

Asset Prices under Random Risk Preferences*

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Abstract

We consider an overlapping-generations model with two types of investors: the *stable* investors have constant risk aversion, but the *unstable* investors have random levels of risk aversion across different generations. Investors are not sure about how risk averse future investors are. We show that *i*) a small amount of randomness in the risk aversion or *ii*) a small population of the unstable investors generates a large deviation from fundamental price and a high price volatility.

Keywords: risk aversion, asset pricing, overlapping generations.

JEL Classifications: D80, D90, G12.

1 Introduction

If investors do not know the risk preferences of future generations of investors, how are asset prices today affected? In a simple overlapping-generations model, we find that a small amount of randomness in the risk aversion for a small group of future investors is enough to generate a large deviation from fundamental price and a high price volatility: as the amount of randomness or the population of the investors with random preferences goes to zero, the price goes to negative infinity and the volatility goes to positive infinity.

Our results imply that there is a discontinuity in the equilibrium price when the randomness goes to zero or the population of investors with randomness goes to zero. One

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interpretation is that a small amount of randomness can have large impacts, and another interpretation is that it is only a puzzling and counterintuitive implication of the model that we are looking at. We do not take a stand on this.

Overlapping generations models have been used to understand the asset markets, and of particular interest are models with generations of heterogeneous investors. The model in this paper follows closely that of De Long et al. (1990). Similar models have been widely used ever since (see Mark and Wu (1998) and Jeanne and Rose (2002) for applications of the model to exchange rates, and see Acharya and Pedersen (2005) and Lewellen and Shanken (2002) for applications to stock prices). In De Long et al. (1990), in each generation there are both rational investors and noise traders, and their interaction has an important effect on asset prices. Investors face the risk that noise in the future is unknown. In this paper, the risk investors face is that they do not know how risk averse are the future investors that they will be selling the assets to.

This paper is also motivated by the literature on history-dependent risk aversion. Guiso, Sapienza, and Zingales (2015) and Malmendier and Nagel (2011) provide empirical evidence on risk aversion being time-varying, and they show that it depends strongly on past experience in the financial markets. Most of the theoretical studies in this literature look at time-dependent risk preferences for representative agents (see Dillenberger and Rozen (2015) and Tserenjigmid (2015) for recent references). This paper is an attempt to find out what happens when investors with constant risk preferences interact with those that have random risk preferences. First we consider the simple case of i.i.d. risk aversion, and then we extend to a case where risk preferences depend on past performance of the asset.

2 Random Risk Aversion

Investors live for 2 periods. They make investment decision when young (and may receive some labor income) and consume when old, and in each period the young investors buy the assets from the old ones. There are two assets. The safe asset s , in perfectly elastic supply, gives a known return of r per unit and its price is normalized to 1. The risky asset u , in a fixed supply of 1, gives the same return of r and its price is p_t .

Investors have mean-variance preference over next (last) period's wealth.¹ In each generation, there are two types of investors. A proportion $1 - \mu$ of the population are *stable* (s) investors, in the sense that their risk preference is constant across generations. A proportion μ of the population are *unstable* (u) investors, in the sense that their risk

¹In the Online Appendix, we show that the results in this paper also hold for CARA utility function under some mild assumptions.

preference is random across generations. The stable investors have a risk aversion coefficient of γ , and the unstable investors have a time-varying degree of risk aversion γ_t which is an i.i.d. random variable defined over the interval $[\underline{\gamma}, \bar{\gamma}]$. In each period, γ_t and its previous values are known by all investors while its future values are unknown.

First, a stable investor maximizes the mean-variance utility

$$\max Ew - \gamma \sigma_w^2$$

where w is the wealth of the investor. The wealth of stable investors at time $t + 1$ is defined by their choice of the two assets: $w_{t+1}^s = \lambda_t^s(r + p_{t+1}) + (w_t - \lambda_t^s p_t)(1 + r)$, where λ_t^s is the demand for the risky asset and $w_t - \lambda_t^s$ is the demand for the safe asset. The initial wealth w_t is given (e.g., an exogenous amount of labor income). Therefore, the investor solves

$$\max_{\lambda_t^s} \lambda_t^s (r + E[p_{t+1}] - p_t(1 + r)) - \gamma (\lambda_t^s)^2 \sigma_{p_{t+1}}^2.$$

The optimal investment for stable investor is $\lambda_t^{s*} = \frac{r + E[p_{t+1}] - p_t(1 + r)}{2\gamma \sigma_{p_{t+1}}^2}$.

