THINNING OF POINT PROCESSES-

COVARIANCE ANALYSES

by

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CHAPTER 1

INTRODUCTION

This dissertation addresses a class of problems in point process theory called 'thinning'. To envision what is meant by thinning of a point process, one imagines the flow of vehicles on a highway. Let us suppose we are observing the flow near an exit ramp. The arrivals of vehicles at the ramp can be modeled as a point process. As each vehicle nears the ramp, one of two things happens. The vehicle may exit the highway or it may continue to remain on the highway. The two resulting traffic flows, namely the ramp traffic and the remaining highway traffic, can also be modeled as point processes. Thus, the original point process (vehicles arriving at the ramp) has been split up or thinned to produce two point processes (vehicles which exit the highway and those that do not). The rule by which each vehicle decides to exit the highway or not is called a thinning rule.

This model may be used to model a wide variety of situations. Consider, for example, jobs arriving at a job shop. Some job might require a turret lathe operation whereas others might require a milling operation. Thus, the point process of incoming jobs is thinned to produce two point processes, one of which is the sequence of jobs going to the turret lathe and the other is the sequence of jobs going to the milling machine. Or imagine aircraft flying over a certain area. The sequence of aircraft can, of course, be modeled as a point process. Each aircraft is either detected by radar or escapes detection. The point process of arriving aircraft is thinned to produce two point process, one of detected and the other of undetected aircraft. All these examples, and others, can be studied as thinning problems. Indeed thinning problems abound in the theory of queueing networks.

Our interest will center around the dependency of the thinned process produced under various thinning rules. The measure we use to study this dependence is the cross covariance between the two thinned processes. This information could be used profitably in many situations. For example, in the road traffic situation ramp design considerations could use such information to advantage. In the job shop we could use the information to improve efficiency of plant operations. The radar detection center could estimate the number of undetected aircraft and also use such information to improve detection techniques.

Ideally, one would prefer to know the joint distribution of the thinned processes. In practice this is usually a difficult problem. However, in many situations, we shall use covariance information to produce sharp results about joint distributions.

1. Overview, Review and Critique of Literature

In Chapter 2 we shall introduce and briefly discuss the concepts of point and marked point process theory. We include this chapter to give the reader a flavor for the area and to provide a theoretical framework for what is to follow. We shall expose the concepts of stationarity and Palm distribution and some operations on point processes. Comprehensive treatments may be found in Daley and Vere-Jones [1972] and Franken, et al. [1981].

In Chapter 3 we shall study the situation where an arbitrary point process is thinned by an independent Bernoulli process. This situation could reasonably model our job shop example. The major known result here is that if the two thinned processes are independent, then all three processes are Poisson. We shall obtain the covariance structure of the thinned processes. For the case where the original process is renewal we shall show that if the thinned processes are uncorrelated then all three processes are Poisson and in fact the thinned processes are independent. This will strengthen the major result mentioned above within the class of renewal processes. We shall also give a counter-intuitive example where the cross covariance of the time between events in the thinned processes is zero, but the counts are correlated.

The assumption that the thinning process and the process being thinned are independent can be exploited to give interesting covergence results. Renyi [1956] found that if every point in the process is deleted with probability q then the resulting process, after suitable scaling, coverges to a Poisson process as q approaches one. For this scaled process Cox and Isham [1980] study auto-covariance properties. Kallenberg [1975] provides necessary and sufficient conditions for the convergence of independently thinned point processes to a point process in which times between units have distributions which are probability mixtures of Erlang distributions. Such a process is called a Cox process. Brown [1979] extended this result to allow for the possibility that the thinning probabilities could depend on the positions of the points in the processes being thinned. Thus, typically, results about thinning are either under the assumption of independence or are asymptotic results or both.

In Chapter 4 we analyze mark dependent thinning of Markov renewal processes. This approach could be used to model the aircraft detection example where the probability of detection could depend on the type of the aircraft. We obtain the time-dependent covariance structure for the case where the mark space of the Markov renewal process is any complete, separable, metric space. In the process we shall use these results along with those of Chapter 3 to solve a non-linear integral equation. Finally, we shall reduce our results to countable mark spaces for use in later chapters.

Chapter 5 uses the results of Chapter 4 to study the overflow queue. These queues were first studied by Erlang [1917]. Later Palm [1943] used them in his development of Palm functions. Khintchin [1960] made rigourous these results and extended them to develop what are now called Palm-Khintchin functions. Since then, the overflow queue has been the target of repeated investigation. Disney, et al. [1973] characterized renewal departures from M/GI/1/N overflow queues. Disney and Simon [1982] obtained a Markov renewal characterization of the input process in overflow queues. Shanbhag and Tambouratzis [1973] showed that the departure processes from these queues are Poisson.

We shall study the time-dependent correlation between the input and overflow processes and the output and overflow processes. We shall show that in equilibrium, the input and overflow processes are uncorrelated as are the output and overflow processes.

The study of the output-overflow correlations produces two interesting insights. First, it provides an example where two uncorrelated but dependent renewal processes, neither of which is Poisson are superposed (see Chapter 2, Section 6) to form a renewal process. Secondly, it demonstrates the feasibility of studying superposition problems within the context of thinning.

Most of the work on superposition assumes that the processes being superposed are independent. The classic result here is that if an infinite number of independent point processes are superposed, the resulting process, under certain regularity conditions, is Poisson. For a proof, see Çinlar [1972]. Cherry and Disney [1973,1981] have studied superposition of independent renewal and Markov renewal processes. Amburtzumian [1965] showed that by observing the process obtained by superposing S identical and independent renewal processes each with distribution function F, it is possible to find S and F. Amburtzumian [1969] also studied the auto-correlation properties of the intervals in the superposed process resulting from two independent renewal processes. He was able to obtain conditions under which some of the processes involved are Poisson. Daley [1973] studied the superposition of a Poisson and an alternating renewal process and gives necessary and sufficient conditions for the superposed process to be a stationary renewal process. Çinlar [1972] contains an excellent bibliography for the superposition problem.

In Chapter 6 we use the tools developed in Chapter 4 to study two problems. We first obtain the time-dependent covariance structure when an alternating Markov process is thinned by an independent Bernoulli process. We show that if the thinned processes are uncorrelated the original process must be Poisson and further that the thinned processes are independent Poisson processes. We then analyze the situation where a renewal process is thinned by an independent Markov chain. We shall show that if the renewal process is Poisson and the chain is in equilibrium, then the thinned process will be uncorrelated if and only if the Markov chain is a Bernoulli process. We shall also demonstrate how to design the chain to produce a pre-specified covariance structure.

Chapter 7 closes out the dissertation. Here we shall outline the areas of future research. We shall also discuss Laplace functionals as a possible approach to studying joint distributions.

2. Editorial Remarks

The numbering scheme in this dissertation uses only Arabic numerals. Lemmas, theorems, propositions, corollaries and equations are not differentiated in their numbering which is in sequence. If reference is made to one of these which is outside the current chapter a three number sequence is used. References within a chapter use a two number scheme. For example, when we say 'Theorem 3.3.12' we mean Chapter 3, Theorem 3.12 and the current chapter is different from Chapter 3. But if we say 'Theorem 3.12' we mean Theorem 3.12 of the current chapter. Similar comments hold for lemmas, propositions and corollaries. Equations are referred to simply by their numbers with a qualifier before the number. Thus, if we say 'from 3.3.8', we mean equation 3.8 in Chapter 3 and the current chapter is not Chapter 3. However, if we say 'from 3.8' we mean equation 3.8 of the current chapter.

Figures are numbered separately and are always referred to by a three number scheme. Thus, Figure 2.5.2 refers to Chapter 2, Figure 2.5.2.

References to the literature are made by quoting the author(s) and year of publication of the reference enclosed in square brackets (for example, Çinlar [1975]). If the reference has three or more authors we quote it as, for example, Franken, et al. [1981], where the name is the first author as listed in the reference.

CHAPTER 2

STOCHASTIC POINT PROCESSES

In this chapter we formally define stochastic (random) point and marked point processes. We give some fundamental results.

Next we give an example from queueing theory where these concepts find a natural application, thus illustrating the usefulness of the concepts discussed. It should be noted that the material in this chapter is not new and for the most part we follow the treatment by Franken, et al. [1981].

1. Point Processes

Let A be a Polish (i.e., a complete, separable, metric) space endowed with metric ρ. Let $\mathcal{A}$ be the σ-field generated by the metric topology on A. For our purposes we consider the case where $A = \mathbb{R}^d$ and $\mathcal{A} = \mathcal{B}^d$ where $\mathcal{B}^d$ is the σ-field of Borel subsets of $\mathbb{R}^d$. Most of the results in this section, however, are valid for any Polish space. No proofs are provided.

<u>(1.1) Definition</u>. A subset B of $\mathbb{R}^d$ is said to be bounded if

$$\sup\{\rho(x,y) \mid x,y \in B \subset \mathbb{R}^d\} < \infty.$$

Let ϕ be a measure on $\mathcal{B}^d$.

<u>(1.2) Definition</u>. ϕ is called a counting measure if the following hold.

a) ϕ is locally finite; i.e., $\phi(B) < \infty$ for all bounded $B \in \mathcal{B}^d$.

b) ϕ takes values in $Z_+ = \{0,1,2,\ldots\}$.

<u>(1.3) Example</u>. Let $x \in \mathbb{R}^d$ be fixed and for $B \in \mathcal{B}^d$ define

$$\delta_x(B) = \begin{cases} 1 \text{ if } x \in B \\ 0 \text{ if } x \notin B. \end{cases}$$

Then $\delta_x(\cdot)$ is a counting measure called the Dirac measure.

Let $S_\phi = \{x \in \mathbb{R}^d | \phi(\{x\}) \geq 1\}$. S_ϕ is called the support of ϕ. For convenience we henceforth write $\phi(x)$ for $\phi(\{x\})$.

<u>(1.4) Theorem</u>. The support, S_ϕ, of every counting measure is at most countable.

As a consequence of Theorem 1.4 we immediately have,

$$\phi(\cdot) = \sum_{x \in S_\phi} \phi(x)\delta_x(\cdot). \tag{1.5}$$

Now let,

$$\mathscr{N} = \{\phi | \phi \text{ is a counting measure}\} \text{ and,}$$

$$N_k(B) = \{\phi | \phi \in \mathscr{N}, \ \phi(B) = k\}. \tag{1.6}$$

Let $\sigma(\mathscr{N})$ be the minimal σ-field generated by sets of the type defined in 1.6. Finally, let $(\Omega, \mathscr{F}, P)$ be a basic probability space.

<u>(1.7) Definition</u>. A (stochastic) point process on $(\Omega, \mathscr{F}, P)$ is a measurable mapping, Φ, from $(\Omega, \mathscr{F}, P)$ to $(\mathscr{N}, \sigma(\mathscr{N}))$.

A measure on $\sigma(\mathscr{N})$ is readily constructed as follows.

$$P_\Phi(M) = P(\Phi^{-1}(M)), \ M \in \sigma(\mathscr{N}) \tag{1.8}$$

$(\mathscr{N}, \sigma(\mathscr{N}), P_\Phi)$ is called a canonical representation and (Φ, P) is referred to as a point process Φ with probability measure $P = P_\Phi$.

<u>(1.9) Example</u>. Let $(X_1, \ldots, X_n)$ be a $\mathbb{R}^n$ valued random vector where $X_1, \ldots, X_n$ are independent and identically distributed with distribution Q. Define,

$$\Phi(\cdot) = \sum_{i=1}^{n} \delta_{X_i}(\cdot).$$

Φ is called a sample process.

(1.10) Definition. A counting measure $\phi \in \mathcal{N}$ is called simple if $\phi(x) \leq 1$ for all $x \in \mathbb{R}^d$.

(1.11) Proposition. Let $\mathcal{N}_1 = \{\phi \in \mathcal{N} | \phi \text{ is simple}\}$. Then $\mathcal{N}_1 \in \sigma(\mathcal{N})$, i.e., $\mathcal{N}_1$ is measurable.

(1.12) Definition. The point process (Φ,P) is simple if $P(\mathcal{N}_1) = 1$.

Informally, a point process is simple if on almost every realization there are no points where two or more events occur simultaneously.

(1.13) Definition. Let $B_1,\ldots,B_m \in \mathcal{B}^d$. Then

$P_{B_1,\ldots,B_m}(\ell_1,\ldots,\ell_m) = P(\Phi(B_j) = \ell_j,\ j = 1,\ldots,m),\ m \in \{1,2,\ldots\}$

are called the finite dimensional distributions (fidi) of (Φ,P).

Let $\mathcal{T}^d$ be a semiring of sets of the type $(a_1,b_1]\times\ldots\times(a_d,b_d]$, and $\mathcal{H}^d$ be a ring over $\mathcal{T}^d$.

(1.14) Theorem. If two point processes (Φ,P) and (Ψ,Q) are such that

$$P_{I_1,\ldots,I_m} = Q_{I_1,\ldots,I_m},\ I_j \in \mathcal{T}^d,\ m = 1,2,\ldots$$

then,

$$P = Q.$$

Thus, if two point processes, Φ and Ψ, have the same fidi, they are stochastically identical. Φ and Ψ are said to be versions of each other.

We now state two important results in the theory of point processes.

(1.15) Theorem. Let (Φ,P) be a point process with fidi $P_{B_1,\ldots,B_m}$. Then the following are true for all $B_i \in \mathcal{B}^d$, $i = 1,\ldots,m$, and

$m = 1,2,\ldots.$

a) Let $\{i_1,\ldots,i_m\}$ be a permutation of $\{1,\ldots,m\}$. Then,

$$P_{B_1,\ldots,B_m}(\ell_1,\ldots,\ell_m) = P_{B_{i_1},\ldots,B_{i_m}}(\ell_{i_1},\ldots,\ell_{i_m}).$$

b) $$P_{B_1,\ldots,B_{m-1}}(\ell_1,\ldots,\ell_{m-1}) = \sum_{\ell_m=0}^{\infty} P_{B_1,\ldots,B_m}(\ell_1,\ldots,\ell_{m-1},\ell_m).$$

c) If $B_m = \bigcup_{i=1}^{m-1} B_i$, then

$$P_{B_1,\ldots,B_m}(\ell_1,\ldots,\ell_m) = \begin{cases} 0 \text{ if } \ell_m \neq \sum_{i=1}^{m-1} \ell_i \\ P_{B_1,\ldots,B_{m-1}}(\ell_1,\ldots,\ell_{m-1}) \text{ if } \ell_m = \sum_{i=1}^{m-1} \ell_i \end{cases}.$$

d) $P_{B_n}(0) \uparrow 1$ as $B_n \downarrow \emptyset$ and,

$$\sum_{\ell=0}^{\infty} P_B(\ell) = 1 \text{ for } B \in \mathscr{B}^d.$$

<u>(1.16) Theorem.</u> If a system of functions $P_{B_1,\ldots,B_m}(\ell_1,\ldots,\ell_m)$ satisfies conditions a through d of Theorem 1.15, then there exists a uniquely determined P on $\sigma(\mathscr{N})$ such that the fidi of P are precisely the given system.

The case where d = 1, namely point processes on the real line, is of special interest as one obtains an equivalent representation of the process which, from an applied prespective, is more appealing. We proceed as follows.

Let $\phi \in \mathscr{N} = \{\phi | \phi \text{ is a counting measure on } \mathbb{R}\}$ be given. Consider the following construction.

$$x_n = x_n(\phi) = \begin{cases} x \geq 0 \text{ if } \phi([0,x)) < n \leq \phi([0,x]) \\ \\ \infty \quad\quad \text{if } 0 \leq \phi([0,\infty)) < n \end{cases} \tag{1.17a}$$

for $n \geq 0$, and

$$x_n = x_n(\phi) = \begin{cases} x < 0 \text{ if } \phi((x,0) \leq -n < \phi([x,0)) \text{ and } \phi((-\infty,0)) > -n \\ \\ -\infty \quad\quad \text{if } 0 \leq \phi((-\infty,0)) \leq -n \end{cases} \tag{1.17b}$$

for $n < 0$.

This yields a measurable mapping from $(\mathcal{N},\sigma(\mathcal{N}))$ to $((\mathbb{R}^c)^Z,(\mathcal{B}^c)^Z)$ where $\mathbb{R}^c = \{-\infty,\infty\} \cup \mathbb{R}$ and $\mathcal{B}^c$ is the σ-field of subsets of $\mathbb{R}^c$. The composition of the above mapping and Φ thus yields a sequence of random variables $X_n = X_n(\phi)$ where

$$\cdots \leq X_{-1} \leq X_0 < 0 \leq X_1 \leq X_2 \leq \cdots \tag{1.18}$$

almost surely and

$$X_n \xrightarrow[n\to\pm\infty]{} \pm\infty \tag{1.19}$$

almost surely.

Thus, a stochastic point process on the real line may be considered as a sequence, $\{X_n\}_{n=-\infty}^{\infty}$, of real-valued random variables satisfying 1.18 and 1.19. Clearly, if 1.18 is satisfied with strict inequalities, the process is simple.

Finally, if for every $\phi \in \mathcal{N}$ we consider the set $(x_0, \{s_n\}_{n=-\infty}^{\infty})$ where $x_0 < 0$, $s_0 \geq -x_0$, $s_n \geq 0$ and $s_n = x_{n+1} - x_n$, the process Φ has the representation $(X_0, \{S_n\}_{n=-\infty}^{\infty})$ such that,

$$X_0 < 0,\ S_0 \geq -X_0,\ S_n \geq 0 \tag{1.20}$$

and

$$\sum_{n=-\infty}^{-1} S_n = \sum_{n=0}^{\infty} S_n = \infty \qquad (1.21)$$

almost surely.

(1.22) Example. Let (Φ,P) be a point process on $(\mathbb{R},\mathscr{B})$ and consider the representations given by 1.20 and 1.21.

a) If (X_0,S_0) is independent of $\{S_n;\ n \neq 0\}$ and $\{S_n;\ n \neq 0\}$ are independent and identically distributed random variables with $P\{S_n \leq x\} = F(x)$ then (Φ,P) is said to form a renewal process.

b) If we restrict (Φ,P) to $\mathbb{R}_+$ then $S_0 = X_1$. Define $F_0(x) = P\{X_1 \leq x\}$.

c) If $F_0(0_+) = 0 = F(0_+)$, then the renewal process is simple.

d) If in c, $F_0(x) = F(x) = 1 - e^{-\lambda x}$, $\lambda > 0$, then (Φ,P) is said to form a Poisson process on the real line. In this case, if $B_1,\ldots,B_m \in \mathscr{B}$ are disjoint, we have

$$P_{B_1,\ldots,B_m}(\ell_1,\ldots,\ell_m) = \prod_{i=1}^{m} P_{B_i}(\ell_i)$$

and

$$P_B(\ell) = \frac{e^{-\lambda|B|}(\lambda|B|)^{\ell}}{\ell!}$$

where $|B|$ is the Lebesgue measure of B.

2. Marked Point Processes

Informally, a marked point process is a point process where for each point in every realization there is assigned some attribute. We formalize this notion as follows.

Let K be a Polish space and $\sigma(K)$ be a σ-field of subsets of K.

K will be called the mark space.

(2.1) Definition. A measure ϕ on $\sigma(\mathbb{R}^d \times K)$ is called a counting measure if the following hold.

a) $\phi(B \times K) < \infty$ for all bounded $B \in \mathscr{B}^d$.

b) $\phi(\cdot)$ takes values in Z_+.

Let

$$\mathscr{N}_K = \{\phi | \phi \text{ is a counting measure on } \sigma(\mathbb{R}^d \times K)\} \tag{2.2a}$$

and

$$N_K^j(B \times L) = \{\phi | \phi \in \mathscr{N}_K,\ \phi(B \times L) = j,\ B \in \mathscr{B}^d,\ L \in \sigma(K),\ j \in Z_+\}. \tag{2.2b}$$

Let $\sigma(\mathscr{N}_K)$ be the minimal σ-field generated by sets of the type in 2.2b. Finally, let $(\Omega,\mathscr{F},P)$ be a basic probability space.

(2.3) Definition. A (stochastic) marked point process on $(\Omega,\mathscr{F},P)$ is a measurable mapping, Φ, from $(\Omega,\mathscr{F},P)$ to $(\mathscr{N}_K,\sigma(\mathscr{N}_K))$.

A measure on $\sigma(\mathscr{N}_K)$, P_Φ, is constructed as follows.

$$P_\Phi(M) = P(\Phi^{-1}(M)),\ M \in \sigma(\mathscr{N}_K). \tag{2.4}$$

$(\mathscr{N}_K,\sigma(\mathscr{N}_K),P_\Phi)$ is called a canonical representation and (Φ,P) is called a marked point process with probability measure $P = P_\Phi$.

(2.5) Remark. For $B \in \mathscr{B}^d$ let $\phi_0(B) = \phi(B \times K)$. Then, $\phi_0 \in \mathscr{N}$ and the mapping from $\mathscr{N}_K$ to $\mathscr{N}$ is a projection of (Φ,P) onto $\mathbb{R}^d$ which gives a point process (Φ_0,P_0) on $\mathbb{R}^d$.

For every $x \in \mathbb{R}^d$, $\phi(\{x\} \times K)$ is the number of points with position x. Let $\{y_{x,1},\ldots,y_{x,\phi(\{x\}\times K)}\}$ be the set of marks of all points with position x. Then a marked point process can be regarded as the sequence

$$\{X, (Y_{X,1}, \ldots, Y_{X,\phi(\{x\}\times K)}),\ X \in S_{\Phi_0}\}. \tag{2.6}$$

Thus, a marked point process can be considered a sequence of points in $\mathbb{R}^d$ with each point having as many marks as there are multiplicities. This immediately leads to the following for every $\phi \in \mathcal{N}_K$.

$$\phi(\cdot) = \sum_{x\in S_{\phi_0}} \sum_{i=1}^{\phi(\{x\}\times K)} \delta_{x,y_{(x,i)}}(\cdot). \tag{2.7}$$

(2.8) Remark. A marked point process on $\mathbb{R}^d$ can be considered as a point process on $\mathbb{R}^d \times K$.

(2.9) Example. A point process with multiplicities can be viewed as a marked point process where the projection onto $\mathbb{R}^d$ is a simple point process and the mark space K is $\{1,2,\ldots\}$. Thus, the mark of a point reveals the multiplicity it has. If we have a renewal process on $\mathbb{R}_+$ with $S_0 = X_1 = 0$ almost surely, and $F(0) = p$, we can construct a simple renewal process on $\mathbb{R}_+$ with inter-event distribution $F^1(x)$ where,

$$F^1(x) = \frac{F(x)-p}{1-p}$$

and

$$P\{K = k\} = p^k(1-p),\ k = 1,2,\ldots$$

(2.10) Definition. $\phi \in \mathcal{N}_K$ is called simple if $\phi(\{x\}\times K) \leq 1$ for all $x \in \mathbb{R}^d$.

Let $\phi \in \mathcal{N}_K^1 = \{\phi | \phi \in \mathcal{N}_K \text{ and } \phi \text{ is simple}\}$.

(2.11) Proposition. $\mathcal{N}_K^1 \in \sigma(\mathcal{N}_K)$, i.e., $\mathcal{N}_K^1$ is measurable.

(2.12) Definition. The marked point process (Φ,P) is simple if $P(\mathcal{N}_K^1) = 1$.

(2.13) Example. Let $K = \{1,2,\ldots\}$ and let the sequence of marks $\{K_n\}_{n=-\infty}^{\infty}$

form a Markov chain with transition matrix $P = [p_{ij}]$. Further, let $\tilde{F}(x) = [F_{ij}(x)]$, $i,j \in K$, be a matrix of probability distribution functions. Let (Φ,P) be a marked point process on $\mathbb{R}$ such that the representing sequence given by 2.6 satisfies the following.

$$P\{X_{n+1}-X_n \leq x, K_{n+1} = j \mid \ldots, X_{-1}, X_0, X_1, \ldots, X_n, K_0, \ldots, K_n = i\}$$

$$= P\{X_{n+1}-X_n \leq x, K_{n+1} = j \mid K_n = i\}$$

$$= p_{ij}F_{ij}(x) = Q(i,j,x).$$

Then (Φ,P) is called a Markov Renewal process. If $\tilde{F}(0) = [0]$, then (Φ,P) is simple.

3. Stationarity

In this section we shall discuss two concepts in point process theory that are of fundamental importance in both theory and application. We shall present these in the context of marked point processes and the restriction to point processes will be obvious.

We start with a marked point process (Φ,P) on $\mathbb{R}^d$ with mark space K. Recall that $\mathcal{N}_K$ was the set of all counting measures on $\sigma(\mathbb{R}^d \times K)$ and $\sigma(\mathcal{N}_K)$ was an appropriate σ-field over $\mathcal{N}_K$. (See 2.2a and 2.2b.) Let $x \in \mathbb{R}^d$ and T_x be a mapping from $\mathcal{N}_K$ to $\mathcal{N}_K$ defined as follows.

$$T_x\phi(B \times L) = \phi((B + x) \times L) \tag{3.1}$$

Thus, the mapping T_x shifts the points in $\mathbb{R}^d$ of every realization ϕ by an amount x while preserving the mark of each point. Hence, if for a given $\phi \in \mathcal{N}_K$ the sequence given by 2.6 is $\{[x_n,k_n], n \in \mathbb{N}\}$, then the for $T_x\phi$ the sequence is $\{[x_n+x,k_n], n \in \mathbb{N}\}$.

(3.2) Definition. A random marked point process is stationary if the distribution, P, of Φ is invariant with respect to T_x for all $x \in \mathbb{R}^d$. Formally, this means the following.

$$P(T_x W) = P(W) \ \forall\, W \in \mathscr{B}^d \times \sigma(K) \text{ and } \forall\, x \in \mathbb{R}^d. \tag{3.3}$$

<u>(3.4) Example</u>. Let (Φ,P) be a renewal process on $\mathbb{R}_+$ as in example 1.22b. Define $F_0(x)$ as follows.

$$F_0(x) = \frac{1}{m} \int_0^x (1-F(u))du \tag{3.5}$$

where

$$m = \int_0^\infty (1-F(u))du.$$

Then (Φ,P) is stationary and P is given by the following.

$$P(\Phi(0,t] = 0) = 1-F_0(t) \tag{3.6a}$$

$$P(\Phi(0,t] = n) = F_0 * (F^{*n-1}-F^{*n})(t), \ n \geq 1 \tag{3.6b}$$

where '*' denotes convolution.

<u>(3.7) Definition</u>. The intensity of a stationary random marked point process, Φ, with respect to a mark set $L \in \sigma(K)$ is defined as follows.

$$\lambda_P(L) = \lambda(L) = E\Phi((0,1]^d \times L) = \Lambda_P((0,1]^d \times L) \tag{3.8}$$

and $\lambda_P(K) = \lambda$ is called the intensity of Φ.

<u>(3.9) Example</u>. In Example 3.4, $\lambda = E\Phi((0,1]) = 1/m$.

<u>(3.10) Proposition</u>. For any $B \in \mathscr{B}^d$, $L \in \sigma(K)$ we have the following.

$$\Lambda_P(B \times L) = \mu(B)\lambda_P(L)$$

where μ is the Lebesgue measure.

4. <u>Palm Distributions</u>

The concept of a Palm distribution will now be developed. We give the development for Φ on $\mathbb{R}$.

