THE MOTION OF A LUNAR SATELLITE UNDER THE INFLUENCE OF THE MOON'S NONCENTRAL FORCE FIELD

by

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IV. INTRODUCTION

The theory of satellite motion about an oblate planet has been discussed by many authors. Early discussions were restricted to low eccentricity and low inclination orbits because these are the type of orbits found in nature. These two approximations greatly simplify the analysis, and after theories have been derived which predicted natural satellite positions within observational accuracy, interest in the subject waned. The recent advent of artificial earth satellites has caused a reemphasis on orbital mechanics because in general artificial satellites do not have small eccentricities or inclinations and hence their motions cannot be analyzed by the older theories.

Recent investigators have used a variety of methods of analysis to find approximate solutions to the problem of the motion of an earth satellite. The first assumption in all of these analyses is that the earth is rotationally symmetric about the polar axis, and in most analyses it is also assumed that the equator is a plane of symmetry. With these assumptions the gravitational potential of the earth is given by MacMillan (5) as a series of even zonal harmonics in the latitude θ,

$$V = \frac{\mu_e}{r}\left[\sum_{n=0}^{\infty} \frac{J_{2n}P_{2n}(\cos\theta)}{r^{2n}}\right]$$

The J_{2n} are experimentally determined parameters except for J_o

which is unity. No closed-form solution is known for a particle moving under this general type of potential. Methods of attacking the problem range from Vinti's (9) exact closed-form solution utilizing an approximate potential to Brouwer's (2) analysis which utilizes first- and second-order perturbation techniques applied to a more or less exact potential. An interesting intermediate theory is given by Garfinkel (4) in which the potential is modified to admit a closed-form solution and then the modification is considered as a perturbation to the first solution. In addition to the obvious need these theories fill by furnishing means of accurately predicting the position of earth satellites, some interesting geodetic applications have been suggested by Whitten (10) and O'Keefe (6) has obtained one of the first important results, namely, that the equatorial plane is not a plane of symmetry, that is, the earth is slightly "pear shaped."

In addition to their obvious utility for gathering lunar physical and environmental data, satellites of the moon may also be used to obtain selenodetic information in much the same manner as discussed above for earth satellites. The feasibility of performing such experiments in the near future is somewhat questionable because of the poor tracking accuracy of present earth-based tracking systems. However, with the inevitable improvement in accuracy, lunar satellites will become important tools for making selenodetic experiments.

A comparison of the various disturbing forces acting on a lunar satellite shows that the two major disturbing forces are the gravitational attraction of the earth and the nonspherical shape of the moon. For lunar satellites within two lunar radii of the moon's center the latter force is predominant, for more distant satellites the earth's gravity represents the major disturbing force. Some recent analyses have been made of the effects of terrestrial gravity on the motion of lunar satellites. A paper of particular interest is one by Schechter (7) who analyzes the problem by using Pontecoulants' solution of the three-body problem. He found "that the maximum decrease in perilunar radius r_p depends on the ratio m of the angular velocity of the perturbing body (earth) to that of the satellite, and on the eccentricity e in the combination $2em(1 + e)r_p$ . . . . The additional perturbation caused by the sun was found to be of negligible magnitude." This paper and others present the major effects of terrestrial gravity on lunar satellites and no attempt will be made here to include terrestrial gravity in the analysis.

The analysis will be mainly concerned with the motion of a lunar satellite governed by the nonspherical gravitational potential of the moon. The moon, unlike the earth, does not have an axis of symmetry and a first approximation to the moon's figure is an ellipsoid. The minor axis of the ellipsoid is the moon's axis of rotation and the major axis points in the direction of the earth. Hence, the lunar gravitation field is more complex than the earth's field in that it is an unsymmetrical rotating field. The potential expansion of the field will thus contain tesseral as well as zonal harmonics and will also depend explicitly on the time. These two complications constitute the major differences between lunar and earth satellite analyses.

V. SYMBOLS

a	semimajor axis of osculating ellipse
e	eccentricity of osculating ellipse
E	eccentric anomaly
g	ω, argument of periselene
μ	gravitational constant times lunar mass
G	total angular momenta, $\sqrt{\mu a(1 - e^2)}$
h	$\Omega - \nu t$, longitude of node in rotating system
H	z component of angular momenta, $\sqrt{\mu a(1 - e^2)}\cos i$
i	inclination of orbital plane to lunar equator
J	oblateness parameter of moon's figure
K	ellipticity parameter of moon's figure
l	mean anomaly
L	$\sqrt{\mu a}$
n	mean angular motion of satellite, $\sqrt{\frac{\mu}{a^3}}$
R	disturbing function
$\bar{r}$	position vector of satellite
t	time
v	true anomaly of satellite
x, y, z	position coordinates of satellite in fixed axis system
x_m, y_m, z_m	position coordinates in rotating system
τ	time of periselenian passage
Ω	longitude of node
ω	argument of periselene
ν	rotational rate of moon on its axis

