THE TOLLMIEN-SCHLICHTING INSTABILITY
OF LAMINAR VISCOUS FLOWS

by

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- o - No suction, experiment
- \(\triangledown\) - Suction, experiment

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- ○ - No suction, experiment
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- ○ - No suction, experiment
- ▼ - Suction, experiment
- ——— Corresponding theory

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- ○ - No suction, experiment
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Chapter One

Introduction

Since 1949 there has been much interest in increasing the range, endurance, and payload of long-range aircraft through Boundary Layer Control, later termed Laminar Flow Control (LFC) in 1960\textsuperscript{1-3}. Applications for submerged bodies are well documented in works such as Reference 4. Overviews of the most current research in stability and transition and LFC are given in References 5 and 6. These efforts have been motivated by the realization that turbulent skin-friction drag can account for up to fifty percent of the total drag of a craft during cruise. Maintaining a laminar boundary layer by delaying transition to a higher Reynolds number substantially reduces this contribution to the overall drag and therefore increases fuel efficiency.

The transition from a laminar boundary layer to a three-dimensional, highly nonlinear turbulent boundary layer is a complicated and not-very-well understood process. Before total breakdown, however, it has been observed on a flat plate that two-dimensional linear velocity and pressure fluctuations occur within the boundary layer\textsuperscript{7}. These disturbances, traveling, harmonic waves called Tollmien-Schlichting waves, either amplify or decay\textsuperscript{8-10}. Those that amplify are observed later to exhibit some three-dimensionality\textsuperscript{11} and nonlinearity\textsuperscript{10,12,13} and eventually there is breakdown to fully developed turbulence. It is apparent that to understand the mechanisms of transition, it is first necessary to understand the initial small disturbances completely and to determine ways to control them.
Linear stability theory, or small disturbance theory, in which disturbances are assumed in the form of linear, harmonic, traveling waves, plays an important role in the design of laminar flow control systems. Using it we can determine the unstable, or amplifying, range of disturbance frequencies and amplification factors. The $e^n$ method\textsuperscript{14-16} and the modified $e^n$ and amplitude methods\textsuperscript{17,18} predict trends for changes in the mean flow that delay transition and are therefore very useful. However, they cannot be expected to predict the exact location of transition because of the strong dependence of transition location on freestream turbulence levels. If the turbulence levels are low, the $n$-factor for predicting transition is about fifteen\textsuperscript{19}; for higher Reynolds numbers and turbulence levels, an $n$-factor of about ten has been suggested\textsuperscript{14-16}. For LFC design Hefner and Bushnell\textsuperscript{20} suggest a value of five.

Suction through porous strips is one approach under consideration for laminar flow control. In the first place the use of suction has been found to be much more effective in delaying transition than heating or cooling the skin of the body. The next consideration is what is the best method of applying this suction. Structurally speaking, using discrete porous strips is better than building a wing totally of porous material. Strips are also more desirable than slots. Because of their much higher suction levels, slots can experience high Reynolds number instabilities such as separation and backflow within the slots themselves.

The effectiveness of using suction through porous strips in delaying transition along with the optimal number, spacing, and mass
flow rate through such strips should be determined by stability calculations of the flow over a body with suction strips. For such calculations the mean flow must first be determined as accurately as possible. In Chapter Two we show that the stability results obtained from using the linearized triple-deck, closed-form solutions of Reed and Nayfeh et al., slightly modified, compare very well with the results obtained from using the interacting boundary-layer solutions. We also show the error in using nonsimilar-boundary-layer solutions for the mean-flow quantities due to the neglect of upstream influence. Our next objective is to develop and demonstrate the practicality of a linear optimization scheme to determine efficient strip configurations for laminar flow control.

In Chapter Three we verify the results of our optimization scheme with experimental data. We find that the theory correctly predicts the experimental results and conclude that the optimization scheme is reliable enough to replace the experiment as a tool in designing efficient strip configurations in so far as two-dimensional incompressible flows are concerned.

The use of suction through porous strips for laminar flow control is also of interest on submerged, streamlined, axisymmetric bodies. Once again, to determine the effectiveness of this approach and to optimize the number, spacing, and flow rate through such strips, one needs to determine the stability of the flow over a body with suction strips. In Chapter Four we develop linear triple-deck, closed-form solutions for the mean-flow quantities, solutions which account for upstream influence. The triple-deck solutions are linear superpositions
of the flow past the body without suction and the perturbations due to
the suction strips. The flow past the suctionless body is calculated
using the Transition Analysis Program System (TAPS)\textsuperscript{23,24}. Then in
Chapter Five we perform a stability analysis on these closed-form solu-
tions and develop an optimization scheme for predicting efficient strip
configurations.

One application of laminar flow control will be to transonic swept
wings with specially-designed cross-sectional airfoil shapes\textsuperscript{25,26}. Because of the sweep, a spanwise pressure gradient exists resulting in a
crossflow profile that is dynamically unstable. The development of
crossflow can lead to the generation of streamwise vortices\textsuperscript{27-29}. Because of this added instability we must now determine the linear
stability of three-dimensional flows.

A correct three-dimensional stability analysis must include both
compressibility and nonparallelism. Nayfeh\textsuperscript{30,31} uses the method of
multiple scales\textsuperscript{32,33} to formulate the problem but presents no numerical
results. He determines the partial differential equations governing
variations of the amplitude and complex wavenumbers and determines con-
ditions on the group velocity components making the problem physically
realistic. El-Hady\textsuperscript{34} presents some results on a 23°-swept wing with a
supercritical airfoil shape.

In our opinion, all past numerical calculations of three-
dimensional boundary layer stability have not satisfactorily answered
the questions about the character of the most unstable disturbance.
Instead of jumping from wave to wave by locally calculating the most
amplified disturbance as you march or specifying some artifical
condition such as constant spanwise or chordwise wavelength, we believe that at some initial point a specific wave must be selected and then that one wave must be followed along its trajectory\textsuperscript{31}. The aim of Chapter Seven is to lay the ground work for the approach of finding the most dangerous frequency and initial spanwise wavenumber for a small-amplitude, three-dimensional disturbance at a specific chord location of a wing in a three-dimensional compressible flow. The mean flow will be that of the X21 wing as generated from the Kaups-Cebeci\textsuperscript{35} computer code discussed in Chapter Six.

Chapter Eight is then a summary of the work in the preceding six chapters.
Chapter Two

Stability of Two-Dimensional, Incompressible Flow

Over Plates With Porous Suction Strips

2.1 Introduction

Suction through porous strips is under consideration for laminar flow control. The effectiveness of this method along with the optimal number, spacing, and mass flow rate through such strips should be determined by stability calculations of the flow over a body with suction strips. For such calculations the mean flow must first be determined as accurately as possible. Nayfeh and El-Hady\textsuperscript{36} used a nonsimilar boundary-layer code. However, nonsimilar boundary-layer calculations fail to account for upstream influence.

The most exact solutions known to the author for the mean-flow quantities are those obtained from the numerical integration of the Navier-Stokes equations or the interacting boundary-layer equations. However, such solutions require prohibitively large amounts of computer time and storage and usually fail to converge for large Reynolds numbers.

In this chapter we show that the stability results obtained from using the linearized triple-deck, closed-form solutions of Reed\textsuperscript{21} and Nayfeh et al.\textsuperscript{22}, slightly modified, compare very well with the results obtained from using the interacting boundary-layer solutions. We also show the error in using nonsimilar-boundary-layer solutions for the mean-flow quantities. Our next objective is to develop and demonstrate
the practicality of a linear optimization scheme to determine efficient strip configurations for laminar flow control.

2.2 Stability Problem Formulation

In this chapter we consider a steady, two-dimensional, incompressible boundary-layer flow over a flat plate at zero angle-of-attack with porous suction strips. The coordinate system we use has x in the streamwise direction along the plate and y normal to the surface. To study the stability of such a basic state, or mean flow, we superpose small disturbances to obtain total flow quantities of the form

\[ q(x,y,t) = Q_0(x,y) + q_1(x,y,t) \]

where \( Q_0(x,y) \) stands for a basic state quantity and \( q_1(x,y,t) \) stands for a small unsteady disturbance. Substituting the total flow quantities \( \hat{u}, \hat{v}, \) and \( \hat{p} \), the total streamwise velocity, normal velocity, and pressure, respectively, into the Navier-Stokes equations, subtracting the basic-state equations, and linearizing in the disturbances, we find to first order that the disturbance equations are given by

\[
\frac{\partial u_1}{\partial t} + U_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial U_0}{\partial x} + V_0 \frac{\partial u_1}{\partial y} + v_1 \frac{\partial U_0}{\partial y} = - \frac{\partial p_1}{\partial x} + \frac{1}{R} \left( \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} \right) \tag{2.1} 
\]

\[
\frac{\partial v_1}{\partial t} + U_0 \frac{\partial v_1}{\partial x} + u_1 \frac{\partial V_0}{\partial x} + V_0 \frac{\partial v_1}{\partial y} + v_1 \frac{\partial V_0}{\partial y} = - \frac{\partial p_1}{\partial y} + \frac{1}{R} \left( \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} \right) \tag{2.2}
\]
These equations are nondimensional with the Reynolds number given by

$$R = \frac{U^* \delta}{\nu^*}$$

where $\delta$ is the reference boundary-layer thickness defined as

$$\delta = \frac{\nu^* x^*/U^*_\infty}{\nu^*}$$

The quantities $\nu^*$, $x^*$, and $U^*_\infty$ are the dimensional freestream kinematic viscosity, streamwise position along the plate, and freestream velocity, respectively.

The boundary conditions for the disturbance equations (2.1)-(2.3) are given by

$$u_1 = v_1 = 0 \text{ at } y = 0 \quad (2.4)$$

$$u_1, v_1 \to 0 \text{ as } y \to \infty \quad (2.5)$$

These were shown to be reasonable approximations by Lekoudis. He assumed the streamwise component of the disturbance $u_1$ at the wall to be zero provided the percentage of the permeable area is small and most of the flow there is directed normal to the surface. He also assumed that at the wall

$$\frac{v_1}{p_1} = f \quad (2.5)$$

where $f$ is an admittance. For closely-spaced perforations and small surface permeability, he found that specifying zero disturbance velocity for $v_1$ at the wall is an acceptable approximation.

Then we use the method of multiple scales to solve Eqs. (2.1)-(2.4). We expand $u_1$, $v_1$, and $p_1$ in the form of traveling, harmonic
Tollmien-Schlichting waves given by

\begin{align}
    u_1 &= [u(x_1,y) + O(\varepsilon)] \exp(i\theta) \\
    v_1 &= [v(x_1,y) + O(\varepsilon)] \exp(i\theta) \\
    p_1 &= [p(x_1,y) + O(\varepsilon)] \exp(i\theta)
\end{align}

(2.6) 

(2.7) 

(2.8)

where the phase function \( \theta \) is defined by

\begin{align}
    \frac{\partial \theta}{\partial x} &= k(x_1) \\
    \frac{\partial \theta}{\partial t} &= -\omega
\end{align}

(2.9) 

(2.10)

These expansions were verified experimentally by Schubauer and Skramstad\(^7\). The quantities \( k \) and \( \omega \) are the dimensionless streamwise wavenumber and frequency, respectively, defined with respect to the dimensional wavenumber \( k^* \) and the dimensional frequency \( \omega^* \) by

\begin{align}
    k &= k^* \delta \\
    \omega &= \frac{\omega^* \delta}{U^*_\infty}
\end{align}

The slow scale is \( x_1 = \varepsilon x \) where \( \varepsilon = \frac{1}{R} \) describes the slow variation of the amplitudes of the waves in the streamwise direction over a wavelength. Also, we consider the case of spatial stability, so that the wavenumber \( k \) is complex and the frequency \( \omega \) is real.

Substituting Eqs. (2.6)-(2.10) into Eqs. (2.1)-(2.4) and considering only mean flows of the form

\begin{align}
    U_0 &= U_0(x_1,y) \\
    V_0 &= \varepsilon \hat{V}_0(x_1,y), \quad \hat{V}_0 = O(1)
\end{align}
\[ P_0 = \delta P_0(x_1) \]

we find to first order that
\[ ik u + \frac{dv}{dy} = 0 \] (2.11)
\[ i(k U_0 - \omega) u + v \frac{dU_o}{dy} = -ikp + \frac{1}{R} \left( \frac{d^2u}{dy^2} - k^2 u \right) \] (2.12)
\[ i(k U_0 - \omega) v = -\frac{dp}{dy} + \frac{1}{R} \left( \frac{d^2v}{dy^2} - k^2 v \right) \] (2.13)

subject to the boundary conditions
\[ u = v = 0 \quad \text{at} \quad y = 0 \] (2.14)
\[ u, v \to 0 \quad \text{as} \quad y \to \infty \] (2.15)

The system of equations (2.11)-(2.15) represents an eigenvalue problem for the parameters \( k, \omega, \) and \( R. \) For known basic state velocity profiles, the equations are integrated numerically using a computer code developed by Scott and Watts\(^{38}\) to handle stiff two-point boundary-value problems such as this. Specifying \( R \) and \( \omega, \) we find the eigenvalue
\[ k = k_r + ik_i, \]
where \( k_r \) and \( k_i \) are the real and imaginary parts of \( k, \) respectively. Then \(-k_i\) is the spatial growth rate of the disturbance.

From \( k_i \) we determine the amplification factor
\[ n = \ln \frac{A}{A_0} = -2 \int_{R_0}^R k_i(\xi) d\xi \] (2.16)

where \( R_0 \) is the square root of the \( x \)-Reynolds number where the constant frequency disturbance first becomes unstable; \( A \) and \( A_0 \) are the amplitudes of the disturbance at \( R \) and \( R_0, \) respectively. The \( e^n \) method\(^{14-16}\)
cannot be used to predict the exact location of transition because of the strong dependence of transition location on freestream turbulence levels. However, linear stability theory is useful, especially for laminar flow control. It can be used as a design tool because it predicts trends for changes in the mean flow that delay transition.

2.3 Basic State

2.3a Linearized Triple Deck

Laminar viscous flow over a flat plate with a disturbance exhibits a triple-deck structure\(^{39-41}\) (see also the review articles of Stewartson\(^{42}\), Adamson and Messiter\(^{43}\), and Smith\(^{44}\)). This is shown schematically in Figure 2.1. In this case the disturbance is a porous strip with center \(x_p^*\) far from the plate's leading edge \(x^* = 0\).

Upstream of the region influenced by the suction is the Prandtl boundary layer. The flow in the neighborhood of the porous strip centered at \(x_p^*\) is described by three decks or nested boundary layers. The small parameter in this problem is \(\varepsilon\) where \(\varepsilon\) is defined as

\[
\varepsilon = Re^{-1/8}, \quad Re = \frac{x^* U^*}{v^*}
\]

According to triple-deck theory the streamwise effects of the disturbance occur in a neighborhood \(O(\varepsilon^3 x_p^*)\) of \(x_p^*\). The middle deck, the displaced Prandtl layer, is \(O(\varepsilon^4 x_p^*)\) thick and characterized by rotational, inviscid disturbances. The upper deck, whose thickness is \(O(\varepsilon^5 x_p^*)\), has inviscid, irrotational disturbances. The lower deck is \(O(\varepsilon^5 x_p^*)\) thick and has viscous, rotational disturbances. The wall boundary conditions are satisfied by the lower-deck governing equations. The linearized
The triple-deck equations we derive are expected to be valid only for strip widths $0(\varepsilon^3 x_p)$ and suction levels $0(\varepsilon^3 U_\infty)$.

For the case of injection through one porous strip, Napolitano and Ressick\textsuperscript{45} developed a closed-form solution for the linearized lower-deck equations by means of Fourier transforms and calculated only the wall pressure coefficient and wall shear stress. Following their work, Reed\textsuperscript{21} and Nayfeh et al.\textsuperscript{22} obtained expressions for the pressure and shear as well as the velocity components and displacement thickness for one finite-length suction strip. Then, they developed closed-form solutions valid at all points in the direction normal to the plate. Finally, noting the linearity of the problem and using superposition, they developed closed-form solutions valid for any number of suction strips.

In this chapter, we examine the stability of the basic state given by slightly modified versions of these linearized triple-deck, closed-form solutions. Here, we consider $n$ porous strips, centered at $x^*, x^*, ..., x^*$ with the leading and trailing edges $x_{LE_1}^*$, $x_{LE_2}^*$, ..., $x_{LE_n}^*$ and $x_{TE_1}^*$, $x_{TE_2}^*$, ..., $x_{TE_n}^*$, respectively. We define the $x$-Reynolds number at $x^*$ and the $i$th strip as

$$
\text{Re}_x = \frac{x^* U_\infty}{v^*}, \quad \text{Re}_{x_i} = \frac{x_i^* U_\infty}{v_i^*}
$$

(2.17)

respectively. The quantity $v^*_{wall_i}$ is the dimensional suction rate at the $i$th strip. The Blasius profile $f'(\eta)$ is defined by

$$
f'''' + ff'' = 0
$$

(2.18)
\[ f(0) = f'(0) = 0, \quad f'(\infty) = 1 \]  \hspace{1cm} (2.19)

where

\[ \eta = y* \frac{U_\infty}{2 \nu x*} \]

The constant \( \lambda \) is defined as

\[ \lambda = \frac{f''(0)}{\sqrt{\nu}} \]  \hspace{1cm} (2.20)

Considering these definitions, for the streamwise component of velocity we use \(^21,22\)

\[
\frac{u^*}{U_\infty}(x^*,y^*) = f'(\eta) + \sum_{i=1}^{n} \frac{V_{wall i}}{U_\infty} \frac{1}{\eta} \left[ \frac{f''(\eta)}{\lambda \sqrt{\nu}} - 1 \right] \delta(x - x_i) + \bar{u}(x - x_i, y) \]

where

\[
\bar{u}(x - x_i, y) = \bar{u}_\infty \left( \frac{\lambda^{5/4}}{x_i^*} \left[ x^* R_x^{3/8} - x_{PE}^* R_x^{3/8} \right], \frac{R_x^{5/8}}{x_i^*} \lambda^{3/4} y^* \right)
\]

\[
- \bar{u}_\infty \left( \frac{\lambda^{5/4}}{x_i^*} \left[ x^* R_x^{3/8} - x_{PE}^* R_x^{3/8} \right], \frac{R_x^{5/8}}{x_i^*} \lambda^{3/4} y^* \right) \]  \hspace{1cm} (2.22)

\[
\bar{v}(x - x_i) = \bar{v}_\infty \left( \frac{\lambda^{5/4}}{x_i^*} \left[ x^* R_x^{3/8} - x_{PE}^* R_x^{3/8} \right], \right)
\]

\[
- \bar{v}_\infty \left( \frac{\lambda^{5/4}}{x_i^*} \left[ x^* R_x^{3/8} - x_{PE}^* R_x^{3/8} \right], \right) \]  \hspace{1cm} (2.23)
The quantities with an asterisk are dimensional.

The difference between our formulation here and that in References 21 and 22 is that we scale \( x \) (not \( x_i \)) in the arguments of \( \delta \) and \( \overline{u} \) with
the \( x \)-Reynolds number at \( x^* \) and scale \( y \) with \( x^* \). We make this correction because we observe experimentally that our original triple-deck solutions predict too much downstream influence. Before the work in References 21 and 22, no one had even considered several disturbances introduced at different points on a flat plate and superposed the resulting individual triple-deck solutions. Each solution is valid in the neighborhood of the corresponding strip and neglects streamtube divergence. In References 21 and 22, for lack of any specific guidance, we just superposed the individual solutions as is. Considering the experiments mentioned in Chapter Three, however, we find that downstream influence is overpredicted with this original formulation giving a too stable flow. But we now find that varying the different parameters in the triple-deck scalings with streamwise position as in Eqs. (2.21)-(2.23) to kill this effect gives excellent agreement with the experiments.

2.3b. Interacting Boundary Layers

Following the work of Ragab\(^46\) and Ragab and Nayfeh\(^47\), we introduce into the dimensionless boundary-layer equations, the Levy-Lees variables

\[
x = \int_0^x U_e(x)dx, \quad \eta = \frac{U_e(x)y\sqrt{Re_x}}{\sqrt{2\xi}} \quad (2.26)
\]

\[
F(\xi, \eta) = \frac{u}{U_e(x)} \quad (2.27a)
\]

\[
V(\xi, \eta) = \frac{2\xi^{\frac{3n+1}{2}}}{U_e(x)} \left( F \frac{\partial n}{\partial x} + \frac{v\sqrt{Re_x}}{\sqrt{2\xi}} \right) \quad (2.27b)
\]

The resulting system of equations is

\[
2\xi F \frac{\partial F}{\partial \xi} + V \frac{\partial F}{\partial n} + F = 0 \quad (2.28)
\]
\[ 2\xi FF_\xi + VF_\eta + \beta(F^2 - 1) - F_{\eta\eta} = 0 \quad (2.29) \]

where
\[ \beta = -\frac{2\xi dp}{U_e^3 dx} \quad (2.30) \]

The corresponding boundary conditions are
\[ F = 0 \quad \text{at} \quad \eta = 0 \quad (2.31a) \]
\[ V = \begin{cases} \sqrt{\frac{2\xi Re}{U_e(x)}} & \text{strip LE} \leq \xi \leq \text{strip TE} \\ 0 & \text{otherwise} \end{cases} \quad \text{at} \quad \eta = 0 \quad (2.31b) \]
\[ F \to 1 \quad \text{as} \quad \eta \to \infty \quad (2.31c) \]
\[ F = F_0(\eta) \quad \text{at} \quad \xi = \xi_0 \quad \text{(far upstream)} \quad (2.31d) \]

Equations (2.28)-(2.31) are solved using a second-order accurate, finite-difference, marching scheme. A Newton-Raphson procedure is used to quasi-linearize the nonlinear terms, giving the momentum and continuity equations coupled in their linearized form, the so-called Davis coupled scheme.\(^4\) The viscous displacement thickness
\[ \delta = \frac{\sqrt{2\xi}}{U_e(x)\sqrt{Re_\infty}} \int_0^\infty (1 - F)d\eta \quad (2.32) \]
is iteratively made to equal the inviscid displacement thickness (i.e., the displacement thickness from boundary-layer interaction with the inviscid flow) given by
\[ \frac{dy_D}{dx} = \frac{1}{\pi} \int_{LE}^\infty \frac{p(t)}{x-t} dt \quad (2.33) \]
so that
The algorithm for the numerical solution of the interacting boundary-layer equations is:

1. Calculate the flow quantities for a Blasius boundary layer all along the plate as an initial guess. Then

\[
\frac{dp}{dx} = 0 = \frac{dp}{dx_{\text{current}}}
\]

\[
y_D = y_D^{\text{Blasius}} = y_D^{\text{current}}
\]

where \(y_D\) is the inviscid displacement thickness.

2. Solve (2.28)-(2.31) numerically using the current pressure gradient. For the backward difference marching procedure, start far enough upstream of the first strip but far enough downstream of the leading edge of the plate to justify using the Blasius flow at the first two points.

3. Calculate the corresponding viscous displacement thickness \(\delta\) using (2.32).

4. If

\[
\sum_{i=1}^{M} |\delta(x_i) - y_D^{\text{curr}}(x_i)| < \text{ERR}
\]

for some specified \(\text{ERR} \ll 1\), then convergence has been achieved and we go to 9. Here, \(M\) is the number of calculated points along the plate.
5. Use the viscous displacement thickness to adjust the current
inviscid displacement thickness according to

\[ \gamma_{D_{\text{NEW}}} = \gamma_{D_{\text{curr}}} (1 - RF) + RF \delta \]

\[ \gamma_{D_{\text{curr}}} = \gamma_{D_{\text{NEW}}} \]

where RF is a relaxation factor (usually .1).

6. Calculate the pressure gradient \( \frac{dp}{dx} \) corresponding to \( \gamma_{D_{\text{curr}}} \)
using (2.34). Then

\[ \frac{dp}{dx_{\text{NEW}}} = \frac{dp}{dx_{\text{curr}}} (1 - RF) + RF \frac{dp}{dx} \]

7. Check the number of iterations. If too many iterations have
been done, then GØ TØ 9.

8. GØ TØ 2.

9. STOP.

To obtain the nonsimilar solution, we set \( \beta = 0 \) in the above
equations and numerically integrate once along the plate. In the
algorithm above, this is just steps 1 and 2. For the rest of this
dissertation we use this definition for nonsimilar calculations.

2.4 Optimization Scheme

Taking advantage of the linearity of the triple-deck formulation,
we develop a linear minimization problem whose solution can be used to
predict efficient strip configurations. The problem is even solvable by
inspection, that is, there is no need for dynamic programming such as
the simplex algorithm\textsuperscript{49}. 
We consider the first-order system of disturbance equations

\[ iku + Dv = 0 \]  
\[ -i\omega u + ikuU + vD U = -i k p + \frac{1}{R} [D^2 u - k^2 u] \]  
\[ -i\omega v + ikvU = -Dp + \frac{1}{R} [D^2 v - k^2 v] \]

where \( D \) has been used as an abbreviation for \( \frac{d}{dy} \). We take the mean flow \( U \) as

\[ U = U_0 + \varepsilon U_1 \quad |\varepsilon| \ll 1, \quad U_1 = \mathcal{O}(1) \]  

This mean flow is exactly as in the triple-deck expression for \( u^*/U_\infty \) where

\[ U_0 = f'(\eta) \text{ (Blasius)} \]  
\[ \varepsilon = \frac{v^*_{\text{wall}}}{U_\infty} \frac{1}{\text{Re}^{1/4} \lambda^{-1/2}} \]

Then, we expand

\[ u = u_0 + \varepsilon u_1 + ... \]  
\[ v = v_0 + \varepsilon v_1 + ... \]  
\[ p = p_0 + \varepsilon p_1 + ... \]  
\[ k = k_0 + \varepsilon k_1 + ... \]

where the quantities with subscript zero represent Blasius disturbance quantities and those with subscript one represent the contributions due to the presence of the suction strips. The zeroth-order problem is

\[ L_1(u_0, v_0, p_0) = i k_0 u_0 + Dv_0 = 0 \]
\[ L_2(u_0, v_0, p_0) = -i\omega u_0 + ik_0 u_0 U_0 + \nu_0 D U_0 + ik_0 p_0 \]
\[-\frac{1}{R} [D^2 u_0 - k^2 u_0] = 0 \] (2.39b)

\[ L_3(u_0, v_0, p_0) = -i\omega v_0 + ik_0 v_0 U_0 + D p_0 - \frac{1}{R} [D^2 v_0 - k^2 v_0] = 0 \] (2.39c)

\[ u_0 = v_0 = 0 \text{ at } y = 0 \] (2.39d)

\[ u_0, v_0 \to 0 \text{ as } y \to \infty \] (2.39e)

This system along with the appropriate boundary conditions determines the eigenvalue problem for \((\omega, k_0, R)\) to determine the disturbances \((u_0, v_0, p_0)\) superposed on an otherwise undisturbed Blasius boundary layer flow.

The first-order problem is

\[ L_1(u_1, v_1, p_1) = ik_1 u_0 \] (2.40a)

\[ L_2(u_1, v_1, p_1) = -ik_1 u_0 U_0 - ik_0 u_0 U_1 - v_0 D U_1 - ik_1 p_0 \]
\[ + \frac{1}{R} [-2k_0 k_1 u_0] \] (2.40b)

\[ L_3(u_1, v_1, p_1) = -ik_1 v_0 U_0 - ik_0 v_0 U_1 + \frac{1}{R} [-2k_0 k_1 v_0] \] (2.40c)

\[ u_1 = v_1 = 0 \text{ at } y = 0 \] (2.40d)

\[ u_1, v_1 \to 0 \text{ as } y \to \infty \] (2.40e)

A solution exists for this system (2.40) only if a solvability condition is satisfied. More is said on this in Chapter Seven. The inhomogeneous terms on the right-hand sides must be orthogonal to every solution of the adjoint homogeneous problem. That is,
or
\[
  k_1 \int_0^\infty f(y)\,dy + \int_0^\infty g(x,y)\,dy = 0 \quad (2.41b)
\]

where
\[
  z_1 = u \\
  z_2 = Du \\
  z_3 = v \\
  z_4 = p
\]

and the starred quantities are the corresponding adjoint solutions.

From Eqs. (2.41) we see that the perturbation \(k_1\) to the Blasius complex wavenumber \(k_0\) can be determined as

\[
k_1 = -\int_0^\infty g(x,y)\,dy/\int_0^\infty f(y)\,dy \quad (2.42)
\]

At each \(x\), since \(k_1\) is independent of \(\epsilon\), the correction to \(k_0\) is a linear function of \(\epsilon\). Remembering \(\epsilon\) to be given by Eq. (2.37b), we find that this implies that \(\epsilon k_1\) is directly proportional to the suction rate.

