BUCKLING

OF

CANTILEVER THIN PLATE

WITH

FREE END SUBJECTED TO UNIFORM SHEAR

by

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I. LISTS OF TABLES, FIGURES, AND SYMBOLS

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x_i	Rectangular coordinates
u_i	Displacements in the x_i directions
σ_{ij}	Stress tensor in x_i system
e_{ij}	Strain tensor in x_i system
F_i	Body forces per unit volume in the x_i directions
Δ	Cubical dilatation
$a_i^{(k)}$, a_k	Parameters
u	Volume density of strain energy
U	Strain energy of the whole elastic system
W	Work done by external forces
P	Potential energy (P = U - W)
$(\sigma_{max})_{cr}$	Maximum stress at buckling state
D	Flexural rigidity of plate $\left[D = \frac{Eb^3}{12(1-\nu^2)} \right]$
E	Modulus of elasticity
ν	Poisson's ratio

II INTRODUCTION

As far back as 1757, the buckling problems of struts under different boundary conditions were investigated by Euler.[01] Lagrange[02] followed and made a more through study to determine the length which a column must attain to be bent by its own or applied weight. In 1845, E. Lamarle[03] found a more accurate differential equation for the buckling load of struts than Lagrange's, and solved this equation by the method of series. This modification introduced into Lagrange's result gives only a factor which is negligible in most practical cases. In the year 1850, G. Kirchhoff[04] introduced the energy method for problems of elastic stability as an extremum principle of mechanics which characterizes the conditions of equilibrium in an elastic body. The first one to apply the energy principle to the solution of buckling problems of plates was Bryan,[05] who applied the energy method to get a differential equation which governs the buckling load of the plate.

To consider the buckling problem as a boundary-value problem has the advantage that the solution obtained is accurate. But it is difficult to write down the governing equation and sometimes harder or even impossible to ascertain and satisfy the boundary conditions. Then the approximate method based on an energy criterion was introduced into the buckling problem. The main credit, apparently, belongs to Rayleigh[06] and Ritz;[07] the former introduced the approximate method for the main purpose of finding the frequencies of vibrating systems, the latter generalized Rayleigh's method into an extremum problem which is widely used in mathematical physics. Timoshenko developed Ritz method into a powerful tool for the treatment of buckling problems under various loading and boundary conditions. This method is frequently used in his book "Theory of Elastic Stability."

The Ritz method leads to an approximate value of buckling load which is larger than the exact one. This is due to the fact that more strain energy is needed to maintain the assumed buckling configuration which deviates from the true one. In 1935, Trefftz[08] supplemented Ritz method by developing a procedure for the determination of a lower bound for the buckling load. Thus the degree of accuracy for the buckling load obtained by the energy method can be judged.

The energy method for solving buckling problems is very effective, since in most cases, only the first term retained in an assumed series for the deflection yields accurate results. But the power of the energy method is mainly rooted in the fact that it can be used as a general approach for the problems of stability.[09][10]

The problem considered in this thesis was first discussed in 1899 by L. Prandtl[11] who obtained a governing differential equation based on the equilibrium conditions of a narrow rectangular cantilever beam. Due to the assumptions, the result obtained by Prandtl is good only when the length of the thin cantilever is much greater than the width. When the length of the cantilever becomes smaller and smaller, then its buckling behavior is more and more like a plate rather than a cantilever beam. Based on this idea, the author has used the energy approach to attack the same problem based on thin plate theory.

The assumption that the deflections and slopes of the plate in the middle plane are small enough to be neglected, is adopted. As this assumption is violated, a procedure based on the energy method in this general case is also discussed.

III. THE INVESTIGATION

1. Subject of Investigation

The buckling problems of thin elastic plates have been fully developed for different boundary conditions and different types of loading. But the cantilever plate, i.e. one edge fixed and the others free, offers some difficulties in finding the critical load. The trouble results from the method of solving such problems. The two tools to investigate both the stability and the buckling problems are the differential equation and the energy method. In the differential equation method, we usually have to solve a boundary-value problem of fourth order in each coordinate x, y, the solution of which contains a particular solution and a homogeneous solution. The form of the particular solution depends on the types of loading, but the homogeneous solution always appears as some combinations of trigonometric and hyperbolic functions. Due to the properties of these functions and their derivatives, it is impossible to make the boundary conditions of both the free and the fixed edges satisfied. The energy method is a general approach to stability problems which can be used as an efficient and economical tool to determine the critical load of some specific structures. In an approximate energy method, we have to assume a buckling configuration instead of solving a differential equation. Therefore the degree of accuracy of the result depends on the adequacy of the assumed function, which is based on suggestions from other solutions, experimental data and even intuition.

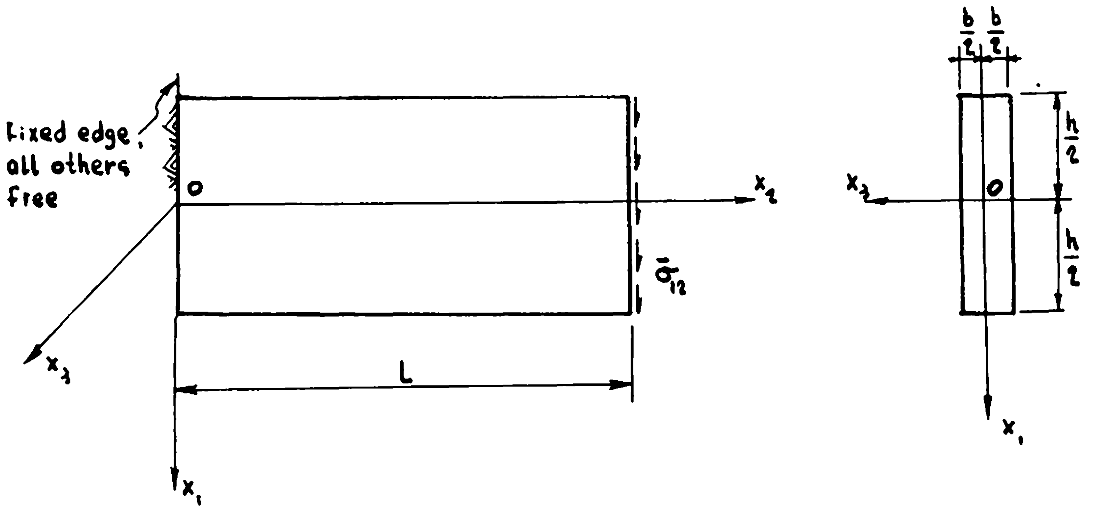


Fig.1 Diagram of the Plate

In this thesis, the author intends to use the energy method to examine the critical load of a cantilevered, thin, rectangular plate of thickness b, lying in the $x_1 x_2$ plane and being fixed along the x_1 axis (Fig.1). The space occupied by the plate before deformation is defined as

$$\left(-\frac{h}{2} ,\ 0,\ -\frac{b}{2} \right) \leq x_i \leq \left(\frac{h}{2} ,\ L,\ \frac{b}{2} \right)$$

The applied stress $\bar{\sigma}_{12}$ is assumed to be a "dead load" and uniformly distributed over the surface $x_2 = L$. The so-called "dead load" means that the applied load remains constant in both magnitude and direction during the buckling process.