VI. EQUATIONS OF MOTION AND COORDINATE SYSTEM

Referred to a nonrotating, noninertial coordinate system with origin at the center of mass of the moon, the equations of motion of a lunar satellite are

$$\ddot{\bar{r}} + \frac{\mu\bar{r}}{\bar{r}^3} = \bar{F} \tag{1}$$

The disturbing force $\bar{F}$ is the vector sum of the forces acting on the satellite that are not included in the central attraction term on the left. $\bar{F}$ is called a force throughout even though it may have the units of an acceleration. For high density satellites the primary disturbing forces are due to noncentral gravitational effects and to pseudoforces of the centrifugal type. These forces are derivable from a potential called the disturbing function, that is,

$$\bar{F} = \bar{\nabla}R$$

For disturbing functions of interest, no complete, exact solution of equation (1) is known and approximate techniques must be utilized to obtain analytical information. To this end it is noted that if the disturbing forces are neglected, equation (1) reduces to the familiar Keplerian two-body problem

$$\ddot{\bar{r}} = -\frac{\mu\bar{r}}{\bar{r}^3} \tag{2}$$

which has a well-known solution and the motion is usually specified by the six Lagrangian orbital elements (a, e, i, ω, Ω, and τ). If $\bar{F}$ is very small in comparison to the central attraction, the solution of equation (2) will be a first approximation to the solution to equation (1). In this case a first-order variation of integration constants technique could be applied to these six orbital elements and would be expected to represent the motion with considerable accuracy.

Instead of utilizing the Lagrangian elements, a set of canonical constants will be used which are generated by solving the Hamilton-Jacobi equation associated with the system of equation (2). For immediate reference this solution is given in appendix A and leads to the usual Delauney elements

$$
\begin{aligned}
L' &= \sqrt{\mu a} & l' &= n(t - \tau) \\
G' &= L'\sqrt{1 - e^2} & g' &= \omega \\
H' &= G' \cos i & h' &= \Omega
\end{aligned}
$$

A slight modification of these parameters will be used here.

The equations for the variation of the elements take a particularly simple form when canonical variables are used; in particular, the equations of motion for the Delauney elements are

$$
\begin{aligned}
\frac{dL'}{dt} &= \frac{\partial F'}{\partial l'} & \frac{dl'}{dt} &= -\frac{\partial F'}{\partial L'} \\
\frac{dG'}{dt} &= \frac{\partial F'}{\partial g'} & \frac{dg'}{dt} &= -\frac{\partial F'}{\partial G'} \\
\frac{dH'}{dt} &= \frac{\partial F'}{\partial h'} & \frac{dh'}{dt} &= -\frac{\partial F'}{\partial H'}
\end{aligned}
$$

and the Hamiltonian is of the form

$$F' = \frac{\mu^2}{2L'^2} + R(L', G', H', l', g', h', t)$$

The development of these equations is outlined in appendix B.

In general, F will be a function of the time and if possible it is convenient to effect a transformation of variables such that the time does not appear explicitly. For this problem, as will be shown later, time appears in the Hamiltonian only in the form $(h' - \nu t)$, that is

$$F'(L', G', H', l', g', h', t) \equiv F'(L', G', H', l', g', (h' - \nu t))$$

The time dependence can be removed by making a transformation from the primed to the unprimed variables defined by the generating function

$$\phi = -\left[L'l + G'g + H'(h + \nu t)\right]$$

Corben (3) gives the relationships between the old (primed) and new (unprimed) variables for a generating function of this form as

$$p_j = -\frac{\partial\phi}{\partial q_j} \qquad q'_j = -\frac{\partial\phi}{\partial p'_j}$$

where p_j and q_j are conjugate coordinates. Applying these formulas to the above generating function leads to

$$
\begin{aligned}
l' &= l & L' &= L \\
g' &= g & G' &= G \\
h' &= h + \nu t & H' &= H
\end{aligned}
$$

These new canonical variables are related to the Lagrangian elements by

$$
\begin{aligned}
a &= \frac{L^2}{\mu} & \tau &= t - \frac{l}{n} \\
e &= \left[1 - \left(\frac{G}{L}\right)^2\right]^{1/2} & \omega &= g \\
i &= \cos^{-1}\left(\frac{H}{G}\right) & \Omega &= h + \nu t
\end{aligned}
$$

The new Hamiltonian takes the form

$$F = F' + \frac{\partial \phi}{\partial t} = F' - H\nu$$

or more explicitly

$$F = \frac{\mu^2}{2L^2} - \nu H + R(L, G, H, l, g, h)$$

which does not depend explicitly on the time and therefore F is a constant of the motion. The new equations of motion retain the canonical form

$$
\left.
\begin{aligned}
\frac{dL}{dt} &= \frac{\partial F}{\partial l} & \frac{dl}{dt} &= -\frac{\partial F}{\partial L} \\
\frac{dG}{dt} &= \frac{\partial F}{\partial g} & \frac{dg}{dt} &= -\frac{\partial F}{\partial G} \\
\frac{dH}{dt} &= \frac{\partial F}{\partial h} & \frac{dh}{dt} &= -\frac{\partial F}{\partial H}
\end{aligned}
\right\} \qquad (3)
$$

It should be noted that these canonical variables are not all constants for the undisturbed motion, since both the mean anomaly l and the variable h would be linear in the time. Equation (3) completely determines the motion of the lunar satellite; however, to utilize these equations the Hamiltonian must be expressed in terms of the canonical constants.

