Reflections on the 150th Anniversary of Winkler's Foundation and its Profound Influence on the Field of Adhesion

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Introduction

This year, 2017, marks the 150th anniversary of Emil Winkler's seminal publication of the beam on elastic foundation (BoEF) solution [1] published in 1867 while a professor at the University of Prague. With wide-ranging interests in analysis of civil engineering structures, he initially proposed the BoEF model for the rather obvious application to sleepers and rails supported by the earth upon which they rest [2]. The essence of the model lies in the simple but profound assumption that the restoring force of an elastic foundation is linearly proportional to the deflection. The important resulting mechanics of materials solution has been applied to a wide range of engineering applications, including a plethora of discrete and continuous loading and boundary conditions, extensions to plates and pontoon bridges, nonlinear behavior, and even the analysis of deflections and stresses in pressurized cylindrical tanks, where the effective restoring force is not supplied by a separate medium but rather by the hoop stresses developed due to stretching of the curved walls. For plane strain bending of a plate, the governing differential equation and the characteristic reciprocal length are:

$$\frac{d^4w}{dx^4} + 4\lambda^4 w = \frac{p}{D} \quad \lambda = \sqrt[4]{\frac{k}{4D}} \tag{0.1}$$

where $D = Et^3 \big/ 12\left(1-\nu^2\right)$ is the plate rigidity, w is the out of plane deflection, p is the applied (spatially-varying) pressure, k is the elastic foundation stiffness, E and ν are Young's modulus and Poisson's ratio of the plate material, and t is the plate thickness.

In 1937 Biot [3] extended the solution to the case where the foundation is a semi-infinite half-space, an analysis that would later find widespread applications for surface layer wrinkling analysis. Hetényi [4] presented solutions for a variety of BoEF geometries applications including buckling in his classic 1946 monograph. Of particular interest to the Adhesion community has been the varied adaptations of this model to the field of adhesion, where it ranks with Volkersen's 1938 shear lag model [5] in both importance and versatility in modeling and explaining stress states in bonded systems.

Applications to Adhesion

The long line of applications of the Winkler foundation to the field of adhesion apparently began with Goland and Reissner's 1944 analysis [6] of the stresses within single lap joints, in which they recognized the significant influence that adherend bending, associated with eccentric loading, imposed on the resulting stress state within the bondline. Considering the adhesive layer as relatively more extensible in the out of plane direction than the adherends, the adhesive layer was effectively modelled as an elastic foundation supporting the two adherends. In spite of the simplifying assumptions made, this model combining shear lag and BoEF effects accurately predicts the bondline shear and peel stresses for a range of practical adhesive joints. In 1956 Lubkin and Reissner [7] extended this lap joint analysis to tubular lap joints, augmenting the adhesive foundation with the radial constraint provided by the cylindrical tubes. Other direct applications of the BoEF model include the analysis of stresses between adherends with a curvature mismatch, including constant [8] and varying [9] mismatch cases, as well as curvature optimization to minimize detrimental peel stresses [10].

Extensions of the BoEF concept to debonding and fracture mechanics applications, including for the above mentioned analyses, can be easily accomplished by effectively shortening the bonded length, as the debonded adherends have no influence in mechanics of materials solutions. Such analyses have found wide applications in modeling the stresses ahead of a growing crack or debond, such as appear in Spies' 1953 classic analysis of peeling tests [11], in Bikerman's 1958 criterion for debonding [12], and soon thereafter in Kaelble's [13-15] papers on the stress distributions within pressure sensitive adhesives undergoing peeling. One interesting application to monolithic materials rather than bonded systems was Kanninen's 1973 double cantilever beam (DCB) solution [16, 17], in which there is no separate layer (e.g. adhesive) upon which to assign a foundation stiffness. Instead, Kanninen modeled the foundation stiffness based on the contributions of the out of plane stiffness of the DCB arms themselves but using half the arm thicknesses to effectively capture the behavior back to the neutral axes. This approach serves as a basis for many later contributions in the field of structural adhesives, such as for asymmetric bonded DCB specimens [18]. Since Stigh's 1988 closed-form solution [19], Winkler's foundation concept has also been extensively applied to the analyses of the separation/splitting of adherends using the cohesive zone model (CZM) methodology [20-25].