Let $\mathcal{N}_L^\infty$, $\sigma(\mathcal{N}_L^\infty)$, $\mathcal{N}_L^0$ and $\sigma(\mathcal{N}_L^0)$ for $L \in \sigma(K)$ be defined as follows.

$$\mathcal{N}_L^\infty = \{\phi | \phi \in \mathcal{N}_K, \ \phi((-\infty,0) \times L) = \infty = \phi((0,\infty) \times L\}, \ L \in \sigma(K). \tag{4.1}$$

$$\sigma(\mathcal{N}_L^\infty) = \sigma(\mathcal{N}_K) \cap \mathcal{N}_L^\infty. \tag{4.2}$$

$$\mathcal{N}_L^0 = \{\phi | \phi \in \mathcal{N}_L^\infty, \ \phi(\{0\} \times L) = 1\}. \tag{4.3}$$

$$\sigma(\mathcal{N}_L^0) = \sigma(\mathcal{N}_K) \cap \mathcal{N}_L^0. \tag{4.4}$$

Thus, $\mathcal{N}_L^0$ is the set of all counting measures such that there are infinite number of points which have marks in the set on both halves of the real line and furthermore have a point at the origin with a mark in L.

Now, for every $\phi \in \mathcal{N}_K$ let $\{t_i^L(\phi)\}_{i=-\infty}^{\infty}$ be the sequence of points with marks in L. Figure 2.4.1 illustrates what this means.

<u>(4.5) Definition</u>. For each $t > 0$, $Y \in \sigma(\mathcal{N}_L^0)$, $L \in \sigma(K)$ with $\lambda_P(L) = E\Phi((0,1] \times L) > 0$ define $P_L^0(Y)$ as follows.

$$P_L^0(Y) = \frac{1}{\lambda_P(L)} \int_{\mathcal{N}_L^\infty} \frac{1}{t} \sum_{i=0}^{\phi((0,t]\times L)-1} I_Y(T_{t_i^L(\phi)}(\phi)) P(d\phi). \tag{4.6}$$

Then P_L^0 is called the Palm distribution of (Φ,P) with respect to L.

<u>(4.7) Proposition</u>. $P_L^0(Y) = \lim\limits_{t \downarrow 0} P\{T_{t_i^L(\phi)}(\phi) \in Y | t_0^L(\phi) < t\}$.

Thus, informally, a Palm distribution is the probability of an event Y assuming that at the origin there is a point with its mark in L.

<u>(4.8) Example</u>. If (Φ,P) is a renewal process on $\mathbb{R}_+$ and $Y = \{\Phi((0,t]) = k\}$ then we have the following.

$$P^0(Y) = F^{*k}(t) - F^{*k+1}(t). \tag{4.9}$$

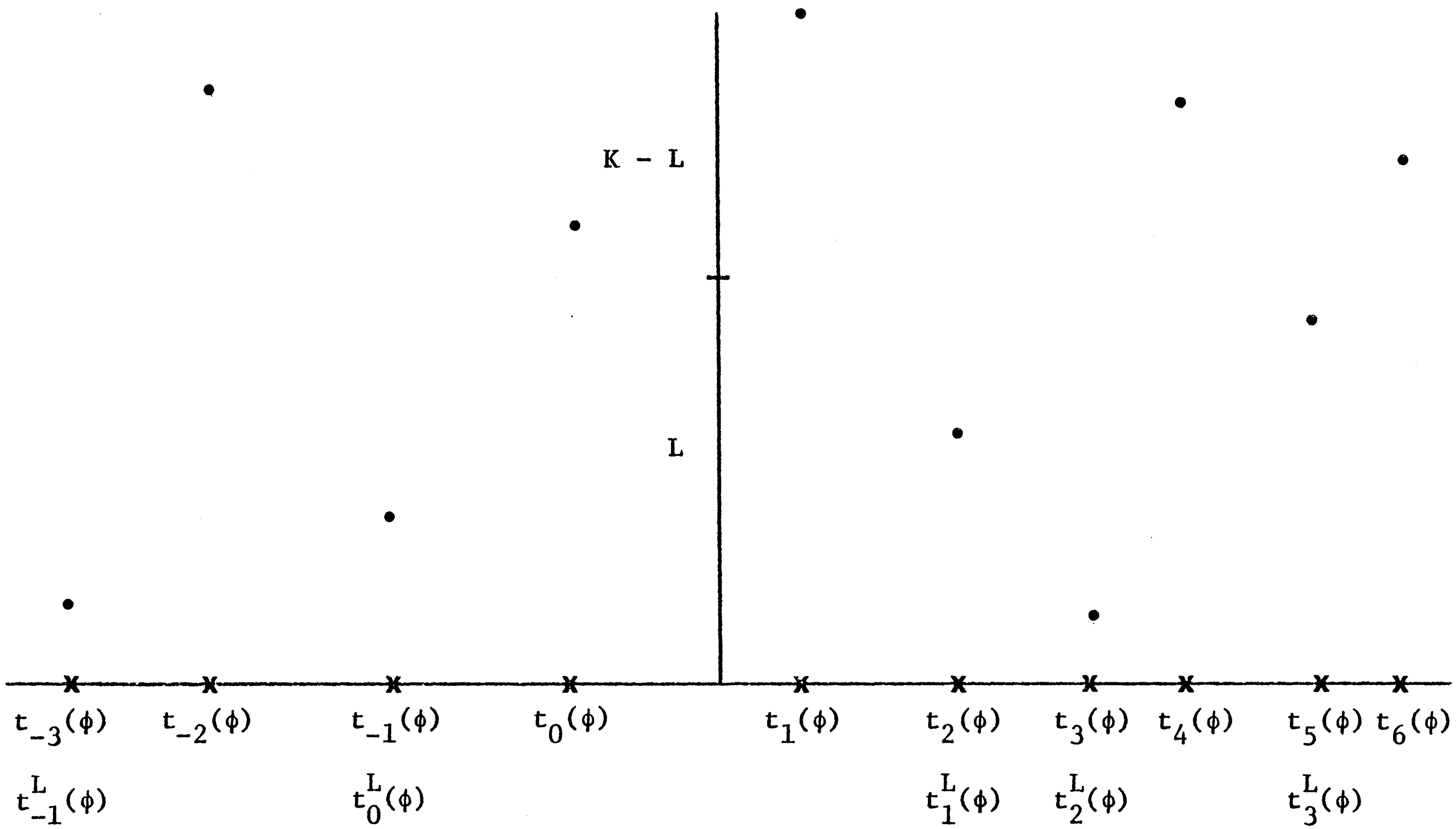

Figure 2.4.1

The Sequence of Points with Mark in L

The expected value of the random variable $\Phi(0,t]$ with respect to P^0 (note that there is no mark space here) is given as follows.

$$E_{P^0}\Phi(0,t] = R(t) \tag{4.10}$$

where R(t) is the so-called renewal function.

(4.11) Example. Consider example 2.13 where (Φ,P) is a Markov Renewal process. Let $L = \{i\}$ and $Y = \{X_1 \leq x,\ K_1 = j\}$. Then we have the following.

$$P_L^0(Y) = p_{ij}F_{ij}(x) = Q(i,j,x) \tag{4.12}$$

Further, if $\Phi^j(B)$ is the random number of points with marks of type j in the Borel set B then we have as follows.

$$E_{P_L^0}\Phi^j(B) = R(i,j,B), \tag{4.13}$$

where R(i,j,B) is the so-called Markov Renewal function.

Palm distributions will be of fundamental importance for the rest of this dissertation. Most of the results in the succeeding chapters will be results involving moments with respect to Palm distributions. The development of these concepts will serve to establish the framework which will support the results of later chapters.

5. Applications to Queueing Theory

The concepts discussed in Sections 1 through 4 form a natural basis for the formal study of queueing theory. In this section we demonstrate this through some illustrative examples. We assume that a basic probability space $(\Omega,\mathcal{F},P)$ is given. For the sake of simplicity we consider all systems in positive time.

(5.1) Example. (The Infinite Server Queue).

Consider the queueing system shown in Figure 2.5.1. It is natural to characterize the process of arriving customers by a sequence of random variables $T_1, T_2, \ldots$. If we invert the mapping 1.17a it is immediately seen that the process of arriving customers forms a point process, Φ, on $\mathbb{R}_+$. Now if we associate with the customer arriving at T_n a service time S_n and consider the sequence $\{(T_n, S_n);\ n \geq 1\}$, we have a marked point process Φ^1 on $\mathbb{R}_+$ with mark space $K = \mathbb{R}_+$. Moreover,

$$\Phi(B) = \Phi^1(B \times \mathbb{R}_+),\ B \varepsilon \mathscr{B}. \tag{5.2}$$

Thus, Φ is simply a projection of Φ^1 onto $\mathbb{R}_+$. Physically, $\Phi(B)$ is the (random) number of arrivals in B and $\Phi^1(B \times L)$ for $L \varepsilon \mathscr{B}_+$ is the (random) number of arrivals in B with service time in the set L.

Let $B = (0,t]$ and $\Phi(t) = \Phi((0,t])$. Consider the following two mappings.

$$\Psi((0,t]) = \sum_{k=1}^{\Phi(t)} I_{(0,t]}(S_k) \tag{5.3}$$

and

$$D_n = T_n + S_n. \tag{5.4}$$

$\Psi(t) = \Psi((0,t])$ is the (random) number of departures in $(0,t]$ and $\{D_n\}_{n=1}^{\infty}$ is the sequence of times when a customer leaves the system. It can be shown that Ψ and $\{D_n\}_{n=1}^{\infty}$ are related through the mapping 1.17a. A quantity of frequent interest is the number of people in the system at any time t, $\Gamma(t)$. This is given by the following.

$$\Gamma(t) = \Gamma(0) + \Phi(t) - \Psi(t). \tag{5.5}$$

(5.6) Example. (The Single Server Queue).

Consider the queueing system with infinite waiting room shown in Figure

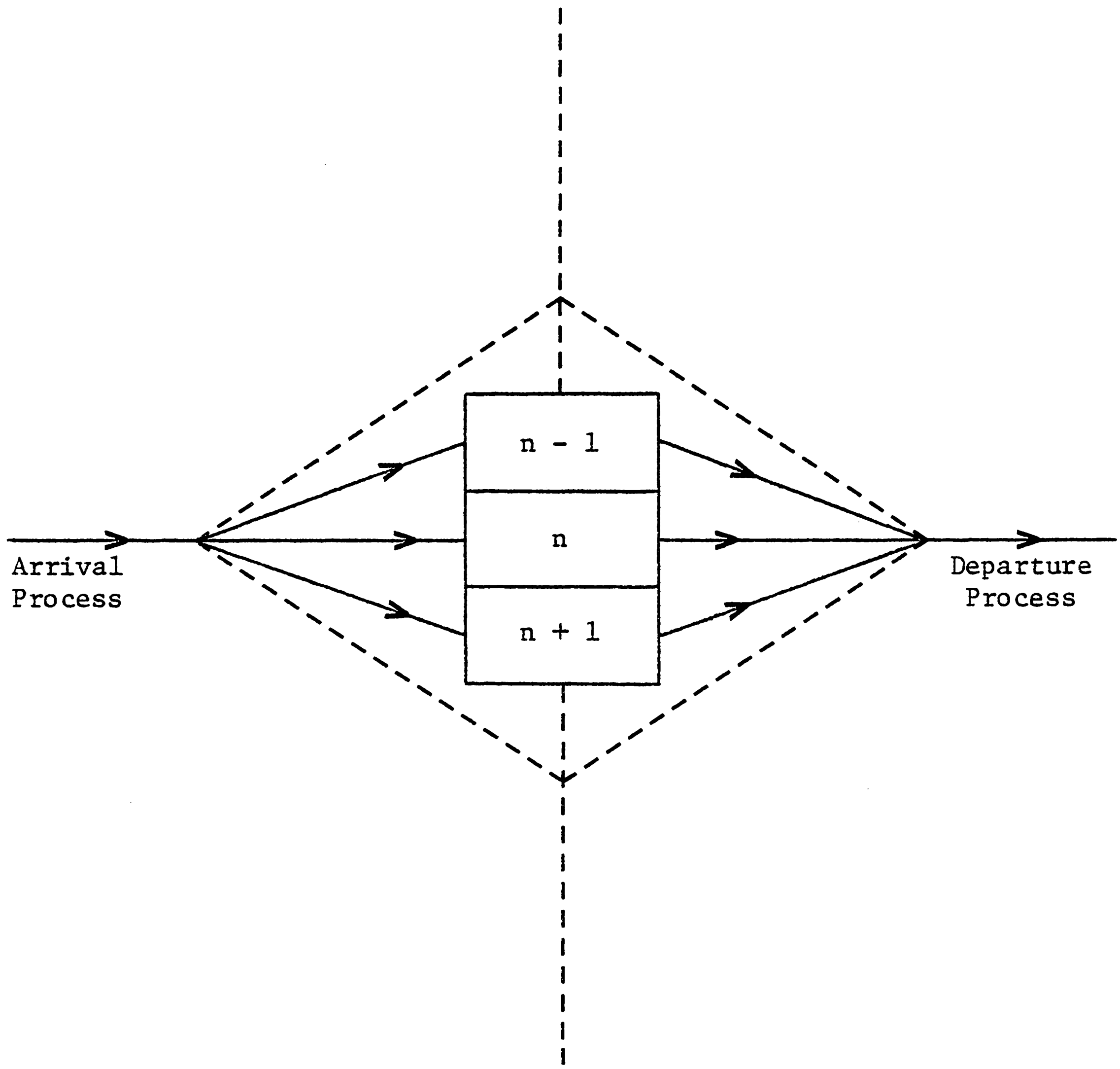

Figure 2.5.1

The Infinite Server Queue

2.5.2. The arrival process in this example is the same as in the previous example. Now consider the following mapping.

$$\Psi((0,t]) = \sup\{N \leq \Phi((0,t]) \mid \sum_{k=1}^{N} S_k \leq t\} \tag{5.7}$$

$\Psi(t) = \Psi((0,t])$ is the number of departures in $(0,t]$. The mapping 1.17a applied to Ψ gives a sequence $\{D_n\}_{n=1}^{\infty}$. This is the sequence of times when a customer leaves the system or departure epochs. Once again, the number in the system at time t, $\Gamma(t)$ is given by 5.5.

(5.8) Remark. In example 5.6 if Φ is a Poisson process on $\mathbb{R}_+$ with rate λ, $\{S_n\}_{n=1}^{\infty}$ is an independent and identically distributed sequence, $P\{S_n \leq x\} = 1 - e^{-\mu x}$ and $\lambda/\mu < 1$, then we have the standard M/M/1 queue and Ψ tends to a Poisson process with rate λ as $t \to \infty$. If the S_n are not exponential, Ψ is a Markov Renewal process with mark space $K = Z_+$.

These two examples illustrate how the theory of point processes finds a natural application in the study of queueing systems. With the methods of point process theory it is possible to obtain a great understanding of very general stochastic systems. An excellent place to find such results is Franken, et al. [1981].

6. Operations on Point Processes

In this section we shall briefly discuss some operations on point processes and marked point processes. We shall also use this section to develop some of the basic notation to be used for the rest of this dissertation.

A) Stretching. Let (N,P) be a random (possibly marked) point process on $\mathbb{R}^d$. If N is a marked point process then we shall take its projection on $\mathbb{R}^d$ to obtain a point process. Using the mappings of

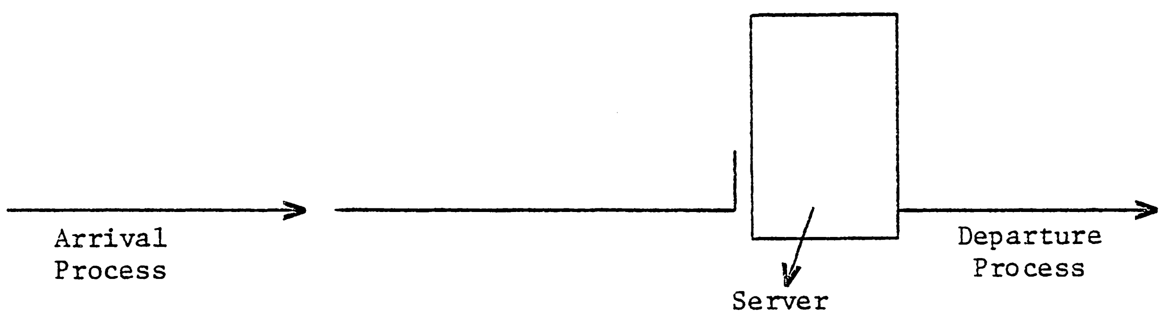

Figure 2.5.2

The Single Server Queue

1.17, for d = 1, we obtain a sequence of random variables $\{T_n\}_{n=-\infty}^{\infty}$. Let $\{W_n\}_{n=-\infty}^{\infty}$ be another sequence of random variables. Consider a third sequence of random variables $\{D_n\}_{n=-\infty}^{\infty}$ constructed by the following mapping.

$$D_n = T_n + W_n. \tag{6A.1}$$

The point process N^1 defined by the sequence $\{D_n\}_{n=-\infty}^{\infty}$ is said to have been obtained by 'stretching' N.

<u>(6A.2) Example</u>. Consider the infinite server queue discussed in example 5.1. Let $\{T_n\}_{n=-\infty}^{\infty}$ be the sequence of arrival times and $\{W_n\}_{n=-\infty}^{\infty}$ be the service times. Then the sequence $\{D_n\}_{n=-\infty}^{\infty}$ defined by 6A.1 gives the departure times and N^1 is the point process of departures.

<u>(6A.3) Example</u>. Consider the single server queue discussed in example 5.6. Now let $\{W_n\}_{n=-\infty}^{\infty}$ be the time spent by the nth customer in the system. Then $\{D_n\}_{n=-\infty}^{\infty}$ is once again the departure times.

B) Superposition. Let $N_1,\ldots,N_n$ be point processes on $\mathbb{R}^d$ with joint distribution Q. Define a point process N as follows.

$$N(\cdot) = \sum_{i=1}^{n} N_i(\cdot). \tag{6B.1}$$

The distribution, Q^s of N, can be obtained from the following.

$$Q_t^s(n) = P\{N(t) = n\} = \sum_{\substack{j_1,j_2,\ldots,j_n \\ j_1+\cdots+j_n=n}} Q_t(j_1,\ldots,j_n) \tag{6B.2}$$

where

$$Q_t(j_1,\ldots,j_n) = P\{N_1(t) = j_1,\ldots,N_n(t) = j_n\}.$$

<u>(6.B3) Example</u>. Consider the queueing system shown in Figure 2.6B.1.

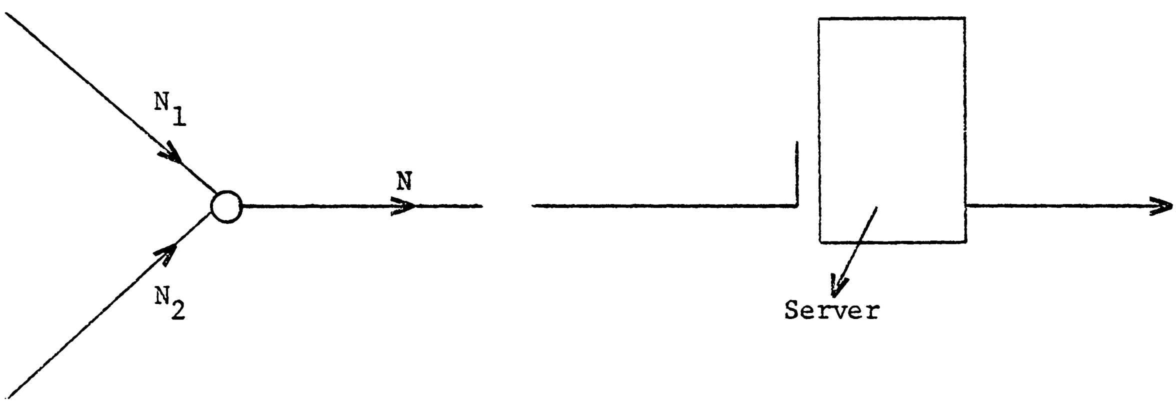

Figure 2.6B.1

The Superposition of Two Arrival Processes to a Queue

Let N_1 and N_2 be two arrival processes. The incoming process seen by the server is given by $N = N_1 + N_2$. Then, if N_1 and N_2 are independent Poisson processes with rates λ_1 and λ_2, N is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$.

(6B.4) Remark. When the processes $N_1,\dots,N_n$ are defined on $\mathbb{R}$ we can obtain an alternate characterization of superposition. Let $T_1,\dots,T_n$ be the sequences obtained from $N_1,\dots,N_n$ by the mapping 1.17. That is, $T_i = \{T_k^i\}_{k=-\infty}^{\infty}$. Consider T defined as follows.

$$T = \bigcup_{i=1}^{n} T_i. \tag{6B.5}$$

T is the sequence obtained from N by the mapping 1.17. The case where $n = 2$ is illustrated by Figure 2.6B.2.

C) Thinning. Let (N,P) be a stochastic process on $\mathbb{R}$ and $\{T_n\}_{n=-\infty}^{\infty}$ be the sequence obtained by 1.17. Let $\{X_n\}_{n=-\infty}^{\infty}$ be another sequence of random variables taking values in a discrete space $\mathscr{D} = \{0,1,\dots,k-1\}$ and $N_0,\dots,N_{k-1}$, be k point processes defined by the following.

$$N_i(\cdot) = \sum_{n=-\infty}^{\infty} I_{\{i\}}(X_n) I_{(\cdot)}(T_n). \tag{6C.1}$$

The processes $N_0,\dots,N_{k-1}$ are said to have been obtained by thinning the process N. Figure 2.6C.1 gives a pictorial representation for the case where $k = 2$.

(6C.2) Example. Let $\mathscr{D} = \{0,1\}$ and $\{X_n\}_{n=-\infty}^{\infty}$ be a Bernoulli process with $p = P\{X_n = 0\} = 1 - q = 1 - P\{X_n = 1\}$ and N be a renewal process. Then N_0 and N_1 are also renewal processes. Further, if N is a Poisson process with rate λ, then so are N_0 and N_1 with rates λp and λq respectively.

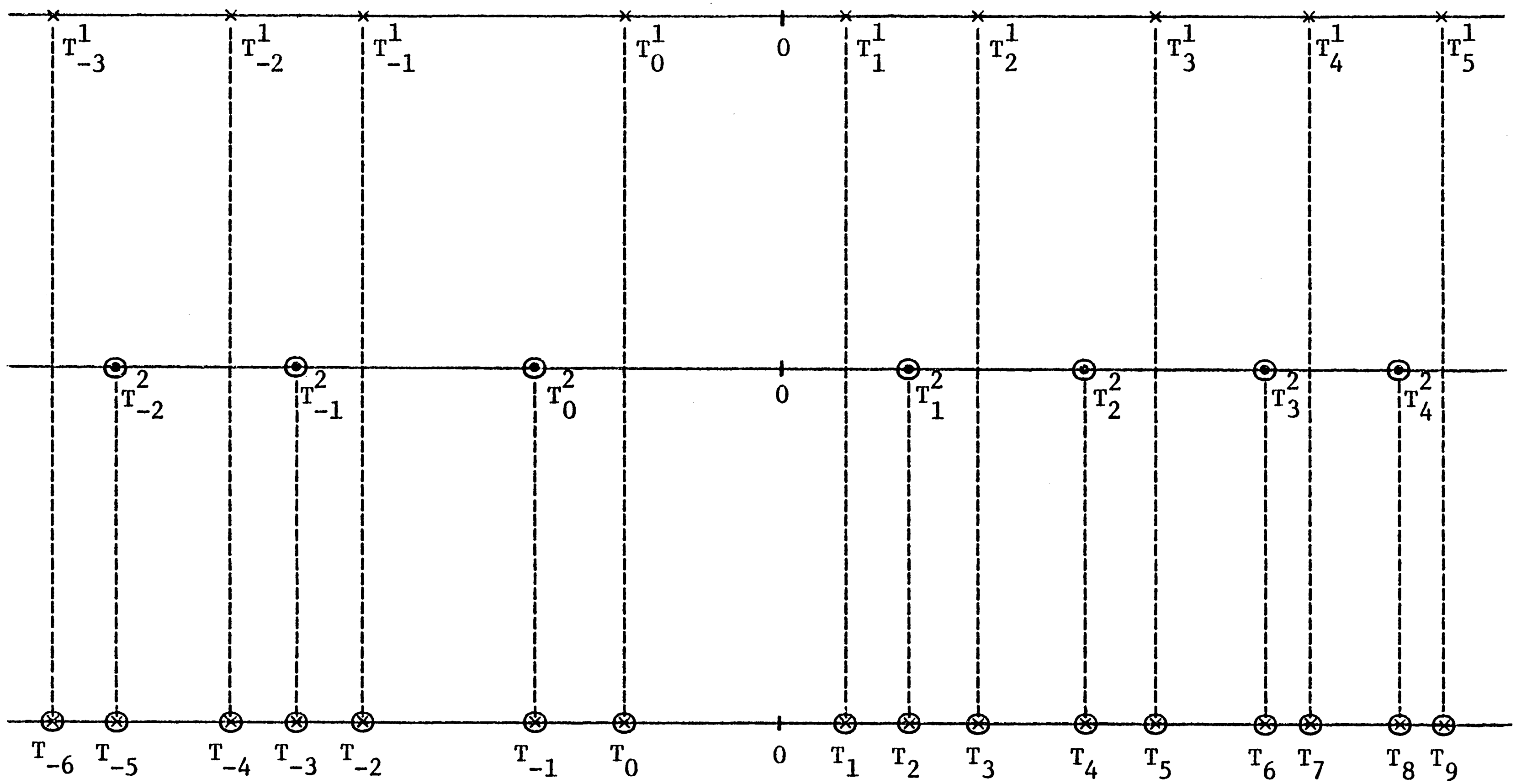

Figure 2.6B.2

Superposition

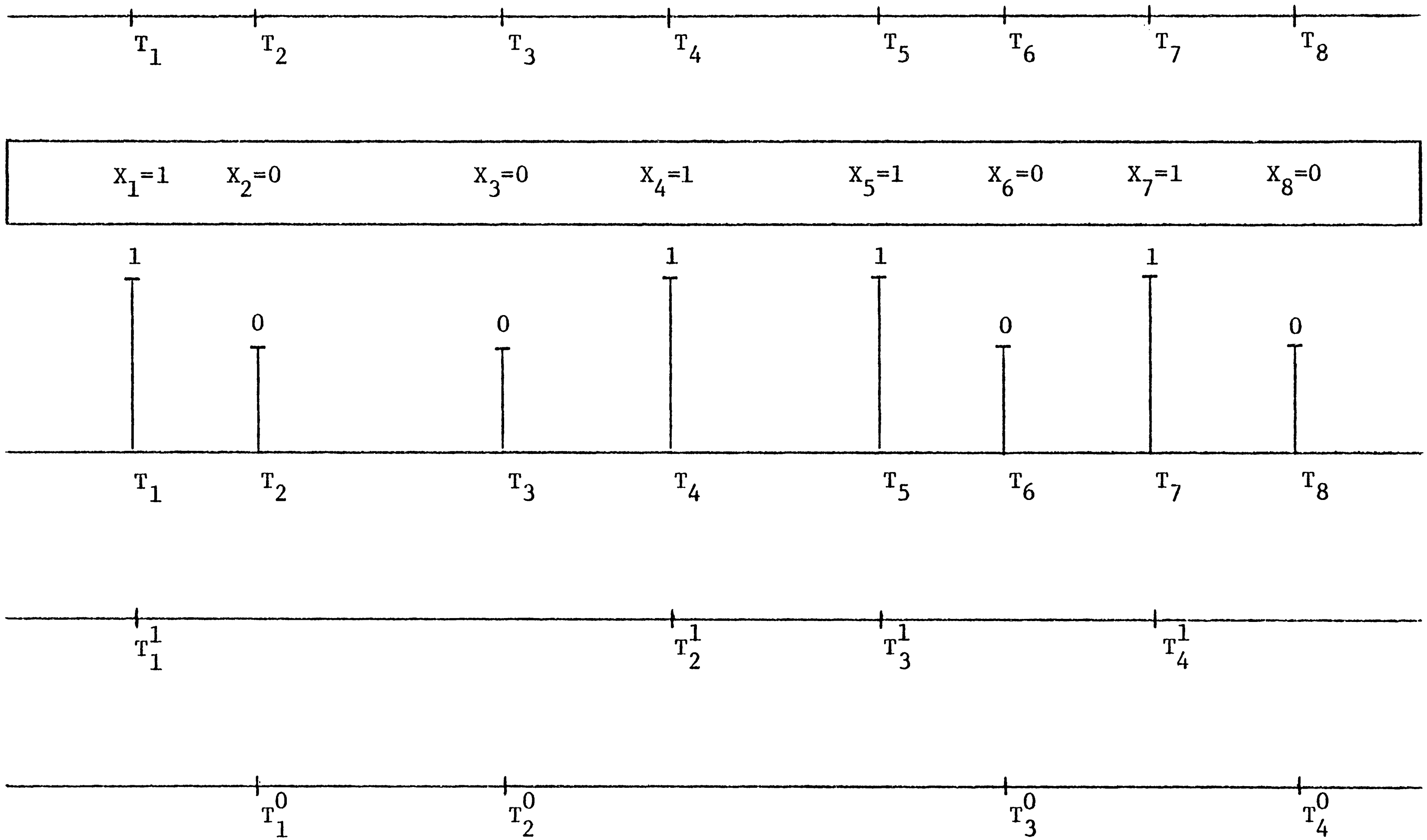

Figure 2.6C.1

Thinning

7. Summary

In this chapter we have introduced the concepts of point process theory and some basic operations which are performed upon them. For the rest of this work we shall concentrate mostly on thinning operations. However, we shall find occasions to reverse the order of things to produce a revelation or two about the problem of superposition.

