

Spectral Covers, Integrality Conditions, and Heterotic/F-theory Duality

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ABSTRACT. In this work we review a systematic, algorithmic construction of dual heterotic/F-theory geometries corresponding to 4-dimensional, $\mathcal{N} = 1$ supersymmetric compactifications. We look in detail at an exotic class of well-defined Calabi-Yau fourfolds for which the standard formulation of the duality map appears to fail, leading to dual heterotic geometry which appears naively incompatible with the spectral cover construction of vector bundles. In the simplest class of examples the F-theory background consists of a generically singular elliptically fibered Calabi-Yau fourfold with E_7 symmetry. The vector bundles arising in the corresponding heterotic theory appear to violate an integrality condition of an $SU(2)$ spectral cover. A possible resolution of this puzzle is explored by studying the most general form of the integrality condition. This leads to the geometric challenge of determining the Picard group of surfaces of general type. We take an important first step in this direction by computing the Hodge numbers of an explicit spectral surface and bounding the Picard number.

1. An algorithm construction of dual heterotic/F-theory geometry

Compactifications of heterotic string theory and F-theory are believed to be dual – that is to lead to the same effective low energy physics – whenever the compactification geometries take the form [9, 10, 11, 12]

$$(1.1) \quad \text{Heterotic on } \pi_h : X_n \xrightarrow{\mathbb{E}} B_{n-1} \Leftrightarrow \text{F-theory on } \pi_f : Y_{n+1} \xrightarrow{K3} B_{n-1}$$

where the $K3$ fiber of Y_{n+1} is itself elliptically fibered over a \mathbb{P}^1 base. The compatibility of these two fibrations leads to the observation that $\rho_f : Y_{n+1} \xrightarrow{\mathbb{E}} \mathcal{B}_n$ and $\sigma_f : \mathcal{B}_n \xrightarrow{\mathbb{P}^1} B_{n-1}$. In recent work [5] this duality was used to systematically enumerate an interesting and finite class of string backgrounds and the properties of the associated 4-dimensional effective theories. As given in (1.1), the choice of geometry in F-theory consists simply of a $K3$ -fibered Calabi-Yau fourfold. For the $E_8 \times E_8$ heterotic string theory the background is determined by an elliptically fibered Calabi-Yau threefold equipped with a pair of poly-stable, holomorphic vector bundles, V_i ($i = 1, 2$) on X_3 with structure groups, $H_i \subset E_8$.

In [5] a program was set out to systematically study the general properties and constraints of the dual effective theories and develop a general and algorithmic

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formalism to build consistent heterotic/F-theory backgrounds. With this goal in mind, the first step in constructing a pair of the form (1.1) is the choice of a twofold base, B_2 appearing in both the heterotic and F-theory geometry. For all smooth threefolds, X_3 , the possible choices for B_2 have been classified [27] (and B_2 must be a generalized del Pezzo surface). Furthermore, to explore and test general structure there is an important dataset of such manifolds consisting of 61,539 toric surfaces systematically constructed by Morrison and Taylor [2, 1, 8].

With these results in place it is possible to begin to build the geometry of (1.1) from the bottom up. In the Calabi-Yau fourfold geometry the next step is to choose a form for the \mathbb{P}^1 -fibration, $\sigma_f : \mathcal{B}_3 \xrightarrow{\mathbb{P}^1} B_2$. As described in Section 2, this can be accomplished for non-degenerate fibrations by building \mathcal{B}_3 as a \mathbb{P}^1 bundle over B_2 , parameterized by a “twist”: a $(1,1)$ -form T in B_2 (see (2.1)). In the heterotic theory this choice of twist corresponds to a piece of the heterotic vector bundle topology (more specifically, a component of the second Chern class $c_2(V)$) [12]. In [5] we established that given a twofold base B_2 , the set of all possible twists is in fact bounded by the conditions imposed by 4-dimensional $N = 1$ supersymmetry. In the heterotic theory this appears through the condition of slope stability of the vector bundles V_i and in F-theory by the condition that the generically singular fourfold Y_4 admits a smooth Calabi-Yau resolution. Finally it should be noted that since we require all fibrations to admit (exactly one) section, each elliptically fibered manifold is birationally equivalent to a Weierstrass model [49] (see (2.2)). Thus, having chosen B_2 and constructed a \mathbb{P}^1 -bundle \mathcal{B}_3 , we have fully specified X_3 and Y_4 .

With consistency conditions in place and a scheme for algorithmically constructing pairs as in (1.1), it remains to extract patterns and structure from the effective theories. Duality here provides a powerful tool to determine otherwise difficult to calculate information on both sides of the theory. While historically heterotic/F-theory duality has been used to determine the effective physics of the mysterious and non-lagrangian F-theory, in [5] we also explored ways in which the singularity structure of the F-theory fourfold could be used to determine non-trivial information about $\mathcal{M}_\omega(c(V))$ – the moduli space of sheaves that are semi-stable with respect to the Kähler form ω with fixed total Chern class $c(V)$. Such information is hard won, since very few techniques exist to determine $\mathcal{M}_\omega(c(V))$ for sheaves/bundles over Calabi-Yau threefolds (or their associated higher-rank Donaldson-Thomas invariants).

As one simple illustration of this correspondence, we note here that the presence of generic symmetries on singular Calabi-Yau fourfolds make it possible to derive correlations between the topology of a slope-stable heterotic vector bundle on a CY threefold and its structure group. Initial investigations of this nature were first undertaken in [15, 14] who constructed “lower bounds” on the second Chern class of a vector bundle with fixed structure group. In [5], we continue to explore the links between structure group and topology, exploring not only these lower bounds but also upper bounds as well (see Section 6 of [5]).

Systematic patterns such as those shown in Table 1 are of use in string phenomenology (for example they could simplify recent algorithmic searches for heterotic Standard Models carried such as those carried out in [20, 21, 22, 18, 17, 4]). In order to fully understand such patterns though, it is necessary to complete the geometric “dictionary” which matches heterotic/F-theory geometry. This includes

Structure Group, H	Topology	Structure Group, H	Topology
$SU(N)$	$\eta \geq N \cdot c_1(B_2)$	E_8	$\eta \geq 5 \cdot c_1(B_2)$
$SO(7)$	$\eta \geq 4 \cdot c_1(B_2)$	E_7	$\eta \geq \frac{14}{3} \cdot c_1(B_2)$
$SO(M)$	$\eta \geq \frac{M}{2} \cdot c_1(B_2)$	E_6	$\eta \geq \frac{9}{2} \cdot c_1(B_2)$
$Sp(K)$	$\eta \geq 2K \cdot c_1(B_2)$	G_2	$\eta \geq \frac{7}{2} \cdot c_1(B_2)$
		F_4	$\eta \geq \frac{7}{2} \cdot c_1(B_2)$

TABLE 1. Constraints linking the topology, $\eta = c_2(V)|_{B_2}$, of an H -bundle V and its structure group on an elliptically fibered CY threefold, $\pi_h : X_3 \rightarrow B_2$. [15, 14].

the inclusion of G -flux in the F-theory background and an understanding of the zero-locus of the induced Gukov-Vafa-Witten superpotential [48]. In this context the quantization conditions on flux and the corresponding constraints in the heterotic theory become particularly important. Indeed, as described in detail [5], in our systematic search, we find many geometries which appear mysterious from the point of view of these commonly assumed integrality conditions.