To test this theory, we consider a flat plate with one porous strip whose center is 197 centimeters from the plate's leading edge and whose width is 1.59 centimeter. The Reynolds number based on the distance from the leading edge of the plate to the center of the strip is 720,000. Into this flow we introduce a disturbance with the constant
dimensionless frequency $F = 59 \times 10^{-6}$, where $F$ is defined as $F = \omega/R = \omega*\delta x/U*^2$. Table 2.1 shows calculated and expected complex wavenumbers as a function of suction level for suction levels at the limit of where the triple-deck solutions can possibly be expected to be valid for these particular Reynolds numbers. The results indicate the linear dependence. However, in actual application we expect to use suction levels an order of magnitude smaller where the agreement is even better.

Using this property of linearity, we consider a plate with $n$ porous strips and $m$ specified points of computation between Branch I and Branch II of the stability curve. We solve the zeroth-order eigenvalue problem to find the Blasius complex wavenumber $k_B$ and the eigenvectors $z_{\ell j}, z_{\ell j}^*, \ell = 1, \ldots, 4$ at each point $j = 1, \ldots, m$. Then, using linear triple-deck theory and considering Eqs. (2.17), (2.41), and (2.42) we calculate the $y$-dependent function

$$\left(\frac{f''(n)}{\lambda \sqrt{2}} - 1\right)\delta(x_j - x_i) + \bar{u}(x_j - x_i, y) \tag{2.43}$$

at each point of computation $j = 1, \ldots, m$ due to each individual strip $i = 1, \ldots, n$, where $\bar{u}$ and $\delta$ are defined in Eqs. (2.18) and (2.19), respectively. Then applying Eqs. (2.41) and (2.42) we determine $k_1$ for each $i$ and $j$ and call this quantity $a_{ij}$. So we propose that the actual complex wavenumber $k_j$ at point $j$ due to the presence of all $n$ strips is

$$k_j = k_{Bj} + \sum_{i=1}^{n} a_{ij} \frac{V_{wall_i}^*}{U*_{\infty}} \text{Re}_{x_i}^{1/4} \lambda^{-1/2} \tag{2.44}$$
To test this formulation (2.44), we consider the problem of 24 1.59-centimeter strips equally spaced between I and II with equal suction levels $v_{wall}^* / U_\infty^* = -2.06 \times 10^{-4}$. We introduce a disturbance with the constant dimensionless frequency $F = 20 \times 10^{-6}$. Figures 2.2 and 2.3 show how the linearization scheme (2.44) compares with the exact solution obtained by performing the stability analysis on the total mean-flow profile given by Eq. (2.17). The agreement between the two schemes is almost perfect in the first half of the region between Branches I and II; however, near the back, displacement thickness effects appear to be lost and the agreement is not as good. The error at Branch II is only about 3% and we expect any results we obtain still to be valid and applicable.

To optimize a suction distribution keeping the total mass-flow rate constant, we should minimize the amplification factor

$$\ln \frac{A}{A_0} = -2 \int_{R_0}^{R} k_i \, d\xi$$

Using the trapezoidal rule to integrate and applying Eq. (2.44), we find that

$$\ln \frac{A}{A_0} = -\text{Imag} \left\{ \sum_{j=2}^{m} \left( k_{B_j} + k_{B_j-1} \right) (R_j - R_{j-1}) \right. \right. + \left. \left. \sum_{j=2}^{m} \sum_{i=1}^{n} (a_{i,j} + a_{i,j-1} \frac{v_{wall}^*}{U_\infty^*} \frac{1}{\lambda^{1/2}} (R_j - R_{j-1})) \right\}$$

(2.46)

Therefore, to minimize $\ln(A/A_0)$, we must find the solution, $\frac{v_{wall}^*}{U_\infty^*}$.
subject to constraints that give a reasonable physical picture. Such constraints can involve upper and lower limits on $|v_{\text{wall}}^*|/U_{\infty}$, a distribution so as to keep $-k_i$ positive after the disturbance first goes unstable, a distribution so as to eliminate displacement thickness effects, and constant specified total mass flow rate. The first of these constraints is important because it considers the physical limitations of the suction system itself. In addition, suction levels typical of slots must be prevented. The second constraint mentioned is especially important near the Branch I neutral point where a subsequent negative growth rate could cause smaller growth of the initially-introduced disturbance and therefore result in the possibility of other more amplified disturbances being introduced. The third constraint involves applying too much suction over a region so as to drastically alter the displacement thickness. The result is that the virtual leading edge of the plate is changed; that is, the flow thinks it is farther forward toward the true leading edge than it really is. These effects usually occur near the Branch II neutral point and cause the flow to experience the larger growth rates of the smaller Reynolds numbers just upstream. Consequently, growth occurs downstream of the Branch II point and results in an increase of the amplification factor. The fourth constraint above is another constraint on the suction system itself; that is, how much total mass the pumping system can withdraw
from the boundary layer in a certain amount of time. To demonstrate
our optimization scheme where we redistribute local mass flow rates in
hopes of reducing the amplification factor, we first must make use of
a nonsimilar code. Even though we know we will not get the most
accurate results because of the neglect of upstream influence, we use
such a code just to give us some rough idea as to the uniform con-
tinuous area suction level $v_w$ that must be applied between Branch I and
Branch II that reduces the amplification factor below nine. From this
we specify our constraint total mass flow rate $M$ as

$$M = v_w(x_{II} - x_I)$$  \hspace{1cm} (2.48)

where $x_I$ and $x_{II}$ are the streamwise coordinates of Branch I and Branch
II, respectively. This number $M$ changes for each frequency, of course.
Then we distribute this total mass flow rate over discrete strips. The
strips will not all have the same suction velocity. However for $n$
strips where the $i$th strip has width $w_i$ and suction level $v_{w_i}$, all the
$v_{w_i}$ must combine linearly so as to satisfy

$$M = \sum_{i=1}^{n} v_{w_i} * w_i$$  \hspace{1cm} (2.49)

where $M$ is the constant defined by Eq. (2.48)

For $i = 1, ..., n$, we evaluate and store the coefficients $c_i$ of
$v_{wall_i}/U_*$. From Eq. (2.47) we see that

$$c_i = -\text{Imag} \sum_{j=2}^{m} (a_{ij} + a_{ij-1}) \text{Re} x_i^{1/4} (R_j - R_{j-1})^{\lambda^{-1/2}}$$  \hspace{1cm} (2.50)

For example, for the same problem indicated in Figures 2.2 and 2.3,
these coefficients are listed in Table 2.2 and plotted in Figure 2.4.
The largest coefficient occurs near the Branch I neutral point indicating that suction should be concentrated there. Obviously, since this is a linear optimization problem, if the only constraint we impose is an upper bound $v_u$ on $|v_{wall}^*/U^*|$, then Eq. (2.47) is satisfied by assigning the $v_{wall}^*/U^*$ multiplying the largest $c_i$ to be $-v_u$, then doing the same for the next largest $c_j$, etc.

Results of this optimization scheme are given in Section 2.5. In each case, we solve the minimization problem by inspection; dynamic programming is not necessary.

2.5 Results and Discussion

In this section our initial objective is to show the validity of our triple-deck formulation by comparing growth rates obtained by using our closed-form solutions with growth rates from considering the interacting boundary-layer mean-flow quantities. The interacting boundary-layer calculations are the most accurate solutions known to the author. In the process we will show in the figures the inaccuracy of the nonsimilar solutions in predicting growth rates. All sketches also show the undisturbed, or Blasius, growth rates so that we may make conclusions about the effects of suction locally and upstream and downstream.

First, we present growth-rate comparisons for the linearized triple-deck, the interacting boundary-layer, and the nonsimilar solutions for the case of a flat plate with one porous strip of width .02 meters centered at a distance of .3 meters from the leading edge of the plate. The x-Reynolds number $Re_x$ at the center of the strip is $1 \times 10^5$.
and the dimensionless mass-flow rate through the suction strip is $-2.3 \times 10^{-4}$. Onto this basic state, we superpose a disturbance with the constant dimensionless frequency $F = 210 \times 10^{-6}$. Figure 2.5 shows the spatial growth rate $-k_i$ plotted versus $R = \sqrt{Re_x}$ in a neighborhood of the strip. This sketch shows the good agreement between the linear triple-deck and the interacting boundary-layer-predicted growth rates. By comparing these two solutions with the undisturbed, or Blasius solutions, we see that even for such a small suction level the growth rates are dramatically decreased in the neighborhood of the strip, thus stabilizing the flow. There is relatively small upstream influence of the suction and it appears to decay quickly while the downstream influence is larger and lingers for quite a distance down the plate. The nonsimilar calculations are very poor in comparison. They predict no upstream influence and then at the leading edge of the strip the solution is discontinuous and jumps to drastically overpredicted local levels. This is due to the impulsive imposition of the wall suction boundary condition as we march along the plate. The smoothing would occur if we would consider interaction with the potential-flow solution.

Next we demonstrate the validity of the linear triple-deck solutions when we increase the Reynolds number. We show the same flat plate but with $Re_x$ equal to $1 \times 10^6$ at the center of the strip. Here, we superpose a disturbance with $F = 40 \times 10^{-6}$. Figure 2.6 shows the comparisons. Again the comparison is very good in a neighborhood of the strip and we can make the same comments about local and upstream
and downstream influence of the relatively small suction levels. As before, the nonsimilar calculations are unacceptable.

Our next consideration is a flat plate with six strips each of width .02 meters whose centers are spaced at .18-meter intervals; the first strip is centered at a distance of .3 meters from the plate's leading edge. The dimensionless mass-flow rate through each strip is \(-2.3 \times 10^{-4}\) and \(\text{Re}_x\) at the center of the first strip is \(1 \times 10^5\). We superpose a disturbance with \(F = 86 \times 10^{-6}\) and show the results in the next six figures. Figures 2.7-2.10 show comparisons of growth rates for the different basic states at the third through sixth strips, respectively. Then Figure 2.11 shows an overall picture of the six strips. We consider this case because of the nature of our closed-form solution. They are linear superpositions of solutions for individual suction strips. These individual solutions are expected to be valid only in a neighborhood of their respective strips according to triple-deck theory, so it was unclear until these calculations what the result would be when we superposed them. It is apparent from Figures 2.7-2.11 that our doubts are unfounded because the comparisons between the interacting boundary layers and linear triple-deck theory are good.

Finally, Figure 2.12 shows results for the same six-strip configuration except that the centers are spaced at .09-meter intervals and \(\text{Re}_x\) at the center of the first strip is \(1 \times 10^6\). Comparisons are shown in a neighborhood of the sixth strip. This is another demonstration of increasing the Reynolds number. Again the results are good. In fact the results are even better than for the lower Reynolds number case.
The linearized triple deck results are in good agreement with those of the interacting boundary layers. Figures 2.5-2.12 show the inaccuracy of the nonsimilar boundary-layer equations due to neglecting the interaction of the viscous and inviscid flows. These results also show that one can use confidently the linearized triple-deck equations to solve accurately for the mean flow over a body with suction through porous strips. Using these closed-form solutions, we now show some optimal strip configurations for laminar flow control.

The flow under consideration for the remainder of this chapter is that of a flat plate with a unit Reynolds number of $1.6 \times 10^6$ /meter. The disturbance we introduce has a constant dimensionless frequency of $20 \times 10^{-6}$. For Blasius flow, the amplification factor reaches a maximum value of 12.04. Using nonsimilar calculations, we reduce the amplification factor to 8.8 with a continuous area suction level of $v^{*}_{wall}/U^{*} = -3 \times 10^{-5}$ applied between Branches I and II. This is consistent with the so-called $e^9$ method. Figures 2.13 and 2.14 show the growth rates and amplification factor, respectively, versus the reference boundary-layer thickness Reynolds number for the case of continuous suction.

When considering porous strips, we keep the total mass flow rate as established by the nonsimilar calculations constant. Our plan is to consider a uniform distribution of suction through strips and then to try to optimize it. We examine a configuration of 24 1.59-centimeter-wide strips distributed evenly between Branches I and II. The leading edge of the first strip lies .66 meters from the plate's leading edge and subsequent strips have their centers spaced approximately 11.43 cm
from each other. Each strip has a suction level of $v_{\text{wall}}^*/U_\infty = -2.06 \times 10^{-4}$. The distribution is shown schematically in Figure 2.15.

Using the linearized scheme discussed in Section 2.4, we calculate the growth rates between Branches I and II and then calculate the amplification factor. These are shown in Figures 2.16 and 2.17, respectively.

Intuitively one would expect that to improve the efficiency of this distribution, one would concentrate more suction in the regions of maximum growth rate and less near the neutral stability points. We show such an example in Figures 2.18-2.20. Figure 2.18 shows a suction distribution for 24 strips, equally spaced with equal widths, but with $v_{\text{wall}}^*/U_\infty$ for the middle third of the strips equal to twice that of the outer thirds. Figure 2.19 shows the growth rate distribution and Figure 2.20 shows the amplification factor calculated up to Branch II. We see that by flattening the growth rate distribution in the middle we increase displacement-thickness effects in the back, resulting in an increase in the growth of the disturbance.

The linear optimization scheme discussed in Section 2.4 tells us how to arrange our strips. If we want 24 equally-spaced strips between Branches I and II and if, for physical reasons, we require an upper bound on $|v_{\text{wall}}^*/U_\infty|$ of .0003 and a lower bound of .0001, then considering the influence coefficients in Table 2.2 and Figure 2.4 we see that the optimization scheme predicts that the suction-level distribution shown in Figure 2.21 is more efficient than the uniform case. This distribution may be surprising since the suction has been concentrated in the neighborhoods first of Branch I and then of Branch II, not where the growth rate is a maximum. Upon calculation of the growth rates and
the amplification factor, we indeed find that \( n \) has been reduced by about 5\%. These results are shown in Figures 2.22 and 2.23. It appears that we must compress the growth rate curve from both ends, making it higher and thinner in the middle, to achieve more efficient distributions.

Similarly, suppose we want 12 1.5-centimeter strips between Branches I and II with equal suction levels. Figures 2.24, 2.25, and 2.26 show the suction levels, growth rates, and amplification factor, respectively, for the case of equal distances between the strips. The amplification factor is about 8.69. Now we impose the constraint that the distance between the centers of the strips cannot be less than 11.43 centimeters. The linear optimization scheme predicts the spacing of the strips shown in Figure 2.27 to be more efficient than the uniform case. The corresponding growth rates and amplification factor \( n \) appear in Figures 2.28 and 2.29, respectively. Again suction is concentrated more in the front and in the back and results in a reduction of \( n \) of about 4\%.

Finally, suppose we are free to put any number of 1.59-centimeter-wide strips on the plate but we require that the maximum absolute value of the suction level be 0.0003 and the minimum distance between strip centers be 11.43 centimeters. The linear optimization scheme predicts that if we use 17 strips with suction levels and spacings as in Figure 2.30, then the amplification factor reduces to 8.2. The growth rates and amplification factor are shown in Figures 2.31 and 2.32, respectively. The reduction in \( n \) is on the order of 6\% from the uniform 24-strip problem discussed above.
These results show the usefulness of our linear optimization scheme in predicting the suction level, spacing, and number of strips to decrease the disturbance growth as much as possible. The surprising, but important result of this work is that the distributions predicted concentrate suction in the regions of neutral stability, not where the growth rates are the greatest. It appears that we must control the disturbances while their growth rates are small.

There are of course limitations on width and suction level for the strips that we may specify. The width must be small enough to confine the strip's major effects to an $O(Re^{-3/8} x^*_p)$ neighborhood. However, looking at the influence coefficients, we expect that as we specify smaller and smaller strip widths, keeping number constant, and concentrate the suction where the largest coefficients are, that the maximum amplification factor will decrease more and more. There are two problems that we recognize immediately here and must be aware of when we use this optimization scheme. Both are associated with the constant total mass-flow-rate constraint. Decreasing width implies increasing the magnitude of the suction level. The first problem is that specifying too high suction levels will result in the strips really acting like slots and having to be treated in such fashion. The second problem is that the suction levels have to stay $O(Re^{-3/8} U^*_*)$ in order for the theory to remain valid.

One good way we can think of to apply the optimization scheme is to overspecify the number of strips, the limit of which is, of course, to consider strips edge to edge all the way from Branch I to Branch II. We would calculate and study all the influence coefficients and then in
the end specify zero suction levels for those strips we didn't really want. With this method we could optimize all three of number of, spacing between, and mass-flow rate through the strips. With the limitation of only being able to consider multiples of the original widths, we could in addition optimize strip width by keeping neighboring strips and considering them as one.

The results of the optimization scheme must be and will be verified by experimental data in the next chapter.
Chapter Three

The Stability of Boundary Layers with Porous Suction Strips:
Experiment and Theory

3.1 Introduction

The stability results of Chapter Two indicating that maximum stabilization occurs when suction is moved toward the Branch I neutral point need to be verified by experiment. Such an experiment investigating the stability of boundary layers with porous strips was performed by Reynolds and Saric\(^5\)\(^1\)-\(^5\)\(^3\) here at VPI & SU beginning in 1980. Numerical calculations using the linear triple-deck theory and stability analysis of Chapter Two were performed corresponding to each experimental strip configuration. In each case, the theory correctly predicts the experimental results.

3.2 Description of the Experiment

This section provides a brief description of the experimental setup. For additional details see References 51-58. The experiments were conducted in the VPI & SU Low Turbulence Wind Tunnel, a closed-circuit facility with a 7.3-meter-long by 1.83-meter-square test section. It is well known that good stability experiments can be done only when freestream turbulence levels are very low. Because of seven screens upstream of the nozzle and a subsequent 9 to 1 contraction ratio, this facility is found to have disturbance levels in a neighborhood of .02% of the freestream velocity at speeds of 15 or 20 meters/sec. This quality of the flow results in a transition Reynolds number
of approximately 3.4 million on a flat plate. These levels are excellent, even when compared with other modern designs.

The model used was a flat plate with a chord, span, and thickness of 4 meters, 1.83 meters, and 0.22 meters, respectively. The plate was hand-polished to a roughness height of approximately 8 microns. The leading edge was chosen to be elliptical with a ratio of major to minor axes of 67:1. This choice was found to reduce the pressure rise in the neighborhood of the stagnation point and at the same time be insensitive to any small changes in angle of attack. The 93 pressure ports embedded in the plate indicated that the flow over the model was indeed a zero-pressure-gradient flow.

The plate was equipped for application of continuous suction at the four panel locations indicated in Figure 3.1. Panels made of a Dynapore porous material were inserted in only two of these four locations. Because the spanwise flutes in the ducting behind the porous material were independent of each other, the suction distribution could be varied from continuous suction over the whole panel down to discrete suction over only one 1.59-centimeter wide flute.

Disturbances of controlled frequency were introduced into the boundary layer by means of a vibrating ribbon. A hot-wire anemometer mounted on a sting extending from a traversing mechanism and calibrated using King's Law then measured both the mean-flow profile and the \( u' \), or chordwise, component of the disturbance profile. The disturbance amplitude was then determined through integration of the \(|u'|\)-profile.
3.3 Comparison of Theory with Experiment

Using the linearized triple-deck closed-form solutions for streamwise velocity components, we performed stability calculations and predicted that optimal strip configurations occur for a given frequency when suction is concentrated close to the streamwise position corresponding to the first neutral point of the stability curve. The next eight figures show comparisons between the linear triple-deck theory and the experiments for different strip configurations as indicated below. All figures show the integrated disturbance amplitude versus $\sqrt{\text{Re}_x}$.

The following is a summary of the suction strip configurations tested. All porous strips are 1.59-centimeters wide. Figures 3.2-3.3 indicate results for a dimensionless frequency of $20 \times 10^{-6}$ and a square root unit Reynolds number per meter of $R_u = 987$. The Branch I and Branch II neutral points occur at $\sqrt{\text{Re}_x} = 1037.7$ and 2315.0, respectively. In Figure 3.2, one suction strip on Panel II at $x = 248$ cm is open with a suction level of $V_0 = 5.5 \times 10^{-3} U_\infty$. This $x$-location corresponds to $\sqrt{\text{Re}_x} = 1554$ so the strip is located just upstream of where the growth rates are a maximum. Figure 3.3 shows the same porous strip moved forward toward the Branch I point to Panel I at $x = 194.3$ cm corresponding to $\sqrt{\text{Re}_x} = 1376$. In both figures the agreement between theory and experiment is remarkable, considering this level of suction is just about at the limit of where the triple deck can be expected to be valid.

Figures 3.4-3.7 are included to show comparisons for a higher dimensionless frequency of $25 \times 10^{-6}$. The Branch I and Branch II
neutral points occur at $\sqrt{\text{Re}_x} = 910.4$ and 1977.1, respectively. The square root unit Reynolds number per meter in Figures 3.4-3.6 is $R_u = 961$. In Figure 3.7, $R_u = 877$. Figure 3.4 shows comparisons for suction concentrated in one strip on Panel I at $x = 194.3$ cm with $V_0 = 5.7 \times 10^{-3} U_\infty$. This x-location corresponds to $\sqrt{\text{Re}_x} = 1340$ so the strip is located just upstream of the point where the growth rates are a maximum. The agreement is again very good. Then we present comparisons for suction distributed over more strips, lowering the local suction levels. Three strips are open on each of Panels I and II in Figure 3.5, each strip having a suction level of $V_0 = 1.0 \times 10^{-3} U_\infty$. The locations of the six strips are $x = 184.8, 194.3, 203.8, 238.1, 247.6, \text{ and } 257.1$ cm. The first panel containing the first three strips is upstream of the maximum growth-rate point while the second panel is just slightly downstream of this point. As before the agreement between theory and experiment is fantastic. In Figure 3.6, the suction on Panel I is distributed through even more strips, lowering the local suction levels still more. There is suction through seven strips on Panel I and three on Panel II, the first seven strips having $V_0 = 4.2 \times 10^{-4} U_\infty$ and the next three having $V_0 = 1.1 \times 10^{-3} U_\infty$. The locations of the ten strips are $x = 184.8, 187.9, 191.1, 194.3, 197.5, 200.6, 203.8, 238.1, 247.6, \text{ and } 257.1$ cm. Our closed-form solutions, linear superpositions of individual strip solutions, have accurately predicted the experimental results. Finally, Figure 3.7 has the same configuration as Figure 3.6 except at a lower unit Reynolds number, to simulate moving the strips more upstream toward Branch I. Again the theory has predicted the experimental results very well.
In Figure 3.8, we superpose the results of Figures 3.6 and 3.7. Figure 3.8 shows not only the agreement between theory and experiment, but also the dramatic difference in the amplitude of a disturbance due to moving the strips upstream toward the Branch I neutral point. It is apparent that controlling the disturbances while their growth rates are still small is the most effective approach to suction efficiency just as the theory predicts. Obviously, our optimization scheme is very reliable in predicting where suction should be located and can replace the experiment as a tool for use in designing efficient strip configurations in so far as two-dimensional incompressible flows are concerned.

We find it very remarkable how introducing such a small level of suction, that is, small with respect to the vertical component of the mean-flow velocity, can influence the stability so dramatically. The physical explanation for this must lie somewhere in the idea that the suction draws the critical layer ever so slightly closer to the wall where there is much higher dissipation. We find it amazing, too, that there can be such a drastic difference in disturbance amplitude just from moving this small amount of suction to different locations on the plate. This is the important message of Figure 3.8.
Chapter Four
Flow Over an Axisymmetric Body with Porous Suction Strips

4.1 Introduction

The use of suction through porous strips on axisymmetric bodies for laminar flow control is also of interest. To determine the effectiveness of such an approach and to optimize the number, spacing, and flow rate through such strips, one needs to determine the stability of the flow over a body with suction strips. The first step in such an approach is the calculation of the mean flow. One such available computer code is the Transition Analysis Program System (TAPS)\textsuperscript{23,24}. This is a digital computer program designed to determine transition using the e\textsuperscript{9} method on axisymmetric and two-dimensional bodies in air and in water. The TAPS code contains four basic components: (1) geometry, (2) potential flow, (3) boundary layer, and (4) stability analysis. The boundary layer calculations are nonsimilar and based on the Cebeci-Smith finite-difference boundary layer program\textsuperscript{59} using the Keller box method for solving the boundary-layer equations\textsuperscript{60-62}. However, nonsimilar boundary-layer calculations fail to account for upstream influence. An interacting boundary layer code would take care of this problem, but such a code usually requires prohibitively large amounts of computer storage and fails to converge for large Reynolds numbers.

In this chapter, we present a linearized triple-deck, closed-form solution, which is more efficient and accounts for upstream influence. Consequently, it is ideal for optimization of the suction distribution for laminar flow control. The linearized triple-deck equations are
expected to be valid for strip widths $O(Re^{-3/8} p^*)$ and suction levels $O(Re^{-3/8} U^*)$ for a disturbance introduced at $s^*$. 

4.2 Basic State

Considering incompressible steady axisymmetric flow past a body of revolution, we define the curvilinear coordinate system $(s^*, n^*, \theta)$ shown in Figure 4.1. This is the coordinate system chosen by Mangler$^{63}$ and used most widely today. The surface distance from the stagnation point is $s^*$, the distance normal to the surface is $n^*$, and the circumferential angle is $\theta$. We denote the local surface radius measured from the axis as $r_\#(s^*)$. All starred quantities are dimensional.

We then define a Reynolds number $Re$ at a point $s^* = s_\#$ as

$$Re = \frac{\rho_{\infty} U_{\infty} s^*_\#}{U_{\infty}^*}$$

and introduce dimensionless variables according to

$$u = \frac{u^*}{U_{\infty}^*}, \quad v = \frac{v^*}{U_{\infty}^*} \sqrt{Re}, \quad p = \frac{p^* - p_{\infty}^*}{\rho_{\infty} U_{\infty}^*}, \quad s = \frac{s^*}{s_\#}, \quad n = \frac{n^*}{n_\#} \sqrt{Re}$$

Quantities with subscript $\infty$ indicate edge quantities. The velocity components $u$ and $v$ are tangent to and normal to the surface, respectively. Assuming $r_\#$ to be much larger than the boundary-layer thickness, that is, transverse-curvature effects are negligible, we find as did Mangler that the boundary layer equations for an axisymmetric flow without suction are

$$\frac{\partial}{\partial s} (r_\# u) + r_\# \frac{\partial}{\partial n} v = 0$$
subject to the boundary conditions

\[ u = v = 0 \quad \text{at} \quad n = 0 \]  
\[ u \to 1 \quad \text{as} \quad n \to \infty \]

Solving the system of equations (4.3)-(4.7) will provide the mean flow in our triple-deck solutions.

4.3 Main Deck

We introduce into the boundary layer of the basic state a disturbance, in this case a porous strip with center \( s^*_p \) far from the plate's leading edge \( s^* = 0 \). We define the small parameter

\[ \varepsilon = \text{Re}_\infty^{-1/8} \ll 1 \]  
\[ \text{Re}_\infty = \rho_\infty U_\infty s^*_p / \mu_\infty \]

where \( \text{Re}_\infty \) is the Reynolds number and \( \infty \) denotes edge quantities. According to triple-deck theory\textsuperscript{39-41}, described already in Chapter Two, the streamwise effects of the disturbance occur in a neighborhood \( O(\varepsilon^3 s^*_p) \) of \( s^*_p \). See also the review articles of Stewartson\textsuperscript{42}, Adamson and Messiter\textsuperscript{43}, and Smith\textsuperscript{44}.

The length scales for the main deck are
where $O(X) = O(Y_m) = 1$. The extent of the middle deck in the $n^*$-direction is the same as the boundary-layer thickness of the basic state. Expanding the flow variables in an asymptotic series, we find that

$$s^* = s_p^* + \varepsilon^3 X s_p^*, \quad n^* = \varepsilon^4 Y_m s_p^* \tag{4.9}$$

$$u^*/U_\infty = U(Y_m) + \varepsilon u_1 + \ldots \tag{4.10a}$$

$$v^*/U_\infty = \varepsilon^2 v_1 + \ldots \tag{4.10b}$$

$$(p^* - p_\infty)/\rho_\infty U_*^2 = \varepsilon^2 p_2 + \ldots \tag{4.10c}$$

where $U(Y_m)$ is the basic state dimensionless streamwise velocity evaluated at $s_p^*$. Streamwise variations of the basic state are assumed to be $O(\varepsilon^3)$, so they do not affect this first-order theory. In other words, local changes in transverse curvature in the streamwise direction are assumed negligible.