CHAPTER 3

BERNOULLI THINNING OF POINT PROCESSES-COVARIANCE STRUCTURE: APPLICATION TO RENEWAL PROCESSES

In this chapter we shall deal with the following situation. We have a point process, (N,P) on $\mathbb{R}_+$, that is being thinned by a Bernoulli process that is independent of (N,P). Our objective will be to completely characterize the covariance structure between the two resulting processes. We shall do this by first developing the pointwise covariance structure and then extending the result to get the complete covariance structure. These results will then be used to study the case when N is a renewal process, either stationary or ordinary. We shall show that in either case, the two resulting processes will be uncorrelated if and only if N is Poisson. Finally, we shall give an example where the thinned processes, N^1 and N^0 will be correlated but the sequences obtained from each by 2.1.17a will be uncorrelated.

1. Pointwise Covariances

In this section we develop the pointwise covariance structure when a point process is thinned by an independent Bernoulli process. We first clarify this notion.

Let (N,P) be a point process on $\mathbb{R}_+$ and $\{T_n\}_{n=1}^{\infty}$ be the sequence obtained from N by the mapping 2.1.17a. Clearly, the following is true.

$$N((0,t]) = \sum_{n=1}^{\infty} I_{(0,t]}(T_n) \tag{1.1}$$

For convenience we shall henceforth write N_t for $N((0,t])$.

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed random variables with distribution given as follows.

$$P\{X_n = 1\} = p. \tag{1.2a}$$

$$P\{X_n = 0\} = q = 1 - p. \tag{1.2b}$$

Using the construction in section 2.6B we obtain two point processes N^1 and N^0 as follows.

$$N^1((0,t]) = \sum_{n=1}^{\infty} I_{\{1\}}(X_n)I_{(0,t]}(T_n) \tag{1.3}$$

$$N^0((0,t]) = \sum_{n=1}^{\infty} I_{\{0\}}(X_n)I_{(0,t]}(T_n) \tag{1.4}$$

Again, for convenience we shall write N_t^i for $N^i((0,t])$ for $i = 0,1$. We shall call N^1 and N^0 the 'thinned' processes obtained by thinning the process N by the Bernoulli process $\{X_n\}_{n=1}^{\infty}$. Until otherwise mentioned we shall assume that N and $\{X_n\}_{n=1}^{\infty}$ are independent.

(1.5) Definition. By the pointwise covariance structure of N^1 and N^0, $R_{10}(t,\tau)$, we mean the following.

$$R_{10}(t,\tau) = \mathrm{cov}(N_t^1, N_{t+\tau}^0) = E(N_t^1 N_{t+\tau}^0) - E(N_t^1)E(N_{t+\tau}^0) \tag{1.6}$$

The remainder of this section will be devoted to determining $R_{10}(t,\tau)$.

(1.7) Remark. In what follows we shall routinely interchange expectations and infinite summations. This can be justified by the monotone covergence theorem.

(1.8) Lemma. $E(N_t^1) = pE(N_t)$ and $E(N_t^0) = qE(N_t)$

Proof: From 1.3 we have,

$$E(N_t^1) = E(\sum_{n=1}^{\infty} I_{\{1\}}(X_n)I_{(0,t]}(T_n))$$

$$= \sum_{n=1}^{\infty} E(I_{\{1\}}(X_n)I_{(0,t]}(T_n))$$

By the independence of N and $\{X_n\}_{n=1}^{\infty}$,

$$E(N_t^1) = \sum_{n=1}^{\infty} E(I_{\{1\}}(X_n))E(I_{(0,t]}(T_n)).$$

Thus,

$$\begin{aligned} E(N_t^1) &= \sum_{n=1}^{\infty} P\{X_n = 1\}E(I_{(0,t]}(T_n)) \\ &= \sum_{n=1}^{\infty} pE(I_{(0,t]}(T_n)) \\ &= pE(\sum_{n=1}^{\infty} I_{(0,t]}(T_n)) \\ &= pE(N_t). \end{aligned}$$

Similarly,

$$E(N_t^0) = qE(N_t). \quad \square$$

<u>(1.9) Lemma</u>. $E(N_t^1 N_t^0) = pq[E(N_t^2) - E(N_t)]$ where

$$N_t^2 = N_t \cdot N_t.$$

<u>Proof</u>: Clearly,

$$E(N_t^1 N_t^0) = E[E(N_t^1 N_t^0 | N_t)].$$

Now,

$$\begin{aligned} P\{N_t^1=k_1, N_t^0=k_2 | N_t\} &= P\{N_t^1=k_1 | N_t^0=k_2, N_t\}P\{N_t^0=k_2 | N_t\} \\ &= \delta_{(N_t-k_2)}(k_1)\binom{N_t}{k_2} q^{k_2} p^{N_t-k_2}, \end{aligned}$$

where δ is the Kronecker delta. Thus,

$$E(N_t^1N_t^0) = \sum_{k_1}\sum_{k_2} k_1k_2\delta_{(N_t-k_2)}(k_1)\binom{N_t}{k_2}q^{k_2}p^{N_t-k_2}$$

$$= \sum_{k_1=0}^{N_t} k_1(N_t-k_1)\binom{N_t}{N_t-k_1}q^{N_t-k_1}p^{k_1}$$

$$= N_t\sum_{k_1=0}^{N_t} k_1\binom{N_t}{k_1}q^{N_t-k_1}p^{k_1} - \sum_{k_1=0}^{N_t} k_1^2\binom{N_t}{k_1}q^{N_t-k_1}p^{k_1}.$$

Note that the summation in the first term is the expected value of a binomal random variable with parameters N_t and p and the summation in the second term is the second moment about 0 for the same random variable. Thus,

$$E(N_t^1N_t^0|N_t) = pN_t^2 - (pqN_t + p^2N_t^2)$$

$$= pq(N_t^2-N_t).$$

Taking expectations on both sides yields the result. □

<u>(1.10) Lemma</u>. $E[N_t^1(N_{t+\tau}^0-N_t^0)] = pqE[N_t(N_{t+\tau}-N_t)]$.

<u>Proof</u>: Note that,

$$N_{t+\tau}^0-N_t^0 = \sum_{j=1}^{\infty} I_{(t,t+\tau]}(T_j)I_{\{0\}}(X_j). \tag{1.11}$$

This and 1.3 gives

$$N_t^1(N_{t+\tau}^0-N_t^0) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} I_{(0,t]}(T_i)I_{\{1\}}(X_i)I_{(t,t+\tau]}(T_j)I_{\{0\}}(X_j).$$

Thus,

$$E[N_t^1(N_{t+\tau}^0-N_t^0)] = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} E(I_{(0,t]}(T_i)I_{(t,t+\tau]}(T_j))E(I_{\{1\}}(X_i)I_{\{0\}}(X_j))$$

by the independence of N and $\{X_n\}_{n=1}^{\infty}$. But,

$$E(I_{\{1\}}(X_i)I_{\{0\}}(X_j)) = pq$$

and

$$I_{(t,t+\tau]}(T_j) = I_{(0,t+\tau]}(T_j) - I_{(0,t]}(T_j)$$

which gives,

$$E[N_t^1(N_{t+\tau}^0 - N_t^0)] = pqE(\sum_{i=1}^{\infty} I_{(0,t]}(T_i) \sum_{j=1}^{\infty} I_{(0,t+\tau]}(T_j) - I_{(0,t]}(T_j))$$

$$= pqE[N_t(N_{t+\tau} - N_t)]. \quad \square$$

<u>(1.12) Theorem</u>.

$$R_{10}(t,\tau) = pq\{\text{cov}(N_t, N_{t+\tau}) - E(N_t)\}.$$

<u>Proof</u>: Note that,

$$E(N_t^1 N_{t+\tau}^0) = E[N_t^1(N_{t+\tau}^0 - N_t^0)] + E(N_t^1 N_t^0)$$

From Lemmas 1.9 and 1.10 we have,

$$E(N_t^1 N_{t+\tau}^0) = pq\{E(N_t N_{t+\tau}) - E(N_t^2) + E(N_t^2) - E(N_t)\}$$

$$= pq\{E(N_t N_{t+\tau}) - E(N_t)\} \tag{1.13}$$

And also from Lemma 1.8,

$$E(N_t^1)E(N_{t+\tau}^0) = pqE(N_t)E(N_{t+\tau}) \tag{1.14}$$

Subtracting 1.14 from 1.13, the result follows. $\square$

<u>(1.15) Corollary</u>. Let $R_{10}'(t_1,t_2) = \text{cov}(N_{t_1}^1, N_{t_2}^0)$. Then

$$R_{10}'(t_1,t_2) = pq\{\text{cov}(N_{t_1}, N_{t_2}) - E(N_{\min(t_1,t_2)})\}.$$

<u>Proof</u>: Let $t_2 > t_1$. Then,

$$R'_{10}(t_1,t_2) = \mathrm{cov}(N^1_{t_1}, N^0_{t_1+(t_2-t_1)})$$

$$= R_{10}(t_1, t_2-t_1)$$

$$= pq\{\mathrm{cov}(N_{t_1}, N_{t_2}) - E(N_{t_1})\}, \tag{1.16}$$

from Theorem 1.12.

Similarly, for $t_2 < t_1$, we have,

$$R'_{10}(t_1,t_2) = pq\{\mathrm{cov}(N_{t_1}, N_{t_2}) - E(N_{t_2})\}. \tag{1.17}$$

1.16 and 1.17 together prove the result. □

<u>(1.18) Example</u>. Let (N,P) be a Poisson process with rate λ. Then,

$$\mathrm{cov}(N_t, N_{t+\tau}) = \mathrm{var}(N_t) = \lambda t = E(N_t).$$

Thus, we have that $\forall\, t, t_1, t_2, \tau \in \mathbb{R}_+$, $R_{10}(t,\tau) = 0 = R'_{10}(t_1,t_2)$ as expected.

2. The Complete Covariance Structure

We now extend the results of the last section, in a straightforward manner, to obtain the complete covariance structure of N^1 and N^0 which is defined as follows.

<u>(2.1) Definition</u>. Let $B_1, B_2 \in \mathscr{B}_+$ where $\mathscr{B}_+$ is the σ-field of Borel subsets of $\mathbb{R}_+$ and B_1, B_2 are bounded. Then, by the complete covariance structure of N^1 and N^0, $R''_{10}(B_1,B_2)$, we mean the following.

$$R''_{10}(B_1,B_2) = \mathrm{cov}(N^1(B_1), N^0(B_2)). \tag{2.2}$$

<u>(2.3) Proposition</u>. There exist $n, m \in \mathbb{N}$, $t^1_i, t^2_i, s^1_j, s^2_j \in \mathbb{R}_+$, $i = 1,\ldots,n$, $j = 1,\ldots,m$ such that for any bounded $B_1, B_2 \in \mathscr{B}_+$ we have the following.

$$R''_{10}(B_1,B_2) = \sum_{i=1}^{n} \sum_{j=1}^{m} R'_{10}(t_i^2,s_j^2) - R'_{10}(t_i^2,s_j^1) - R'_{10}(t_i^1,s_j^2) + R'_{10}(t_i^1,s_j^1).$$

Proof: Since $B_1, B_2 \in \mathscr{B}_+$ and are bounded, they can be written as a finite union of disjoint intervals. Thus, we may write B_1 and B_2 as,

$$B_1 = \bigcup_{i=1}^{n} (t_i^1, t_i^2]$$

and

$$B_2 = \bigcup_{j=1}^{m} (s_j^1, s_j^2]$$

where the intervals in each of the unions are mutually disjoint.

Clearly,

$$N^1(B_1) = \sum_{i=1}^{n} N^1(t_i^2) - N^1(t_i^1) \tag{2.4}$$

and

$$N^0(B_2) = \sum_{j=1}^{m} N^0(s_j^2) - N^0(s_j^1). \tag{2.5}$$

And so,

$$N^1(B_1)N^0(B_2) = \sum_{i=1}^{n} \sum_{j=1}^{m} [N^1(t_i^2) - N^1(t_i^1)][N^0(s_j^2) - N^0(s_j^1)]. \tag{2.6}$$

Taking expectations across equations 2.4, 2.5 and 2.6 and using the definition of covariance, the result follows. □

Corollary 1.15 and Proposition 2.3 show that the pointwise covariance structure, $R_{10}(t,\tau)$, provides all the information needed about the complete covariance structure. Thus, all we need to study is $R_{10}(t,\tau)$.

In the next two sections we shall examine the case where N is a stationary renewal process and an ordinary renewal process respectively.

Since it is computationally more tractable to work with $R_{10}(t,\tau)$, the following result will be useful.

(2.7) Proposition. The following statements are equivalent.

a) $R''_{10}(B_1,B_2) = 0 \quad \forall\, B_1,B_2 \in \mathscr{B}_+$

b) $R'_{10}(t_1,t_2) = 0 \quad \forall\, t_1,t_2 \in \mathbb{R}_+$

c) $R_{10}(t,\tau) \quad = 0 \quad \forall\, t,\tau \in \mathbb{R}_+$.

Proof: By Corollary 1.15, c $\Longrightarrow$ b, by Proposition 2.3, b $\Longrightarrow$ a, and by setting $B_1 = (0,t]$, $B_2 = (0,t+\tau]$, a $\Longrightarrow$ c. □

To conclude this section, we note a symmetry in the various covariance structures discussed so far. These are as follows.

$$\mathrm{cov}(N^1_t, N^0_{t+\tau}) = \mathrm{cov}(N^1_{t+\tau}, N^0_t) \qquad t,\tau \in \mathbb{R}_+ \tag{2.8}$$

$$\mathrm{cov}(N^1_{t_1}, N^0_{t_2}) = \mathrm{cov}(N^1_{t_2}, N^0_{t_1}) \qquad t_1,t_2 \in \mathbb{R}_+ \tag{2.9}$$

$$\mathrm{cov}(N^1(B_1), N^0(B_2)) = \mathrm{cov}(N^1(B_2), N^0(B_1)) \qquad B_1,B_2 \in \mathscr{B}_+ \tag{2.10}$$

3. Stationary Renewal Processes

Recall (Example 2.1.22 a & b) that a renewal process on $\mathbb{R}_+$ can be specified as a sequence of random variables $\{T_n\}_{n=1}^{\infty}$ where $\{T_{n+1}-T_n\}_{n=1}^{\infty}$ are independent and identically distributed random variables. Let (N^s,P) be such a process and let $F(t)$, m and $G(t)$ be defined as follows.

$$F(t) = P\{T_{n+1}-T_n \leq t\} \qquad n \geq 1 \tag{3.1}$$

$$m = \int_0^{\infty} (1-F(t))\,dt \tag{3.2}$$

$$G(t) = P\{T_1 \leq t\}. \tag{3.3}$$

Further, let (N^s,P) be stationary in the sense of definition 2.3.2.

(3.4) Proposition. The process (N^s,P) is stationary iff

$$G(t) = \frac{1}{m} \int_0^t (1-F(u))du.$$

Proof: See Çinlar [1975] pg. 304, Proposition 9.3.7. □

Now, let N_t^s, $H_0(t)$ and $H(t)$ be defined as follows.

$$N_t^s = N^s((0,t]) \tag{3.5}$$

$$H_0(t) = E(N_t^s) \tag{3.6}$$

$$H(t) = \sum_{n=1}^{\infty} F^{*n}(t) \tag{3.7}$$

$F^{*n}(t)$ is the n-fold convolution of $F(t)$ with itself and is determined by the following.

$$F^{*n}(t) = \int_0^t F^{*n-1}(t-x)dF(x). \tag{3.8}$$

(3.9) Lemma. $H_0(t) = \frac{t}{m}$.

Proof: See Çinlar [1975] pg. 305, Proposition 9.3.9. □

(3.10) Lemma. $E[(N_t^s)(N_t^s-1)\ldots(N_t^s-k+1)] = k!\ (H_0 * H^{*(k-1)})(t)$.

Proof: See Franken [1963]. □

Let $\{X_n\}_{n=1}^{\infty}$ be a Bernoulli process as in 1.2. Using 1.3 and 1.4 we construct two processes, N^{1s} and N^{0s}. Finally, let $R_{10}^s(t,\tau) = \mathrm{cov}(N_t^{1s}, N_{t+\tau}^{0s})$. Theorem 1.12 then gives us the following.

$$R_{10}^s(t,\tau) = pq\{\mathrm{cov}(N_t^s, N_{t+\tau}^s) - E(N_t^s)\}. \tag{3.11}$$

We now present the main result of this section.

(3.12) Theorem. $R_{10}^s(t,\tau) = 0$ for $t \in (0,\infty)$, $\tau \in [0,\infty)$ if and only if $F(t) = 1 - e^{-t/m}$ for $t \in (0,\infty)$.

Proof: If $F(t) = 1 - e^{-t/m}$ for $t \in (0,\infty)$, (N^s,P) is a Poisson process.

Then,

$$\mathrm{cov}(N_t^s, N_{t+\tau}^s) = \mathrm{Var}(N_t^s) = E(N_t^s). \tag{3.13}$$

This and 3.11 give $R_{10}^s(t,\tau) = 0$ for $t \in (0,\infty)$, $\tau \in [0,\infty)$. Now, let $R_{10}^s(t,\tau) = 0$ for $t \in (0,\infty)$, $\tau \in [0,\infty)$. Consider now the case $\tau = 0$. Then 3.11 gives,

$$pq\{\mathrm{var}(N_t^s) - E(N_t^s)\} = 0 \qquad t \in (0,\infty). \tag{3.14}$$

If $pq = 0$ then $p = 0$ or 1 in which case one of the processes, N^{1s} or N^{0s}, does not exist. Hence, we assume $p \in (0,1)$. Thus,

$$\mathrm{var}(N_t^s) = E(N_t^s) \qquad t \in (0,\infty)$$

$$\Longrightarrow \quad E[N_t^s(N_t^s - 1)] = [E(N_t^s)]^2 \qquad t \in (0,\infty).$$

Lemmas 3.9 and 3.10 give,

$$2\,(H_0 * H)(t) = t^2/m^2$$

$$\Longrightarrow \quad \frac{2}{m}\int_0^t H(t-u)\,du = t^2/m^2$$

$$\Longrightarrow \quad \frac{2}{m}\int_0^t H(u)\,du = t^2/m^2.$$

Differentiating both sides gives,

$$H(t) = \frac{t}{m}$$

and hence,

$$F(t) = 1 - e^{-t/m}. \quad \square$$

Theorem 3.12 leads to the following corollary.

<u>(3.15) Corollary</u>. The following are equivalent.

a) $\mathrm{cov}(N_t^{1s}, N_t^{0s}) = 0$ for $t \in (0,\infty)$.

b) The process N^s is Poisson.

c) The processes N^{1s} and N^{0s} are independent Poisson processes.

The last result is of statistical importance since we have a situation where zero correlation implies independence. This is of practical significance since testing for zero correlation is easier than testing for independence. Further, we need to check only for pointwise covariances with $\tau = 0$.

4. Ordinary Renewal Processes

In this section we shall extend the results of the last section to ordinary renewal processes on $\mathbb{R}_+$. An ordinary renewal process is defined as follows.

(4.1) Definition. Let (N,P) be a renewal process on $\mathbb{R}_+$ in the sense of Example 2.1.22b. N is said to be an ordinary renewal process if $F_0(x) = F(x)$ for all $x \in [0,\infty)$. Thus, all intervals, including the first one have the same distribution function.

Let m and $H(t)$ be as in 3.2 and 3.7 respectively and N^s be the stationary version of N. Let N^{1s}, N^{0s} be the processes obtained by thinning the process N^s with a Bernoulli process $\{X_n\}_{n=1}^{\infty}$ and N^1 and N^0 be obtained from N by the same operation.

(4.2) Proposition. If $F(\infty) = 1$ and F is not periodic then for all $x,t \in \mathbb{R}_+$ we have that,

$$N_{t+x} - N_x \xrightarrow[x\to\infty]{\mathscr{D}} N_t^s$$

where the symbol $\xrightarrow{\mathscr{D}}$ denotes convergence in distribution.

Proof: It is clearly sufficient to show that,

$$\lim_{x\to\infty} P\{N_{t+x}-N_x=k\} = P\{N_t^s=k\} \tag{4.3}$$

for $k \in Z_+$.

Let $x \in \mathbb{R}_+$ and $V_x = T_{N_x+1} - x$ be the time until the next renewal after x. See Figure 3.4.1.

Let,

$$G_x(v) = P\{V_x \leq v\}$$

and

$$S_{N_x} = T_{N_x+1} - T_{N_x}.$$

Now, for k = 0,

$$P\{N_{t+x}-N_x=0\} = 1-G_x(t)$$

and hence,

$$\lim_{x\to\infty} P\{N_{t+x}-N_x=0\}\ 1-\lim_{x\to\infty} G_x(t).$$

But from renewal theory (Cox [1962] pg. 63) we know that,

$$\lim_{x\to\infty} G_x(v) = G(v) = \frac{1}{m}\int_0^v (1-F(u))du.$$

Thus,

$$\lim_{x\to\infty} P\{N_{t+x}-N_x=0\} = 1-G(t) = P\{N_t^s=0\}.$$

For $k \geq 1$ observe that,

$$\{N_{t+x}-N_x=k\} \iff \{V_x + \sum_{n=0}^{k-1} S_{N_x+1+n} \leq t\} \cap \{V_x + \sum_{n=0}^{k} S_{N_x+1+n} > t\}.$$

Further, notice that,

$$\{V_x + \sum_{n=0}^{k} S_{N_x+1+n} > t\} \iff \{V_x + \sum_{n=0}^{k} S_{N_x+1+n} \leq t\}^c$$

and,

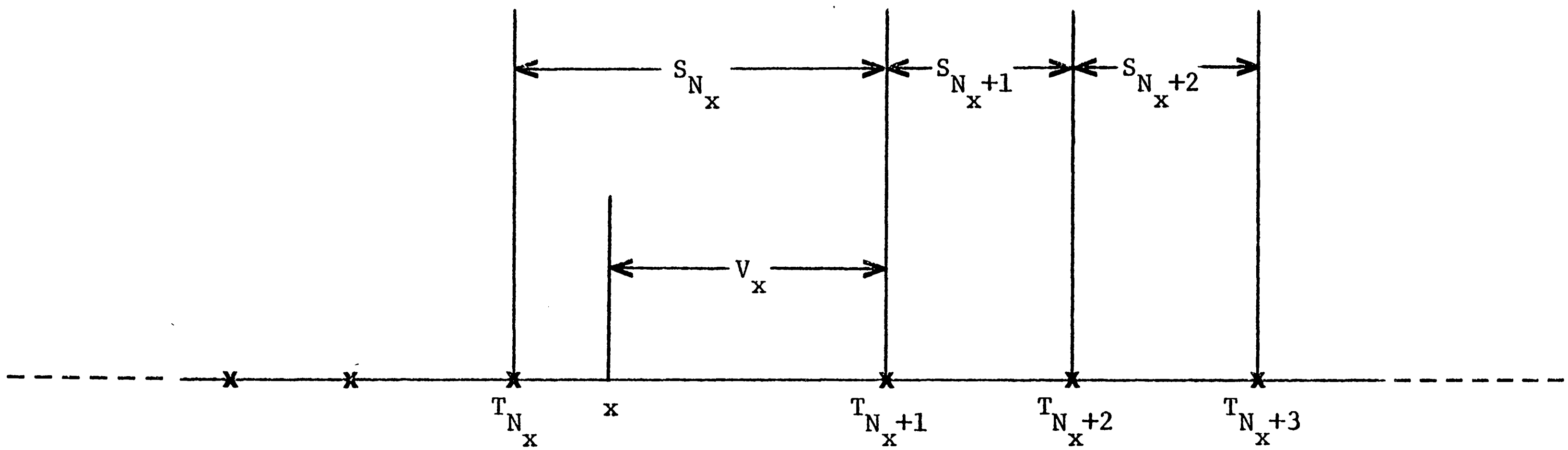

Figure 3.4.1

The Time Until the Next Renewal After x

$$\{V_x + \sum_{n=0}^{k} S_{N_x+1+n} \leq t\} \subset \{V_x + \sum_{n=0}^{k-1} S_{N_x+1+n} \leq t\}.$$

Hence,

$$P\{N_{t+x}-N_x=k\} = P\{V_x + \sum_{n=0}^{k-2} S_{N_x+1+n} \leq t\} - P\{V_x + \sum_{n=0}^{k-1} S_{N_x+1+n} \leq t\}.$$

To avoid difficulties in the first summation when k = 1 we assume that,

$$\sum_{n=0}^{-1} S_{N_x+1+n} = 0.$$

Now by the independence of V_x and S_{N_x+1+n} for n = 0,1,2,... and the fact that N is a renewal process we have,

$$\begin{aligned} P\{N_{t+x}-N_x=k\} &= G_x*F^{*k-1}(t)-G_x*F^{*k}(t) \\ &= G_x*(F^{*k-1}-F^{*k})(t) \end{aligned}$$

and finally,

$$\begin{aligned} \lim_{x\to\infty} P\{N_{t+x}-N_x=k\} &= G*(F^{*k-1}-F^{*k})(t) \\ &= P\{N_t^s=k\}. \quad \square \end{aligned}$$

<u>(4.4) Lemma</u>. $\lim_{x\to\infty} E(N_{t+x}-N_x) = E(N_t^s)$ and $\lim_{x\to\infty} \mathrm{var}(N_{t+x}-N_x) = \mathrm{var}(N_t^s)$.

