TRANSIENT VIBRATIONS OF A CANTILEVER BEAM ROTATING AT A CONSTANT ANGULAR ACCELERATION

by

LEONARD FREDERICK ERMOLD, JR.

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SYMBOL	QUANTITY	UNITS
A	Cross-sectional area of the beam	in^2
γ	Weight per unit volume of the beam	lb/in^3
Q	Shear force acting on the beam	lb
M	Moment acting on the beam	in-lb
α	Angular Acceleration of hub	rad/sec^2
g	Acceleration of gravity	in/sec^2
L	Total length of the beam	in
r	Radius of hub	in
t	Time	sec
x	Distance from the fixed-end support to an arbitrary point on the beam	in
y	Deflection at an arbitrary point on the beam	in
G	Modulus of rigidity	lb/in^2
E	Modulus of elasticity	lb/in^2

SYMBOL	QUANTITY	UNITS
K'	Shape factor for shear	--
I	Area moment of inertia of the beam's cross section about an axis parallel to the axis of rotation	in^4
Ψ	Slope of the deflection curve due to bending	rad
P	Axial force acting on the beam	lb
ϕ	Angle between undeflected beam axis and a radial line to a point on the beam axis	rad
β	Slope of the deflection curve due to shear deformation	rad
ω	Angular velocity of the hub at any time t	rad/sec
λ'	Distance from the fixed-end support along a radial line to any point on the beam axis	in
c	Velocity of wave propagation	in/sec

SYMBOL	QUANTITY	UNITS
c_o	$\sqrt{E/\rho}$, where E is the modulus of elasticity for the material and ρ is its mass density	in/sec
a	Radius of circular bar	in
λ	Wave length	in
H	Finite difference interval along the beam axis	--
H1	Finite difference interval of time	--
AB	Dimensionless constant = $K'G/\gamma L$	--
AB1	Dimensionless constant = $L\alpha/g$	--
AB2	Dimensionless constant = I/AL^2	--
AB3	Dimensionless constant = $EI/\gamma AL^3$	--
$\bar{R}$	Dimensionless constant = r/L	--
$\bar{Q}$	Dimensionless constant = $Q/\gamma AL$	--
$\bar{X}$	Dimensionless constant = X/L	--
$\bar{Y}$	Dimensionless constant = y/L	--

SYMBOL	QUANTITY	UNITS
$\bar{T}$	Dimensionless constant $= \sqrt{g/L}\ t$	--
$\bar{M}$	Dimensionless constant $= M/\gamma AL^2$	--
$\bar{P}$	Dimensionless constant $= P/\gamma AL$	--

III. INTRODUCTION

The problem of a cantilever beam rotating at a constant angular velocity about an axis perpendicular to its base and vibrating in its plane of rotation has been previously investigated and solved by numerous authors. Two of the more recent papers dealing with this topic were written by Huang and Wu, reference (3), and Milner, reference (5). Both papers employ a method similar to that originally developed by Myklestad, but include the effects of shear deformation and rotary inertia.

The necessity for investigations of this nature has been greatly amplified in recent years due to modern design trends. It is interesting to note, however, that no one to date has considered a beam's transient reaction to a constant angular acceleration. Being able to predict and accurately determine such effects would greatly aid the future design of rotating members. As an example, consider a turbine blade rotating at a constant angular velocity. The inertia forces and their effects on this member are well established. However, prior to obtaining constant angular velocity, the turbine blade must first pass through a period of angular acceleration. During this period it will be loaded with additional forces over and above those of constant angular velocity. If the blade is assumed initially at rest, the application of an angular acceleration will produce an impulsive-type loading. Being able to accurately determine the maximum displacements caused by such loads and the time interval over which they occur will obviously enhance subsequent stress-strain calculations on rotating members.

The purpose of this thesis will then be to investigate the transient vibrations of a cantilever beam rotating at a constant angular acceleration about an axis perpendicular to the base of the beam. It is to be understood that:

1. This is not merely an extension of the constant angular velocity problem; for, the basic assumption that harmonic motion exists at every point on the beam cannot be justified for constant angular acceleration; and

2. All beams considered here will be of constant cross-sectional area.

The governing differential equations and the general governing differential equation of this problem will include the effects of bending, shear deformation, and rotary inertia. Coriolis' acceleration, however, will be neglected.

Subsequent to presenting the general development of the above equations, the governing differential equations will be converted to a non-dimensional form, and solved by means of numerical analysis.

Actual numerical results, obtained by use of an I.B.M. 1620 Digital Computer, will be presented in tabular and graphical form.

IV. INVESTIGATION

A. Development of the Governing Differential Equation

The study of any mechanics problem should, for purposes of clarity, be preceded by an adequate description of the system to be considered. In this problem, the general system consists of a cantilever beam rigidly attached to a hub which in turn is mounted on a shaft. A front view illustrating this shaft, hub and beam combination is shown in Figure 1. Next, the assumptions pertinent to the solution of the problem should be established. These are as follows:

1. The cross-sectional area of the beam is constant.
2. The system starts initially from rest.
3. Vibrations will occur only in the plane of rotation, i.e., the x, y plane of Figure 1.
4. The weight of the beam is negligible in comparison to the body forces.
5. The deflections of the beam from its undeformed axis are small.
6. The elastic properties of the beam are at all times preserved.
7. The hub acts as a fixed-end support for the beam.
8. The shaft supporting the hub and beam combination will not deflect nor vibrate.
9. Air resistance is negligible.

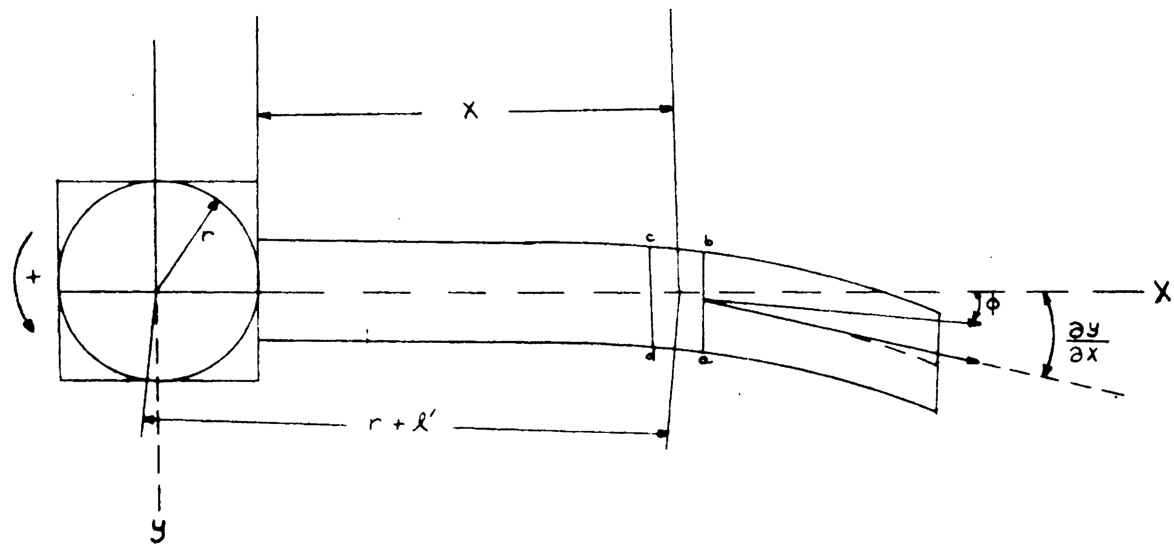

Figure 1. Front View of Hub and Beam Combination

The governing differential equations will now be developed. An enlarged view of the element abcd (Figure 1) is shown in Figure 2; acting on this element are the following loads:

M -- internal bending moment acting on the face cd.

$M + \frac{\partial M}{\partial x} dx$ -- the internal bending moment acting on the face ab.

Q -- shear force acting on face cd.

$Q + \frac{\partial Q}{\partial x} dx$ -- shear force acting on face ab.

$\frac{\gamma A}{g} \frac{\partial^2 y}{\partial t^2} dx$ -- inertia force due to vibratory motion of the element in the y direction.

$\frac{\gamma A}{g}(r+\ell') \omega^2 dx$ -- inertia force due to centrifugal acceleration about the axis of rotation.

$\frac{\gamma A}{g}(r+\ell') \alpha \, dx$ -- inertia force due to tangential acceleration about the axis of rotation.

$\frac{I \gamma}{g} \alpha \, dx$ -- inertia couple acting opposite to the angular acceleration

$\frac{I \gamma}{g} \frac{\partial^2 \psi}{\partial t^2} dx$ -- Rayleigh's Inertia Term - inertia couple opposing rotation of the element about an axis through its center and parallel to the axis of rotation.

$\frac{2 \gamma A}{g} \omega \frac{\partial y}{\partial t} dx$ -- Coriolis' force

Since all body and surface forces are shown acting on the element of Figure 2, a set of governing differential equations will be developed by direct application of the basic equations of statics.

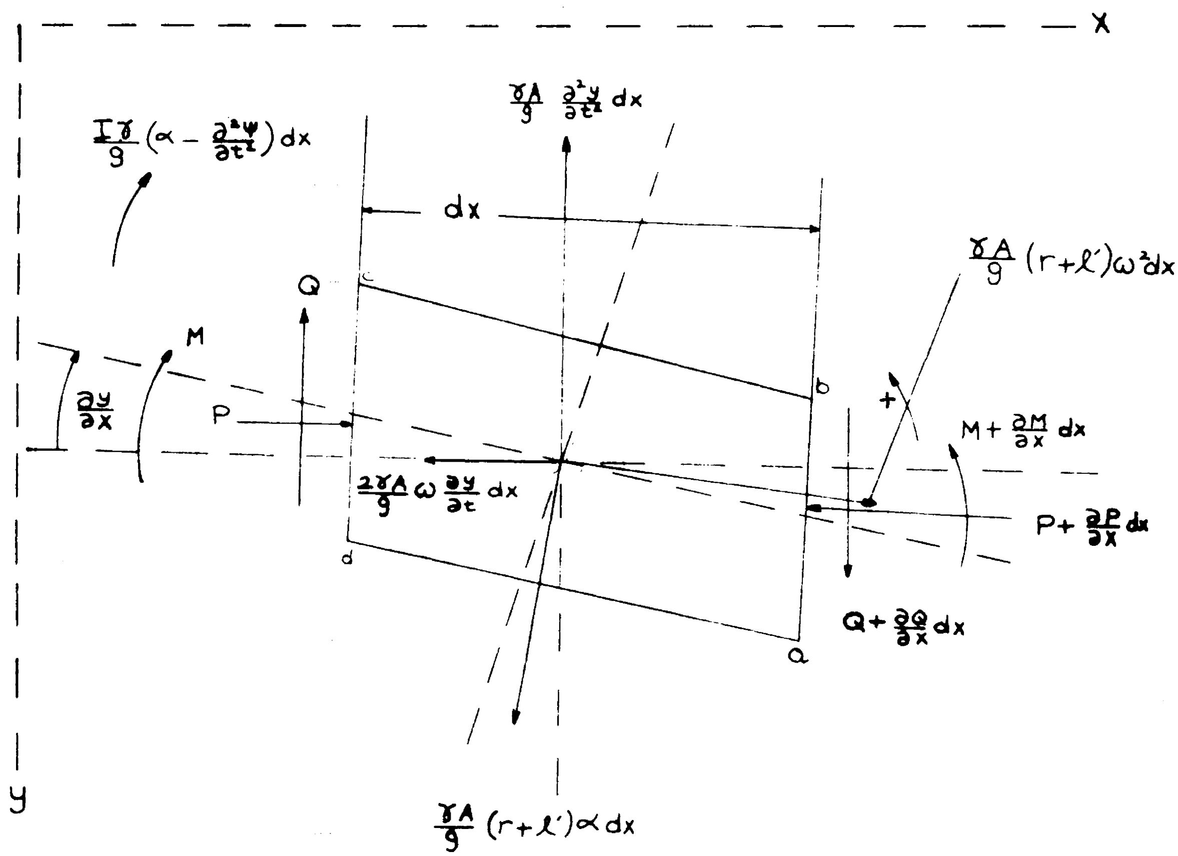

Figure 2. Free Body Diagram for an Arbitrary Beam Element

Summation of forces in the negative y direction:

$$Q - \left(\frac{\gamma A}{g}(r+\ell')\alpha\right) dx \cos \phi + \frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} dx$$

$$- \left(\frac{\gamma A}{g}(r+\ell')\omega^2\right) dx \sin \phi - Q - \frac{\partial Q}{\partial x} dx = 0$$

Introducing the assumptions $\cos \phi \approx 1$; $\sin \phi \approx \tan \phi = \frac{y}{r+\ell'}$; and $r+\ell' \approx r+x$; plus the relationship $\omega = \alpha t$, the final equation

$$\frac{\partial Q}{\partial x} = - \left(\frac{\gamma A}{g}(r+x)\alpha\right) + \frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} - \left(\frac{\gamma A}{g}\alpha^2 t^2 y\right) \qquad (1)$$

is obtained.