Similarly, the wealth of an unstable investor at time $t + 1$ is $w_{t+1}^u = \lambda_t^u(r + p_{t+1}) + (w_t - \lambda_t^u p_t)(1 + r)$. Therefore, the unstable investors solve

$$\max_{\lambda_t^u} \lambda_t^u (r + E[p_{t+1}] - p_t(1 + r)) - \gamma_t (\lambda_t^u)^2 \sigma_{p_{t+1}}^2.$$

The optimal investment for the unstable investor is $\lambda_t^{u*} = \frac{r + E[p_{t+1}] - p_t(1 + r)}{2\gamma_t \sigma_{p_{t+1}}^2}$.

To clear the market, the total demand for the risky asset has to sum up to 1: $(1 - \mu)\lambda_t^{s*} + \mu \lambda_t^{u*} = 1$, or, equivalently

$$\begin{aligned} 1 &= (1 - \mu) \frac{r + E[p_{t+1}] - p_t(1 + r)}{2\gamma \sigma_{p_{t+1}}^2} + \mu \frac{r + E[p_{t+1}] - p_t(1 + r)}{2\gamma_t \sigma_{p_{t+1}}^2} \\ &= \left(\frac{r + E[p_{t+1}] - p_t(1 + r)}{2\sigma_{p_{t+1}}^2} \right) \left(\frac{1 - \mu}{\gamma} + \frac{\mu}{\gamma_t} \right). \end{aligned}$$

Let $\Gamma_t = \frac{1}{\frac{1-\mu}{\gamma} + \frac{\mu}{\gamma_t}}$, which is the weighted harmonic mean of the two risk aversion parameters. The equilibrium price is

$$(1) \quad p_t = \frac{r + E[p_{t+1}] - 2\sigma_{p_{t+1}}^2 \Gamma_t}{1 + r}$$

Note that when $\gamma_t = \gamma$, we have $\Gamma_t = \gamma$ and the asset price will be constant at 1. We now ready to state our results (the proofs are in the Appendix). At the steady state, we need to have $E[p_t] = E[p_{t+1}]$ and $V[p_t] = V[p_{t+1}]$. Therefore, we can characterize the steady state price.

Proposition 1. *The steady state price is given by the equation*

$$(2) \quad p_t = 1 - \frac{1 + r}{2r V[\Gamma_t]} (E[\Gamma_t] + r \Gamma_t).$$

Moreover, the variance of the price is given by the equation

$$(3) \quad \sigma_{p_t}^2 = \frac{(1 + r)^2}{4V[\Gamma_t]}.$$

Moreover, in the steady state, the demands for the risky asset can be written as $\lambda_t^{s*} = \frac{\Gamma_t}{\gamma}$ and $\lambda_t^{u*} = \frac{\Gamma_t}{\gamma_t}$. Therefore, when the unstable investors become more risk averse (i.e., high γ_t), price goes down and the stable investors hold more of the risky asset; that is, $\lambda_t^{s*} = \frac{\Gamma_t}{\gamma} > 1 > \lambda_t^{u*} = \frac{\Gamma_t}{\gamma_t}$. However, when the unstable investors become less risk averse (i.e., low γ_t), price goes up and the stable investors hold less of the risky asset; that is, $\lambda_t^{s*} = \frac{\Gamma_t}{\gamma} < 1 < \lambda_t^{u*} = \frac{\Gamma_t}{\gamma_t}$.

The main result of the paper explores relations between p_t and the proportion of unstable investors μ and the variance of γ_t . In particular, we show that *i*) a small amount of randomness in the risk aversion or *ii*) a small population of unstable investors generates a large deviation from fundamental price and a high price volatility. Let $\sigma_{\gamma_t}^2$ be the variance of γ_t .

Proposition 2. *As $\mu \sigma_{\gamma_t} \rightarrow 0$, $V[\Gamma_t] \rightarrow 0$. Consequently, $p_t \rightarrow -\infty$ and $\sigma_{p_t}^2 \rightarrow \infty$.*

The intuition behind the two results is as follows: demand for the risky asset decreases with the risk aversion parameter and variance of price, and price itself depends positively on demand. As a result, when the variance of the risk aversion parameter decreases, the variance of price increases. This increase in the variance of price in turn drives down the price of the risky asset. On the other hand, a smaller population of the unstable investors magnifies the the impact of the variance of the risk aversion parameter, leading to the same outcomes on price.

In Appendix A.3, we also provide a specific example to illustrate Proposition 2.