Since this paper discusses the buckling of a thin plate, all assumptions used in thin plate theory will be adopted in the subsquent discussions unless noted.

2. Mathematical Procedures of Investigation

As long as the applied load is smaller than the critical one, there will be no displacement in the x_3 direction. But when the load increases to a certain value, the plate will be slightly twisted. The buckling load here is defined as one at which the plate starts twisting and which keeps the plate at its first equilibrium configuration.

Let u_3 (x_i, a_n) be the assumed displacement in the x_3 direction which satisfies the geometrical boundary conditions at least, where a_n are undetermined parameters. Then the strain energy and the work done by the external forces can be formulated as $U(a_n)$ and $W(a_n)$ respectively. The function $P(a_n)$ will be defined as

$$P(a_n) = U(a_n) - W(a_n) \qquad (3.2.1)$$

which is the potential energy. Since the buckling load keeps the buckled plate in an equilibrium position, this requires that the potential energy assume a minimum value. Thus we have

$$\frac{\partial P}{\partial a_n} = 0 \qquad (3.2.2a)$$

or

$$\frac{\partial P}{\partial a_n} = \frac{\partial}{\partial a_n}\left[\frac{W}{\bar{\sigma}_{12}} \left(\frac{U}{W}\bar{\sigma}_{12} - \bar{\sigma}_{12} \right)\right] = 0 \qquad (3.2.2b)$$

The split in Eq.(3.2.2b) seems meaningless, but actually, the term $\frac{U}{W}\bar{\sigma}_{12}$ does not contain $\bar{\sigma}_{12}$, if W is a linear and homogeneous function of $\bar{\sigma}_{12}$. However, this is true within the elastic limit. By carrying out the differentiation of Eq.(3.2.2b), we get

$$\left(\frac{U}{W}\bar{\sigma}_{12} - \bar{\sigma}_{12} \right) \cdot \frac{\partial}{\partial a_n}\left(\frac{W}{\bar{\sigma}_{12}} \right) + \frac{W}{\bar{\sigma}_{12}} \cdot \frac{\partial}{\partial a_n}\left(\frac{U}{W}\bar{\sigma}_{12} \right) = 0$$

(3.2.3.)

since $\bar{\sigma}_{12}$ is assumed to be constant during deformation. Because the system is equilibrium, the strain energy stored in the system must be equal to the work done by the external forces acting on the same system , that is

$$U - W = \frac{W}{\bar{\sigma}_{12}}\left(\frac{U}{W}\bar{\sigma}_{12} - \bar{\sigma}_{12} \right) = 0$$

or

$$\frac{U}{W}\bar{\sigma}_{12} - \bar{\sigma}_{12} = 0 \qquad (3.2.4)$$

since $\frac{W}{\bar{\sigma}_{12}}$ does not vanish at all. Eq. (3.2.3) will be satisfied if and only if

$$\frac{\partial}{\partial a_n}\left(\frac{U}{W}\bar{\sigma}_{12} \right) = 0$$

And it follows from Eq.(3.2.4) that

$$\frac{\partial}{\partial a_n}\left(\bar{\sigma}_{12} \right) = 0 \qquad (3.2.5)$$

Now the conclusion has been reached that the potential energy to be a minimum is equivalent to saying that

$$U - W = 0 \tag{3.2.6a}$$

and

$$\frac{\partial}{\partial a_n} (\bar{\sigma}_{12}) = 0 \tag{3.2.6b}$$

For an isotropic, homogeneous elastic body, we have

$$\sigma_{ij} = \frac{E}{1+\nu} \left(\frac{\nu}{1-2\nu} \Delta \delta_{ij} + e_{ij} \right) \tag{3.2.7}$$

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \qquad \begin{matrix} i = 1, 2, 3. \\ j = 1, 2, 3. \end{matrix} \tag{3.2.8}$$

where $\Delta = e_{ii}$, the cubical dilatation.

Here and in the sequel, summation convention is used. That is, when a latin suffix is repeated in one term, summation over the range of 1,2,3 with respect to that suffix is understood. And the partial differentiation is denoted by a comma.

For a narrow plate, that is, a plate whose thickness is very small compared with the other dimensions, the stresses are assumed to be constant in the x_3 direction. In the loading condition under consideration, we have

$$\sigma_{3i} = 0 \tag{3.2.9}$$

In Eq.(3.2.9) and all subsequent equations, the range of a latin suffix being 1,2,3 is understood.

The statement, $\sigma_{3i} = 0$, can be verified in the following way. Since the faces of the plate at $x_3 = \pm\frac{b}{2}$ are free of external loads,

$$\sigma_{3i} (x_1 , x_2 , \pm \frac{b}{2}) = 0 \qquad (3.2.10)$$

and these equations associated with the equilibrium equation in the x_3 direction,

$$\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0$$

demand that

$$\sigma_{33,3} (x_1 , x_2 , \pm \frac{b}{2}) = 0 \qquad (3.2.11)$$

The stress $\sigma_{33} (x_1 , x_2 , \pm \frac{b}{2}) = 0$ and its derivative $\sigma_{33,3} (x_1, x_2, \pm \frac{b}{2}) = 0$ means that the stress σ_{33} differs from zero very slightly through the plate if the thickness is small. Eas.(3.2.9) have been obtained under the assumption of a thin plate.

From here on, a Greek suffix has a range of 1,2 and its summation is expanded over 1 and 2.