Summation of forces in the positive x direction:

$$P - \frac{2\gamma A}{g}\omega\frac{\partial y}{\partial t} dx + \left(\frac{\gamma A}{g}(r+\ell')\omega^2 dx\right) \cos \phi$$

$$- \left(\frac{\gamma A}{g}(r+\ell')\alpha dx\right) \sin \phi - P - \frac{\partial P}{\partial x} = 0$$

Again introducing the assumptions $\cos \phi \approx 1$; $\sin \phi \approx \tan \phi = \frac{y}{r+\ell'}$; and $r+\ell' \approx r+x$; plus the relationship $\omega = \alpha t$, we obtain

$$\frac{\partial P}{\partial x} = - \frac{2\gamma A}{g}\alpha t \frac{\partial y}{\partial t} - \frac{\gamma A}{g}\alpha y + \frac{\gamma A}{g}(r+x)\alpha^2 t^2$$

Neglecting Coriolis' force and the x component of the inertia force due to tangential acceleration reduces the above equation to

$$\frac{\partial P}{\partial x} = \frac{\gamma A}{g}(r+x)\alpha^2 t^2$$

Integration of this expression yields

$$P = \frac{\gamma A}{g}(rx + \frac{x^2}{2})\alpha^2 t^2 + C$$

Applying the boundary condition P = 0 at the free end of the beam, the final expression

$$P = \frac{\gamma A}{g}(rx + \frac{x^2}{2})\alpha^2 t^2 - \frac{\gamma A}{g}(rL + \frac{L^2}{2})\alpha^2 t^2 \qquad (2)$$

is obtained.

Summing moments about the center of the element abcd,

$$M + \frac{I\gamma}{g}(\alpha - \frac{\partial^2 \psi}{\partial t^2})\, dx - M - \frac{\partial M}{\partial x}\, dx + P \frac{\partial y}{\partial x}\, dx$$

$$+ \frac{\partial P}{\partial x}\, dx\, (\frac{dx}{2}) + Q\, dx + \frac{\partial Q}{\partial x}\, dx\, (\frac{dx}{2}) = 0$$

reducing and neglecting higher-order infinitesimals, the final equation

$$\frac{\partial M}{\partial x} = \frac{I\gamma}{g}\,(\alpha - \frac{\partial^2 \psi}{\partial t^2}) + P\,\frac{\partial y}{\partial x} + Q \qquad (3)$$

is obtained.

From elementary strength of materials we may write the following equation for shear:

$$Q = K'\beta AG$$

However, at any point on the beam the total slope of the deflection curve will be equal to the slope caused by bending plus the slope due to shear.

Therefore,

$$\frac{\partial y}{\partial x} = \beta + \psi$$

Solving for β and substituting into the shear equation, we obtain

$$Q = K'AG\left(\frac{\partial y}{\partial x} - \psi\right). \tag{4}$$

Finally from elementary strength of materials we note that the bending moment equation

$$M = -EI\frac{\partial \psi}{\partial x} \tag{5}$$

must also be satisfied. With the writing of this equation we now have a set of five linear partial differential equations. This set of equations will be referred to as the governing differential equations.

To obtain the general governing differential equation for the beam in terms of the dependent variable "y" and the independent variables "x" and "t", a series of lengthy mathematical manipulations must be performed. Since these manipulations are of an elementary nature, they are placed in the appendix and only the final results presented here:

$$\begin{aligned}
&EI\frac{\partial^4 y}{\partial x^4} + \left(\frac{I\gamma}{g} - \frac{EI\gamma}{K'Gg}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2} + \left(\frac{EI\gamma}{K'Gg}\alpha^2 t^2 + \frac{\gamma A}{g}\left(rx\right.\right. \\
&\left.\left. + \frac{x^2}{2}\right)\alpha^2 t^2\right) - \frac{\gamma A}{g}\left(rL + \frac{L^2}{2}\right)\alpha^2 t^2\Big)\frac{\partial^2 y}{\partial x^2} + \frac{\gamma A}{g}(r + x)\alpha^2 t^2\left(\frac{\partial y}{\partial x}\right) \\
&- \frac{I\gamma^2}{g^2K'G}\frac{\partial^4 y}{\partial t^4} + \left(\frac{I\gamma^2}{g^2K'G}\alpha^2 t^2 + \frac{\gamma A}{g}\right)\frac{\partial^2 y}{\partial t^2} + \frac{4I\gamma^2\alpha^2 t}{g\,K'G}\frac{\partial y}{\partial t} \\
&+ \left(\frac{2I\gamma^2\alpha^2}{g^2K'G} - \frac{\gamma A}{g}\alpha^2 t^2\right)y - \frac{\gamma A}{g}(r + x)\alpha = 0
\end{aligned} \tag{6}$$

By examining Equation 6, it is obvious that this equation does not lend itself readily to an exact solution; hence, necessitating the use of finite differences -- or some other approximate method. However, prior to applying the finite differences, the boundary conditions will be given and non-dimensional forms of the above equations determined.

B. Presentation of Boundary Conditions

The boundary conditions obtained from physical considerations are as follows:

At time $t = 0$:

1. $y = 0$ at all points on the beam
2. $\Psi = 0$ at all points on the beam
3. $M = 0$ at all points on the beam
4. $Q = 0$ at all points on the beam
5. $\frac{\partial y}{\partial t} = 0$ at all points on the beam
6. $\frac{\partial \Psi}{\partial t} = 0$ at all points on the beam

At any time t:

7. $y = 0$ at the fixed end of the beam
8. $\Psi = 0$ at the fixed end of the beam
9. $Q = 0$ at the free end of the beam
10. $M = 0$ at the free end of the beam

C. Non-Dimensional Form of the Governing Differential Equations

The conversion of Equations 1 through 5 to a non-dimen-- sional form entails a series of routine mathematical operations.

For this reason, they are placed in the appendix and only the final results presented here:

$$\frac{\partial \overline{Q}}{\partial \overline{x}} = -AB1\ (\overline{R} + \overline{x}) + \frac{\partial^2 \overline{y}}{\partial \overline{T}^2} - (AB1)^2\, \overline{T}^2\, \overline{y} \tag{1a}$$

$$\overline{P} = (AB1)^2\ (\overline{Rx} + \frac{\overline{x}^2}{2})\overline{T}^2 - (AB1)^2 (\overline{R} + .5)\overline{T}^2 \tag{2a}$$

$$\frac{\partial \overline{M}}{\partial \overline{x}} = (AB2)\ (AB1 - \frac{\partial^2 \psi}{\partial \overline{T}^2}) + \overline{P}\,\frac{\partial \overline{y}}{\partial \overline{x}} + \overline{Q} \tag{3a}$$

$$\overline{Q} = AB\ (\frac{\partial \overline{y}}{\partial \overline{x}} - \psi) \tag{4a}$$

$$\overline{M} = -AB3\ \frac{\partial \psi}{\partial \overline{x}} \tag{5a}$$

Definitions for the dimensionless quantities used in the above equations are given in the List of Symbols, and will not be repeated here.

The non-dimensional form of Equation 6 will not be presented since several factors (discussed in the following section) make its use in the solution to this problem impractical.

D. Discussion of Various Methods of Solution.

Initially the method of finite difference was applied to the governing differential equation (Equation 6). The resulting finite difference pattern for this equation is as shown in Figure 3a. Imposing this pattern on a finite difference network, or a net similar to the one shown in Figure 3b, will, due to insufficient boundary conditions, always result in a system of more unknowns than

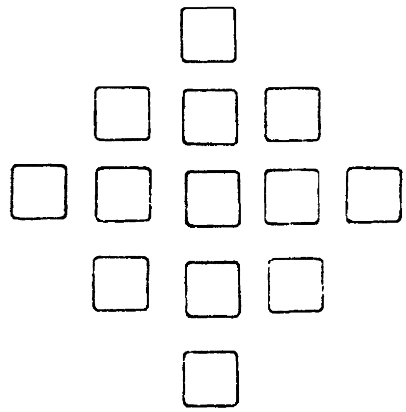

Figure 3a. Finite Difference Pattern for the
Governing Differential Equation

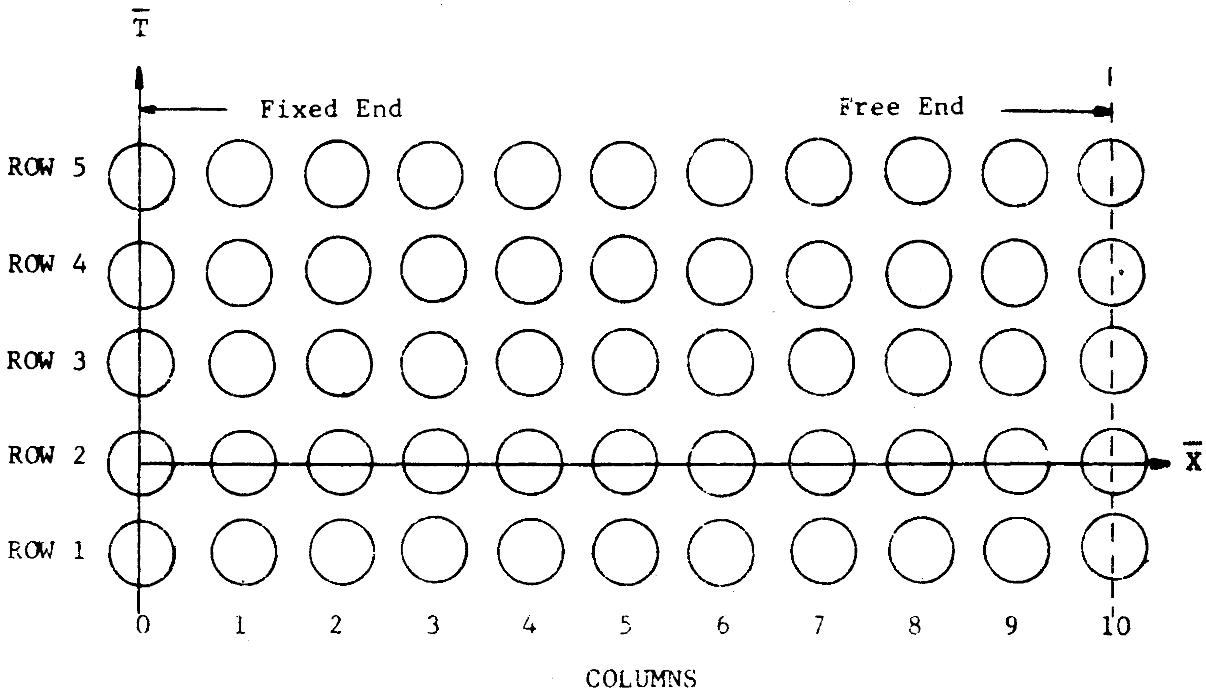

Figure 3b. Finite Difference Field

equations. This situation could be corrected by establishing additional relationships at the boundaries by using differential forms of Equations (1), (2), (3), (4), and (5). However, the error involved in approximating a fourth-order derivative with finite difference will always be greater than that for a second or first-order derivative. Therefore, due to the increased error and the necessity of additional boundary conditions, it was decided to abandon Equation 6 and seek a solution in terms of the lower-order governing differential equations (Equations 1-5.)

The use of the governing differential equations will not in itself completely solve the problem; however, it will minimize the error involved in a finite difference solution. Certain inevitable difficulties are still present at the boundaries. This may be attributed to the fact that this particular problem is an initial--value problem with respect to time, and a boundary-value problem with respect to the beam's axis. The split boundary conditions of the beam's axis present no significant difficulty when applying finite difference; however, an initial-value problem does.

As a result of this, the first approach using the governing differential equations applied the concept that a derivative at a particular point could be expressed in terms of subsequent points (forward difference) or preceding points (backward difference) in contrast to ordinary or central difference. By applying backward difference to all derivatives with respect to time and central difference to all derivatives with respect to the beam's axis, a solution is obtained.

As a starting point, the beam of Figure 1 was divided arbitrarily into ten equal sections. A diagramatic representation of this is shown in Figure 3b, for five successive values of $\overline{T}$. Let it be assumed, arbitrarily, that Row 2 of Figure 3b corresponds to $\overline{T} = 0$. Through application of boundary conditions 1 and 2, we find that all values of displacement (y) and bending slope (ψ) are zero on row 2. In order to apply boundary conditions 5 and 6 next, a backward difference expression must be written for the first derivative with respect to time. Such an expression may be written as

$$\frac{\partial f}{\partial t} = \frac{1}{H1}\left(f(I,J) - f(I-1,J)\right)$$

Applying this equation to the second row of Figure 3b, boundary conditions 5 and 6 may now be written as follows:

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{1}{H1}\left(y(2,J) - y(1,J)\right) = 0 \qquad (7)$$

$$\left(\frac{\partial \psi}{\partial t}\right)_{t=0} = \frac{1}{H1}\left(\psi(2,J) - (1,J)\right) = 0 \qquad (8)$$

By combining these equations with boundary conditions 1 and 2, we find all values of y and ψ on row 1 are zero. Having now established the values of y and ψ for rows 1 and 2, the number of unknowns present on row 3 are totalled. Subsequent to this, the governing equations and boundary conditions are applied. The governing equations require a solution for the displacement, horizontal force, bending slope, shear, and moment at each point. This then gives a total of 55 unknowns for Row 3. However, by application of governing Equation 2a, and boundary conditions 7, 8, 9, and 10, the total number of unknowns is reduced to 40.