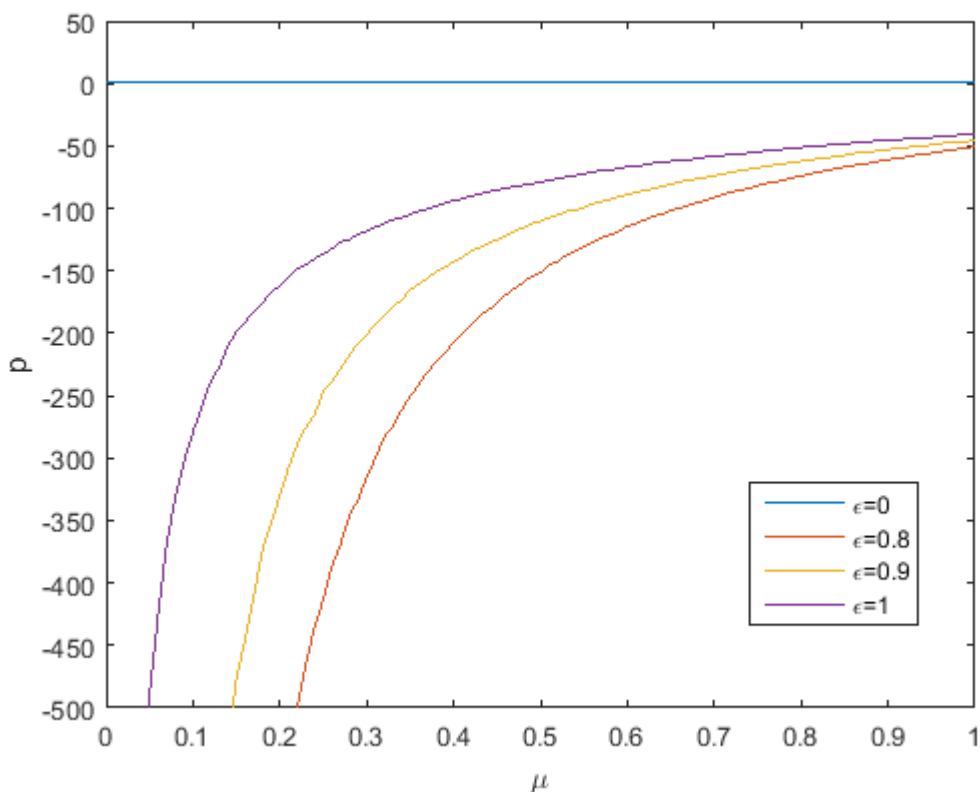


Figure 1: **Randomness and price.** We plot the relationship between unconditional mean price (p) and population size of unstable investors (μ) for different degrees of randomness (ϵ). The benchmark price is constant at 1.

To present our results graphically, we consider the case of $\gamma = 1$ for the stable investors and $\gamma_t \sim U(\gamma - \epsilon, \gamma + \epsilon)$ for the unstable investors, with the rate of return r equal to 5%. We plot the average price for different values of μ , the population of the unstable investors in Figure 1.

When all investors are stable ($\epsilon = 0$), price is equal to 1. When there is some randomness in preference, price of risky asset drops significantly below 1, especially when μ is close to zero. As the amount of randomness in risk preferences increases, price moves closer to 1.

On the other hand, wealth comparisons for the two types of investors are ambiguous. In particular, we can show that $E[w_t^s - w_t^u] > 0$ when μ is close to 1 and $E[w_t^s - w_t^u] < 0$ when γ is small enough.² Please see Appendix A.4 for details.

²Notice that two of our results are similar to those in De Long et al. (1990): that rational investors are not able to drive the asset price to its fundamental value, and that the welfare comparison between the two types of investors is ambiguous.

3 Price-Dependent Risk Aversion

Why are risk preference random? As mentioned in the introduction, risk aversion can be time-dependent in the sense that a good history of returns reduces its level.³ With the simple model we consider here, we can allow risk aversion to depend on past price as $\gamma(p_{t-1})$. While the results below go through for dependence of any sign, it may be more intuitive to consider the case of a negative relationship: the past price is what the past old generation receives by selling their asset, and if the current young observe that the old gets a high price, they become “optimistic” and less risk averse.

We simplify the discussion by looking at a binary case.⁴ Suppose there are two possible prices p_t^H and p_t^L , and when p_{t-1}^S is realized where $S \in \{L, H\}$, the unstable investors’ degree of risk aversion is γ_t^S .⁵ Price follows a first-order Markov process with the probability of staying at the same price being equal, $\rho(H|H) = \rho(L|L) = \pi$. Following similar steps as in the previous section, we can obtain the equilibrium price as a function of p_{t-1} , through the price-dependent risk aversion parameter γ_t^S :

$$p_t(p_{t-1}) = \frac{r + E_t[p_{t+1}] - 2V_t[p_{t+1}]\Gamma_t(\gamma(p_{t-1}))}{1 + r},$$

In Appendix A.5, we prove that as the difference between γ_t^H and γ_t^L goes to zero, the difference between the equilibrium prices p_t^H and p_t^L goes to infinity. The crucial step in the proof is to express the difference in price as the product of the variance of price and the difference in Γ_t :

$$p_t^S(p_{t-1}^H) - p_t^S(p_{t-1}^L) = \frac{2\pi(1 - \pi)(p_{t+1}^H(p_{t-1}^S) - p_{t+1}^L(p_{t-1}^S))^2(\Gamma_t(\gamma_t^L) - \Gamma_t(\gamma_t^H))}{1 + r}.$$

When the difference in Γ_t goes to zero, the difference in price explodes. In Appendix A.5 we show that the same results also apply to the case where risk aversion depends on current price as $\gamma(p_t)$, and the proof is similar.