VII. DERIVATION OF THE DISTURBING FUNCTION

Observation of the moon's motions and physical features indicate that the moon's figure can be approximated by an ellipsoid with the major axis pointing in the general direction of the earth. The minor axis is the axis of rotation and makes an angle of 1.6° with the normal to the ecliptic. The axis of rotation regresses with a period of 18-2/3 years. These facts are embodied in Cassini's laws as given by Plummer (7).

Alexandrov (1) has reviewed the current data on the lunar gravitational field and has shown that the potential can be approximated with one-half of 1 percent by assuming the moon to be a homogeneous ellipsoid. Taking a rotating coordinate system with z_m measured along the moon's axis of rotation and the x_m axis pointing in the direction of the earth and coincident with the principle axis, Alexandrov has derived the following values for the semiaxis lengths of the moon

$$R = 1738.57 \pm 0.07 \text{ km}$$

$$b = 1738.21 \pm 0.07 \text{ km}$$

$$c = 1737.49 \pm 0.07 \text{ km}$$

where R, b, and c are measured along the x_m, y_m, and z_m axes, respectively.

MacMillan (5) shows that the disturbing potential for an ellipsoid can be written as

$$R = \frac{\mu R^2}{10r^3}\left\{\epsilon_1{}^2\left[1 - 3\left(\frac{y_m}{r}\right)^2\right] + \epsilon_2{}^2\left[1 - 3\left(\frac{z_m}{r}\right)^2\right] + \begin{bmatrix}\text{terms of the} \\ \text{order } \ \epsilon_1{}^4,\ \epsilon_2{}^4\end{bmatrix}\right\}$$

where

$$\epsilon_1{}^2 = \frac{R^2 - b^2}{R^2}, \quad \epsilon_2{}^2 = \frac{R^2 - c^2}{R^2}$$

Using the values of R, b, and c from above yields

$$\epsilon_1{}^2 = 0.000414 \quad \text{and} \quad \epsilon_2{}^2 = 0.001242$$

Hence the disturbing potential is of the order of $\left[\frac{\mu}{r}\left(\frac{R}{r}\right)^2 \times 10^{-4}\right]$ while the potential energy of the satellite due to the central attraction is of the order of $\left[\frac{\mu}{r}\right]$. Thus for close lunar satellites the disturbing forces will be smaller than the central force by a factor of at most 10^{-3}. Since the disturbing forces are small in comparison to the central force, a first-order perturbation technique should adequately represent the motion. Also, the terms in the potential expansion of order $\epsilon_1{}^4$ and $\epsilon_2{}^4$ can be neglected unless a second-order solution is required. Hence, the disturbing potential considered here will be

$$R = \frac{\alpha_1}{r^3} + \frac{\alpha_2}{r^3}\left(\frac{y_m}{r}\right)^2 + \frac{\alpha_3}{r^3}\left(\frac{z_m}{r}\right)^2 \tag{4}$$

where

$$\alpha_1 = \frac{1}{10} \mu R^2 \left(\epsilon_1{}^2 + \epsilon_2{}^2\right)$$

$$\alpha_2 = - \frac{3}{10} \mu R^2 \epsilon_1{}^2$$

$$\alpha_3 = - \frac{3}{10} \mu R^2 \epsilon_2{}^2$$

The disturbing function must now be expanded in terms of the elements L, G, H, l, g, h. To this end

$$\begin{vmatrix} x_m \\ y_m \\ z_m \end{vmatrix} = \begin{vmatrix} \cos \nu t & \sin \nu t & 0 \\ -\sin \nu t & \cos \nu t & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix}$$

where x, y, z are the nonrotating coordinates shown in figure 1 and ν is the angular rotational rate of the moon on its axis. But in terms of the orbital parameters

$$\begin{vmatrix} x \\ y \\ z \end{vmatrix} = r \begin{vmatrix} \cos(v + g)\cos h' - \sin(v + g)\sin h' \cos i \\ \cos(v + g)\sin h' + \sin(v + g)\cos h' \cos i \\ \sin(v + g)\sin i \end{vmatrix}$$

so that the position components relative to the moon take the form

$$\begin{vmatrix} x_m \\ y_m \\ z_m \end{vmatrix} = r \begin{vmatrix} \cos(v + g)\cos h - \sin(v + g)\sin h \cos i \\ \cos(v + g)\sin h + \sin(v + g)\cos h \cos i \\ \sin(v + g)\sin i \end{vmatrix}$$