<u>Proof</u>: First observe that,

$$P\{N_{t+x}-N_x < \infty\} = 1 \text{ for } t \in (0,\infty)$$

and

$$P\{(N_{t+x}-N_x)^2 < \infty\} = 1 \text{ for } t \in (0,\infty).$$

This and Proposition 4.2 gives the result. See Breiman [1968] pg. 163, Proposition 8.12. $\square$

We now give the main result of this section.

(4.5) Theorem. If N^1 and N^0 are uncorrelated then

$$F(t) = 1 - e^{-t/m}.$$

Proof: We first show that if N^1 and N^0 are uncorrelated then N_t^s is a Poisson process. Let,

$$R_{10}^{s}(t,0) = \mathrm{cov}(N_t^{1s}, N_t^{0s}).$$

Now,

$$R_{10}^{s}(t,0) = pq\{\mathrm{var}(N_t^s) - E(N_t^s)\}$$

$$= pq \lim_{x\to\infty} \{\mathrm{var}(N_{t+x}-N_x) - E(N_{t+x}-N_x)\}$$

by Lemma 4.4. Thus,

$$R_{10}^{s}(t,0) = pq \lim_{x\to\infty} \{\mathrm{var}(N_{t+x}) + \mathrm{var}(N_x) - 2\mathrm{cov}(N_{t+x},N_x) - E(N_{t+x}) + E(N_x)\}$$

$$= \lim_{x\to\infty} \mathrm{cov}(N_{t+x}^1 - N_x^1, N_{t+x}^0 - N_x^0).$$

But since N^1 and N^0 are uncorrelated by assumption, we have

$$R_{10}^{s}(t,0) = \lim_{x\to\infty} 0 = 0.$$

From the proof of Theorem 3.12 and Corollary 3.15 we know that N_t^s is thus a Poisson process. And since $F(t)$ denotes the generic inter-renewal distribution for both N^s and N we have that,

$$F(t) = 1 - e^{-t/m}. \quad \square$$

Theorem 4.5 leads immediately to the following corollary.

(4.6) Corollary. Let $F(\infty) = 1$ and let F be aperiodic. Then, the following are equivalent.

a) $\mathrm{cov}(N_t^1, N_t^0) = 0 \qquad t \in (0,\infty).$

b) The process N is Poisson.

c) The processes N^1 and N^0 are independent Poisson processes.

5. Interval-Count Relationships

In this section we give a counter-intuitive example to a reasonable conjecture. Recall that with every point process N on $\mathbb{R}_+$ we may associate a sequence of random variables $\{T_n\}_{n=1}^{\infty}$. Now let $\{T_n^1\}_{n=1}^{\infty}$ and $\{T_n^0\}_{n=1}^{\infty}$ be the sequences associated with the processes N^1 and N^0 generated by thinning N with an independent Bernoulli switch. Since the sequences $\{T_n^1\}_{n=1}^{\infty}$ and $\{T_n^0\}_{n=1}^{\infty}$ determine N^1 and N^0 respectively, it might be reasonable to assume that if $\{T_n^1\}_{n=1}^{\infty}$ and $\{T_n^0\}_{n=1}^{\infty}$ are uncorrelated, so would N^1 and N^0. We now construct a counter-example to show that this is not true.

Let N be a renewal process on $\mathbb{R}_+$ and let F(t) be as before. Since the processes N^1 and N^0 are also renewal process it follows that the sequences $\{T_n^1\}_{n=1}^{\infty}$ and $\{T_n^0\}_{n=1}^{\infty}$ will be uncorrelated if and only if T_1^1 and T_1^0 are uncorrelated. In a private note, Franken [1981] has obtained a condition for T_1^1 and T_1^0 be uncorrelated. We now give this result. Let,

$$S_n = T_{n+1} - T_n \qquad n \geq 1.$$

<u>(5.1) Proposition.</u> $\operatorname{cov}(T_1^1, T_1^0) = 0$ if and only if

$$E(T_1^2) = 2[E(T_1)]^2 \tag{5.2}$$

<u>Proof:</u>

$$\operatorname{cov}(T_1^1, T_1^0) = E(T_1^1 T_1^0) - E(T_1^1)E(T_1^0).$$

Observe that T_1^1 is given by,

$$T_1^1 = S_1 + \cdots + S_{J_1}$$

where J_1 is a geometrically distributed random variable with distribution

$$P\{J_1=j_1\} = q^{j_1-1}p \qquad j_1 \geq 1.$$

J_1 is determined by the Bernoulli process $\{X_n\}_{n=1}^{\infty}$ which is independent of $\{T_n\}_{n=1}^{\infty}$. Hence, by Wald's lemma,

$$E(T_1^1) = E(S_1)E(J_1)$$

$$= E(T_1) \cdot \frac{1}{p} . \tag{5.3a}$$

Similarly,

$$E(T_1^0) = E(T_1) \cdot \frac{1}{q} . \tag{5.3b}$$

Now,

$$E(T_1^1T_1^0) = E[E(T_1^1T_1^0)|X_1]. \tag{5.4}$$

If $X_1 = 1$ we have,

$$T_1^1 = T_1 = S_1 \tag{5.5a}$$

and,

$$T_1^0 = S_1+\cdots+S_{J_0+1} \tag{5.5b}$$

where J_0 is a geometrically distributed random variable with distribution

$$P\{J_0=j_0\} = p^{j_0-1}q .$$

If $X_1 = 0$ we have,

$$T_1^1 = S_1+\cdots+S_{J_1+1} \tag{5.6a}$$

and,

$$T_1^0 = T_1 = S_1 \tag{5.6b}$$

From 5.4, 5.5 and 5.6, we have

$$E(T_1^1T_1^0) = p\sum_{j_0=1}^{\infty} E[S_1(S_1+\cdots+S_{j_0+1})]p^{j_0-1}q + q\sum_{j_1=1}^{\infty} E[S_1(S_1+\cdots+S_{j_1+1})]q^{j_1-1}p.$$

Noting that $S_1 = T_1$ and that S_i and S_j are independent for $i \neq j$ we have,

$$E(T_1^1T_1^0) = p\sum_{j_0=1}^{\infty} E(S_1^2)p^{j_0-1}q + \sum_{j_0=1}^{\infty} j_0p^{j_0}q[E(S_1)]^2$$

$$+ q\sum_{j_1=1}^{\infty} E(S_1^2)q^{j_1-1}p + \sum_{j_1=1}^{\infty} j_1q^{j_1}p[E(S_1)]^2.$$

This gives,

$$E(T_1^1T_1^0) = E(S_1)^2 + [E(S_1)]^2[\frac{p}{q} + \frac{q}{p}]. \tag{5.7}$$

Thus, from 5.3 and 5.7 we have,

$$\text{cov}(T_1^1,T_1^0) = E(S_1^2) + [E(S_1)]^2[\frac{q^2+p^2-1}{pq}]$$

and finally,

$$\text{cov}(T_1^1,T_1^0) = E(T_1^2) - 2[E(T_1)]^2$$

which proves the result. □

Now, let the inter-renewal distribution of N, F(t) be given as follows.

$$F(t) = r_1 \int_0^t (1-e^{-\mu_1(t-x)})\mu_2 e^{-\mu_2 x}dx + r_2(1-e^{-\mu_3 t})$$

where $r_1 + r_2 = 1$ and $r_1, r_2 \geq 0$. It can then be shown that 5.2 is equivalent to

$$\left[\frac{1}{\mu_1} + \frac{1}{\mu_2} - \frac{1}{\mu_3}\right]^2 = \frac{1}{r_2\mu_1\mu_2} \tag{5.8}$$

If we set $\mu_1 = 1/4$, $\mu_2 = 4$, $\mu_3 = 4/9$ and $r_1 = 3/4$ we can satisfy 5.2. Hence, the sequences $\{T_n^1\}_{n=1}^{\infty}$ and $\{T_n^0\}_{n=1}^{\infty}$ will be uncorrelated. But since the process N is clearly not Poisson N^1 and N^0 will be correlated by Corollary 4.6.

Finally, we note that when N is a renewal process the zero correlation of N^1 and N^0 implies the zero correlation of $\{T_n^1\}_{n=1}^{\infty}$ and $\{T_n^0\}_{n=1}^{\infty}$. This is because if N^1 and N^0 are uncorrelated, N is Poisson. Hence, F(t) is an exponential distribution which satisfies 5.2.

6. Summary

We have dealt with, in this chapter, the covariance properties when a point process is thinned by an independent Bernoulli process. We shall now take another direction. In subsequent chapters the independence assumption will be dropped and we shall assume a specific but useful structure for N. We shall indicate, where appropriate, how the results in sections 3 and 4 may be retrieved from the more general results to be developed.

CHAPTER 4

MARK DEPENDENT THINNING OF MARKOV RENEWAL PROCESSES-COVARIANCE STRUCTURES

In this chapter we deal with a Markov renewal process (Example 2.2.13) being thinned into two processes by a mark dependent process. Our objective will be to develop the structure of the covariance between the thinned processes.

This chapter differs from Chapter 3 in that the thinning process will be allowed to depend on the process being thinned. Consequently, the thinning process will no longer be a Bernoulli process in general. Subsequent chapters will show that this allows for very general thinning rules.

Of course what is lost here is the generality of the process that is being thinned. We are putting a special structure on the process, namely a Markov renewal structure. However, Markov renewal processes find a natural application in many situations and have been particularly useful in modeling queueing situations and providing useful insights (cf. Disney et al. (1973), Disney et al. (1981)). Further, the special structure of Markov renewal processes allows for computations without undue difficulty.

Results will be developed where the mark space is a Polish space. These will then be reduced to the case where the mark space is countable enabling us to put our results in matrix form. Throughout our discussion all processes will be simple in the sense of Definition 2.2.12.

1. Arbitrary Mark Spaces

Let (N,P) be a Markov renewal process on $\mathbb{R}_+$ with mark space E. E is a Polish space. We assume that N is simple. Recall that (2.2.6) any marked point process can be regarded as a sequence of pairs of random variables. Accordingly, we henceforth consider the process (N,P) as the following sequence.

$$(X,T) = \{(X_n,T_n)\}_{n=1}^{\infty}. \tag{1.1}$$

We call X_n the mark of the event at T_n. Note that we consider the mark as the first element in the pair (X_n,T_n) and not the other way around as in Chapter 2. We first define Markov renewal processes on an arbitrary Polish space, E.

<u>(1.2) Definition</u>. Let $\sigma(E)$ be the σ-field generated by the metric topology on E. Then, (X,T) is said to form a Markov renewal process if for any $S \varepsilon \sigma(E)$, the following holds almost surely.

$$P\{X_{n+1} \varepsilon S, T_{n+1} - T_n \le t | X_0,\ldots,X_n,T_0,\ldots,T_n\} \tag{1.3}$$

$$= P\{X_{n+1} \varepsilon S, T_{n+1} - T_n \le t | X_n\}$$

<u>(1.4) Definition</u>. Let $Q_{n+1}(x,S,t) = P\{X_{n+1} \varepsilon S, T_{n+1} - T_n \le t | X_n = x\}$. Then the collection $Q = \{Q_n(x,S,t),\ x \varepsilon E,\ S \varepsilon \sigma(E),\ t \varepsilon \mathbb{R}_+,\ n \varepsilon \mathbb{N}\}$ is called the semi-Markov kernel for the process (X,T).

<u>(1.5) Definition</u>. If $Q_n(\cdot,\cdot,\cdot) = Q(\cdot,\cdot,\cdot)$ for all n, then the process (X,T) is said to be time homogeneous.

Henceforth, we shall consider only time homogeneous processes.

<u>(1.6) Remark</u>. If (X,T) is a Markov renewal process, then the sequence $\{X_n\}_{n=1}^{\infty}$ is a Markov chain on E. Further, the $\{T_{n+1} - T_n\}_{n=1}^{\infty}$ forms a sequence of mutually conditionally independent random variables given $\{X_n\}_{n=1}^{\infty}$. Formally this means that the following holds almost surely.

$$P\{T_1 \le t_1, T_2 - T_1 \le t_2, \ldots, T_{n+1} - T_n \le t_{n+1} | X_0, \ldots, X_{n+1}\}$$

$$= P\{T_1 \le t_1 | X_0, X_1\} P\{T_2 - T_1 \le t_2 | X_1, X_2\} \ldots P\{T_{n+1} - T_n \le t_{n+1} | X_n, X_{n+1}\} \quad (1.7)$$

for $t_1, \ldots, t_{n+1} \in \mathbb{R}_+$, and $n \in \mathbb{N}$.

<u>(1.8) Assumption</u>. $Q(\cdot,\cdot,\cdot)$ is a regular conditional probability on $\sigma(E) \times \mathscr{B}_+$. That is, the following hold.

a) For fixed $x \in E$, $Q(x,\cdot,\cdot)$ is a probability measure on $\sigma(E) \times \mathscr{B}_+$.

b) For fixed $S \in \sigma(E)$, $t \in \mathbb{R}_+$, $Q(\cdot,S,t)$ is Borel measurable.

<u>(1.9) Proposition</u>. Let $Q^{*n}(x,S,t) = P\{X_n \in S, T_n \le t | X_0 = x\}$. Then the following holds.

$$Q^{*n+1}(x,S,t) = \int_0^t \int_E Q^{*n}(x,da,dv) Q(a,S,t-v) \quad (1.10)$$

<u>Proof</u>: $Q^{*n+1}(x,S,t) = P\{X_{n+1} \in S, T_{n+1} \le t | X_0 = x\}$

$$= \int_0^t P\{X_{n+1} \in S, T_{n+1} \le t, X_n \in E, T_n \le v | X_0 = x\}$$

$$= \int_0^t \int_E P\{X_{n+1} \in S, T_{n+1} \le t | X_n = a, T_n = v, X_0 = x\}$$

$$\cdot P\{X_n \in (a,a+da], T_n \in (v,v+dv] | X_0 = x\}$$

The result follows by the Markov renewal property and the definition of $Q^{*n}(\cdot,\cdot,\cdot)$. □

It can be shown that $Q^{*n}(\cdot,\cdot,\cdot)$ for $n \in \mathbb{N}$ are regular conditional probabilities on $\sigma(E) \times \mathscr{B}_+$. $Q^{*n}(\cdot,\cdot,\cdot)$ is called the n-fold convolution of $Q(\cdot,\cdot,\cdot)$.

Now, let π_0 be a probability distribution on $\sigma(E)$ defined as follows.

<u>(1.11) Definition</u>. $\pi_0(B) = P\{X_0 \in B\}$, $B \in \sigma(E)$. $\pi_0(\cdot)$ is called the

initial distribution of the process.

(1.12) Proposition. Let $\pi_n(B) = P\{X_n \in B\}$. Then we have the following.

$$\pi_n(B) = \int_E \int_0^\infty Q^{*n}(a,B,dt)\pi_0(da). \tag{1.13}$$

Proof: $\pi_n(B) = P\{X_n \in B\}$

$$= \int_E \int_0^\infty P\{X_n \in B, T_n \in (t,t+dt] | X_0 = a\}\pi_0(da)$$

$$= \int_E \int_0^\infty Q^{*n}(a,B,dt)\pi_0(da). \quad \square$$

We now address the main problem in this section. Let A, B be two subsets of E such that the following holds.

$$A,B \in \sigma(E), \qquad A \cap B = \emptyset, \qquad A \cup B = E. \tag{1.14}$$

Thus $\mathscr{D} = \{A,B\}$ is a partition of E. Now define $N(t)$, $N_A(t)$ and $N_B(t)$ as follows.

$$N(t) = \sum_{n=1}^{\infty} I_{(0,t]}(T_n) \tag{1.15}$$

$$N_A(t) = \sum_{n=1}^{\infty} I_A(X_n) I_{(0,t]}(T_n) \tag{1.16}$$

$$N_B(t) = \sum_{n=1}^{\infty} I_B(X_n) I_{(0,t]}(T_n) \tag{1.17}$$

We say that $N_A(\cdot)$ and $N_B(\cdot)$ are the thinned processes obtained by the mark dependent thinning of the Markov renewal process N. The structures of N_A and N_B are both Markov renewal on the mark spaces A and B respectively. What we are interested in is the structure of the covariance between the two Markov renewal processes N_A and N_B. Analogous to Definition 3.1.5 we have the following.

(1.18) Definition. By the pointwise covariance structure of N_A and N_B

we mean the following.

$$R_{AB}(t,\tau) = \mathrm{cov}(N_A(t),N_B(t+\tau)). \tag{1.19}$$

To determine $R_{AB}(t,\tau)$ we have to compute $E[N_A(t)N_B(t+\tau)]$, $E(N_A(t)]$ and $E[N_B(t+\tau)]$. We do the last two first.

<u>(1.20) Proposition</u>. Let $R(a,A,t) = E[N_A(t)|X_0 = a]$ and, $R(b,B,t) = E[N_B(t)|X_0 = b]$. Then,

$$R(a,A,t) = \sum_{n=1}^{\infty} Q^{*n}(a,A,t) \tag{1.21a}$$

and,

$$R(b,B,t) = \sum_{n=1}^{\infty} Q^{*n}(b,B,t) \tag{1.21b}$$

<u>Proof</u>: $R(a,A,t) = E[N_A(t)|X_0 = a]$

$$= \sum_{n=1}^{\infty} E[I_A(X_n)I_{(0,t]}(T_n)|X_0 = a]$$

$$= \sum_{n=1}^{\infty} P\{X_n \in A, T_n \le t|X_0 = a\}$$

$$= \sum_{n=1}^{\infty} Q^{*n}(a,A,t).$$

1.21b can be established in similar fashion. □

<u>(1.22) Corollary</u>. Let $R(A,t) = E[N_A(t)]$ and $R(B,t) = E[N_B(t)]$. Then,

$$R(A,t) = \int_E R(a,A,t)\pi_0(da) \tag{1.23a}$$

and,

$$R(B,t) = \int_E R(b,B,t)\pi_0(db). \tag{1.23b}$$

<u>Proof</u>: $R(A,t) = E[N_A(t)]$

$$= \int_E E[N_A(t)|X_0 = a]P\{X_0 \in (a,a+da]\}$$

$$= \int_E R(a,A,t)\pi_0(da).$$

1.23b can be established in similar fashion. □

<u>(1.24) Proposition.</u>

$$E[N_A(t)N_B(t+\tau)] = \sum_{n=1}^{\infty}\sum_{m=1}^{n-1} \int_E \int_0^t \int_B Q^{*n-m}(b,A,t-u)Q^{*m}(c,db,du)\pi_0(dc)$$

$$+ \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} \int_E \int_0^{t+\tau} \int_A Q^{*m-n}(a,B,t+\tau-u)Q^{*n}(c,da,du)\pi_0(dc)$$

$$- \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} \int_E \int^{\tau} \int_A Q^{*m-n}(a,B,\tau-u)Q^{*n}(c,da,du)\pi_0(dc).$$

<u>Proof</u>: From 1.16 and 1.17 we have,

$$N_A(t)N_B(t+\tau) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} I_A(X_n)I_B(X_m)I_{(0,t]}(T_n)I_{(0,t+\tau]}(T_m).$$

Thus,

$$E[N_A(t)N_B(t+\tau)] = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} P\{X_n \varepsilon A, X_m \varepsilon B, T_n \le t, T_m \le t+\tau\}.$$

Since A and B are disjoint,

$$P\{X_n \varepsilon A, X_n \varepsilon B, T_n \le t, T_n \le t+\tau\} = 0.$$

Thus,

$$E[N_A(t)N_B(t+\tau)] = \sum_{n=1}^{\infty}\sum_{m=1}^{n-1} P\{X_n \varepsilon A, X_m \varepsilon B, T_n \le t, T_m \le t+\tau\}$$

$$+ \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} P\{X_n \varepsilon A, X_m \varepsilon B, T_n \le t, T_m \le t+\tau\}.$$

Now for $m < n$ we have almost surely,

$$T_m < T_n$$

since the process is simple. Thus,

$$\{T_n \le t\} \Longrightarrow \{T_m \le t\} \Longrightarrow \{T_m \le t+\tau\}.$$

Hence,

$$P\{X_n \varepsilon A, X_m \varepsilon B, T_n \leq t, T_m \leq t+\tau\} = P\{X_n \varepsilon A, X_m \varepsilon B, T_n \leq t\}$$

$$= P\{X_n \varepsilon A, X_m \varepsilon B, T_n \leq t, T_m \leq t\}.$$

Now,

$$P\{X_n \varepsilon A, X_m \varepsilon B, T_n \leq t, T_m \leq t\}$$

$$= \int_E P\{X_n \varepsilon A, X_m \varepsilon B, T_n \leq t, T_m \leq t | X_0 = c\} P\{X_0 \varepsilon (c, c+dc]\}$$

$$= \int_E \int_0^t \int_B P\{X_n \varepsilon A, T_n \leq t | X_m = b, T_m = u, X_0 = c\}$$

$$\cdot P\{X_m \varepsilon (b, b+db], T_m \varepsilon (u, u+du] | X_0 = c\} P\{X_0 \varepsilon (c, c+dc]\}$$

$$= \int_E \int_0^t \int_B Q^{*n-m}(b, A, t-u) Q^{*m}(c, db, du) \pi_0(dc).$$

The last step follows from the Markov renewal property.

For $m > n$,

$$P\{X_n \varepsilon A, X_m \varepsilon B, T_n \leq t, T_m \leq t+\tau\}$$

$$= \int_E P\{X_n \varepsilon A, X_m \varepsilon B, T_n \leq t, T_m \leq t+\tau | X_0 = c\} P\{X_0 \varepsilon (c, c+dc]\}$$

$$= \int_E \int_0^t \int_A P\{X_m \varepsilon B, T_m \leq t+\tau | X_n = a, T_n = u, X_0 = c\}$$

$$\cdot P\{X_n \varepsilon (a, a+da], T_n \varepsilon (u, u+du] | X_0 = c\} P\{X_0 \varepsilon (c, c+dc]\}$$

$$= \int_E \int_0^t \int_A Q^{*m-n}(a, B, t+\tau-u) Q^{*n}(c, da, du) \pi_0(dc)$$

$$= \int_E \int_0^{t+\tau} \int_A Q^{*m-n}(a, B, t+\tau-u) Q^{*n}(c, da, du) \pi_0(dc)$$

$$- \int_E \int_t^{t+\tau} \int_A Q^{*m-n}(a, B, t+\tau-u) Q^{*n}(c, da, du) \pi_0(dc).$$

Letting $v = u - t$ in the second term we have,

$$\int_E \int_t^{t+\tau} \int_A Q^{*m-n}(a,B,t+\tau-u)Q^{*n}(c,da,du)\pi_0(dc)$$

$$= \int_E \int_0^{\tau} \int_A Q^{*m-n}(a,B,\tau-v)Q^{*n}(c,da,dv)\pi_0(dc).$$

This completes the proof. □

Corollary 1.22 and Proposition 1.24 thus complete the covariance analysis.

We now use these developments to study a problem already dealt with in Chapter 3. In the process, however, we shall find that the concepts developed here enable us to solve a non-linear integral equation. It provides an interesting application for the results developed in this section. We proceed as follows.

Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of independent and identically distributed random variables taking values in a Polish space E with distribution π. Further, let $\{T_n\}_{n=1}^{\infty}$ form an ordinary renewal process independent of $\{X_n\}_{n=0}^{\infty}$. Then, we have the following.

(1.25) Proposition. a) The sequence $\{(X_n, T_n)\}_{n=1}^{\infty}$ forms a Markov renewal process.

b) $Q^{*n}(x,S,t) = \pi(S)F^{*n}(t)$ where $S \in \sigma(E), X \in E$ and $F(t) = P\{T_1 \leq t\}$.

c) $R(S,t) = \pi(S)R(t)$ where, $R(t) = \sum_{n=1}^{\infty} F^{*n}(t)$, the renewal function for the renewal process.

Proof: a) This is obvious.

b) $Q^{*n}(x,S,t) = \int_E P\{X_n \in S, T_n \leq t | X_0 = x\}\pi(dx)$

$$= \int_E P\{X_n \in S | X_0 = x\}P\{T_n \leq t | X_0 = x\}\pi(dx)$$

by the independence of X_n and T_n. Thus,

$$Q^{*n}(x,S,t) = \int_E \pi(S)F^{*n}(t)\pi(dx)$$

$$= \pi(S)F^{*n}(t) \int_E \pi(dx)$$

$$= \pi(S)F^{*n}(t).$$

c) $R(S,t) = \int_E R(a,S,t)\pi(da)$ by 1.23a. Thus, from 1.21a and b above we have,

$$R(S,t) = \int_E \sum_{n=1}^{\infty} Q^{*n}(a,S,t)\pi(da)$$

$$= \int_E \sum_{n=1}^{\infty} \pi(S)F^{*n}(t)\pi(da)$$

$$= \pi(S) \sum_{n=1}^{\infty} F^{*n}(t) \int_E \pi(da)$$

$$= \pi(S)R(t). \quad \square$$

Now by using Proposition 1.24, we can compute $E[N_A(t)N_B(t+\tau)]$.

$$E[N_A(t)N_B(t+\tau)] = \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \int_E \int_0^t \int_B F^{*n-m}(t-u)\pi(A)F^{*m}(du)\pi(db)\pi(dc)$$

$$+ \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \int_E \int_0^{t+\tau} \int_A F^{*m-n}(t+\tau-u)\pi(B)F^{*n}(du)\pi(da)\pi(dc)$$

$$- \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \int_E \int^{\tau} \int_A F^{*m-n}(\tau-u)\pi(B)F^{*n}(du)\pi(da)\pi(dc).$$

Simplifying we have the following.

$$E[N_A(t)N_B(t+\tau)] = \pi(A)\pi(B)[\sum_{n=1}^{\infty} \sum_{m=1}^{n-1} F^{*n}(t) + \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} F^{*m}(t+\tau)$$

$$- \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} F^{*m}(\tau)]. \tag{1.26}$$

To complete the picture we give the following result.

<u>(1.27) Proposition</u>. $\sum_{n=1}^{\infty} \sum_{m=1}^{n-1} F^{*n}(t) = \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} F^{*m}(t) = R * R(t).$

<u>Proof</u>: $\sum_{n=1}^{\infty} \sum_{m=1}^{n-1} F^{*n}(t) = \sum_{n=1}^{\infty} (n-1)F^{*n}(t)$

$$= F^{*2}(t) + 2F^{*3}(t) + \cdots$$

$$= \sum_{n=1}^{\infty} F^{*(n+1)}(t) + \sum_{n=1}^{\infty} F^{*(n+2)}(t) + \cdots$$

$$= (F * \sum_{n=1}^{\infty} F^{*n})(t) + (F^{*2} * \sum_{n=1}^{\infty} F^{*n})(t) + \cdots$$

$$= [(F + F^{*2} + \cdots) * \sum_{n=1}^{\infty} F^{*n}](t)$$

$$= R * R(t).$$

Also,

$$\sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} F^{*m}(t) = \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} F^{*m}(t),$$

where the equality is obtained by interchanging summations on the left hand side. Thus,

$$\sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} F^{*m}(t) = \sum_{m=2}^{\infty} (m-1)F^{*m}(t)$$

$$= R * R(t)$$

(see above). □

From Propositions 1.25 and 1.27 we have the following.

$$\operatorname{cov}(N_A(t), N_B(t+\tau)) =$$

$$\pi(A)\pi(B)[R^{*2}(t) + R^{*2}(t+\tau) - R^{*2}(\tau) - R(t)R(t+\tau)]. \tag{1.28}$$

Notice that the thinning operation described here is simply a Bernoulli thinning of a renewal process. Thus, by Theorem 3.1.12, the following is true.

$$\mathrm{cov}(N(t),N(t+\tau)) - E[N(t)] = R^{*2}(t) + R^{*2}(t+\tau) - R^{*2}(\tau) - R(t)R(t+\tau). \qquad (1.29)$$

But by Corollary 3.4.6 we know that the left hand side is zero if and only if $R(t) = t/m$. We have thus proved the following result.