Considering next the finite difference pattern for Equations 4a and 5a (Figure 4a), and comparing this with the overall network, it is readily verified that these equations may be applied at points 1 through 9 on the third row. However, in order to apply Equations 1a and 3a to the finite difference net defined by Figure 3b, it will be necessary to write a backward difference expression for the second derivative with respect to $\bar{T}$. Such an expression may be written as follows:

$$\frac{\partial^2 f}{\partial t^2} = \frac{1}{(H1)^2} \left(f(I-2,J) - 2f(I-1,J) + f(I,J)\right)$$

Therefore, using this expression for second derivatives with respect to $\bar{T}$, and central difference for all derivatives with respect to $\bar{x}$, the finite difference pattern for Equations 1a and 3a will be as shown in Figure 4b. Once again, comparing this finite difference pattern with the overall network, we find that Equations 1a and 3a may be applied at points 1 through 9 on the third row. We now have a system consisting of 40 unknowns and only 36 equations. To obtain the four additional expressions, Equations 1a, 3a, 4a, and 5a must be applied either at the free end or the fixed end of the beam, expressing

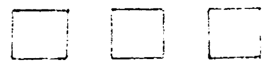

Figure 4a. Finite Difference Pattern for the Fourth and Fifth Governing Differential Equations

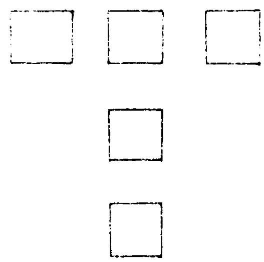

Figure 4b. Finite Difference Pattern for the First and Third Governing Differential Equations

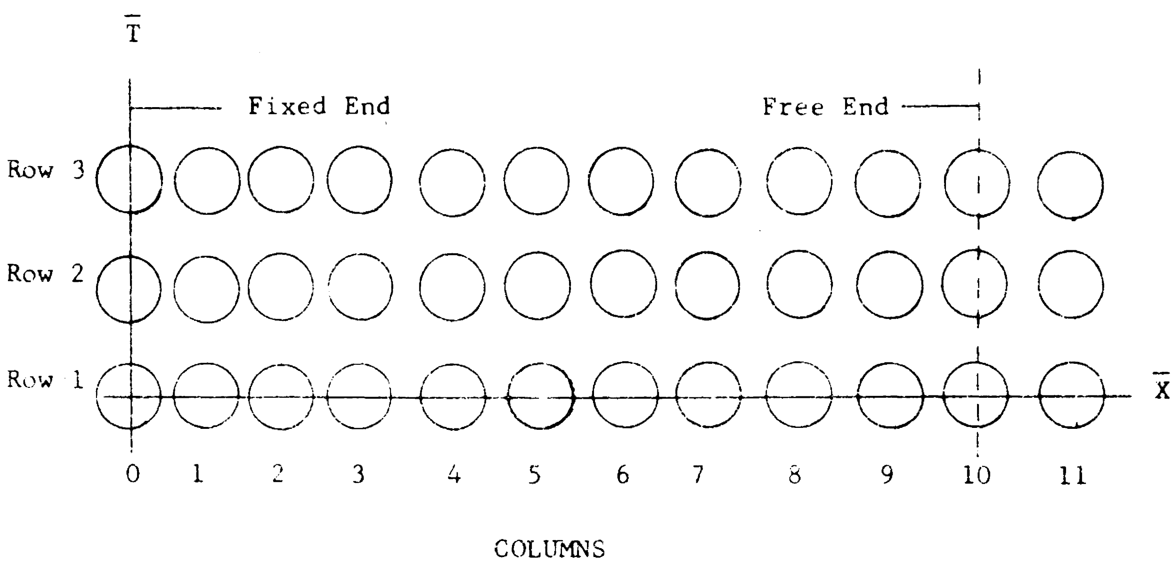

Figure 4c. Finite Difference Field for Final Method of Solution

all derivatives with respect to $\bar{x}$ in terms of succeeding or preceding points. The fixed end of the beam was selected as the point of application, and the following interpolation formula used to obtain the first derivative with respect to $\bar{x}$:

$$\frac{\partial y}{\partial x} = \frac{1}{H}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \text{-----------------------} - \frac{1}{10}\Delta^{10} y_0\right) \qquad (9)$$

With the application of Equations 1a, 3a, 4a, and 5a at the fixed end of the beam, a system of 40 equations and 40 unknowns is finally obtained.

It should be noted that initially the first derivative with respect to $\bar{x}$ was not written in terms of all eleven points on the beam, but only in terms of points zero and 1. With this method, a discontinuity developed at the free end of the beam. In order to eliminate this discontinuity it was necessary to apply the relationship given by Equation 9.

This solution was first tested using a $\bar{T}$ increment of 0.1 between successive rows of calculations. The resulting displacement curve converged immediately to a shape similar to that which would be obtained by elementary considerations based on a linearly increasing load. However, as the $\bar{T}$ interval was reduced in an attempt to study the transient vibrations, the calculated displacements approached infinity. It was felt at first that this might be caused by the effect of Equation 9 at low values of $\bar{T}$. Exhaustive testing proved this to be incorrect and the approach was finally abandoned.

E. Final Method of Solution.

A careful study of the governing differential equations and the available boundary conditions will reveal that the most direct approach to this problem consists of eliminating two unknowns at each point. To do this, both Equations 4 and 5 are differentiated once with respect to $\bar{x}$, substituted into Equations 1a and 3a respectively, and the following results obtained:

$$AB\left(\frac{\partial^2 \bar{y}}{\partial \bar{x}^2} - \frac{\partial \psi}{\partial \bar{x}}\right) = -AB1\,(\bar{R} + \bar{X}) + \frac{\partial^2 \bar{y}}{\partial \bar{T}^2} - (AB1)^2\,\bar{T}^2\,\bar{y}$$

$$-AB3\,\frac{\partial^2 \psi}{\partial \bar{x}^2} = (AB2)\left(AB1 - \frac{\partial^2 \psi}{\partial \bar{T}^2}\right) + \bar{P}\,\frac{\partial \bar{y}}{\partial \bar{x}} + \bar{Q} \qquad (10)$$

but

$$\bar{Q} = AB\left(\frac{\partial \bar{y}}{\partial \bar{x}} - \psi\right)$$

Therefore,

$$-AB3\,\frac{\partial^2 \psi}{\partial \bar{x}^2} = (AB2)\left(AB1 - \frac{\partial^2 \psi}{\partial \bar{T}^2}\right) + \bar{P}\,\frac{\partial \bar{y}}{\partial \bar{x}} + AB\left(\frac{\partial \bar{y}}{\partial \bar{x}} - \psi\right)$$

If all derivatives with respect to $\bar{x}$ are expressed in terms of central difference and second derivatives with respect to $\bar{T}$ in terms of backward difference, the finite difference pattern for Equations 10 and 11 will be as shown in Figure 4b. The general finite difference network required by these equations is as shown in Figure 4c. Applying the boundary conditions for $t = 0$ once again indicates that all values of y and ψ in rows 1 and 2 are zero. Since there are two unknowns at each point, neglecting the horizontal force since it may be evaluated, and twelve columns present in the overall network, row 3 now contains 24 unknowns.

Through a comparison of the finite difference pattern and the general network, we find that Equations 10 and 11 may be applied on the third row at points 1 through 10. This will yield a set of 20 equations. Application of the boundary conditions

$$\bar{y} = 0 \text{ (fixed end)}$$

$$\Psi = 0 \text{ (fixed end)}$$

$$\bar{Q} = AB\left(\frac{\partial \bar{y}}{\partial \bar{x}} - \Psi\right) = 0 \text{ (free end)}$$

$$\bar{M} = -AB3\frac{\partial \Psi}{\partial \bar{x}} = 0 \text{ (free end)}$$

results in a system of 24 equations and 24 unknowns.

A program was written and compiled to solve the above system of equations on a 60K, I.B.M. 1620 Digital Computer (a copy of the program is included in the Appendix.) This program is of a general nature and will handle a minimum of six and a maximum of 26 points on a beam. The $\bar{T}$ increment is completely arbitrary and is entered as input data. Equations 10 and 11 are automatically arranged by the program and placed in a matrix. The matrix is then triangulated and a simultaneous solution affected. The time required to complete a row of calculations is dependent upon the number of points considered on the beam.

Due to limited computer facilities, the small $\bar{T}$ increment necessary for studying transient vibrations, and the large number of calculations required for a detailed study, all numerical results presented in this thesis will be based on 12 points per row.

F. Determination of the $\bar{T}$ Increment.

Prior to an actual study of the transient effects, the time required for a wave to propagate the length of the beam must be determined. This is necessary to insure the selection of an appropriate $\bar{T}$ increment for use in subsequent calculations. The Pochhammer--Chree Theory for a bar of circular cross-section will be used. This theory employs Poisson's ratio as a parameter, and establishes a relationship between c/co and a/Λ . If this relationship is plotted for Poisson's ratio = .29, it is found that c/co for the first and second modes will approach a constant value of approximately 0.8 for a/Λ greater than 0.5. The ratio c/c_o will, therefore, be assumed equal to 0.8. Since for all cases treated in this thesis the modulus of elasticity, weight per unit volume, and length of beam remain constant, the $\bar{T}$ increment will be established at this time.

Letting

$$E = 30 \times 10^6 \text{ lb/in}^2$$

$$\rho = .283/386 \text{ lb-sec}^2/\text{in}^4$$

$$L = 6 \text{ in}$$

and writing the following relations

$$c = \frac{L}{t} = .8c_o = .8\sqrt{\frac{E}{\rho}}$$

or

$$t = \frac{L}{.8}\sqrt{\frac{\rho}{E}}$$

we find that t equals 36.9×10^{-6} seconds.

Noting the relationship

$$\bar{T} = \sqrt{\frac{g}{L}}\, t$$

we find $\bar{T}$ equals 298×10^{-6}.

V. PRESENTATION OF DATA

The physical parameters assigned to the beam for the purpose of obtaining actual numerical results are as follows:

Area = .2028 in^2

Length = 6 in

Weight per unit volume = .2833 lb/in^3

Shape factor = .536

Shear modulus = 11.15×10^6 lb/in^2

Modulus of elasticity = 30×10^6 lb/in^2

Moment of inertia = $.231 \times 10^{-2}$ in^4

These parameters approximately represent a turbine bucket and were (except for the modulus of elasticity) taken directly from Reference 5.

The first run was made with an angular acceleration of 62.8 $rads/sec^2$, and a hub radius of 45 inches.

Based on the previous calculations of $\bar{T}$ and the fact that it is generally better to obtain conclusive results for one problem rather than inconclusive results for several problems, a $\bar{T}$ increment of 1×10^{-6} was selected and 1100 rows of calculations made. The results of these calculations for row 1, 10, 50, and every 50 rows thereafter are shown in tabular and graphical form, with the results of runs 2 and 3 still to be discussed, in the latter portion of this section.

It should be noted at this time that all tables and graphs presented in this section were produced by an I.B.M. 1620 digital computer. Below the title on each graph is printed the table from which the numerical values for that graph were obtained. Printed on the right hand side of the graph is the actual numerical value of the plotted point. The necessity for this stems from the fact that the computer drops any significant figures after the first two prior to plotting a number. This then introduces the possibility of two unequal numbers being plotted in the same vertical column. However, providing the actual value to be plotted gives an adequate clarification.

The second run was made with an angular acceleration of 62.8 rads/sec^2 , a hub radius of 45 inches, and a $\overline{T}$ increment of 1. 250 rows were computed. The purpose of this was to determine what effects the centrifugal force would have at large values of time. The results of these calculations are shown for rows 1, 10, 50, 100, 150, 200, 225,and 250, in the latter portion of this section.

The third run was made with an angular acceleration of 62.8 rads/sec^2 , a hub radius of 5 inches, and a $\overline{T}$ increment of 1×10^{-6} . 275 rows were computed. The purpose of this was to study the effects produced by a reduction in the radius of the hub. Tabular and graphical results for rows 1, 50, 100, 150, 200, 225, 250, and 275 are given in the latter portion of this section.

TABLE 1

INITIAL VALUES

LENGTH = .60000000E+01 IN.
AREA = .20280000E+00 SQ.IN.
RADIUS OF HUB = .45000000E+02 IN.
SHAPE FACTOR = .53600000E+00 - - -
GRAVITY = .38600000E+03 IN./SEC.SQ.
BEAM INCREMENT = .10000000E+00 - - -
TIME INCREMENT = .10000000E-05 - - -
SHEAR MODULUS = .11150000E+08 LB./CU.IN.
MOMENT OF INERTIA = .23100000E-02 IN.4TH
MODULUS OF ELASTICITY = .30000000E+08 LB./SQ.IN.
WEIGHT PER UNIT VOLUME = .28332400E+00 LB./CU.IN.
ANGULAR ACCELERATION = .62800000E+02 RAD./SEC.SQ.