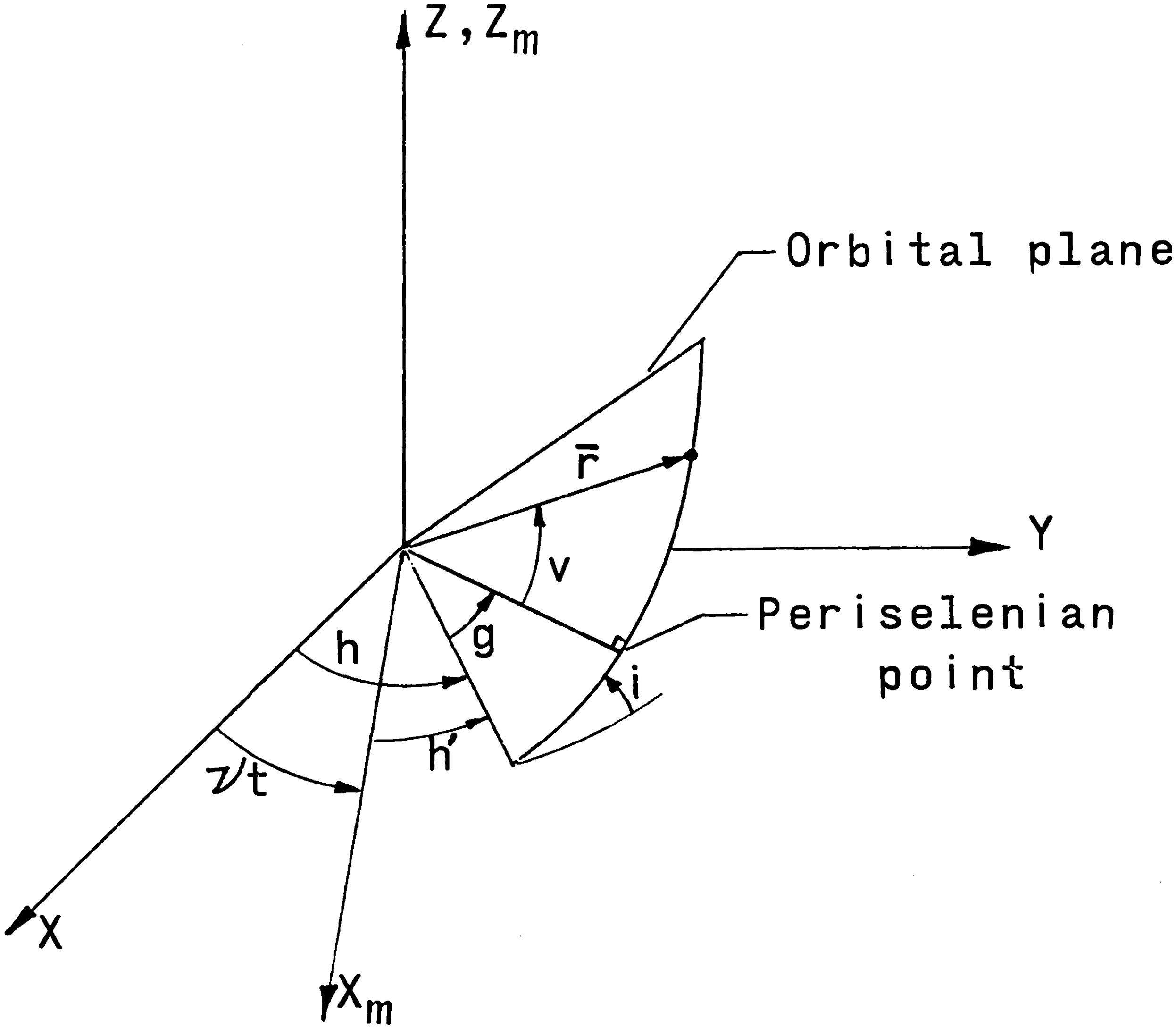

Figure 1.- Illustration of some angular parameters.

where $h' - \nu t$ has been replaced by h according to the transformation in the previous section. The physical meaning of this transformation is now seen to represent a transformation from a nonrotating frame to a coordinate system fixed in the rotating moon.

Substitution for y_m and z_m into the disturbing function (4) gives after some simplification

$$R = \frac{1}{r^3}\left[S_1 + S_2\cos 2v + S_3\sin 2v\right]$$

where

$$S_1 = S_1(H, G, h) = \alpha_1 + \alpha_2\beta_1 + \alpha_3\beta_4$$

$$S_2 = S_2(H, G, h, g) = (\alpha_2\beta_2 - \alpha_3\beta_4)\cos 2g + \alpha_2\beta_3\sin 2g$$

$$S_3 = S_3(H, G, h, g) = (\alpha_3\beta_4 - \alpha_2\beta_2)\sin 2g + \alpha_2\beta_3\cos 2g$$

and recalling that $\cos i = \frac{H}{G}$

$$\beta_1 = \frac{1}{2}\left[\sin^2 h + \left(\frac{H}{G}\right)^2\cos^2 h\right]$$

$$\beta_2 = \frac{1}{2}\left[\sin^2 h - \left(\frac{H}{G}\right)^2\cos^2 h\right]$$

$$\beta_3 = \left(\frac{H}{G}\right)\sin h\cos h$$

$$\beta_4 = \frac{1}{2}\left[1 - \left(\frac{H}{G}\right)^2\right]$$

The disturbing function is now expressed in terms of the canonical elements except for the true anomaly v and the radius r. These two parameters always present the most difficulty because they are functions of the mean anomaly l which is related to v through Kepler's equation

$$l = E - e \sin E$$

with the auxiliary relation

$$\tan \frac{E}{2} = \sqrt{\frac{1 - e}{1 + e}} \tan \frac{v}{2}$$

Inverting these equations to give the true anomaly as a function of the mean anomaly results in a Fourier-type series. Historically, an associated expansion was the origin of the Bessel functions.