Expanding Eqs.(3.2.9) in terms of strains by Eq.(3.2.7), we get

$$e_{3\alpha} = 0 \qquad (3.2.12a)$$

$$e_{33} = - \frac{\nu}{1 - 2\nu} \Delta \qquad (3.2.12b)$$

Substituting Eq.(3.2.8) into Eqs.(3.2.12), we have

$$u_{\alpha,3} = -u_{3,\alpha} \qquad (3.2.13a)$$

$$u_{3,3} = - \frac{\nu}{1-\nu} u_{\alpha,\alpha} \qquad (3.2.13b)$$

By integrating Eq.(3.2.13a) with respect to x_3, we have the displacements as

$$u_\alpha = - \int u_{3,\alpha} dx_3 + C_\alpha$$

Since u_3 is assumed to be a function of x_1 and x_2 only,

$$u_\alpha = -x_3 u_{3,\alpha} + C_\alpha$$

For a thin plate, the displacements along the x_1 and x_2 axis are comparatively very small, so we have the conditions that $u_\alpha = 0$ as $x_3 = 0$ to evaluate the integration constants, which results in $C_\alpha = 0$. Thus we find

$$u_\alpha = -x_3 u_{3,\alpha} \qquad (3.2.14)$$

By differentiation of Eqs.(3.2.14) with respect to x_β,

$$u_{\alpha,\beta} = -x_3 u_{3,\alpha\beta} \qquad (3.2.15)$$

Substituting the results of Eq.(3.2.13b) and Eq.(3.2.15) into Eq.(3.2.8), we get all strain components in terms of u_3 and x_3.

$$e_{\alpha\beta} = -x_3 u_{3,\alpha\beta} \qquad (3.2.16a)$$

$$e_{33} = \frac{\nu}{1-\nu} x_3 u_{3,\alpha\alpha} \qquad (3.2.16b)$$

$$e_{3\alpha} = 0 \qquad (3.2.16c)$$

The general expression of volume density of strain energy is

$$u = \frac{1}{2} \sigma_{ij} e_{ij}$$

$$= \frac{E}{2(1+\nu)} \left(e_{ij} e_{ij} + \frac{\nu}{1-2\nu} \Delta^2 \right)$$

where

$$e_{ij}e_{ij} = e_{11}^2 + e_{22}^2 + e_{33}^2 + 2e_{12}^2$$
$$= x_3^2 \left(1 + \frac{\nu}{(1-\nu)^2}\right)\left(u_{3,11}^2 + u_{3,22}^2\right)$$
$$+ \frac{2\nu^2}{(1-\nu)^2}\left(u_{3,11}\,u_{3,22} + 2u_{3,12}^2\right)$$

$$\Delta^2 = (e_{ii})^2$$
$$= \frac{(2\nu-1)^2}{(1-\nu)^2}\,x_3^2\,(u_{3,\alpha\alpha})^2$$

Inserting the values of Δ^2 and $e_{ij}\,e_{ij}$ into the expression for volume density of strain energy, we get

$$u = \frac{E x_3^2}{1+\nu}\left[\frac{1}{2(1-\nu)}(u_{3,\alpha\alpha})^2 + u_{3,12}^2 - u_{3,11}\,u_{3,22}\right] \tag{3.2.17}$$

The total energy of the plate is obtained by integrating Eq.(3.2.17) over the whole deformed volume.

$$U = \int_{V'} u\,dv$$
$$= \frac{E}{1+\nu}\int_{V'} x_3^2\left[\frac{1}{2(1-\nu)}(u_{3,\alpha\alpha})^2 + u_{3,12}^2 - u_{3,11}\,u_{3,22}\right]dv \tag{3.2.18}$$

where v' denotes the deformed volume.

The element of the deformed volume can with sufficient accuracy be written as $dx_1\,dx_2\,dx_3$, since the displacements are small. And all terms in the bracket in Eq.(3.2.18) are functions free of x_3. After carrying out the integration over x from the limit $-\frac{b}{2}$ to $+\frac{b}{2}$, we get

$$U = \frac{D}{2}\int_A \left[(u_{3,\alpha\alpha})^2 + 2(1-\nu)(u_{3,12}^2 - u_{3,11}\,u_{3,22})\right]dA \tag{3.2.19}$$

where $dA = dx_1 dx_2$, the elemental area of the plate before deformation, and

$$D = \frac{Eb^3}{12(1-\nu^2)}$$, the flexural rigidity of the plate.

In order to compute the critical load by Eqs.(3.2.6), we have to find the expression for the work done by the external forces.

By Clapeyron's theory[14], we have

$$\int_V F_i u_i \, dv + \int_S \sigma_i u_i \, ds = 2 \int_V u dv \qquad (3.2.20)$$

where $u = \frac{1}{2} \sigma_{ij} e_{ij}$, the volume density of strain energy, and
F_i = components of body force per unit volume in the x_i direction.

The left hand side of Eq.(3.2.20) is the work done by the external forces and can be expressed by

$$W = \int_V \sigma_{ij} e_{ij} \, dv$$

The nonlinear terms of displacements must be included in the expression for the strain tensor, otherwise the function $W(e_{ij})$ will vanish identically. Therefore, we put

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i} + u_{r,i} u_{r,j})$$

And since

$\sigma_{ij} = 0$, except σ_{12}

$$W = \int_V \sigma_{12} (u_{1,2} + u_{2,1} + u_{1,1} u_{1,2} + u_{2,1} u_{2,2} + u_{3,1} u_{3,2}) dv$$

By the assumption that the deflections and slopes in the x , x directions are small, the third and fourth terms in the parenthesis can be neglected. Then a simpler form of work done by the external forces is obtained,

$$W = \int_V \sigma_{12} (u_{1,2} + u_{2,1} + u_{3,1} u_{3,2})dv$$

from Eq.(3.2.15),

$$u_{\alpha,\beta} = -x_3 u_{3,\alpha\beta}$$

the function W can be formulated as follows:

$$W = -2 \int_V \sigma_{12} x_3 u_{3,12}\, dv + \int_V \sigma_{12} u_{3,1} u_{3,2}\, dv \qquad (3.2.21)$$

Assume the applied load $\sigma_{12} = \bar{\sigma}_{12}$ at the free end is uniformly distributed and remains constant during the buckling process. Therefore the integration can be carried out over x_3, since $u_3 = u_3 (x_1 , x_2)$ only.

