelastic models or other standard techniques.

Once we have obtained all the $J_{ijk\ell}$ , the $G_{ijk\ell}$ may be obtained by relaxation tests where a constant strain is applied and the variation of stress with time is recorded, or by use of the inverse matrix relation given by equation (4.19).

Very little research has been conducted on the ductile fracture of polymers and the anisotropic effects on the ductile fracture of metals. It is hoped this work contributes in some way to a better understanding of these processes.

In conclusion it is hoped that it will be possible to verify experimentally, at least within engineering accuracy, the results presented here. An experimental program to go along with the analytical treatment given here would give an even better insight into ductile fracture. The general scope of this paper should prove its usefulness in applications to problems that have previously been ignored because of their complex nature. Finally, a large bibliography is included at the end of this paper to serve those who wish to do further investigation in this area.

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# A STUDY OF ANISOTROPIC AND VISCOELASTIC DUCTILE FRACTURE

by

Henry Gonzalez Jr.

ABSTRACT

The problem of ductile fracture in anisotropic and viscoelastic solids is an important engineering problem. In this paper the Dugdale model is assumed, that is, the yielded zone is replaced by a constant yield stress. Solutions are presented for anisotropic static, orthotropic dynamic and anisotropic viscoelastic solids.

The anisotropic static and orthotropic dynamic solutions were obtained by the complex variable approach. Stress functions which must satisfy a generalized bi-harmonic equation, are represented in terms of two analytic functions of two different complex variables. In this way boundary value problems can be reduced to problems of complex function theory. The anisotropic and dynamic solutions are thus seen to be completely analogous and thus similar relations are obtained for both cases.

The yield stress is assumed to follow a Von Mises' yield criterion which was adopted to the anisotropic-dynamic case. For the static anisotropic and the orthotropic dynamic cases the following results were obtained;

1. The plastic zone is given by the same relation as in the isotropic - static case,

$$\frac{\ell}{a} = \cos \left[ \frac{\pi T}{2Y} \right]$$

2. Any anisotropic or corresponding dynamic function may be obtained from the corresponding isotropic or static function by simply multiplying the isotropic or static function by a coefficient.
3. A limit on yielding along the line of the crack, and therefore a limit on the anisotropy and the velocity are derived.

The anisotropic viscoelastic solution is obtained from the static solution from Graham's extension of the correspondence principle after it is shown that the problem fits the restriction set by this technique. The effects of anisotropy in the material are handled by the inclusion of the generalized creep compliance and relaxation modulus. Once these terms are evaluated for each material an approximate inversion method for the Laplace transform may have to be used.

For the case of constant external load it is shown that the $\sigma_{xx}$ stress may be separated into a time variation and a space variation. Several different viscoelastic materials are assumed and the stresses solved for.

In the limit all results reduce to the isotropic and static solutions. Finally, a large bibliography is included to serve those who wish to investigate the area further.