The essential differences between satellite analyses which utilize a variation of parameters technique is in the manner of handling the terms containing the true anomaly. A number of methods were attempted and the most satisfactory method for this problem was found to be von Zeipel's method as used by Brouwer (2). The functions of interest are first expanded in Fourier series

$$\left(\frac{a}{r}\right)^3 = \left(\frac{L}{G}\right)^3 + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos nl \int_0^{2\pi} \left(\frac{a}{r}\right)^3 \cos nl \, dl = \left(\frac{L}{G}\right)^3 + P_1$$

$$\left(\frac{a}{r}\right)^3 \cos 2v = \frac{1}{\pi} \sum_{n=1}^{\infty} \cos nl \int_0^{2\pi} \left(\frac{a}{r}\right)^3 \cos 2v \cos nl \, dl = P_2$$

$$\left(\frac{a}{r}\right)^3 \sin 2v = \frac{1}{\pi} \sum_{n=1}^{\infty} \sin nl \int_0^{2\pi} \left(\frac{a}{r}\right)^3 \sin 2v \cos nl \, dl = P_3$$

where the integrals are performed by utilizing Kepler's equation and the resulting series are infinite series in ascending powers of the eccentricity where the eccentricity must, of course, be replaced by $\sqrt{1 - \left(\frac{G}{L}\right)^2}$. Hence, the disturbing function takes the form

$$R = \left(\frac{\mu}{LG}\right)^3 S_1 + \left(\frac{\mu}{L^2}\right)^3 (S_1P_1 + S_2P_2 + S_3P_3)$$

The Hamiltonian of the system can therefore be written as

$$F = F_s + F_p$$

where

$$F_s = \frac{\mu^2}{2L^2} - \nu H + \left(\frac{\mu}{LG}\right)^3 S_1$$

and

$$F_p = \left(\frac{\mu}{L^2}\right)^3 (S_1P_1 + S_2P_2 + S_3P_3)$$

F_s is a function of all the orbital elements save one, namely, the mean anomaly, l. Hence, F_s will be a slowly varying quantity and the secular and long-period variations will come from this part of the Hamiltonian. F_p depends explicitly on the mean anomaly and in fact is periodic with a fundamental period of $\frac{2\pi}{n}$, that is, the satellites orbital period. The short-period perturbations will arise from this part of the Hamiltonian.

At this point it is convenient to analyze the order of magnitude of the various terms which constitute the Hamiltonian. Considering only F_s, since F_p is the same order as the last term of F_s, we have

$$F_s \sim nL \left\{ \frac{1}{2} - \left(\frac{\nu}{n}\right)\left(\frac{H}{L}\right) + \left(\frac{L}{G}\right)^3\left(\frac{R}{a}\right)^2\left(\frac{\epsilon_1{}^2 + \epsilon_2{}^2}{10}\right)\right\}$$

where use has been made of the definition of S_1 and the identity $n = \frac{\mu^2}{L^3}$ has been used twice. For close satellites of the moon (L/G) and (R/a) are of order unity, while (H/L) is approximately equal to the cosine of the inclination of the orbital plane. Now ν is the angular rotational rate of the moon about its polar axis while n is the mean angular rate of the satellite in its orbit. The ratio of these two terms for close satellites is about 3×10^{-3}. The order of the individual terms is therefore

$$F_s \sim nL \left\{ (1) + (10^{-3}) + (10^{-4}) \right\}$$

Hence, in this first-order analysis, the disturbance due to the rotation of the moon on its axis can be considered as a first-order perturbation. And the Hamiltonian is

$$F = \frac{\mu^2}{2L^2} + \left[R - \nu H\right]$$

where the term in brackets is the first-order perturbation term. With this assumption in mind, we turn to the integration of the equations of motion using von Zeipel's method.

VIII. INTEGRATION OF THE EQUATION OF MOTION

von Zeipel's method is very similar to the procedure used in chapter VI to eliminate the time from the Hamiltonian. In that case, a canonical transformation was found such that an undesirable variable did not appear in the new Hamiltonian. If such a transformation can be found, then the new conjugate coordinate is a constant of the motion. For the example mentioned above, eliminating the time meant that the new Hamiltonian was constant. Finding the generating function which eliminated the time could be done almost by inspection. In general, finding the generating function which eliminates some variable means that one must be able to separate the Hamiltonian-Jacobi equation into a part depending on that variable only and another part depending on the other variables. von Zeipel's method is a procedure for constructing such a generating function and a new Hamiltonian by successive approximations. The Kth order approximation will have a coefficient of order $(\epsilon_1{}^2)^K$ or $(\nu/n)^K$. Since only first-order terms in $\epsilon_1{}^2$ were maintained in the potential, only first-order forms will be kept here.