<u>(1.30) Theorem</u>. Let $R(t)$ be a renewal function. Then the unique solution to the non-linear integral equation

$$R^{*2}(t) + R^{*2}(t+\tau) - R^{*2}(\tau) = R(t)R(t+\tau) \qquad (1.31)$$

is linear and unique up to a multiplicative constant.

<u>(1.32) Remark</u>. If $\tau = 0$, 1.31 reduces to,

$$2R * R(t) = [R(t)]^2.$$

2. <u>Countable Mark Spaces</u>

We now reduce the main results (Corollary 1.22 and Proposition 1.24) to the important special case where E is a countable space. In this case the semi-Markov kernel (Definition 1.4) can be regarded as a matrix of functions $\mathbf{Q}(t)$. We can then use the partition of the state space to partition this matrix and the initial probability vector. We now clarify these ideas. Let,

$$Q(i,j,t) = P\{X_{n+1} = j, T_{n+1} - T_n \le t \mid X_n = i\} \qquad (2.1)$$

$$\mathbf{Q}(t) = \left[\begin{array}{c|c} \mathbf{Q}_{AA}(t) & \mathbf{Q}_{AB}(t) \\ \hline \mathbf{Q}_{BA}(t) & \mathbf{Q}_{BB}(t) \end{array}\right] \qquad (2.2)$$

$$\pi_0 = (\pi_0(j);\ j \in E) = (\pi_A \mid \pi_B) \qquad (2.3)$$

The n-fold convolutions of the elements in the kernel are given by the following.

$$Q^{*n+1}(i,j,t) = P\{X_{n+1} = j, T_{n+1} \le t | X_0 = i\}$$

$$= \sum_{k\varepsilon E} \int_0^t Q^{*n}(i,k,dx)Q(k,j,t-x). \qquad (2.4)$$

Clearly, the n-step transition matrix can also be partitioned analogous to $\mathbf{Q}(t)$.

$$\mathbf{Q}^{*n}(t) = \left[\begin{array}{c|c} \mathbf{Q}_{AA}^{*n}(t) & \mathbf{Q}_{AB}^{*n}(t) \\ \hline \mathbf{Q}_{BA}^{*n}(t) & \mathbf{Q}_{BB}^{*n}(t) \end{array}\right]$$

It should be noted that $\mathbf{Q}_{AA}^{*n}$ is not the n-fold convolution of $\mathbf{Q}_{AA}$, but rather the restriction of $\mathbf{Q}^{*n}(t)$ to the set A. Similar remarks hold for the other elements in the partition. Finally, let U_A and U_B be column vectors all of whose elements are unity and the number of such elements is the same as the number of elements in A and B respectively.

<u>(2.5) Proposition</u>. $R(A,t) = E[N_A(t)] = \sum_{n=1}^{\infty} (\pi_A \mathbf{Q}_{AA}^{*n} + \pi_B \mathbf{Q}_{BA}^{*n})(t)U_A$.
$R(B,t+\tau) = E[N_B(t+\tau)] = \sum_{m=1}^{\infty} (\pi_A \mathbf{Q}_{AB}^{*m} + \pi_B \mathbf{Q}_{BB}^{*m})(t+\tau)U_B$.

<u>Proof</u>: From 1.23a

$$R(A,t) = \int_E R(i,A,t)\pi_0(di)$$

$$= \sum_{i\varepsilon E} \sum_{j\varepsilon A} R(i,j,t)\pi_0(i)$$

$$= \sum_{n=1}^{\infty} \sum_{i\varepsilon E} \sum_{j\varepsilon A} Q^{*n}(i,j,t)\pi_0(i)$$

$$= \sum_{n=1}^{\infty} [\sum_{i\varepsilon A} \sum_{j\varepsilon A} Q^{*n}(i,j,t)\pi_0(i)$$

$$+ \sum_{i\varepsilon B} \sum_{j\varepsilon A} Q^{*n}(i,j,t)\pi_0(i)]$$

$$= \sum_{n=1}^{\infty} (\pi_A \mathbf{Q}_{AA}^{*n} + \pi_B \mathbf{Q}_{BA}^{*n})(t)U_A.$$

$R(B,t)$ can be established in similar fashion. □

(2.6) Proposition.

$$E[N_A(t)N_B(t+\tau)] = \sum_{n=1}^{\infty}\sum_{m=1}^{n-1} [(\pi_A \mathbf{Q}_{AB}^{*m} + \pi_B \mathbf{Q}_{BB}^{*m}) * \mathbf{Q}_{BA}^{*n-m}](t)U_A$$

$$+ \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} [(\pi_A \mathbf{Q}_{AA}^{*n} + \pi_B \mathbf{Q}_{BA}^{*n}) * \mathbf{Q}_{AB}^{*m-n}](t+\tau)U_B$$

$$- \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} [(\pi_A \mathbf{Q}_{AA}^{*n} + \pi_B \mathbf{Q}_{BA}^{*n}) * \mathbf{Q}_{AB}^{*m-n}](\tau)U_B.$$

Proof: Consider the first term of $E[N_A(t)N_B(t+\tau)]$ as given in Proposition 1.24.

$$\int_E \int_0^t \int_B Q^{*n-m}(b,A,t-u)Q^{*m}(c,db,du)\pi_0(dc)$$

$$= \sum_{c\varepsilon E} \int_0^t \sum_{b\varepsilon B}\sum_{a\varepsilon A} Q^{*n-m}(b,a,t-u)Q^{*m}(c,b,du)\pi_0(c)$$

$$= \sum_{c\varepsilon A}\sum_{b\varepsilon B}\sum_{a\varepsilon A} \int_0^t Q^{*n-m}(b,a,t-u)Q^{*m}(c,b,du)\pi_0(c)$$

$$+ \sum_{c\varepsilon B}\sum_{b\varepsilon B}\sum_{a\varepsilon A} \int_0^t Q^{*n-m}(b,a,t-u)Q^{*m}(c,b,du)\pi_0(c)$$

$$= \pi_A \mathbf{Q}_{AB}^{*m} * \mathbf{Q}_{BA}^{*n-m}(t)U_A + \pi_B * \mathbf{Q}_{BB}^{*m} * \mathbf{Q}_{BA}^{*n-m}(t)U_A$$

$$= [(\pi_A \mathbf{Q}_{AB}^{*m} + \pi_B \mathbf{Q}_{BB}^{*m}) * \mathbf{Q}_{BA}^{*n-m}](t)U_A.$$

The other terms may be obtained in the same way. □

Propositions 2.5 and 2.6 complete the covariance structure.

We now have the tools necessary to analyse a more general version of the problem studied in Chapter 3. We first introduce the notion of a Markov renewal kernel.

(2.7) Definition. Let $\mathbf{R}(t) = \sum_{n=1}^{\infty} \mathbf{Q}^{*n}(t)$. The matrix $\mathbf{R}(t)$ is called the Markov renewal kernel. Its elements, $R(i,j,t)$ are given by,

$$R(i,j,t) = E[N_{\{j\}}(t)|X_0=i] = \sum_{n=1}^{\infty} Q^{*n}(i,j,t).$$

Now let $\{(X_n,T_n)\}_{n=0}^{\infty}$ be a Markov renewal process with a countable mark space E, semi-Markov kernel $\mathbf{Q}(\cdot)$ and Markov renewal kernel $\mathbf{R}(\cdot)$. Further, let $\{Y_n\}_{n=0}^{\infty}$ be a Bernoulli process with $P\{Y_n=0\} = p = 1 - q = 1 - P\{Y_n=1\}$. Finally, let $\{Y_n\}_{n=0}^{\infty}$ and $\{(X_n,T_n)\}_{n=0}^{\infty}$ be independent.

<u>(2.8) Proposition</u>. $\{(Y_n,X_n,T_n)\}_{n=0}^{\infty}$ is a Markov renewal process with mark space $\hat{E} = \{0,1\} \times E$.

<u>Proof</u>: Is trivial.

The semi-Markov kernel and initial distribution for $\{(Y_n,X_n,T_n)\}_{n=0}^{\infty}$ is given as follows.

$$\hat{\mathbf{Q}}(t) = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \left[\begin{array}{c|c} p\,\mathbf{Q}(t) & q\,\mathbf{Q}(t) \\ \hline p\,\mathbf{Q}(t) & q\,\mathbf{Q}(t) \end{array}\right] \end{array} \tag{2.9}$$

$$\hat{\pi} = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ (p\pi \mid q\pi) \end{array} \tag{2.10}$$

If we let $\hat{A} = \{0\} \times E$ and $\hat{B} = \{1\} \times E$, then $N_{\hat{A}}(\cdot)$ and $N_{\hat{B}}(\cdot)$ are Markov renewal processes on E and are the thinned processes resulting from an independent Bernoulli thinning of the process $\{(X_n,T_n)\}_{n=0}^{\infty}$. From Propositions 2.5 and 2.6, letting $U_{\hat{A}} = U_{\hat{B}} = U = (1,1,\ldots)^T$, we deduce the covariance structure of $N_{\hat{A}}$ and $N_{\hat{B}}$.

$$\begin{aligned} \operatorname{cov}(N_{\hat{A}}(t),N_{\hat{B}}(t+\tau)) = {} & \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} pq\pi\, \mathbf{Q}^{*n}(t)U \\ & + \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \pi\, \mathbf{Q}^{*m}(t+\tau)U - \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} pq\pi\, \mathbf{Q}^{*n}(\tau) \\ & - \sum_{n=1}^{\infty} p\pi\, \mathbf{Q}^{*n}(t)U \sum_{m=1}^{\infty} q\pi\, \mathbf{Q}^{*m}(t+\tau)U. \end{aligned}$$

It is a trivial matter to show that,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{n-1} \mathbf{Q}^{*n}(t) = \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} \mathbf{Q}^{*m}(t) = \mathbf{R} * \mathbf{R}(t) = \mathbf{R}^{*2}(t). \quad (2.11)$$

The proof of Proposition 1.27 goes through unchanged if we replace F(t) by $\mathbf{Q}(t)$. Thus, we have the following.

$$\begin{aligned}\operatorname{cov}(N_{\hat{A}}(t), N_{\hat{B}}(t+\tau)) = pq\{&\pi \mathbf{R}^{*2}(t)U + \pi \mathbf{R}^{*2}(t+\tau)U - \pi \mathbf{R}^{*2}(\tau)U \\ &- (\pi \mathbf{R}(t)U)(\pi \mathbf{R}(t+\tau)U)\}. \quad (2.12)\end{aligned}$$

Note the similarity with 1.28. It is, of course, an interesting question if $\operatorname{cov}(N_{\hat{A}}(t), N_{\hat{B}}(t+\tau)) = 0$ for all t, τ implies that the Markov renewal process $\{(X_n, T_n)\}_{n=0}^{\infty}$ is Poisson. To satisfactorily resolve the issue would involve questions about lumpability of Markov renewal processes which are beyond the scope of the document. We leave the question unanswered at this juncture and as an interesting avenue for future research.

3. Variances of $N_A(t)$ and $N_B(t)$

In applications it is almost always more useful to study correlations than covariances. This is especially true for studying asymptotic behaviour. To get correlations we have to be able to compute variances. Thus, for the problem studied in the previous two sections, we need $\operatorname{var}[N_A(t)]$ and $\operatorname{var}[N_B(t+\tau)]$, where A and B are as in 1.14. Then, of course,

$$\begin{aligned} r_{AB}(t,\tau) &= \operatorname{corr}(N_A(t), N_B(t+\tau)) \\ &= \frac{\operatorname{cov}(N_A(t), N_B(t+\tau))}{\sqrt{\operatorname{var}[N_A(t)]\operatorname{var}[N_B(t+\tau)]}} . \quad (3.1)\end{aligned}$$

In this section we compute $\operatorname{var}[N_A(t)]$ and $\operatorname{var}[N_B(t)]$ for the Markov renewal process $\{(X_n, T_n)\}_{n=1}^{\infty}$ with mark space E. We do this only for the case where E is countable.

$$\operatorname{var}[N_A(t)] = E[N_A^2(t)] - \{E[N_A(t)]\}^2$$

$$= E[N_A^2(t)] - R(A,t) \tag{3.2}$$

where $R(A,t)$ is given by Proposition 2.5. Thus, we need only compute $E[N_A^2(t)]$.

Now,

$$N_A(t) = \sum_{n=1}^{\infty} I_A(X_n) I_{(0,t]}(T_n)$$

$$= \sum_{m=1}^{\infty} I_A(X_m) I_{(0,t]}(T_m).$$

Thus,

$$N_A^2(t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} I_A(X_n) I_A(X_m) I_{(0,t]}(T_n) I_{(0,t]}(T_m).$$

Taking expectations gives,

$$E[N_A^2(t)] = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P\{X_n \varepsilon A, X_m \varepsilon B, T_n \leq t, T_m \leq t\}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{i\varepsilon A} \sum_{j\varepsilon B} P\{X_n = i, X_m = j, T_n \leq t, T_m \leq t\}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \sum_{i\varepsilon A} \sum_{j\varepsilon B} P\{X_n = i, X_m = j, T_n \leq t, T_m \leq t\}$$

$$+ \sum_{n=1}^{\infty} \sum_{i\varepsilon A} \sum_{j\varepsilon B} P\{X_n = i, X_n = j, T_n \leq t, T_n \leq t\}$$

$$+ \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \sum_{i\varepsilon A} \sum_{j\varepsilon B} P\{X_n = i, X_m = j, T_n \leq t, T_m \leq t\}.$$

We compute each summation separately.

Case 1. $m < n$.

$$P\{X_n = i, X_m = j, T_n \le t, T_m \le t\}$$

$$= \int_0^t P\{X_n = i, T_n \le t \mid X_m = j, T_m = x\} dP\{X_m = j, T_m \le x\}$$

$$= \int_0^t Q^{*n-m}(j,i,t-x) \sum_{k \in E} \pi(k) Q^{*m}(k,j,dx)$$

$$= \int_0^t Q^{*n-m}(j,i,t-x) \sum_{k \in A} \pi(k) Q^{*m}(k,j,dx)$$

$$+ \int_0^t Q^{*n-m}(j,i,t-x) \sum_{k \in B} \pi(k) Q^{*m}(k,j,dx).$$

Thus,

$$\sum_{i \in A} \sum_{j \in B} P\{X_n = i, X_m = j, T_n \le t, T_m \le t\} = [\pi_A \mathbf{Q}_{AA}^{*n}(t) + \pi_B \mathbf{Q}_{BA}^{*m} * \mathbf{Q}_{AA}^{*n-m}(t)] U_A.$$

Case 2. $m = n$.

$$P\{X_n = i, X_m = j, T_n \le t, T_m \le t\} = \delta_{\{i\}}(j) P\{X_n = i, T_n \le t\}$$

where δ is the Kronecker delta. Then,

$$P\{X_n = i, X_m = j, T_n \le t, T_m \le t\} = \delta_{\{i\}}(j) \sum_{k \in E} \pi(k) Q^{*n}(k,j,t)$$

$$= \delta_{\{i\}}(j) [\sum_{k \in A} \pi(k) Q^{*n}(k,j,t)$$

$$+ \sum_{k \in B} \pi(k) Q^{*n}(k,j,t)].$$

Thus,

$$\sum_{i \in A} \sum_{j \in A} P\{X_n = i, X_m = j, T_n \le t, T_m \le t\} = [\pi_A \mathbf{Q}_{AA}^{*n}(t) + \pi_B \mathbf{Q}_{BB}^{*n}(t)] U_A$$

and from 2.5,

$$\sum_{n=1}^{\infty} [\pi_A \mathbf{Q}_{AA}^{*n}(t) + \pi_B \mathbf{Q}_{BB}^{*n}(t)] U_A = R(A,t).$$

Case 3. $m > n$.

An analysis similar to Case 1 produces,

$$\sum_{i\varepsilon A}\sum_{j\varepsilon B} P\{X_n = i, X_m = j, T_n \le t, T_m \le t\} = [\pi_A \mathbf{Q}_{AA}^{*m}(t) + \pi_B \mathbf{Q}_{BA}^{*n} * \mathbf{Q}_{AA}^{*m-n}(t)]U_A.$$

Putting the pieces together we have from 3.1,

$$\begin{aligned} \mathrm{var}[N_A(t)] = &\sum_{n=1}^{\infty}\sum_{m=1}^{n-1} [\pi_A \mathbf{Q}_{AA}^{*n} + \pi_B \mathbf{Q}_{BA}^{*m} * \mathbf{Q}_{AA}^{*n-m}](t)U_A \\ &+ R(A,t) + \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} [\pi_A \mathbf{Q}_{AA}^{*m} + \pi_B \mathbf{Q}_{BA}^{*n} * \mathbf{Q}_{AA}^{*m-n}](t)U_A \\ &- [R(A,t)]^2. \end{aligned} \tag{3.3}$$

The analysis for $\mathrm{var}[N_B(t)]$ is similar. We simply state the result and refrain from giving the computational details.

$$\begin{aligned} \mathrm{var}[N_B(t)] = &\sum_{n=1}^{\infty}\sum_{m=1}^{n-1} [\pi_A \mathbf{Q}_{AB}^{*m} * \mathbf{Q}_{BB}^{*n-m} + \pi_B \mathbf{Q}_{BB}^{*n}](t)U_B + R(B,t) \\ &+ \sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty} [\pi_A \mathbf{Q}_{AB}^{*m} * \mathbf{Q}_{BB}^{*n-m} + \pi_B \mathbf{Q}_{BB}^{*m}](t)U_B \\ &- [R(B,t)]^2. \end{aligned} \tag{3.4}$$

Propositions 2.5, 2.6 and Equations 3.3 and 3.4 now complete the analysis required for $r_{AB}(t,\tau)$ as defined in 3.1.

4. Summary

This chapter has substantially resolved the issue of the covariance properties resulting upon a mark dependent thinning of a Markov renewal process. In subsequent chapters we apply these tools to analyse some interesting queueing problems. We shall also take up some problems in renewal theory and Markov processes and see that interesting results can indeed be obtained from what seems a hopelessly intractable working tool.

CHAPTER 5

APPLICATIONS TO QUEUEING THEORY-THE OVERFLOW QUEUE

In this chapter we shall use the tools of the last chapter to study a queueing problem, the overflow queue. We shall study the covariance structure between the input and overflow processes and between the output and overflow processes. In both cases we shall present graphs of the correlation functions and give some interesting asymptotic results. An interesting feature of the analysis for the covariance between output and overflow streams is that we shall in fact be studying a superposition problem with the tools developed for a thinning problem. As a consequence, we shall be able to get a surprising result involving superposition of renewal processes.

1. The Overflow Queue

Informally, an overflow queue is one in which the system has a finite number, N, of waiting places. Thus, when a customer arrives to find all N waiting places occupied, he leaves without service or "overflows". Figure 5.1.1 is a schematic representation of the system.

We now formalise the system. Let $(\Omega,\mathscr{F},P)$ be a probability space. Let the arrival process be the point process N_a and $\{A_n\}_{n=1}^{\infty}$ be the sequence of arrival times. Let $\{S_m\}_{m=1}^{\infty}$ be the sequence of service times.

Let J_ℓ be the time of the ℓ-th successful entry into the system. This means that the customer arriving at J_ℓ does not overflow. The sequence $\{J_\ell\}_{\ell=1}^{\infty}$ is called the <u>input sequence</u>.

Let C_i be the time of the i-th overflow. $\{C_i\}_{i=1}^{\infty}$ is called the <u>overflow sequence</u>. Clearly,

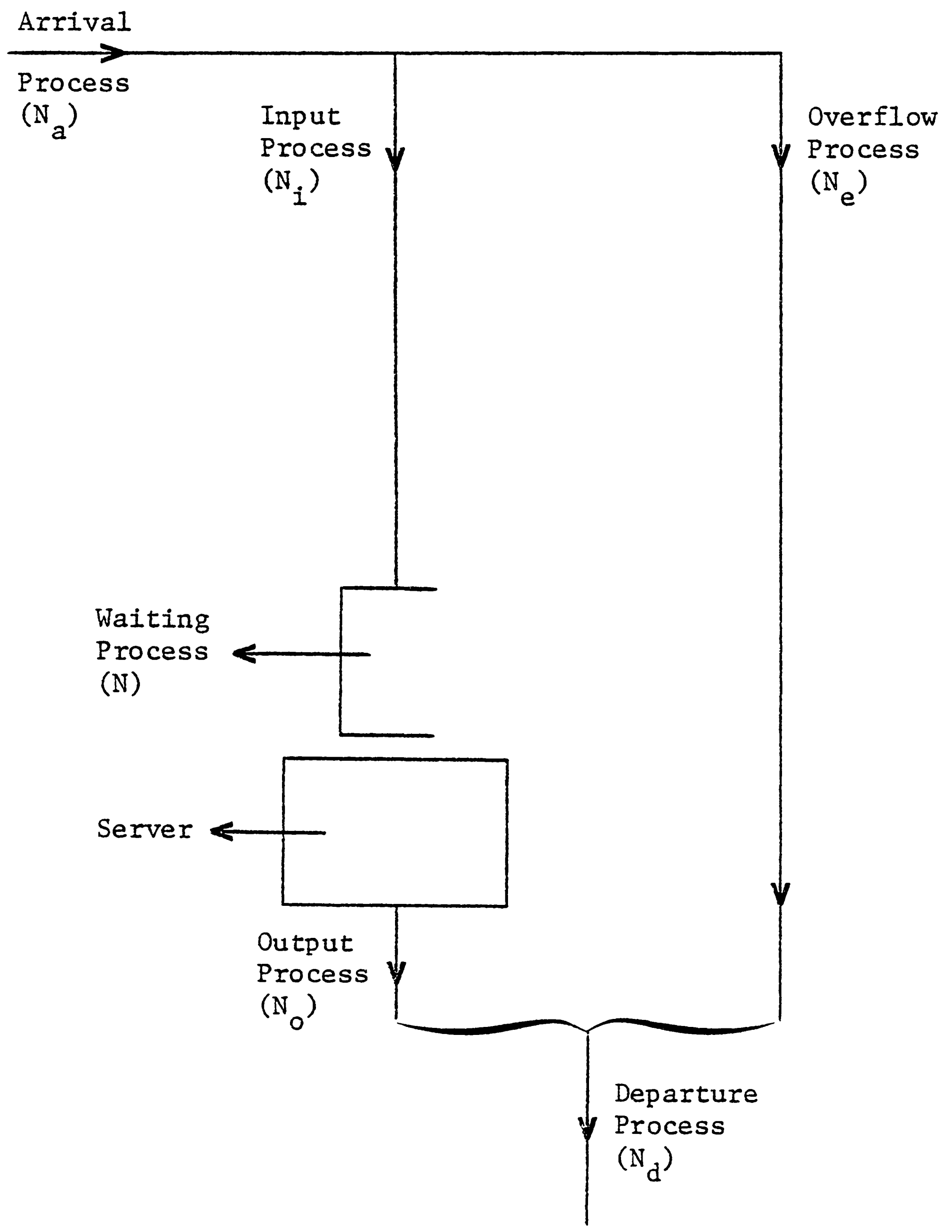

Figure 5.1.1

An Overflow Queue

$$\{A_n\}_{n=1}^{\infty} = \{J_\ell\}_{\ell=1}^{\infty} \cup \{C_i\}_{i=1}^{\infty} . \tag{1.1}$$

Let D_k be the time of the k-th service completion. The sequence $\{D_k\}_{k=1}^{\infty}$ is called the <u>output sequence</u>.

Now, if T_j is the time when the j-th customer leaves the system, either as an overflow or after being served, the sequence $\{T_j\}_{j=1}^{\infty}$ can be written as follows.

$$\{T_j\}_{j=1}^{\infty} = \{D_k\}_{k=1}^{\infty} \cup \{C_i\}_{i=1}^{\infty} . \tag{1.2}$$

$\{T_j\}_{j=1}^{\infty}$ is called the <u>departure sequence</u>.

Now, if $N_i(t)$, $N_a(t)$, $N_o(t)$, $N_c(t)$ and $N_d(d)$ are the number of inputs, arrivals, outputs, overflows and departures in $(0,t]$ respectively, then they are related to the appropriate sequences by 2.1.17a.

<u>(1.3) Definition</u>. Let the queue described by Figure 5.1.1 satisfy the following.

a) N_a is a renewal process with $F(t) = P\{A_{n+1} - A_n \leq t\}$, $n \geq 0$

b) $\{S_m\}_{m=1}^{\infty}$ is a sequence of independent and identically distributed random variables with exponential distributions. Let $E(S_m) = 1/\mu$.

c) $N = 0$.

The system is called a GI/M/1/0 queue.

In the next section we study the covariance between N_i and N_c, the input and overflow processes for the GI/M/1/0 queue.

2. The Covariance Between N_i and N_c for the GI/M/1/0 Queue

From 1.1 and Figure 5.1.1 it is clear that the input and overflow processes are obtained by thinning the arrival process. We can use this observation to model the situation in a form appropriate enough to use the results of Chapter 4 to get the information needed. We proceed as follows.

Let,

$$I_n = \begin{cases} 1 & \text{if arrival at } A_n \text{ is an input} \\ 0 & \text{if arrival at } A_n \text{ is an overflow.} \end{cases} \tag{2.1}$$

(2.2) Definition. Let $(I,A) = \{(I_n,A_n)\}_{n=1}^{\infty}$. Then (I,A) is called the arrival process.

(2.3) Theorem. The arrival process, (I,A), is a Markov renewal process with mark space $E = \{0,1\}$.

Proof: See Disney and Simon [1982]. □

Now, partition E into $\hat{A}$ and $\hat{B}$ as follows.

$$\hat{A} = \{0\},\ \hat{B} = \{1\}$$

Then clearly, $N_{\hat{A}}(t)$ and $N_{\hat{B}}(t)$ are the number of overflows and inputs in $(0,t]$ respectively. Thus, $N_{\hat{A}} = N_c$ and $N_{\hat{B}} = N_i$ and,

$$\operatorname{cov}(N_{\hat{A}}(t),N_{\hat{B}}(t+\tau)) = \operatorname{cov}(N_c(t),N_i(t+\tau)).$$

Let $\mathbf{Q}(t) = \{Q(i_1,i_2,t),\ i_1,i_2 \in \{0,1\},\ t \geq 0\}$ be the semi-Markov kernel for the process (I,A).