Addressing Coupling Contributions

The ubiquitous applications of Winkler's BoEF model have been based in part on the simplicity of the model – that the restoring force on the beam is linearly proportional to the deflection at that point only. This simplicity, however, belies the complications that can arise due to several coupling effects. Although Winkler's model is derived and typically expressed for continuous systems, illustrations typically show the foundation as a series of discrete axial springs, which lends to the idea that each spring acts independently of the other. The linear model inherently neglects resistances arising for continuous foundations from both the derivative of the deflection, which imposes shear deformation in the foundation layer (i.e. assumes foundation shear modulus is zero), and the integral of the nearby deflections, through which foundation compressibility enters the solution (i.e. assumes foundation bulk modulus is zero). Kerr [26] reviews and critiques multiple approaches to include shear coupling to effectively incorporate the shear modulus of a continuous foundation on the behavior, including through the addition of either a fictitious pre-stressed membrane (Filonenko-Borodich foundation) or intermediate beam or plate (Hetényi foundation), along with numerous refinements suggested by multiple authors. None of the models proposed to incorporate foundation shear stiffness, however, appear to have addressed the role of foundation compressibility. In essence, all of the models continued to ignore the bulk stiffness of the foundation, a problem that becomes especially important for "incompressible" elastomeric foundations with shear moduli much lower than their bulk moduli.

Lefebvre et al. [27] hinted at this issue when the measured stiffness of DCB specimens involving steel beams bonded with a neoprene interlayer suggested discrepancies with their BoEF analysis, leading them to speculate on the role that elastomer "incompressibility" might play on the analysis. This finding provided the impetus for Dillard's subsequent analysis [28] in 1989 of a plate supported by a continuous elastomeric foundation. The resulting 6th order differential equation built on the 4th order Winkler solution to include the coupling resulting from the constraint of an incompressible foundation layer and the characteristic reciprocal length are:

$$\frac{d^6 w}{dx^6} - \lambda^6 w = \frac{1}{D}\frac{d^2 p}{dx^2} \quad \lambda = \sqrt[6]{\frac{12\mu}{Dh^3}} \qquad (0.2)$$

where μ is the shear modulus of the elastomer and h its thickness. (Interestingly, a 6th order differential equation also results in the Reissner foundation [29], though resulting from a foundation's shear contributions modeled by a fictitious intermediate plate suspended by two layers of spring foundation, so bulk stiffness was still not considered.) Bert [30] extended this analysis to the case of foundations of arbitrary Poisson's ratio.

Beginning in 2003, Chaudhury and his group presented their pioneering work in a series of papers [31-33] where this elastomeric foundation analysis has become very important for significant advancements in the understanding of stresses and debonding for soft matter adhesion. Of particular interest has been the delineation of the mechanics of fingering phenomena occurring in the debonding of soft matter. Building on this work, our group has recently analytically and numerically investigated debonding of soft matter layers between rigid and flexible adherends [34-36]. In a recent publication, Cabello et al. [37] proposed a model to account for the compressibility of a flexible adhesive interlayer in a DCB geometry subjected to Mode-I loading.

Finally, we would be remiss in discussing BoEF applications to adhesion without mentioning the important extension to coatings. Lacking the discrete foundation layer evident in adhesive bond applications, stress analysis for coatings considerably softer than the substrate, Kanninen-like assumptions for the coating have been employed to estimate interfacial peel and shear stresses in bonded coatings [38]. Alternatively, for the case where the coating is much stiffer than the substrate, periodic wrinkling can occur [39], in which the foundation stiffness depends on the substrate modulus acting over an effective length equal to the characteristic wrinkling wavelength divided by π [3].

Conclusions

This presentation is meant as a tribute to Emil Winkler and a recognition of the significance of his 1867 BoEF formulation for the field of adhesion science. Numerous applications are discussed, providing some historical perspective along with some recent extensions to our field. The simple mechanics of materials solution and its numerous extensions have clearly made a profound impact on the mechanics of adhesion.

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