First, the short-period terms will be eliminated; that is, a generating function $\phi(L', G', H', l, g, h)$ will be sought such that under ϕ the old Hamiltonian $F(L, G, H, l, g, h)$ becomes $F'(L', G', H', g', h')$. The old and new coordinates are related as before by

$$L = \frac{\partial\phi}{\partial l} \qquad G = \frac{\partial\phi}{\partial g} \qquad H = \frac{\partial\phi}{\partial h}$$

$$l' = \frac{\partial\phi}{\partial L'} \quad g' = \frac{\partial\phi}{\partial G'} \quad h' = \frac{\partial\phi}{\partial H'} \tag{5}$$

The generating function to first order (the subscripts will denote the order of each form) is

$$\phi = \phi_0 + \phi_1$$

where ϕ_0 is, of course, the identity transformation

$$\phi_0 = L'l + G'g + H'h$$

Since ϕ does not depend explicitly on the time the old and new Hamiltonians can be equated

$$F_0(L) + F_1(L, G, H, l, g, h) = F_0'(L', G', H' - g', h') + F_1'(L', G', H' - g', h')$$

Substituting for L, G, H, g', h' using equation (5) gives

$$F_0\left(\frac{\partial\phi}{\partial l}\right) + F_1\left(\frac{\partial\phi}{\partial l}, \frac{\partial\phi}{\partial g}, \frac{\partial\phi}{\partial h}, l, g, h\right)$$

$$= F_0'\left(L', G', H' - \frac{\partial\phi}{\partial G'}, \frac{\partial\phi}{\partial H'}\right) + F_1'\left(L', G', H' - \frac{\partial\phi}{\partial G'}, \frac{\partial\phi}{\partial H'}\right)$$

But

$$\frac{\partial\phi}{\partial l} = \frac{\partial\phi_0}{\partial l} + \frac{\partial\phi_1}{\partial l} = L' + \frac{\partial\phi_1}{\partial l}; \text{ etc.}$$

A Taylor expansion can be made of both sides of the above equation. For example, to first order

$$F_0\left(\frac{\partial \phi}{\partial l}\right) = F_0\left(L' + \frac{\partial \phi_1}{\partial l}\right) \sim F\,(L') + \frac{\partial F_0}{\partial L'}\frac{\partial \phi_1}{\partial l}$$

After the expansion is completed, equating forms of equal order lead to two equations

$$F_0 = F_0' \qquad (6\text{-a})$$

and

$$\frac{\partial F_0}{\partial L'}\frac{\partial \phi_1}{\partial l} + F_1 = F_1' \qquad (6\text{-b})$$

Now F_0 represents the Hamiltonian of the undisturbed system and F_1 represents the first-order perturbation to the Hamiltonian, so from the previous section

$$F_0 = \frac{\mu^2}{2L^2}$$

$$F_1 = R - \nu H$$

Now $\frac{\partial F_0}{\partial L'} = \frac{\partial F_0'}{\partial L'} = -\frac{\mu^2}{L'^3} = -n'$. Thus equation (6-b) becomes

$$n'\frac{\partial \phi_1}{\partial l} = F_1 - F_1' \qquad (7)$$

$F_1 = R - \nu H$ can be written as the sum of two terms

$F_{1_p} = \left(\frac{\mu}{L^2}\right)^3 (S_1P_1 + S_2P_2 + S_3P_3)$ and $F_{1_s} = \left(\frac{\mu}{L^2}\right)^3 S_1 - \nu H$, where the term is periodic in l and the latter term is independent of l. Likewise, equation (7) can be divided into

$$F_{1_s} = F_1'$$

and

$$n' \frac{\partial \phi_1}{\partial l} = F_{1_p} \tag{8}$$

The first equation here together with equation (6-a) gives the new Hamiltonian

$$F' = F_0 + F_{1_s} \tag{8-a}$$

where in the right-hand side the old coordinates are to be replaced by the new coordinates using equation (5). But first equation (8) must be solved for ϕ_1. Equation (8) is to be considered as an ordinary differential equation in the single variable, l; hence,

$$\phi_1 = \frac{1}{n'} \int F_{1_p}\, dl$$

Examining the form of F_{1_p} shows that three integrals will be required.

$$\int P_1\, dl = \int \left[\left(\frac{a}{r}\right)^3 - \left(\frac{L}{G}\right)^3\right] dl$$

$$\int P_2\, dl = \int \left(\frac{a}{r}\right)^3 \cos 2v\, dl$$

$$\int P_3\, dl = \int \left(\frac{a}{r}\right)^3 \sin 2v\, dl$$

These integrals are readily evaluated by changing the variable of integration from l to v by means of $dl = (L/G)(r/a)^2 dv$ and then substituting $(a/r) = (L/G)^2(1 + e\cos v)$. The results are

$$\int P_1\, dl = \left(\frac{L}{G}\right)^3 \left[v + e \sin v - l\right]$$

$$\int P_2\, dl = \left(\frac{L}{G}\right)^3 \left[\frac{1}{2}\sin 2v + \frac{e}{2}\sin v + \frac{e}{6}\sin 3v\right]$$

$$\int P_3\, dl = -\left(\frac{L}{G}\right)^3 \left[\frac{1}{2}\cos 2v + \frac{e}{2}\cos v + \frac{e}{6}\cos 3v\right] \qquad (9)$$