By noticing that $\sigma_{12} x_3 u_{3,12}$ is an odd function of x_3, and thus vanishes after integration,

$$W = b\, \bar{\sigma}_{12} \int_A u_{3,1} u_{3,2}\, dA \qquad (3.2.22)$$

Substituting Eq.(3.2.22) and Eq.(3.2.19) into Eq.(3.2.6a) yields

$$\bar{\sigma}_{12} = \frac{D}{2b} \frac{\int_A \left[(u_{3,\alpha\alpha})^2 + 2 (1 - \nu) (u_{3,12}^2 - u_{3,11} u_{3,22}) \right] dA}{\int_A u_{3,1} u_{3,2}\, dA} \qquad (3.2.23)$$

From Eq.(3.2.23), we notice that the load $\bar{\sigma}_{12}$ will be completely determined if the function $u_3(x_i)$ is known. The disadvantage of the energy method is that we do not know the true expression of $u_3(x_i)$ which describes the buckling deformation. But the benefit of this method is that we can obtain an approximate result by an assumed function which satisfies the geometrical boundary conditions at least.

Let $u_3(x_i)$ be expressed as

$$u_3(x_i) = \sum_{n=0}^{\infty} a_n f(x_1) X_n(x_2) \qquad (3.2.24)$$

This expression does not satisfy all the boundary conditions of the plate, however, the geometrical boundary condition at the fixed edge will be satisfied if the functions $f(x_1)$ and $X_n(x_2)$ are suitably chosen. When comparing buckling and vibration phenomena, many similarities between them(15) will be noticed. Thus it is natural to choose

$$X_n = \cos(k_n x_2) - \cosh(k_n x_2) + \alpha_n \left[\sin(k_n x_2) - \sinh(k_n x_2) \right] \qquad (3.2.25)$$

the normal functions of a cantilever beam vibrating in its transverse direction.(16) The values of $k_n L$ and α_n are to be determined from the following equations,

$$\cos(k_n L)\cosh(k_n L) + 1 = 0 \qquad (3.2.26a)$$

$$\alpha_n = - \frac{\cos(k_n L) + \cosh(k_n L)}{\sin(k_n L) + \sinh(k_n L)} \qquad (3.2.26b)$$

Since the boundary conditions of the vibrating beam are satisfied by X_n, the geometrical boundary conditions of the cantilever plate are satisfied also. Thus the choice of the function $f(x_1)$ will be guided by the following considerations:

(a) The product of $f(x_1)$ and X_n must not violate the geometrical boundary conditions.

(b) The closer the actual buckling configuration the function describes, the better the result that will be obtained.

(c) If we take x_2 as constant temporarily, then $u_3(x_i) = u_3(x_1)$ which depends on the geometrical dimensions and the loading conditions. However, it is reasonable to assume that it is a linear function of x_1 as the slope in the x_1 direction is not large.

According to the above considerations, the function $f(x_1)$ will be chosen as

$$f(x_1) = 1 - \frac{x_1}{h} \qquad (3.2.27)$$

where

$$-\frac{h}{2} \leq x_1 \leq \frac{h}{2}$$

Substituting Eq.(3.2.25) and Eq.(3.2.27) into Eq.(3.2.24) yields

$$u_3(x_i) = \sum_{n=1}^{\infty} a_n\left(1 - \frac{x_1}{h} \right)X_n(x_2) \qquad (3.2.28)$$

Due to the assumed $u_3(x_i)$ as given in Eq.(3.2.28), Eq.(3.2.23) will be simplified to

$$\bar{\sigma}_{12} = \frac{D}{2b} \frac{\int_A \left[(u_{3,22})^2 + 2(1-\nu)(u_{3,12})^2 \right] dA}{\int_A u_{3,1} u_{3,2} dA} \qquad (3.2.29)$$

since $u_{3,11} = 0$

After differentiating $u_3(x_i)$ in Eq.(3.2.28), we get the following expressions:

$$u_{3,1} = -\frac{1}{h} \sum_{n=1}^{\infty} a_n X_n$$

$$u_{3,2} = \left(\frac{x_1}{h} - 1 \right) \sum_{n=1}^{\infty} a_n k_n S_n$$

$$u_{3,12} = \frac{1}{h} \sum_{n=1}^{\infty} a_n k_n S_n$$

$$u_{3,22} = \left(\frac{x_1}{h} - 1 \right) \sum_{n=1}^{\infty} a_n k_n^2 C_n$$

where

$$S_n = -\frac{1}{k_n} \frac{dX_n}{dx_2}$$

$$= \sin(k_n x_2) + \sinh(k_n x_2) - \alpha_n \left[\cos(k_n x_2) - \cosh(k_n x_2) \right]$$

$$C_n = \frac{1}{k_n} \frac{dS_n}{dx_2}$$

$$= \cos(k_n x_2) + \cosh(k_n x_2) + \alpha_n \left[\sin(k_n x_2) + \sinh(k_n x_2) \right]$$

For the sake of compactness in handling, the following notations are introduced.

$$C_{mn} = C_m C_n$$

$$S_{mn} = S_m S_n$$

$$Z_{mn} = X_m S_n$$

$$a_{mn} = a_m a_n$$

Attention should be paid to the fact that all expressions are symmetric with respect to m and n except Z_{mn}. By use of these notations, the following expressions will be shortened.

$$(u_{3,22})^2 = (1 - \frac{x_1}{h})^2 (\sum_{n=1}^{\infty} a_n^2 k_n^4 C_{nn} + 2 \sum_{m \neq n} a_{mn} k_{mn}^2 C_{mn})$$

$$(u_{3,12})^2 = \frac{1}{h^2} (\sum_{n=1}^{\infty} a_n^2 k_n^2 S_{nn} + 2 \sum_{m \neq n}^{\infty} a_{mn} k_{mn} S_{mn})$$

$$(u_{3,1} u_{3,2}) = \frac{1}{h} (1 - \frac{x_1}{h}) (\sum_{n=1}^{\infty} a_n^2 k_n Z_{nn} + \sum_{m \neq n}^{\infty} a_{mn} k_n Z_{mn})$$

By substituting these expressions into Eq.(3.2.29) and carrying out the integration with respect to x, over the range from $-\frac{h}{2}$ to $+\frac{h}{2}$, the final expression for $\bar{\sigma}_{12}$ is obtained.

$$\bar{\sigma}_{12} = \frac{D}{2b} \frac{N(a_n)}{M(a_n)} \qquad (3.2.30)$$

where

$$N = \frac{13}{12} h(\sum_{n=1}^{\infty} a_n^2 k_n^4 C_{nn}^{\circ} \Big|_0^L + 2 \sum_{m \neq n}^{\infty} a_{mn} k_{mn}^2 C_{mn}^{\circ} \Big|_0^L)$$

$$+ \frac{2(1-\nu)}{h} (\sum_{n=1}^{\infty} a_n^2 k_n^2 S_{nn}^{\circ} \Big|_0^L + 2 \sum_{m \neq n}^{\infty} a_{mn} k_{mn} S_{mn}^{\circ} \Big|_0^L)$$

$$M = \sum_{n=1}^{\infty} a_n^2 k_n Z_{nn}^{\circ} \Big|_0^L + \sum_{m \neq n}^{\infty} a_{mn} k_n Z_{mn}^{\circ} \Big|_0^L$$

For the sake of simplicity, the notations

$$C^{o}_{mn} = \int C_{mn}\, dx_{1}$$

$$S^{o}_{mn} = \int S_{mn}\, dx_{1}$$

$$Z^{o}_{mn} = \int Z_{mn}\, dx_{1}$$

have been used in Eq.(3.2.30).