(2.4) Proposition.

$$\mathbf{Q}(t) = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \int_0^t e^{-\mu x}dF(x) & \int_0^t (1-e^{-\mu x})dF(x) \\ \int_0^t e^{-\mu x}dF(x) & \int_0^t (1-e^{-\mu x})dF(x) \end{bmatrix} \end{array}$$

Proof: $Q(i_1,i_2,t) = P\{I_{n+1} = i_2, A_{n+1}-A_n \leq t | I_n = i_1\}$

$$= \int_0^t P\{I_{n+1} - i_2 | A_{n+1}-A_n = x, I_n = i_1\}$$

$$\cdot\ dP\{A_{n+1}-A_n \leq x | I_n = i_1\}.$$

Clearly, $A_{n+1}-A_n$ is independent of I_n. Thus,

$$P\{A_{n+1}-A_n \leq x | I_n = i_1\} = F(x),\ i_1 = 0,1.$$

Case 1. Let $i_1 = 0$, $I_2 = 0$. That is, the arrivals at both A_n and A_{n+1} overflow. Thus, for I_{n+1} to be 0 the service time of the customer present at A_n must be bigger than $A_{n+1}-A_n$. This gives.

$$P\{I_{n+1} = 0 | A_{n+1}-A_n = x, I_n = 0\} = e^{-\mu x}$$

and hence,

$$Q(0,0,t) = \int_0^t e^{-\mu x} dF(x)$$

Case 2. Let $i_1 = 0$, $i_2 = 1$. That is, the arrival at A_n overflows but the one at A_{n+1} enters service. But since,

$$P\{I_{n+1} = 1 | A_{n+1}-A_n = x, I_n = 0\} = 1 - P\{I_n = 0 | A_{n+1}-A_n = x, I_n = 0\},$$

we have,

$$Q(0,1,t) = \int_0^t (1-e^{-\mu x}) dF(x)$$

Case 3. Let $i_1 = 1$, $i_2 = 0$. That is, the arrival at A_n enters service but the one at A_{n+1} overflows. An argument similar to case 1 gives,

$$Q(1,0,t) = \int_0^t e^{-\mu x} dF(x).$$

Case 4. Let $i_1 = 1$, $i_2 = 1$. That is, the arrivals at both A_n and A_{n+1} enter service. The argument here is similar to case 2 and we have,

$$Q(1,1,t) = \int_0^t (1-e^{-\mu x})dF(x)$$

This completes the proof. □

Let

$\tilde{F}(s)$ = Laplace-Stieltjes transform of $F(t)$, (2.5)

$G(t) = \int_0^t e^{-\mu x}dF(x)$, (2.6)

$W(t) = F(t) - G(t)$, (2.7)

$\tilde{G}(s)$ = Laplace-Stieltjes transform of $G(t)$, (2.8)

$\tilde{W}(s)$ = Laplace-Stieltjes transform of $W(t)$. (2.9)

Then, if $\tilde{\mathbf{Q}}(s)$ is the Laplace-Stieltjes transform of $Q(t)$ we have,

$$\tilde{\mathbf{Q}}(s) = \begin{bmatrix} \tilde{G}(s) & \tilde{W}(s) \\ \tilde{G}(s) & \tilde{W}(s) \end{bmatrix} . \tag{2.10}$$

<u>(2.11) Proposition</u>.

$$[\tilde{\mathbf{Q}}(s)]^n = \begin{bmatrix} \tilde{G}(s)[\tilde{F}(s)]^{n-1} & \tilde{W}(s)[\tilde{F}(s)]^{n-1} \\ \tilde{G}(s)[\tilde{F}(s)]^{n-1} & \tilde{W}(s)[\tilde{F}(s)]^{n-1} \end{bmatrix} .$$

<u>Proof</u>: Clearly, the result is true for $n = 1$. Now let it be true for $n = k$. Then,

$$[\tilde{\mathbf{Q}}(s)]^{k+1} = \tilde{\mathbf{Q}}(s)[\tilde{\mathbf{Q}}(s)]^k.$$

By the induction hypothesis,

$$[\tilde{\mathbf{Q}}(s)]^{k+1} = \begin{bmatrix} \tilde{G}(s) & \tilde{W}(s) \\ \tilde{G}(s) & \tilde{W}(s) \end{bmatrix} \begin{bmatrix} \tilde{G}(s)[\tilde{F}(s)]^{k-1} & \tilde{W}(s)[\tilde{F}(s)]^{k-1} \\ \tilde{G}(s)[\tilde{F}(s)]^{k-1} & \tilde{W}(s)[\tilde{F}(s)]^{k-1} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{G}(s)[\tilde{F}(s)]^k & \tilde{W}(s)[\tilde{F}(s)]^k \\ \tilde{G}(s)[\tilde{F}(s)]^k & \tilde{W}(s)[\tilde{F}(s)]^k \end{bmatrix},$$

and the result follows by induction. □

Let $\pi = (\pi_0, \pi_1)$ be the initial distribution of the system. That is,

$$\pi_0 = P\{I_0 = 0\}$$

and

$$\pi_1 = P\{I_0 = 1\} = 1 - \pi_0.$$

We are now in a position to use 4.1.23 and 4.1.24 to get the result of interest. If we take Laplace-Stieltjes tranforms of 4.1.23 and 4.1.25, compute the results in transforms and invert, we can get $\mathrm{cov}(N_c(t), N_i(t+\tau))$. We give the result for $F(t) = 1-e^{-\lambda t}$. Computational details are omitted.

Let, E_β^1 and E_β^2 be exponential and Erlang-2 distributions respectively with parameter β. Define $H(t)$ as follows.

$$H(t) = At + \frac{B}{2}t^2 + \frac{c}{(\lambda+\mu)}E^1_{(\lambda+\mu)}(t) + \frac{D}{(\lambda+\mu)^2}E^2_{(\lambda+\mu)}(t) \qquad (2.12)$$

where,

$$A = \frac{\lambda^2\mu(\mu-\lambda)}{(\lambda+\mu)^3}$$

$$B = \frac{\lambda^3\mu}{(\lambda+\mu)^2}$$

$$C = \frac{\lambda^2\mu(\lambda-\mu)}{(\lambda+\mu)^3}$$

$$D = -\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 .$$

Then,

$$E[N_c(t)N_i(t+\tau)] = H(t) + H(t+\tau) - H(\tau). \tag{2.13}$$

Further,

$$E[N_c(t)] = \frac{\lambda^2}{\lambda+\mu}t + \frac{\lambda\mu}{(\lambda+\mu)^2}E^1_{(\lambda+\mu)}(t) \tag{2.14}$$

and

$$E[N_i(t)] = \frac{\lambda\mu}{\lambda+\mu}t - \frac{\lambda\mu}{(\lambda+\mu)^2}E^1_{(\lambda+\mu)}(t). \tag{2.15}$$

Notice that,

$$E[N_c(t)] + E[N_i(t)] = \frac{\lambda(\lambda+\mu)}{(\lambda+\mu)}t = \lambda t$$

as expected. Further notice that the covariance function is independent of the initial distribution of the system.

To find asymptotic properties, we would like to find the correlation between the processes N_c and N_i. Since,

$$\operatorname{corr}(N_c(t),N_i(t+\tau)) = \frac{\operatorname{cov}(N_c(t),N_i(t+\tau))}{\sqrt{\operatorname{var}(N_c(t))\operatorname{var}(N_i(t+\tau))}} \tag{2.16}$$

we need to compute the variances of $N_c(t)$ and $N_i(t+\tau)$. This we do using 4.3.3 and 4.3.4. Applying those results to the problem at hand we can find the following. Once again, we dispense with the computational details.

Let,

$$H_1(t) = A_1t + \frac{B_1}{2}t^2 + \frac{C_1}{(\lambda+\mu)}E^1_{(\lambda+\mu)}(t) + \frac{D_1}{(\lambda+\mu)^2}E^2_{(\lambda+\mu)}(t)$$

where,

$$A_1 = \frac{\lambda}{(\lambda+\mu)^4}[\lambda(\lambda+\mu)^2(\pi_0\mu+2\lambda) - 2\lambda^2(\lambda+\mu)(\lambda+\pi_0\mu)]$$

$$B_1 = \frac{\lambda_3}{(\lambda+\mu)^2}[\lambda+\mu\pi_0]$$

$$C_1 = \frac{\lambda}{(\lambda+\mu)^4}[\lambda\mu(\lambda+\mu)^2(\pi_0-2) + 2\lambda\mu^2(\lambda+\mu)\pi_1]$$

$$D_1 = (\frac{\lambda\mu}{\lambda+\mu})^2\pi_1 .$$

Further, let

$$H_2(t) = A_2 t + \frac{B_2}{2}t^2 + \frac{C_2}{(\lambda+\mu)}E^1_{(\lambda+\mu)}(t) + \frac{D_2}{(\lambda+\mu)^2}E^2_{(\lambda+\mu)}(t)$$

where,

$$A_2 = \frac{\lambda\mu}{(\lambda+\mu)^4}[\lambda\pi_1(\lambda+\mu)^2 - 2\lambda(\mu+\lambda\pi_1)(\lambda+\mu)]$$

$$B_2 = \frac{\lambda^2\mu}{(\lambda+\mu)^2}[\mu+\lambda\pi_1]$$

$$C_2 = \frac{\lambda\mu}{(\lambda+\mu)^4}[\lambda\pi_1(\lambda+\mu)^2 + 2\lambda\mu(\lambda+\mu)\pi_0]$$

$$D_2 = (\frac{\lambda\mu}{\lambda+\mu})^2\pi_0 .$$

Then,

$$\mathrm{Var}[N_c(t)] = 2H_1(t) + E[N_c(t)] - \{E[N_A(t)]\}^2 \qquad (2.17)$$

and,

$$\mathrm{Var}[N_i(t)] = 2H_2(t) + E[N_i(t)] - \{E[N_i(t)]\}^2. \qquad (2.18)$$

Figures 5.2.1,...5.2.3 show the correlation function for various values of λ, μ and π_0. We now give the following asymptotic results.

(2.19) Theorem.

$$\lim_{t\to\infty} \operatorname{corr}(N_c(t), N_i(t+\tau)) = 0.$$

Proof:

$$\operatorname{cov}(N_c(t), N_i(t+\tau)) = H(t) + H(t+\tau) - H(\tau) - E[N_c(t)]E[N_i(t+\tau)].$$

Notice that,

$$H(t) + H(t+\tau) - H(\tau) = Bt^2 + o(t^2)$$

$$E[N_c(t)]E[N_i(t+\tau)] = \frac{\lambda^3\mu}{(\lambda+\mu)^2}t^2 + o(t^2)$$

$$\operatorname{var}[N_c(t)] = K_1 t^2 + o(t^2)$$

$$\operatorname{var}[N_i(t+\tau)] = K_2 t^2 + o(t^2)$$

where K_1 and K_2 are non-zero and can be found from 2.17 and 2.18.

Thus, from 2.16

$$\operatorname{corr}(N_c(t), N_i(t+\tau)) = \frac{Bt^2 - \frac{\lambda^3\mu}{(\lambda+\mu)^2}t^2 + o(t^2)}{\sqrt{(K_1t^2+o(t^2))(K_2t^2+o(t^2))}} .$$

Dividing numerator and denominator by t^2 and taking limits gives us

$$\lim_{t\to\infty} \operatorname{corr}(N_c(t), N_i(t+\tau)) = \frac{B - \frac{\lambda^3\mu}{(\lambda+\mu)^2}}{\sqrt{K_1K_2}} = 0$$

since $B = \lambda^3\mu/(\lambda+\mu)^2$. □

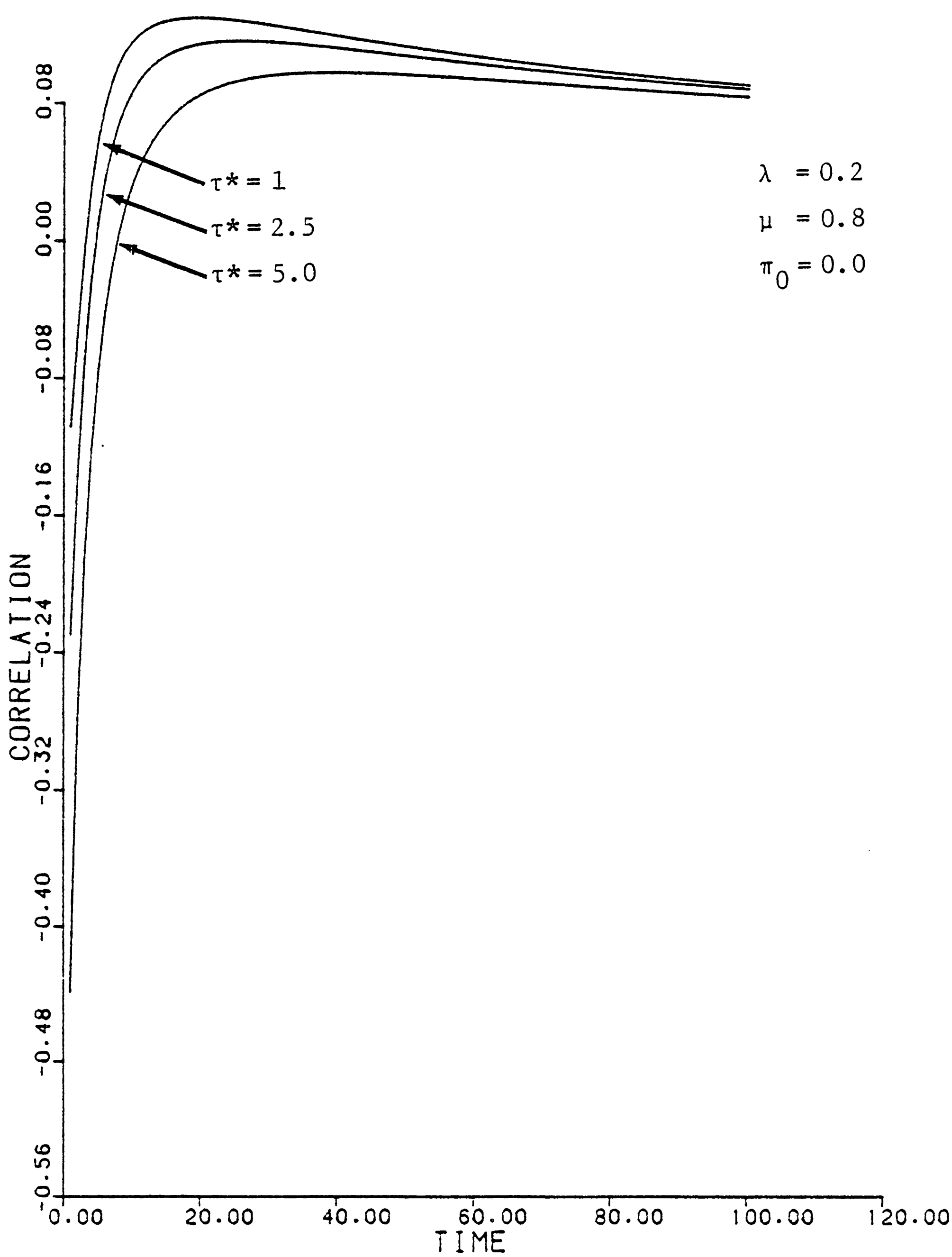

Figure 5.2.1

Overflow-Input Correlations for the M/M/1/0 Queue

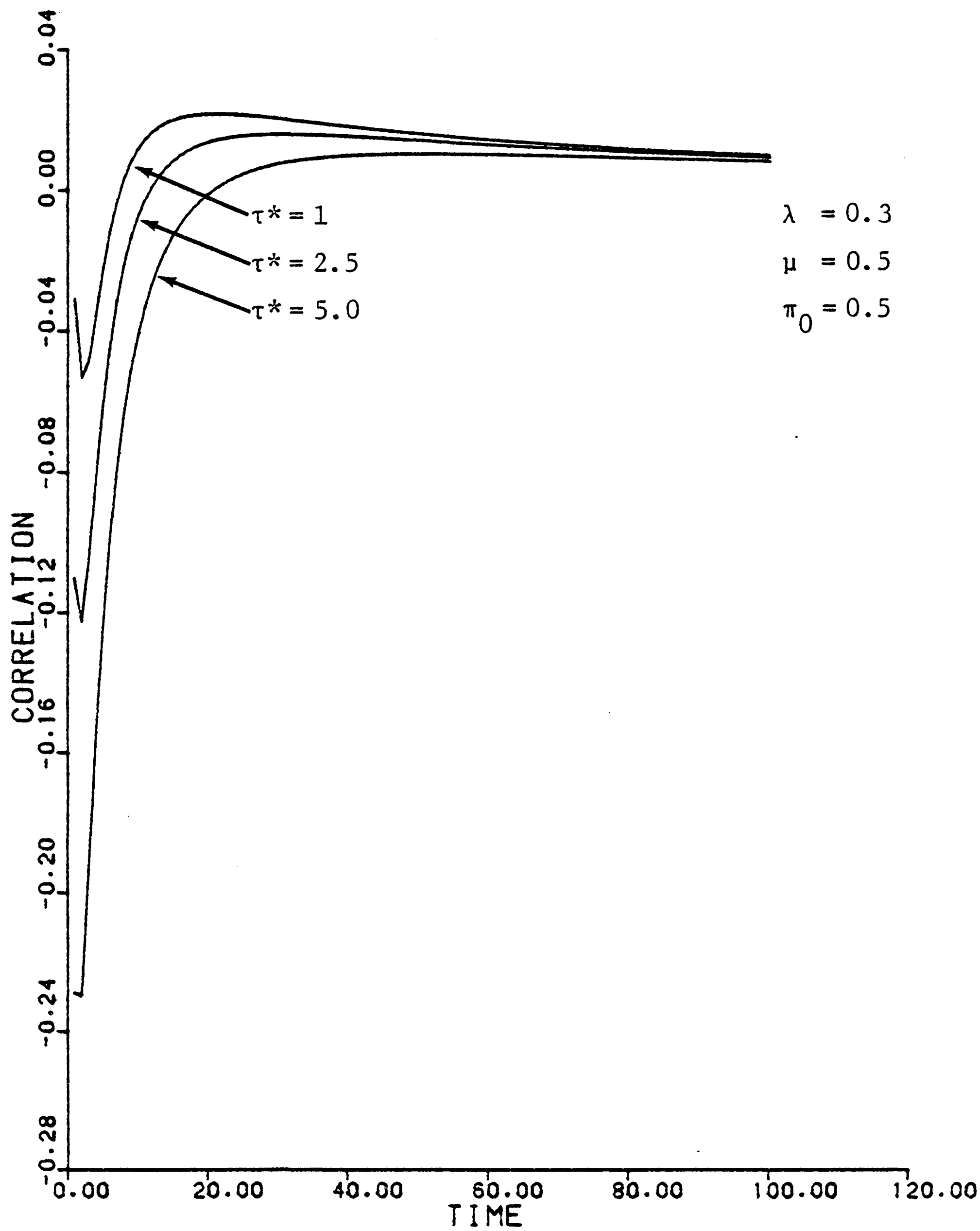

Figure 5.2.2

Overflow-Input Correlations for the M/M/1/0 Queue

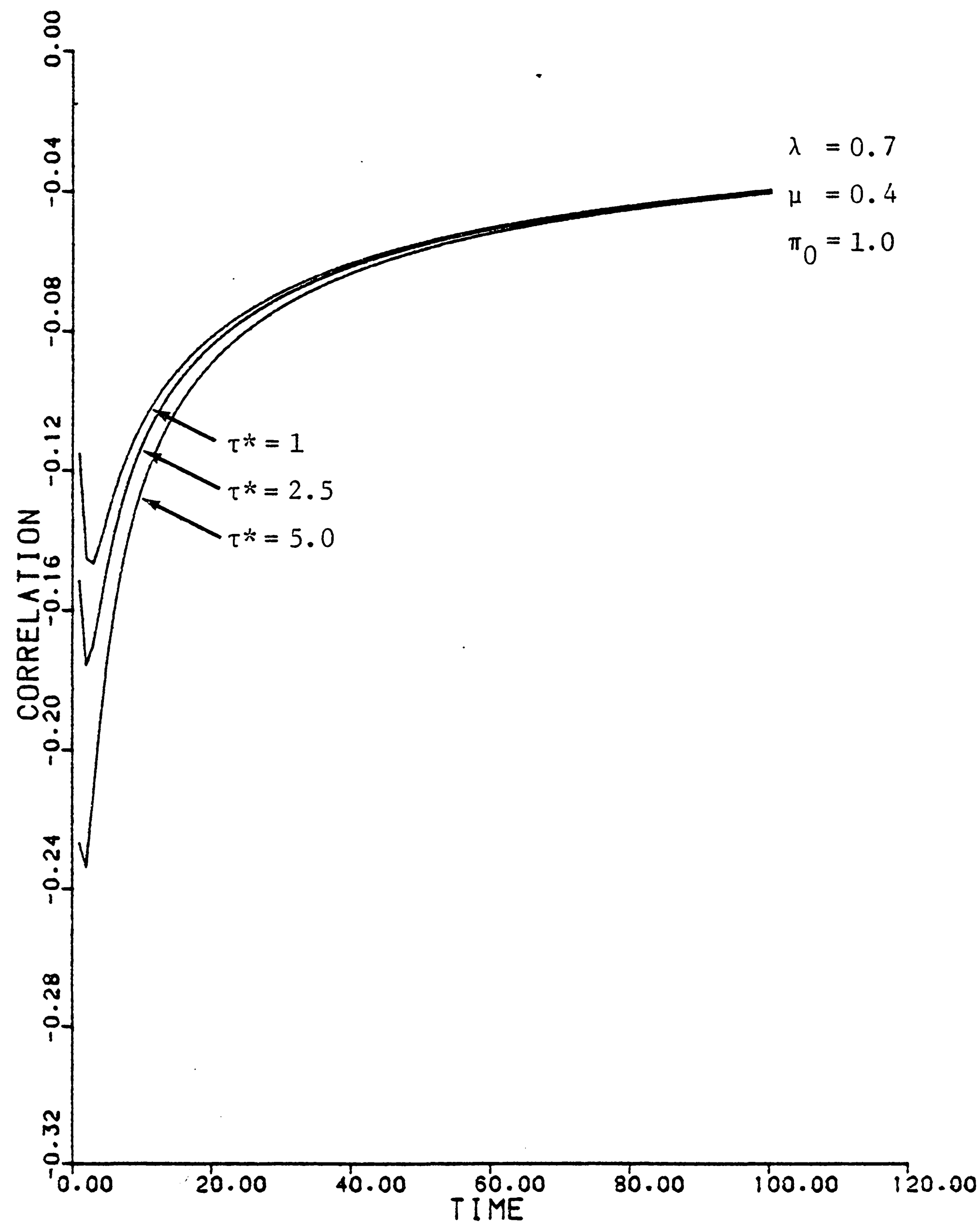

Figure 5.2.3

Overflow-Input Correlations for the M/M/1/0 Queue

(2.20) Theorem.

$$\lim_{\tau\to\infty} \operatorname{corr}(N_c(t),N_i(t+\tau)) = \frac{Bt - E[N_c(t)]\frac{\lambda\mu}{\lambda+\mu}}{\sqrt{\operatorname{var}[N_c(t)](B_2 - (\frac{\lambda\mu}{\lambda+\mu})^2)}} \quad .$$

Proof: Note that

$$H(t) + H(t+\tau) - H(t) = Bt\tau + o(\tau)$$

$$E[N_i(t+\tau)] = \frac{\lambda\mu}{\lambda+\mu}\tau + o(\tau)$$

$$\operatorname{var}[N_i(t+\tau)] = (B_2 - (\frac{\lambda\mu}{\lambda+\mu})^2)\tau^2 + o(\tau^2).$$

Then, using 2.16 we have,

$$\operatorname{corr}(N_c(t),N_i(t+\tau)) = \frac{Bt\tau + o(\tau) - E[N_c(t)]\frac{\lambda\mu}{\lambda+\mu}\tau}{\sqrt{\operatorname{var}[N_c(t)][(B_2 - (\frac{\lambda\mu}{\lambda+\mu})^2)\tau^2 + o(\tau^2)]}} \quad .$$

Dividing numerator and denominator by τ and taking limits, the result follows. □

Finally, we have,

(2.21) Corollary.

$$\lim_{\tau\to\infty} \lim_{t\to\infty} \operatorname{corr}(N_c(t),N_i(t+\tau)) = 0 = \lim_{t\to\infty} \lim_{\tau\to\infty} \operatorname{corr}(N_c(t),N_i(t+\tau)) \ .$$

Proof: The first equality follows from Theorem 2.19. For the second inequality we have from Theorem 2.20,

$$\lim_{\tau\to\infty} \operatorname{corr}(N_c(t),N_i(t+\tau)) = \frac{B - \frac{\lambda^3\mu}{(\lambda+\mu)^2}\, t + o(t)}{\sqrt{K_3 t^2 + o(t^2)}} \ ,$$

where we use 2.14 for $E[N_c(t)]$ and K_3 is some non-zero constant. Dividing numerator and denominator by t, taking limits and observing that $B = \lambda^3\mu/(\lambda+\mu)^2$, the result follows. □

This is an interesting result for the following reason. It has been shown that neither the input nor the overflow processes are ever Poisson (cf. Disney and Simon [1982]). In general, the input process is Markov renewal. However, in our case it is renewal since it is a Markov renewal process with one state. Similarly, so is the overflow process. Thus, what we have is the Poisson process, N_a, being thinned by the Markov chain $\{I_n\}_{n=1}^{\infty}$ to produce in equilibrium two renewal processes neither of which is Poisson but which are uncorrelated. An intuitive explanation could be that, in equilibrium, the Markov chain behaves like a Bernoulli process. Thus, zero correlation could be expected. However, neither of the thinned processes is Poisson. This is because $\{A_n\}_{n=1}^{\infty}$ and $\{I_n\}_{n=1}^{\infty}$ are not independent. What is happening here is that for any n, I_n is independent of $\{A_{n+k}, k \geq 1\}$ but not independent of $\{A_{n-k}, k \geq 0\}$. An interesting conjecture is whether N_c and N_i are independent in equilibrium. The answer to that would almost certainly be negative since $N_a(\cdot) = N_c(\cdot) + N_i(\cdot)$.

In the next section we shall study the covariance between the processes N_o and N_c and find similar results.

3. The Covariance Between N_o and N_c for the M/M/1/0 Queue

From 1.1, we see that the departure sequence contains information about both the output and overflow sequences. Thus, if we can identify each departure point as an overflow or an output, we have the ingredients necessary to implement the analysis of the covariance between

N_o and N_c. Define a zero-one random variable, I_n, as follows. Once again we let $F(t) = 1-e^{-\lambda t}$.

$$I_n = \begin{cases} 1 & \text{if the customer leaving the system at } C_n \text{ is an output} \\ 0 & \text{if the customer leaving the system at } C_n \text{ is an overflow.} \end{cases} \tag{3.1}$$

(3.2) Definition. Let $(I,C) = \{(I_n,C_n)\}_{n=1}^{\infty}$. (I,C) is called the departure process.

(3.3) Theorem. The departure process, (I,C), is a Markov renewal process with mark space $E = \{0,1\}$.

Proof: Let Z_n be the queue length at C_n^+. Then,

$$Z_n = 1 - I_n. \tag{3.4}$$

But Z_n is clearly a Markov chain on $E = \{0,1\}$. Also, $C_{n+1} - C_n$ depends only on Z_{n+1} and Z_n and the result follows. □

Now partition E into A and B as follows.

$$A = \{0\}, \quad B = \{1\}.$$

Then clearly, $N_A(t)$ and $N_B(t)$ are the numbers of overflows and outputs respectively. Thus, $N_A = N_c$, $N_B = N_o$ and,

$$\text{cov}(N_A(t),N_B(t+\tau)) = \text{cov}(N_c(t),N_o(t+\tau)).$$

Therefore, the process (I,C) has the information we need.

(3.5) Remark. (I,C) is in fact a Markov process. However, we shall find it convenient to refer to it as a Markov renewal process.