In the following sections we will review the standard construction of heterotic/F-theory dual pairs. In its most explicit form, the duality map depends on a particular method of constructing Mumford poly-stable vector bundles – namely, the *spectral cover construction* [12]. In recent work [5, 6] it has been observed that many apparently consistent F-theory fourfolds have topology which appears to be inconsistent with a naive construction of spectral cover bundles. We will explore this discrepancy further in concrete examples in the following Sections.

Out of the dataset generated in [5], we consider one of the simplest examples of such an exotic heterotic/F-theory dual pair. In particular, we explore the so-called “integrality” condition on the spectral data (see (2.16)) and set out to determine whether it is really correct/necessary as frequently implemented in the literature. In addition, we lay out the necessary geometric questions that must be addressed if this criterion is to be refined or improved. We will argue that in many cases the surface forming the $SU(2)$ spectral cover can have a larger Picard group than is generically assumed and that the heterotic bundle can in fact be described by a consistent spectral cover pair $(\mathcal{S}, \mathcal{L}_{\mathcal{S}})$, consisting of a 2-sheeted cover $\pi_{\mathcal{S}} : \mathcal{S} \rightarrow B_2$ and a line bundle over it $\mathcal{L}_{\mathcal{S}}$ over it. We begin with a brief review of heterotic/F-theory duality in 4-dimensions to set the stage for these investigations.

2. Heterotic/F-theory Duality in 4-dimensions

In this section we will provide a rough outline of the geometric correspondence that arises in heterotic and F-theory dual pairs. Many excellent reviews exist in the literature and we refer the reader to classic sources such as [12, 16] and modern summaries such as [5] for a more complete treatment. In recent work, [5] a constructive algorithm was developed to consistently build and enumerate dual heterotic/F-theory geometries. As a tractable starting point for that work, heterotic backgrounds were considered consisting of a smooth elliptically fibered Calabi-Yau threefold X_3 (with a single section¹) over a base B_2 , together with two

¹For geometries without section and some of the physics of these more general genus-1 fibrations see recent progress in [45, 46, 47].

holomorphic, Mumford poly-stable vector bundles [7]. In such cases, the dual F-theory compactification geometry can be built beginning with a rationally fibered threefold base \mathcal{B}_3 that is a \mathbb{P}^1 bundle over B_2 (the same surface used to define the heterotic Calabi-Yau threefold). The F-theory compactification space is then an elliptically and K3-fibered fourfold, $\rho_f : Y_4 \rightarrow \mathcal{B}_3$. Following [12], without loss of generality, the non-degenerate \mathbb{P}^1 -fibered base (\mathcal{B}_3) can be defined as a \mathbb{P}^1 bundle through the projectivization of a sum of two line bundles

$$(2.1) \quad \mathcal{B}_3 = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}),$$

where \mathcal{L} is a general line bundle on the base B_2 . Over \mathcal{B}_3 , the classes $R = c_1(\mathcal{O}(1)), T = c_1(\mathcal{L})$, can be defined, where $\mathcal{O}(1)$ is a bundle that restricts to the usual $\mathcal{O}(1)$ on each \mathbb{P}^1 fiber. The \mathbb{P}^1 fibration is equipped with sections Σ_- and $\Sigma_+ = \Sigma_- + T$ of \mathcal{B}_3 satisfying $\Sigma_- \cdot \Sigma_+ = 0$, corresponding to the relation $R(R + T) = 0$ in cohomology.

Finally, then the fourfold itself can be described in Weierstrass form as

$$(2.2) \quad y^2 = x^3 + fx + g$$

where y, x are (affine) coordinates along the elliptic fiber and $f \in H^0(\mathcal{B}_3, K_3^{-4}), g \in H^0(\mathcal{B}_3, K_3^{-6})$. As usual the position of singular fibers is encoded in the discriminant locus, $\Delta = 4f^3 + 27g^2$.

For this choice of an F-theory model on Y_4 and a heterotic theory on X_3 , it is now possible to begin by matching topology [12, 16]. Starting with the $E_8 \times E_8$ heterotic theory, the bundle decomposes as $V_1 \oplus V_2$, and without loss of generality, the curvatures split as

$$(2.3) \quad \frac{1}{30} \text{Tr } F_i^2 = \eta_i \wedge \omega_0 + \zeta_i, \quad i = 1, 2$$

where η_i, ζ_i are (pullbacks of) 2-forms and 4-forms on B_2 and ω_0 is Poincaré dual to the zero-section of the elliptic fibration. The heterotic Bianchi identity [7] gives $\eta_1 + \eta_2 = 12c_1(B_2)$. Thus, it is possible to parameterize a solution as

$$(2.4) \quad \eta_{1,2} = 6c_1(B_2) \pm T', \quad (E_8 \times E_8)$$

where T' is a $\{1, 1\}$ form on B_2 . Next, returning to the F-theory geometry described above in (2.1), the canonical class of \mathcal{B}_3 is determined by adjunction to be

$$(2.5) \quad -K_3 = 2\Sigma_- - K_2 + T,$$

By studying the 4-dimensional effective theories of these dual heterotic/F-theory compactifications it is straightforward to determine that the defining $\{1, 1\}$ forms T, T' in B_2 are in fact identical: $T = T'$ [12, 3]. The $\{1, 1\}$ -form T is referred to as the “twist” (of the \mathbb{P}^1 -fibration) and is the crucial defining data of the simplest classes of heterotic/F-theory dual pairs.

2.1. The spectral cover construction. To explicitly match the degrees of freedom – including the geometric moduli – of a heterotic/F-theory dual pair, it is necessary to modify our description of the slope-stable holomorphic vector bundles arising as part of the heterotic background. A powerful tool to this end is the description of vector bundles known as the “spectral cover construction²” [12, 24, 25, 26]. In the simplest cases it is possible to form a $1 - 1$, onto map from a

²More generally, the “cameral” cover construction [25, 24].

suitable³ slope-stable, holomorphic, rank N vector bundle $\pi : V \rightarrow X_3$ to a pair $(\mathcal{S}, \mathcal{L}_{\mathcal{S}})$ (referred to as the “spectral data”) where \mathcal{S} is a smooth divisor in X_3 (forming an N -fold cover of the base B_2 and referred to as the “spectral cover”) and $\mathcal{L}_{\mathcal{S}}$ is a line bundle⁴ over \mathcal{S} .