Examining the expression of $\bar{\sigma}_{12}$ in Eq.(3.2.30), we see that $\bar{\sigma}_{12}$ is a function of n parameters which will be determined by the critical equilibrium condition.

Substituting $\bar{\sigma}_{12}$ defined by Eq.(3.2.30) into Eq.(3.2.6b) yields

$$\frac{\partial}{\partial a_n}(\bar{\sigma}_{12}) = \frac{D}{2bM} \; \frac{M\frac{\partial N}{\partial a_n} - N\frac{\partial M}{\partial a_n}}{M} = 0$$

Since $\frac{D}{2bM} \neq 0$ for non-trivial solution of a_n,

$$\frac{\partial N}{\partial a_n} - \frac{N}{M}\frac{\partial M}{\partial a_n} = 0$$

We recall that $\frac{N}{M} = \frac{2b}{D}\bar{\sigma}_{12}$, hence

$$\frac{\partial N}{\partial a_n} - \frac{2b}{D}\bar{\sigma}_{12}\frac{\partial M}{\partial a_n} = 0 \qquad n=1,2,3,\ldots\ldots\ldots\ldots m' \tag{3.2.31}$$

where m' is the number of parameters retained in the assumed displacement $u_j(x_i, a_n)$. Since M and N are functions of a_n which are in quadratic form, Eq.(3.2.31) gives m' linear, homogeneous equations for a_n. The necessary and sufficient condition that we have a non-trivial solution for a_n is that the determinant of a_n is zero. That is

$$\nabla = 0 \qquad (3.2.32)$$

This is the characteristic determinant for buckling stress $(\bar{\sigma}_{12})_{cr}$ which is the only unknown quantity. The degree of $\bar{\sigma}_{12}$ in Eq. (3.2.32) will be equal to the number of terms retained in the series for $u_3(x_i, a_n)$. Therefore, it is evident that if more terms are retained, more work is needed to solve this equation. A numerical calculation will be presented in the next section as an illustration.

IV NUMERICAL RESULTS

For specific problems, certain difficulties are encountered in finding the definite integrals for the functions C_{mn}, S_{mn} and Z_{mn} that appear in Eq.(3.2.30). After a series of tedious calculations, two tables for these integrals were prepared and are presented in the appendix for application.

The values of $k_n L$ and α_n which make $u_j(x_i)$ in Eq.(3.2.28) satisfy the geometrical boundary conditions can be determined by Eq.(3.2.26).

Table 1 Values of $k_n L$ and α_n from n = 1 to n = 5

n	1	2	3	4	5
k L	1.875	4.694	7.855	10.996	14.137
α_n	-0.734	-1.108	-1.000	-1.000	- 1.000

As n becomes large, then very sensibly

$$k_n L = (2n - 1) \frac{\pi}{2} \qquad (4.1.1)$$

or

$$\lim_{n \to \infty} (k_n L - k_{n-1} L) = \pi$$

By noticing the fact that

$$\lim_{k_n L \to \infty} \left[\cosh(k_n L) - \sinh(k_n L) \right] = \lim_{k_n L \to \infty} \frac{1}{e^{k_n L}} = 0,$$

we have

$\cosh(k_n L) \approx \sinh(k_n L)$

and

$$\alpha_n = -\frac{\cos(k_n L) + \cosh(k_n L)}{\sin(k_n L) + \sinh(k_n L)} \approx -1$$

for large $k_n L$, since $\sin(k_n L)$ and $\cos(k_n L)$ vary between zero and absolute unity which is comparatively small.

Due to the properties of hyperbolic functions and α_n as n becomes large, the following simplifications are observed.

$$C_n \approx \cos(k_n x_2) - \sin(k_n x_2) \qquad (4.1.2a)$$

$$S_n \approx \cos(k_n x_2) + \sin(k_n x_2) - 2\sinh(k_n x_2) \qquad (4.1.2b)$$

Then the functions C^{o}_{mn}, S^{o}_{mn}, and Z^{o}_{mn} for large values of $k_n x_2$ can be easily obtained by the following approximate expressions.

$$C^{o}_{mn} \approx \frac{\sin(k_m - k_n)x_2}{k_m - k_n} + \frac{\cos(k_m + k_n)x_2}{k_m + k_n} \qquad (4.1.3a)$$

$$\begin{aligned} S^{o}_{mn} \approx\ & \frac{\sin(k_m - k_n)x_2}{k_m - k_n} - \frac{\cos(k_m + k_n)x_2}{k_m + k_n} \\ & - \frac{2}{k_m^2 + k_n^2}\Big[k_m \sin(k_m x_2)\sinh(k_n x_2) + k_n \cos(k_m x_2)\cosh(k_n x_2)\Big] \\ & - \frac{2}{k_m^2 + k_n^2}\Big[k_n \sin(k_m x_2)\cosh(k_n x_2) - k_m \cos(k_m x_2)\sinh(k_n x_2)\Big] \\ & - \frac{2}{k_m^2 + k_n^2}\Big[k_n \sin(k_n x_2)\sinh(k_m x_2) + k_m \cos(k_n x_2)\cosh(k_m x_2)\Big] \\ & - \frac{2}{k_m^2 + k_n^2}\Big[k_m \sin(k_n x_2)\cosh(k_m x_2) - k_n \cos(k_n x_2)\sinh(k_n x_2)\Big] \end{aligned}$$

$$- \frac{4}{k_m^2 - k_n^2} \left[k_m \sinh(k_n x_2) \cosh(k_m x_2) - k_n \cosh(k_n x_2) \sinh(k_m x_2) \right]$$

(4.1.3b)

$$Z^{o}_{mn} \approx \frac{\sin(k_m + k_n) x_2}{k_m + k_n} + \frac{\cos(k_m - k_n) x_2}{k_m - k_n}$$

$$+ \frac{2}{k_m^2 + k_n^2} \left[k_n \sin(k_m x_2) \cosh(k_n x_2) - k_m \cos(k_m x_2) \sinh(k_n x_2) \right]$$

$$- \frac{2}{k_m^2 + k_n^2} \left[k_m \sin(k_m x_2) \sinh(k_n x_2) + k_n \cos(k_m x_2) \cosh(k_n x_2) \right]$$

(4.1.3c)

For $k_m = k_n$, the functions C_{mn}, S_{mn} and Z_{mn} can be obtained from Eqs.(4.1.3) by using the limit process.