Once again a small amount of randomness can generate large impacts on price and volatility. Notice that for investors in time t , the risk aversion parameter for the unstable investors is actually known. The source of randomness comes from not knowing the risk

³See Guiso, Sapienza, and Zingales (2015) and Malmendier and Nagel (2011) for empirical evidence. There is an alternative interpretation: investors have belief on the variance of price $\sigma_{p_{t+1}}^2$ instead, believing that it has a low or high value with equal probabilities (the empirical evidence of Malmendier and Nagel (2011) is also consistent with this interpretation). Algebraically we cannot distinguish the two interpretations.

⁴The results in this section can be extended to more general settings, e.g., more possible levels of prices and Markov processes of higher orders.

⁵This type of state-dependent risk preferences is widely used in the literature. For example, see Karni (1985), Gordon and St-Amour (2000), and Melino and Yang (2003).

preferences of *future* unstable investors.

4 Conclusion

In a simple overlapping-generations model, we show that a small amount of randomness in the risk preferences of a small population of investors is enough to have a large impact on price. We leave it to future research to find out if the results hold in more general settings.

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A Appendix

A.1 Proof of Proposition 1

We assume that $E[p_t] = E[p_{t+1}]$ and $V[p_t] = V[p_{t+1}]$. First, from (1), we have

$$E[p_t] = E \left[\frac{r + E[p_{t+1}] - 2\sigma_{p_{t+1}}^2 \Gamma_t}{1+r} \right] = \frac{r + E[p_{t+1}] - 2\sigma_{p_{t+1}}^2 E[\Gamma_t]}{1+r}.$$

It is equivalent to

$$(4) \quad E[p_t] = 1 - \frac{2\sigma_{p_{t+1}}^2 E[\Gamma_t]}{r}.$$

Similarly we have from (1)

$$\sigma_{p_t}^2 = V[p_t] = V \left[\frac{r + E[p_{t+1}] - 2\sigma_{p_{t+1}}^2 \Gamma_t}{1+r} \right] = V \left[\frac{-2\sigma_{p_{t+1}}^2 \Gamma_t}{1+r} \right] = \frac{4\sigma_{p_{t+1}}^4 V[\Gamma_t]}{(1+r)^2}.$$

Since $V[p_t] = V[p_{t+1}]$, we obtain (3) due to

$$\sigma_{p_t}^2 = \frac{\sigma_{p_{t+1}}^4}{\sigma_{p_t}^2} = \frac{(1+r)^2}{4V[\Gamma_t]}.$$

By (3), (4) is equivalent to

$$(5) \quad E[p_t] = 1 - \frac{2\sigma_{p_{t+1}}^2 E[\Gamma_t]}{r} = 1 - \frac{2 \cdot \frac{(1+r)^2}{4V[\Gamma_t]} E[\Gamma_t]}{r} = 1 - \frac{(1+r)^2 E[\Gamma_t]}{2r V[\Gamma_t]}.$$

Finally, by (3) and (5), (1) is equivalent to

$$\begin{aligned} p_t &= \frac{r + E[p_{t+1}] - 2\sigma_{p_{t+1}}^2 \Gamma_t}{1+r} = \frac{r + \left(1 - \frac{(1+r)^2 E[\Gamma_t]}{2r V[\Gamma_t]}\right) - 2 \frac{(1+r)^2}{4V[\Gamma_t]} \Gamma_t}{1+r} \\ &= 1 - \frac{1+r}{2r V[\Gamma_t]} (E[\Gamma_t] + r \Gamma_t). \end{aligned}$$

Therefore, we have (2).

A.2 Proof of Proposition 2

We want to prove that $V[\Gamma_t]$ goes to zero as $\mu \sigma_{\gamma_t}$ goes to zero. We use the following well-known inequality:

Chebyshev's Inequality: for any random variable X and $\delta > 0$,

$$Pr(|X - E[X]| > \delta) \leq \frac{V[X]}{\delta^2}.$$

Let $\gamma^* \equiv E[\gamma_t]$. By setting $X = \gamma_t$ and $\delta = \sqrt{\sigma_{\gamma_t}}$, Chebyshev's Inequality implies that

$$(6) \quad Pr(|\gamma_t - \gamma^*| > \sqrt{\sigma_{\gamma_t}}) \leq \sigma_{\gamma_t}.$$