Considering these expansions, we find that the governing equations for this deck are inviscid but rotational to $O(\varepsilon^3)$, that is,

**Continuity**

$$\frac{\partial u^*}{\partial s^*} + \frac{\partial v^*}{\partial n^*} = 0 \tag{4.11}$$

**$s^*$-Momentum**

$$\rho^* (u^* \frac{\partial u^*}{\partial s^*} + v^* \frac{\partial u^*}{\partial n^*}) = - \frac{\partial p^*}{\partial s^*} \tag{4.12}$$
\[ n^*-\text{Momentum} \]
\[ p^*(u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial n^*}) = - \frac{\partial p^*}{\partial n^*} \]  
(4.13)

Hence, the equations governing the first-order perturbations are

\[ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y_m} = 0 \]  
(4.14a)

\[ U \frac{\partial u_1}{\partial x} + v_1 \frac{dU}{dY_m} = 0 \]  
(4.14b)

\[ \frac{\partial p_2}{\partial Y_m} = 0 \]  
(4.14c)

These give

\[ u_1 = \frac{dU}{dY_m} A_1(X) \]  
(4.15a)

\[ v_1 = -U \frac{dA_1}{dx} \]  
(4.15b)

\[ p_2 = p_2(Y) \]  
(4.15c)

where \( A_1(X) \to 0 \) as \( X \to -\infty \) because the disturbances decay upstream of the porous strip. The functions \( A_1(X) \) and \( p_2(Y) \) must be determined by matching with the upper and lower decks.

4.4 Upper Deck

The middle deck will not satisfy the boundary conditions that demand the flow quantities return to the basic state far away from the body. So we must have an upper deck to match with the middle deck as \( Y_m \to \infty \) that satisfies the boundary conditions far away from the body.

The length scales for this deck are

\[ s^* = s_p^* + \varepsilon^3 X s_p^* \quad n^* = \varepsilon^3 Y s_p^* \]  
(4.16)
where \( O(X) = O(Y_u) = 1 \).

To see how the variables in the upper-deck flow behave, we take the limit of the middle-deck quantities as \( Y_m \to \infty \). The result is

\[
\lim_{Y_m \to \infty} u_1(X, Y_m) = \lim_{Y_m \to \infty} \frac{dU}{dY_m} A_1(X) = 0
\]

(4.17a)

\[
\lim_{Y_m \to \infty} v_1(X, Y_m) = \lim_{Y_m \to \infty} (-U \frac{dA_1}{dX}) = -\frac{dA_1}{dX} = 0
\]

(4.17b)

\[
\lim_{Y_m \to \infty} p_2(X) = p_2(X)
\]

(4.17c)

Equations (4.17) imply that the expansions in this deck are of the form

\[
\frac{u^*}{U^\infty} = 1 + \epsilon^2 \hat{u}_2 + ...
\]

(4.18a)

\[
\frac{v^*}{U^\infty} = \epsilon^2 \hat{v}_2 + ...
\]

(4.18b)

\[
\frac{p^* - p^*_{\infty}}{\rho^\infty U_{\infty}^2} = \epsilon^2 \hat{p}_2 + ...
\]

(4.18c)

From these expansions we see that to \( O(\epsilon^3) \) the governing equations for the upper deck are inviscid and irrotational. They are the same as Eqs. (4.11)-(4.13). Substituting Eq. (4.18) into Eqs. (4.11)-(4.13) gives the following equations governing the first-order perturbations:

\[
\frac{\partial \hat{u}_2}{\partial X} + \frac{\partial \hat{v}_2}{\partial Y_u} = 0
\]

(4.19a)

\[
\frac{\partial \hat{u}_2}{\partial X} = -\frac{\partial \hat{p}_2}{\partial X}
\]

(4.19b)

\[
\frac{\partial \hat{v}_2}{\partial X} = -\frac{\partial \hat{p}_2}{\partial Y_u}
\]

(4.19c)

Matching the middle and upper decks provides the boundary conditions for
these equations. Equations (4.17) and (4.18) imply that

\[ \hat{p}_2(X,0) = p_2(X) \]  
(4.20)

\[ \hat{v}_2(X,0) = - \frac{dA_1}{dX} \]  
(4.21)

To solve Eqs. (4.19)-(4.21), we differentiate Eq. (4.19a) with respect to \( Y_u \), Eq. (4.19b) with respect to \( Y_u \), Eq. (4.19c) with respect to \( X \), and obtain

\[ \frac{\partial^2 \hat{u}_2}{\partial X \partial Y_u} + \frac{\partial^2 \hat{v}_2}{\partial Y_u^2} = 0 \]  
(4.22a)

\[ \frac{\partial^2 \hat{u}_2}{\partial X \partial Y_u} = - \frac{\partial^2 p_2}{\partial X \partial Y_u} \]  
(4.22b)

\[ \frac{\partial^2 \hat{v}_2}{\partial X^2} = - \frac{\partial^2 p_2}{\partial X \partial Y_u} \]  
(4.22c)

These lead to

\[ \frac{\partial^2 \hat{v}_2}{\partial X^2} + \frac{\partial^2 \hat{v}_2}{\partial Y_u^2} = 0 \]  
(4.23)

subject to the condition (4.21) and the following conditions at \( \infty \):

\[ \hat{v}_2(X,\infty) = 0, \quad \hat{v}_2(\pm \infty, Y_u) = 0 \]  
(4.24)

The solution of Eqs. (4.23), (4.21), and (4.24) is

\[ \hat{v}_2 = - \frac{Y_u}{\pi} \int_{-\infty}^{\infty} \frac{dA_1(t)/dt}{(X - t)^2 + Y_u^2} dt \]  
(4.25)

See, for example, Chester\(^64\) for details. Equations (4.19c) and (4.25) imply that
\[
\frac{3v_2}{aX} = \frac{Y_u}{\pi} \int_{-\infty}^{\infty} \frac{2(X - t) dA_1(t)/dt}{[Y_u^2 + (X - t)^2]} dt = -\frac{d\hat{p}_2}{dY_u}
\]

Hence
\[
\hat{p}_2(X, Y_u) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2(Y_u) dA_1(t)/dt}{(X - t)^2 + Y_u^2} dt
\]

or
\[
p_2(X) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dA_1(t)/dt}{(X - t)} dt \tag{4.26}
\]

The integral in Eq. (4.26) denotes the Cauchy principal value, that is
\[
p_2(X) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dA_1(t)/dt}{X - t} dt + \frac{1}{\pi} \int_{X+\Delta}^{\infty} \frac{dA_1(t)/dt}{X - t} dt
\]

where \(0 < \Delta \ll 1\) \tag{4.27}

We now have a relation between the two unknown functions \(p_2(X)\) and \(A_1(X)\), but we must go to the lower deck to determine \(p_2(X)\) or \(A_1(X)\) explicitly.

4.5 Lower Deck

The middle deck will satisfy neither the no-slip condition nor the specified suction velocities at the wall. So we must have a lower deck to match with the middle deck as \(Y_m \rightarrow 0\) that satisfies the boundary conditions at the wall.

The length scales for this deck are
\[ s^* = s^* + \varepsilon^3 X s^* \quad n^* = \varepsilon^5 Y s^* \] (4.28)

where \( O(X) = O(Y, Q) = 1 \).

To see how the variables in the lower-deck flow behave, we take the limit of the middle-deck quantities as \( Y_m \to 0 \). The result is

\[ \lim_{Y_m \to 0} u_1(X, Y_m) = \lim_{Y_m \to 0} \left( \frac{dU}{dY_m} A_1(X) \right) = \lambda A_1(X) \] (4.29a)

\[ \lim_{Y_m \to 0} v_1(X, Y_m) = \lim_{Y_m \to 0} \left( -U \frac{dA_1}{dX} \right) \]

\[ = - \frac{dA_1}{dX} \lim_{Y_m \to 0} \left[ U(0) + \frac{dU}{dY_m} Y_m + ... \right] = -\lambda \frac{dA_1}{dX} Y_m \] (4.29b)

where

\[ \lambda = \left. \frac{dU}{dY_m} \right|_{Y_m=0} \] (4.29c)

Equations (4.29) imply that for this layer

\[ \frac{u^*}{U^*} = \varepsilon \tilde{u} + ... \] (4.30a)

\[ \frac{v^*}{U^*} = \varepsilon^3 \tilde{v} + ... \] (4.30b)

\[ \frac{p^* - \rho\tilde{u}^2}{\rho U^*} = \varepsilon^2 \tilde{p} + ... \] (4.30c)

From these expansions we see that to \( O(\varepsilon^3) \) the governing equations for the lower deck are viscous and rotational. Following Stewartson and Williams 39, we introduce
\[ X = \lambda^{-5/4} x \]
\[ Y_{\lambda} = \lambda^{-3/4} y \]
\[ \bar{p}(X,Y_{\lambda}) = \lambda^{1/2} p(x,y) \quad (4.31) \]
\[ \bar{u}(X,Y_{\lambda}) = \lambda^{1/4} u(x,y) \]
\[ \bar{v}(X,Y_{\lambda}) = \lambda^{3/4} v(x,y) \]
\[ A_1(X) = \lambda^{-3/4} A(x) \]

These transformations keep leading-order terms independent of free-stream variables. Using Eqs. (4.28), (4.30) and (4.31), we find the equations governing the first-order perturbations to be

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.32a) \]
\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \quad (4.32b) \]
\[ \frac{\partial p}{\partial y} = 0 \quad \text{or} \quad p = p(x) \quad (4.32c) \]

To determine the boundary conditions for Eqs. (4.32), we first match the lower and middle decks. To this end, we note that

\[ \lim_{Y_{\lambda} \to \infty} u = y + A(x) \quad (4.33) \]

The wall boundary conditions are

\[ u(x,0) = 0 \quad (4.34a) \]
\[ v(x,0) = \begin{cases} v_{\text{wall}} & x_{\text{LE}} \leq x \leq x_{\text{TE}} \\ 0 & \text{otherwise} \end{cases} \quad (4.34b) \]
where \( x_{LE} \) and \( x_{TE} \) are the leading and trailing edges of the porous strip, respectively, and \( v_{\text{wall}} \) is the specified suction rate in terms of lower-deck variables. As \( x \to -\infty \), we have the initial condition

\[
u \rightarrow y \tag{4.35}\]

to match far upstream with the basic state.

The pressure in the lower deck is related to \( A_1 \) by

\[
\bar{p} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dA_1}{dx} \frac{dt}{x - t} \tag{4.36}
\]

This is found from matching the lower-deck \( \bar{p} \) to the middle-deck \( p_2 \) and then using Eqs. (4.20) and (4.26). Relation (4.36) gives

\[
\lambda^{1/2} p(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \lambda^{1/2} \frac{dA}{dt} \frac{dt}{x - t}
\]

or

\[
p(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dA}{dt} \frac{dt}{x - t}
\]

Finally, we use the condition that

\[
p(\pm\infty) = 0 \tag{4.37}
\]

4.6 Linearized Solution

Following Napolitano and Ressick, Reed, and Nayfeh et al. and for \( |v_{\text{wall}}| \ll 1 \), we assume expansions in the form

\[
u(x,y) = y + v_{\text{wall}} \bar{u}(x,y) + O(v_{\text{wall}}^2) \tag{4.38a}
\]

\[
v(x,y) = v_{\text{wall}} \bar{v}(x,y) + O(v_{\text{wall}}^2) \tag{4.38b}
\]

\[
A(x) = v_{\text{wall}} \bar{A}(x) + O(v_{\text{wall}}^2) \tag{4.38c}
\]

\[
p(x) = v_{\text{wall}} \bar{p}(x) + O(v_{\text{wall}}^2) \tag{4.38d}
\]
Substituting these expansions into Eqs. (4.32)-(4.35), (4.36), and (4.37), we find that the equations governing the first-order perturbations are

\[
\begin{align*}
\bar{u}_x + \bar{v}_y &= 0 \\
y\bar{u}_x + \bar{v} &= -\frac{dp}{dx} + \bar{u}_{yy}
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
\bar{u}(x,0) &= 0 \\
\bar{v}(x,0) &= \begin{cases} 
1 & x_{LE} \leq x \leq x_{TE} \\
0 & \text{otherwise}
\end{cases} \\
\bar{u}(x,\infty) &= \delta \\
\bar{u}(-\infty,y) &= 0 \\
\bar{\rho} &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\delta(t)}{x-t} \, dt \\
\bar{\rho}(\pm\infty) &= 0
\end{align*}
\]

Still following Napolitano and Ressick\textsuperscript{45}, Reed\textsuperscript{21}, and Nayfeh et al.\textsuperscript{22}, first we consider a semi-infinite strip with leading edge at \( x = 0 \). Because the system of equations (4.39)-(4.41) is linear, solutions for finite-length strips can be obtained by translations and superpositions of semi-infinite solutions. We will use the subscript \( \infty \) on barred quantities to indicate semi-infinite solutions. For the semi-infinite problem, boundary condition (4.41b) becomes

\[
\bar{v}_\infty(x,0) = H(x)
\]

where \( H(x) \) is the Heaviside function. To solve the resulting problem, we differentiate the \( x \)-momentum equation (4.40) with respect to \( y \), apply the continuity equation (4.39), and obtain
If we let \( w = \underline{u}_{\lambda y} \), then

\[
w_{yy} - yw_x = 0 \tag{4.44}
\]

We define the Fourier transform in \( x \) of any dependent variable \( r(x) \) by

\[
R(\omega) = \lim_{\alpha \to 0} \int_{-\infty}^{\infty} r(x)e^{-(i\omega + \alpha)x} dx \tag{4.45}
\]

Since \( \underline{u}_\infty, \underline{v}_\infty, \underline{p}_\infty, \) and \( \underline{\delta}_\infty \) all \( \to 0 \) as \( x \to -\infty \), this integral always converges at the lower limit. Using a limit process shows that the integral converges at the upper limit when \( 0 < \alpha < \infty \). The corresponding capital letter will indicate the Fourier transform of the function denoted by the small letter. Using complex variables and defining the polar form of \( \omega \) as \( |\omega|e^{i\phi} \), we find that the only non-integral powers of \( i\omega \) involve the cubic root. The most convenient branch for these turns out to be

\[
[i\omega]^{1/3} = |\omega|^{1/3} \exp\left[\frac{1}{3} i(\phi + \frac{1}{2} \pi)\right]
\]

where \(-3\pi/2 < \phi < \pi/2\), giving the barrier indicated in Figure 4.2.

Applying the Fourier transform in \( x \) to Eq. (4.44) gives Airy's equation

\[
w_{yy} - i\omega y W = 0 \tag{4.46}
\]

One pair of linearly independent solutions of Eq. (4.46) is

\[
Ai[(i\omega)^{1/3} y], \quad Bi[(i\omega)^{1/3} y]
\]

where \( Ai \) and \( Bi \) are Airy functions of the first and second kind, respectively. So the general solution of Eq. (4.46) can be expressed as
\[ W = c_1 \text{Ai}[\omega^{1/3} y] + c_2 \text{Bi}[\omega^{1/3} y] \] (4.47)

Since we are using the branch \( \omega^{1/3} = \exp(i\pi/6) \) and the cut defined by \(-3\pi/2 < \phi < \pi/2\), then \(|\text{arg}[\omega^{1/3} y]| < \pi/3\) and as \( y \to \infty \)

\( \text{Ai}[\omega^{1/3} y] \to 0 \) exponentially

\( \text{Bi}[\omega^{1/3} y] \to \infty \) exponentially

(e.g., Abramowitz and Stegun\(^6\)). Hence, Eq. (4.41c) demands that \( c_2 = 0 \)

so that

\[ W = c_1 \text{Ai}[\omega^{1/3} y] \] (4.48)

Integrating Eq. (4.48) with respect to \( y \), using Eq. (4.41a), and recalling that \( \omega = \omega_0 y' \)

we obtain

\[ U = c_1 \int_0^y \text{Ai}[\omega^{1/3} t] \text{d}t \] (4.49)

Using Eq. (4.49) in the Fourier transform of Eq. (4.41c) gives

\[ \Delta = c_1 \int_0^\infty \text{Ai}[\omega^{1/3} t] \text{d}t = \frac{c_1}{3(\omega)^{1/3}} \] (4.50)

Hence

\[ c_1 = 3(\omega)^{1/3} \Delta \] (4.51)

Taking the Fourier transform of Eq. (4.41e) gives

\[ P = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \delta_t(t) \frac{e^{-i\omega x}}{x-t} \text{d}x \right] \text{d}t \]

\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} \delta_t(t) e^{-i\omega t} \left[ \int_{-\infty}^{\infty} \frac{e^{-i\omega z}}{z} \text{d}z \right] \text{d}t \]

\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} \delta_t(t) e^{-i\omega t}[\pi \text{sgn}\omega] \text{d}t = \omega \text{sgn}\omega \]
so that

$$\omega \Delta = P \text{sgn} \omega$$  \hspace{1cm} (4.52)

where

$$\text{sgn} \omega = \begin{cases} 1, & \omega > 0 \\ -1, & \omega < 0 \end{cases}$$

When \( y = 0 \), Eqs. (4.40) and (4.42) give

$$H(x) + \frac{p_{\infty}}{\rho} = \overline{u_{\infty y}}(x, 0)$$  \hspace{1cm} (4.53)

whose Fourier transform yields

$$\frac{1}{i\omega} + i\omega P = U_{yy}(\omega, 0)$$  \hspace{1cm} (4.54)

Equations (4.49) and (4.54) give

$$i\omega P + \frac{1}{i\omega} = c_1 A_1(0)(i\omega)^{1/3}$$  \hspace{1cm} (4.55)

Equations (4.51), (4.52), and (4.55) constitute three equations in the three unknowns \( c_1, \Delta, \) and \( P \) whose solution is

$$P = -\frac{(i\omega)^{-2/3}}{D}$$  \hspace{1cm} (4.56)

$$c_1 = -3i(i\omega)^{-4/3} \frac{\text{sgn} \omega}{D}$$  \hspace{1cm} (4.57)

$$\Delta = -\frac{i(i\omega)^{-5/3}}{D} \text{sgn} \omega$$  \hspace{1cm} (4.58)

where

$$D = (i\omega)^{4/3} + \theta^{4/3} \text{isgn} \omega$$

and

$$\theta = [-3A_1(0)]^{3/4}$$

is Lighthill's constant. The above solution of the lower-deck equations
for the semi-infinite strip is almost exactly like that found by Napolitano and Ressick\textsuperscript{45}, Reed\textsuperscript{21}, and Nayfeh et al.\textsuperscript{22}.

In the next three sections, we invert the Fourier transforms to find $\overline{p}_\infty$, $\overline{u}_\infty$, and $\overline{\delta}_\infty$, respectively. When integrating from $\omega = -\infty$ to $\omega = +\infty$, we integrate along the contour shown in Figure 4.3 and take the limit as $R \to 0$.

4.7 Lower-Deck Pressure

The inverse transform of Eq. (4.56) gives

\[
\overline{p}_\infty = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(i\omega)^{2/3}}{(i\omega)^{2/3} + \theta^{2/3}} e^{i\omega x} d\omega
\]

\[
= -\frac{1}{2\pi} \int_{0}^{\infty} \frac{e^{i\omega x} d\omega}{-\omega^2 + \theta^{2/3} \omega^{2/3} e^{i5\pi/6}}
\]

\[
- \frac{1}{2\pi} \int_{-\infty}^{0} \frac{e^{i\omega x} d\omega}{-\omega^2 + \theta^{2/3} \omega^{2/3} e^{i5\pi/6}}
\]

\[
- \frac{1}{2\pi} \int_{0}^{\infty} \frac{e^{-i\omega x} d\omega}{-\omega^2 + \theta^{2/3} \omega^{2/3} e^{-i5\pi/6}}
\]

which upon letting $\omega = r^3 \theta$ becomes

\[
\overline{p}_\infty = -\frac{3}{2\pi \theta} \int_{0}^{\infty} \frac{e^{ir^3 \theta x} dr}{-r^4 + e^{i5\pi/6}} - \frac{3}{2\pi \theta} \int_{0}^{\infty} \frac{e^{-ir^3 \theta x} dr}{-r^4 + e^{-i5\pi/6}}
\]

\[\text{(4.59)}\]

The poles of $(r^6 - e^{i5\pi/6})^{-1}$ are

\[r = e^{-i7\pi/24}, \ e^{-i19\pi/24}, \ e^{-i31\pi/24}, \ \text{and} \ e^{-i43\pi/24}.\]  

\[\text{(4.60)}\]

They are shown together with the barrier in Figure 4.4. The poles of
(r^4 - e^{-i5\pi/6})^{-1} are
\[ r = e^{i7\pi/24}, \ e^{i19\pi/24}, \ e^{i31\pi/24}, \ \text{and} \ e^{i43\pi/24} \quad (4.61) \]

They are shown together with the barrier in Figure 4.5.

For \( x < 0 \), Eq. (4.59) can be rewritten as.
\[
\bar{p}_\infty = -\frac{3}{2\pi\theta} \int_0^{\infty} \frac{e^{-ir\theta|x|}}{-r^4 + e^{i5\pi/6}} \, dr - \frac{3}{2\pi\theta} \int_0^{\infty} \frac{e^{ir\theta|x|}}{-r^4 + e^{-i5\pi/6}} \, dr \quad (4.62)
\]

To evaluate each of these two integrals, we use Cauchy's Residue Theorem:

Let \( D \) be the interior domain bounded by a closed contour \( C \) and let \( f(z) \) be regular at all points of \( \overline{D} = D \cup \text{C} \) with the exception of a finite number of singular points \( a_1, a_2, \ldots, a_n \) contained in \( D \). Then the integral of \( f(z) \) around \( C \) is \( 2\pi i \) times the sum of its residues at the points \( a_k \), that is
\[
\oint_C f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{res} \, f(a_k)
\]
where
\[
\text{res} \, f(a_k) = \frac{1}{(m - 1)!} \lim_{z \to a_k} \left[ (z - a_k)^m \cdot f(z) \right]
\]
where \( m \) is the order of the pole at \( a_k \). (See, for example Fuchs and Shabat.)

For the first integral in Eq. (4.62), we consider the closed contour \( C = C_1 \cup C_2 \cup C_3 \) shown in Figure 4.6, where \( C_1 = [0,R] \), \( C_2 = \{ r | r \in \text{circular arc from R to Rexp(-i\pi/6)} \} \), and \( C_3 = [Rexp(-i\pi/6),0] \). Thus, we have
because none of the poles in Eq. (4.60) lies within the closed contour C. As $R \to \infty$,

$$\oint_{C_2} \frac{e^{-ir^3\theta|x|}}{-r^4 + e^{i5\pi/6}} \, dr \to 0,$$

so that

$$\oint_{0}^{\infty} \frac{e^{-ir^3\theta|x|}}{-r^4 + e^{i5\pi/6}} \, dr = \oint_{0}^{\infty} e^{-i\pi/6} \frac{e^{-ir^3\theta|x|}}{-r^4 + e^{i5\pi/6}} \, dr \tag{4.63}$$

For the second integral in Eq. (4.62), we consider the closed contour consisting of $[0,R]$, \{r|r \in\text{circular arc from } R \text{ to } R\exp(i\pi/6)\}, and $[R\exp(i\pi/6),0]$ where $R \to \infty$. None of the poles in Eq. (4.61) lies within this closed contour so that

$$\oint_{0}^{\infty} \frac{e^{ir^3\theta|x|}}{-r^4 + e^{-i5\pi/6}} \, dr = \oint_{0}^{\infty} e^{i\pi/6} \frac{e^{ir^3\theta|x|}}{-r^4 + e^{-i5\pi/6}} \, dr \tag{4.64}$$

Using Eqs. (4.63) and (4.64) in Eq. (4.59), we have

$$\overline{\rho}_{\infty} = -\frac{3}{2\pi\theta} \oint_{0}^{\infty} e^{-i\pi/6} \frac{e^{-ir^3\theta|x|}}{-r^4 + e^{i5\pi/6}} \, dr - \frac{3}{2\pi\theta} \oint_{0}^{\infty} e^{i\pi/6} \frac{e^{ir^3\theta|x|}}{-r^4 + e^{-i5\pi/6}} \, dr \tag{4.65}$$

Letting $\rho = r\exp(i\pi/6)$ in the first integral in Eq. (4.65) and $\rho = r\exp(-i\pi/6)$ in the second integral gives
\[
\overline{\rho}_\infty = -\frac{3}{2\pi \theta} \int_0^\infty \frac{e^{-i\pi/6} e^{-\rho^3 \theta |x|}}{-\rho^4 e^{-i2\pi/3} + e^{i5\pi/6}} d\rho - \frac{3}{2\pi \theta} \int_0^\infty \frac{e^{i\pi/6} e^{-\rho^3 \theta |x|}}{-\rho^4 e^{-i2\pi/3} + e^{-i5\pi/6}} d\rho
\]

Combining these two integrals and observing that there are no poles at \(\rho = 0\), we obtain

\[
\overline{\rho}_\infty = \frac{3}{\pi \theta} \int_0^\infty \frac{1}{\rho^4 + 1} e^{-\rho^3 \theta |x|} d\rho \quad \text{for} \quad x < 0 \quad (4.66)
\]

For \(x > 0\), Eq. (4.59) can be rewritten as

\[
\overline{\rho}_\infty = -\frac{3}{2\pi \theta} \int_0^\infty \frac{e^{i\theta \rho |x|}}{-r^4 + e^{i5\pi/6}} dr - \frac{3}{2\pi \theta} \int_0^\infty \frac{e^{-i\theta \rho |x|}}{-r^4 + e^{-i5\pi/6}} dr \quad (4.67)
\]

For the first integral, we consider the closed contour consisting of \([0,R], \{r|re \text{ circular arc from } R \text{ to } R e^{i\pi/6}\}, \text{ and } [R e^{i\pi/6}, 0]\); none of the poles calculated in Eq. (4.60) lies within this closed contour. For the second integral, we consider the closed contour consisting of \([0,R], \{r|re \text{ circular arc from } R \text{ to } R e^{-i\pi/6}\}, \text{ and } [R e^{-i\pi/6}, 0]\); none of the poles calculated in Eq. (4.61) lies within this closed contour. Following a reasoning similar to that in the case \(x < 0\) and letting \(R \to \infty\), we rewrite Eq. (4.67) as

\[
\overline{\rho}_\infty = -\frac{3}{2\pi \theta} \int_0^\infty \frac{e^{i\pi/6} e^{i\theta \rho |x|}}{-r^4 + e^{i5\pi/6}} dr - \frac{3}{2\pi \theta} \int_0^\infty \frac{e^{-i\pi/6} e^{-i\theta \rho |x|}}{-r^4 + e^{-i5\pi/6}} dr \quad (4.68)
\]

Letting \(\rho = re^{i\pi/6}\) in the first integral and \(\rho = re^{-i\pi/6}\) in the
second, we have

\[
\overline{p}_\infty = -\frac{3}{2\pi} \int_0^\infty \frac{e^{i\pi/6} e^{-\rho^3 \theta |x|}}{-\rho^e e^{-i(2\pi/3) + e^{i5\pi/6}}} + \frac{3}{2\pi} \int_0^\infty \frac{e^{-i\pi/6} e^{-\rho^3 \theta |x|}}{-\rho^4 e^{i2\pi/3} + e^{-i5\pi/6}} dp
\]

Combining these two integrals and observing that there are no poles at \( \rho = 0 \), we obtain

\[
\overline{p}_\infty = \frac{3}{2\pi} \int_0^\infty \frac{1}{\rho^8 - \sqrt{3} \rho^4 + 1} e^{-\rho^3 \theta |x|} dp \quad \text{for} \quad x > 0 \quad (4.69)
\]

Summing up what has been done in this section, we find that

\[
\overline{p}_\infty = \frac{3}{2H(x)} \int_0^\infty \frac{e^{-\rho^3 \theta |x|}}{\rho^8 - \sqrt{3} \rho^4 H(x) + 1} dp \quad (4.70)
\]

which agrees with that of Napolitano and Ressick\(^{45}\), Reed\(^{21}\), and Nayfeh et al.\(^{22}\). This in turn implies that

\[
\overline{p}_\infty = \left(-\frac{1}{2}\right)H(x) \frac{3}{\pi} \int_0^\infty \frac{\rho^e e^{-\rho^3 \theta |x|}}{\rho^8 - \sqrt{3} \rho^4 H(x) + 1} dp \quad (4.71)
\]

4.8 **Lower Deck \( \overline{u}_\infty \)**

Taking the inverse Fourier transform of Eq. (4.49) and using Eq. (4.57), we find that
\[
\overline{u}_\infty = -\frac{3}{2\pi} \int_{-\infty}^{\infty} \frac{i(\omega)^{-1/3} \text{sgn} \omega e^{i\omega x}}{\text{sgn} \omega (i\omega)^{1/3} + \theta^{1/3}} \int_0^\infty \text{Ai}[i(\omega)^{1/3} t] dt d\omega
\]

\[
= -\frac{3}{2\pi} \int_{\infty}^{0} e^{i\omega x} e^{-i\omega |x|} \int_{-\infty}^{\infty} e^{-i\pi/6 \omega^{1/3} + \theta^{1/3}} e^{-i2\pi/3 \omega^{1/3}} \text{Ai}(e^{-i\pi/6 \omega^{1/3} t} dt d\omega
\]

\[
\times \int_{0}^{\infty} e^{-i\pi/6 \omega^{1/3} t} dt d\omega = \frac{3}{2\pi}
\]

\[
\times \int_{0}^{\infty} e^{i\omega x} e^{-i\omega |x|} \int_{-\infty}^{0} e^{-i\pi/6 \omega^{1/3} + \theta^{1/3}} e^{-i2\pi/3 \omega^{1/3}} \text{Ai}(e^{i\pi/6 \omega^{1/3} t} dt d\omega)
\]

\[
= -\frac{3}{2\pi} \int_{0}^{\infty} e^{-i\omega x} e^{-i\omega |x|} \int_{0}^{\infty} e^{-i\pi/6 \omega^{1/3} + \theta^{1/3}} e^{-i2\pi/3 \omega^{1/3}} \text{Ai}(e^{-i\pi/6 \omega^{1/3} t} dt d\omega
\]

\[
\times \int_{0}^{\infty} e^{-i\pi/6 \omega^{1/3} t} dt d\omega
\]

We let \(\eta = \omega^{1/3} t \exp(i\pi/6)\) and \(\eta = \omega^{1/3} t \exp(-i\pi/6)\) in the first and second integrals, respectively, and obtain

\[
\overline{u}_\infty = -\frac{3}{2\pi} \int_{0}^{\infty} e^{i\omega x} \int_{-\omega^{1/3} + \theta^{1/3}}^{0} e^{i\pi/6 \omega^{1/3} y} \text{Ai}(\eta) d\eta d\omega
\]

\[
-\frac{3}{2\pi} \int_{0}^{\infty} e^{-i\omega x} \int_{-\omega^{1/3} + \theta^{1/3}}^{0} e^{-i\pi/6 \omega^{1/3} y} \text{Ai}(\eta) d\eta d\omega
\]

Letting \(\omega = r^3 \theta\) in Eq. (4.73), we have
The poles excluding $r = 0$ for these two integrals are the same as in Eqs. (4.60) and (4.61), respectively. Hence, we consider the same closed contours we used for the pressure.