It will be recalled that Eqs.(4.1.3) are only applicable for large values of $k_n L$.

For simplicity, only one term in Eq.(3.2.28) is retained. Then the values of the functions $C^{o}_{11}\big|_0^L$, $S^{o}_{11}\big|_0^L$ and $Z^{o}_{11}\big|_0^L$ are obtained from Table A-1 as follows:

$$C^{o}_{11}\big|_0^L = 0.981L$$

$$S^{o}_{11}\big|_0^L = 1.316L$$

$$Z^{o}_{11}\big|_0^L = 1.430L$$

From Eq.(3.2.30) for n = 1, we have

$$N = \frac{13}{12} h\, a_1^2 k_1^4 C^{o}_{11}\big|_0^L + \frac{2(1-\nu)}{h} a_1^2 k_1^2 S^{o}_{11}\big|_0^L$$

$$M = a_1^2 k_1 Z_{11}^{\circ}\Big|_0^L$$

Substituting the values of k_1, $C_{11}^{\circ}\Big|_0^L$, $S_{11}^{\circ}\Big|_0^L$ and $Z_{11}^{\circ}\Big|_0^L$ into and rearranging the expressions for N and M yield

$$N = \frac{a_1^2}{hL^3}\left[13.141 h^2 + 9.254(1-\nu)L^2\right]$$

$$M = 2.681 a_1$$

By differentiation, we get

$$\frac{\partial N}{\partial a_1} = \frac{a_1}{hL^3}\left[26.282 h^2 + 18.508(1-\nu)L^2\right]$$

$$\frac{\partial M}{\partial a_1} = 5.362\, a_1$$

From Eq.(3.2.31) for n = 1, the equation for determining the buckling stress becomes

$$\frac{1}{hL^3}\left[26.282 h^2 + 18.508(1-\nu)L^2\right] - \frac{2b}{D}\left[5.362(\bar{\sigma}_{12})_{cr}\right] = 0$$

By rearranging, we obtain

$$(\bar{\sigma}_{12})_{cr} = \frac{Dh}{bL^3}\left[2.451 + 1.726(1-\nu)\frac{L^2}{h^2}\right] \qquad (4.1.4)$$

For convenience in comparison with other results, the maximum stress which occurs at the remotest fibre at the fixed edge can be obtained from the following equation.

$$(\sigma_{max})_{cr} = \frac{6L}{h}(\bar{\sigma}_{12})_{cr} \qquad (4.1.5)$$

Substituting $(\bar{\sigma}_{12})_{cr}$ defined in Eq.(4.1.4) into Eq.(4.1.5) and recalling $D = \dfrac{Eb^3}{12(1-\nu^2)}$ yields

$$(\sigma_{max})_{cr} = \frac{2.487\,Eb^2}{hL}\left(\frac{0.493}{1-\nu^2}\,\frac{1}{\lambda} - \frac{0.347}{1+\nu}\,\lambda\right)$$

(4.1.6)

where $\lambda = \dfrac{L}{h}$.

$\nu = 0.3$ for most structural steels, then Eq. (4.1.6) becomes

$$(\sigma_{max})_{cr} = \frac{2.487\,Eb^2}{hL}\left(0.542\,\frac{1}{\lambda} + 0.267\,\lambda\right)$$

(4.1.7)

V. DISCUSSION AND CONCLUSIONS

For the sake of comparison, Eq.(4.1.7) and the result obtained originally by Prandtl[17] are plotted as curves C_1 and C_2 respectively in Fig.2, p.33. Due to the lack of experimental data, the author cannot give a definite valid range of λ in his result, however, some qualitative conclusions can still be derived by reasoning.

The maximum stress at the buckling state is a function of h and L to first order in C_2, but to higher order in C_1. The curve C_1 is divided into three parts by the intersection points with C_2 at $\lambda_1 = 0.656$ and $\lambda_2 = 3.090$.

The left part of C_1 goes to infinity as λ goes to zero. It shows that the buckling stress obtained from Eq.(4.1.4) is much larger than that obtained from Prandtl's result.

In the middle part of C_1, the maximum stress is less than that in curve C_2. Since the buckling stress obtained by the energy method is always greater than the true value, curve C_1 gives a more satisfactory result than C_2 which is obtained by Prandtl. The values of λ_1 and λ_2 may shift more or less as we retain more terms in the series $u_\lambda(x_i)$. But, anyhow, we can say that the energy method based on thin plate theory gives satisfactory result for λ between and around 0.656 and 3.090.

In the right part, C_1 deviates from C_2 further and further as λ becomes larger and larger. Then the curve C_1 gives a less satisfactory approximation. But as λ becomes greater than a certain value, the maximum stresses defined by both Eq.(4.1.7) and Prandtl's result will be beyond the elastic limits for most structural materials. Thus curves C_1 and C_2 do not apply in the case that λ is larger than a certain value which varies according to the material and the dimensions of the plate under consideration.

As λ becomes large, Eq.(4.1.7) does not hold, even if $(\sigma_{max})_{cr}$ remains in the elastic range, since the deflections and slopes in the x_1 and x_2 directions are no longer small enough to be neglected. Then the potential energy will be formulated in terms of u_i rather than u_3 only. In this general case, we may assume the displacements in the form

$$u_i = \sum_{n=1}^{\infty} a_i^{(n)} X_n \qquad \text{where} \quad i = 1, 2, 3. \qquad (5.1.1)$$

In these series, X_n are certain admissible functions of x_i which satisfy the geometrical boundary conditions; $a_i^{(n)}$ are unknown parameters, independent of x_i, which will be determined by the critical equilibrium condition.