Now let us bound $V[\Gamma_t]$ by using (6). Let us explicitly write Γ_t as a function of γ_t , i.e., $\Gamma_t(\gamma_t)$. Notice that $\Gamma_t(\gamma_t)$ is an increasing function of γ_t . Let G be the cdf of γ_t . Then we have

$$\begin{aligned} V[\Gamma_t] &= E[(\Gamma_t - E[\Gamma_t])^2] = \int_{\underline{\gamma}}^{\bar{\gamma}} (\Gamma_t(\gamma_t) - E[\Gamma_t])^2 dG(\gamma_t) \\ &= \int_{\gamma^* - \delta}^{\gamma^* + \delta} (\Gamma_t(\gamma_t) - E[\Gamma_t])^2 dG(\gamma_t) + \int_{|\gamma_t - \gamma^*| > \delta} (\Gamma_t(\gamma_t) - E[\Gamma_t])^2 dG(\gamma_t) \\ &\leq \int_{\gamma^* - \delta}^{\gamma^* + \delta} (\Gamma_t(\gamma_t) - E[\Gamma_t])^2 dG(\gamma_t) + Pr(|\gamma_t - \gamma^*| > \delta) (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^2 \\ &\leq \int_{\gamma^* - \delta}^{\gamma^* + \delta} (\Gamma_t(\gamma_t) - E[\Gamma_t])^2 dG(\gamma_t) + \sigma_{\gamma_t} (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^2, \text{ by (6)}. \end{aligned}$$

Now let us bound $\int_{\gamma^* - \delta}^{\gamma^* + \delta} (\Gamma_t(\gamma_t) - E[\Gamma_t])^2 dG(\gamma_t)$. Since $(a - b)^2 \leq (a - c)^2 + (c - b)^2$ for any $a, b, c \in \mathbb{R}$, we have

$$\begin{aligned} V[\Gamma_t] &\leq \int_{\gamma^* - \delta}^{\gamma^* + \delta} (\Gamma_t(\gamma_t) - E[\Gamma_t])^2 dG(\gamma_t) + \sigma_{\gamma_t} (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^2 \\ (7) \quad &\leq \int_{\gamma^* - \delta}^{\gamma^* + \delta} (\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*))^2 + (\Gamma_t(\gamma^*) - E[\Gamma_t])^2 dG(\gamma_t) + \sigma_{\gamma_t} (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^2. \end{aligned}$$

Next, let us bound $(E[\Gamma_t] - \Gamma_t(\gamma^*))^2$. Note that

$$\begin{aligned} |E[\Gamma_t] - \Gamma_t(\gamma^*)| &= \left| \int_{\underline{\gamma}}^{\bar{\gamma}} \Gamma_t(\gamma_t) - \Gamma_t(\gamma^*) dG(\gamma_t) \right| \\ &= \left| \int_{\gamma^* - \delta}^{\gamma^* + \delta} \Gamma_t(\gamma_t) - \Gamma_t(\gamma^*) dG(\gamma_t) + \int_{|\gamma_t - \gamma^*| > \delta} \Gamma_t(\gamma_t) - \Gamma_t(\gamma^*) dG(\gamma_t) \right| \\ &\leq \int_{\gamma^* - \delta}^{\gamma^* + \delta} |\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*)| dG(\gamma_t) + Pr(|\gamma_t - \gamma^*| > \delta) (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma})) \end{aligned}$$

$$\leq \int_{\gamma^*-\delta}^{\gamma^*+\delta} |\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*)| dG(\gamma_t) + \sigma_{\gamma_t} (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma})), \text{ by (6).}$$

By the Cauchy-Schwarz inequality, we have $(a + b)^2 \leq 2(a^2 + b^2)$ for any $a, b \in \mathbb{R}$ and

$$\left(\int_{\gamma^*-\delta}^{\gamma^*+\delta} |\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*)| dG(\gamma_t) \right)^2 \leq \int_{\gamma^*-\delta}^{\gamma^*+\delta} |\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*)|^2 dG(\gamma_t).$$

Therefore, we have

$$\begin{aligned} (E[\Gamma_t] - \Gamma_t(\gamma^*))^2 &\leq \left(\int_{\gamma^*-\delta}^{\gamma^*+\delta} |\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*)| dG(\gamma_t) + \sigma_{\gamma_t} (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma})) \right)^2 \\ &\leq 2 \left(\left(\int_{\gamma^*-\delta}^{\gamma^*+\delta} |\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*)| dG(\gamma_t) \right)^2 + \sigma_{\gamma_t}^2 (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^4 \right) \\ (8) \quad &\leq 2 \left(\int_{\gamma^*-\delta}^{\gamma^*+\delta} |\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*)|^2 dG(\gamma_t) + \sigma_{\gamma_t}^2 (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^4 \right). \end{aligned}$$