Now let $\mathbf{Q}(t) = \{Q(i_1,i_2,t),\ i_1,i_2 \in \{0,1\},\ t \geq 0\}$ be the semi-Markov kernel of the process (I,C) and let $\tilde{\mathbf{Q}}(s)$ be the Laplace-Stieltjes transform of $\mathbf{Q}(t)$. That is,

$$\tilde{Q}(i_1,i_2,s) = \int_0^\infty e^{-st} Q(i_1,i_2,t). \tag{3.6}$$

We now work with $\tilde{\mathbf{Q}}(s)$ till we are ready to invert our results back into the time domain.

<u>(3.7) Theorem.</u>

$$\tilde{\mathbf{Q}}(s) = \begin{bmatrix} \dfrac{\lambda}{\lambda+\mu+s} & \dfrac{\mu}{\lambda+\mu+s} \\ \\ \dfrac{\lambda^2}{(\lambda+s)(\lambda+\mu+s)} & \dfrac{\lambda\mu}{(\lambda+s)(\lambda+\mu+s)} \end{bmatrix} .$$

<u>Proof</u>: $Q(i_1,i_2,t) = P\{I_{n+1} = i_2, C_{n+1}-C_n \le t | I_n = i_1\}$

$$= \int_0^t P\{I_{n+1} = i_2 | C_{n+1}-C_n = x, I_n = i_1\}$$

$$\cdot\ dP\{C_{n+1}-C_n \le x | I_n = i_1\}.$$

Case 1. Let $i_1 = 0 = i_2$. Thus, the customers departing at C_n and C_{n+1} are both overflows. Then $C_{n+1}-C_n$ is the minimum of an arrival time and the forward recurrence time of a service time. Hence,

$$P\{C_{n+1}-C_n \le x\} = 1-e^{-(\lambda+\mu)x}.$$

Also, if $I_n = 0$ for $I_{n+1} = 0$, the next arrival must occur before the service on the customer present is completed. Hence,

$$P\{I_{n+1} = 0 | C_{n+1}-C_n = x, I_n = 0\} = \frac{\lambda}{\lambda+\mu} .$$

Thus,

$$Q(0,0,t) = \int_0^t \frac{\lambda}{\lambda+\mu} (\lambda+\mu)e^{-(\lambda+\mu)x}dx$$

$$= \frac{\lambda}{\lambda+\mu} (1-e^{-(\lambda+\mu)t}).$$

Taking Laplace-Stieltjes transforms gives,

$$\tilde{Q}(0,0,s) = \frac{\lambda}{\lambda+\mu+s} .$$

Case 2. Let $i_1 = 0$, $i_2 = 1$. That is, the customers departing at C_n and C_{n+1} are overflows and outputs respectively. Now,

$$P\{I_{n+1} = 1 | C_{n+1}-C_n = x, I_n = 0\} = 1 - P\{I_{n+1} = 0 | C_{n+1}-C_n = x, I_n = 0\}.$$

From case 1 we have,

$$P\{I_{n+1} = 1 | C_{n+1}-C_n = x, I_n = 0\} = 1 - \frac{\lambda}{\lambda+\mu} = \frac{\mu}{\lambda+\mu} .$$

Thus,

$$Q(0,1,t) = \int_0^t \frac{\mu}{\lambda+\mu} (\lambda+\mu) e^{-(\lambda+\mu)x} dx$$

$$= \frac{\mu}{\lambda+\mu} (1-e^{-(\lambda+\mu)t}) .$$

Taking Laplace-Stieltjes transforms gives,

$$\tilde{Q}(0,1,s) = \frac{\mu}{\lambda+\mu+s} .$$

Case 3. Let $i_1 = 1$, $i_2 = 0$. That is, the customers departing at C_n and C_{n+1} are outputs and overflows respectively. Let X_1 be the time until the next arrival after C_n and S_1 be the service time of this arrival. Finally, let X_2 be time between this arrival and the next. Then, given that $I_n = 1$, $I_{n+1} = 0$, clearly,

$$C_{n+1}-C_n = \min(X_1+S_1, X_1+X_2)$$

$$= X_1 + \min(S_1, X_2).$$

Thus,

$$P\{C_{n+1}-C_n \leq x|I_n = 1\} = (\frac{\lambda+\mu}{\mu})(1-e^{-\lambda x}) - \frac{\lambda}{\mu}(1-e^{-(\lambda+\mu)x}).$$

Arguments similar to cases 1 and 2 given

$$P\{I_{n+1} = 0|C_{n+1}-C_n = x, I_n = 1\} = \frac{\lambda}{\lambda+\mu}.$$

Thus,

$$Q(1,0,t) = \int_0^t \frac{\lambda}{\lambda+\mu}(\frac{\lambda}{\mu}(\lambda+\mu)e^{-\lambda x} - \frac{\lambda}{\mu}(\lambda+\mu)e^{-(\lambda+\mu)x})dx.$$

Taking Laplace-Stieltjes transforms gives

$$Q(1,0,t) = \frac{\lambda^2}{(\lambda+s)(\lambda+\mu+s)}.$$

Case 4. Let $i_1 = 1 = i_2$. That is, the customers departing at C_n and C_{n+1} are outputs. From cases 1, 2, and 3 we have,

$$Q(1,1,t) = \int_0^t \frac{\mu}{\lambda+\mu}(\frac{\lambda}{\mu}(\lambda+\mu)e^{-\lambda x} - \frac{\lambda}{\mu}(\lambda+\mu)e^{-(\lambda+\mu)x})dx.$$

Taking Laplace-Stieltjes transforms gives,

$$\tilde{Q}(1,1,s) = \frac{\lambda\mu}{(\lambda+s)(\lambda+\mu+s)}.$$

This completes the proof. □

Note that $\mathbf{Q}_{AA}(t) = Q(0,0,t)$, $\mathbf{Q}_{AB}(t) = Q(0,1,t)$, $\mathbf{Q}_{BA}(t) = Q(1,0,t)$ and $\mathbf{Q}_{BB}(t) = Q(1,1,t)$. The n-fold convolution of $\mathbf{Q}(t)$ can be determined by the n-th power of $\tilde{\mathbf{Q}}(s)$. We now give this result.

(3.8) Proposition.

$$[\tilde{\mathbf{Q}}(s)]^n = \begin{bmatrix} \frac{\lambda}{\lambda+\mu+s}(\frac{\lambda}{\lambda+s})^{n-1} & \frac{\mu}{\lambda+\mu+s}(\frac{\lambda}{\lambda+s})^{n-1} \\ \\ \frac{\lambda}{\lambda+\mu+s}(\frac{\lambda}{\lambda+s})^{n} & \frac{\mu}{\lambda+\mu+s}(\frac{\lambda}{\lambda+s})^{n} \end{bmatrix}. \tag{3.9}$$

Proof: The result is obviously true for n = 1. Now, let 5.9 be true for

$n = k$. Then,

$$[\tilde{\mathbf{Q}}(s)]^{k+1} = [\tilde{\mathbf{Q}}(s)]^k[\tilde{\mathbf{Q}}(s)].$$

This yields, by the induction hypothesis,

$$[\tilde{\mathbf{Q}}(s)]^{k+1} = \begin{bmatrix} \frac{\lambda}{\lambda+\mu+s}(\frac{\lambda}{\lambda+s})^k & \frac{\mu}{\lambda+\mu+s}(\frac{\lambda}{\lambda+s})^k \\ \\ \frac{\lambda}{\lambda+\mu+s}(\frac{\lambda}{\lambda+s})^{k+1} & \frac{\mu}{\lambda+\mu+s}(\frac{\lambda}{\lambda+s})^{k+1} \end{bmatrix}$$

and hence the result is true for $n = k + 1$. The result for all n follows by induction. □

Finally, let $\pi = (\pi_0,\pi_1)$ be the initial distribution of the system. We assume that at time zero a customer left the system. Let I_0 be zero (one) if this customer is an overflow (output). Then,

$$\pi_0 = P\{I_0 = 0\} = 1 - P\{I_0 = 1\} = 1 - \pi_1. \qquad (3.10)$$

As in the previous section, we can use the Laplace-Stieltjes transforms of 4.1.23 and 4.1.25 to find $\mathrm{cov}(N_c(t),N_o(t+\tau))$. Once again, computational details are omitted.

Let $H_i(t)$, $i \in \{1,2\}$ be defined as follows.

$$H_i(t) = A_i t + \frac{B_i}{2}t^2 + \frac{C_i}{\lambda+\mu}E^1_{(\lambda+\mu)}(t) + \frac{D_i}{(\lambda+\mu)^2}E^2_{(\lambda+\mu)}(t).$$

A_i, B_i, C_i and D_i for $i \in \{1,2\}$ are given below.

$$A_1 = \frac{\pi_0\lambda^2\mu(\lambda+\mu)^2 - 2\lambda^3\mu(\lambda+\mu)}{(\lambda+\mu)^4}$$

$$B_1 = \frac{\lambda^3\mu}{(\lambda+\mu)^2}$$

$$C_1 = \frac{\pi_0\lambda^2\mu(\lambda+\mu)^2 + 2\lambda^2(\lambda+\mu)(\pi_1\lambda\mu - \pi_0\mu^2)}{(\lambda+\mu)^4}$$

$$D_1 = \frac{\lambda^2}{(\lambda+\mu)^2}(\pi_1\lambda\mu - \pi_0\mu^2)$$

$$A_2 = \frac{\lambda^2\mu(\lambda+\mu)^2(1+\pi_0) - 2\lambda^3\mu(\lambda+\mu)}{(\lambda+\mu)^4}$$

$$B_2 = \frac{\lambda^3\mu}{(\lambda+\mu)^2}$$

$$C_2 = \frac{(\lambda+\mu)^2(\mu(\pi_1\lambda^2-\pi_0\lambda\mu) - \pi_0\lambda\mu) - 2\mu^2(\pi_1\lambda^2-\pi_0\lambda\mu)(\lambda+\mu)}{(\lambda+\mu)^4}$$

$$D_2 = \frac{\mu(\pi_0\lambda\mu-\pi_1\lambda^2)}{(\lambda+\mu)^2} .$$

Also, let $H_i(t)$, $i \in \{3,4\}$ be defined as follows.

$$H_i(t) = A_i t + \frac{B_i}{\lambda+\mu} E^1_{(\lambda+\mu)}(t).$$

A_i, B_i, $i \in \{3,4\}$ are given below.

$$A_3 = \frac{\lambda^2}{\lambda+\mu}$$

$$B_3 = \frac{\lambda(\pi_0\mu-\pi_1\lambda)}{(\lambda+\mu)}$$

$$A_4 = \frac{\lambda\mu}{\lambda+\mu}$$

$$B_4 = \frac{\mu(\pi_0\mu-\pi_1\lambda)}{(\lambda+\mu)} .$$

Then,

$$\text{cov}(N_c(t),N_o(t+\tau)) = H_1(t) + H_2(t+\tau) - H_2(\tau) - H_3(t)H_4(t+\tau) \ .$$

To compute correlations, we find var$[N_c(t)]$ and var$[N_o(t)]$ from 4.3.3 and 4.3.4. Define $H_i(t)$, $i \in \{5,6\}$ exactly as $H_i(t)$, $i \in \{1,2\}$. A_i, B_i, C_i, D_i, $i \in \{5,6\}$ are given below.

$$A_5 = \frac{(\lambda+\mu)^2(\lambda^2(\lambda+\pi_0\mu+\lambda\pi_0)) - 2\lambda^3(\lambda+\pi_0\mu)(\lambda+\mu)}{(\lambda+\mu)^4}$$

$$B_5 = \frac{\lambda^3(\lambda+\pi_0\mu)}{(\lambda+\mu)^2}$$

$$C_5 = \frac{(\lambda+\mu)^2(\lambda^2(\pi_1\lambda-\pi_0\mu) - 2\pi_1\lambda^3\mu(\lambda+\mu))}{(\lambda+\mu)^4}$$

$$D_5 = \frac{\lambda^3\mu\pi_1}{(\lambda+\mu)^2}$$

$$A_6 = \frac{\lambda\mu(\lambda+\mu)^2(\pi_0\mu+\pi_1\lambda) - 2\lambda\mu(\lambda\mu+\pi_1\lambda^2)(\lambda+\mu)}{(\lambda+\mu)^4}$$

$$B_6 = \frac{\lambda\mu(\lambda\mu+\pi_1\lambda^2)}{(\lambda+\mu)^2}$$

$$C_6 = \frac{\lambda\mu(\lambda+\mu)^2(\pi_0\mu+\pi_1\lambda) - 2\lambda\mu^3\pi_0(\lambda+\mu)}{(\lambda+\mu)^4}$$

$$D_6 = -\frac{\lambda\mu^3\pi_0}{(\lambda+\mu)^2} \ .$$

Then,

$$\text{var}[N_c(t)] = 2H_5(t) + H_3(t) - [H_3(t)]^2$$

and,

$$\mathrm{var}[N_o(t)] = 2H_6(t) + H_4(t) - [H_4(t)]^2.$$

Finally, we have,

$$\mathrm{corr}(N_c(t),N_o(t+\tau)) = \frac{\mathrm{cov}(N_c(t),N_o(t+\tau))}{\sqrt{\mathrm{var}[N_c(t)]\mathrm{var}[N_o(t+\tau)]}} .$$

Figures 5.3.1,...,5.3.3 give plots of $\mathrm{corr}(N_c(t),N_o(t+\tau))$ for various values of λ, μ, π_0 and τ.

Analogous to Theorems 2.19, 2.20 and Corollary 2.21, we obtain the following results. The proofs are identical and we refrain from giving them.

(3.11) Theorem.

$$\lim_{t\to\infty} \mathrm{corr}(N_c(t),N_o(t+\tau)) = 0.$$

(3.12) Theorem.

$$\lim_{\tau\to\infty} \mathrm{corr}(N_c(t),N_o(t+\tau)) = \frac{B_2 t - A_4 E[N_c(t)]}{\sqrt{\mathrm{var}[N_c(t)](B_6 - (\frac{\lambda\mu}{\lambda+\mu})^2)}} .$$

(3.13) Corollary.

$$\lim_{\tau\to\infty}\lim_{t\to\infty} \mathrm{corr}(N_c(t),N_o(t+\tau)) = 0 = \lim_{t\to\infty}\lim_{\tau\to\infty} \mathrm{corr}(N_c(t),N_o(t+\tau)).$$

This is a significant result. Since N_d is a Markov renewal process, both N_o and N_c are Markov renewal processes with one state each. Thus, N_o and N_c are both renewal processes. Moreover, neither of these processes is Poisson (cf. Disney et al. [1973], Disney and Simon [1982]). Thus, we have the interesting situation where two renewal processes that

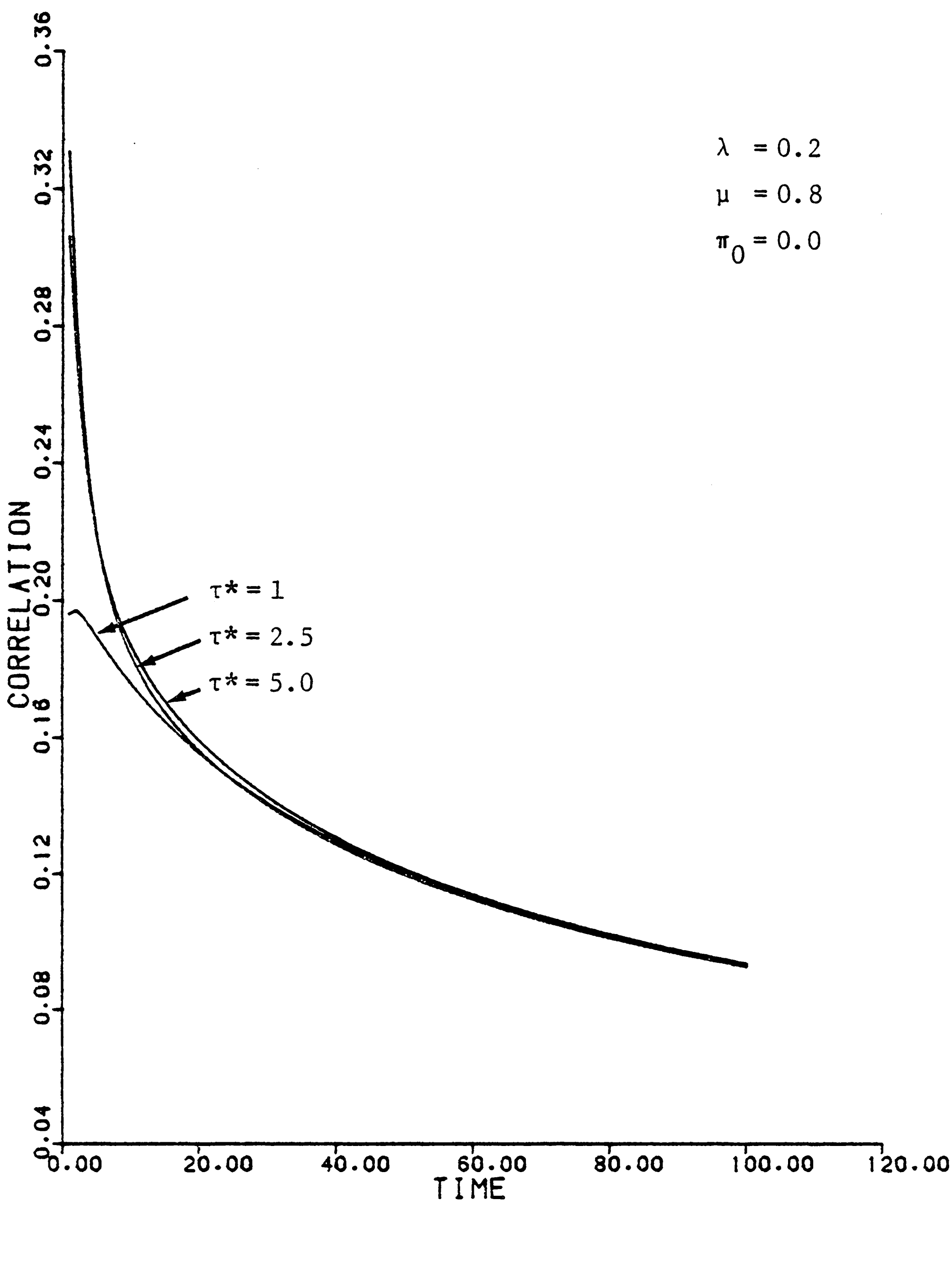

Figure 5.3.1

Overflow-Output Correlations for the M/M/1/0 Queue

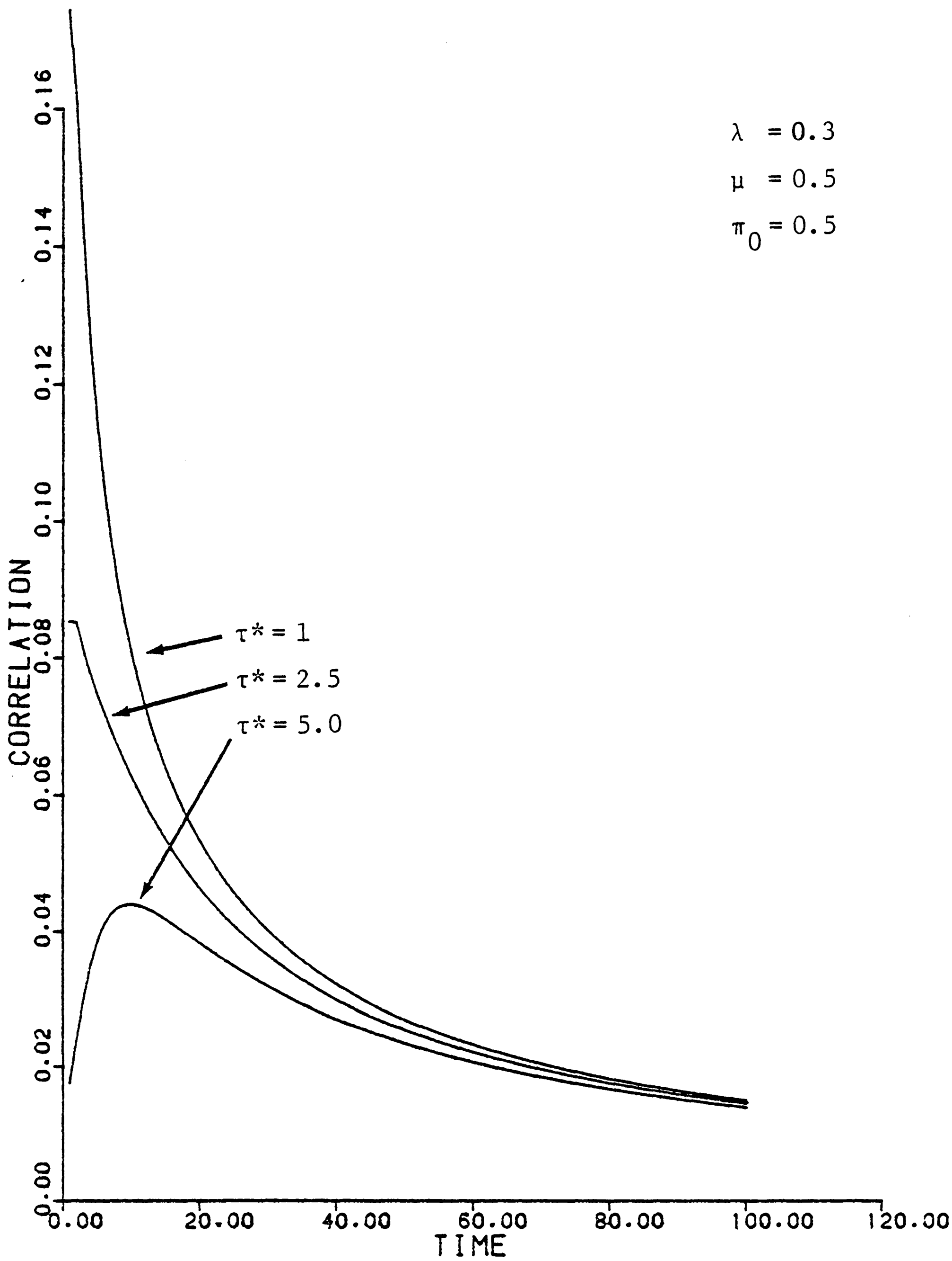

Figure 5.3.2

Overflow-Output Correlations for the M/M/1/0 Queue

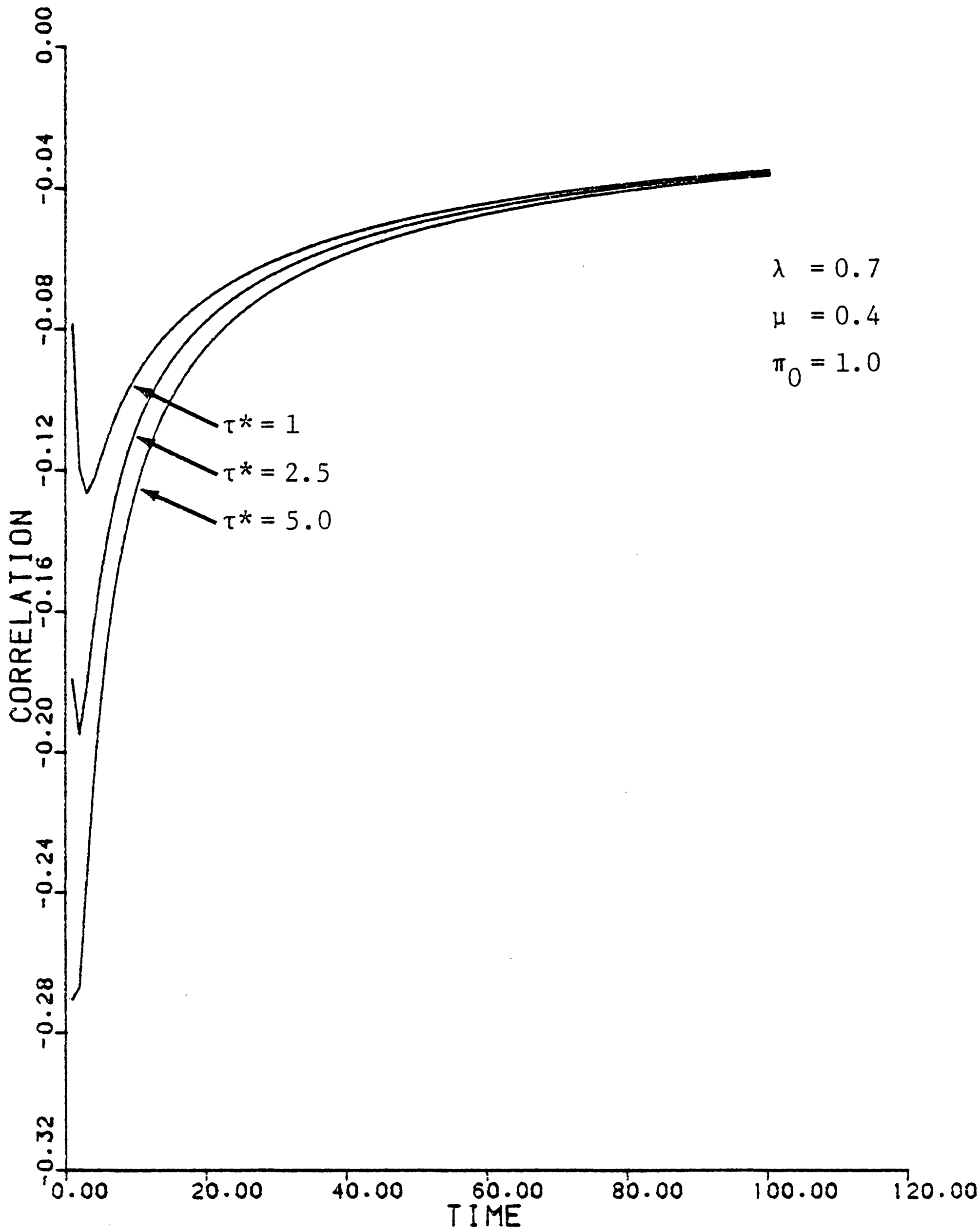

Figure 5.3.3

Overflow-Output Correlations for the M/M/1/0 Queue

are not Poisson, are uncorrelated and yet when superposed produce, in equilibrium, the Poisson process N_o. (cf. Shanbhag and Tambouratzis [1973]). Even more interestingly, if we reverse time, N_c and N_i can be represented as thinned versions of N_d. From the proof of Theorem 3.7, the thinning process $\{I_n\}_{n=1}^{\infty}$ is a Bernoulli process. Thus, we have the Poisson process N_o (in equilibrium) being thinned by a Bernoulli process to produce two uncorrelated but dependent renewal processes neither of which is Poisson. This seems to contradict Corollary 3.3.13. However, from the proof of Theorem 3.7 it is quite obvious that $\{C_n\}_{n=1}^{\infty}$ and $\{I_n\}_{n=1}^{\infty}$ are not independent, a prerequisite for Corollary 3.3.13. This resolves the apparent contradiction.

4. Summary

In this chapter we have used the thinning approach of Chapter 4 to study some properties of overflow queues. As a consequence we have seen how to analyse a superposition problem with the same tools. We have also seen some interesting results in renewal theory. In the next chapter we study two problems in renewal theory.

CHAPTER 6

BERNOULLI THINNING OF ALTERNATING MARKOV PROCESSES; MARKOV CHAIN THINNING OF RENEWAL PROCESSES

In this chapter we use the tools of Chapter 4 to study two problems. We shall analyse the case where an alternating Markov process is thinned by an independent Bernoulli process. We then look at the situation where an ordinary renewal process is being thinned by a Markov chain. In both cases we shall present some uniqueness results. In the latter situation we shall also solve a design problem for the special case where the renewal process is Poisson.