Since the displacements u_i are assumed, the potential energy for the whole elastic body after integration will be a function of $a_i^{(n)}$. Thus

$$P = P(\, a_i^{(n)} \,)$$

where $i = 1, 2, 3$

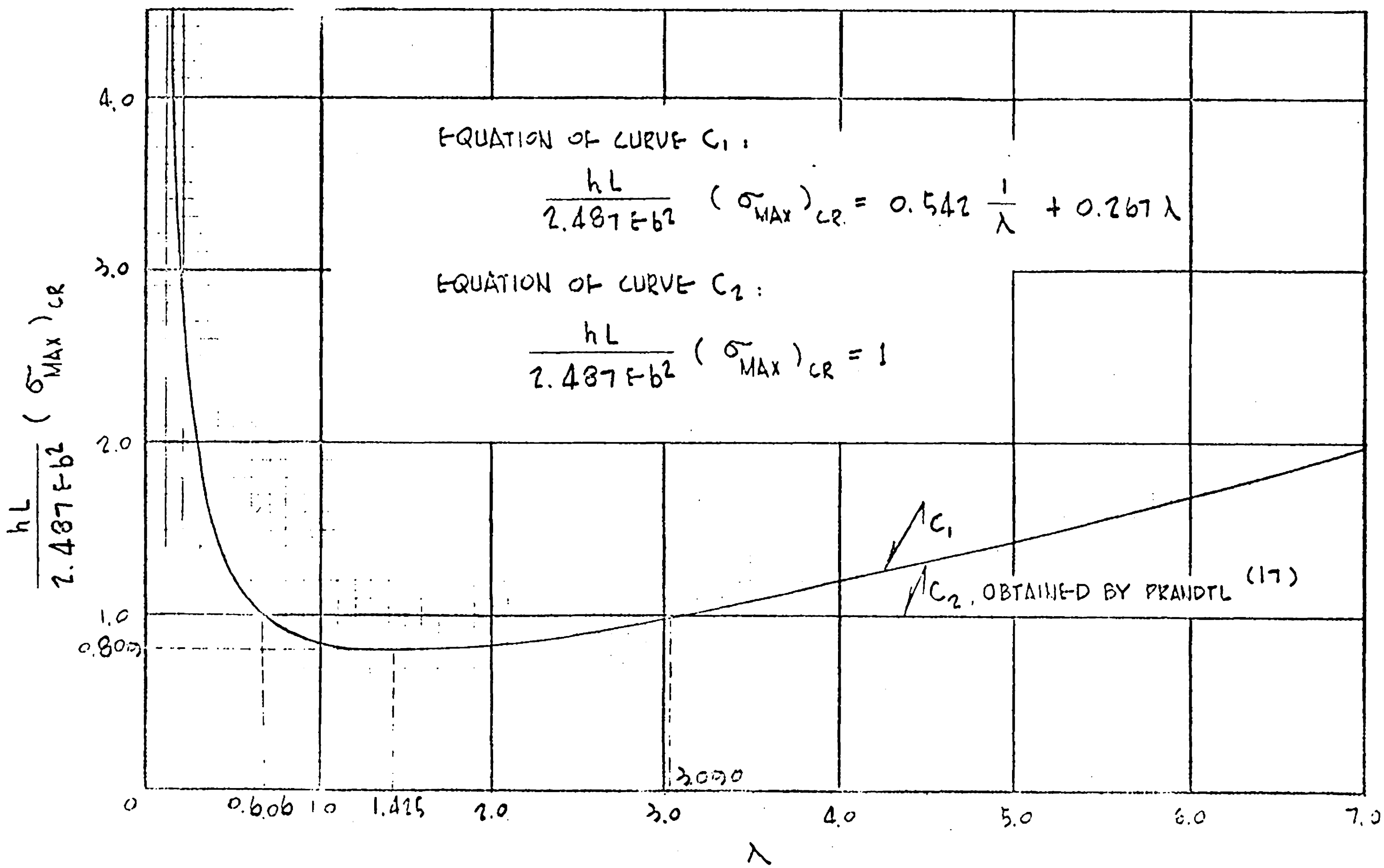
$n = 1, 2, 3, \ldots\ldots\ldots\ldots, n$

The unknown parameters $a_i^{(n)}$ corresponding to the displacements in Eq.(5.1.1) which will produce equilibrium can be found by minimizing the expression for $P(a_i^{(n)})$,

$$\frac{\partial P(a_i^{(n)})}{\partial a_i^{(n)}} = 0 \qquad (5.1.2)$$

From Eq.(5.1.2), we obtain a system of 3n linear homogeneous equations, where n is the number of terms retained in each of the series defined by Eq.(5.1.1). In order to guarantee that the system has a non-trivial solution, the determinant of the coefficients must vanish. The coefficients are functions of geometrical dimensions, elastic properties and the external load which is the only unknown to be found. Therefore, the lowest value of the external load, for which the characteristic determinant vanishes, and which corresponds the critical value of the load, will be determined.

FIG. 1 CURVES FOR MAXIMUM STRESS AT BUCKLING STATE



VI APPENDIXES

TABLE A-1 FUNCTIONS C°_{nn}, S°_{nn} & Z°_{nn} FOR ARBITRARY VALUE OF n

FUNCTIONS	C°_{nn}	S°_{nn}	Z°_{nn}
$\frac{1}{K_n} \sin(2K_n X_2)$	$\frac{1-\alpha^2}{4}$	$-\frac{1-\alpha^2}{4}$	$-\frac{\alpha}{2}$
$\frac{1}{K_n} \cos(2K_n X_2)$	$-\frac{\alpha}{2}$	$\frac{\alpha}{2}$	$-\frac{1-\alpha^2}{4}$
$\frac{1}{K_n} \sinh(2K_n X_2)$	$\frac{1+\alpha^2}{4}$	$\frac{1+\alpha^2}{4}$	0
$\frac{1}{K_n} \cosh(2K_n X_2)$	$\frac{\alpha}{2}$	$\frac{\alpha}{2}$	$\frac{1-\alpha^2}{4}$
$\frac{1}{K_n} \sin(K_n X_2) \sinh(K_n X_2)$	2α	0	$-(1-\alpha^2)$
$\frac{1}{K_n} \sin(K_n X_2) \cosh(K_n X_2)$	$1+\alpha^2$	$1-\alpha^2$	0
$\frac{1}{K_n} \cos(K_n X_2) \sinh(K_n X_2)$	$1-\alpha^2$	$-(1+\alpha^2)$	2α
$\frac{1}{K_n} \cos(K_n X_2) \cosh(K_n X_2)$	0	-2α	$1-\alpha^2$
$\frac{1}{K_n} (K_n X_2)$	1	α^2	$-\alpha$