By combining (7) and (8), we have

$$\begin{aligned} V[\Gamma_t] &\leq \int_{\gamma^*-\delta}^{\gamma^*+\delta} (\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*))^2 + (\Gamma_t(\gamma^*) - E[\Gamma_t])^2 dG(\gamma_t) + \sigma_{\gamma_t} (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^2 \\ &\leq \int_{\gamma^*-\delta}^{\gamma^*+\delta} (\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*))^2 dG(\gamma_t) + (\Gamma_t(\gamma^*) - E[\Gamma_t])^2 + \sigma_{\gamma_t} (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^2 \\ (9) \quad &\leq 3 \int_{\gamma^*-\delta}^{\gamma^*+\delta} (\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*))^2 dG(\gamma_t) + 2\sigma_{\gamma_t}^2 (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^4 + \sigma_{\gamma_t} (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^2. \end{aligned}$$

Finally, we need to bound $\int_{\gamma^*-\delta}^{\gamma^*+\delta} (\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*))^2 dG(\gamma_t)$. Note that

$$\begin{aligned} \int_{\gamma^*-\delta}^{\gamma^*+\delta} (\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*))^2 dG(\gamma_t) &= \int_{\gamma^*}^{\gamma^*+\delta} (\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*))^2 dG(\gamma_t) + \int_{\gamma^*-\delta}^{\gamma^*} (\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*))^2 dG(\gamma_t) \\ &\leq \int_{\gamma^*}^{\gamma^*+\delta} (\Gamma_t(\gamma^* + \delta) - \Gamma_t(\gamma^*))^2 dG(\gamma_t) + \int_{\gamma^*-\delta}^{\gamma^*} (\Gamma_t(\gamma^*) - \Gamma_t(\gamma^* - \delta))^2 dG(\gamma_t) \\ &\leq (\Gamma_t(\gamma^* + \delta) - \Gamma_t(\gamma^*))^2 + (\Gamma_t(\gamma^*) - \Gamma_t(\gamma^* - \delta))^2 \\ (10) \quad &= \mu^2 \delta^2 \left(\frac{\Gamma_t^2(\gamma^* + \delta) \Gamma_t^2(\gamma^*)}{(\gamma^* + \delta)^2 (\gamma^*)^2} + \frac{\Gamma_t^2(\gamma^* - \delta) \Gamma_t^2(\gamma^*)}{(\gamma^* - \delta)^2 (\gamma^*)^2} \right). \end{aligned}$$

By combining (9) and (10), we have

$$\begin{aligned}
V[\Gamma_t] &\leq 3 \int_{\gamma^*-\delta}^{\gamma^*+\delta} (\Gamma_t(\gamma_t) - \Gamma_t(\gamma^*))^2 dG(\gamma_t) + 2\sigma_{\gamma_t}^2 (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^4 + \sigma_{\gamma_t} (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^2 \\
&\leq 3\mu^2\sigma_{\gamma_t} \left(\frac{\Gamma_t^2(\gamma^* + \delta) \Gamma_t^2(\gamma^*)}{(\gamma^* + \delta)^2(\gamma^*)^2} + \frac{\Gamma_t^2(\gamma^* - \delta) \Gamma_t^2(\gamma^*)}{(\gamma^* - \delta)^2(\gamma^*)^2} \right) + 2\sigma_{\gamma_t}^2 (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^4 + \sigma_{\gamma_t} (\Gamma_t(\bar{\gamma}) - \Gamma_t(\underline{\gamma}))^2 \\
(11) \quad &= \mu^2\sigma_{\gamma_t} \left(\frac{3\Gamma_t^2(\gamma^* + \delta) \Gamma_t^2(\gamma^*)}{(\gamma^* + \delta)^2(\gamma^*)^2} + \frac{3\Gamma_t^2(\gamma^* - \delta) \Gamma_t^2(\gamma^*)}{(\gamma^* - \delta)^2(\gamma^*)^2} + 2\mu^2\sigma_{\gamma_t} \frac{\Gamma_t^4(\bar{\gamma}) \Gamma_t^4(\underline{\gamma})(\bar{\gamma} - \underline{\gamma})^4}{\bar{\gamma}^4 \underline{\gamma}^4} + \right. \\
&\quad \left. + \frac{\Gamma_t^2(\bar{\gamma}) \Gamma_t^2(\underline{\gamma})(\bar{\gamma} - \underline{\gamma})^2}{\bar{\gamma}^2 \underline{\gamma}^2} \right).
\end{aligned}$$

From (11), we can see that $V[\Gamma_t]$ goes to zero as $\mu\sigma_{\gamma_t}$ goes to zero.