The spectral cover construction has been used extensively in heterotic theories to construct rank N bundles with structure group $SU(N)$ or $Sp(2N)$ that are slope-stable in some region of Kähler moduli space. As shown in [12], the class of the divisor \mathcal{S} is given by

$$(2.6) \quad [\mathcal{S}] = N[\sigma] + \pi^*(\eta)$$

where σ is the zero section of $\pi : X_3 \rightarrow B_2$ and η is defined as in (2.3) and (2.4).

It is helpful to once again describe the elliptically fibered heterotic threefold in Weierstrass form:

$$(2.7) \quad \hat{Y}^2 = \hat{X}^2 + f(u)\hat{X}\hat{Z}^4 + g(u)\hat{Z}^6$$

where $\{\hat{X}, \hat{Y}, \hat{Z}\}$ are coordinates on the elliptic fiber (described as a degree six hypersurface in \mathbb{P}_{123}) and $\{u\}$ are coordinates on the base B_2 . Here $\hat{Z} = 0$ defines the section σ . For $SU(N)$ bundles, the *spectral cover*, \mathcal{S} , can be represented as the zero set of the polynomial

$$(2.8) \quad s = a_0\hat{Z}^N + a_2\hat{X}\hat{Z}^{N-2} + a_3\hat{Y}\hat{Z}^{N-3} + \dots$$

ending in $a_N\hat{X}\hat{Z}^{\frac{N}{2}}$ for N even and $a_N\hat{X}\hat{Z}^{\frac{N-3}{2}}\hat{Y}$ for N odd [12]. The polynomials a_i are sections of line bundles over the base B_2

$$(2.9) \quad a_i \in H^0(B_2, K_{B_2}^{\otimes i} \otimes \mathcal{O}(\eta)),$$

In order for the spectral cover to be an actual algebraic surface in X_3 (a necessary condition for the associated vector bundle to be Mumford slope-stability) it is necessary that \mathcal{S} be an effective class in $H_4(X_3, \mathbb{Z})$. There is a further condition – that the spectral cover must be *indecomposable* – that must be imposed in order for the spectral cover bundle V to be slope stable. It can be seen that \mathcal{S} is indecomposable if η is base-point free (*i.e.*, has no base locus in a Zariski-type decomposition and $\eta - Nc_1(B_2)$ is effective (see [29] for example)).

All that remains to fully determine the holomorphic bundle V is the data of the rank 1 sheaf, $\mathcal{L}_{\mathcal{S}}$. As described in [12], given the projection $\pi_{\mathcal{S}} : \mathcal{S} \rightarrow B_2$, the Grothendieck-Riemann-Roch theorem [30] indicates that

$$(2.10) \quad \pi_{\mathcal{S}*} \left(e^{c_1(\mathcal{L}_{\mathcal{S}})} Td(\mathcal{S}) \right) = ch(\pi_*(V)) Td(B_2)$$

At the level of the first Chern class this yields

$$(2.11) \quad \pi_{\mathcal{S}*} \left(c_1(\mathcal{L}_{\mathcal{S}}) + \frac{1}{2}c_1(\mathcal{S}) \right) = \frac{N}{2}c_1(B_2) + c_1(V)$$

At this point, the condition that $c_1(V) = 0$ (necessary for our choice of $SU(N)$ bundle $V \rightarrow X_3$) fixes $c_1(\mathcal{L}_{\mathcal{S}}) \in H^{1,1}(\mathcal{S}) \cap H^2(\mathcal{S}, \mathbb{Z})$ up to a class $\gamma \in \ker(\pi_{\mathcal{S}*})$. Since $\pi_{\mathcal{S}}$ is an N -sheeted cover of B_2 , $\pi_{\mathcal{S}*}\pi_{\mathcal{S}}^*(c_1(B_2)) = Nc_1(B_2)$ and hence

$$(2.12) \quad c_1(\mathcal{L}_{\mathcal{S}}) = \frac{N\sigma + \eta + c_1(B_2)}{2} + \gamma$$

³Here suitability is rigorously defined via the concept of “regularity” [13, 41].

⁴More generally, a rank 1 sheaf. For interesting physical examples where this distinction is crucial see [43, 42, 44].

with

$$(2.13) \quad \pi_{\mathcal{S}*}(\gamma) = 0$$

Here we are faced with the generally difficult problem of determining γ . We will return to this in the next section, but for now we simply review the observations made in [12]: $c_1(\mathcal{L}_{\mathcal{S}})$ must be an integral $(1, 1)$ -class on \mathcal{S} . For the cases of interest, such classes may be scarce since it can be verified that frequently $h^{2,0}(\mathcal{S}) \neq 0$. As a result, the only obvious $(1, 1)$ -classes on \mathcal{S} are those inherited from X_3 , namely the restriction of the zero section of the elliptic fibration, σ , and pullbacks $\pi_{\mathcal{S}}^*(\beta)$ of integral $(1, 1)$ classes on B_2 .

Since $\pi_{\mathcal{S}*}\sigma|_{\mathcal{S}} = \eta - Nc_1(B)$ one finds [12] that a description of $\gamma \in \ker(\pi_{\mathcal{S}*})$ in this “obvious” basis is

$$(2.14) \quad \gamma = \lambda(N\sigma|_{\mathcal{S}} - \pi_{\mathcal{S}}^*(\eta - Nc_1(B)))$$

where λ must be either integer or half integer according to

$$(2.15) \quad \lambda = \begin{cases} m + \frac{1}{2}, & \text{if } N \text{ is odd} \\ m, & \text{if } N \text{ is even} \end{cases}$$

When N is even it is clear that this **integrality condition** imposes

$$(2.16) \quad \eta = c_1(B_2) \pmod{2}$$

where “mod 2” indicates that η and $c_1(B_2)$ differ only by an even element of $H^2(B_2, \mathbb{Z})$. This leads to the form most commonly assumed in the literature [12]:

$$(2.17) \quad c_1(L_{\mathcal{S}}) = N \left(\frac{1}{2} + \lambda \right) \sigma + \left(\frac{1}{2} - \lambda \right) \pi_{\mathcal{S}}^* \eta + \left(\frac{1}{2} + N\lambda \right) \pi_{\mathcal{S}}^* c_1(B_2)$$

Having fully specified the topology of the spectral cover, it is possible to infer the full topology of V itself. The Chern classes of a spectral cover bundle V , specified by η and the integers n and λ is [12, 13, 28, 23]

$$(2.18) \quad c_1(V) = 0$$

$$(2.19) \quad c_2(V) = \eta\sigma - \frac{N^3 - N}{24} c_1(B_2)^2 + \frac{N}{2} \left(\lambda^2 - \frac{1}{4} \right) \eta \cdot (\eta - Nc_1(B_2))$$

$$(2.20) \quad c_3(V) = 2\lambda\sigma\eta \cdot (\eta - Nc_1(B_2))$$

Note that since $c_1(V) = 0$, $\text{Ind}(V) = ch_3(V) = \frac{1}{2}c_3(V)$.