NOTE. THE FUNCTION IS EQUAL TO THE SUM OF THE PRODUCTS OF THE TERMS IN THE FIRST COLUMN AND THE CORRESPONDING TERMS IN THE COLUMN UNDER THAT FUNCTION. FOR EXAMPLE:

$$C^{\circ}_{mn} = \frac{1}{K_n}\left[\frac{1-\alpha^2}{4}\sin(2K_n X_2) - \frac{\alpha}{2}\cos(2K_n X_2) + \text{---------}\right]$$

TABLE A-2   FUNCTIONS C^{o}_{mn}, S^{o}_{mn} & Z^{o}_{mn} FOR ARBITRARY VALUES OF m & n

FUNCTIONS	C^{c}_{mn}	S^{o}_{mn}	Z^{o}_{mn}
$\frac{\sin(K_m - K_n)x_2}{2(K_m - K_n)}$	$1 + \alpha_m \alpha_n$	$1 + \alpha_m \alpha_n$	$\alpha_m - \alpha_n$
$\frac{\sin(K_m + K_n)x_2}{2(K_m + K_n)}$	$1 - \alpha_m \alpha_n$	$-(1 - \alpha_m \alpha_n)$	$-(\alpha_m - \alpha_n)$
$\frac{\cos(K_m - K_n)x_2}{2(K_m - K_n)}$	$-(\alpha_m - \alpha_n)$	$-(\alpha_m - \alpha_n)$	$1 + \alpha_m \alpha_n$
$\frac{\cos(K_m + K_n)x_2}{2(K_m + K_n)}$	$-(\alpha_m + \alpha_n)$	$\alpha_m + \alpha_n$	$-(1 - \alpha_m \alpha_n)$
$\frac{\sinh(K_m x_2)\cdot\sinh(K_n x_2)}{K_m^2 - K_n^2}$	$K_m \alpha_n - K_n \alpha_m$	$-K_n \alpha_n + K_m \alpha_m$	$K_m + K_n \alpha_m \alpha_n$
$\frac{\cosh(K_m x_2)\cdot\cosh(K_n x_2)}{K_m^2 - K_n^2}$	$K_m \alpha_m - K_n \alpha_n$	$-K_n \alpha_m + K_m \alpha_n$	$-K_n - K_m \alpha_m \alpha_n$
$\frac{\sin(K_m x_2)\cdot\sinh(K_n x_2)}{K_m^2 + K_n^2}$	$K_m \alpha_n + K_n \alpha_m$	$-K_m \alpha_m + K_n \alpha_n$	$-K_m + K_n \alpha_m \alpha_n$
$\frac{\sin(K_n x_2)\cdot\sinh(K_m x_2)}{K_m^2 + K_n^2}$	$K_m \alpha_n + K_n \alpha_m$	$K_m \alpha_m - K_n \alpha_n$	$-K_m + K_n \alpha_m \alpha_n$
$\frac{\sin(K_m x_2)\cosh(K_n x_2)}{K_m^2 + K_n^2}$	$K_m + K_n \alpha_m \alpha_n$	$K_n - K_m \alpha_m \alpha_n$	$K_m \alpha_n - K_n \alpha_m$
$\frac{\sin(K_n x_2)\cdot\cosh(K_m x_2)}{K_m^2 + K_n^2}$	$K_n + K_m \alpha_m \alpha_n$	$K_m - K_n \alpha_m \alpha_n$	$K_n \alpha_n - K_m \alpha_m$
$\frac{\cos(K_m x_2)\cdot\sinh(K_n x_2)}{K_m^2 + K_n^2}$	$K_n - K_m \alpha_m \alpha_n$	$-K_m - K_n \alpha_m \alpha_n$	$K_m \alpha_m + K_n \alpha_n$
$\frac{\cos(K_n x_2)\cdot\sinh(K_m x_2)}{K_m^2 + K_n^2}$	$K_m - K_n \alpha_m \alpha_n$	$-K_n - K_m \alpha_m \alpha_n$	$K_m \alpha_n + K_n \alpha_m$
$\frac{\cos(K_m x_2)\cdot\cosh(K_n x_2)}{K_m^2 + K_n^2}$	$-K_m \alpha_m + K_n \alpha_n$	$-K_n \alpha_m - K_m \alpha_n$	$K_n - K_m \alpha_m \alpha_n$
$\frac{\cos(K_n x_2)\cdot\cosh(K_m x_2)}{K_m^2 + K_n^2}$	$-K_n \alpha_n + K_m \alpha_m$	$-K_m \alpha_n - K_n \alpha_m$	$-K_n + K_m \alpha_m \alpha_n$
$\frac{\sinh(K_m x_2)\cdot\cosh(K_n x_2)}{K_m^2 - K_n^2}$	$K_m - K_n \alpha_m \alpha_n$	$-K_n + K_m \alpha_m \alpha_n$	$-K_m \alpha_n - K_n \alpha_m$
$\frac{\sinh(K_n x_2)\cdot\cosh(K_m x_2)}{K_m^2 - K_n^2}$	$-K_n + K_m \alpha_m \alpha_n$	$K_m - K_n \alpha_m \alpha_n$	$K_m \alpha_m + K_n \alpha_n$

NOTE: THE FUNCTION IS EQUAL TO THE SUM OF THE PRODUCTS OF THE TERMS IN THE FIRST COLUMN AND THE CORRESPONDING TERMS IN THE COLUMN UNDER THAT FUNCTION.

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BUCKLING OF CANTILEVER THIN PLATE
WITH FREE END SUBJECTED TO UNIFORM SHEAR

by

James Chie Meng Yu

Abstract

This thesis is concerned with the buckling problem of a cantilever thin plate with its free end subjected to uniform shear. The same problem was originally solved by Prandtl in 1899, based on the equilibrium condition of a deep beam. The author has used the energy method based on the thin plate theory to attack the problem.

After the displacement is assumed, the potential energy can be formulated. From the condition that the potential energy assumes a minimum value in an equilibrium configuration, results a system of n linear homogeneous algebraic equations of n parameters which are introduced in the assumed displacement. For a non-trivial solution, the determinant of the coefficients must vanish. This gives a characteristic equation from which the buckling load is determined. The author has obtained a curve for maximum stress at buckling state, which shows that the result is better than that obtained by Prandtl in certain cases.

The energy method has been generalized to a three dimensional problem to consider the displacements in all directions.