For $x < 0$, Eq. (4.74) can be rewritten as

$$\bar{u}_\infty = -\frac{9}{2\pi \theta^2} \int_0^\infty \frac{e^{-i\rho^3|x|}}{\rho^2 + \rho^{3/2}} \int_0^{\pi/6} e^{\pi/6} \rho^{1/3} y \mathrm{Ai}(\eta) \mathrm{d}\eta \mathrm{d}\rho$$

$$(4.75)$$

Rotating the contour of integration by $-\pi/6$ in the first integral and by $\pi/6$ in the second integral and letting $r = \rho \exp(-i\pi/6)$ in the first integral and $r = \rho \exp(i\pi/6)$ in the second, we obtain

$$\bar{u}_\infty = -\frac{9}{2\pi \theta^2} \int_0^\infty \frac{e^{-\rho^3\theta|x|}}{\rho^2 + \rho^{3/2}} \int_0^{\pi/6} e^{\pi/6} \rho^{1/3} y \mathrm{Ai}(\eta) \mathrm{d}\eta \mathrm{d}\rho$$

$$(4.76)$$

We note that there are poles at $\rho = 0$ to consider. Splitting the integrands into partial fractions, however, yields

$$\bar{u}_\infty = -\frac{9}{\pi \theta^2} \int_0^\infty \frac{\rho^{\theta^{1/3} y}}{\rho^2 + 1} \int_0^{\rho^{\theta^{1/3} y}} \mathrm{Ai}(\eta) \mathrm{d}\eta \mathrm{e}^{-\rho^3 \theta |x|} \mathrm{d}\rho$$

$$(4.77)$$
Differentiating Eq. (4.77) yields

\[
\overline{u}_{\infty y} = -\frac{9}{\pi^3} \int_0^\infty \frac{\rho^2}{\rho^3 + 1} \text{Ai}(\rho \theta^{1/3}) e^{-\rho^3 \theta} |x| d\rho \tag{4.78}
\]

\[
\overline{u}_{\infty y} = -\frac{9}{\pi^3} \int_0^\infty \frac{\rho^3}{\rho^3 + 1} \text{Ai}'(\rho \theta^{1/3}) e^{-\rho^3 \theta} |x| d\rho \tag{4.79}
\]

For \(x > 0\), Eq. (4.74) gives

\[
\overline{u}_{\infty} = -\frac{9}{2\pi^2} \int_0^\infty \frac{e^{i\rho^3 \theta} |x|}{-r^7 + r^3 e^{i5\pi/6}} \int_0^{\infty} e^{i\pi/6} r^1 \theta^{1/3} y \text{Ai}(\eta) dn d\rho
\]

\[
= -\frac{9}{2\pi^2} \int_0^\infty \frac{e^{-i\rho^3 \theta} |x|}{-r^7 + r^3 e^{-i5\pi/6}} \int_0^{\infty} e^{-i\pi/6} r^1 \theta^{1/3} y \text{Ai}(\eta) dn d\rho
\]

\[
= -\frac{9}{2\pi^2} \int_0^\infty \frac{e^{i\pi/6}}{-r^7 + r^3 e^{i5\pi/6}} \int_0^{\infty} e^{i\pi/6} r^1 \theta^{1/3} y \text{Ai}(\eta) dn d\rho
\]

\[
-\frac{9}{2\pi^2} \int_0^\infty \frac{e^{-i\pi/6}}{-r^7 + r^3 e^{-i5\pi/6}} \int_0^{\infty} e^{-i\pi/6} r^1 \theta^{1/3} y \text{Ai}(\eta) dn d\rho
\]

\[
\overline{u}_{\infty} = -\frac{9}{2\pi^2} \int_0^\infty \left[ \frac{1}{\rho^7 + \rho^3 e^{i17\pi/6}} \int_0^{\infty} e^{i\pi/3} \rho^1 \theta^{1/3} y \text{Ai}(\eta) dn 
\right]
\]

\[
+ \frac{1}{\rho^7 + \rho^3 e^{-i17\pi/6}} \int_0^{\infty} e^{-i\pi/3} \rho^1 \theta^{1/3} y \text{Ai}(\eta) dn \right] e^{-\rho^3 \theta} |x| d\rho \tag{4.81}
\]

according to Cauchy's theorem. Letting \(r = \rho \exp(i\pi/6)\) in the first integral and \(r = \rho \exp(-i\pi/6)\) in the second integral yields

Again there are poles at \(\rho = 0\) to deal with. Splitting the integrands into partial fractions and using the transformations \(\tau = \rho^3 \theta |x| \exp(i\pi)\)
and $\tau = \rho^{3/2} |x| \exp(-i\pi)$ in the integrals containing the poles, we find that

$$\bar{u}_{\infty} = -\frac{3}{2\pi i} \frac{|x|^{2/3}}{\theta^{1/3}} \left\{ \int_{\tau}^{\infty} e^{i\pi \tau} \int_{0}^{\infty} \frac{e^{\tau y}}{|x|^{1/3}} \text{Ai}(n) \text{d}n \text{d}\tau \right\}$$

$$+ \int_{-\infty}^{0} e^{\tau \frac{1}{3}} \int_{0}^{\infty} (\tau^{1/3} y) \frac{e^{|x|^{1/3}}}{|x|^{1/3}} \text{Ai}(n) \text{d}n \text{d}\tau$$

$$+ \frac{9}{2\pi \theta^2} \int_{0}^{\infty} \frac{e^{-i7\pi/6}}{\rho^4 + e^{-i7\pi/6}} \int_{0}^{\infty} e^{i\pi/3 \rho \theta^{1/3}} y \text{Ai}(n) \text{d}n \text{d}\rho$$

$$+ \frac{9}{2\pi \theta^2} \int_{0}^{\infty} \frac{e^{i7\pi/6}}{\rho^4 + e^{i7\pi/6}} \int_{0}^{\infty} e^{-i\pi/3 \rho \theta^{1/3}} y \text{Ai}(n) \text{d}n \text{d}\rho$$

for $x > 0$ \hspace{1cm} (4.82)

It follows from Eq. (4.82) that

$$\bar{u}_{\infty} = -\frac{3}{2\pi i} \frac{|x|^{1/3}}{\theta^{1/3}} \left\{ \int_{\tau}^{\infty} e^{i\pi \tau} \int_{0}^{\infty} \frac{e^{\tau y}}{|x|^{1/3}} \text{Ai}(\tau^{1/3} \frac{y}{|x|^{1/3}}) \text{d}n \text{d}\tau \right\}$$

$$+ \int_{-\infty}^{0} e^{\tau \frac{1}{3}} \int_{0}^{\infty} (\tau^{1/3} y) \frac{e^{|x|^{1/3}}}{|x|^{1/3}} \text{Ai}(\tau^{1/3} \frac{y}{|x|^{1/3}}) \text{d}n \text{d}\tau$$

$$+ \frac{9}{2\pi \theta^2} \int_{0}^{\infty} \frac{e^{-i5\pi/6}}{\rho^4 + e^{-i5\pi/6}} \text{Ai}(e^{i\pi/3 \rho \theta^{1/3}} y) e^{-\rho^{3/2} |x|} \text{d} \rho$$

$$+ \frac{9}{2\pi \theta^2} \int_{0}^{\infty} \frac{e^{i5\pi/6}}{\rho^4 + e^{i5\pi/6}} \text{Ai}(e^{-i\pi/3 \rho \theta^{1/3}} y) e^{-\rho^{3/2} |x|} \text{d} \rho$$

\hspace{1cm} (4.83)
\[
\bar{u}_{\infty y} = -\frac{3}{2\pi i} \frac{1}{\theta^{1/3}} \left\{ \int_0^\infty e^{i\pi} \frac{e^{\frac{\pi}{\tau}}}{|x|^{1/3}} Ai(\tau^{1/3} \frac{y}{|x|^{1/3}}) d\tau + \int_{\infty}^0 e^{-i\pi} \frac{e^{\frac{\pi}{\tau}}}{|x|^{1/3}} Ai(\tau^{1/3} \frac{y}{|x|^{1/3}}) d\tau \right\} - \frac{9i}{2\pi \theta^{4/3}} \int_0^\infty \frac{\rho^{3}}{\rho^4 + e^{i7\pi/6}} \times Ai(e^{i\pi/3 \rho^{1/3} y})e^{-\rho^3 \theta |x|} d\rho + \frac{9i}{2\pi \theta^{4/3}} \int_0^\infty \frac{\rho^{3}}{\rho^4 + e^{-i7\pi/6}} \times Ai(e^{-i\pi/3 \rho^{1/3} y})e^{-\rho^3 \theta |x|} d\rho \quad (4.84)
\]

for \( x > 0 \). Because of the transformations we have used, the path of integration for the integrals of \( \bar{u}_{\infty} \) and its derivatives containing the poles is shown in Figure 4.7 for the case \( x > 0 \). We note that the original branch cut has moved from \( \pi/2 \) to \( \pi \).

4.9 Lower Deck \( \bar{\delta}_{\infty} \)

The inverse transform of Eq. (4.58) gives

\[
\bar{\delta}_{\infty} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-i(i\omega)^{-s/3} \text{sgn} \omega}{(i\omega)^{s/3} + \theta^{s/3} \text{isgn} \omega} e^{i\omega x} d\omega
\]

\[
= \frac{1}{2\pi} \int_0^{\infty} \frac{e^{i\omega x} d\omega}{\omega^3 - \theta^{s/3} e^{15\pi/6} |\omega|^{5/3}}
\]

\[
+ \frac{1}{2\pi} \int_{-\infty}^0 \frac{e^{i\omega x} d\omega}{\omega^3 - \theta^{s/3} e^{-15\pi/6} |\omega|^{5/3}}
\]

\[
= \frac{1}{2\pi} \int_0^{\infty} \frac{e^{i\omega x} d\omega}{\omega^3 - \theta^{s/3} \omega^{5/3} e^{15\pi/6}}
\]

\[
+ \frac{1}{2\pi} \int_0^{\infty} \frac{e^{-i\omega x} d\omega}{\omega^3 - \theta^{s/3} \omega^{5/3} e^{-15\pi/6}}
\]

Letting \( \omega = r^3 \theta \), we rewrite Eq. (4.85) as
For $x < 0$, following steps similar to those used earlier, we rewrite Eq. (4.86) as

$$\bar{\delta}_\infty = \frac{3}{2\pi^2} \int_0^\infty \frac{e^{i\pi/6}}{r^7 - r^3 e^{i5\pi/6}} + \frac{3}{2\pi^2} \int_0^\infty \frac{e^{-i\pi/6}}{r^7 - r^3 e^{-i5\pi/6}}$$

(4.86)

For $x < 0$, following steps similar to those used earlier, we rewrite Eq. (4.86) as

$$\bar{\delta}_\infty = -\frac{3}{\pi\theta^2} \int_0^\infty \frac{\rho}{\rho^8 + 1} e^{-\rho^3 |x|} d\rho$$

(4.87)

The pole at $r = 0$ again does not give any trouble.

For $x > 0$, following steps similar to those used above, we obtain from Eq. (4.86) that

$$\bar{\delta}_\infty = \frac{3}{2\pi^2} \int_0^\infty \frac{e^{-i\pi/6}}{\rho^7 - \rho^3 e^{i5\pi/6}} + \frac{3}{2\pi^2} \int_0^\infty \frac{e^{i\pi/6}}{\rho^7 - \rho^3 e^{-i5\pi/6}}$$

(4.88)

Letting $\tau = \rho^3 |x| \exp(i\pi)$ in the first integral of Eq. (4.88) and $\tau = \rho^3 |x| \exp(-i\pi)$ in the second integral yields

$$\bar{\delta}_\infty = -\frac{1}{2\pi^2} \frac{|x|^{2/3}}{\theta^{4/3}} \left[ \int_0^\infty e^{\tau/\theta^{4/3}} d\tau + \int_0^\infty e^{i\pi/\theta} d\tau + \frac{e^{i\pi/\theta}}{\theta^{4/3}} d\tau \right]$$

(4.89)

The transformations have shifted the original branch cut for the first two integrals in Eq. (4.89) from $\pi/2$ to $\pi$ and the original contour to the one shown in Figure 4.7. Using the Hankel contour integral formula, we find that

$$\bar{\delta}_\infty = -\frac{|x|^{2/3}}{\theta^{4/3} \Gamma(5/3)} - \frac{3}{2\pi^2} \int_0^\infty \frac{\sqrt{3} \rho^5 - \rho}{\rho^8 - \sqrt{3} \rho^4 + 1} e^{-\rho^3 |x|} d\rho$$

(4.90)
for $x > 0$, where $\Gamma$ is the gamma function.$^{65\text{-}67}$.

**4.10 Constant-Suction, Finite-Length Strip Solutions in the Lower Deck**

Let $x_{LE}$ and $x_{TE}$ denote the leading and trailing edges of the strip, respectively, in triple-deck variables; $s^*_p$ will be taken to be the middle of the strip $\frac{1}{2} (s^*_L + s^*_E)$.

To obtain solutions for finite-length strips, we appeal to the linearity of Eqs. (4.39)-(4.41) and use translations and superpositions of semi-infinite solutions. Then, the boundary condition (4.42) implies that

\[ u(x, y) = u_\infty(x - x_{LE}, y) - u_\infty(x - x_{TE}, y) \quad (4.91) \]

\[ p(x) = p_\infty(x - 0_00 - x_{LE}) - p_\infty(x - 0_00 - x_{TE}) \quad (4.92) \]

\[ \delta(x) = \delta_\infty(x - 0_00 - x_{LE}) - \delta_\infty(x - 0_00 - x_{TE}) \quad (4.93) \]

and similarly for their derivatives, where $u_\infty$ is given by either Eq. (4.77) or Eq. (4.82), depending on the signs of the arguments given above; $p_\infty$ is given by either Eq. (4.66) or Eq. (4.69); and $\delta_\infty$ is given by either Eq. (4.87) or Eq. (4.90). Here, we have assumed that $v^*_{wall}$, the dimensional suction rate, is constant over the whole strip. To accommodate varying suction over the strip, we would approximate the suction distribution by steps and just use more translations and superpositions of semi-infinite solutions.

**4.11 Composite Solution in the $y^*$-direction for One Strip**

In the previous sections, we used the method of matched asymptotic expansions to obtain solutions valid in the lower deck, solutions valid in the middle deck, and solutions valid in the upper deck. In order to
combine all of these solutions into one set of solutions valid everywhere, we form composite expansions (see, for example, Nayfeh\textsuperscript{32,33}). To this end, we obtain, in the original dimensional variables \( s^* \) and \( n^* \),

\[
\frac{u^*}{U^*_\infty} = U(n) + \text{Re}_{\infty}^{1/4} \lambda^{-1/2} \text{v}_{\text{wall}}^{*} \frac{U^*}{U^*_\infty} \left[ \frac{\left(U'(0) - 1\right)}{U'(n) - 1} \right] \text{Re}_{\infty}^{3/8} \lambda^{5/4} \frac{s^* - s^*_p}{s^*_p} \frac{U^*}{U^*_\infty} \left[ \frac{\lambda^{5/4} s^* - s^*_p}{s^*_p} \text{Re}_{\infty}^{3/8} \lambda^{3/4} \right] + \frac{\lambda}{u} \left[ \frac{s^* - s^*_p}{s^*_p} \text{Re}_{\infty}^{3/8} \lambda^{5/4} \right] \tag{4.94}
\]

valid to first order, where \( u \) and \( \delta \) are given by Eqs. (4.91) and (4.93), respectively. These are valid in a neighborhood of \( s^*_p \), the center of the strip. As before,

\[
\text{Re}_{\infty} = \frac{s^*_p U^*}{U^*_\infty}, \quad \lambda = \left. \frac{dU}{dT} \right|_{Y_m=0} \tag{4.96a}
\]

and \( v_{\text{wall}}^{*} \) is the suction rate (which must be input as a negative number). The variable \( n \) is defined as

\[
\eta = n^* \sqrt{\frac{U^*}{S^* U^*_\infty}} \tag{4.96b}
\]

We note that in the expression (4.94) for \( u^*/U^*_\infty \) we find forms of the Airy function contained in \( u \). One could either integrate these expressions numerically or appeal to asymptotic expansions. 4.12

4.12 Composite Solution for \( n \) Strips

We consider now \( n \) porous strips centered at \( s^*_1, s^*_2, \ldots, s^*_n \) ordered so that \( s^*_1 < s^*_2 < \ldots < s^*_n \). We define the Reynolds number at strip \( i \) as
where \( U^* \) and \( \nu^* \) are the dimensional edge velocity and kinematic viscosity at the strip center \( s^* \). Neglecting the influence of all downstream strips, we propose the dimensional flow quantities, denoted by *, in the neighborhood of the nth strip to be

\[
\frac{u^*}{U^*} = U(\eta) + \sum_{i=1}^{n} \frac{1}{Re^*_i} \frac{1}{\lambda_i^{1/2}} \frac{v^*_{wall_i}}{U^*_i} \left[ \left( U'(\eta) - 1 \right) \right] \\
\times \left[ \frac{\lambda_i^{5/4}}{s^*_i} \left[ s^*Re^*_i^{3/8} - s^*_iRe^*_i^{3/8} \right] \right] + \left( \frac{\lambda_i^{5/4}}{s^*_i} \left[ s^*Re^*_i^{3/8} - s^*_iRe^*_i^{3/8} \right] \right)
\]

\[
\times n^* \frac{Re^*_i^{5/8}}{s^*_i^{3/4}} (4.97)
\]

\[
\frac{p^*-p^*_\infty}{\rho^*U^*_{\infty}^2} = \sum_{i=1}^{n} \frac{1}{Re^*_i} \frac{1}{\lambda_i^{1/4}} \frac{v^*_{wall_i}}{U^*_i} \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{\lambda_i^{5/4}}{s^*_i} \left[ tRe^*_i^{3/8} - s^*_iRe^*_i^{3/8} \right] \right) \\
\times \left( s^* - t \right)^5 \left( s^* - t \right)^2 + n^*^2 \right) \frac{dt}{(4.98)}
\]

where \( \bar{u} \) and \( \bar{v} \) are defined by Eqs. (4.91) and (4.93), respectively. As before, \( v^*_{wall_i} \), the dimensional suction rate at the ith strip, is negative for suction.

All the mean flow profiles involved in Eq. (4.97) are functions of

\[
\eta = n^* \frac{U^*}{\nu^*_{s^*}} (4.99)
\]

where \( s^* \), \( U^*_\infty \), and \( \nu^*_\infty \) are the dimensional surface distance, streamwise
edge velocity, and kinematic viscosity for the position at which each flow quantity is to be evaluated. The Reynolds number \( Re \) is then defined as

\[
Re = \frac{U^*s^*}{v^*}
\]  

(4.100)

Expanding the equations in triple-deck variables in the neighborhood of the \( ith \) strip, we evaluated the mean flow \( U \) and its derivatives at the center of the strip \( s^*_i \). However, the streamwise variable \( s^* \) is a slow scale in the mean flow solution and therefore, for consistency, when we move far away from strip \( i \), the \( s^* \)-dependence should be restored to the mean flow profile. We also believe that as in the two-dimensional case the linear triple-deck theory will overpredict the downstream influence. We correct this again by scaling the \( x \) (not \( x_i \)) in the arguments of \( \delta \) and \( \bar{u} \) with the \( s^* \)-Reynolds number at \( s^* \) and the \( y \) with \( s^* \).

In the next chapter we will perform a stability analysis on these closed-form solutions and develop an optimization scheme for predicting efficient strip configurations.

We neglect transverse curvature to this order so that locally the flow sees a flat plate making the resulting equations of this formulation the same as in the two-dimensional problem in References 21 and 22. Transverse curvature and its streamwise variations larger than \( O(\varepsilon^3) \) would violate the local triple-deck assumptions at the disturbance point rendering this triple-deck analysis invalid. For bodies with large curvatures near the disturbance points we would have to solve the mean flow by some interacting boundary-layer technique with the equations including curvature terms.
Also, in an adverse pressure gradient situation, if the boundary layer separates without suction and remains detached over some distance, the triple-deck closed-form solutions would break down. This is because of the dependence of the solutions on the mean flow profile of the body without suction. Again we suggest using some interacting boundary layer technique.
Chapter Five
Stability of Flow Over an Axisymmetric Body
With Porous Strips

5.1 Introduction

Laminar flow control is a concept being considered for submerged, streamlined axisymmetric bodies, too. Having developed linearized triple-deck, closed form solutions for the mean-flow quantities in Chapter Four, we now determine the effectiveness of applying suction through porous strips by stability analysis. We develop a linear optimization scheme to find optimal number, spacing, and mass-flow rate through such strips.

5.2 Stability Problem Formulation

In this chapter we consider a steady axisymmetric incompressible boundary-layer flow past a body of revolution at zero angle-of-attack with porous suction strips. We consider the same curvilinear coordinate system $(s^*, n^*, \theta)$ shown in Figure 4.1 and assume the body's transverse curvature and its streamwise variations to be negligible to first order. The Navier-Stokes equations in these coordinates for an axisymmetric flow under these assumptions reduce to

**Continuity**

\[
\frac{\partial u^*}{\partial s^*} + \frac{\partial v^*}{\partial n^*} = 0 \tag{5.1a}
\]

**s*-Momentum**

\[
\rho^*(\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial s^*} + v^* \frac{\partial u^*}{\partial n^*}) = - \frac{\partial p^*}{\partial s^*} + \mu^*(\frac{\partial^2 u^*}{\partial s^*^2} + \frac{\partial^2 u^*}{\partial n^*^2}) \tag{5.1b}
\]
n*-Momentum

\[ p^*(\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial s^*} + v^* \frac{\partial v^*}{\partial n^*}) = - \frac{\partial p^*}{\partial n^*} + \mu^* (\frac{\partial^2 v^*}{\partial s^* n^*} + \frac{\partial^2 v^*}{\partial n^* n^*}) \]  \hspace{1cm} (5.1c)

All starred quantities are dimensional.

To study the stability of an axisymmetric flow, we first define a Reynolds number \( R \) as

\[ R = \frac{\rho_{\infty} U_{\infty} \delta^*}{\mu_{\infty}} \]  \hspace{1cm} (5.2)

and introduce dimensionless variables into Eqs. (5.1) according to

\[ u^* = \frac{u}{U_{\infty}}, \quad v^* = \frac{v}{U_{\infty}}, \quad p^* = \frac{p - p_{\infty}}{\rho_{\infty} U_{\infty}^2} \]  \hspace{1cm} (5.3a)

\[ x^* = \frac{x}{\delta^*}, \quad y^* = \frac{y}{\delta^*}, \quad t^* = \frac{t}{U_{\infty} \delta^*} \]  \hspace{1cm} (5.3b)

Variables with subscript \( \infty \) indicate edge conditions. The quantity \( \delta^* \) is the dimensional displacement thickness.

Then we superpose small disturbances onto the mean flow quantities to obtain total flow quantities of the form

\[ \hat{q}(x,y,t) = q_0(x,y) + q_1(x,y,t) \]

where \( q_0(x,y) \) stands for a basic state quantity and \( q_1(x,y,t) \) stands for a small unsteady disturbance. Substituting the total flow quantities \( \hat{u}, \hat{v}, \) and \( \hat{p} \) into the nondimensionalized Navier-Stokes equations, subtracting the basic state, and linearizing, we find to first order that the disturbance equations are given by

\[ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \]  \hspace{1cm} (5.4a)
The boundary conditions for the disturbance equations (5.4) are given by

\[ u_1 = v_1 = 0 \quad \text{at} \quad y = 0 \] (5.5a)

\[ u_1, v_1 \to 0 \quad \text{as} \quad y \to \infty \] (5.5b)

These were shown to be reasonable approximations by Lekoudis\(^{37}\).

Then we use the method of multiple scales\(^{32,33}\) to solve Eqs. (5.4)-(5.5). As in the two-dimensional case, we expand \(u_1, v_1,\) and \(p_1\) in the form of traveling, harmonic Tollmien-Schlichting waves given by

\[ u_1 = \left[ u(x_1, y) + O(\epsilon) \right] \exp(i\theta) \] (5.6a)

\[ v_1 = \left[ v(x_1, y) + O(\epsilon) \right] \exp(i\theta) \] (5.6b)

\[ p_1 = \left[ p(x_1, y) + O(\epsilon) \right] \exp(i\theta) \] (5.6c)

where

\[ \frac{\partial \theta}{\partial x} = k(x_1) \] (5.7a)
\[
\frac{\partial \theta}{\partial t} = -\omega \tag{5.7b}
\]

The slow scale \( x_1 = \varepsilon x \) where \( \varepsilon = \frac{1}{R} \), describes the slow variation of the amplitudes of the waves in the streamwise direction over a wavelength. Also, we consider the case of spatial stability, so that the wavenumber \( k \) is complex and the frequency \( \omega \) is real.

Substituting Eqs. (5.6)-(5.7) into Eqs. (5.4)-(5.5), and considering only mean flows of the form

\[
U_0 = U_0(x_1,y) \tag{5.8a}
\]

\[
V_0 = \varepsilon \hat{V}_0(x_1,y), \quad \hat{V}_0 = O(1) \tag{5.8b}
\]

\[
P_0 = P_0(x_1) \tag{5.8c}
\]

we find to first order that

\[
i(\kappa U_0 - \omega)u + v \frac{dU_0}{dy} = -ikp + \frac{1}{R} \left( \frac{d^2u}{dy^2} - k^2u \right) \tag{5.9b}
\]

\[
i(\kappa U_0 - \omega)v = -\frac{dp}{dy} + \frac{1}{R} \left( \frac{d^2v}{dy^2} - k^2v \right) \tag{5.9c}
\]

subject to the boundary conditions

\[
u = \nu = 0 \quad \text{at} \quad y = 0 \tag{5.10a}
\]

\[
u, \nu \to 0 \quad \text{as} \quad y \to \infty \tag{5.10b}
\]

The system of equations (5.9)-(5.10) determines an eigenvalue problem for the parameters \( k, \omega, \) and \( R \). For known basic state velocity profiles, the equations are integrated numerically using a computer code developed by Scott and Watts\(^{38} \) to handle stiff two-point boundary value problems such as this. Specifying \( R \) and \( \omega \), we find the eigenvalue \( k \).
Knowing $k$, we find the dimensional wavenumber $k^* = k^* + i k_\parallel^*$ to be

$$k^* = \frac{k}{\delta^*} \tag{5.11}$$

Then $-k_\parallel^*$ is the spatial growth rate of the disturbance.