The spectral cover construction provides a powerful tool in explicitly matching the geometric moduli of heterotic/F-theory dual pairs. For the details of the duality map and the necessary stable degeneration limit, we refer the reader to the classic references [12, 16] and conclude here with only a rough hint in Table 2 of how the degrees of freedom associated to $(\mathcal{S}, \mathcal{L}_{\mathcal{S}})$ correspond to the moduli of a Calabi-Yau fourfold in F-theory. In later investigations, we will further compare the structure of an $SU(2)$ spectral cover with its F-theory dual consisting of a generically singular fourfold with E_7 symmetry.

3. A database of Heterotic/F-theory dual pairs

In [5] a systematic algorithm was laid out for constructing heterotic/F-theory dual pairs in which \mathcal{B}_3 (the base of the elliptically fibered fourfold geometry) is constructed as a \mathbb{P}^1 bundle over B_2 . To illustrate the methods of construction, the complete dataset of Calabi-Yau fourfolds with smooth heterotic duals and *toric*

Het/Bundle	Het/Spec. Cov.	F-theory
$H^1(\text{End}_0(V))$	$H^{2,0}(\mathcal{S}) \sim \text{Def}(\mathcal{S})$	$H^{3,1}(\tilde{Y}_4)$
	$H^{1,0}(\mathcal{S}) \sim \text{Pic}_0(\mathcal{S})$	$H^{2,1}(\tilde{Y}_4)$
	$H^{1,1}(\mathcal{S}) \sim \text{Discrete data of } \mathcal{L}_{\mathcal{S}}$	$H^{2,2}(\tilde{Y}_4, \mathbb{Z})$

TABLE 2. A rough, schematic matching of the heterotic vector bundle moduli, encoded as spectral data $(\mathcal{S}, \mathcal{L}_{\mathcal{S}})$, and geometric moduli of the (resolved) F-theory fourfold in the stable degeneration limit [12, 16].

twofold bases were enumerated. This consisted of 4962 Calabi-Yau fourfolds, dual to heterotic threefold/bundle geometry. Of these, 947 were found to be generically singular with an E_7 symmetry (in at least one heterotic E_8 factor, equivalently F-theory coordinate patch). In the heterotic theory the E_7 gauge symmetry is realized by the commutant structure within E_8 , via an $SU(2)$ vector bundle over the dual Calabi-Yau threefold. These rank 2 vector bundles provide one of the simplest windows into the generic properties of the bundle moduli space $\mathcal{M}_{\omega}(c(V))$. Because of the fact that these E_7 symmetries are un-Higgsable – that is the fourfolds are generically singular for all values of the complex structure moduli, the results of Table 1 indicate that for this choice of η the moduli space of stable sheaves contains only $SU(2)$ bundles.

Since the heterotic/F-theory duality map is most clearly understood in the case that the heterotic bundles can be described via spectral covers, it is natural to ask whether we can use this formalism to explicitly match the full degrees of freedom in dual E_7 effective theories described above.

As described in [5], the three conditions on the defining topological data, η , for consistent spectral covers are

- η effective
- η base-point-free within B_2
- $\eta = c_1(B_2) \bmod 2$

In [5], it was explored how these conditions compare to those arising in defining good Calabi-Yau fourfold backgrounds for F-theory. It can be shown that the first of these conditions is true for all K3-fibered fourfolds arising as F-theory backgrounds. Moreover, it can be shown that if the second condition is violated for a fourfold with a generic E_7 singularity, then the Calabi-Yau manifold is too singular to admit a Kähler resolution. To that point, the geometric consistency conditions on an F-theory fourfold and an $SU(2)$ heterotic spectral cover bundle are identical. However, as we will see, at the final condition, this agreement appears to end.

The condition $\eta = c_1(B_2) \bmod 2$ is required for the integrality of $\mathcal{L}_{\mathcal{S}}$ in (2.16). However, a direct construction of the dataset in [5] shows immediately that this is violated for most fourfolds with generic E_7 symmetries – in fact, 897 of the 947! How then are we to make sense of these dual pairs?

One obvious resolution to the puzzle could occur if none of the 897 moduli spaces of $SU(2)$ bundles could admit any bundle built via the spectral cover construction. While possible, this seems unlikely from experience of how generic spectral cover bundles appear to be in known moduli spaces [41]. Another possible

answer is that the integrality condition placed on $c_1(\mathcal{L}_S)$ in (2.16) may be artificially restrictive. This will clearly be the case whenever the Picard number of \mathcal{S} is greater than $1 + h^{1,1}(B_2)$ as assumed by [12].

One class of examples in which the Picard group of \mathcal{S} is larger than the generic case was outlined in [23]. There, it was pointed out that if the matter curve $a_2 = 0$ in (4.12) (in the class $[\eta - 2c_1(B_2)]$) is reducible in B_2 , its components may in fact pull back to distinct, new divisors in \mathcal{S} . That is, if the curve $\bar{\eta} \in [\eta - 2c_1(B_2)]$ can be written as $\bar{\eta} = D + D' \subset B_2$, then its pullback can be described as

$$(3.1) \quad \pi_{\mathcal{S}}^*(\bar{\eta}) = \mathcal{D} + \mathcal{D}'$$

and even if D, D' are well-understood divisors in B_2 , the class \mathcal{D} in \mathcal{S} may not be a simple linear combination of the divisors $\sigma|_{\mathcal{S}}$ and $\pi_{\mathcal{S}}^*(\phi)$ (with ϕ an effective curve class in B_2) assumed in the generic formula (2.12). In [6] we explored whether or not this observation could alleviate the disparity of the mysterious 897 E_7 theories found in [5]. While a handful of the examples found over Hirzebruch bases could be resolved by this mechanism, the majority of them remained unexplained [6]. To really resolve this puzzle and decide whether or not these geometries consist of valid heterotic/F-theory dual pairs, it is necessary to go further and attempt to study the integrality condition in detail. We turn to this now in the context a simple example of an $SU(2)$ bundle defined over $\pi_h : X_3 \rightarrow \mathbb{P}^2$.