VALUES OF DISPLACEMENT

TIME =	.100000E-05	.100000E-04	.500000E-04	.100000E-03
POINT 1	.000000E-99	.000000E-99	.000000E-99	.000000E-99
POINT 2	.741627E-11	.406193E-09	.874343E-08	.282834E-07
POINT 3	.751648E-11	.413434E-09	.965509E-08	.384821E-07
POINT 4	.761409E-11	.418774E-09	.971911E-08	.389299E-07
POINT 5	.771170E-11	.424143E-09	.983273E-08	.390005E-07
POINT 6	.780932E-11	.429512E-09	.995667E-08	.394326E-07
POINT 7	.790694E-11	.434881E-09	.100810E-07	.399235E-07
POINT 8	.800455E-11	.440249E-09	.102055E-07	.404176E-07
POINT 9	.810217E-11	.445618E-09	.103300E-07	.409111E-07
POINT 10	.819979E-11	.450987E-09	.104544E-07	.414032E-07
POINT 11	.829740E-11	.456356E-09	.105789E-07	.418967E-07
POINT 12	.839502E-11	.461725E-09	.107033E-07	.423890E-07

```
11X/L

11  I                                      X      .8395E-11

10  I                                      X      .8297E-11

 9  I                                      X      .8199E-11

 8  I                                     X       .8102E-11

 7  I                                     X       .8004E-11

 6  I                                    X        .7906E-11

 5  I                                    X        .7809E-11

 4  I                                   X         .7711E-11

 3  I                                   X         .7614E-11

 2  I                                  X          .7516E-11

 1  I                                  X          .7416E-11

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .2098E-12

PLOT OF DEFLECTIONS T = .10000000E-05

TABLE 1

```
11X/L

11  I                                      X      .4617E-09

10  I                                      X      .4563E-09

 9  I                                      X      .4509E-09

 8  I                                     X       .4456E-09

 7  I                                     X       .4402E-09

 6  I                                    X        .4348E-09

 5  I                                    X        .4295E-09

 4  I                                   X         .4241E-09

 3  I                                   X         .4187E-09

 2  I                                  X          .4134E-09

 1  I                                  X          .4061E-09

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .1154E-10

PLOT OF DEFLECTIONS T = .10000000E-04

TABLE 1

```
11X/L

11  I                                       X     .1070E-07

10  I                                      X      .1057E-07

 9  I                                      X      .1045E-07

 8  I                                     X       .1033E-07

 7  I                                     X       .1020E-07

 6  I                                    X        .1008E-07

 5  I                                    X        .9956E-08

 4  I                                   X         .9832E-08

 3  I                                   X         .9719E-08

 2  I                                   X         .9655E-08

 1  I                               X             .8743E-08

 0  X                                             .0000E-99

    0    5    10   15   20   25   30   35   40
```

SCALE FACTOR = .2675E-09

PLOT OF DEFLECTIONS T = .50000000E-04

TABLE 1

```
11X/L

11  I                                      X       .4238E-07

10  I                                      X       .4189E-07

 9  I                                      X       .4140E-07

 8  I                                     X        .4091E-07

 7  I                                     X        .4041E-07

 6  I                                    X         .3992E-07

 5  I                                    X         .3943E-07

 4  I                                   X          .3900E-07

 3  I                                   X          .3892E-07

 2  I                                   X          .3848E-07

 1  I                         X                    .2828E-07

 0  X                                              .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .1059E-08

PLOT OF DEFLECTIONS T = .10000000E-03

TABLE 1

TABLE 1A

VALUES OF DISPLACEMENT

TIME =	.150000E-03	.200000E-03	.250000E-03	.300000E-03
POINT 1	.000000E-99	.000000E-99	.000000E-99	.000000E-99
POINT 2	.486105E-07	.668471E-07	.849560E-07	.103340E-06
POINT 3	.830208E-07	.134506E-06	.186569E-06	.239302E-06
POINT 4	.891111E-07	.159903E-06	.245223E-06	.337029E-06
POINT 5	.881137E-07	.159264E-06	.254456E-06	.371077E-06
POINT 6	.884429E-07	.157441E-06	.248493E-06	.365061E-06
POINT 7	.895014E-07	.158703E-06	.247622E-06	.357804E-06
POINT 8	.906262E-07	.160761E-06	.250585E-06	.359799E-06
POINT 9	.917379E-07	.162779E-06	.253941E-06	.364941E-06
POINT 10	.928408E-07	.164735E-06	.257051E-06	.369743E-06
POINT 11	.939453E-07	.166695E-06	.260107E-06	.374214E-06
POINT 12	.950503E-07	.168652E-06	.263203E-06	.378895E-06

TIME =	.350000E-03	.400000E-03	.450000E-03	.500000E-03
POINT 1	.000000E-99	.000000E-99	.000000E-99	.000000E-99
POINT 2	.121178E-06	.138788E-06	.156609E-06	.174293E-06
POINT 3	.293879E-06	.349245E-06	.404757E-06	.461055E-06
POINT 4	.432259E-06	.531750E-06	.634943E-06	.740214E-06
POINT 5	.501714E-06	.640692E-06	.787268E-06	.941354E-06
POINT 6	.507985E-06	.672380E-06	.851504E-06	.104243E-05
POINT 7	.493114E-06	.657074E-06	.848281E-06	.106083E-05
POINT 8	.489071E-06	.641676E-06	.822564E-06	.103380E-05
POINT 9	.495130E-06	.644192E-06	.814122E-06	.101029E-05
POINT 10	.502531E-06	.654632E-06	.825175E-06	.101483E-05
POINT 11	.509139E-06	.664870E-06	.840097E-06	.103066E-05
POINT 12	.516070E-06	.674203E-06	.849454E-06	.103362E-05

```
11X/L

11  I                                       X       .9505E-07

10  I                                      X        .9394E-07

 9  I                                      X        .9284E-07

 8  I                                     X         .9173E-07

 7  I                                     X         .9062E-07

 6  I                                    X          .8950E-07

 5  I                                    X          .8844E-07

 4  I                                    X          .8811E-07

 3  I                                    X          .8911E-07

 2  I                                 X             .8302E-07

 1  I                   X                           .4861E-07

 0  X                                               .0000E-99

    0    5    10   15   20   25   30   35   40
```

SCALE FACTOR = .2376E-08

PLOT OF DEFLECTIONS T = .15000000E-03

TABLE 1A

```
11X/L

11  I                                       X     .1686E-06

10  I                                       X     .1666E-06

 9  I                                       X     .1647E-06

 8  I                                      X      .1627E-06

 7  I                                      X      .1607E-06

 6  I                                     X       .1587E-06

 5  I                                     X       .1574E-06

 4  I                                     X       .1592E-06

 3  I                                     X       .1599E-06

 2  I                               X             .1345E-06

 1  I               X                             .6684E-07

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .4216E-08

PLOT OF DEFLECTIONS T = .20000000E-03

TABLE 1A

```
11X/L

11  I                                       X      .2632E-06

10  I                                       X      .2601E-06

 9  I                                       X      .2570E-06

 8  I                                      X       .2539E-06

 7  I                                      X       .2505E-06

 6  I                                     X        .2476E-06

 5  I                                     X        .2484E-06

 4  I                                      X       .2544E-06

 3  I                                     X        .2452E-06

 2  I                           X                  .1865E-06

 1  I           X                                  .8495E-07

 0  X                                              .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .6580E-08

PLOT OF DEFLECTIONS T = .25000000E-03

TABLE 1A

```
11X/L

11 I                                      X      .3788E-06

10 I                                      X      .3742E-06

 9 I                                      X      .3697E-06

 8 I                                     X       .3649E-06

 7 I                                    X        .3597E-06

 6 I                                    X        .3578E-06

 5 I                                     X       .3650E-06

 4 I                                      X      .3710E-06

 3 I                                  X          .3370E-06

 2 I                        X                    .2393E-06

 1 I         X                                   .1033E-06

 0 X                                             .0000E-99

   0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .9472E-08

PLOT OF DEFLECTIONS T = .30000000E-03

TABLE 1A

```
11X/L

11  I                                       X     .5160E-06

10  I                                       X     .5091E-06

 9  I                                      X      .5025E-06

 8  I                                      X      .4951E-06

 7  I                                     X       .4890E-06

 6  I                                      X      .4931E-06

 5  I                                       X     .5079E-06

 4  I                                      X      .5017E-06

 3  I                                 X           .4322E-06

 2  I                     X                       .2938E-06

 1  I        X                                    .1211E-06

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .1290E-07

PLOT OF DEFLECTIONS T = .35000000E-03

TABLE 1A

```
11X/L

11  I                                       X      .6742E-06

10  I                                       X      .6648E-06

 9  I                                      X       .6546E-06

 8  I                                      X       .6441E-06

 7  I                                      X       .6416E-06

 6  I                                      X       .6570E-06

 5  I                                       X      .6723E-06

 4  I                                      X       .6406E-06

 3  I                              X               .5317E-06

 2  I                   X                          .3492E-06

 1  I       X                                      .1387E-06

 0  X                                              .0000E-99

    0    5    10   15   20   25   30   35   40
```

SCALE FACTOR = .1685E-07

PLOT OF DEFLECTIONS T = .40000000E-03

TABLE 1A

```
11X/L

11  I                                       X         .8494E-06

10  I                                       X         .8400E-06

 9  I                                      X          .8251E-06

 8  I                                      X          .8141E-06

 7  I                                      X          .8225E-06

 6  I                                       X         .8482E-06

 5  I                                       X         .8515E-06

 4  I                                    X            .7872E-06

 3  I                             X                   .6349E-06

 2  I                  X                              .4047E-06

 1  I      X                                          .1566E-06

 0  X                                                 .0000E-99

    0    5    10   15   20   25   30   35   40
```

SCALE FACTOR = .2128E-07

PLOT OF DEFLECTIONS T = .45000000E-03

TABLE 1A

```
11X/L

11  I                                      X         .1033E-05

10  I                                      X         .1030E-05

 9  I                                      X         .1014E-05

 8  I                                      X         .1010E-05

 7  I                                      X         .1033E-05

 6  I                                        X       .1060E-05

 5  I                                       X        .1042E-05

 4  I                                   X            .9413E-06

 3  I                           X                    .7402E-06

 2  I                 X                              .4610E-06

 1  I     X                                          .1742E-06

 0  X                                                .0000E-99

    0    5    10   15   20   25   30   35   40
```

SCALE FACTOR = .2652E-07

PLOT OF DEFLECTIONS T = .50000000E-03

TABLE 1A

TABLE 1B

VALUES OF DISPLACEMENT

TIME =	.550000E-03	.600000E-03	.650000E-03	.700000E-03
POINT 1	.000000E-99	.000000E-99	.000000E-99	.000000E-99
POINT 2	.191751E-06	.209317E-06	.227004E-06	.244653E-06
POINT 3	.518111E-06	.575433E-06	.633111E-06	.691126E-06
POINT 4	.847489E-06	.957030E-06	.106783E-05	.117907E-05
POINT 5	.110083E-05	.126410E-05	.143140E-05	.160182E-05
POINT 6	.124474E-05	.145656E-05	.167533E-05	.190107E-05
POINT 7	.128971E-05	.153408E-05	.179396E-05	.206875E-05
POINT 8	.127207E-05	.153328E-05	.181723E-05	.212403E-05
POINT 9	.123832E-05	.149879E-05	.178682E-05	.209606E-05
POINT 10	.122676E-05	.146436E-05	.172722E-05	.201233E-05
POINT 11	.123040E-05	.143752E-05	.166037E-05	.191334E-05
POINT 12	.121851E-05	.140491E-05	.160542E-05	.183715E-05

TIME =	.750000E-03	.800000E-03	.850000E-03	.900000E-03
POINT 1	.000000E-99	.000000E-99	.000000E-99	.000000E-99
POINT 2	.262205E-06	.279392E-06	.295892E-06	.311268E-06
POINT 3	.748857E-06	.805994E-06	.861918E-06	.915890E-06
POINT 4	.129108E-05	.140301E-05	.151462E-05	.162824E-05
POINT 5	.177403E-05	.194941E-05	.213090E-05	.232016E-05
POINT 6	.213550E-05	.237978E-05	.263410E-05	.289623E-05
POINT 7	.235846E-05	.266202E-05	.297397E-05	.328739E-05
POINT 8	.244898E-05	.278429E-05	.312553E-05	.347304E-05
POINT 9	.242179E-05	.276246E-05	.311910E-05	.349342E-05
POINT 10	.232038E-05	.265641E-05	.302354E-05	.342106E-05
POINT 11	.220554E-05	.253829E-05	.291232E-05	.333114E-05
POINT 12	.211426E-05	.244416E-05	.282968E-05	.327232E-05