From $k_\parallel^*$ we determine the amplification factor

$$n = \ln \frac{A}{A_0} = -\int_{s_\parallel^*}^{s^*} k_1^* d\zeta \tag{5.12}$$

where $s_\parallel^*$ is the surface distance from the stagnation point of the location where the constant frequency disturbance first becomes unstable; $A$ and $A_0$ are the amplitudes of the disturbance at $s^*$ and $s_\parallel^*$, respectively. The $e^n$ method cannot be used to predict the exact location of transition because of the strong dependence of transition location on freestream turbulence levels. However, linear stability theory is useful, especially for laminar flow control. It can be used as a design tool because it predicts trends for changes in the mean flow that delay transition.

5.3 **Basic State**

5.3a **Linearized Triple Deck**

In this chapter, we examine the stability of the basic state given by the linearized triple-deck, closed-form solutions developed in Chapter Four. For the streamwise component of velocity we use

$$\frac{u^*}{U^*_\infty} (s^*, n^*) = U(n) + \sum_{i=1}^{n} \text{Re}^{1/4} \lambda_1^{-1/2} \frac{V_{wall_i}}{U^*_\infty} \left( \frac{U'(n)}{U'(0)} - 1 \right) \overline{\delta}(x - x_i) + \overline{u}(x - x_i, y) \tag{5.13}$$
where

\[
\overline{u}(x - x_i, y) = \overline{u}_\infty \left( \frac{\lambda_{i}^{5/4}}{s_{i}^{3/4}} \left[ s \cdot \text{Re}_{\infty}^{3/8} - s \cdot \text{Re}_{\infty}^{3/8} \right], \frac{\text{Re}_{\infty}^{5/8}}{s_{i}^{3/4}} \lambda_{i}^{3/4} n^* \right)
\]

\[\overline{u}(x - x_i) = \overline{u}_\infty \left( \frac{\lambda_{i}^{5/4}}{s_{i}^{3/4}} \left[ s \cdot \text{Re}_{\infty}^{3/8} - s \cdot \text{Re}_{\infty}^{3/8} \right], \frac{\text{Re}_{\infty}^{5/8}}{s_{i}^{3/4}} \lambda_{i}^{3/4} n^* \right) (5.14)\]

\[
\overline{d}(x - x_i) = \overline{d}_\infty \left( \frac{\lambda_{i}^{5/4}}{s_{i}^{3/4}} \left[ s \cdot \text{Re}_{\infty}^{3/8} - s \cdot \text{Re}_{\infty}^{3/8} \right] \right)
\]

\[\overline{d}(x - x_i) = \overline{d}_\infty \left( \frac{\lambda_{i}^{5/4}}{s_{i}^{3/4}} \left[ s \cdot \text{Re}_{\infty}^{3/8} - s \cdot \text{Re}_{\infty}^{3/8} \right] \right) (5.15)\]

\[
\overline{u}_\infty(x, y) = -\frac{3}{2\pi i} |x|^{2/3} \int_0^\infty e^{i\pi} \frac{e^{\tau}}{\tau^{3/3}} \int_0^1 (x^{1/3} y)/|x|^{1/3} A_1(\eta) d\eta d\tau
\]

\[+ \int_{-\infty}^0 e^{-i\pi} \frac{e^{\tau}}{\tau^{3/3}} \int_0^1 (x^{1/3} y)/|x|^{1/3} A_1(\eta) d\eta d\tau
\]

\[+ \frac{9}{2\pi^2 \pi} \int_0^\infty e^{-i\pi/6} \rho \int_0^1 e^{i\pi/3} \rho^{3/3} \frac{e^{i\pi/3} \rho \theta^{1/3} y}{e^{-i\pi/3} \rho \theta^{1/3} y} A_1(\eta) d\eta d\rho
\]

\[+ \frac{9}{2\pi^2 \pi} \int_0^\infty e^{i\pi/6} \rho \int_0^1 e^{-i\pi/3} \rho \theta^{1/3} y A_1(\eta) d\eta d\rho
\]

for \( x > 0 \)

\[(5.16a)\]

\[
\overline{u}_\infty(x, y) = -\frac{g}{\pi \theta^\pi} \int_0^\infty \rho \theta^{\rho^3} y A_1(\eta) d\eta d\rho
\]

for \( x < 0 \)

\[(5.16b)\]
\[ \bar{c}_{\infty}(x,y) = -\frac{|x|^{2/3}}{\Gamma(5/3)} \frac{3}{2\pi^{2/3}} \int_{0}^{\infty} \frac{\sqrt{\pi} \rho^{5} - \rho}{\sqrt{3} \rho^{4} + 1} e^{-\rho^{3/2}|x|} d\rho \]

for \( x > 0 \) \hfill (5.17a)

\[ \bar{c}_{\infty}(x,y) = -\frac{3}{\pi^{2/3}} \int_{0}^{\infty} \frac{\rho^{5} - \rho}{\sqrt{3} \rho^{4} + 1} e^{-\rho^{3/2}|x|} d\rho \quad \text{for} \quad x < 0 \] \hfill (5.17b)

The quantities with asterisk are dimensional. Here, we consider \( n \) porous strips, centered at \( s_{1}^{*}, s_{2}^{*}, \ldots, s_{n}^{*} \) with the leading and trailing edges \( s_{LE_{1}}^{*}, s_{LE_{2}}^{*}, \ldots, s_{LE_{n}}^{*} \) and \( s_{TE_{1}}^{*}, s_{TE_{2}}^{*}, \ldots, s_{TE_{n}}^{*} \), respectively. We define the \( s \)-Reynolds number at \( s^{*} \) and the \( i \)th strip as

\[ Re_{s} = \frac{s^{*} U_{s}^{*}}{v_{s}^{*}}, \quad Re_{i} = \frac{s_{i}^{*} U_{i}^{*}}{v_{i}^{*}} \quad (5.18) \]

respectively, as in Chapter Four. The quantity \( v_{wall}^{*} \) is the dimensional suction rate at the \( i \)th strip.

**5.3b Mean Flow over a Suctionless Body**

As seen from Eq. (5.13), we need to specify some suctionless mean flow profile \( U \) in the triple-deck formulation. For this work, we choose \( U \) as the boundary layer profile along a Rankine body\(^{68} \), because it will be easy to work with a closed-form solution for the edge-velocity.

Superposing a source and a sink of equal strength \( q \) in a uniform stream \( U_{s} \), we find, in the coordinate system of Figure 5.1, that the velocity potential at any point \( P \) is

\[ \phi(P) = U_{s} x + A \left[ \frac{1}{\sqrt{(x-a)^{2} + r^{2}}} - \frac{1}{\sqrt{(x+a)^{2} + r^{2}}} \right] \quad (5.19) \]

Here \( A \) is the source strength divided by \( 4\pi \) and \( a \) is one half the
distance between the source and the sink. Choosing $u_r$, $u_\phi$, $u_x$ to be
the velocity components in the $r$-, $\phi$-, and $x$-directions, respectively,
we have from Eq. (5.19) that

$$u_r = -A \left\{ \frac{r}{[(x-a)^2 + r^2]^{3/2}} - \frac{r}{[(x+a)^2 + r^2]^{3/2}} \right\}$$  \hspace{1cm} (5.20a)

$$u_\phi = 0$$  \hspace{1cm} (5.20b)

$$u_x = U_s - A \left\{ \frac{x-a}{[(x-a)^2 + r^2]^{3/2}} - \frac{x+a}{[(x+a)^2 + r^2]^{3/2}} \right\}$$  \hspace{1cm} (5.20c)

The stagnation points are then found from Eqs. (5.20) to be at the two
coordinate locations $(r_s, x_s)$ described by

$$r_s = 0$$  \hspace{1cm} (5.21a)

$$U_s - A \left\{ \frac{x_s-a}{|x_s-a|^3} - \frac{x_s+a}{|x_s+a|^3} \right\} = 0$$  \hspace{1cm} (5.21b)

In any axial plane, a streamline is described by

$$d\psi = ru_x \, dr - ru_r \, dx$$  \hspace{1cm} (5.22)

Upon integration of Eq. (5.22), we see that streamlines are given by

$$A \left[ \frac{x-a}{\sqrt{(x-a)^2 + r^2}} - \frac{x+a}{\sqrt{(x+a)^2 + r^2}} \right] + \frac{1}{2} U_s r^2 = C$$  \hspace{1cm} (5.23)

where $C$ is a constant. In particular, considering Eq. (5.21), we find
that the stagnation streamline, the streamline describing the closed
body, is given by

$$A \left[ \frac{x-a}{\sqrt{(x-a)^2 + r^2}} - \frac{x+a}{\sqrt{(x+a)^2 + r^2}} \right] + \frac{1}{2} U_s r^2 = 0$$  \hspace{1cm} (5.24)
If we set \( x = 0 \) in Eq. (5.24), we see that the maximum radius \( r = r_c \) of the body is found by solving the relation

\[
r_c^2 \sqrt{a^2 + r_c^2} = 4a \frac{A}{U_s}
\]  

(5.25)

Specifying the length of the body, the maximum radius of the body \( r_c \), and a freestream velocity \( U_s \), we can determine the source strength \( q \) and the distance between source and sink \( 2a \). Knowing these quantities we immediately find the surface, or edge, speed \( U_e \) from Eq. (5.20) to be

\[
U_e = (u_r^2 + u_x^2)^{1/2}
\]  

(5.26)

where \( u_r \) and \( u_x \) are given by Eqs. (5.20a) and (5.20c), respectively.

To generate the boundary-layer velocity profile \( U \) knowing \( U_e \) and the geometry, we then use the boundary-layer option of the Transition Analysis Program System (TAPS)\(^{23,24}\) mentioned in Chapter Four. The boundary-layer calculations are nonsimilar and based on the Cebeci-Smith finite-difference boundary-layer program\(^ {59}\) that uses the Keller box method for solving the boundary-layer equations\(^ {60-62}\).

5.4 Optimization Scheme

Taking advantage of the linearity of the triple-deck formulation, we develop a linear minimization problem whose solution can be used to predict efficient strip configurations. The problem is even solvable by inspection, that is, there is no need for dynamic programming such as the simplex algorithm\(^ {49}\).

We consider the first-order system of disturbance equations

\[
i ku + Dv = 0
\]  

(5.27a)
\[-i\omega u + ikU + vDU = -ikp + \frac{1}{R} [D^2 u - k^2 u] \]  
\[-i\omega v + ikvU = -Dp + \frac{1}{R} [D^2 v - k^2 v] \]  

where \( D \) has been used as an abbreviation for \( \frac{d}{dy} \). We take the mean flow \( U \) as

\[ U = U_0 + \epsilon U_1 \quad |\epsilon| \ll 1, \quad U_1 = O(1) \]  

This mean flow is exactly as in the triple-deck expression for \( u^*/U_\infty \) where

\[ U_0 = U(\eta) \quad \text{(suctionless body)} \]  

\[ \epsilon = \frac{v_{\text{wall}}}{u^*_\infty} \frac{1}{Re_{\infty}^{1/4}} \]  

Then, we expand

\[ u = u_0 + \epsilon u_1 + \ldots \]  

\[ v = v_0 + \epsilon v_1 + \ldots \]  

\[ p = p_0 + \epsilon p_1 + \ldots \]  

\[ k = k_0 + \epsilon k_1 + \ldots \]  

where the quantities with subscript zero represent suctionless-body disturbance quantities and those with subscript one represent the contributions due to the presence of the suction strips. The zeroth-order problem is

\[ L_1(u_0, v_0, p_0) = ik_0 u_0 + Dv_0 = 0 \]  

\[ L_2(u_0, v_0, p_0) = -i\omega u_0 + ik_0 u_0 U_0 + v_0 DU_0 + ik_0 p_0 
- \frac{1}{R} [D^2 u_0 - k^2 u_0] = 0 \]
This system along with the appropriate boundary conditions constitutes the eigenvalue problem for \((\omega, k_0, R)\) to determine the disturbances \((u_0, v_0, p_0)\) superposed on an otherwise undisturbed suctionless-body boundary layer flow.

The first-order problem is

\[
L_1(u_1, v_1, p_1) = i k_1 u_0 \tag{5.32a}
\]

\[
L_2(u_1, v_1, p_1) = -i k_1 u_0 U_0 - i k_0 u_0 U_1 - v_0 U_1 - i k_1 p_0 + \frac{1}{R} [-2 k_0 k_1 u_0] \tag{5.32b}
\]

\[
L_3(u_1, v_1, p_1) = -i k_1 v_0 U_0 - i k_0 v_0 U_1 + \frac{1}{R} [-2 k_0 k_1 v_0] \tag{5.32c}
\]

\[
u_1 = v_1 = 0 \text{ at } y = 0 \tag{5.32d}
\]

\[
u_1, v_1 \to 0 \text{ as } y \to \infty \tag{5.32e}
\]

A solution exists for this system (5.32) only if a solvability condition is satisfied\(^{33}\). More is said on this in Chapter Seven. The inhomogeneous terms on the right-hand sides must be orthogonal to every solution of the adjoint homogeneous problem. That is,

\[
k_1 \oint_{0} \left\{ [2 k_0 + i R U_0] z \dd z_1 + i R z \dd z_4 - i z \dd z_1 - \frac{1}{R} z \dd z_2 - \frac{2 k_0}{R} z \dd z_3 ight. \\
- i U_0 z \dd z \bigg\} dy + \oint_{0} \left\{ i R k_0 U_1 z \dd z_1 + R U_1 z \dd z_3 - i k_0 U_1 z \dd z_3 \right\} dy = 0
\]

or

\[
L_3(u_0, v_0, p_0) = -i v_0 + i k_0 v_0 U_0 + D p_0 - \frac{1}{R} [D^2 v_0 - k_0 v_0] \tag{5.31c}
\]

\[
u_0 = v_0 = 0 \text{ at } y = 0 \tag{5.31d}
\]

\[
u_0, v_0 \to 0 \text{ as } y \to \infty \tag{5.31e}
\]
$$k_1 \int_0^\infty f(y) dy + \int_0^\infty g(x,y) dy = 0 \quad (5.34)$$

where

$$z_1 = u \quad (5.35a)$$
$$z_2 = Du \quad (5.35b)$$
$$z_3 = v \quad (5.35c)$$
$$z_4 = p \quad (5.35d)$$

and the starred quantities are the corresponding adjoint solutions.

From Eq. (5.34) we see that the perturbation $k_1$ to the suctionless-body complex wavenumber $k_0$ can be determined by

$$k_1 = -\int_0^\infty g(x,y) dy/\int_0^\infty f(y) dy \quad (5.36)$$

At each $x$, since $k_1$ is independent of $\varepsilon$, the correction to $k_0$ is a linear function of $\varepsilon$. Remembering $\varepsilon$ to be given by Eq. (5.29), we find that this implies that $\varepsilon k_1$ is directly proportional to the suction rate.

Using this property of linearity, we consider an axisymmetric body with $n$ porous strips and $m$ specified points of computation between Branch I and Branch II of the stability curve. We solve the zeroth-order eigenvalue problem to find the suctionless-body complex wavenumber $k_{Bj}$ and the eigenvectors $z_{\lambda j}$, $z_{\lambda j}^*$, $\lambda = 1, \ldots, 4$ at each point $j = 1, \ldots, m$. Then, using linear triple-deck theory and considering Eqs. (5.13), (5.33), and (5.36) we calculate the $y$-dependent function

$$U' j(n) - 1 \delta(x_j - x_i) + \bar{u}(x_j - x_i, y) \quad (5.37)$$

at each point of computation $j = 1, \ldots, m$ due to each individual strip $i = 1, \ldots, n$, where $\bar{u}$ and $\delta$ are defined in Eqs. (5.14) and (5.15),
respectively. Then applying Eqs. (5.33)-(5.36) we determine \( k_i \) for each \( i \) and \( j \) and call this quantity \( a_{ij} \). So we propose that the actual complex wavenumber \( k_j \) at point \( j \) due to the presence of all \( n \) strips is

\[
k_j = k_{B_j} + \sum_{i=1}^{n} a_{ij} \frac{v_{\text{wall}i}}{U_{\infty i}^{*}} \operatorname{Re}_{0i}^{1/4} \lambda_i^{-1/2}
\]

(5.38)

To optimize a suction distribution keeping the dimensional total mass-flow rate constant, we should minimize the amplification factor

\[
\ln \frac{A}{A_0} = -\int_{s_j}^{s_j^*} k_i^* d\xi
\]

(5.39)

Using the trapezoidal rule to perform the integration in Eqs. (5.39) and applying Eq. (5.38), we find that

\[
\ln \frac{A}{A_0} = -\operatorname{Imag} \left\{ \sum_{j=2}^{m} \left( \frac{k_{B_j}}{\delta_j^*} + \frac{k_{B_{j-1}}}{\delta_{j-1}^*} \right) \left( s_j^* - s_{j-1}^* \right) + \sum_{j=2}^{m} \sum_{i=1}^{n} \left( \frac{a_{ij}}{\delta_j^*} + \frac{a_{ij-1}}{\delta_{j-1}^*} \right) \frac{v_{\text{wall}j}}{U_{\infty i}^{*}} \operatorname{Re}_{0i}^{1/4} \lambda_i^{-1/2} \left( s_j^* - s_{j-1}^* \right) \right\}
\]

(5.40)

Therefore, to minimize \( \ln(A/A_0) \), we must find the solution, \( \frac{v_{\text{wall}j}^{*}}{U_{\infty i}^{*}} \), \( i = 1, \ldots, n \) for

\[
\min \left\{ -\operatorname{Imag} \sum_{i=1}^{n} \left[ \sum_{j=2}^{m} \left( \frac{a_{ij}}{\delta_j^*} + \frac{a_{ij-1}}{\delta_{j-1}^*} \right) \operatorname{Re}_{0i}^{1/4} \lambda_i^{-1/2} \left( s_j^* - s_{j-1}^* \right) \right] \frac{v_{\text{wall}j}^{*}}{U_{\infty i}^{*}} \right\}
\]

(5.41)

subject to constraints that give a reasonable physical picture\(^{50}\). Such constraints involve a ceiling on \( |v_{\text{wall}j}^{*}/U_{\infty i}^{*}| \), a lower limit on
\[ |v_{\text{wall}}^*/U_{\infty}^*| \], a distribution so as to keep \(-k_i^*\) positive after the disturbance first goes unstable, and a distribution so as to eliminate distribution thickness effects.

For \( i = 1, ..., n \), we evaluate and store the coefficients \( c_i \) of \( v_{\text{wall}}^*/U_{\infty}^* \). From Eq. (5.40) we see that

\[
c_i = -\text{Imag} \sum_{j=2}^{m} \left( \frac{a_{i,1}}{\delta_{j}^*} + \frac{a_{i,j-1}}{\delta_{j-1}^*} \right) \text{Re}_{\infty}^{1/4} \lambda_i^{-1/2} (s_j^* - s_{j-1}^*)
\]

(5.42)

Obviously, if the only constraint we impose is an upper bound \( v_u \) on \( |v_{\text{wall}}^*/U_{\infty}^*| \), then Eq. (5.41) is satisfied by assigning the \( v_{\text{wall}}^*/U_{\infty}^* \) multiplying the largest \( c_i \) the value of \(-v_u\), then doing the same for the next largest \( c_j \), etc. We expect that the optimization scheme will predict that we concentrate the suction at the Branch I neutral point as in the two-dimensional case.

5.5 Results and Conclusions

In this section we show results of our optimization scheme. The axisymmetric body we choose on which to demonstrate the theory is a Rankine body 4.88 meters long with a maximum body radius of .3 meters. The freestream unit Reynolds number per meter is 6.6 million and the freestream velocity is approximately 6.4 meters per second. The body and the edge velocity distribution are shown plotted in Figure 5.2.

We superpose a disturbance of constant dimensionless frequency \( F = 15 \times 10^{-6} \) into the boundary layer of this body. Here dimensionless means with respect to freestream variables, not edge variables. Figures 5.3 and 5.4 show the growth rate and corresponding amplification factor curve, respectively, between Branch I and Branch II of the stability
curve for the suctionless body. They are shown plotted versus the
dimensional surface distance from the stagnation point.

Our plan is to consider a uniform distribution of suction through
strips and then to try to optimize it, keeping the dimensional total
mass flow rate a constant. We examine a configuration of fifteen
.01-meter-wide strips distributed evenly between Branch I and Branch II.
Strip centers are spaced at .09-meter intervals and suction levels are
\(-4.93 \times 10^{-4}\) times the freestream velocity. The distribution is shown
schematically in Figure 5.5. Using the linearized scheme discussed in
Section 5.4, we calculate the growth rates between Branches I and II and
then the amplification factor. These are shown in Figures 5.6 and 5.7,
respectively.

Considering the optimization scheme, we find the influence
coefficients to be as given in Table 5.1. It is suggested from these
numbers that we concentrate suction near the Branch I neutral point.
Suppose we are free to put any number of .01-meter-wide strips on the
body but we require that the minimum distance between strip centers be
.09 meters. The linear minimization scheme with the physical con-
straints taken into account predicts that if we use eleven strips with
suction levels and spacings as in Figure 5.8, then the amplification
factor is reduced significantly from that of the uniform problem above.
The growth rates and amplification factor are shown in Figures 5.9 and
5.10, respectively.

These results show the usefulness of our linear optimization scheme
in predicting the suction level, spacing, and number of strips to
decrease the disturbance growth as much as possible on an axisymmetric
body. At present, we are finishing the development of and documentation for a computer code for official distribution that will interface with TAPS and suggest efficient configurations using our theory.

Calculations for this dissertation were done in the adverse pressure-gradient region near the nose of the body. Deceleration of a flow is another destabilizing effect adding to the amplification factor. But we find the influence coefficients to be an order of magnitude higher here in the adverse pressure-gradient area near Branch I than for the two-dimensional problem where there is no pressure gradient. This implies that suction concentrated near Branch I in an unfavorable situation will dramatically decrease the maximum amplification factor, as we indeed find in Figures 5.8-5.10.

Other adverse pressure-gradient cases must be and will be investigated by us. We will accomplish this by varying both frequency and Reynolds number, thus moving all over the body.
Chapter Six
Compressible Mean Flow

6.1 Introduction

Before proceeding to compressible three-dimensional stability calculations, we need to establish a mean flow. The basic state we use is one obtained from the computer code of Kaups and Cebeci. Under the assumption of a negligible spanwise pressure gradient, the code solves the compressible laminar boundary-layer equations for a wing with wall mass transfer, sweep, and tapering. However because of the conical flow assumption, twist and tip and wing-body effects must be neglected. The governing equations are converted to a two-dimensional form by similarity transformations and then solved numerically by Keller's box method. The Kaups-Cebeci code has the limitation of constant Prandtl number and specific heat at constant pressure, two quantities that must be allowed to vary with temperature to give the most accurate stability results.

6.2 Governing Equations

Considering the coordinate system shown in Figure 6.1 and using the conical flow assumption that \( \frac{\partial p}{\partial x} = 0 \), the governing equations for a compressible three-dimensional boundary-layer flow are

**Continuity**

\[
\frac{\partial}{\partial x} (\rho u x) + \frac{\partial}{\partial y} (\rho w y) + \frac{\partial}{\partial y} (\rho x v) = 0 \tag{6.1}
\]

**x-Momentum**

\[
\rho u \frac{\partial u}{\partial x} + \rho \frac{w}{x} \frac{\partial u}{\partial \theta} + \rho v \frac{\partial u}{\partial y} - \rho \frac{w^2}{x} = \frac{\partial}{\partial y} (\mu \frac{\partial u}{\partial y}) \tag{6.2}
\]
θ-Momentum

\[ \rho u \frac{\partial w}{\partial x} + \rho \frac{w}{x} \frac{\partial w}{\partial \theta} + \rho v \frac{\partial w}{\partial y} + \rho \frac{uw}{x} + \rho \frac{w}{y} = - \frac{1}{x} \frac{\partial p}{\partial \theta} + \frac{\partial}{\partial y} \left( \mu \frac{\partial w}{\partial y} \right) \quad (6.3) \]

Energy

\[ \rho u \frac{\partial H}{\partial x} + \rho \frac{w}{x} \frac{\partial H}{\partial \theta} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y} \left[ \frac{u}{\Pr} \frac{\partial H}{\partial y} + \mu (1 - \frac{1}{\Pr}) \frac{\partial}{\partial y} \left( \frac{u^2 + w^2}{2} \right) \right] \quad (6.4) \]

In this system (6.1)-(6.4), the polar coordinates \( x, \theta, \) and \( y \) represent the distance along the generator from the origin, the angle in the plane from the stagnation line \( \theta = 0 \), and the normal distance from the surface of the wing, respectively. The quantities \( u, w, \) and \( v \) correspond to \( x, \theta, \) and \( y \), respectively; \( \rho \) is the density; \( \mu \) is the dynamic viscosity determined from Sutherland's Law; \( p \) is the pressure; \( \Pr \) is the constant Prandtl number; and \( H \) is the total enthalpy. The code assumes the gas to be perfect.

The appropriate boundary conditions are

\[ u = 0, \quad v = v_w, \quad w = 0, \quad \frac{\partial H}{\partial y} = 0 \quad \text{at} \quad y = 0 \quad (6.5a) \]

\[ u \rightarrow u_e, \quad w \rightarrow w_e, \quad H \rightarrow H_e \quad \text{as} \quad y \rightarrow \delta \quad (6.5b) \]

Because \( w = 0 \) along the stagnation line, the θ-momentum equation (6.3) is singular. Kaups and Cebeci differentiate this equation with respect to \( \theta \), set \( w = 0 \), and obtain the following equation to replace the θ-momentum equation along this line:

\[ \rho u \frac{\partial w}{\partial x} + \frac{\partial}{\partial x} \left( \frac{w^2}{\theta} \right) + \rho v \frac{\partial w}{\partial y} + \rho \frac{u w}{x} + \rho \frac{w}{y} = \frac{1}{x} \frac{\partial p}{\partial \theta} + \frac{\partial}{\partial y} \left( \mu \frac{\partial w}{\partial y} \right) \quad (6.6) \]

This is subject to the same boundary conditions as the original system except for the edge condition on \( w \) in Eq. (6.5b) which is now replaced
by
\[ w_\theta + w_\theta e \] as \( y + \delta \) \hspace{1cm} (6.7)

The inviscid velocity components \( u_e \) and \( w_e \) appearing in the boundary conditions are determined from the input pressure coefficient distribution \( C_p \) using the conical-flow relationship

\[ \frac{d}{d\theta} \bar{u}_e = -w_e = -\sqrt{u^2_s - \bar{u}_e^2} \] \hspace{1cm} (6.8)

In this equation \( \bar{u}_e = -u_e \) and \( u_s \) is the resultant inviscid velocity obtained from the input pressure-coefficient distribution. Equation (6.8) is nonlinear in \( \bar{u}_e \) and is integrated numerically by a fourth-order Runge-Kutta scheme. The initial conditions are taken to be \( u_s = \bar{u}_e \) at \( \theta = 0 \).

6.3 Similarity Transformation

Kaups and Cebeci convert the governing equations to a two-dimensional form by way of a similarity transformation. For a conical flow they define the following dimensionless variable

\[ dn = \sqrt{\frac{\bar{u}_e}{\rho u_e x}} \rho dy \] \hspace{1cm} (6.9)

along with a potential vector having two components satisfying Eq. (6.1) according to

\[ \rho u x = \frac{\partial \psi}{\partial y} \] \hspace{1cm} (6.10a)

\[ \rho w x = \frac{\partial \phi}{\partial y} \] \hspace{1cm} (6.10b)

\[ \rho v x = \frac{\partial \psi}{\partial x} - \frac{1}{x} \frac{\partial \phi}{\partial \theta} + (\rho v x)_{y=0} \] \hspace{1cm} (6.10c)
They then define the following expressions for \( \psi \) and \( \phi \) using the dimensionless functions \( f \) and \( g \):

\[
\psi = x^{3/2} \sqrt{\rho_e u_e \bar{u}_e} f(\eta, \theta)
\]

(6.11a)

\[
\phi = x^{3/2} \sqrt{\rho_e u_e \bar{u}_e} \frac{w_e}{u_e} g(\eta, \theta)
\]

(6.11b)

Here \( \bar{u} = -u \) and \( \bar{u}_e = -u_e \). The computer code then solves the resulting governing equations and boundary conditions for \( f \), \( g \), and \( E \), where \( E = \frac{H}{H_e} \), by Keller's box method, an accurate and efficient method for solving parabolic partial-differential equations. This scheme is well documented\(^{59-62,70}\).