4. A case study: bounding the Picard number $\rho(\mathcal{S})$

To begin, it is useful to summarize the discussion of the previous sections in the context of an $SU(2)$ spectral cover. To fully specify the $SU(2)$ gauge bundle appearing in the heterotic compactification, it is not enough to choose a spectral cover of the form given in (2.6) and (2.8), we must also fully describe the line bundle, \mathcal{L}_S over \mathcal{S} . A priori, we can describe the 1st Chern class of \mathcal{L}_S via (2.12) as

$$(4.1) \quad c_1(\mathcal{L}_S) = \frac{N\sigma + \eta + c_1(B_2)}{2} + \gamma$$

where $\pi_{\mathcal{S}*}(\gamma) = 0$. By the construction of $\mathcal{S} \subset X_3$ there are $1 + h^{1,1}(B_2)$ natural integral $(1, 1)$ -classes on \mathcal{S} (consisting of the restriction of σ , the section of the elliptic fibration, and the pullback of classes from the base). Using these as basis (and ignoring any other possibilities for γ) the integrality condition given in (2.14) and (2.16) were obtained in [12]. However, recent work on the F-theory side of the duality [5] indicates that this integrality condition appears to be violated in the vast majority of known examples (897 of 947 generic E_7 models enumerated in [5]) we must now ask whether or not it is possible to derive a more general integrality condition for $c_1(\mathcal{L}_S)$? To accomplish this, \mathcal{L}_S must be expressed in a complete basis. We are thus led to the following question:

QUESTION 4.1. *For a general surface $\mathcal{S} \subset X_3$ (as described above) which is a ramified, N -sheeted cover of B_2 in the class $[N\sigma + \pi_h^*(\eta)]$ what is the rank of the Picard group of \mathcal{S} ?*

As illustrated in the next Section, generically $h^{1,0}(\mathcal{S}) = 0$ [12] and \mathcal{S} is a surface of general type. Unfortunately, determining the Picard number of such complex surfaces is a notoriously difficult problem (see [33, 34, 35] and references therein

for some recent advances). To begin, it is enough to consider ways to bound the Picard number $\rho(\mathcal{S})$ as an important first step.

Let us briefly recall a few standard definitions regarding the Picard group (see [30, 36] for example). To define divisors (and hence line bundles), one begins with the exponential sequence

$$(4.2) \quad 0 \rightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{exp} \mathcal{O}^* \rightarrow 0$$

where the map i is an inclusion and exp is the exponential map. With vanishing $Pic_0(\mathcal{S})$ (i.e. with $h^{1,0}(\mathcal{S}) = h^1(\mathcal{S}, \mathcal{O}) = 0$ there are no continuous degrees of freedom in the Picard group), the associated long exact sequence in cohomology takes the form

$$(4.3) \quad 0 \rightarrow H^1(\mathcal{S}, \mathcal{O}^*) \rightarrow H^2(\mathcal{S}, \mathbb{Z}) \rightarrow H^2(\mathcal{S}, \mathcal{O})$$

The image of $H^1(\mathcal{S}, \mathcal{O}^*)$ (modulo torsion) in $H^2(\mathcal{S}, \mathbb{Z})$ parametrizes the Neron-Severi group, $NS(\mathcal{S})$, of the surface and its rank is the Picard number (i.e. $\rho(\mathcal{S})$, the number of discrete parameters which we can use to construct $\mathcal{L}_{\mathcal{S}}$). The Picard group is given by the kernel of the map from $H^2(\mathcal{S}, \mathbb{Z})$ to $H^2(\mathcal{S}, \mathcal{O}) = H^{0,2}$. The Hodge decomposition and Lefschetz' theorem [30] demonstrate that it is also zero in $H^{2,0}$ and hence must be a subset of $H^{1,1}$:

$$(4.4) \quad NS(\mathcal{S}) \simeq H^2(\mathcal{S}, \mathbb{Z}) \cap H^{1,1}(\mathcal{S})$$

Stated simply, divisors on \mathcal{S} are determined by how the complex subspace $H^{1,1}$ of $H^2(\mathcal{S}, \mathbb{C})$ intersects the discrete subgroup $H^2(\mathcal{S}, \mathbb{Z})$. For surfaces with vanishing geometric genus ($p_g = h^{0,2} = 0$) this is a trivial identification, but few tools exist to address the general case with $p_g \neq 0$. To begin, it should be observed that there is at least a bound:

$$(4.5) \quad \rho(\mathcal{S}) \leq h^{1,1}(\mathcal{S})$$

Since in the present work we are focused on the case of 2-sheeted spectral covers and the mysterious E_7 cases described in the previous section, here we will try to make a first step towards answering this question. We will consider a simple example appearing in [5], with $\pi_{\mathcal{S}} : \mathcal{S} \rightarrow \mathbb{P}^2$. As we will see, even here determining the full Neron-Severi group is a non-trivial problem in algebraic geometry and for this brief work, we content ourselves with simply bounding the Picard number, $\rho(\mathcal{S})$ as described above.

4.1. A double cover of \mathbb{P}^2 . In an explicit example we can explore in detail the possible form of the spectral line bundle, $\mathcal{L}_{\mathcal{S}}$. We consider here a 2-sheeted spectral cover, \mathcal{S} , and one of the simplest examples arising in the dataset of [5]. Let $\pi : X_3 \rightarrow \mathbb{P}^2$ be a Calabi-Yau threefold described via the generic (smooth) Weierstrass model over \mathbb{P}^2 :

$$(4.6) \quad \hat{Y}^2 = \hat{X}^2 + f(u)\hat{X}\hat{Z}^4 + g(u)\hat{Z}^6$$

where u_i ($i = 1, 2, 3$) are homogeneous coordinates of \mathbb{P}^2 and

$$(4.7) \quad f \in H^0(\mathbb{P}^2, \mathcal{O}(12H)) \text{ , } g \in H^0(\mathbb{P}^2, \mathcal{O}(18H))$$

where H is the hyperplane divisor in \mathbb{P}^2 . This Weierstrass model can be realized as hypersurface inside a toric variety. In a language more familiar to physicists, this threefold also be written via a GLSM-style charge matrix (see [32] for example): The hodge numbers of this threefold are well-known to be $h^{1,1} = 2$, $h^{2,1} = 272$.

$$\begin{array}{c|cccccc} & \hat{Y} & \hat{X} & \hat{Z} & u_1 & u_2 & u_3 \\ \hline 6 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 & 1 \end{array}$$

Furthermore, a basis of divisors on X_3 is given by D_1, D_2 where $D_2 = \pi^*(H)$ is the pullback of the hyperplane in \mathbb{P}^2 and D_1 is related to the elliptic fiber such that the class of the zero section ($\hat{Z} = 0$) is given in this basis as $\sigma = D_1 - 3D_2$. The tangent bundle of X_3 is described via adjunction as

$$(4.8) \quad 0 \rightarrow TX_3 \rightarrow T\mathcal{A}|_{X_3} \rightarrow \mathcal{O}(6D_1)|_{X_3} \rightarrow 0$$

where $T\mathcal{A}$ denotes the tangent sheaf of the toric ambient space. This in turn is defined by an Euler sequence [30]:

$$(4.9) \quad 0 \rightarrow \mathcal{O}^{\oplus 2} \rightarrow \mathcal{O}(3D_1) \oplus \mathcal{O}(2D_1) \oplus \mathcal{O}(D_1 - 2D_2) \oplus \mathcal{O}(D_2)^{\oplus 3} \rightarrow T\mathcal{A} \rightarrow 0$$