A.3 Approximation for a Binary Case

In this subsection, we illustrate Proposition 2 for a special case in which γ_t returns $\gamma - \epsilon$ with probability $\frac{1}{2}$ and $\gamma + \epsilon$ with probability $\frac{1}{2}$. First, note that as $\mu\epsilon \rightarrow 0$, $V[\Gamma_t] \rightarrow 0$ since

$$V[\Gamma_t] = \frac{1}{4} \left(\frac{1}{\frac{1-\mu}{\gamma} + \frac{\mu}{\gamma+\epsilon}} - \frac{1}{\frac{1-\mu}{\gamma} + \frac{\mu}{\gamma-\epsilon}} \right)^2 = \frac{\mu^2 \epsilon^2}{\left(1 + (1-\mu)\frac{\epsilon}{\gamma}\right)^2 \left(1 - (1-\mu)\frac{\epsilon}{\gamma}\right)^2}.$$

Moreover, note that when $\mu\epsilon$ is small enough, Γ_t is close around γ . In other words, $E[\Gamma_t] + r\Gamma_t \approx (1+r)\gamma$. Consequently,

$$p_t = 1 - \frac{1+r}{2r V[\Gamma_t]} (E[\Gamma_t] + r\Gamma_t) \approx 1 - \frac{(1+r)^2 \gamma}{2r \mu^2 \epsilon^2} \left(1 + (1-\mu)\frac{\epsilon}{\gamma}\right)^2 \left(1 - (1-\mu)\frac{\epsilon}{\gamma}\right)^2.$$

Therefore, as $\mu\epsilon \rightarrow 0$, $p_t \rightarrow -\infty$.

A.4 Wealth Comparisons

We want to show that wealth comparisons for the stable and unstable investors are ambiguous. In particular, we show that $E[w_t^s - w_t^u] > 0$ when μ is close to 1 and $E[w_t^s - w_t^u] < 0$ when γ is small enough. Since

$$w_t^s = \lambda_t^{s*} (r + p_{t+1} - p_t(1+r)) + w_t(1+r)$$

and

$$w_t^u = \lambda_t^{u*} (r + p_{t+1} - p_t(1+r)) + w_t(1+r),$$

we have

$$w_t^s - w_t^u = (\lambda_t^{s*} - \lambda_t^{*u}) (r + p_{t+1} - p_t(1+r)).$$

Moreover, since

$$\lambda_t^{s*} - \lambda_t^{*u} = \frac{r + E[p_{t+1}] - p_t(1+r)}{2\sigma_{p_t}^2} \left(\frac{1}{\gamma} - \frac{1}{\gamma_t} \right),$$

we have

$$E[w_t^s - w_t^u] = E \left[(r + p_{t+1} - p_t(1+r)) \frac{r + E[p_{t+1}] - p_t(1+r)}{2\sigma_{p_t}^2} \left(\frac{1}{\gamma} - \frac{1}{\gamma_t} \right) \right].$$

By direct calculations, at the steady state we have

$$E[w_t^s - w_t^u] = \frac{(1+r)^2}{2V[\Gamma_t]} E \left[\Gamma_t^2 \left(\frac{1}{\gamma} - \frac{1}{\gamma_t} \right) \right].$$

Therefore, it is enough to discuss whether $E \left[\Gamma_t^2 \left(\frac{1}{\gamma} - \frac{1}{\gamma_t} \right) \right] > 0$. For an illustration, consider a case in which γ_t returns $\gamma + \epsilon$ with probability $\frac{1}{2}$ and $\gamma - \epsilon$ with probability $\frac{1}{2}$. Then by direct calculations, we obtain that

$$E \left[\Gamma_t^2 \left(\frac{1}{\gamma} - \frac{1}{\gamma_t} \right) \right] = \frac{\epsilon^2 \gamma^2 (\gamma + \frac{\epsilon^2}{\gamma}(1-\mu) - 2(1-\mu))}{(\gamma - \epsilon(1-\mu))^2 (\gamma + \epsilon(1-\mu))^2}.$$

Therefore, $E \left[\Gamma_t^2 \left(\frac{1}{\gamma} - \frac{1}{\gamma_t} \right) \right] > 0$ iff $\gamma + \frac{\epsilon^2}{\gamma}(1-\mu) > 2(1-\mu)$. It is easy to see that when μ is close to 1, $\gamma + \frac{\epsilon^2}{\gamma}(1-\mu) \approx \gamma > 2(1-\mu) \approx 0$, or equivalently, $E \left[\Gamma_t^2 \left(\frac{1}{\gamma} - \frac{1}{\gamma_t} \right) \right] > 0$. Moreover, when γ is small enough, since $\epsilon < \gamma$, we have $\gamma + \frac{\epsilon^2}{\gamma}(1-\mu) \approx 0 < 2(1-\mu)$, or equivalently, $E \left[\Gamma_t^2 \left(\frac{1}{\gamma} - \frac{1}{\gamma_t} \right) \right] < 0$.

A.5 Price-Dependent Risk Aversion: A Binary Example

Suppose there are two possible prices at each time t : a high price p_t^H and a low price p_t^L . Let the state be $S \in \{H, L\}$ and the conditional probabilities of staying at the same price be equal at $\rho(H|H) = \rho(L|L) = \pi$.