6.4 Results Applied to Stability Coordinates

In our stability analysis in Chapter 7 we will employ the xyz-coordinate system shown in Figure 6.1. The x-coordinate of the stability analysis is taken along the \( \theta \)-direction of the Kaups-Cebeci code, the y-coordinate is normal to the surface in the direction out from the page and is the same as \( \eta \) in the Kaups-Cebeci code, and the z-coordinate is along a wing generator in the direction toward the tip. Keeping this coordinate system in mind, we obtained the basic state flow quantities required in the stability analysis from \( f \) and \( g \), and their derivatives with respect to \( \eta \) and \( \theta \).

6.5 The Airfoil

For the compressible three-dimensional stability calculations in Chapter 7, we intend to use as the mean flow the boundary layer velocity
profiles on the upper surface of the X-21 wing. This wing was designed for laminar flow control and derived from the NACA 65A210 airfoil. The sweeps at the leading and trailing edges are 33.2 and 19.1 degrees, respectively.

Figures 6.2 and 6.3 show the input pressure coefficient distribution and suction distribution, respectively. The displacement thickness Reynolds number is then shown plotted in Figure 6.4. The chord length is 14.66 feet and the freestream velocity is 774.4 feet per second.
Chapter Seven
Stability of Compressible Three-Dimensional Flows

7.1 Introduction

One application of laminar flow control will be to transonic swept wings with specially-designed cross-sectional airfoil shapes\textsuperscript{25,26}. Because of the sweep, a spanwise pressure gradient exists resulting in a crossflow profile that is dynamically unstable. The development of crossflow can lead to the generation of streamwise vortices\textsuperscript{27-29}. Because of this added instability we must now determine the linear stability of three-dimensional flows.

For parallel three-dimensional incompressible flows, Gregory, Stuart, and Walker\textsuperscript{28} derive the three-dimensional linear stability equations including boundary-layer growth and streamline curvature. Then they determine a transformation reducing the three-dimensional temporal problem to a two-dimensional one. For flows over a rotating disk and a sweptback wing, Brown\textsuperscript{71} solves these equations numerically.

An excellent reference giving details of the incompressible parallel stability problem formulation, results obtained, and physical mechanisms behind the instabilities is that of Mack\textsuperscript{72}. Nayfeh and Padhye\textsuperscript{73} present a method for calculating neutral stability points for a flat-plate flow. Because calculations of neutral stability points, points separating stable and unstable flows, are extremely tedious and difficult in three dimensions, their work is significant. From their iterative scheme they derive equations relating neutral and nonneutral disturbances. Cebeci and Stewartson\textsuperscript{74} identify an absolute neutral curve, called zarf, for
the rotating disk. For given dimensionless frequency $\omega$, they find a neutral curve on which the growth rate is zero. That is, $\alpha_i + \beta_i^* \tan \phi = 0$, $y = x \tan \phi$, where $\alpha_i$ and $\beta_i$ are the imaginary parts of the complex streamwise and spanwise wavenumbers, respectively, and $\phi$ is the direction of propagation. They then define the absolute neutral curve $\gamma_{zf}$ on which both $\alpha_i$ and $\beta_i$ are zero.

For parallel three-dimensional incompressible stability calculations a computer code SALLY has been developed by Srokowski and Orszag$^{75}$ that uses the $e^n$-method$^{14-16}$ for correlating the transition location. They calculate the maximum temporal amplification rate for a given dimensional frequency from the parallel incompressible stability equations. Then they use the real part of the group velocity to convert the temporal amplification rate into a spatial one and integrate along the path defined by the real part of the group velocity. Mack in his spatial calculations for the rotating disk$^{18}$ and Falkner-Skan-Cooke yawed wedges$^{76,77}$ also defines the direction of growth as that of the real part of the group velocity. Cebeci and Stewartson$^{74}$ use the condition that $\frac{d\alpha}{d\beta}$ be real, a condition also found by Nayfeh$^{30,31}$, to calculate an $n$-factor for the rotating disk. They start at a point of $\gamma_{zf}$ and fix the group velocity angle as a constant before marching. Nayfeh and Padhye$^{78}$ establish a relation between three-dimensional temporal and spatial stabilities and a relation between spatial stabilities using the complex group velocity. Malik$^{79}$ and Malik and Orszag$^{80}$ compare several methods of transition prediction using incompressible stability theory and conclude that the SALLY code is the most efficient. For the rotating disk, Malik, Wilkinson, and Orszag$^{81}$ then use SALLY to calculate
temporal eigenvalues, which they convert to spatial using a group velocity transformation. They then calculate n-factors using the real part of the group velocity.

The basic equations for the linear stability analysis of parallel-flow compressible boundary layers were first derived by Lees and Lin\textsuperscript{82}, Lin\textsuperscript{83}, and Dunn and Lin\textsuperscript{84} using small disturbance theory. The papers of Mack\textsuperscript{17,72,85-87} are also excellent references for the compressible parallel stability analysis. Lekoudis\textsuperscript{88} and Mack\textsuperscript{89} evaluate the effects of compressibility on the stability of the boundary-layer flow over an infinite-span swept wing. They find that the inclusion of compressibility significantly reduces the maximum amplification rate and changes the most unstable wave's orientation. El-Hady\textsuperscript{34} and Mack\textsuperscript{90} both report on the parallel compressible stability of the flow over a 23°-swept wing with a supercritical airfoil shape. In their work, Malik and Orszag\textsuperscript{91} describe the computer code COSAL they have developed that efficiently computes temporal eigenvalues by finite differences.

The effects of nonparallelism must also be considered. Bouthier\textsuperscript{92,93}, Nayfeh et al.\textsuperscript{94}, Gaster\textsuperscript{95}, Eagles and Weissman\textsuperscript{96}, Saric and Nayfeh\textsuperscript{97,98}, El-Hady and Nayfeh\textsuperscript{99,100} and Nayfeh and El-Hady\textsuperscript{101} for two-dimensional mean flows, and El-Hady\textsuperscript{34} and Padhye and Nayfeh\textsuperscript{102} for three-dimensional mean flows, in their calculations, all find nonparallel effects to be important to the accuracy of stability results.

It is apparent that a correct three-dimensional stability analysis must include both compressibility and nonparallelism. Nayfeh\textsuperscript{30,31} uses the method of multiple scales\textsuperscript{32,33} to formulate the problem but presents no numerical results. He determines the partial differential equations
governing variations of the amplitude and complex wavenumbers and determines conditions on the group velocity components making the problem physically realistic. El-Hady\textsuperscript{34} presents some results on a 23°-swept wing with a supercritical airfoil shape.

However, in our opinion, all past numerical calculations of three-dimensional boundary layer stability have not satisfactorily answered the questions about the character of the most unstable disturbance. Instead of jumping from wave to wave by locally calculating the most amplified disturbance as you march or specifying some artificial condition such as constant spanwise or chordwise wavelength, we believe that at some initial point a specific wave must be selected and then that one wave must be followed along its trajectory\textsuperscript{31}. The aim of this chapter is to lay the groundwork for this approach to find the most dangerous frequency and initial spanwise wavenumber for a small-amplitude, three-dimensional disturbance at a specific chord location of a wing in a three-dimensional compressible flow.

7.2 Stability Equations

In this section we consider a three-dimensional, compressible boundary-layer flow such as the flow over the X21 wing described in Chapter Six. To study the stability of such a basic state, or mean flow, we superpose small disturbances to obtain total flow quantities of the form

\[ \tilde{q}(x,y,z,t) = Q_s(x,y,z) + q(x,y,z,t) \] (7.1)

Here \( Q_s(x,y,z) \) stands for a basic state quantity and \( q(x,y,z,t) \) stands for a small unsteady disturbance. Substituting the total flow
quantities \( \hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{\mu}, \) and \( \hat{T} \) into the Navier-Stokes equations, subtracting the basic state, and linearizing, we find to first order that the disturbance equations are given by

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho_s u + \rho u_s) + \frac{\partial}{\partial y} (\rho_s v + \rho v_s) + \frac{\partial}{\partial z} (\rho_s w + \rho w_s) = 0 \tag{7.2}
\]

\[
\rho_s \frac{\partial u}{\partial t} + u_s \frac{\partial u}{\partial x} + u \frac{\partial u_s}{\partial x} + v_s \frac{\partial u}{\partial y} + v \frac{\partial u_s}{\partial y} + w_s \frac{\partial u}{\partial z} + w \frac{\partial u_s}{\partial z} + \rho (u_s \frac{\partial u_s}{\partial x} + \frac{\partial u}{\partial x}) = -\frac{\partial p}{\partial x} + \frac{1}{\rho S} \left[ \frac{\partial}{\partial x} \left[ \mu (r \frac{\partial u}{\partial x} + m \frac{\partial v}{\partial y} + m \frac{\partial w}{\partial z}) \right] + \frac{\partial}{\partial y} \left[ \mu (r \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial x}) \right] + \frac{\partial}{\partial z} \left[ \mu (r \frac{\partial u}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u_s}{\partial y} + \frac{\partial w_s}{\partial y}) \right] \right] \tag{7.3}
\]

\[
\rho_s \frac{\partial v}{\partial t} + u_s \frac{\partial v}{\partial x} + u \frac{\partial v_s}{\partial x} + v_s \frac{\partial v}{\partial y} + v \frac{\partial v_s}{\partial y} + w_s \frac{\partial v}{\partial z} + w \frac{\partial v_s}{\partial z} + \rho (v_s \frac{\partial v_s}{\partial x} + \frac{\partial v}{\partial x}) = -\frac{\partial p}{\partial y} + \frac{1}{\rho S} \left[ \frac{\partial}{\partial y} \left[ \mu (r \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial x}) \right] + \frac{\partial}{\partial x} \left[ \mu (r \frac{\partial u}{\partial x} + m \frac{\partial v}{\partial y} + m \frac{\partial w}{\partial z}) \right] + \frac{\partial}{\partial z} \left[ \mu (r \frac{\partial u}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u_s}{\partial y} + \frac{\partial w_s}{\partial y}) \right] \right] \tag{7.4}
\]

\[
\rho_s \frac{\partial w}{\partial t} + u_s \frac{\partial w}{\partial x} + u \frac{\partial w_s}{\partial x} + v_s \frac{\partial w}{\partial y} + v \frac{\partial w_s}{\partial y} + w_s \frac{\partial w}{\partial z} + w \frac{\partial w_s}{\partial z} + \rho (w_s \frac{\partial w_s}{\partial x} + \frac{\partial w}{\partial x}) = -\frac{\partial p}{\partial z} + \frac{1}{\rho S} \left[ \frac{\partial}{\partial z} \left[ \mu (r \frac{\partial u}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u_s}{\partial y} + \frac{\partial w_s}{\partial y}) \right] + \frac{\partial}{\partial x} \left[ \mu (r \frac{\partial u}{\partial x} + m \frac{\partial v}{\partial y} + m \frac{\partial w}{\partial z}) \right] + \frac{\partial}{\partial y} \left[ \mu (r \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial x}) \right] \right] \tag{7.5}
\]
\[
\rho_s \left[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right] \\
+ \rho \left[ Us \frac{\partial T}{\partial x} + Vs \frac{\partial T}{\partial y} + Ws \frac{\partial T}{\partial z} \right] = (\gamma - 1)M_e^2 \left[ \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z} \right] \\
+ V_s \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z} + \frac{1}{\text{RePr}} \left[ \frac{\partial}{\partial x} \left( \mu_s \frac{\partial T}{\partial x} + u \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \mu_s \frac{\partial T}{\partial y} + v \frac{\partial T}{\partial y} \right) \right]
\]

\[
\frac{p}{p_s} = \frac{T}{T_s} + \frac{\rho}{\rho_s}
\]

where \( \phi \) is the perturbation dissipation function defined by

\[
\phi = \mu_s \left\{ 2r \left( \frac{\partial U_s}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial V_s}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial W_s}{\partial z} \frac{\partial w}{\partial z} \right) + 2m \left( \frac{\partial U_s}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial V_s}{\partial y} \frac{\partial u}{\partial x} \right) \right. \\
+ \left. 2 \left( \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} \right) \left( \frac{\partial U_s}{\partial x} + \frac{\partial W_s}{\partial z} \right) + 2 \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \left( \frac{\partial V_s}{\partial y} + \frac{\partial W_s}{\partial z} \right) \right\} + \mu \left\{ \frac{\partial U_s}{\partial x} \right. \\
+ \left. \left( \frac{\partial V_s}{\partial y} \right)^2 + \left( \frac{\partial W_s}{\partial z} \right)^2 \right\} 2 \left( \frac{\partial U_s}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial V_s}{\partial y} \frac{\partial u}{\partial x} \right) \right. \\
+ \left. \left( \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} \right) \left( \frac{\partial U_s}{\partial x} + \frac{\partial W_s}{\partial z} \right) + \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \left( \frac{\partial V_s}{\partial y} + \frac{\partial W_s}{\partial z} \right) \right\} 
\]

\[
(7.8)
\]

The constants \( r \) and \( m \) are given by

\[
r = 2 + \frac{\lambda}{\mu} = \frac{5.6}{3}
\]

\[
m = \frac{\lambda}{\mu} = -\frac{4}{3}
\]

(see Mack\textsuperscript{85}). Equations (7.2)-(7.8) are nondimensional with the Reynolds number and Prandtl number given by

\[
R = \frac{U^* e^*}{v^*}
\]

(7.10a)
respectively. Here $U_e^*$ is the dimensional edge freestream velocity, $v_e^*$ is the dimensional edge kinematic viscosity, $\delta^*$ is the dimensional displacement thickness, $\mu_e^*$ is the dimensional edge dynamic viscosity, $C_p^*$ is the dimensional specific heat at constant pressure and $\kappa_e^*$ is the dimensional edge thermal conductivity. In this formulation, $\text{Pr}$ and $C_p^*$ are taken as constants. This will affect the accuracy of the stability results; however, the mean flow we will consider in this chapter has this limitation built in.

The boundary conditions for the disturbance equations (7.2)-(7.8) are given by

\begin{align}
\text{at } y = 0 & \quad u = v = w = T = 0 \quad (7.11) \\
\text{as } y \to \infty & \quad u, v, w, T \to 0 \quad (7.12)
\end{align}

We will confine our analysis to mean flows that are only slightly nonparallel; that is, the normal velocity component $V_s$ is small compared with the other components $U_s$ and $W_s$. This in turn implies that all the mean-flow variables must be weak functions of the streamwise and spanwise coordinates compared with the normal coordinate. In other words

\begin{align}
U_s &= U_s(x_1, y, z_1) \quad (7.13a) \\
V_s &= eV_s^*(x_1, y, z_1), \quad V_s^* = 0(1) \quad (7.13b) \\
W_s &= W_s(x_1, y, z_1) \quad (7.13c) \\
p_s &= p_s(x_1, z_1) \quad (7.13d) \\
T_s &= T_s(x_1, y, z_1) \quad (7.13e)
\end{align}
We describe the relatively slow variations of the mean-flow quantities in the streamwise and spanwise directions by the slow scales \( x_1 = \varepsilon x \) and \( z_1 = \varepsilon z \), respectively. Here \( \varepsilon = \frac{1}{R} \).

We also assume that the viscosity is a function of the temperature only, so that, with \( T/T_s \) small
\[
\hat{\mu}(T_s + T) = \mu_s + \frac{d\mu_s}{dT} \bigg|_{T_s} T
\]
or
\[
\mu_s = T \frac{d\mu_s}{dT} \bigg|_{T_s}
\] (7.14)

### 7.3 Method of Multiple Scales

Next we use the method of multiple scales to solve Eqs. (7.2)-(7.7), (7.11) and (7.12). Onto the basic state we superpose a Tollmien-Schlichting wave, a traveling harmonic wave. We describe the disturbance quantities in the form
\[
q(x,y,z,t;\varepsilon) = [q_0(x_1,y,z_1,t_1) + \varepsilon q_1(x_1,y,z_1,t_1) + ...] \exp(i\theta)
\] (7.15)

where \( t_1 = \varepsilon t \), a slow scale for time, and
\[
\frac{\partial \theta}{\partial x} = \alpha(x_1,z_1)
\] (7.16a)
\[
\frac{\partial \theta}{\partial z} = \beta(x_1,z_1)
\] (7.16b)
\[
\frac{\partial \theta}{\partial t} = -\omega
\] (7.16c)

Here \( \alpha \) and \( \beta \) are the dimensionless quasi-parallel components of the wavenumber in the chordwise and spanwise directions, respectively, and \( \omega \) is the dimensionless frequency of the disturbance. Assuming the phase \( \theta \) to be continuously differentiable, we have
If we consider a spatial stability analysis, then $\alpha$ and $\beta$ are complex and $\omega$ is real. On the other hand, if we consider a temporal stability analysis, then $\alpha$ and $\beta$ are real and $\omega$ is complex. In either case, we can define a quasi-parallel growth rate. For spatial stability, $-\alpha$ and $-\beta$ are the $x$- and $z$-components of the spatial growth rate, respectively. For temporal stability, $\omega$ is the temporal growth rate. The subscript $i$ here means the imaginary part of the complex number.

Applying Eqs. (7.13)-(7.16) to Eqs. (7.2)-(7.7), (7.11), and (7.12) and equating coefficients of powers of $\varepsilon$, we obtain the zeroth- and first-order problems describing the disturbances.

### 7.4 The Zeroth-Order Problem

We find the equations governing the zeroth-order to be

**Continuity**

\[
L_1(u_0,v_0,w_0,p_0,T_0) = -i\rho_s(\omega - \alpha u + \beta w)(P_0 - T_0) + i\rho_s(\alpha u_0 + \beta w_0) + D(\rho_s v_0) = 0
\]

\[
(7.18)
\]

**x-Momentum**

\[
L_2(u_0,v_0,w_0,p_0,T_0) = -i\rho_s(\omega - \alpha u - \beta w)u_0 + \rho_s v_0 DU_s + i\alpha p_0
\]

\[
- \frac{1}{R} \left\{-\mu_s(\alpha^2 + \beta^2)u_0 + D(\mu_s Du_0) + i\mu_s(1 + m)Dv_0 + i\alpha v_0 Du_s
\right.

\[
- \alpha \beta \mu_s(1 + m)W_0 + D\left(\frac{d\mu_s}{dT} T_0\right) DU_s + \frac{d\mu_s}{dT} T_0^2 U_s\right\} = 0
\]

\[
(7.19)
\]
y-Momentum

\[ L_3(u_0, v_0, w_0, p_0, T_0) = -i\rho S(\omega - \alpha u_S - \beta w_S)v_0 + dp_0 \]
\[ -\frac{1}{R} \left\{ i\mu S(1 + m)(\alpha Du_0 + \beta Dw_0) + im(\alpha u_0 + \beta w_0)Du_S - (\alpha^2 + \beta^2)\mu S v_0 + D(\mu S rDv_0) + i(\alpha Du_S + \beta Dw_S) \frac{d\mu S}{dT} T_0 \right\} = 0 \] (7.20)

z-Momentum

\[ L_4(u_0, v_0, w_0, p_0, T_0) = -i\rho S(\omega - \alpha u_S - \beta w_S)w_0 + \rho_S v_0 Dw_S + i\beta p_0 \]
\[ -\frac{1}{R} \left\{ -\alpha \beta \mu S(1 + m)u_0 + i\beta \mu S(1 + m)Dv_0 + i\beta v_0 Du_S - \mu S(\alpha^2 + \beta^2)w_0 + D(\mu S Dw_0) + D(\frac{d\mu S}{dT} T_0) Dw_S + \frac{d\mu S}{dT} T_0 D^2 w_S \right\} = 0 \] (7.21)

Energy

\[ L_5(u_0, v_0, w_0, p_0, T_0) = -i\rho S(\omega - \alpha u_S - \beta w_S)T_0 + \rho S v_0 DT_S \]
\[ + i(\gamma - 1)M_e p_0 (\omega - \alpha u_S - \beta w_S) - \frac{(\gamma - 1)M_e}{R} \left\{ 2\mu S (Du_S Du_0 + Dw_S Dw_0) + 2i\mu S (\alpha Du_S + \beta Dw_S)v_0 + \frac{d\mu S}{dT} [(Du_S)^2 + (Dw_S)^2] T_0 \right\} - \frac{1}{R^p} \left\{ 2D\mu S DT_0 + \mu S D^2 T_0 - (\alpha^2 + \beta^2)\mu S T_0 + D^2 \mu S T_0 \right\} = 0. \] (7.22)

The symbol D stands for \( \frac{d}{dy} \). This system (7.18)-(7.22) is subject to the boundary conditions

\[ u_0 = v_0 = w_0 = T_0 = 0 \quad \text{at} \quad y = 0 \] (7.23)
\[ u_0, v_0, w_0, T_0 \to 0 \quad \text{as} \quad y \to \infty \] (7.24)

If we adopt the convention
we find that the zeroth-order equations can be written as a system of
eight linear first-order ordinary differential equations of the form
\[
Dz_\text{on} - \sum_{m=1}^{8} a_{nm} z_\text{om} = 0
\]
for \( n = 1, 2, \ldots, 8 \). The \( a_{nm} \) are the elements of an \( 8 \times 8 \) variable-
coefficient matrix whose nonzero elements are given in Appendix A. The
boundary conditions for (7.26) become
\[
\begin{align*}
    z_1 &= z_3 = z_5 = z_7 = 0 \quad \text{at} \quad y = 0 \\
    z_1, z_3, z_5, z_7 &\rightarrow 0 \quad \text{as} \quad y \rightarrow \infty
\end{align*}
\]
7.5 Numerical Procedure for Solving the Zeroth-Order Problem

The system (7.26)-(7.28) constitutes an eigenvalue problem. Given
the Reynolds number \( R \) and mean-flow profiles, we determine the
dispersion relation
\[
\omega = \omega(\alpha, \beta, R)
\]
numerically. For a temporal problem the procedure is to specify $\alpha$ and $\beta$ and then determine $\omega$ as an eigenvalue. For a spatial problem we specify $\omega$ and two relations among $\alpha_i$, $\alpha_r$, $\beta_r$, and $\beta_i$ and then determine the remaining two relations using (7.29).

Since our problem is linear, it has eight linearly independent solutions. The general solution is then a linear combination of these linearly independent solutions where the coefficients are determined from the boundary conditions. The procedure we use is to guess the eigenvalues and then numerically integrate the system (7.26) from the boundary conditions for $y \to \infty$ to the wall at $y = 0$. In our analysis we employ the computer code SUPORT developed by Scott and Watts\textsuperscript{38}. This code, based on the method of Godunov\textsuperscript{103}, handles stiff two-point boundary-value problems such as this. It integrates with a Runge-Kutta-Fehlburg scheme and uses Gram-Schmidt orthonormalization to keep the solution vectors linearly independent. In general, our guess of the eigenvalues is wrong and therefore one of the boundary conditions at $y = 0$ is left unsatisfied. We then use a Newton-Raphson scheme on this unsatisfied boundary condition to iterate and try to converge to the correct eigenvalue.

Once the correct eigenvalue is determined, we recover the eigensolutions $\xi_j$, $j = 1, \ldots, 8$. Then recognizing that the $a_{ij}$ are only slowly-varying functions of $x$ and $z$, we write the solution of (7.26)-(7.28) as

$$z_{0j} = A(x_1, z_1, t_1)\xi_j(x_1, y, z_1) \quad (7.30)$$
The amplitude function $A$ is left to be determined by imposing a solvability condition on the first-order problem.

7.6 Concept of the Adjoint

7.6a The Adjoint System

Before proceeding to the first-order problem, we want to establish the idea of and how we use the adjoint system of (7.26)-(7.28). We write the original system (7.26) for the moment as

$$Dz - A_0 z = 0$$

where the variable-coefficient matrix $A_0$ whose elements are the $a_{ij}$, defined in Appendix A, is not to be confused with the amplitude function $A$ in the preceding section. The eight-component vector $\tilde{z}$ represents the elements $Z_{01}, \ldots, Z_{08}$. Equation (7.31) is of course subject to the boundary conditions

$$\tilde{z} \bigg|_{y=0} = \begin{bmatrix} 0, Z_{02}, 0, Z_{04}, 0, Z_{06}, 0, Z_{08} \end{bmatrix}^T$$

$$\tilde{z} \bigg|_{y=\infty} \to 0$$

The adjoint equation corresponding to Eq. (7.31) is found by left-multiplying Eq. (7.31) by the transpose of the adjoint solution and then integrating the result over the interval $y = 0$ to $y = \infty$. We then use integration by parts to find the adjoint equations. That is,

$$\int_0^\infty \tilde{z}^T Dz dy - \int_0^\infty \tilde{z}^T A_0 \tilde{z} dy = 0$$
These equations defining the adjoint solution \( z^* \) are found by requiring that the coefficient of \( \tilde{z} \) in the integrand of Eq. (7.34) be zero. That is,

\[
Dz^* + z^* A_0 = 0^T
\]

or, taking the transpose,

\[
Dz^* + A_0^T z^* = 0 \tag{7.35}
\]

Then from Eq. (7.34) we see that

\[
\left. z^* T \right|_\infty - \left. z^* T \right|_0 = 0 \tag{7.36}
\]

We choose the adjoint boundary conditions such that all the terms in Eq. (7.36) vanish independently. Applying Eqs. (7.32) and (7.33) to Eq. (7.36) we see that the boundary conditions for the adjoint problem are

\[
z^* \left. \begin{array}{c} y=0 \\ y=\infty \end{array} \right| = \left\{ \begin{array}{c} z_{y}^* \ 0 \ z_{z}^* \ 0 \ z_{y}^* \ 0 \ z_{z}^* \ 0 \end{array} \right\}^T \tag{7.37}
\]

\[
z^* \left. \begin{array}{c} y=\infty \end{array} \right| \to 0 \tag{7.38}
\]

7.6b Solvability Conditions

In applying the method of multiple scales, we obtain problems that need to be solved in succession. The zeroth-order problem in Section 7.5 is linear and homogeneous, whereas the higher-order problems are linear and inhomogeneous, as we will see in Section 7.7. To determine the dependence of our problem on the slow scales introduced, we must
examine these higher-order problems. Our expansion, however, leads to inconsistencies between the zeroth- and higher-orders, unless we impose solvability conditions. The rule is that a solution exists to these higher-order problems only if the vector composed of the inhomogeneous terms on the right-hand sides is orthogonal to the linear space spanned by the solutions of the adjoint homogeneous problem.

So we must solve the adjoint problem

\[ Dz^* + \sum_{m=1}^{8} a_{mn}^m z^* = 0 \quad n = 1, 2, \ldots, 8 \quad (7.39) \]

\[ z^*_y = z^*_z = z^*_z = z^*_z = 0 \quad \text{at} \quad y = 0 \quad (7.40) \]

\[ z^*_y, z^*_z, z^*_z, z^*_z \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty \quad (7.41) \]

Equations (7.39)-(7.41) are just Eqs. (7.35), (7.37), and (7.38) rewritten in different notation. We use the same numerical procedure as mentioned above; however, there is no need to iterate on the eigenvalues as the adjoint and zeroth-order problems have the same ones. As in the zeroth-order problem we express \( z^*_i \) in the form

\[ z^*_i = A^*(x_1, z_1, t_1) \xi^*_i(x_1, y, z_1) \quad (7.42) \]

where the \( \xi^*_i \) are the adjoint eigensolutions and \( A^* \) is an amplitude function.

7.6c Application of Boundary Conditions at Infinity

Using the concept of the adjoint, we next establish what infinite boundary conditions we impose to integrate numerically both the zeroth-order and adjoint problems. If we apply Eq. (7.28) or Eq. (7.41)
directly to some finite value of $y$ and then integrate to the wall, the results may be inaccurate.

We use the adjoint to impose the asymptotic boundary conditions in the following way (see Ragab and Nayfeh\textsuperscript{104}). Let us consider the zeroth-order problem first. For $y$ larger than the boundary layer thickness $\delta$, the variable-coefficient matrix $[a_{ij}]$ of Eq. (7.26) becomes a constant matrix, because the mean-flow quantities become independent of $y$. We call this constant matrix $C$ and list its nonzero components $c_{ij}$ in Appendix B. In other words, outside the boundary layer, Eq. (7.26) becomes

$$Dz - Cz = 0 \quad (7.43)$$

which has the general solution

$$z_{0i} = \sum_{j=1}^{8} \Lambda_{ij} d_j \exp(\lambda_j y) \quad i = 1, 2, \ldots, 8 \quad (7.44)$$

where the $\lambda_j$ are the corresponding eigenvectors and the $d_j$ are arbitrary constants.