1. Bernoulli Thinning of an Alternating Markov Process

Let $(X,T) = \{(X_n,T_n)\}_{n=1}^{\infty}$ be a marked point process on the mark space $E = \{a,b\}$. Further, let (X,T) be a Markov process with the following transition probabilities.

$$Q(i,j,t) = P\{X_{n+1} = j, T_{n+1}-T_n \leq t \mid X_n = i\} = \begin{cases} 1-e^{-\lambda_0 t}, & i=a,\ j=b \\ 1-e^{-\lambda_1 t}, & i=b,\ j=a \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

Then (X,T) is called an alternating Markov process. Informally, this process is a simple point process with time between events being independent and having their distributions alternating between exponential distributions with parameters λ_0 and λ_1 respectively.

Clearly, (X,T) is also a Markov renewal process. If $\mathbf{Q}(t) = \{Q(i,j,t),\ i,j \in \{a,b\},\ t \geq 0\}$ is the semi-Markov kernel of (X,T) then $\mathbf{Q}(t)$ is given by the following.

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & 1-e^{-\lambda_0 t} \\ 1-e^{-\lambda_1 t} & 0 \end{bmatrix}. \tag{1.2}$$

Now let $Y = \{Y_n\}_{n=1}^{\infty}$ be a Bernoulli process independent of (X,T) and let $P\{Y_n = 0\} = p = 1-q = 1 - P\{Y_n = 1\}$. We now thin the process (X,T) by the process Y and consider the two processes thus obtained N_1 and N_0. This situation was analyzed for the case where (X,T) is Markov renewal on any mark space E in Chapter 4. The covariance between the processes N_1 and N_0 is given by 4.2.12, which we reproduce here.

$$\begin{aligned}\mathrm{cov}(N_0(t),N_1(t+\tau)) = pq\{&\pi\mathbf{R}^{*2}(t)U + \pi\mathbf{R}^{*2}(t+\tau)U - \pi\mathbf{R}^{*2}(\tau)U \\ &- (\pi\mathbf{R}(t)U)(\pi\mathbf{R}(t+\tau)U)\}\end{aligned} \tag{1.3}$$

π is the initial distribution and $\mathbf{R}(t)$ is the Markov renewal kernel for the process (X,T). Let $\hat{\mathbf{Q}}(s)$ and $\hat{\mathbf{R}}(s)$ be the Laplace-Stieltjes transform of $\mathbf{Q}(t)$ and $\mathbf{R}(t)$. Since the mark space of (X,T) is finite we have, from Disney [1982], the following relationship.

$$\hat{\mathbf{R}}(s) = (\mathbf{I} - \hat{\mathbf{Q}}(s))^{-1} \tag{1.4}$$

where $\mathbf{I}$ is the identity matrix. This gives,

$$\hat{\mathbf{R}}(s) = \begin{bmatrix} \dfrac{(\lambda_0+s)(\lambda_1+s)}{s(s+\lambda_0+\lambda_1)} & \dfrac{\lambda_0(\lambda_1+s)}{s(s+\lambda_0+\lambda_1)} \\ \\ \dfrac{\lambda_1(\lambda_0+s)}{s(s+\lambda_0+\lambda_1)} & \dfrac{(\lambda_0+s)(\lambda_1+s)}{s(s+\lambda_0+\lambda_1)} \end{bmatrix}. \tag{1.5}$$

It is now a matter of computational detail to compute the right hand side of 1.3. We give the result for $\tau = 0$. From 1.3 we have,

$$\operatorname{cov}(N_0(t),N_1(t)) = pq\{2\pi \mathbf{R} * \mathbf{R}(t)U - (\pi \mathbf{R}(t)U)^2\}. \tag{1.6}$$

Let

$$\pi = (\pi_a, \pi_b)$$

where,

$$\pi_a = P\{X_0 = a\} \tag{1.7a}$$

$$\pi_b = P\{X_0 = b\} = 1 - \pi_a . \tag{1.7b}$$

It can be shown that the following hold.

$$\begin{aligned}\pi \mathbf{R} * \mathbf{R}(t)U = {} & 2\left(\frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}\right)^2 t^2 + \frac{\lambda_0\lambda_1(\lambda_0-\lambda_1)}{(\lambda_0+\lambda_1)^3}\,(3(\pi_a\lambda_0-\pi_b\lambda_1) - (\pi_a\lambda_1-\pi_b\lambda_0))t \\ & + \frac{\lambda_0\lambda_1}{(\lambda_0+\lambda_1)^4}\,(6\lambda_0\lambda_1-5(\pi_a\lambda_0^2+\pi_b\lambda_1^2) - (\pi_a\lambda_1^2+\pi_b\lambda_0^2))(1-e^{-(\lambda_0+\lambda_1)t}) \\ & + \frac{2\lambda_0\lambda_1}{(\lambda_0+\lambda_1)^3}\,(\lambda_0-\lambda_1)(\pi_a\lambda_0-\pi_b\lambda_1)te^{-(\lambda_0+\lambda_1)t}.\end{aligned} \tag{1.7}$$

$$\begin{aligned}(\pi \mathbf{R}(t)U)^2 = {} & 4\left(\frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}\right)^2 t^2 + \frac{4\lambda_0\lambda_1}{(\lambda_0+\lambda_1)^3}\,(\lambda_0-\lambda_1)(\pi_a\lambda_0-\pi_b\lambda_1)t(1-e^{-(\lambda_0+\lambda_1)t}) \\ & + \frac{(\pi_a\lambda_0-\pi_b\lambda_1)(\lambda_0-\lambda_1)}{(\lambda_0+\lambda_1)^2}^{2}\,(1-2e^{-(\lambda_0+\lambda_1)t}+e^{-2(\lambda_0+\lambda_1)t}).\end{aligned} \tag{1.8}$$

The next result shows that N_0 and N_1 are uncorrelated if and only if $\{T_n\}_{n=1}^{\infty}$ forms a Poisson process.

<u>(1.9) Theorem.</u> The following are equivalent.

a) $\mathrm{cov}(N_0(t), N_1(t)) = 0$ for $t \in (0,\infty)$.

b) $\{T_n\}_{n=1}^{\infty}$ forms a Poisson process.

c) N_0 and N_1 are independent Poisson processes.

<u>Proof</u>: a $\Longrightarrow$ b) If $\mathrm{cov}(N_0(t), N_1(t)) = 0$, then from 1.6 we must have,

$$2\pi \mathbf{R} * \mathbf{R}(t)U = (\pi \mathbf{R}(t)U)^2.$$

Multipling 1.7 by 2 and comparing coefficients with 1.8 we get the following conditions.

$$\pi_a \lambda_0 = \pi_b \lambda_1 \quad \text{or} \quad \lambda_0 = \lambda_1 \tag{1.10}$$

and,

$$\pi_a \lambda_1 = \pi_b \lambda_0. \tag{1.11}$$

1.10 and 1.11 imply that $\lambda_0 = \lambda_1$ always and hence $\{T_n\}_{n=1}^{\infty}$ forms a Poisson process.

b $\Longrightarrow$ c) See Çinlar (Theorem 4.5.3, p. 89 [1975]).

c $\Longrightarrow$ a) Is obvious. □

Thus, we have, as in 3.4.6, a situation where zero correlation implies independence. In fact, in this case, all we need is the equality of two mean values to give us the powerful conclusion that the processes N_0 and N_1 are independent.

We now consider the case where a renewal process is thinned by a Markov chain.

2. <u>Markov Chain Thinning of Renewal Processes</u>

Let $\{T_n\}_{n=1}^{\infty}$ be an ordinary renewal process and define the following quantities.

$$F(t) = P\{T_{n+1}-T_n \leq t\},\ n \geq 0. \tag{2.1}$$

$$\tilde{F}(s) = \text{Laplace-Stieltjes transform of } F(t). \tag{2.2}$$

$$R(t) = \sum_{n=1}^{\infty} F^{*n}(t), \text{ the renewal function.} \tag{2.3}$$

$$\tilde{R}(s) = \text{Laplace-Stieltjes transform of } R(t). \tag{2.4}$$

From 2.3 we have,

$$\tilde{R}(s) = \frac{\tilde{F}(s)}{1-\tilde{F}(s)} . \tag{2.5}$$

Now, let $\{X_n\}_{n=1}^{\infty}$ be a Markov chain on $E = \{0,1\}$, and transition matrix $\mathbf{P}$ given by

$$\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} . \tag{2.6}$$

We assume that $a,b \in (0,1)$. Further, if $\{X_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ are independent, we have,

<u>(2.7) Proposition</u>. The sequence $(X,T) = \{(X_n,T_n)\}_{n=1}^{\infty}$ is a Markov renewal process with mark space $E = \{0,1\}$ and semi-Markov kernel $\mathbf{Q}(t)$ given by,

$$\mathbf{Q}(t) = \begin{bmatrix} (1-a)F(t) & aF(t) \\ bF(t) & (1-b)F(t) \end{bmatrix} . \tag{2.8}$$

<u>Proof</u>: The result follows from the fact that $\{X_n\}_{n=1}^{\infty}$ is a Markov chain and $\{T_n\}_{n=1}^{\infty}$ is a renewal process independent of $\{X_n\}_{n=1}^{\infty}$. □

Let,

$$y = \frac{b}{a+b}\,, \tag{2.9}$$

$$z = 1-a-b\,. \tag{2.10}$$

Then we have,

<u>(2.11) Proposition</u>. The n-step transition matrix for the process (X,T) is given by,

$$\mathbf{Q}^{*n}(t) = \begin{bmatrix} (y+(1-y)z^n)F^{*n}(t) & (1-y-(1-y)z^n)F^{*n}(t) \\ \\ (y-yz^n)F^{*n}(t) & (1-y+yz^n)F^{*n}(t) \end{bmatrix}. \tag{2.12}$$

<u>Proof</u>: By the independence of $\{X_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$,

$$\mathbf{Q}^{*n}(t) = \mathbf{P}^n \times F^{*n}(t).$$

But from Parzen [1962] p. 197,

$$\mathbf{P}^n = \begin{bmatrix} y+(1-y)z^n & 1-y-(1-y)z^n \\ \\ y-yz^n & 1-y+yz^n \end{bmatrix}$$

and the result follows. □

Now if E is partitioned into the sets A and B where,

$$A = \{0\} \quad \text{and} \quad B = \{1\},$$

then the situation outlined above represents one where the renewal process $\{T_n\}_{n=1}^{\infty}$ is being thinned by the Markov chain $\{X_n\}_{n=1}^{\infty}$. The n-step transition matrix contains the information needed to compute the covariance between the processes N_A and N_B from Propositions 4.2.5 and 4.2.6. Let,

$$\tilde{R}_z(s) = \frac{z\tilde{F}(s)}{1-z\tilde{F}(s)} . \tag{2.13}$$

$$R_z(t) = \text{Laplace-Stieltjes inverse of } \tilde{R}_z(s). \tag{2.14}$$

$$\pi_0 = P\{X_0 = 0\},\ \pi_1 = P\{X_0 = 1\} = 1 - \pi_0. \tag{2.15}$$

It can then be shown that,

$$\begin{aligned} \operatorname{cov}(N_A(t),N_B(t)) &= y(1-y)[2R * R(t) - (R(t))^2] \\ &\quad + (y(1+2\pi_0) + 1 - \pi_0 - 3y^2)R * R_z(t) \\ &\quad + y(\pi_0-y)R_z * R_z(t) + (y(1+2\pi_0)-2y^2-\pi_0) \\ &\quad \cdot R(t)R_z(t) + (y-\pi_0)^2(R_z(t))^2. \end{aligned} \tag{2.16}$$

We omit the computational details. 2.16 leads to the following partial converse to Corollary 4.2.5.

<u>(2.17) Theorem</u>. Let $F(t) = 1-e^{-\lambda t}$. Thus, $\{T_n\}_{n=1}^{\infty}$ forms a Poisson process. Further, let π_0 and π_1 be the invariant distribution for the Markov chain $\{X_n\}_{n=1}^{\infty}$. Then the following are equivalent.

a) $\operatorname{cov}(N_A(t),N_B(t)) = 0$ for $t \in (0,\infty)$.

b) $\{X_n\}_{n=1}^{\infty}$ is a Bernoulli process.

c) N_A and N_B are independent Poisson processes.

<u>Proof</u>: a $\Longrightarrow$ b) Since $F(t) = 1-e^{-\lambda t}$ we have $R(t) = \lambda t$. Then from Theorem 4.1.30 we have,

$$2R * R(t) = (R(t))^2.$$

Further, $\pi_0 = y$ by assumption. Then 2.16 reduces to

$$\operatorname{cov}(N_A(t),N_B(t)) = (1-y)^2 R * R_z(t).$$

Now if $\mathrm{cov}(N_A(t),N_B(t)) = 0$ for $t > 0$, we have, since $y < 1$,

$$R * R_z(t) = 0 \qquad \text{for } t > 0.$$

Taking Laplace-Stieltjes transforms gives

$$R(s)R_z(s) = 0 \qquad \text{for } s > 0$$

$$\Longrightarrow \quad \frac{\lambda}{s}\,\frac{z\tilde{F}(s)}{1-z\tilde{F}(s)} = 0 \qquad \text{for } s > 0$$

$$\Longrightarrow \quad z = 0$$

$$\Longrightarrow \quad a + b = 1.$$

Thus,

$$\mathbf{P} = \begin{bmatrix} 1-a & a \\ 1-a & a \end{bmatrix}$$

which proves that $\{X_n\}_{n=1}^{\infty}$ is a Bernoulli process.

b $\Longrightarrow$c) See Çinlar (Theorem 4.5.3 p. 89 [1975]).

c $\Longrightarrow$a) Is obvious. □

Note that this is a result about the uniqueness of the thinning process whereas 4.2.5 was a result about the uniqueness of the process being thinned.

From 2.16 we can see that the covariance function is restricted to certain forms. Given this restriction, 2.16 can be used to design a system to produce a pre-determined covariance function. We do this now for the case of a Poisson process being thinned by a Markov chain whose initial distribution is the invariant distribution of the chain.

<u>(2.18) Proposition</u>. If $F(t) = 1-e^{-\lambda t}$ and $\pi_0 = y$, then

$$\mathrm{cov}(N_A(t),N_B(t)) = C(t - \frac{1}{\nu}(1-e^{-\nu t}))$$

where,

$$C = \frac{\lambda(1-a-b)}{a+b} (1 - (\frac{b}{a+b})^2) \tag{2.19}$$

and,

$$\nu = \lambda(a+b). \tag{2.20}$$

<u>Proof</u>: Let $\tilde{R}_{AB}(s)$ = Laplace-Stieltjes transform of $\mathrm{cov}(N_A(t),N_B(t))$. Then from the proof of 2.17,

$$\tilde{R}_{AB}(s) = (1-y)^2\tilde{R}(s)\tilde{R}_z(s). \tag{2.21}$$

Since $\tilde{R}(s) = \lambda/s$ and $\tilde{F}(s) = \lambda/\lambda+s$, 2.19 reduces to,

$$\tilde{R}_{AB}(s) = (1-y)^2 \frac{\lambda^2 z}{s(s+\lambda(1-z))} .$$

If we expand this in partial fractions we have,

$$\tilde{R}_{AB}(s) = \frac{(1-y^2)z\lambda}{1-z} [\frac{1}{s} + \frac{1}{s+(1-z)\lambda}]. \tag{2.22}$$

Inverting this gives,

$$\mathrm{cov}(N_A(t),N_B(t)) = \frac{z(1-y^2)}{1-z} (t - \frac{1}{\lambda(1-z)} (1-e^{-\lambda(1-z)t})) \tag{2.23}$$

which is the result. □

Now, if we specify C and ν we can design the transition probabilities of the Markov chain to obtain the desired covariance function. This involves solving 2.19 and 2.20 for a and b. This gives,

$$a = \frac{\nu}{\lambda} \sqrt{1 - \frac{\nu C}{\lambda(\lambda-\nu)}} \tag{2.24}$$

and,

$$b = \frac{\nu}{\lambda}\left(1 - \sqrt{1 - \frac{\nu C}{\lambda(\lambda-\nu)}}\right) \tag{2.25}$$

The design is valid only if a and b are in the interval (0,1). Further since we have,

$$\frac{\nu}{\lambda} = a + b$$

we must also have,

$$\frac{\nu}{\lambda} < 2.$$

(2.26) Example. Let $\lambda = 2$, $C = 0.5$ and $\nu = 1$. Then, from 2.24 and 2.25 we have,

$$a = \frac{1}{2}\sqrt{1 - \frac{0.5}{2(2-1)}} = 0.433$$

$$b = \frac{1}{2}\left(1 - \sqrt{1 - \frac{0.5}{2(2-1)}}\right) = 0.067$$

and,

$$\mathrm{cov}(N_A(t), N_B(t)) = 0.5(t - (1-e^{-t})).$$

We can realize more general covariance functions by using more of the features present in 2.16.

3. Summary

In this chapter we used our results of Chapter 5 to show two things. Firstly, if an alternating Markov process is thinned by a Bernoulli process, the thinned process will be uncorrelated if and only if the Markov process is Poisson. Secondly, a Poisson process being thinned by an invariant Markov chain produces uncorrelated processes if and only if the chain is a Bernoulli process. We also gave the covariance function obtained when an ordinary renewal process is thinned by a Markov chain. Finally, we showed how one could design the chain to obtain a specified covariance function.

The next chapter concludes this dissertation. There we shall have some closing remarks and point the way to future research in the area covered by the document.

CHAPTER 7

CONCLUSION

In this dissertation we have examined thinning problems for point and marked point processes. Specifically, we have studied covariance structures between the thinned processes. In Chapter 1 we informally introduced the problem and discussed relevant literature on the topic. Chapter 2 was a brief discussion of the concepts of point and marked point process theory which formed the theoretical basis for what followed. We then studied, in Chapter 3, the case where a point process is thinned by an independent Bernoulli process. These results were used to study the case where the point process was a renewal process, which gave us uniqueness results about the nature of the process. Chapter 4 studied mark dependent thinning of Markov renewal processes. The results were used to solve a non-linear integral equation. In Chapter 5 we used the results of Chapter 4 to study the overflow queue which produced useful insights into renewal theory. Also, we demonstrated that it is possible to study superposition problems within the context of thinning. Finally, in Chapter 6 we studied two specific thinning problems which produced more uniqueness results about the processes involved. We were also able to invert the analysis to design the process doing the thinning to obtain a pre-specified covariance function.

We now outline areas for further research that arise as a consequence of the work in this document.

1. Covariance and Independence

We have seen that in many cases the zero covariance of thinned processes in fact implied their independence. We specifically refer to Corollaries 3.3.15 and 3.4.6 and Theorems 6.1.9 and 6.2.17. In all these results we assumed that the thinning process was independent of the process being thinned. It would be useful to explore how far the independence assumption could be relaxed and still obtain similar results. Further, the results obtained all referred to renewal processes. We would like to ascertain if the four results referred to above hold without the renewal assumption. We believe the first step would be to solve the matrix non-linear integral equation 4.2.12 for the situation where a Markov renewal process is thinned by an independent Bernoulli process.

2. General Thinning

The problems studied here involved thinning a possibly marked point process into two processes. Moreover, the thinning decision at each event depends only on the mark of this event. The next step would be to considerably generalize the problem as follows.

Let $(X,T) = \{(X_n,T_n)\}_{n=1}^{\infty}$ be an arbitrary marked point process, with mark space E, where X_n is the mark of the event at T_n. Let $Y = \{Y_n\}_{n=1}^{\infty}$ be a sequence of random variables taking values in $\mathscr{D} = \{0,\ldots,k\}$. Let the distribution function of Y_n be given as follows.

$$P\{Y_n = i|X_0,\ldots,X_n,T_0,\ldots,T_n\} = h(i,X_0,\ldots,X_n,T_0,\ldots,T_n) \qquad (2.1)$$

Let N^i, $i \in \mathscr{D}$ be k + 1 point process which are defined as,

$$N_i((0,t]) = \sum_{n=1}^{\infty} I_{\{i\}}(Y_n) I_{(0,t]}(T_n). \tag{2.2}$$

Then we say that the processes $N^0,\ldots,N^k$ are obtained by thinning (X,T) with the process Y.

(2.3) Example. Let (X,T) be a Markov renewal process on E, $\mathcal{D} = \{0,1\}$, $\{A,B\}$ a partition of E, and

$$h(1,X_0,\ldots,X_n,T_0,\ldots,T_n) = h(1,X_n) = I_A(X_n)$$

and

$$h(0,X_0,\ldots,X_n,T_0,\ldots,T_n) = h(1,X_n) = I_B(X_n).$$

Then, this is exactly the situation discussed in Chapter 4.

With the general structure given by (2.1) we would like to study the various covariance structures defined below. Let $N^i(t) = N^i((0,t])$.

$$R_{ij}(t,\tau) = \mathrm{cov}(N^i(t),N^j(t+\tau)) \qquad i,j \in \mathcal{D}. \tag{2.4}$$

The interesting question here is that if $R_{ij}(t,\tau) = 0$ for some $i,j \in \mathcal{D}$, are N^i and N^j independent?

3. Joint Distributions

While we have seen (cf. Section 1) that covariance structures can lead to results about joint distributions, it would be of great interest to study joint distributions for the general thinning problem discussed in Section 2. This would be of use in understanding queueing networks. This is largely an unsolved problem. Serfozo [1981] has outlined a possible approach to the problem which we now discuss briefly.

Let N, N^0 and N^1 be (possibly marked) point processes where N^0 and N^1 are obtained from N by some thinning rule.

<u>(3.1) Definition</u>. Let $w,f,g \in \mathcal{F}_c$ where $\mathcal{F}_c$ is the set of all real-valued continuous functions on $\mathbb{R}_+$ with compact support. Define,

$$L_N(w) = E[e^{-\int_{\mathbb{R}_+} w(x)\,dN(x)}], \qquad w \in \mathcal{F}_c \tag{3.2}$$

and,

$$L_{N^0,N^1}(f,g) = E[e^{-\int_{\mathbb{R}_+} f(x)\,dN^0(x) - \int_{\mathbb{R}_+} g(x)\,dN^1(x)}], \qquad f,g \in \mathcal{F}_c. \tag{3.3}$$

Then $L_N(w)$ is called the Laplace functional of N and $L_{N^0,N^1}(f,g)$ is called the joint Laplace functional of N^0 and N^1.

Now, if we define

$$f(x) = \begin{cases} s_0 & 0 < x \leq t \\ 0 & \text{elsewhere} \end{cases} \tag{3.4}$$

and

$$g(x) = \begin{cases} s_1 & 0 < x \leq t + \tau \\ 0 & \text{elsewhere} \end{cases} \tag{3.5}$$

we have,

$$L_{N^0,N^1}(f,g) = L_{N^0,N^1}(s_0,s_1) = E[e^{-s_0 N^0(t) - s_1 N^1(t+\tau)}] \tag{3.6}$$

which is the joint Laplace function of the random variables $N^0(t)$ and $N^1(t+\tau)$. We would like to compute 3.6 for various structures of N and the thinning rule. An interesting question here is the conditions under which we have

$$L_{N^0,N^1}(s_1,s_2) = E[e^{-s_0 N^0(t)}]E[e^{-s_1 N^0(t)}] \tag{3.7}$$

which is the situation where $N^0(t)$ and $N^1(t+\tau)$ are independent random variables. Then we could examine if it is possible to extend such results to say something about the independence of the processes N^0 and N^1.

Covariance structures may, in principle, be also obtained from 3.6 as follows.

$$E[N^0(t)] = - \frac{\partial L_{N^0,N^1}(s_0,s_1)}{\partial s_0} \Bigg|_{\substack{s_0 = 0 \\ s_1 = 0}} . \tag{3.8}$$

$$E[N^1(t+\tau)] = - \frac{\partial L_{N^0,N^1}(s_0,s_1)}{\partial s_1} \Bigg|_{\substack{s_0 = 0 \\ s_1 = 0}} . \tag{3.9}$$

$$E[N^0(t)N^1(t+\tau)] = - \frac{\partial^2 L_{N^0,N^1}(s_0,s_1)}{\partial s_0 \partial s_1} \Bigg|_{\substack{s_0 = 0 \\ s_1 = 0}} . \tag{3.10}$$

Now, for the problem studied in Chapter 4 we have,

$$\begin{aligned} L_{N^0,N^1}(s_0,s_1) &= E[\exp\{-s_0 N_A(t)) - s_1 N_B(t+\tau)\}] \\ &= E[\exp\{-s_0 \sum_{n=1}^{\infty} I_A(X_n) I_{(0,t]}(T_n) \\ &\quad - s_1 \sum_{n=1}^{\infty} I_B(X_n) I_{(0,t+\tau]}(T_n)\}] \\ &= E[\prod_{n=1}^{\infty} \exp\{-s_0 I_A(X_n) I_{(0,t]}(T_n) \\ &\quad - s_1 I_B(X_n) I_{(0,t+\tau]}(T_n)\}]. \end{aligned}$$

To get joint distributions, one has to invert a joint Laplace-Stieltjes transform. For covariances, to get equations to which 3.8, 3.9 and 3.10 may be fruitfully applied requires a level of effort at least as much as that required in Chapter 4. Thus, we refrained from using this approach for this purpose. But, for studying joint distributions it seems that 3.6 would be more useful than a direct approach. The algebra promises to be tedious but, regretably, unavoidable.

4. Superposition

We saw, in Chapter 5, that it was possible to study a superposition problem within the context of thinning. It would be of interest to pursue cases where this approach could be fruitfully exploited. Such a technique would be especially useful in situations where the superposed process is easier to study than the processes being superposed. Here, we feel that the concepts of stochastic reversibility, as discussed by Kelly [1979] would be of immense value.

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THINNING OF POINT PROCESSES-COVARIANCE ANALYSES

by

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(ABSTRACT)

This dissertation addresses a class of problems in point process theory called 'thinning'. By thinning we mean an operation whereby a point process is split into two point processes by some rule. We obtain the covariance structure between the thinned processes under various thinning rules. We first obtain this structure for independent Bernoulli thinning of an arbitrary point process. We show that if the point process is a renewal (stationary or ordinary) process, the thinned processes will be uncorrelated if and only if the renewal process is Poisson in which case the thinned processes are independent. Thus, we have a situation where zero correlation implies independence. We also show that while the intervals between events in the thinned processes may be uncorrelated, the counts need not be.

Next, we obtain the covariance structure between the thinned processes resulting from a mark dependent thinning of a Markov renewal process with a Polish mark space. These results are used to study the overflow queue where we show that in equilibrium the input and overflow processes are uncorrelated as are the output and overflow processes. We thus provide an example where two uncorrelated but dependent renewal processes, neither of which is Poisson but which produce a Poisson process when superposed.

Next, we study Bernoulli thinning of an alternating Markov process and show that the thinned process are uncorrelated if and only if the

process is Poisson in which case the thinned processes are independent. Finally, we obtain the covariance structure obtained when a renewal process is thinned by an independent Markov chain. We show that if the renewal process is Poisson and the chain is stationary, the thinned processes will be uncorrelated if and only if the Markov chain is a Bernoulli process. We also show how to design the chain to obtain a pre-specified covariance function.

We then close the dissertation by summarizing the work presented and indicating areas of further research including a short discussion of the use of Laplace functionals in the determination of joint distributions of thinned processes.