```
11X/L

11 I                                    X        .1218E-05

10 I                                     X       .1230E-05

 9 I                                     X       .1226E-05

 8 I                                     X       .1238E-05

 7 I                                      X      .1272E-05

 6 I                                      X      .1289E-05

 5 I                                     X       .1244E-05

 4 I                                 X           .1100E-05

 3 I                         X                   .8474E-06

 2 I               X                             .5181E-06

 1 I    X                                        .1917E-06

 0 X                                             .0000E-99

   0    5    10   15   20   25   30   35   40
```

SCALE FACTOR = .3224E-07

PLOT OF DEFLECTIONS T = .55000000E-03

TABLE 1B

```
11X/L

11  I                                   X         .1404E-05

10  I                                    X        .1437E-05

 9  I                                     X       .1464E-05

 8  I                                      X      .1498E-05

 7  I                                      X      .1533E-05

 6  I                                      X      .1534E-05

 5  I                                    X        .1456E-05

 4  I                               X             .1264E-05

 3  I                       X                     .9570E-06

 2  I              X                              .5754E-06

 1  I    X                                        .2093E-06

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .3835E-07

PLOT OF DEFLECTIONS T = .60000000E-03

TABLE 1B

```
11X/L

11  I                                  X            .1605E-05

10  I                                   X           .1660E-05

 9  I                                     X         .1727E-05

 8  I                                      X        .1786E-05

 7  I                                      X        .1817E-05

 6  I                                      X        .1793E-05

 5  I                                   X           .1675E-05

 4  I                              X                .1431E-05

 3  I                      X                        .1067E-05

 2  I            X                                  .6331E-06

 1  I   X                                           .2270E-06

 0  X                                               .0000E-99

    0    5    10   15   20   25   30   35   40
```

SCALE FACTOR = .4543E-07

PLOT OF DEFLECTIONS T = .65000000E-03

TABLE 1B

```
11X/L

11  I                                  X          .1837E-05

10  I                                    X        .1913E-05

 9  I                                     X       .2012E-05

 8  I                                       X     .2096E-05

 7  I                                       X     .2124E-05

 6  I                                      X      .2068E-05

 5  I                                   X         .1901E-05

 4  I                              X              .1601E-05

 3  I                     X                       .1179E-05

 2  I            X                                .6911E-06

 1  I   X                                         .2446E-06

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .5310E-07

PLOT OF DEFLECTIONS T = .70000000E-03

TABLE 1B

```
11X/L

11  I                                 X           .2114E-05

10  I                                   X         .2205E-05

 9  I                                    X        .2320E-05

 8  I                                      X      .2421E-05

 7  I                                      X      .2448E-05

 6  I                                     X       .2358E-05

 5  I                                 X           .2135E-05

 4  I                           X                 .1774E-05

 3  I                    X                        .1291E-05

 2  I           X                                 .7488E-06

 1  I   X                                         .2622E-06

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .6122E-07

PLOT OF DEFLECTIONS T = .75000000E-03

TABLE 1B

```
11X/L

11  I                                  X            .2444E-05

10  I                                   X           .2538E-05

 9  I                                     X         .2656E-05

 8  I                                      X        .2762E-05

 7  I                                       X       .2784E-05

 6  I                                     X         .2662E-05

 5  I                                 X             .2379E-05

 4  I                           X                   .1949E-05

 3  I                   X                           .1403E-05

 2  I          X                                    .8059E-06

 1  I   X                                           .2793E-06

 0  X                                               .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .6960E-07

PLOT OF DEFLECTIONS T = .80000000E-03

TABLE 1B

```
11X/L

11   I                                    X             .2829E-05

10   I                                     X            .2912E-05

 9   I                                      X           .3023E-05

 8   I                                       X          .3119E-05

 7   I                                       X          .3125E-05

 6   I                                      X           .2973E-05

 5   I                                 X                .2634E-05

 4   I                           X                      .2130E-05

 3   I                  X                               .1514E-05

 2   I          X                                       .8619E-06

 1   I  X                                               .2958E-06

 0   X                                                  .0000E-99

     0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .7813E-07

PLOT OF DEFLECTIONS T = .85000000E-03

TABLE 1B

```
11X/L

11  I                                    X         .3272E-05

10  I                                     X        .3331E-05

 9  I                                      X       .3421E-05

 8  I                                      X       .3493E-05

 7  I                                      X       .3473E-05

 6  I                                    X         .3287E-05

 5  I                                X             .2896E-05

 4  I                         X                    .2320E-05

 3  I                 X                            .1628E-05

 2  I         X                                    .9158E-06

 1  I  X                                           .3112E-06

 0  X                                              .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .8733E-07

PLOT OF DEFLECTIONS T = .90000000E-03

TABLE 1B

TABLE 1C

VALUES OF DISPLACEMENT

TIME =	.950000E-03	.100000E-02	.105000E-02	.110000E-02
POINT 1	.000000E-99	.000000E-99	.000000E-99	.000000E-99
POINT 2	.324833E-06	.337196E-06	.351982E-06	.373897E-06
POINT 3	.969186E-06	.102566E-05	.108913E-05	.116143E-05
POINT 4	.174734E-05	.187357E-05	.200531E-05	.213796E-05
POINT 5	.251666E-05	.271639E-05	.291364E-05	.310548E-05
POINT 6	.316012E-05	.342039E-05	.367653E-05	.392956E-05
POINT 7	.360005E-05	.391349E-05	.422796E-05	.454238E-05
POINT 8	.382808E-05	.419083E-05	.456137E-05	.494058E-05
POINT 9	.388601E-05	.429659E-05	.472602E-05	.517567E-05
POINT 10	.384879E-05	.430850E-05	.480091E-05	.532455E-05
POINT 11	.379663E-05	.430715E-05	.486136E-05	.545962E-05
POINT 12	.377336E-05	.433296E-05	.494992E-05	.562150E-05

```
11X/L

11  I                                     X       .3773E-05

10  I                                      X      .3796E-05

 9  I                                      X      .3848E-05

 8  I                                      X      .3886E-05

 7  I                                      X      .3828E-05

 6  I                                    X        .3600E-05

 5  I                               X             .3160E-05

 4  I                        X                    .2516E-05

 3  I                X                            .1747E-05

 2  I        X                                    .9691E-06

 1  I  X                                          .3248E-06

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .9715E-07

PLOT OF DEFLECTIONS T = .95000000E-03

TABLE 1C

```
11X/L

11  I                                      X        .4332E-05

10  I                                      X        .4307E-05

 9  I                                      X        .4308E-05

 8  I                                      X        .4296E-05

 7  I                                     X         .4190E-05

 6  I                                   X           .3913E-05

 5  I                              X                .3420E-05

 4  I                        X                      .2716E-05

 3  I                X                              .1873E-05

 2  I        X                                      .1025E-05

 1  I  X                                            .3371E-06

 0  X                                               .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .1083E-06

PLOT OF DEFLECTIONS T = .10000000E-02

TABLE 1C

```
 11X/L

 11  I                                       X     .4949E-05

 10  I                                       X     .4861E-05

  9  I                                     X       .4800E-05

  8  I                                     X       .4726E-05

  7  I                                   X         .4561E-05

  6  I                                 X           .4227E-05

  5  I                            X                .3676E-05

  4  I                      X                      .2913E-05

  3  I               X                             .2005E-05

  2  I       X                                     .1089E-05

  1  I X                                           .3519E-06

  0  X                                             .0000E-99

     0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .1237E-06

PLOT OF DEFLECTIONS T = .10500000E-02

TABLE 1C

```
11X/L

11  I                                       X     .5621E-05

10  I                                      X      .5459E-05

 9  I                                     X       .5324E-05

 8  I                                    X        .5175E-05

 7  I                                  X          .4940E-05

 6  I                               X             .4542E-05

 5  I                           X                 .3929E-05

 4  I                     X                       .3105E-05

 3  I              X                              .2137E-05

 2  I       X                                     .1161E-05

 1  I  X                                          .3738E-06

 0  X                                             .0000E-99

    0    5    10   15   20   25   30   35   40
```

SCALE FACTOR = .1405E-06

PLOT OF DEFLECTIONS T = .11000000E-02

TABLE 1C

TABLE 2

INITIAL VALUES

LENGTH = .60000000E+01 IN.
AREA = .20280000E+00 SQ.IN.
RADIUS OF HUB = .45000000E+02 IN.
SHAPE FACTOR = .53600000E+00 - - -
GRAVITY = .38600000E+03 IN./SEC.SQ.
BEAM INCREMENT = .10000000E+00 - - -
TIME INCREMENT = .10000000E+01 - - -
SHEAR MODULUS = .11150000E+08 LB./CU.IN.
MOMENT OF INERTIA = .23100000E-02 IN.4TH
MODULUS OF ELASTICITY = .30000000E+08 LB./SQ.IN.
WEIGHT PER UNIT VOLUME = .28332400E+00 LB./CU.IN.
ANGULAR ACCELERATION = .62800000E+02 RAD./SEC.SQ.

VALUES OF DISPLACEMENT

TIME =	.100000E+01	.100000E+02	.500000E+02	.100000E+03
POINT 1	.000000E-99	.000000E-99	.000000E-99	.000000E-99
POINT 2	.146221E-05	.145551E-05	.131418E-05	.103826E-05
POINT 3	.515829E-05	.513186E-05	.457448E-05	.348890E-05
POINT 4	.106219E-04	.105640E-04	.934575E-05	.697956E-05
POINT 5	.174389E-04	.173400E-04	.152566E-04	.112216E-04
POINT 6	.252485E-04	.251008E-04	.219934E-04	.159895E-04
POINT 7	.337429E-04	.335411E-04	.292945E-04	.211066E-04
POINT 8	.426694E-04	.424097E-04	.369484E-04	.264356E-04
POINT 9	.518296E-04	.515101E-04	.447907E-04	.318728E-04
POINT 10	.610812E-04	.607009E-04	.527039E-04	.373448E-04
POINT 11	.703383E-04	.698969E-04	.606172E-04	.428071E-04
POINT 12	.795718E-04	.790693E-04	.685064E-04	.482447E-04

```
11X/L

11  I                                       X     .7957E-04

10  I                                  X          .7033E-04

 9  I                             X               .6108E-04

 8  I                         X                   .5182E-04

 7  I                    X                        .4266E-04

 6  I               X                             .3374E-04

 5  I           X                                 .2524E-04

 4  I       X                                     .1743E-04

 3  I    X                                        .1062E-04

 2  I X                                           .5158E-05

 1  X                                             .1462E-05

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .1989E-05

PLOT OF DEFLECTIONS T = .10000000E+01

TABLE 2

```
11X/L

11  I                                       X     .7906E-04

10  I                                  X          .6989E-04

 9  I                             X               .6070E-04

 8  I                         X                   .5151E-04

 7  I                    X                        .4240E-04

 6  I               X                             .3354E-04

 5  I           X                                 .2510E-04

 4  I       X                                     .1734E-04

 3  I    X                                        .1056E-04

 2  I X                                           .5131E-05

 1  X                                             .1455E-05

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .1976E-05

PLOT OF DEFLECTIONS T = .10000000E+02

TABLE 2

```
11X/L

11  I                                       X     .6850E-04

10  I                                   X         .6061E-04

 9  I                              X              .5270E-04

 8  I                         X                   .4479E-04

 7  I                    X                        .3694E-04

 6  I                X                            .2929E-04

 5  I           X                                 .2199E-04

 4  I       X                                     .1525E-04

 3  I    X                                        .9345E-05

 2  I X                                           .4574E-05

 1  X                                             .1314E-05

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .1712E-05

PLOT OF DEFLECTIONS T = .50000000E+02

TABLE 2

```
11X/L

11  I                                      X      .4824E-04

10  I                                  X          .4280E-04

 9  I                             X               .3734E-04

 8  I                         X                   .3187E-04

 7  I                    X                        .2643E-04

 6  I                X                            .2110E-04

 5  I            X                                .1598E-04

 4  I        X                                    .1122E-04

 3  I    X                                        .6979E-05

 2  I X                                           .3488E-05

 1  X                                             .1038E-05

 0  X                                             .0000E-99

    0    5    10   15   20   25   30   35   40
```

SCALE FACTOR = .1206E-05

PLOT OF DEFLECTIONS T = .10000000E+03

TABLE 2

TABLE 2A

VALUES OF DISPLACEMENT

TIME =	.150000E+03	.200000E+03	.225000E+03	.250000E+03
POINT 1	.000000E-99	.000000E-99	.000000E-99	.000000E-99
POINT 2	.810857E-06	.654381E-06	.596034E-06	.547324E-06
POINT 3	.259919E-05	.199287E-05	.176904E-05	.158361E-05
POINT 4	.505334E-05	.375559E-05	.328209E-05	.289322E-05
POINT 5	.795811E-05	.578306E-05	.499811E-05	.435860E-05
POINT 6	.111615E-04	.797417E-05	.683474E-05	.591276E-05
POINT 7	.145539E-04	.102621E-04	.873987E-05	.751508E-05
POINT 8	.180550E-04	.126007E-04	.106784E-04	.913874E-05
POINT 9	.216059E-04	.149571E-04	.126254E-04	.107645E-04
POINT 10	.251655E-04	.173080E-04	.145629E-04	.123780E-04
POINT 11	.287085E-04	.196386E-04	.164791E-04	.139695E-04
POINT 12	.322262E-04	.219435E-04	.183695E-04	.155351E-04