Past-Price Dependence: Note that the equilibrium price, which depends on the previous price, can be derived as

$$p_t(p_{t-1}) = \frac{r + E_t[p_{t+1}] - 2V_t[p_{t+1}]\Gamma_t(\gamma(p_{t-1}))}{1+r},$$

and it is, once we explicitly write out the two possible states $S \in \{H, L\}$, equivalent to

$$p_t^S(p_{t-1}) = \frac{r + (\rho(H|S)p_{t+1}^H(p_t^S) + \rho(L|S)p_{t+1}^L(p_t^S)) - 2\pi(1 - \pi)(p_{t+1}^H(p_t^S) - p_{t+1}^L(p_t^S))^2 \Gamma_t(\gamma(p_{t-1}))}{1 + r}.$$

Therefore, we can write the difference in price under two difference past prices as

$$p_t^S(p_{t-1}^H) - p_t^S(p_{t-1}^L) = \frac{2\pi(1 - \pi)(p_{t+1}^H(p_t^S) - p_{t+1}^L(p_t^S))^2 (\Gamma_t(\gamma_t^L) - \Gamma_t(\gamma_t^H))}{1 + r}.$$

Then we can write the difference for the two states as

$$\begin{aligned} & (p_t^H(p_{t-1}^H) - p_t^H(p_{t-1}^L)) - (p_t^L(p_{t-1}^H) - p_t^L(p_{t-1}^L)) = \\ & = \frac{2\pi(1 - \pi)(\Gamma_t(\gamma_t^L) - \Gamma_t(\gamma_t^H)) \left((p_{t+1}^H(p_t^H) - p_{t+1}^L(p_t^H))^2 - (p_{t+1}^H(p_t^L) - p_{t+1}^L(p_t^L))^2 \right)}{1 + r}. \end{aligned}$$

In the steady state, we need the following condition to hold:

$$(p_t^H(p_{t-1}^H) - p_t^H(p_{t-1}^L)) - (p_t^L(p_{t-1}^H) - p_t^L(p_{t-1}^L)) = (p_{t+1}^H(p_t^H) - p_{t+1}^H(p_t^L)) - (p_{t+1}^L(p_t^H) - p_{t+1}^L(p_t^L)).$$

Therefore, assuming that $(p_t^H(p_{t-1}^H) - p_t^H(p_{t-1}^L)) - (p_t^L(p_{t-1}^H) - p_t^L(p_{t-1}^L)) \neq 0$, we have

$$1 = \frac{2\pi(1 - \pi)(\Gamma_t(\gamma_t^L) - \Gamma_t(\gamma_t^H)) \left((p_{t+1}^H(p_t^H) - p_{t+1}^L(p_t^H)) + (p_{t+1}^H(p_t^L) - p_{t+1}^L(p_t^L)) \right)}{1 + r}.$$

Now we have the final result that as $\Gamma_t(\gamma_t^L) - \Gamma_t(\gamma_t^H) \rightarrow 0$,

$$(p_{t+1}^H(p_t^H) - p_{t+1}^L(p_t^H)) + (p_{t+1}^H(p_t^L) - p_{t+1}^L(p_t^L)) \rightarrow +\infty.$$

Current-Price Dependence: Let us illustrate that the above argument also works when risk aversion depends on the current price as $\gamma(p_t)$. The equilibrium price can be derived as

$$p_t = \frac{r + E_t[p_{t+1}] - 2V_t[p_{t+1}]\Gamma_t(\gamma(p_t))}{1 + r},$$

and it is equivalent to

$$p_t^S = \frac{r + (\rho(H|S)p_{t+1}^H + \rho(L|S)p_{t+1}^L) - 2\pi(1 - \pi)(p_{t+1}^H - p_{t+1}^L)^2 \Gamma_t(\gamma_t^S)}{1 + r}.$$

Therefore, we can write the difference in price as

$$p_t^H - p_t^L = \frac{(2\pi - 1)(p_{t+1}^H - p_{t+1}^L) + 2\pi(1 - \pi)(p_{t+1}^H - p_{t+1}^L)^2 (\Gamma_t(\gamma_t^L) - \Gamma_t(\gamma_t^H))}{1 + r}.$$

In the steady state the following condition has to hold: $p_t^H - p_t^L = p_{t+1}^H - p_{t+1}^L$. Therefore,

we have

$$2 + r - 2\pi = 2\pi(1 - \pi)(p_t^H - p_t^L)(\Gamma_t(\gamma_t^L) - \Gamma_t(\gamma_t^H)).$$

Again we have the result that $p_t^H - p_t^L \rightarrow +\infty$ as $\Gamma(\gamma_t^L) - \Gamma(\gamma_t^H) \rightarrow 0$.