For substitution into ϕ, L, G, and H must be replaced by L', G', and H'. Thus

$$\begin{aligned}\phi = L'l + G'g + H'h + \frac{1}{10}\left(\frac{\mu^2R^2}{G'^3}\right)\Bigg[\Bigg[&\left\{\epsilon_1{}^2 + \epsilon_2{}^2 - \frac{3}{2}\epsilon_1{}^2\left[\sin^2 h + \left(\frac{H'}{G'}\right)^2\cos^2 h\right]\right.\\ &\left.- \frac{3}{2}\epsilon_2{}^2\left[1 - \left(\frac{H'}{G'}\right)^2\right]\right\}\left\{v' - l + e'\sin v'\right\} + \left\{\frac{3}{2}\epsilon_2{}^2\left[1 - \left(\frac{H'}{G'}\right)^2\right]\right.\\ &\left.- \frac{3}{2}\epsilon_1{}^2\left[\sin^2 h - \left(\frac{H'}{G'}\right)^2\cos^2 h\right]\right\}\left\{\frac{1}{2}\sin(2v' + 2g) + \frac{e'}{2}\sin(v' + 2g)\right.\\ &\left.+ \frac{e'}{6}\sin(3v' + 2g)\right\} + \left\{3\epsilon_1{}^2\left(\frac{H'}{G'}\right)\sin h\cos h\right\}\left\{\frac{1}{2}\cos(2v' + 2g)\right.\\ &\left.+ \frac{e'}{2}\cos(v' + 2g) + \frac{e'}{6}\cos(3v' + 2g)\right\}\Bigg]\Bigg]\end{aligned} \qquad (10)$$

The primes on e and v indicate that these quantities must be considered as functions of L' and G' when the partials of ϕ are taken. The dependence of e on L and G follows immediately from $e^2 = 1 - \left(\frac{G}{L}\right)^2$, so

$$\frac{\partial e}{\partial L} = \frac{G^2}{eL^3} \quad \text{and} \quad \frac{\partial e}{\partial G} = -\frac{G}{eL^2}$$

The dependence of v on L and G is slightly more complicated; however, if v is first considered to be a function of e and l we have from Kepler's equation, $l = E - e \sin E$, that

$$0 = \frac{\partial E}{\partial e}(1 - e \cos E) - \sin E$$

or

$$\frac{\partial E}{\partial e} = \frac{\sin E}{1 - e \cos E} = \frac{a}{r} \sin E$$

The eccentric and true anomalies are related by

$$\cos E = \frac{\cos v + e}{1 + e \cos v} \quad \text{and} \quad \sin E = \frac{\sqrt{1 - e^2}\sin v}{1 + e \cos v}$$

Taking the partial of the first expression with respect to e gives

$$-\sin E \frac{\partial E}{\partial e} = \frac{\sin^2 v - (1 - e^2)\sin v \frac{\partial v}{\partial e}}{(1 + e \cos v)^2}$$

Substituting for $\frac{\partial E}{\partial e}$ and $\sin E$ from above and simplifying yields

$$\frac{\partial v}{\partial e} = \frac{\sin v(2 + e \cos v)}{1 - e^2}$$

Now

$$\frac{\partial v}{\partial L} = \frac{\partial v}{\partial e}\frac{\partial e}{\partial L} = \frac{\sin v(2 + e \cos v)}{eL}$$

and likewise

$$\frac{\partial v}{\partial G} = -\frac{\sin v(2 + e\cos v)}{eG}$$

The relation between the old and new coordinates can now be derived by a straightforward application of equations (5) to the generating function, ϕ. The results after some simplification are equations (11).

$$L = L' + \frac{n'R^2}{10}\Bigg[\Bigg[\Big\{\epsilon_1{}^2 + \epsilon_2{}^2 - \tfrac{3}{2}\epsilon_1{}^2\Big[\sin^2 h + \Big(\frac{H'}{G'}\Big)^2\cos^2 h\Big] - \tfrac{3}{2}\epsilon_2{}^2\Big[1 - \Big(\frac{H'}{G'}\Big)^2\Big]\Big\}\Big\{\Big(\frac{a}{r}\Big)^3 - \Big(\frac{L'}{G'}\Big)^3\Big\} + \Big\{\tfrac{3}{2}\epsilon_1{}^2\Big[\Big(\frac{H'}{G'}\Big)^2\cos^2 h - \sin^2 h\Big]$$

$$+ \tfrac{3}{2}\epsilon_2{}^2\Big[1 - \Big(\frac{H'}{G'}\Big)^2\Big]\Big\}\Big\{\Big(\frac{a}{r}\Big)^3\cos(2v + 2g)\Big\} - \Big\{\tfrac{3}{2}\epsilon_1{}^2\Big(\frac{H'}{G'}\Big)\Big(\frac{a}{r}\Big)^3\sin 2h\,\sin(2v + 2g)\Big\}\Bigg]\Bigg]$$