```
11X/L

11  I                                       X     .3222E-04

10  I                                   X         .2870E-04

 9  I                              X              .2516E-04

 8  I                          X                  .2160E-04

 7  I                     X                       .1805E-04

 6  I                 X                           .1455E-04

 5  I             X                               .1116E-04

 4  I         X                                   .7958E-05

 3  I     X                                       .5053E-05

 2  I  X                                          .2599E-05

 1  IX                                            .8108E-06

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .8056E-06

PLOT OF DEFLECTIONS T = .15000000E+03

TABLE 2A

```
11X/L

11  I                                       X      .2194E-04

10  I                                   X          .1963E-04

 9  I                              X               .1730E-04

 8  I                          X                   .1495E-04

 7  I                     X                        .1260E-04

 6  I                 X                            .1026E-04

 5  I             X                                .7974E-05

 4  I         X                                    .5783E-05

 3  I     X                                        .3755E-05

 2  I  X                                           .1992E-05

 1  IX                                             .6543E-06

 0  X                                              .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .5485E-06

PLOT OF DEFLECTIONS T = .20000000E+03

TABLE 2A

```
11X/L

11  I                                       X     .1836E-04

10  I                                  X          .1647E-04

 9  I                              X              .1456E-04

 8  I                          X                  .1262E-04

 7  I                      X                      .1067E-04

 6  I                  X                          .8739E-05

 5  I             X                               .6834E-05

 4  I         X                                   .4998E-05

 3  I      X                                      .3282E-05

 2  I  X                                          .1769E-05

 1  IX                                            .5960E-06

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .4592E-06

PLOT OF DEFLECTIONS T = .22500000E+03

TABLE 2A

```
11X/L

11  I                                       X      .1553E-04

10  I                                  X           .1396E-04

 9  I                              X               .1237E-04

 8  I                          X                   .1076E-04

 7  I                      X                       .9138E-05

 6  I                  X                           .7515E-05

 5  I              X                               .5912E-05

 4  I          X                                   .4358E-05

 3  I      X                                       .2893E-05

 2  I   X                                          .1583E-05

 1  IX                                             .5473E-06

 0  X                                              .0000E-99

    0    5    10   15   20   25   30   35   40
```

SCALE FACTOR = .3883E-06

PLOT OF DEFLECTIONS T = .25000000E+03

TABLE 2A

TABLE 3

INITIAL VALUES

LENGTH = .60000000E+01 IN.
AREA = .20280000E+00 SQ.IN.
RADIUS OF HUB = .50000000E+01 IN.
SHAPE FACTOR = .53600000E+00 - - -
GRAVITY = .38600000E+03 IN./SEC.SQ.
BEAM INCREMENT = .10000000E+00 - - -
TIME INCREMENT = .10000000E-05 - - -
SHEAR MODULUS = .11150000E+08 LB./CU.IN.
MOMENT OF INERTIA = .23100000E-02 IN.4TH
MODULUS OF ELASTICITY = .30000000E+08 LB./SQ.IN.
WEIGHT PER UNIT VOLUME = .28332400E+00 LB./CU.IN.
ANGULAR ACCELERATION = .62800000E+02 RAD./SEC.SQ.

VALUES OF DISPLACEMENT

TIME =	.100000E-05	.500000E-04	.100000E-03	.150000E-03
POINT 1	.000000E-99	.000000E-99	.000000E-99	.000000E-99
POINT 2	.910785E-12	.107795E-08	.352684E-08	.616755E-08
POINT 3	.100870E-11	.129353E-08	.513922E-08	.111076E-07
POINT 4	.110632E-11	.141176E-08	.563913E-08	.128403E-07
POINT 5	.120393E-11	.153503E-08	.608611E-08	.137223E-07
POINT 6	.130155E-11	.165942E-08	.657212E-08	.147395E-07
POINT 7	.139917E-11	.178388E-08	.706481E-08	.158388E-07
POINT 8	.149678E-11	.190833E-08	.755778E-08	.169456E-07
POINT 9	.159440E-11	.203279E-08	.805084E-08	.180518E-07
POINT 10	.169202E-11	.215726E-08	.854378E-08	.191569E-07
POINT 11	.178963E-11	.228171E-08	.903668E-08	.202617E-07
POINT 12	.188725E-11	.240617E-08	.952965E-08	.213671E-07

```
11X/L

11  I                                      X      .1887E-11

10  I                                    X        .1789E-11

 9  I                                  X          .1692E-11

 8  I                                X            .1594E-11

 7  I                              X              .1496E-11

 6  I                            X                .1399E-11

 5  I                          X                  .1301E-11

 4  I                        X                    .1203E-11

 3  I                      X                      .1106E-11

 2  I                    X                        .1008E-11

 1  I                  X                          .9107E-12

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .4718E-13

PLOT OF DEFLECTIONS T = .10000000E-05

TABLE 3

```
11X/L

11  I                                      X      .2406E-08

10  I                                    X        .2281E-08

 9  I                                  X          .2157E-08

 8  I                                X            .2032E-08

 7  I                              X              .1908E-08

 6  I                            X                .1783E-08

 5  I                          X                  .1659E-08

 4  I                        X                    .1535E-08

 3  I                      X                      .1411E-08

 2  I                    X                        .1293E-08

 1  I                X                            .1077E-08

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .6015E-10

PLOT OF DEFLECTIONS T = .50000000E-04

TABLE 3

```
11X/L

11  I                                       X      .9529E-08

10  I                                     X        .9036E-08

 9  I                                   X          .8543E-08

 8  I                                 X            .8050E-08

 7  I                               X              .7557E-08

 6  I                             X                .7064E-08

 5  I                           X                  .6572E-08

 4  I                         X                    .6086E-08

 3  I                       X                      .5639E-08

 2  I                     X                        .5139E-08

 1  I              X                               .3526E-08

 0  X                                              .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .2382E-09

PLOT OF DEFLECTIONS T = .10000000E-03

TABLE 3

```
11X/L

11  I                                        X     .2136E-07

10  I                                     X        .2026E-07

 9  I                                   X          .1915E-07

 8  I                                 X            .1805E-07

 7  I                               X              .1694E-07

 6  I                             X                .1583E-07

 5  I                           X                  .1473E-07

 4  I                         X                    .1372E-07

 3  I                       X                      .1284E-07

 2  I                   X                          .1110E-07

 1  I          X                                   .6167E-08

 0  X                                              .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .5341E-09

PLOT OF DEFLECTIONS T = .15000000E-03

TABLE 3

TABLE 3A

VALUES OF DISPLACEMENT

TIME =	.200000E-03	.225000E-03	.250000E-03	.275000E-03
POINT 1	.000000E-99	.000000E-99	.000000E-99	.000000E-99
POINT 2	.864283E-08	.988699E-08	.111622E-07	.124611E-07
POINT 3	.181470E-07	.217948E-07	.254894E-07	.292530E-07
POINT 4	.229450E-07	.288759E-07	.352143E-07	.418393E-07
POINT 5	.246765E-07	.314819E-07	.391660E-07	.476620E-07
POINT 6	.262131E-07	.332560E-07	.412355E-07	.502059E-07
POINT 7	.280919E-07	.355231E-07	.438379E-07	.530671E-07
POINT 8	.300616E-07	.380110E-07	.468809E-07	.566663E-07
POINT 9	.320296E-07	.405081E-07	.499764E-07	.604292E-07
POINT 10	.339926E-07	.429926E-07	.530470E-07	.641553E-07
POINT 11	.359523E-07	.454721E-07	.561085E-07	.678635E-07
POINT 12	.379151E-07	.479564E-07	.591768E-07	.715809E-07

```
11X/L

11  I                                      X      .3791E-07

10  I                                    X        .3595E-07

 9  I                                  X          .3399E-07

 8  I                                X            .3202E-07

 7  I                              X              .3006E-07

 6  I                            X                .2809E-07

 5  I                          X                  .2621E-07

 4  I                         X                   .2467E-07

 3  I                       X                     .2294E-07

 2  I                  X                          .1814E-07

 1  I        X                                    .8642E-08

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .9478E-09

PLOT OF DEFLECTIONS T = .20000000E-03

TABLE 3A

```
11X/L

11  I                                        X          .4795E-07

10  I                                     X             .4547E-07

 9  I                                   X               .4299E-07

 8  I                                 X                 .4050E-07

 7  I                               X                   .3801E-07

 6  I                            X                      .3552E-07

 5  I                          X                        .3325E-07

 4  I                         X                         .3148E-07

 3  I                       X                           .2887E-07

 2  I                 X                                 .2179E-07

 1  I       X                                           .9886E-08

 0  X                                                   .0000E-99

    0    5    10   15   20   25   30   35   40
```

SCALE FACTOR = .1198E-08

PLOT OF DEFLECTIONS T = .22500000E-03

TABLE 3A

```
11X/L

11  I                                      X       .5917E-07

10  I                                    X         .5610E-07

 9  I                                  X           .5304E-07

 8  I                                X             .4997E-07

 7  I                              X               .4688E-07

 6  I                            X                 .4383E-07

 5  I                          X                   .4123E-07

 4  I                         X                    .3916E-07

 3  I                      X                       .3521E-07

 2  I                X                             .2548E-07

 1  I      X                                       .1116E-07

 0  X                                              .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .1479E-08

PLOT OF DEFLECTIONS T = .25000000E-03

TABLE 3A

```
11X/L

11  I                                        X    .7158E-07

10  I                                     X       .6786E-07

 9  I                                   X         .6415E-07

 8  I                                X            .6042E-07

 7  I                              X              .5666E-07

 6  I                            X                .5306E-07

 5  I                           X                 .5020E-07

 4  I                         X                   .4766E-07

 3  I                      X                      .4183E-07

 2  I               X                             .2925E-07

 1  I     X                                       .1246E-07

 0  X                                             .0000E-99

    0    5   10   15   20   25   30   35   40
```

SCALE FACTOR = .1789E-08

PLOT OF DEFLECTIONS T = .27500000E-03

TABLE 3A

VI. DISCUSSION OF NUMERICAL RESULTS

The results of Run 1 showed a definite wave propagation with a continuous increase in the displacements. It should be noted that the seemingly large discontinuity present in the initial plots is caused by the small change between Points 1 and 11. This, however, serves to illustrate that the initial deformation, caused by the impulsive load, is primarily due to shear. The effects of bending become apparent, however, as the wave travels out the beam.

The second run was made to determine the effect of the centrifugal force. Since this force is dependent upon the angular velocity, its effects will not be significant at low values of time. This necessitated the use of a large time increment. As shown by graphs for Tables 2 and 2A, the effects of this force is a tendency to return the beam to its undeformed axis.

As a final consideration, the hub radius was reduced in an attempt to study the effect of bringing the beam closer to its axis of rotation. By so doing, it was found that due to the appreciable reduction in the tangential acceleration forces, the wave propagation became less distinct, and a marked reduction in the displacements was noted.

VII. CONCLUSIONS

The problem of a cantilever beam rotating at a constant angular acceleration about an axis perpendicular to its base and vibrating in its plane of rotation has been solved for cases of constant cross-sectional area. The theoretical development and solution include the effects of bending, shear deformation, and rotary inertia.

Application of this solution showed that as the beam was instantaneously loaded with a large angular acceleration, a wave propagated out the beam. Accompanying the wave propagation were continually increasing displacements, but a definite trend to a return to the undeformed shape gradually appeared after the initial disturbance. The initial displacements appeared to be the result of shear deformation. Upon reducing the tangential acceleration, it was observed that the wave propagation became less distinct.

Being able to determine the deflected shape of the beam as a function of time will undoubtedly enhance subsequent investigations of dynamic stresses in rotating members of this type. Since the theory is of a general nature, it also lays the foundation for considering, for instance, a complete cycle of angular acceleration, constant angular velocity, and finally, deceleration. It would also be of value as a check for future developments considering the case of non-uniform cross-sectional area. The extension of the governing differential equations to the case of non-uniform cross-sectional area would necessitate modifying the free body diagram used in formulating the governing equations. This would result in the addition of terms containing area derivatives which would be known in terms of the variable x. It is the author's opinion, however, that the best approach to a consideration of non-uniform cross-sectional area could be achieved through a lumping of the masses at various points on the beam.

As a final point of interest, it should be noted that by putting $\alpha = 0$, the constant ω case is obtained, and this affords a method of checking the accuracy of the solution as found by comparing with previous solutions.

VIII. ACKNOWLEDGEMENTS

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IX. BIBLIOGRAPHY

1. Allen, D. N. de G., Relaxation Methods, McGraw-Hill, Co., Inc. 1954, pp. 69-72

2. Browne, Edward Tankard, Introduction to the Theory of Determinants and Matrices, The University of North Carolina Press, 1958, pp. 38-41.