For this geometry we specify vector bundles and a dual F-theory geometry by making a choice of twist as in Section 2, eq.(2.4). Here we select

$$(4.10) \quad T = 10H$$

In the heterotic theory this leads to an $SU(2)$ bundle $V \rightarrow X_3$ with

$$(4.11) \quad \eta = 6c_1(\mathbb{P}^2) - T = 8H$$

In the dual F-theory geometry this corresponds to a Calabi-Yau fourfold with generic E_7 singularity [5]. From (2.6), the spectral cover is in the class $[\mathcal{S}] = [2\sigma + 8\pi^*(H)]$ which in the basis given above corresponds to a section of the line bundle $N_{\mathcal{S}} = \mathcal{O}(2D_1 + 2D_2)$. Explicitly \mathcal{S} is given by (2.8) as the zero locus of

$$(4.12) \quad a_0\hat{Z}^2 + a_2\hat{X} = 0$$

with $a_0 \in H^0(\mathbb{P}^2, \mathcal{O}(8H))$ and $a_2 \in H^0(\mathbb{P}^2, \mathcal{O}(2H))$. Let us now take a closer look at \mathcal{S} . The complex, Kähler surface is a ramified double cover of \mathbb{P}^2 and we can directly compute its three independent Hodge numbers

$$(4.13) \quad h^{2,0}(\mathcal{S}), h^{1,0}(\mathcal{S}), h^{1,1}(\mathcal{S})$$

To explicitly determine these numbers, we can once again make use of an adjunction formula, this time for \mathcal{S} itself as a hypersurface inside X_3 :

$$(4.14) \quad 0 \rightarrow T\mathcal{S} \rightarrow TX_3|_{\mathcal{S}} \rightarrow \mathcal{O}(2D_1 + 2D_2)|_{\mathcal{S}} \rightarrow 0$$

Furthermore, to determine the cohomology of vector bundles restricted to \mathcal{S} , the Koszul sequence for hypersurfaces

$$(4.15) \quad 0 \rightarrow \mathcal{O}_{X_3}(-2D_1 - 2D_2) \rightarrow \mathcal{O}_{X_3} \rightarrow \mathcal{O}_{\mathcal{S}} \rightarrow 0$$

and its associated long exact sequence in cohomology plays a useful role (see [37] for a review). In the case at hand, all the relevant cohomology groups on X_3 can be determined by considering the defining sequences (4.14), (4.8) and (4.15) and line bundle cohomology on X_3 . For this geometry we employed the techniques of [32] to compute line bundle cohomology on X_3 (as implemented in [31]).

To begin, we note that $h^{2,0}(\mathcal{S}) = H^0(\mathcal{S}, \mathcal{O}(2D_1 + 2D_2)|_{\mathcal{S}})$. Twisting (4.15) by $\mathcal{O}(2D_1 + 2D_2)$ we obtain

$$(4.16) \quad 0 \rightarrow \mathcal{O}_{X_3} \rightarrow \mathcal{O}_{X_3}(2D_1 + 2D_2) \rightarrow \mathcal{O}_{\mathcal{S}}(2D_1 + 2D_2) \rightarrow 0$$

The associated long exact sequence in cohomology leads to

$$(4.17) \quad H^0(\mathcal{S}, \mathcal{O}(2D_1 + 2D_2)|_{\mathcal{S}}) = H^0(X_3, \mathcal{O}(2D_1 + 2D_2))/\mathbb{C}$$

Which can be directly calculated to yield $h^0(\mathcal{S}, \mathcal{O}(2D_1 + 2D_2)|_{\mathcal{S}}) = h^0(X_3, \mathcal{O}(2D_1 + 2D_2)) - 1 = 51 - 1$. This provides the first of three independent hodge numbers (the geometric genus):

$$(4.18) \quad h^{2,0}(\mathcal{S}) = 50$$

Note that this is expected via the description of \mathcal{S} in (4.12). By inspection of that formula it can be noted that there are 51 degrees of freedom in the coefficients a_0, a_2 over \mathbb{P}^2 . Subtracting 1 for the overall scale, we see that this agrees with the expectation of the embedding moduli of $\mathcal{S} \subset X_3$.

Next, note that $h^{1,0} = h^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ (the ‘‘irregularity’’ of the surface). Here the long exact sequence in cohomology associated to (4.15) yields

$$(4.19) \quad h^{1,0}(\mathcal{S}) = 0$$

Finally, to determine $h^{1,1}(\mathcal{S})$, consider the dual sequence

$$(4.20) \quad 0 \rightarrow \mathcal{O}(-2D_1 - 2D_2)|_{\mathcal{S}} \rightarrow TX_3^{\vee}|_{\mathcal{S}} \rightarrow T\mathcal{S}^{\vee} \rightarrow 0$$

To evaluate this it should first be noted that the Koszul sequence for $\mathcal{O}(-2D_1 - 2D_2)$ produces the following short exact sequence

$$(4.21) \quad 0 \rightarrow \mathcal{O}_{X_3}(-4D_1 - 4D_2) \rightarrow \mathcal{O}_{X_3}(-2D_1 - 2D_2) \rightarrow \mathcal{O}_{\mathcal{S}}(-2D_1 - 2D_2) \rightarrow 0$$

and from the associated sequence in cohomology

$$(4.22) \quad \begin{aligned} h^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-2D_1 - 2D_2)) &= h^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-2D_1 - 2D_2)) = 0 \\ h^2(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-2D_1 - 2D_2)) &= 219 \end{aligned}$$

This gives the full cohomology of the first term bundle in (4.20). But what is $H^*(\mathcal{S}, TX_3^{\vee}|_{\mathcal{S}})$? The last necessary pieces can be obtained by considering (4.15) twisted by TX_3^{\vee} :

$$(4.23) \quad 0 \rightarrow TX_3 \otimes \mathcal{O}_{X_3}(-2D_1 - 2D_2) \rightarrow TX_3^{\vee} \rightarrow TX_3^{\vee}|_{\mathcal{S}} \rightarrow 0$$

Here the long exact sequence in cohomology produces

$$(4.24) \quad \begin{aligned} h^0(\mathcal{S}, TX_3^{\vee}|_{\mathcal{S}}) &= 0 \\ h^1(\mathcal{S}, TX_3^{\vee}|_{\mathcal{S}}) &= h^1(X_3, TX^{\vee}) + \dim(\ker(\phi)) = 2 + \dim(\ker(\phi)) \\ h^2(\mathcal{S}, TX_3^{\vee}|_{\mathcal{S}}) &= \dim(\operatorname{coker}(\phi)) \\ \phi : H^2(X, TX_3 \otimes \mathcal{O}_{X_3}(-2D_1 - 2D_2)) &\rightarrow H^2(X, TX_3^{\vee}) \end{aligned}$$

Since $h^2(X, TX_3 \otimes \mathcal{O}_{X_3}(-2D_1 - 2D_2)) = 393$ and $h^2(X, TX_3^{\vee}) = 272$, it follows that $\dim(\ker(\phi)) = 121 + m$ for some $m \geq 0$, and $\dim(\operatorname{coker}(\phi)) = m$ by exactness. In fact, for generic choices of spectral cover in (4.12), we expect the induced map ϕ to be surjective and $h^1(\mathcal{S}, TX_3^{\vee}|_{\mathcal{S}}) = 123$.