We find $\lambda_j$ and $\Lambda_{ij}$ by rewriting the eight first-order equations in Eq. (7.43) as four second-order equations as follows:

$$D^2 E_i - \sum_{j=1}^{4} b_{ji} E_i = 0 \quad i = 1, 2, 3, 4 \quad (7.45)$$

where

$$E_1 = z_{01} \quad (7.46a)$$
$$E_2 = z_{04} \quad (7.46b)$$
$$E_3 = z_{05} \quad (7.46c)$$
$$E_4 = z_{07} \quad (7.46d)$$
The $b_{ij}$ are given in Appendix C. The solution of Eq. (7.45) is of the general form

$$E_i = \sum_{j=1}^{8} B_{ij} e_j \exp(\lambda_j y) \quad i = 1, 2, 3, 4$$

(7.47)

where the $\lambda_j$ are the same as the eigenvalues of Eq. (7.43), the $B_{ij}$ are the corresponding eigenvectors, and the $e_j$ are arbitrary constants.

Considering the characteristic determinant and the characteristic equation, we find the eigenvalues $\lambda_j$ and the corresponding eigenvectors $B_{ij}$ to be given as in Appendix C. The $\Lambda_{ij}$ are then found from Eqs. (7.44)-(7.47) and also listed in Appendix C.

Next we write the matrix $C$ of Eq. (7.43) in its Jordan canonical form as follows:

$$J = P^{-1} CP$$

(7.48)

Here $J$ is a diagonal matrix whose nonzero elements are the eigenvalues of $C$. That is,

$$J = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_8)$$

(7.49)

We order the eigenvalues so that the positive eigenvalues appear in the last four rows of $J$. Then the matrix $P$ is the orthogonal matrix whose $j$th column is the eigenvector $A_{ij}$ corresponding to the $j$th eigenvalue.

Letting

$$z = P \xi$$

(7.50)

in Eq. (7.43), we obtain

$$D\xi = J\xi$$

(7.51)

Because of the ordering of the eigenvalues, the only way we will have
decaying solutions at infinity is if

\[ \xi_5 = \xi_6 = \xi_7 = \xi_8 = 0 \]  

(7.52)

Recalling Eq. (7.36) and remembering that each term must vanish independently, we see that we must satisfy the condition

\[ \lim_{y \to \infty} z^T z = 0 \]  

(7.53)

If we substitute Eq. (7.50) into Eq. (7.53) we have

\[ \lim_{y \to \infty} z^T p \xi = 0 \]  

or

\[ \lim_{y \to \infty} (p^T z^*) \xi = 0 \]  

(7.55)

Letting

\[ \xi^* = p^T z^* \]  

(7.56)

we rewrite Eq. (7.55) as

\[ \lim_{y \to \infty} \xi^T \xi = 0 \]  

(7.57)

Applying the boundary conditions (7.52) to Eq. (7.57) gives us

\[ \lim_{y \to \infty} (\xi_1 \xi_1 + \xi_2 \xi_2 + \xi_3 \xi_3 + \xi_4 \xi_4) = 0 \]  

(7.58)

Because \( \xi_1, \xi_2, \xi_3, \) and \( \xi_4 \) are all linearly independent, each of their coefficients must vanish. That is,

\[ \xi_1^* = \xi_2^* = \xi_3^* = \xi_4^* = 0 \]  

(7.59)

Then it follows from Eqs. (7.56) and (7.59) that the boundary conditions at infinity are replaced with

\[ T^* z^* = 0 \quad \text{at} \quad y = y_e \]  

(7.60)
where the four rows of $T^*$ are the first four rows of $P^T$ and $y_e$ is larger than the boundary-layer thickness.

Next we consider the adjoint problem. Outside the boundary layer, the system (7.39) becomes

$$\mathbf{D}\mathbf{z}^* + \mathbf{c}^T\mathbf{z}^* = 0 \quad (7.61)$$

which has the general solution

$$\mathbf{z}^* = \sum_{j=1}^{8} \Lambda^*_{ij} d_j^* \exp(\lambda_j y) \quad i = 1, 2, \ldots, 8 \quad (7.62)$$

where the $\lambda_j$ are the same as for the first-order problem, but the corresponding eigenvectors $\Lambda^*_{ij}$ are different from the $\Lambda_{ij}$. We obtain $\Lambda^*_{ij}$ analytically the same way we obtained $\Lambda_{ij}$ and give the results in Appendix D.

We write $\mathbf{C}^* = -\mathbf{C}^T$ in its Jordan canonical form as follows:

$$\mathbf{J}^* = \mathbf{P}^{-1} \mathbf{C}^* \mathbf{P}^* \quad (7.63)$$

Here $\mathbf{J}^*$ is a diagonal matrix whose nonzero elements are the eigenvalues of $\mathbf{C}^*$, ordered so that the positive eigenvalues appear in the last four rows of $\mathbf{J}^*$. The matrix $\mathbf{P}^*$ is the orthogonal matrix whose columns are the eigenvectors $\Lambda^*_{ij}$ of $\mathbf{C}^*$.

Letting

$$\mathbf{z}^* = \mathbf{P}^{-1} \mathbf{\xi}^* \quad (7.64)$$

in Eq. (7.61), we obtain

$$\mathbf{D}\mathbf{\xi}^* = \mathbf{J}^* \mathbf{\xi}^* \quad (7.65)$$

For decaying solutions at infinity, we require that
If we follow steps similar to those in Eqs. (7.53)-(7.60), we find that the boundary conditions at infinity are replaced with

\[ T_z = 0 \quad \text{at} \quad y = y_e \]  

where the four rows of \( T \) are the first four rows of \( P^T \).

### 7.7 The First-Order Problem--Amplitude Modulation Equation

Using Eq. (7.30), we find that the first-order problem can be written as a first-order system for eight differential equations of the form

\[
\frac{\partial z_{1n}}{\partial y} = \sum_{m=1}^{8} a_{nm} z_{1m} = D_n \frac{\partial A}{\partial t_1} + E_n \frac{\partial A}{\partial x_1} + F_n \frac{\partial A}{\partial t_1} + G_n A
\]

for \( n = 1, 2, \ldots, 8 \)  

\[ (7.68) \]

where

\[ z_{11} = u_1 \]  

\[ z_{12} = \frac{\partial u_1}{\partial y} \]  

\[ z_{13} = v_1 \]  

\[ z_{14} = p_1 \]  

\[ z_{15} = T_1 \]  

\[ z_{16} = \frac{\partial T_1}{\partial y} \]  

\[ z_{17} = W_1 \]  

\[ z_{18} = \frac{\partial W_1}{\partial y} \]  

and the \( a_{nm} \) are the same as in Eq. (7.26). The \( D_n, E_n, F_n, \) and \( G_n \) are
known or easily-determined functions of the zeroth-order eigensolutions $\zeta_n$ of Eq. (7.30), $\alpha, \beta, \omega$, and the mean-flow quantities. We define these functions in Appendix E. The system (7.68) is subject to the boundary conditions

$$Z_{11} = Z_{13} = Z_{15} = Z_{17} = 0 \text{ at } y = 0$$  \hspace{1cm} (7.70)

$$Z_{11}, Z_{13}, Z_{15}, Z_{17} \to 0 \text{ as } y \to \infty$$  \hspace{1cm} (7.71)

As discussed in Section 7.6b, because the homogeneous parts of Eqs. (7.68)-(7.71) are the same as Eqs. (7.25)-(7.28) and because we found a nontrivial solution to Eqs. (7.25)-(7.28), the system (7.68)-(7.71) will have a nontrivial solution only if a solvability condition is satisfied. This solvability condition is

$$\sum_{n=1}^{\infty} \int_0^\infty \left[ D_n \frac{\partial A}{\partial t_1} + E_n \frac{\partial A}{\partial x_1} + F_n \frac{\partial A}{\partial z_1} + G_n A \right] \zeta_n^* dy = 0$$  \hspace{1cm} (7.72)

where the $\zeta_n^*$ are the eigensolutions of the adjoint homogeneous problem. This condition (7.72) implies that

$$\frac{\partial A}{\partial t_1} + \omega_\alpha \frac{\partial A}{\partial x_1} + \omega_\beta \frac{\partial A}{\partial z_1} = h_1 A$$  \hspace{1cm} (7.73)

Here $\omega_\alpha$ and $\omega_\beta$, the group velocity components in the $x$- and $z$-directions, respectively, and $h_1$ are given in Appendix F. It is easy to see that $\omega_\alpha$ and $\omega_\beta$ are defined this way by differentiating Eq. (7.26) with respect to $\alpha$ and $\beta$, respectively, and then applying the solvability conditions 33.

Equation (7.73) describes the evolution of the amplitude $A$. The function $h_1$ reflects the effect of nonparallelism of the basic state on the growth of the disturbance amplitude and therefore contains $x_1$ and $z_1$. 
derivatives of all of the basic-state variables, the wavenumbers $\alpha$ and $\beta$, and the eigensolutions of the zeroth-order problem. Finding derivatives of the basic-state variables is straightforward. In the next section we indicate how to determine the derivatives of the wavenumbers and eigensolutions.

7.8 Wavenumber Modulation Equations

To determine the wavenumber modulation equations, we replace $z_0 n$ by the eigensolutions $\zeta_n$ in Eqs. (7.25)-(7.28), differentiate the resulting system with respect to $x_1$, and obtain

$$D \left( \frac{\partial \zeta_n}{\partial x_1} \right) - \sum_{m=1}^{8} a_{mn} \left( \frac{\partial \zeta_m}{\partial x_1} \right) = i F_n \frac{\partial \alpha}{\partial x_1} + i F_n \frac{\partial \beta}{\partial x_1} + \sum_{m=1}^{8} \frac{\partial a_{nm}}{\partial x_1} \bigg|_{\alpha, \beta} \zeta_m$$

subject to the boundary conditions

$$\frac{\partial \zeta_1}{\partial x_1} = \frac{\partial \zeta_3}{\partial x_1} = \frac{\partial \zeta_5}{\partial x_1} = \frac{\partial \zeta_7}{\partial x_1} = 0 \quad \text{at} \quad y = 0$$

$$\frac{\partial \zeta_1}{\partial x_1}, \frac{\partial \zeta_3}{\partial x_1}, \frac{\partial \zeta_5}{\partial x_1}, \frac{\partial \zeta_7}{\partial x_1} \to 0 \quad \text{as} \quad y \to \infty$$

We apply the solvability condition to Eqs. (7.74)-(7.76) and obtain

$$\omega \frac{\partial \alpha}{\partial x_1} + \omega \frac{\partial \beta}{\partial x_1} = h_2$$

where $h_2$ is given in Appendix F. Similarly, considering the $z_1$-derivative, we find that

$$\omega \frac{\partial \alpha}{\partial z_1} + \omega \frac{\partial \beta}{\partial z_1} = h_3$$

where $h_3$ is given in Appendix F. The quantities $h_2$ and $h_3$ reflect the effect of the nonparallelism of the mean flow on the growth of the wavenumbers. Applying the consistency condition (7.17) to Eqs. (7.77)
and (7.78), we have

\[ \omega \frac{\partial \alpha}{\partial x_1} + \omega \frac{\partial \alpha}{\partial z_1} = h_2 \quad (7.79) \]

\[ \omega \frac{\partial \alpha}{\partial x_1} + \omega \frac{\partial \beta}{\partial z_1} = h_3 \quad (7.80) \]

For a monofrequency disturbance, Nayfeh$^{30,31}$ showed that, for this formulation to be realistic physically, \( \omega_\beta/\omega_\alpha \) must be real. So we consider the equations

\[ \frac{\partial \alpha}{\partial x_1} + \frac{\omega_\beta}{\omega_\alpha} \frac{\partial \alpha}{\partial z_1} = \frac{h_2}{\omega_\alpha} \quad (7.81) \]

\[ \frac{\partial \beta}{\partial x_1} + \frac{\omega_\beta}{\omega_\alpha} \frac{\partial \beta}{\partial z_1} = \frac{h_3}{\omega_\alpha} \quad (7.82) \]

The most logical approach to solving this Cauchy problem is to specify the spanwise wavenumber distribution \( \beta \) of a disturbance for all \( z_1 \) at a given \( x_1 \) as an initial condition and then determine the solution to Eqs. (7.81)-(7.82) analytically$^6$. That is, we specify

\[ \beta(x_1 = a, z_1) = \beta_0(z_1) \]

The characteristic equations of Eqs. (7.81) and (7.82) are

\[ \frac{dx_1}{ds} = 1 \quad (7.83a) \]

\[ \frac{dz_1}{ds} = \frac{\omega_\beta}{\omega_\alpha} \quad (7.83b) \]

The solution of Eqs. (7.83) is a one-parameter family of curves. We call this parameter \( \tau \) and write the solution in the form

\[ x_1 = \tilde{x}_1(s, \tau) \quad (7.84a) \]

\[ z_1 = \tilde{z}_1(s, \tau) \quad (7.84b) \]
Each of these curves has at every point \((x_1, z_1)\) the direction \((1, \omega_\beta/\omega_\alpha)\).

The initial data curve \(\Gamma\) in parametric form is

\[
\Gamma: \begin{cases} 
  x_1 = a \\
  z_1 = \tau
\end{cases}
\]  

(7.85a)  

(7.85b)

giving initial conditions for Eqs. (7.83) in the parametric form

\[
\begin{align*}
  \dot{x}_1(a, \tau) &= a \\
  \dot{z}_1(a, \tau) &= \tau
\end{align*}
\]  

(7.86a)  

(7.86b)

We take \(s = a\) for convenience since Eqs. (7.83) are invariant under a translation of \(s\).

The solution to Eqs. (7.83) satisfying Eqs. (7.86) is

\[
\begin{align*}
  x_1 &= s \\
  z_1 &= \int_a^s \frac{\omega_\beta}{\omega_\alpha} \, dt + \tau
\end{align*}
\]  

(7.87a)  

(7.87b)

Eliminating \(s\) we find the following equation for the characteristics of Eqs. (7.81) and (7.82):

\[
\begin{align*}
  z_1 &= \int_a^{x_1} \frac{\omega_\beta}{\omega_\alpha} \, dt + \tau
\end{align*}
\]  

(7.88)

in terms of \(x_1, z_1,\) and \(\tau\).

The partial-differential equation (7.82) reduces to

\[
\frac{d\beta}{ds} = \frac{h_3}{\omega_\alpha}
\]  

(7.89)

along the characteristics (7.88). The initial data for \(\beta\) along \(\Gamma\) is

\[
\beta = \beta_0(\tau)
\]  

(7.90)
The solution of Eq. (7.89) satisfying Eq. (7.90) is

$$\beta = \int_a^s \frac{h_3}{\omega_\alpha} \, dt + \beta_0(\tau) \quad (7.91)$$

Eliminating $s$ with Eq. (7.87) we find that

$$\beta = \int_a^{x_1} \frac{h_3}{\omega_\alpha} \, dt + \beta_0(\tau) \quad (7.92)$$

We recall that

$$\tau = z_1 - \int_a^{x_1} \frac{\omega}{\omega_\alpha} \, dt \quad (7.93)$$

is a constant along a characteristic, so that as we follow one wave only, $\beta_0(\tau)$ will be a constant, the initial condition of the point from which we begin to march.

To find $\frac{\partial \beta}{\partial x_1}$, we differentiate Eq. (7.92) with respect to $x_1$ and obtain

$$\frac{\partial \beta}{\partial x_1} = \frac{h_3}{\omega_\alpha} - \frac{\omega}{\omega_\alpha} \beta_0'(\tau) \quad (7.94)$$

Then $\frac{\partial \beta}{\partial z_1}$ is found to be

$$\frac{\partial \beta}{\partial z_1} = \beta_0'(\tau) \quad (7.95)$$

From Eqs. (7.17) and (7.81), we then determine $\frac{\partial \alpha}{\partial x_1}$ and $\frac{\partial \alpha}{\partial z_1}$ to be

$$\frac{\partial \alpha}{\partial x_1} = \frac{h_2}{\omega_\alpha} - \frac{\omega}{\omega_\alpha} \frac{h_3}{\omega_\alpha} + \frac{\omega}{\omega_\alpha} \beta_0'(\tau) \quad (7.96)$$

$$\frac{\partial \alpha}{\partial z_1} = \frac{h_3}{\omega_\alpha} - \frac{\omega}{\omega_\alpha} \beta_0'(\tau) \quad (7.97)$$

Knowing these wavenumber variations, we now numerically solve Eqs.
(7.74)-(7.76) and the corresponding system for \( z_1 \)-derivatives. From this point we are able to determine the \( h_1 \) function in the amplitude-modulation equation (7.73).

7.9 Method of Finding the Most Unstable Disturbance

In this section we propose a method of finding the most unstable disturbance propagating from some given initial chord location on a wing. In our opinion, past numerical calculation of three-dimensional boundary-layer stability have not satisfactorily answered the questions about the character of the most unstable disturbance. Instead of jumping from wave to wave by locally calculating the most amplified disturbance as you march or specifying some artificial condition such as constant spanwise or chordwise wavelength or whatever as all our predecessors have done, we believe that, after careful consideration\(^{31,105}\) the correct way to handle this three-dimensional stability analysis is to select one specific wave at an initial point and then follow only that one wave along its trajectory. To find the most unstable disturbance, then, we change the initial conditions.

Before proceeding we wish to establish our convention. The quantities \( \omega_r \) and \( \omega_i \) are the real and imaginary parts of the dimensionless complex frequency \( \omega \). The quantities \( \alpha_r \) and \( \alpha_i \) are the real and imaginary parts of the chordwise wavenumber \( \alpha \) and the quantities \( \beta_r \) and \( \beta_i \) are the real and imaginary parts of the spanwise wavenumber.

According to Eq. (7.90), we need to specify a spanwise wavenumber distribution \( \beta_r \) along some initial curve and then march along the characteristics (7.88). Because we are interested in a spatial stability
analysis, we also specify a dimensional frequency in hertz, a frequency that remains constant as we march. We note that \( \omega_i = 0 \). As we march, we calculate the mean flow profiles using the code of Kaups and Cebeci described in Chapter Six.

To demonstrate the method we choose \( \beta^* \), the dimensional form of \( \beta_r \), to be a constant at some initial chordwise location \( x_1 = a \) for all \( z_1 \). At this initial point we still have three unspecified values \( \beta_i, \alpha_r, \) and \( \alpha_i \). Two of these are determined through numerical integration of the disturbance equations using SUPORT and satisfaction of the boundary conditions through Newton-Raphson iteration. The third is found by requiring that \( \omega_\beta/\omega_\alpha \) be real as mentioned above in Section 7.8. This fixes the direction of marching. From here we evaluate

\[
\frac{d\alpha}{ds} = \frac{h_2}{\omega_\alpha} \tag{7.98a}
\]

\[
\frac{d\beta}{ds} = \frac{h_3}{\omega_\alpha} \tag{7.98b}
\]

To stay on the initial wave, we increment the characteristic variable \( s \) by \( ds \), evaluate \( dx_1 \) and \( dz_1 \) from Eqs. (7.87), and correct \( \alpha \) and \( \beta \) by

\[
\alpha = \alpha + \frac{h_2}{\omega_\alpha} \, ds \tag{7.99a}
\]

\[
\beta = \beta + \frac{h_3}{\omega_\alpha} \, ds \tag{7.99b}
\]

We then renondimensionalize with respect to local edge variables at the new \( x_1 \) and \( z_1 \). To test that we are on the same wave, we integrate the disturbance equations at the new \( x_1 \) and \( z_1 \) with the new \( \alpha \) and \( \beta \) to see if the boundary conditions are satisfied and \( \omega_\beta/\omega_\alpha \) is real. If they are, we evaluate \( \frac{d\alpha}{ds} \) and \( \frac{d\beta}{ds} \), increment \( s \), correct \( \alpha \) and \( \beta \), and continue
marching along the trajectory. If these conditions are not all satisfied, we decrease the step size $ds$ until they are and then proceed as above.

Letting $A_0$ be the amplitude of the disturbance initially, we find that according to Eq. (7.73) for a monochromatic wave (i.e. $\frac{\partial}{\partial t} = 0$), the amplitude function $A$ of Eq. (7.30) at $(x_1,z_1)$ is given by

$$A = A_0 \exp\left(\int_a^s \frac{h_1}{\omega} ds\right)$$  \hspace{1cm} (7.100)

From Eqs. (7.15), (7.30), (7.83), and (7.100) we then see that

$$u = A_0 \xi_1(x_1,y,z_1) \exp\left\{\int_a^x \left[i(\alpha + \frac{\omega \beta}{\omega \alpha}) + \varepsilon \frac{h_1}{\omega} \right] dx - i\omega t\right\}$$  \hspace{1cm} (7.101)

We define an $n$-factor from Eq. (7.101) as

$$n = -\int_a^x \left[\alpha_i + \frac{\omega \beta}{\omega \alpha} \beta_i - \varepsilon \left(\frac{h_1}{\omega}\right)_i \right] dx$$

As we march we compute the $n$-factor from where the disturbance first goes unstable ($\frac{dn}{dx} > 0$) to where it becomes stable again ($\frac{dn}{dx} < 0$). Once we compute $n$ we change the initial conditions on $\beta_r^*$ to determine which $\beta_r^*$ is associated with the largest value of $n$. This tells us the initial spanwise wavenumber of the most dangerous wave for a given frequency from a given chord location. Then we vary the dimensional frequency to determine the most dangerous frequency.

Numerical results of our proposed method will be published when they become available.
Chapter Eight
Summary

In this work, an analysis of the Tollmien-Schlichting instability of laminar viscous flows has been presented. The following is a summary of our accomplishments:

(i) For two-dimensional incompressible flow past a flat plate with porous suction strips, we used linear triple-deck, closed-form solutions for the mean flow to do a linear, parallel, spatial stability analysis. We developed a simple linear optimization scheme to determine the number, spacing, and mass-flow rate through the strips and concluded, surprisingly, that suction should be concentrated near the Branch I neutral point of the stability curve.

(ii) We verified the results of our optimization scheme with experimental data. We found that the theory correctly predicts the experimental results and concluded that the optimization scheme is reliable enough to replace the experiment as a tool in designing efficient strip configurations in so far as two-dimensional incompressible flows are concerned.

(iii) For axisymmetric incompressible flow past a body with porous strips, we developed linear triple deck, closed-form solutions for the mean-flow quantities, solutions which account for upstream influence. These solutions are linear superpositions of the flow past the body without suction plus the perturbations due to the suction strips. The flow past the
suctionless body was calculated using the Transition Analysis Program System (TAPS).

(iv) Using the linear triple deck, closed-form solutions mentioned in (iii) we then developed a simple linear optimization scheme to determine number, spacing, and mass flow rate through the strips on an axisymmetric body. At present, we are finishing the development of and documentation for a computer code for official distribution that will interface with TAPS and suggest efficient configurations using our theory.

(v) For compressible three-dimensional flow, we used the method of multiple scales to formulate the three-dimensional stability problem and determine the partial-differential equations governing variations of the amplitude and complex wavenumbers. We then proposed a method for following one specific wave along its trajectory to ascertain the characteristics of the most unstable disturbance. Numerical results using the flow over the X-21 wing as calculated from the Kaups-Cebeci code will be published when they become available.
References


APPENDICES
Appendix A

\[ a_{12} = a_{56} = a_{78} = 1 \]

\[ a_{21} = \alpha^2 + \beta^2 - i\omega R/\mu_S T_S \]

\[ a_{22} = - D\mu_S/\mu_S \]

\[ a_{23} = - i\alpha(m + 1)DT_S/T_S - i\alpha D\mu_S/\mu_S + RDU_s/\mu_s T_s \]

\[ a_{24} = i\alpha R/\mu_S - (m + 1)\gamma M_e^2 \omega \]

\[ a_{25} = - \alpha(m + 1)\hat{\omega}/T_S - D(u_s DU_s)/\mu_s \]

\[ a_{26} = - \mu_s DU_s/\mu_s \]

\[ a_{31} = - i\alpha \]

\[ a_{33} = DT_S/T_S \]

\[ a_{34} = i\gamma M_e^2 \hat{\omega} \]

\[ a_{35} = - i\omega/T_S \]

\[ a_{37} = - i\beta \]

\[ a_{41} = - i\gamma r(rDU_S/T_s + 2D\mu_s/\mu_s) \]

\[ a_{42} = - i\chi \alpha \]

\[ a_{43} = \chi[- \alpha^2 - \beta^2 + i\omega R/\mu_S T_S + rD^2 T_S/T_S + rD\mu_S DT_S/\mu_S T_S] \]

\[ a_{44} = - i\chi r\gamma M_e^2 [\alpha DU_S + \beta DW_S - \hat{\omega} DT_S/T_S - \hat{\omega} D\mu_S/\mu_S] \]

\[ a_{45} = i\chi [r(\alpha DU_S + \beta DW_S)/T_S + \mu_s'(\alpha DU_S + \beta DW_S)/\mu_s - r\hat{\omega} D\mu_S/\mu_s T_S] \]

\[ a_{46} = - i\chi r\omega/T_S \]

\[ a_{47} = - i\chi rBDT_S/T_S - 2i\chi BD\mu_S/\mu_S \]

\[ a_{48} = - i\chi \beta \]
\[ a_{62} = -2(\gamma - 1)M_e^2 Pr Du_s \]
\[ a_{63} = -2i(\gamma - 1)M_e^2 Pr (\alpha Du_s + i\beta Dw_s) + RPr DT_s/\mu_s T_s \]
\[ a_{64} = i(\gamma - 1)M_e^2 Pr R\omega/\mu_s \]
\[ a_{65} = \alpha^2 + \beta^2 - iRPr\omega/\mu_s T_s - (\gamma - 1)M_e^2 Pr\mu'_s[(Du_s)^2 + (Dw_s)^2]/\mu_s \]
\[ - D^2\mu_s/\mu_s \]
\[ a_{66} = -2Du_s/\mu_s \]
\[ a_{68} = -2(\gamma - 1)M_e^2 Pr Dw_s \]
\[ a_{83} = -i(m + 1)\beta DT_s/T_s - i\beta Du_s/\mu_s + RDw_s/\mu_s T_s \]
\[ a_{84} = (m + 1)\gamma M_e^2 \hat{\omega} + i\beta R/\mu_s \]
\[ a_{85} = -(m + 1)\beta \hat{\omega}/T_s - D(\mu'_s Dw_s)/\mu_s \]
\[ a_{86} = -\mu'_s Dw_s/\mu_s \]
\[ a_{87} = \alpha^2 + \beta^2 - i\omega R/\mu_s T_s \]
\[ a_{88} = -Du_s/\mu_s \]

where

\[ \mu'_s = d\mu_s/dT_s, \quad DF = \partial F/\partial y, \]

and

\[ \hat{\omega} = \omega - \alpha Du_s - \beta Dw_s , \quad \chi = [R/\mu_s - i\gamma M_e^2 \hat{\omega}]^{-1} \]
\[ C_{12} = C_{56} = C_{78} = 1 \]
\[ C_{21} = -iR \hat{\omega} + \alpha^2 + \beta^2 \]
\[ C_{24} = iR\alpha + \gamma M_e^2 \hat{\omega}(1 + m) \]
\[ C_{25} = -\alpha \hat{\omega}(1 + m) \]
\[ C_{31} = -i\alpha \]
\[ C_{34} = i\gamma M_e^2 \hat{\omega} \]
\[ C_{35} = -i \hat{\omega} \]
\[ C_{37} = -i\beta \]
\[ C_{42} = -i\chi \alpha \]
\[ C_{43} = -\chi [-iR \hat{\omega} + \alpha^2 + \beta^2] \]
\[ C_{46} = -i\chi \omega \hat{r} \]
\[ C_{48} = -i\chi \beta \]
\[ C_{64} = iR Pr(\gamma - 1)M_e^2 \hat{\omega} \]
\[ C_{65} = -iR Pr \hat{\omega} + \alpha^2 + \beta^2 \]
\[ C_{84} = i\beta + \gamma M_e^2 \beta \hat{\omega}(1 + m) \]
\[ C_{85} = -\beta \hat{\omega}(1 + m) \]
\[ C_{87} = -iR \omega + \alpha^2 + \beta^2 \]

where
\[ \hat{\omega} = \omega - \alpha - \beta w_e \]
\[ x = \left[ R - i\gamma M e^\omega \right]^{-1} \]

\[ W_e = \lim_{y \to \infty} W_s \]
Appendix C

\[ b_{11} = C_{21} \]
\[ b_{12} = C_{24} \]
\[ b_{13} = C_{25} \]
\[ b_{22} = C_{24}C_{42} + C_{34}C_{43} + C_{46}C_{64} + C_{48}C_{84} \]
\[ b_{23} = C_{25}C_{42} + C_{35}C_{43} + C_{46}C_{65} + C_{48}C_{85} \]
\[ b_{32} = C_{64} \]
\[ b_{33} = C_{65} \]
\[ b_{42} = C_{84} \]
\[ b_{43} = C_{85} \]
\[ b_{44} = C_{21} \]

\[ \lambda_{1,5} = \sqrt{b_{11}}^{1/2} \]
\[ \lambda_{2,6} = \sqrt{\frac{1}{2} (b_{22} + b_{33}) + \left[ \frac{1}{4} (b_{22} - b_{33})^2 + b_{23}b_{32} \right]^{1/2}} \]
\[ \lambda_{3,7} = \sqrt{\frac{1}{2} (b_{22} + b_{33}) - \left[ \frac{1}{4} (b_{22} - b_{33})^2 + b_{23}b_{32} \right]^{1/2}} \]
\[ \lambda_{4,8} = \sqrt{b_{11}}^{1/2} \]

\[ B_{1j} = 1, \quad B_{2j} = 0, \quad B_{3j} = 0, \quad B_{4j} = 0 \quad \text{for} \quad j = 1, 5 \]
\[ B_{1j} = \frac{(\lambda_j^2 - C_{65})C_{24} + C_{25}C_{64}}{(C_{21} - \lambda_j^2)}, \quad B_{2j} = C_{65} - \lambda_j^2, \quad B_{3j} = -C_{64}, \]
\[ B_{4j} = \frac{C_{64}C_{85} + (\lambda_j^2 - C_{65})C_{84}}{(C_{21} - \lambda_j^2)} \quad \text{for} \quad j = 2, 3, 6, 7 \]

and
\( B_{1j} = 0, \ B_{2j} = 0, \ B_{3j} = 0, \ B_{4j} = 1 \) for \( j = 4, 8 \)

\[ \Lambda_{1j} = 1, \ \Lambda_{2j} = \lambda_j, \ \Lambda_{3j} = \frac{c_{31B_{1j}} + c_{34B_{2j}} + c_{35B_{3j}} + c_{37B_{4j}}}{\lambda_jB_{1j}}, \]

\[ \Lambda_{4j} = \frac{B_{2j}}{B_{1j}}, \ \Lambda_{5j} = \frac{B_{3j}}{B_{1j}}, \ \Lambda_{6j} = \frac{\lambda_jB_{3j}}{B_{1j}}, \ \Lambda_{7j} = \frac{B_{4j}}{B_{1j}}, \]

\[ \Lambda_{8j} = \frac{c_{84B_{2j}} + c_{85B_{3j}} + c_{87B_{4j}}}{\lambda_jB_{1j}} \]