3. Huang, T. C. and Wu, N. C., Approximate Analysis of Flexural Vibrations of Beams, Volume 1, Developments in Mechanics, Plenum Press, 1961, pp. 73-86.

4. Kunz, Kaiser S., Numerical Analysis, McGraw-Hill Co., Inc., 1957, pp. 127-128.

5. Milner, James LaMar, An Approximate Method for Determining the Natural Frequencies of a Rotating Beam Vibrating in the Plane of Rotation, Including the Effects of Rotary Inertia and Transverse Shear, Master's Thesis, Virginia Polytechnic Institute, 1962.

6. Morrell, Bernard, Mechanical Vibrations, The Ronald Press Company, 1957

7. Shaw, F. S., An Introduction to Relaxation Methods, Dover Publications, Inc., 1953, pp. 26-37.

8. Stanton, Ralph G., Numerical Methods for Science and Engineering, Prentice-Hall, Inc., 1961, pp. 8-17; 226-240.

9. Timoshenko, Stephen, and MacCullough, Gleason H., Elements of Strength of Materials, D. Van Nostrand Company, Inc., 1941.

10. Timoshenko, Stephen, Vibration Problems in Engineering, D. Van Nostrand Company, Inc., 1937, pp. 331-338.

11. Timoshenko, Stephen, Strength of Materials, Part I, D. Van Nostrand Company, Inc., 1951.

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XI. APPENDICES

A. Mathematical Development of the General Governing Differential Equation.

The governing differential equations previously developed are repeated here for convenience.

$$\frac{\partial Q}{\partial x} = -\left(\frac{\gamma A}{g}(r + x)\alpha\right) + \frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} - \left(\frac{\gamma A}{g}\alpha^2 t^2 y\right) , \qquad \text{A-1-a}$$

$$P = \frac{\gamma A}{g}\left(rx + \frac{x^2}{2}\right)\alpha^2 t^2 - \frac{\gamma A}{g}\left(rL + \frac{L^2}{2}\right)\alpha^2 t^2 , \qquad \text{A-2-a}$$

$$\frac{\partial M}{\partial x} = \frac{I\gamma}{g}\left(\alpha - \frac{\partial^2 \psi}{\partial t^2}\right) + P\frac{\partial y}{\partial x} + Q , \qquad \text{A-3-a}$$

$$Q = K'AG\left(\frac{\partial y}{\partial x} - \psi\right) , \qquad \text{A-4-a}$$

$$M = -EI\frac{\partial \psi}{\partial x} \qquad \text{A-5-a}$$

Differentiating Equation A-1-a twice with respect to x:

$$\frac{\partial^3 Q}{\partial x^3} = \frac{\gamma A}{g}\frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\gamma A}{g}\alpha^2 t^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{A-1-b}$$

Differentiating Equation A-1-a twice with respect to t:

$$\frac{\partial^3 Q}{\partial t^2 \partial x} = \frac{\gamma A}{g}\frac{\partial^4 y}{\partial t^4} - \frac{\gamma A}{g}\alpha^2\left(2y + 4t\frac{\partial y}{\partial t} + t^2\frac{\partial^2 y}{\partial t^2}\right) \qquad \text{A-1-c}$$

Differentiating Equation A-2-a once with respect to x:

$$\frac{\partial P}{\partial x} = \frac{\gamma A}{g}(r + x)\alpha^2 t^2 \qquad \text{A-2-b}$$

Differentiating Equation A-3-a once with respect to x:

$$\frac{\partial^2 M}{\partial x^2} = \frac{I\gamma}{g}\left(\frac{\partial^3\psi}{\partial x\partial t^2}\right) + \frac{\partial P}{\partial x}\left(\frac{\partial y}{\partial x}\right) + P\frac{\partial^2 y}{\partial x^2} + \frac{\partial Q}{\partial x} \qquad \text{A-3-b}$$

Differentiating Equation A-4-a once with respect to x and solving for $\frac{\partial\psi}{\partial x}$ we obtain:

$$\frac{\partial\psi}{\partial x} = \frac{\partial^2 y}{\partial x^2} - \frac{1}{K'AG}\left(\frac{\partial Q}{\partial x}\right) \qquad \text{A-4-b}$$

Differentiating Equation A-4-b twice with respect to x:

$$\frac{\partial^3\psi}{\partial x^3} = \frac{\partial^4 y}{\partial x^4} - \frac{1}{K'AG}\left(\frac{\partial^3 Q}{\partial x^3}\right) \qquad \text{A-4-c}$$

Differentiating Equation A-4-b twice with respect to t:

$$\frac{\partial^3\psi}{\partial t^2\partial x} = \frac{\partial^4 y}{\partial t^2\partial x^2} - \frac{1}{K'AG}\left(\frac{\partial^3 Q}{\partial t^2\partial x}\right)$$

If we assume that all derivatives of ψ, and y are continuous functions with respect to x and t, we may rewrite the above expression as follows:

$$\frac{\partial^3\psi}{\partial x\partial t^2} = \frac{\partial^4 y}{\partial x^2\partial t^2} - \frac{1}{K'AG}\left(\frac{\partial^3 Q}{\partial t^2\partial x}\right) \qquad \text{A-4-d}$$

Differentiating Equation A-5-a twice with respect to x:

$$-EI\frac{\partial^3\psi}{\partial x^3} = \frac{\partial^2 M}{\partial x^2} \qquad \text{A-5-b}$$

Using A-5-b and substituting into it A-4-c and A-3-b, we obtain

$$EI \left(\frac{\partial^4 y}{\partial x^4} - \frac{1}{K'AG}\left(\frac{\partial^3 Q}{\partial x^3}\right)\right) = -\frac{I\gamma}{g}\left(\frac{\partial^3 \psi}{\partial x \partial t^2}\right)$$
$$-\frac{\partial P}{\partial x}\left(\frac{\partial y}{\partial x}\right) - P\frac{\partial^2 y}{\partial x^2} - \frac{\partial Q}{\partial x} \qquad \text{A-6}$$

Using Equation A-6 and substituting into it Equations A-1-b, A-4-d, A-1-a, A-2-a, and A-2-b, we obtain

$$EI\left(\frac{\partial^4 y}{\partial x^4} - \frac{1}{K'AG}\left(\frac{\gamma A}{g}\frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\gamma A}{g}\alpha^2 t^2 \frac{\partial^2 y}{\partial x^2}\right)\right) =$$
$$-\frac{I\gamma}{g}\left(\frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{1}{K'AG}\left(\frac{\partial^3 Q}{\partial t^2 \partial x}\right)\right) - \left(\frac{\gamma A}{g}(r + x)\alpha^2 t^2\right)\left(\frac{\partial y}{\partial x}\right)$$
$$-\left(\frac{\gamma A}{g}\left(rx + \frac{x^2}{2}\right)\alpha^2 t^2 - \frac{\gamma A}{g}\left(rL + \frac{L^2}{2}\right)\alpha^2 t^2\right)\left(\frac{\partial^2 y}{\partial x^2}\right) \qquad \text{A-7}$$
$$-\left(\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} - \left(\frac{\gamma A}{g}(r + x)\alpha\right) - \frac{\gamma A}{g}\alpha^2 t^2 y\right)$$

Using A-7 and A-1-c, and rearranging we obtain finally

$$EI\frac{\partial^4 y}{\partial x^4} + \left(\frac{I\gamma}{g} - \frac{EI\gamma}{K'gG}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2} + \left(\frac{EI\gamma}{K'gG}\alpha^2 t^2\right.$$
$$\left. + \frac{\gamma A}{g}\left(rx + \frac{x^2}{2}\right)\alpha^2 t^2 - \frac{\gamma A}{g}\left(rL + \frac{L^2}{2}\right)\alpha^2 t^2\right)\frac{\partial^2 y}{\partial x^2}$$
$$+ \frac{\gamma A}{g}(r + x)\alpha^2 t^2 \left(\frac{\partial y}{\partial x}\right) - \frac{I\gamma^2}{g^2 K'G}\frac{\partial^4 y}{\partial t^4} + \left(\frac{I\gamma^2}{g^2 K'G}\alpha^2 t^2\right.$$
$$\left. + \frac{\gamma A}{g}\right)\frac{\partial^2 y}{\partial t^2} + \frac{4I\gamma^2\alpha^2 t}{g^2 K'G}\frac{\partial y}{\partial t} + \left(\frac{2I\gamma^2\alpha^2}{g^2 K'G} - \frac{\gamma A}{g}\alpha^2 t^2\right) y$$
$$-\frac{\gamma A}{g}(r + x)\alpha = 0$$

B. Mathematical Development of the Dimensionless General Governing Differential Equations.

Equation 1a:

$$\frac{\partial Q}{\partial x} = -\left(\frac{\gamma A}{g}(r + x)\alpha\right) + \frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} - \left(\frac{\gamma A}{g}\alpha^2 t^2 y\right)$$

dividing through by γA

$$\frac{\partial \frac{Q}{\gamma AL}}{\partial \frac{x}{L}} = -\left(\frac{\gamma A}{\gamma A}\left(\frac{r}{L} + \frac{x}{L}\right)\frac{\alpha L}{g}\right) + \frac{\gamma A}{\gamma A}\frac{\partial^2 \frac{y}{L}}{\left(\partial\sqrt{\frac{g}{L}}\, t\right)^2}$$

$$- \frac{\gamma A}{\gamma A}\frac{L^2\alpha^2}{g^2}\left(\sqrt{\frac{g}{L}}\, t\right)^2 \frac{y}{L}$$

cancelling and defining the terms

$$\overline{Q} = \frac{Q}{\gamma AL} \qquad\qquad AB1 = \frac{L\alpha}{g}$$

$$\overline{x} = \frac{x}{L} \qquad\qquad \overline{y} = \frac{y}{L}$$

$$\overline{R} = \frac{r}{L} \qquad\qquad \overline{T} = \sqrt{\frac{g}{L}}\; t$$

We obtain finally

$$\frac{\partial \overline{Q}}{\partial \overline{x}} = -AB1\,(\overline{R} + \overline{x}) + \frac{\partial^2 \overline{y}}{\partial \overline{T}^2} - (AB1)^2\, \overline{T}^2\, \overline{y}$$

Equation 2a:

$$P = \frac{\gamma A}{g}\left(rx + \frac{x^2}{2}\right)\alpha^2 t^2 - \frac{\gamma A}{g}\left(rL + \frac{L^2}{2}\right)\alpha^2 t^2$$

Dividing through by γ AL

$$\frac{P}{\gamma AL} = \frac{\gamma A}{\gamma A}\frac{L^2\alpha^2}{g^2}\left(\left(\frac{r}{L}\right)\left(\frac{x}{L}\right) + \frac{1}{2}\left(\frac{x}{L}\right)^2\right)\left(\sqrt{\frac{g}{L}}\right)^2 t^2$$

$$-\frac{\gamma A}{\gamma A}\left(\frac{L^2\alpha^2}{g^2}\right)\left(\left(\frac{r}{L}\right)\left(\frac{L}{L}\right) + \frac{1}{2}\left(\frac{L}{L}\right)^2\right)\left(\sqrt{\frac{g}{L}}t\right)^2$$

cancelling and defining the term

$$\bar{P} = \frac{P}{\gamma AL}$$

we obtain finally

$$\bar{P} = (AB1)^2\left(\bar{R}\,\bar{x} + \frac{\bar{x}^2}{2}\right)\bar{T}^2 - (AB1)^2\left(\bar{R} + .5\right)\bar{T}^2$$

Equation 3a:

$$\frac{\partial M}{\partial x} = \frac{I\gamma}{g}\left(\alpha - \frac{\partial^2\psi}{\partial t^2}\right) + P\frac{\partial Y}{\partial x} + Q$$

Dividing through by γ AL

$$\frac{\partial \frac{M}{\gamma AL^2}}{\partial \frac{x}{L}} = \left(\frac{I}{AL^2}\right)\left(\frac{L\alpha}{g}\right) - \frac{I}{AL^2}\frac{\partial^2\psi}{\left(\partial\sqrt{\frac{g}{L}}\,t\right)^2} + \frac{P}{\gamma AL}\frac{\partial\left(\frac{Y}{L}\right)}{\partial\left(\frac{x}{L}\right)} + \frac{Q}{\gamma AL}$$

cancelling and defining the terms

$$\bar{M} = \frac{M}{\gamma AL^2} \qquad\qquad AB2 = \frac{I}{AL^2}$$

we obtain finally

$$\frac{\partial \overline{M}}{\partial \overline{x}} = AB2\ (AB1 - \frac{\partial^2 \psi}{T}) + \overline{P}\,\frac{\partial \overline{Y}}{\partial \overline{x}} + \overline{Q}$$