With this in hand, we are now in a position to put the pieces together to determine $H^1(\mathcal{S}, T\mathcal{S}^{\vee})$. Using (4.22) and (4.24), and returning to the long exact sequence in cohomology associated to (4.20) gives the following long exact sequence:

$$(4.25) \quad 0 \rightarrow H^2(\mathcal{S}, TX_3^{\vee}|_{\mathcal{S}}) \rightarrow H^{1,1}(\mathcal{S}) \rightarrow H^2(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-2D_1 - 2D_2)) \rightarrow H^2(\mathcal{S}, TX_3^{\vee}|_{\mathcal{S}}) \rightarrow 0$$

It is helpful to note that $h^2(\mathcal{S}, T\mathcal{S}^\vee) = h^{1,0} = 0$, and the alternating sum of the dimensions in (4.25) leads at last to

$$(4.26) \quad h^{1,1}(\mathcal{S}) = (123 + m) + (219 - m) = 342$$

Thus, in summary we have determined that \mathcal{S} is a complex surface with $h^{1,0} = 0$, $h^{2,0} = 50$ and $h^{1,1} = 342$. It follows that the Euler number of \mathcal{S} is $e = 2 + 2p_g + h^{1,1} - 4h^{1,0} = 444$ (with $e = c_2(T\mathcal{S})$) and the holomorphic Euler characteristic is $\chi = 51$ (leading to $K_{\mathcal{S}}^2 = 168$). According to Kodaira's classification, \mathcal{S} is a surface of general type (Kodaira dimension 2).

Taking a step back, one can now ask what we have learned from this example? The first observation is that in this case

$$(4.27) \quad 2 \leq \rho(\mathcal{S}) \leq 342$$

where the lower bound arises from concrete construction of divisors [12] and the upper bound is obtained from $h^{1,1}$ as described in the previous Subsection. It should be noted here that there are in principle hugely more parameters in the spectral data than are commonly assumed in the physics literature. While the full computation of $\rho(\mathcal{S})$ is beyond the scope of the present work, tools exist to analyze the intersection structure of curves in \mathcal{S} and can be used to further constrain $\rho(\mathcal{S})$ in many cases. We hope to explore this in future work. For the moment, in the example above, we expect that $H^2(\mathcal{S}, \mathbb{Z}) \cap H^{1,1}(\mathcal{S})$ will generically be large. Indeed, despite the fact that $p_g = 50$, $h^{1,1}$ is sufficiently big that contrary to the expectations of [12], it may be that the Picard number $\rho(\mathcal{S})$ is considerably above its minimum value of 2. In this case, there are certainly more general choices available for the line bundle, $\mathcal{L}_{\mathcal{S}}$, and the integrality condition in (2.16) is manifestly incorrect and too restrictive.

To proceed further with this explicit example, it might be possible to consider the branch locus of the two-sheeted cover in detail. Such an analysis was undertaken in [50] for certain double covers of \mathbb{P}^2 . There for special choices of topology, the resolution of singularities in the branch curve led to concrete descriptions of the Neron-Severi group of the double cover (which was in fact maximal in those cases). It would be interesting in the future to explore the application of these techniques to heterotic spectral covers.

Finally it should be noted that as \mathcal{S} is varied within the 50-parameter family given in (4.12), the Picard number can surely change. While difficult to compute, these special, higher codimensional ‘‘Noether-Lefschetz Loci’’ [38, 39] may be especially significant for the underlying physics, determining for example, where the complex structure moduli of the dual F-theory geometry, Y_4 are stabilized by G -flux [40].

To conclude, the example above was provided as a simple illustration of the fact that the integrality condition for spectral cover bundles given in (2.16) may be too restrictive in many cases. Furthermore, it serves to highlight the interesting and frequently difficult geometric questions that arise in fully determining the geometry of dual heterotic/F-theory pairs. As a final comment on the mysterious 897 E_7 examples highlighted in [5], the arguments presented above indicate to us that in fact there is more to understand about integrality conditions in spectral covers and that this may provide a resolution to the seeming discrepancy in all the exotic heterotic/F-theory pairs. We hope in future work to build upon the simple examples

considered here and to fully compute the Picard group of \mathcal{S} systematically in the full dataset. By addressing these remaining geometric puzzles we hope it will be possible to complete the program laid out in [5] and fully enumerate all consistent heterotic/F-theory dual pairs.

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References

- [1] V. Kumar, D. R. Morrison and W. Taylor, “Mapping 6D $N = 1$ supergravities to F-theory,” *JHEP* **1002**, 099 (2010) [arXiv:0911.3393 [hep-th]].
- [2] D. R. Morrison and W. Taylor, “Classifying bases for 6D F-theory models,” *Central Eur. J. Phys.* **10**, 1072 (2012) [arXiv:1201.1943 [hep-th]].
- [3] T. W. Grimm and W. Taylor, “Structure in 6D and 4D $N=1$ supergravity theories from F-theory,” arXiv:1204.3092 [hep-th].
- [4] L. B. Anderson, A. Constantin, J. Gray, A. Lukas and E. Palti, “A Comprehensive Scan for Heterotic $SU(5)$ GUT models,” *JHEP* **1401**, 047 (2014) [arXiv:1307.4787 [hep-th]].
- [5] L. B. Anderson and W. Taylor, “Geometric constraints in dual F-theory and heterotic string compactifications,” *JHEP* **1408**, 025 (2014) [arXiv:1405.2074 [hep-th]].
- [6] L. B. Anderson, “Exploring Novel Geometry in Heterotic/F-theory Dual Pairs,” *String-Math 2013*, Proc. of Symp. in Pure Math., 2014, Vol. 88, AMS, 2014.
- [7] M. B. Green, J. H. Schwarz and E. Witten, “Superstring Theory, Vols. 1, 2,” *Cambridge Univ. Pr.* (1987)
- [8] D. R. Morrison and W. Taylor, “Toric bases for 6D F-theory models,” *Fortsch. Phys.* **60**, 1187 (2012) [arXiv:1204.0283 [hep-th]].
- [9] C. Vafa, “Evidence for F-Theory,” *Nucl. Phys. B* **469**, 403 (1996) arXiv:hep-th/9602022.
- [10] D. R. Morrison and C. Vafa, “Compactifications of F-Theory on Calabi–Yau Threefolds – I,” *Nucl. Phys. B* **473**, 74 (1996) arXiv:hep-th/9602114.
- [11] D. R. Morrison and C. Vafa, “Compactifications of F-Theory on Calabi–Yau Threefolds – II,” *Nucl. Phys. B* **476**, 437 (1996) arXiv:hep-th/9603161.
- [12] R. Friedman, J. Morgan and E. Witten, “Vector bundles and F theory,” *Commun. Math. Phys.* **187**, 679 (1997) [hep-th/9701162].
- [13] R. Friedman, J. W. Morgan and E. Witten, “Vector bundles over elliptic fibrations,” alg-geom/9709029.
- [14] P. Berglund and P. Mayr, “Stability of vector bundles from F theory,” *JHEP* **9912**, 009 (1999) [hep-th/9904114].
- [15] G. Rajesh, “Toric geometry and F theory / heterotic duality in four-dimensions,” *JHEP* **9812**, 018 (1998) [hep-th/9811240].
- [16] G. Curio and R. Y. Donagi, “Moduli in $N=1$ heterotic / F theory duality,” *Nucl. Phys. B* **518**, 603 (1998) [hep-th/9801057].
- [17] L. B. Anderson, J. Gray, A. Lukas and E. Palti, “Heterotic Line Bundle Standard Models,” *JHEP* **1206**, 113 (2012) [arXiv:1202.1757 [hep-th]].
- [18] L. B. Anderson, J. Gray, A. Lukas and E. Palti, “Two Hundred Heterotic Standard Models on Smooth Calabi-Yau Threefolds,” *Phys. Rev. D* **84**, 106005 (2011) [arXiv:1106.4804 [hep-th]].
- [19] W. Taylor, “On the Hodge structure of elliptically fibered Calabi-Yau threefolds,” *JHEP* **1208**, 032 (2012) [arXiv:1205.0952 [hep-th]].
- [20] L. B. Anderson, J. Gray, Y. -H. He and A. Lukas, “Exploring Positive Monad Bundles And A New Heterotic Standard Model,” *JHEP* **1002**, 054 (2010) [arXiv:0911.1569 [hep-th]].
- [21] L. B. Anderson, Y. -H. He and A. Lukas, “Monad Bundles in Heterotic String Compactifications,” *JHEP* **0807**, 104 (2008) [arXiv:0805.2875 [hep-th]].
- [22] L. B. Anderson, Y. -H. He and A. Lukas, “Heterotic Compactification, An Algorithmic Approach,” *JHEP* **0707**, 049 (2007) [hep-th/0702210 [HEP-TH]].
- [23] G. Curio, “Moduli restriction and Chiral Matter in Heterotic String Compactifications,” *JHEP* **1201**, 015 (2012) [arXiv:1110.6315 [hep-th]].