The \( c_{ij} \) are given in Appendix B.
Appendix D

\[ B_{\mathbf{j}} = 1, \quad B_{\mathbf{j}} = \frac{(\lambda_j^2 - c_{65})c_{24} + c_{25}c_{64}}{(c_{65} - \lambda_j^2)(b_{22} - \lambda_j^2) - c_{64}b_{23}}, \]

\[ B_{\mathbf{j}} = \frac{c_{24}b_{23} - (b_{22} - \lambda_j^2)c_{25}}{(c_{65} - \lambda_j^2)(b_{22} - \lambda_j^2) - c_{64}b_{23}}, \quad B_{\mathbf{j}} = 0 \quad \text{for} \quad j = 1,5 \]

\[ B_{\mathbf{j}} = 0, \quad B_{\mathbf{j}} = -c_{65}, \quad B_{\mathbf{j}} = b_{22} - \lambda_j^2, \quad B_{\mathbf{j}} = 0 \quad \text{for} \quad j = 2,3,6,7 \]

and

\[ B_{\mathbf{j}} = 0, \quad B_{\mathbf{j}} = \frac{(\lambda_j^2 - c_{65})c_{84} + c_{95}c_{64}}{(c_{65} - \lambda_j^2)(b_{22} - \lambda_j^2) - c_{64}b_{23}}, \]

\[ B_{\mathbf{j}} = \frac{c_{84}b_{23} - (b_{22} - \lambda_j^2)c_{85}}{(c_{65} - \lambda_j^2)(b_{22} - \lambda_j^2) - c_{64}b_{23}}, \quad B_{\mathbf{j}} = 1 \quad \text{for} \quad j = 4,8 \]

\[ \Lambda_{\mathbf{j}} = 1, \quad \Lambda_{\mathbf{j}} = \frac{B_{\mathbf{j}} + c_{42}B_{\mathbf{j}}}{\lambda_j B_{\mathbf{j}}}, \quad \Lambda_{\mathbf{j}} = \frac{c_{42}B_{\mathbf{j}}}{\lambda_j B_{\mathbf{j}}}, \quad \Lambda_{\mathbf{j}} = \frac{B_{\mathbf{j}}}{B_{\mathbf{j}}}, \]

\[ \Lambda_{\mathbf{j}} = \frac{B_{\mathbf{j}}}{B_{\mathbf{j}}}, \quad \Lambda_{\mathbf{j}} = \frac{c_{46}B_{\mathbf{j}} + B_{\mathbf{j}}}{\lambda_j B_{\mathbf{j}}}, \quad \Lambda_{\mathbf{j}} = \frac{B_{\mathbf{j}}}{B_{\mathbf{j}}}, \]

\[ \Lambda_{\mathbf{j}} = \frac{c_{48}B_{\mathbf{j}} + B_{\mathbf{j}}}{\lambda_j B_{\mathbf{j}}} \]

The \( c_{ij} \) are given in Appendix B. The \( b_{ij} \) are given in Appendix C.
Appendix E

\[ D_n = i \sum_{m=1}^{8} \frac{\partial a_{nm}}{\partial \omega} \zeta_m \]

\[ E_n = -i \sum_{m=1}^{8} \frac{\partial a_{nm}}{\partial \alpha} \zeta_m \]

\[ F_n = -i \sum_{m=1}^{8} \frac{\partial a_{nm}}{\partial \beta} \zeta_m \]

where small terms \( O(R^{-1}) \) can be neglected. The \( a_{nm} \) are given in Appendix A.

\[ G_1 = G_5 = G_7 = 0 \]

\[ G_2 = R \frac{\mu_s}{T_s} \left[ \frac{U_s}{T_s} \frac{\partial \xi_1}{\partial x_1} + \frac{\xi_1}{T_s} \frac{\partial u_s}{\partial x_1} + \frac{v_s}{T_s} \frac{\partial \xi_1}{\partial y} + \frac{z_7}{T_s} \frac{\partial u_s}{\partial z_1} + \frac{w_s}{T_s} \frac{\partial \xi_1}{\partial z_1} \right. \]

\[ + \frac{\partial \xi_4}{\partial x_1} + \left( \frac{\gamma m^2 \xi_4}{T_s} - \frac{\xi_5}{T_s} \right) \left( \frac{\partial u_s}{\partial x_1} + \frac{v_s}{T_s} \frac{\partial u_s}{\partial y} + \frac{w_s}{T_s} \frac{\partial u_s}{\partial z_1} \right) \] + \( O(1) \)

\[ G_3 = -\frac{\xi_1}{T_s} \frac{\partial t_s}{\partial x_1} - \frac{\partial \xi_1}{\partial x_1} - u_s \gamma m^2 \left( \frac{\partial \xi_4}{\partial x_1} - \frac{\xi_5}{T_s} \right) + \frac{v_s}{T_s} \frac{\partial \xi_5}{\partial x_1} - \frac{2u_s \xi_5}{T_s} \frac{\partial t_s}{\partial x_1} \]

\[ - \left( \frac{\partial u_s}{\partial x_1} + \frac{\partial v_s}{\partial y} + \frac{\partial w_s}{\partial z_1} \right) \left( \gamma m^2 \xi_4 - \frac{\xi_5}{T_s} \right) - v_s \left( \gamma m^2 \frac{\partial \xi_4}{\partial y} - \frac{\gamma m^2 \xi_4}{T_s} \frac{\partial t_s}{\partial y} \right) \]

\[ - \frac{1}{T_s} \frac{\partial \xi_5}{\partial y} + \frac{2 \xi_5}{T_s} \frac{\partial t_s}{\partial y} \] - \( \gamma m^2 w_s \left( \frac{\partial \xi_4}{\partial z_1} - \frac{\xi_5}{T_s} \frac{\partial t_s}{\partial z_1} \right) + \frac{z_7}{T_s} \frac{\partial t_s}{\partial z_1} + \frac{w_s}{T_s} \frac{\partial \xi_5}{\partial z_1} \]

\[ - \frac{2w_s \xi_5}{T_s} \frac{\partial t_s}{\partial z_1} - \frac{\partial z_7}{\partial z_1} + u_s \frac{\xi_5}{T_s} \gamma m^2 \left( \frac{\partial p_s}{\partial x_1} + \frac{\partial p_s}{\partial y} \right) - \gamma m^2 \left( \frac{\partial \xi_1}{\partial x_1} + \frac{\partial \xi_7}{\partial z_1} \right) \]

\[ + O(R^{-1}) \]

\[ G_4 = - \frac{1}{T_s} \left[ u_s \frac{\partial \xi_3}{\partial x_1} + v_s \frac{\partial \xi_3}{\partial y} + \frac{\partial v_s}{\partial y} + w_s \frac{\partial \xi_3}{\partial z_1} \right] + O(R^{-1}) \]
\[ G_6 = -\frac{RPr}{\mu_s} \left[ -\frac{\zeta_1}{T_s} \frac{\partial T_s}{\partial x_1} - \frac{U_s}{T_s} \frac{\partial \zeta_5}{\partial x_1} - \frac{V_s}{T_s} \frac{\partial \zeta_5}{\partial y} - \frac{\zeta_7}{T_s} \frac{\partial T_s}{\partial z_1} - \frac{W_s}{T_s} \frac{\partial \zeta_5}{\partial z_1} \right. \\
\left. - \left( \frac{\gamma M^2 \zeta_4}{T_s} - \frac{\zeta_5}{T_s} \right) \left( \frac{\partial T_s}{\partial x_1} + \frac{\partial T_s}{\partial y} + \frac{\partial T_s}{\partial z_1} \right) + (\gamma - 1)M^2_{e} \left( \frac{\partial \eta}{\partial x_1} \eta \right) \\
\right. + U_s \frac{\partial \zeta_4}{\partial x_1} + V_s \frac{\partial \zeta_4}{\partial y} \left. + \frac{\partial \zeta_4}{\partial z_1} \eta + W_s \frac{\partial \zeta_4}{\partial z_1} \right] + O(1) \]

\[ G_8 = \frac{R}{\mu_s} \left[ \frac{U_s}{T_s} \frac{\partial \zeta_7}{\partial x_1} + \frac{\zeta_1}{T_s} \frac{\partial W_s}{\partial x_1} + \frac{V_s}{T_s} \frac{\partial \zeta_7}{\partial y} + \frac{W_s}{T_s} \frac{\partial \zeta_7}{\partial z_1} + \frac{\zeta_7}{T_s} \frac{\partial W_s}{\partial z_1} \right. \\
\left. + \frac{\partial \zeta_8}{\partial z_1} + \left( \frac{\gamma M^2 \zeta_4}{T_s} - \frac{\zeta_5}{T_s} \right) \left( \frac{\partial W_s}{\partial x_1} + V_s \frac{\partial W_s}{\partial y} + W_s \frac{\partial W_s}{\partial z_1} \right) \right] + O(1) \]
Appendix F

\[ g_1 = i \sum_{m,n=1}^{8} \int_0^{\infty} \frac{\partial a_{nm}}{\partial \omega} \frac{\zeta_m}{\zeta_n} dy \]

\[ g_2 = -i \sum_{m,n=1}^{8} \int_0^{\infty} \frac{\partial a_{nm}}{\partial \alpha} \frac{\zeta_m}{\zeta_n} dy \]

\[ g_3 = -i \sum_{m,n=1}^{8} \int_0^{\infty} \frac{\partial a_{nm}}{\partial \beta} \frac{\zeta_m}{\zeta_n} dy \]

\[ \omega_\alpha = g_2/g_1, \quad \omega_\beta = g_3/g_1 \]

\[ h_1 = -g_1^{-1} \sum_{m=1}^{8} \int_0^{\infty} G_m \frac{\zeta_m}{\zeta_n} dy \]

\[ h_2 = ig_1^{-1} \sum_{m,n=1}^{8} \int_0^{\infty} \left. \frac{\partial a_{nm}}{\partial x_1} \right|_{\alpha, \beta} \frac{\zeta_m}{\zeta_n} dy \]

\[ h_3 = ig_1^{-1} \sum_{m,n=1}^{8} \int_0^{\infty} \left. \frac{\partial a_{nm}}{\partial z_1} \right|_{\alpha, \beta} \frac{\zeta_m}{\zeta_n} dy \]

The \( a_{nm} \) are given in Appendix A. The \( G_m \) are given in Appendix E.
Table 2.1
Linear Dependence of Complex Wavenumber on $v_{wall}^*$

<table>
<thead>
<tr>
<th>$v_{wall}^<em>/U_{</em>\infty}$</th>
<th>x*(in.)</th>
<th>Calculated k</th>
<th>Expected k</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.004</td>
<td>76.875</td>
<td>(.1467, -.0030)</td>
<td>(.1466, -.0029)</td>
</tr>
<tr>
<td></td>
<td>77.1875</td>
<td>(.1461, -.0024)</td>
<td>(.1460, -.0023)</td>
</tr>
<tr>
<td></td>
<td>77.5</td>
<td>(.1448, -.0015)</td>
<td>(.1446, -.0014)</td>
</tr>
<tr>
<td></td>
<td>77.8125</td>
<td>(.1438, -.0003)</td>
<td>(.1435, -.0002)</td>
</tr>
<tr>
<td></td>
<td>78.125</td>
<td>(.1438, .0007)</td>
<td>(.1435, .0009)</td>
</tr>
<tr>
<td>-0.006</td>
<td>76.875</td>
<td>(.1454, -.0016)</td>
<td>(.1452, -.0015)</td>
</tr>
<tr>
<td></td>
<td>77.1875</td>
<td>(.1444, -.0008)</td>
<td>(.1442, -.0007)</td>
</tr>
<tr>
<td></td>
<td>77.5</td>
<td>(.1424, .0006)</td>
<td>(.1420, .0007)</td>
</tr>
<tr>
<td></td>
<td>77.8125</td>
<td>(.1407, .0023)</td>
<td>(.1402, .0025)</td>
</tr>
<tr>
<td></td>
<td>78.125</td>
<td>(.1405, .0039)</td>
<td>(.1400, .0041)</td>
</tr>
</tbody>
</table>
Table 2.2

Typical Coefficients $C_i$ of the Optimization Scheme

<table>
<thead>
<tr>
<th>Strip Number $i$</th>
<th>$C_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>555.7</td>
</tr>
<tr>
<td>2</td>
<td>1096.5</td>
</tr>
<tr>
<td>3</td>
<td>1109.7</td>
</tr>
<tr>
<td>4</td>
<td>1038.1</td>
</tr>
<tr>
<td>5</td>
<td>960.8</td>
</tr>
<tr>
<td>6</td>
<td>885.2</td>
</tr>
<tr>
<td>7</td>
<td>809.2</td>
</tr>
<tr>
<td>8</td>
<td>740.3</td>
</tr>
<tr>
<td>9</td>
<td>674.7</td>
</tr>
<tr>
<td>10</td>
<td>618.9</td>
</tr>
<tr>
<td>11</td>
<td>569.5</td>
</tr>
<tr>
<td>12</td>
<td>531.5</td>
</tr>
<tr>
<td>13</td>
<td>505.8</td>
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<td>14</td>
<td>488.1</td>
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<td>15</td>
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<tr>
<td>16</td>
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<td>661.6</td>
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<td>20</td>
<td>724.4</td>
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<tr>
<td>21</td>
<td>732.7</td>
</tr>
</tbody>
</table>
## Table 2.2 (Con't)

**Typical Coefficients $C_i$ of the Optimization Scheme**

<table>
<thead>
<tr>
<th>Strip Number $i$</th>
<th>$C_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
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<td>23</td>
<td>129.7</td>
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<td>24</td>
<td>47.9</td>
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Table 5.1
Typical Influence Coefficients $C_i$ for an Axisymmetric Body

<table>
<thead>
<tr>
<th>Strip Number $i$</th>
<th>$C_i$</th>
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</thead>
<tbody>
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</tr>
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<td>6</td>
<td>2850</td>
</tr>
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</tr>
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<td>2416</td>
</tr>
<tr>
<td>9</td>
<td>623</td>
</tr>
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<td>14</td>
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<tr>
<td>15</td>
<td>41</td>
</tr>
</tbody>
</table>
Figure 2.1  Schematic of the triple deck structure near a disturbance point.
Figure 2.2  Comparison of approximate and exact linear triple deck theory. Twenty four equally-spaced strips between Branch I and Branch II, 5/8-inch wide, equal suction, $F = 20 \times 10^{-6}$. 
Figure 2.3  Comparison of approximate and exact linear triple deck theory. Twenty four equally-spaced strips between Branch I and Branch II, 5/8-inch wide, equal suction, $F = 20 \times 10^{-6}$.
Figure 2.4 Influence coefficients $C_i$ for 5/8-inch wide strips and a dimensionless frequency of $20 \times 10^{-6}$. 
Figure 2.5  Spatial growth rate $-k_i$ versus $\sqrt{Re_x}$ for one strip. $v_{wall} = -2.3 \times 10^{-4}$, $F = 210 \times 10^{-6}$.

--- Linear Triple Deck (LTD)  --- Interacting Boundary Layer (IBL)

--- Nonsimilar (NS)  --- Blasius (B)
Figure 2.6 Spatial growth rate $-k_i$ versus $\sqrt{Re_x}$ for one strip. $v_{wall} = -2.3 \times 10^{-4}$, $F = 40 \times 10^{-6}$.

--- LTD
--- IBL
--- NS
--- B
Figure 2.7  Spatial growth rate $-k_i$ versus $\sqrt{Re_x}$ in a neighborhood of the third of six strips.

$v_{wall} = -2.3 \times 10^{-4}$, $F = 86 \times 10^{-6}$.

--- LTD
--- IBL
• • • NS
--- B
Figure 2.8  Spatial growth rate $-k_i$ versus $\sqrt{Re_x}$ in a neighborhood of the fourth of six strips. $v_{wall} = -2.3 \times 10^{-6}$, $F = 86 \times 10^{-6}$.

--- LTD
--- IBL
--- NS
--- B
Figure 2.9  Spatial growth rate $-k_i$ versus $\sqrt{Rex}$ in a neighborhood of the fifth of six strips.

$v_{wall} = -2.3 \times 10^{-4}$, $\tau = 86 \times 10^{-6}$.

--- LTD  --- IBL

--- NS  --- B
Figure 2.10  Spatial growth rate $-k_1$ versus $\sqrt{Re_x}$ in a neighborhood of the sixth of six strips. $v_{wall} = -2.3 \times 10^{-4}$, $F = 86 \times 10^{-6}$.

--- LTD          --- IBL
- • - NS          --- B
Figure 2.11 Overall view of the spatial growth rate $-k_i$ versus $\sqrt{Re_x}$ for the third through six strips. $v_{wall} = -2.3 \times 10^{-4}$, $F = 86 \times 10^{-6}$.

--- LTD

--- --- IBL

--- * --- NS

--- --- B
Figure 2.12 Spatial growth rate $-k_i$ versus $\sqrt{Re_x}$ in a neighborhood of the sixth of six strips.

$v_{wall} = -2.3 \times 10^{-4}$, $F = 20 \times 10^{-6}$.

--- LTD
--- IBL
--- NS
--- B
Figure 2.13 Spatial growth rate curve for continuous area suction. \( v_{\text{wall}} = -3 \times 10^{-5}, F = 20 \times 10^{-6}. \)
Figure 2.14 Amplification factor curve for continuous area suction. \( v_{\text{wall}} = -3 \times 10^{-5} \), \( F = 20 \times 10^{-6} \).
Figure 2.15  Suction distribution for twenty four equally-spaced strips between Branch I and Branch II, 5/8-inch wide, equal suction, $F = 20 \times 10^{-6}$. 
Figure 2.16 Spatial growth rate curve for twenty four equally-spaced strips between Branch I and Branch II, 5/8-inch wide, equal suction, $F = 20 \times 10^{-6}$. 
Figure 2.17 Amplification factor curve for twenty four equally-spaced strips between Branch I and Branch II, 5/8-inch wide, equal suction, $F = 20 \times 10^{-6}$. 
Figure 2.18 Suction distribution for twenty four equally-spaced strips between Branch I and Branch II, 5/8-inch wide, double suction for middle third of strips, $F = 20 \times 10^{-6}$. 
Figure 2.19 Spatial growth rate curve for twenty four equally-spaced strips between Branch I and Branch II, 5/8-inch wide, double suction for middle third of strips, $F = 20 \times 10^{-6}$. 
Figure 2.20 Amplification factor curve for twenty four equally-spaced strips between Branch I and Branch II, 5/8-inch wide, double suction for middle third of strips, $F = 20 \times 10^{-6}$. 
Figure 2.21 Suction distribution for twenty four equally-spaced strips between Branch I and Branch II, 5/8-inch wide, suction level so as to decrease $\ln(A/A_0)$, $F = 20 \times 10^{-6}$.
Figure 2.22 Spatial growth rate curve for twenty four equally-spaced strips between Branch I and Branch II, 5/8-inch wide suction level so as to decrease $\ln(A/A_0), F = 20 \times 10^{-6}$. 
Figure 2.23 Amplification factor curve for twenty four equally-spaced strips between Branch I and Branch II, 5/8-inch wide, suction level so as to decrease \( \ln(A/A_0) \), \( F = 20 \times 10^{-6} \).
Figure 2.24 Suction distribution for twelve equally-spaced strips between Branch I and Branch II, 5/8-inch wide, equal suction, $F = 20 \times 10^{-6}$. 
Figure 2.25 Spatial growth rate curve for twelve equally-spaced strips between Branch I and Branch II, 5/8-inch wide, equal suction, $F = 20 \times 10^{-6}$. 
Figure 2.26 Amplification factor curve for twelve equally-spaced strips between Branch I and Branch II, 5/8-inch wide, equal suction, $F = 20 \times 10^{-6}$. 
Figure 2.27 Suction distribution for twelve strips between Branch I and Branch II, 5/8-inch wide, equal suction, spacing so as to decrease $\ln(A/A_0)$, 
$F = 20 \times 10^{-6}$. 
Figure 2.28 Spatial growth rate curve for twelve strips between Branch I and Branch II, 5/8-inch wide, equal suction, spacing so as to decrease $\ln(A/A_0)$, $F = 20 \times 10^{-6}$. 
Figure 2.29 Amplification factor curve for twelve strips between Branch I and Branch II, 5/8-inch wide, equal suction, spacing so as to decrease $\ln(A/A_0)$, $F = 20 \times 10^{-5}$. 
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open with a suction level of \( V_0 = 5.5 \times 10^{-3} U_\infty \).
The dimensionless frequency is
\( F = 20 \times 10^{-6} \) and the square root of the unit
Reynolds number per meter is \( R_u = 987 \). Strips
are 5/8-inch wide
- \( \circ \) - No suction, experiment
- \( \nabla \) - Suction, experiment
--- Corresponding theory
Figure 3.3  Comparison of theory with experiment: one suction strip on Panel I at $x = 194.3$ cm is open with a suction level of $V_0 = 5.5 \times 10^{-3} U_\infty$. The dimensionless frequency is $F = 20 \times 10^{-6}$ and the square root of the unit Reynolds number per meter is $R_u = 987$. Strips are 5/8-inch wide.

- ○ - No suction, experiment
- ▽ - Suction, experiment
- —— Corresponding theory
Figure 3.4 Comparison of theory with experiment: one suction strip on Panel I at $x = 194.3$ cm is open with a suction level of $V_0 = 5.7 \times 10^{-3} U_\infty$. The dimensionless frequency is $F = 25 \times 10^{-6}$ and the square root of the unit Reynolds number per meter is $R_u = 961$. Strips are 5/8-inch wide.

- ○ - No suction, experiment
- ▽ - Suction, experiment

--- Corresponding theory
Figure 3.5  Comparison of theory with experiment: three strips are open on each of Panels I and II at $x = 184.8, 194.3, 203.8, 238.1, 247.6, \text{ and } 257.1 \text{ cm}$ with the same suction level of $V_0 = 1.0 \times 10^{-3} U_\infty$. The dimensionless frequency is $F = 25 \times 10^{-6}$ and the square root of the unit Reynolds number per meter is $R_u = 961$. Strips are 5/8-inch wide.

- $\circ$ - No suction, experiment

- $\triangledown$ - Suction, experiment

- - - Corresponding theory
Figure 3.6 Comparison of theory with experiment: seven strips are open on Panel I at $x = 184.8, 187.9, 191.1, 194.3, 197.5, 200.6,$ and $203.8$ cm with $V_0 = 4.2 \times 10^{-4} U_\infty$ and three strips are open on Panel II at $x = 238.1, 247.6,$ and $257.1$ cm with $V_0 = 1.1 \times 10^{-3} U_\infty$. The dimensionless frequency is $F = 25 \times 10^{-6}$ and the square root of the unit Reynolds number per meter is $R_u = 961$. Strips are 5/8-inch wide.

- $\circ$ - No suction, experiment
- $\triangledown$ - Suction, experiment
- --- Corresponding theory
Figure 3.7  Comparison of theory with experiment: seven strips are open on Panel I at $x = 184.8, 187.9, 191.1, 194.3, 197.5, 200.6, \text{ and } 203.8 \text{ cm with } V_0 = 4.2 \times 10^{-6} U_\infty$ and three strips are open on Panel II at $x = 238.1, 247.6, \text{ and } 257.1 \text{ cm with } V_0 = 1.1 \times 10^{-3} U_\infty$. The dimensionless frequency is $F = 25 \times 10^{-6}$ and the square root of the unit Reynolds number per meter is $R_u = 877$. Strips are 5/8-inch wide.

- ○ - No suction, experiment
- ▽ - Suction, experiment
- — — Corresponding theory
Figure 3.8 Effect of moving the strips upstream toward the Branch I neutral point. We observe a significant decrease in the growth of a disturbance.

- ○ - No suction, experiment
- O - No suction, experiment
- ▼ - Suction, experiment (indicated on upper scale)
- ◇ - Suction moved upstream toward Branch I, experiment (indicated on lower scale)
- — — Corresponding theory
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\[ \phi = -\frac{31}{24}\pi \]
\[ \phi = -\frac{43}{24}\pi \]
\[ \phi = -\frac{7}{24}\pi \]
\[ \phi = \frac{\pi}{6} \]

\[ (1,0) \]

X POLES
Figure 4.5  Poles of $(r^4 - e^{-i5\pi/6})^{-1}$. 

\[ \phi = \frac{19}{24} \pi, \quad \phi = \frac{7}{24} \pi, \quad \phi = \frac{31}{24} \pi, \quad \phi = \frac{43}{24} \pi, \quad \phi = -\frac{\pi}{6} \] 

X POLES
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THE TOLLMIEN-SCHLICHTING INSTABILITY
OF LAMINAR VISCOUS FLOWS

Helen L. Reed

(ABSTRACT)

In this work, an analysis of the Tollmien-Schlichting instability of laminar viscous flows is presented. For two-dimensional incompressible flow past a flat plate with porous suction strips, we use linear triple-deck, closed-form solutions for the mean flow to do a linear, parallel, spatial stability analysis. We develop a simple linear optimization scheme to determine the number, spacing, and mass-flow rate through the strips and conclude, surprisingly, that suction should be concentrated near the Branch I neutral point of the stability curve.

We then verify the results of our optimization scheme with experimental data. We find that the theory correctly predicts the experimental results and conclude that the optimization scheme is reliable enough to replace the experiment as a tool in designing efficient strips configurations in so far as two-dimensional, incompressible flows are concerned.

For axisymmetric incompressible flow past a body with porous strips, we develop linear triple deck, closed-form for the mean-flow quantities, solutions which account for upstream influence. These solutions are linear superpositions of the flow past the body without suction plus the perturbations due to the suction strips. The flow past the suctionless body is calculated using the Transition Analysis Program System (TAPS).
Using these linear triple deck, closed-form solutions we then develop a simple linear optimization scheme to determine number, spacing, and mass flow rate through the strips on an axisymmetric body. At present, we are finishing the development of and documentation for a computer code for official distribution that will interface with TAPS and suggest efficient configurations using our theory.

For compressible three-dimensional flow, we use the method of multiple scales to formulate the three-dimensional stability problem and determine the partial-differential equations governing variations of the amplitude and complex wavenumbers. We then propose a method for following one specific wave along its trajectory to ascertain the characteristics of the most unstable disturbance. Numerical results using the flow over the X-21 wing as calculated from the Kaups-Cebeci code will be published when they become available.