Equation 4a:

$$Q = K'AG\ (\frac{\partial Y}{\partial x} - \psi)$$

Dividing through by γ AL

$$\frac{Q}{\gamma AL} = \frac{K'AG}{\gamma AL}\left(\frac{\partial(\frac{Y}{L})}{\partial(\frac{X}{L})} - \psi\right)$$

cancelling and defining the term

$$AB = \frac{K'G}{\gamma L}$$

we obtain finally

$$\overline{Q} = -\ AB\ (\frac{\partial \overline{Y}}{\partial \overline{x}} - \psi)$$

Equation 5a:

$$M = -\ EI\ \frac{\partial \psi}{\partial x}$$

Dividing through by γ AL^2

$$\frac{M}{\gamma AL^2} = -\ \frac{EI}{\gamma AL^3}\,\frac{\partial \psi}{\partial(\frac{X}{L})}$$

defining the term

$$AB3 = \frac{EI}{\gamma AL^2}$$

We obtain finally

$$\bar{M} = -AB3 \frac{\partial \psi}{\partial \bar{X}}$$

IBM 1620 PROGRAM

```
C   C  MAIN LINE PROGRAM
       DIMENSION A(52,53),SOLN(53),Y(4,27)
       COMMON A,SOLN,Y,ABC,ABC1,ABC2,ABC3,ABC4,ABC5,ABC6,ABC7,ABC8,ABC9,
      1IJ,KD,H4,H1,T,CAT,MLL,MKL,MNM,H,AB,AB1,AB2,AB3,R,H8,H9,H10
       CALL READ A
       IF(CAT)1,2,1
     2 CALL IN VAL A
     1 T=T+H1
       CALL ZERO M
       CALL SET M
       CALL SIM EQ
       CALL RESET
       CALL PUNCH
       IF(SENSE SWITCH 2)3,1
     3 CALL F OUT
       GO TO 1
       END

       SUBROUTINE READ A
       DIMENSION A(52,53),SOLN(53),Y(4,27 )
       COMMON A,SOLN,Y,ABC,ABC1,ABC2,ABC3,ABC4,ABC5,ABC6,ABC7,ABC8,ABC9,
      1IJ,KD,H4,H1,T,CAT,MLL,MKL,MNM,H,AB,AB1,AB2,AB3,R,H8,H9,H10
       READ 20,ABC,ABC1,ABC2,ABC3,ABC4
    20 FORMAT(5E14.8)
       READ 21,ABC5,ABC6,ABC7,ABC8,ABC9
    21 FORMAT(5E14.8)
       READ 22,IJ,KD,H4,H1,T,CAT
    22 FORMAT(2I2,4E14.8)
       MLL=(IJ/2)+1
       MKL=IJ+1
       MNM=IJ-3
       H=1.0/((H4/2.0)-1.0)
       AB=(ABC3*ABC4)/(ABC1*ABC2)
       AB1=(ABC2*ABC5)/(ABC6)
       AB2=(ABC7)/(ABC*ABC2*ABC2)
       AB3=(ABC8*ABC7)/(ABC1*ABC*ABC2*ABC2*ABC2)
       R=ABC9/ABC2
       H8=H*H
       H9=H1*H1
       H10=AB1*AB1
       IF(CAT)26,24,26
    24 DO 25 I=1,4
       DO 25 J=1,MLL
    25 Y(I,J)=0.
       GO TO 29
    26 READ 508,((Y(I,J),J=1,MLL),I=1,4)
   508 FORMAT(5E14.8)
    29 RETURN
       END

       SUBROUTINE INVALA
       DIMENSION A(52,53),SOLN(53),Y(4,27)
       COMMON A,SOLN,Y,ABC,ABC1,ABC2,ABC3,ABC4,ABC5,ABC6,ABC7,ABC8,ABC9,
      1IJ,KD,H4,H1,T,CAT,MLL,MKL,MNM,H,AB,AB1,AB2,AB3,R,H8,H9,H10
```

```
      PUNCH 25
   25 FORMAT(32X,15H INITIAL VALUES)
      PUNCH 26
   26 FORMAT(//)
      PUNCH 29,ABC2
   29 FORMAT(27X,8HLENGTH =E14.8,4H IN.)
      PUNCH 27,ABC
   27 FORMAT(27X,6HAREA =E14.8,7H SQ.IN.)
      PUNCH 36,ABC9
   36 FORMAT(24X,15HRADIUS OF HUB =E14.8,4H IN.)
      PUNCH 30,ABC3
   30 FORMAT(24X,14HSHAPE FACTOR =E14.8,6H - - -)
      PUNCH 33,ABC6
   33 FORMAT(23X,9HGRAVITY =E14.8,12H IN./SEC.SQ.)
      PUNCH 37,H
   37 FORMAT(22X,16HBEAM INCREMENT =E14.8,6H - - -)
      PUNCH 38,H1
   38 FORMAT(22X,16HTIME INCREMENT =E14.8,6H - - -)
      PUNCH 31,ABC4
   31 FORMAT(20X,15HSHEAR MODULUS =E14.8,11H LB./CU.IN.)
      PUNCH 34,ABC7
   34 FORMAT(20X,19HMOMENT OF INERTIA =E14.8,7H IN.4TH)
      PUNCH 35,ABC8
   35 FORMAT(16X,23HMODULUS OF ELASTICITY =E14.8,11H LB./SQ.IN.)
      PUNCH 28,ABC1
   28 FORMAT(16X,24HWEIGHT PER UNIT VOLUME =E14.8,11H LB./CU.IN.)
      PUNCH 32,ABC5
   32 FORMAT(16X,22HANGULAR ACCELERATION =E14.8,13H RAD./SEC.SQ.)
      RETURN
      END

      SUBROUTINE ZERO M
      DIMENSION A(52,53),SOLN(53),Y(4,27 )
      COMMON A,SOLN,Y,ABC,ABC1,ABC2,ABC3,ABC4,ABC5,ABC6,ABC7,ABC8,ABC9,
     1IJ,KD,H4,H1,T,CAT,MLL,MKL,MNM,H,AB,AB1,AB2,AB3,R,H8,H9,H10
      DO 113 I=1,MKL
      SOLN(I)=0.0
      DO 113 J=1,IJ
  113 A(J,I)=0.0
      RETURN
      END

      SUBROUTINE SET M
      DIMENSION A(52,53),SOLN(53),Y(4,27)
      COMMON A,SOLN,Y,ABC,ABC1,ABC2,ABC3,ABC4,ABC5,ABC6,ABC7,ABC8,ABC9,
     1IJ,KD,H4,H1,T,CAT,MLL,MKL,MNM,H,AB,AB1,AB2,AB3,R,H8,H9,H10
      X=H
      I=1
      DO 110 J=1,MNM,2
      A(I,J)=(2.0*AB/(H8))+(1.0/H9)-(H10*T*T)
      A(I,J+2)=-AB/H8
      A(I,J+3)=AB/(2.0*H)
      A(I+1,J+1)=(AB2/H9)+(2.0*AB3/H8)+AB
      A(I+1,J+2)=-(H10*T*T*(R*X+(X*X/2.0)-R-.5)+AB)/(2.0*H)
      A(I+1,J+3)=-AB3/H8
      X=X+H
  110 I=I+2
      J=1
```

```
      DO 111 I=3,MNM,2
      A(I,J)=A(I,J+4)
      A(I,J+1)=-A(I,J+5)
      A(I+1,J)=-A(I+1,J+4)
      A(I+1,J+1)=A(I+1,J+5)
  111 J=J+2
      J=2
      X=H
      DO 112 I=1,MNM,2
      A(I,MKL)=-AB1*(R+X)-((2.0*Y(2,J)-Y(1,J))/(H9))
      A(I+1,MKL)=-AB2*(AB1+((2.0*Y(4,J)-Y(3,J))/H9))
      X=X+H
  112 J=J+1
      A(IJ-1,IJ-1)=1.0
      A(IJ-1,IJ-2)=-2.0*H
      A(IJ-1,IJ-5)=-1.0
      A(IJ,IJ)=-1.0
      A(IJ,IJ-4)=1.0
      RETURN
      END

      SUBROUTINE SIM EQ
      DIMENSION A(52,53),SOLN(53),Y(4,27)
      COMMON A,SOLN,Y,ABC,ABC1,ABC2,ABC3,ABC4,ABC5,ABC6,ABC7,ABC8,ABC9,
     1IJ,KD,H4,H1,T,CAT,MLL,MKL,MNM,H,AB,AB1,AB2,AB3,R,H8,H9,H10
      M=IJ-1
      MKL=IJ+1
      DO 16 J=1,M
      IF(A(J,J))10,9,10
    9 TYPE 2,J
    2 FORMAT(29H ZERO ELEMENT ON DIAGONAL,ROWI3)
      GO TO 21
   10 IJJ=J+KD
      IF(M-IJJ)12,13,13
   12 IJJ=M
   13 DO 16 I=J,IJJ
      K=I+1
      IF(A(K,J))11,16,11
   11 NN=J+KD
      IF(MKL-NN)19,3,15
   19 NN=MKL
      GO TO 3
   15 A(K,MKL)=A(K,MKL)-((A(J,MKL)*A(K,J))/A(J,J))
    3 CI=A(K,J)/A(J,J)
      DO 14 L=J,NN
   14 A(K,L)=A(K,L)-(A(J,L)*CI)
   16 CONTINUE
      IF(A(IJ,IJ))4,5,4
    5 J=IJ
      GO TO 9
    4 DO 18 I=1,IJ
      K=IJ+1-I
      SOLN(K)=0.
      SOLN(MKL)=1.
      DO 17 J=1,I
      L=IJ+2-J
   17 SOLN(K)=SOLN(K)+A(K,L)*SOLN(L)
   18 SOLN(K)=-SOLN(K)/A(K,K)
```

```
 21 RETURN
    END

    SUBROUTINE RESET
    DIMENSION A(52,53),SOLN(53),Y(4,27 )
    COMMON A,SOLN,Y,ABC,ABC1,ABC2,ABC3,ABC4,ABC5,ABC6,ABC7,ABC8,ABC9,
   1IJ,KD,H4,H1,T,CAT,MLL,MKL,MNM,H,AB,AB1,AB2,AB3,R,H8,H9,H10
    DO 114 J=1,MLL
    Y(1,J)=Y(2,J)
114 Y(3,J)=Y(4,J)
    I=2
    DO 116 J=1,IJ,2
    Y(2,I)=SOLN(J)
    Y(4,I)=SOLN(J+1)
116 I=I+1
    RETURN
    END

    SUBROUTINE PUNCH
    DIMENSION A(52,53),SOLN(53),Y(4,27 )
    COMMON A,SOLN,Y,ABC,ABC1,ABC2,ABC3,ABC4,ABC5,ABC6,ABC7,ABC8,ABC9,
   1IJ,KD,H4,H1,T,CAT,MLL,MKL,MNM,H,AB,AB1,AB2,AB3,R,H8,H9,H10
    CAT=CAT+1.
    IF(SENSE SWITCH 4)515,516
515 TYPE 514,CAT
514 FORMAT(F10.0)
516 PUNCH 79,T
 79 FORMAT(28H VALUES OF DISPLACEMENT TIMEE14.8)
    PUNCH 115,(Y(2,J),J=1,MLL)
115 FORMAT(5E14.8)
    RETURN
    END

    SUBROUTINE F OUT
    DIMENSION A(52,53),SOLN(53),Y(4,27 )
    COMMON A,SOLN,Y,ABC,ABC1,ABC2,ABC3,ABC4,ABC5,ABC6,ABC7,ABC8,ABC9,
   1IJ,KD,H4,H1,T,CAT,MLL,MKL,MNM,H,AB,AB1,AB2,AB3,R,H8,H9,H10
    PUNCH 20,ABC,ABC1,ABC2,ABC3,ABC4
 20 FORMAT(5E14.8)
    PUNCH 21,ABC5,ABC6,ABC7,ABC8,ABC9
 21 FORMAT(5E14.8)
    PUNCH 22,IJ,KD,H4,H1,T,CAT
 22 FORMAT(2I2,4E14.8)
    PUNCH 408,((Y(I,J),J=1,MLL),I=1,4)
408 FORMAT(5E14.8)
    PAUSE
    RETURN
    END
```

ABSTRACT

A method for determining the transient vibrational effects produced by application of a constant angular acceleration to a cantilever beam initially at rest was determined. This method is applicable to beams of uniform cross-sectional area vibrating in their planes of rotation. The governing differential equations include the effects of bending, shear deformation and rotary inertia. Coriolis' acceleration, however, is neglected.

These governing equations were non-dimensionalized and solved by numerical means using a finite difference approach and dividing the beam's length into 12 sections, since their complexity made an exact solution appear impossible. This was done by the aid of a 1620 I.B.M. Computer.

Application of this solution to an actual beam indicated that a wave propagation type of response becomes more clearly evident as the hub radius is increased. The numerical results also indicate that the initial displacements are a direct result of shear deformation.

The effect of centrifugal force was also analyzed. At large values of time this force caused the beam to return to its undeformed axis.

Solutions for a short time interval extending over 1100 steps, a large time interval extending over 275 steps, and another using

the short time interval together with a reduced radius, 250 steps, were obtained.