- [24] R. Donagi, "Heterotic / F theory duality: ICMP lecture," In *Brisbane 1997, Mathematical physics* 206-213 [hep-th/9802093].
- [25] R. Donagi, "Spectral Covers", in "Current topics in complex algebraic geometry", MSRI pub. **28** (1992), 65-86. [alg-geom/9505009].
- [26] R. Donagi, "Principal Bundles on elliptic fibrations", Asian J. Math. **1** (1997), 214-223. arXiv:alg-geom/9702002.
- [27] M. Gross, "A finiteness theorem for elliptic Calabi-Yau threefolds," Duke Math. J. **74**, 2, 271-299 (1994) arXiv:alg-geom/9305002.
- [28] G. Curio, "Chiral matter and transitions in heterotic string models," Phys. Lett. B **435**, 39 (1998) [hep-th/9803224].
- [29] R. Donagi, Y. -H. He, B. A. Ovrut and R. Reinbacher, "The Particle spectrum of heterotic compactifications," JHEP **0412**, 054 (2004) [hep-th/0405014].
- [30] R. Hartshorne, "Algebraic Geometry," Graduate Texts in Mathematics, 1977, Springer.
- [31] cohomCalc package, Download link, 2010, URL <http://wwwth.mppmu.mpg.de/members/blumenha/cohomcal> high-performance line bundle cohomology computation based on [32].
- [32] R. Blumenhagen, B. Jurke, T. Rahn and H. Roschy, "Cohomology of Line Bundles: A Computational Algorithm," J. Math. Phys. **51**, 103525 (2010) [arXiv:1003.5217 [hep-th]].
- [33] B. Poonen, D. Testa, and R. van Luijk, "Computing Neron-Severi groups and cycle class groups," [arXiv:1210.3720 [alg-geom]]. To appear in Compositio Math.
- [34] M. Schutt, "Picard numbers of quintic surfaces," [arXiv:1308.2525 [alg-geom]].
- [35] A. Beauville, "Some surfaces with maximal Picard number," J. de l'Ecole Poly. Math. **1**, (1014), pg. 101-116.
- [36] R. Miranda, "An overview of algebraic surfaces," Notes for Lectures at the 1995 Summer School on Algebraic Geometry, BICAS, Ankara, Turkey. <http://www.math.tamu.edu/~ptretkoff/miranda.ps>.
- [37] L. B. Anderson, "Heterotic and M-theory Compactifications for String Phenomenology," arXiv:0808.3621 [hep-th].
- [38] S. O. Kim, "Noether-Lefschetz Locus for Surfaces," Tran. of Amer. Math. Soc., Vol 324, No. **1**, (March 1991). pg. 369-384.
- [39] M. L. Green, "Components of maximal dimension in the Noether-Lefschetz locus", J. Diff. Geom. Vol 29, No. **2** (1989), 295-302.
- [40] R. Donagi and M. Wijnholt, "Higgs Bundles and UV Completion in F-Theory," Commun. Math. Phys. **326**, 287 (2014) [arXiv:0904.1218 [hep-th]].
- [41] M. Bershadsky, T. M. Chiang, B. R. Greene, A. Johansen and C. I. Lazaroiu, "F theory and linear sigma models," Nucl. Phys. B **527**, 531 (1998) [hep-th/9712023].
- [42] P. S. Aspinwall and R. Y. Donagi, "The Heterotic string, the tangent bundle, and derived categories," Adv. Theor. Math. Phys. **2**, 1041 (1998) [hep-th/9806094].
- [43] R. Donagi and E. Markman, "Spectral curves, algebraically completely integrable Hamiltonian systems, and moduli of bundles," alg-geom/9507017.
- [44] L. B. Anderson, J. J. Heckman and S. Katz, "T-Branes and Geometry," JHEP **1405**, 080 (2014) [arXiv:1310.1931 [hep-th]].
- [45] V. Braun and D. R. Morrison, "F-theory on Genus-One Fibrations," JHEP **1408**, 132 (2014) [arXiv:1401.7844 [hep-th]].
- [46] D. R. Morrison and W. Taylor, "Sections, multisections, and U(1) fields in F-theory," arXiv:1404.1527 [hep-th].
- [47] L. B. Anderson, I. Garca-Etxebarria, T. W. Grimm and J. Keitel, "Physics of F-theory compactifications without section," arXiv:1406.5180 [hep-th].
- [48] S. Gukov, C. Vafa and E. Witten, "CFT's from Calabi-Yau four folds," Nucl. Phys. B **584**, 69 (2000) [Erratum-ibid. B **608**, 477 (2001)] [hep-th/9906070].
- [49] N. Nakayama, "On Weierstrass models, Algebraic geometry and commutative algebra," Vol. **II**, Kinokuniya, Tokyo, (1988), pp. 405?431. MR 977771 (90m:14030).
- [50] U. Persson, "Horikawa Surfaces with Maximal Picard Numbers", Math. Ann. 259, 287-312 (1982).

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