$$G = G' + \frac{3n'R^2}{20}\Big(\frac{L'}{G'}\Big)^3\Bigg[\Bigg[\Big\{\epsilon_1{}^2\Big[\Big(\frac{H'}{G'}\Big)^2\cos^2 h - \sin^2 h\Big] + \epsilon_2{}^2\Big[1 - \Big(\frac{H'}{G'}\Big)^2\Big]\Big\}\Big\{\cos(2v + 2g) + e\cos(v + 2g) + \frac{e}{3}\cos(3v + 2g)\Big\}$$

$$- \Big\{\epsilon_1{}^2\Big(\frac{H'}{G'}\Big)\sin 2h\Big\}\Big\{\sin(2v + 2g) + e\sin(v + 2g) + \frac{e}{3}\sin(3v + 2g)\Big\}\Bigg]\Bigg]$$

$$H = H' - \frac{3\epsilon_1{}^2 n'R^2}{20}\Big(\frac{L'}{G'}\Big)^3\Bigg[\Bigg[\sin 2h\Big\{\Big[1 - \Big(\frac{H'}{G'}\Big)^2\Big]\Big[v - l + e\sin v\Big] + \Big[1 + \Big(\frac{H'}{G'}\Big)^2\Big]\Big[\tfrac{1}{2}\sin(2v + 2g) + \frac{e}{2}\sin(v + 2g) + \frac{e}{6}\sin(3v + 2g)\Big]\Big\}$$

$$+ \cos 2h\Big\{\Big(\frac{H'}{G'}\Big)\Big[\cos(2v + 2g) + e\cos(v + 2g) + \frac{e}{3}\cos(3v + 2g)\Big]\Big\}\Bigg]\Bigg]$$

$$l = l' - \frac{n'R^2}{10}\Big(\frac{L'}{G'}\Big)^3\Bigg[\Bigg[\Big\{\epsilon_1{}^2 + \epsilon_2{}^2 - \tfrac{3}{2}\epsilon_1{}^2\Big[\sin^2 h + \Big(\frac{H'}{G'}\Big)^2\cos^2 h\Big] - \tfrac{3}{2}\epsilon_2{}^2\Big[1 - \Big(\frac{H'}{G'}\Big)^2\Big]\Big\}\Big\{\frac{\sin v}{eL'}\Big\}\Big(3 + 3e\cos v - e^2\sin^2 v\Big) + \Big\{\tfrac{3}{2}\epsilon_1{}^2\Big[\Big(\frac{H'}{G'}\Big)^2\cos^2 h - \sin^2 h\Big]$$

$$+ \tfrac{3}{2}\epsilon_2{}^2\Big[1 - \Big(\frac{H'}{G'}\Big)^2\Big]\Big\}\Big\{\Big[\frac{\sin v(2 + e\cos v)}{eL'}\Big]\Big[\cos 2(v + 2g) + \frac{e}{2}\cos(v + 2g) + \frac{e}{2}\cos(3v + 2g)\Big] + \Big[\tfrac{1}{2}\sin(v + 2g) + \tfrac{1}{6}\sin(3v + 2g)\Big]\Big[\frac{1}{eL'}\Big(\frac{G'}{L'}\Big)^2\Big]\Big\}$$

$$- \Big\{3\epsilon_1{}^2\Big(\frac{H'}{G'}\Big)\sin h\cos h\ \Big\{\Big[\sin(2v + 2g) + \frac{e}{2}\sin(v + 2g) + \frac{e}{2}\sin(3v + 2g)\Big]\Big[\frac{\sin v}{eL'}(2 + e\cos v)\Big] - \Big[\frac{1}{eL'}\Big(\frac{G'}{L'}\Big)^2\Big]\Big[\tfrac{1}{2}\cos(v + 2g) + \tfrac{1}{6}\cos(3v + 2g)\Big]\Big\}\Big\}\Bigg]\Bigg]$$

$$g = g' + \frac{3n'R}{100'}\Big(\frac{L'}{G'}\Big)^3\Bigg[\Bigg[\Big\{\epsilon_1{}^2 + \epsilon_2{}^2 - \tfrac{3}{2}\epsilon_1{}^2\Big[\sin^2 h + \Big(\frac{H'}{G'}\Big)^2\cos^2 h\Big] - \tfrac{3}{2}\epsilon_2{}^2\Big[1 - \Big(\frac{H'}{G'}\Big)^2\Big]\Big\}\Big(v - l + e\sin v\Big) + \Big\{\tfrac{3}{2}\epsilon_2{}^2\Big[1 - \Big(\frac{H'}{G'}\Big)^2\Big] - \tfrac{3}{2}\epsilon_1{}^2\Big[\sin^2 h$$

$$- \Big(\frac{H'}{G'}\Big)^2\cos^2 h\Big]\Big\}\Big\{\tfrac{1}{2}\sin(2v + 2g) + \frac{e}{2}\sin(v + 2g) + \frac{e}{6}\sin(3v + 2g)\Big\} + \Big\{3\epsilon_1{}^2\Big(\frac{H'}{G'}\Big)\sin h\cos h\Big\}\Big\{\tfrac{1}{2}\cos(2v + 2g) + \frac{e}{2}\cos(v + 2g) + \frac{e}{6}\cos(3v + 2g)\Big\}\Big]$$

$$- \frac{n'R^2}{10}\Big(\frac{L'}{G'}\Big)^3\Big[\Big\{\frac{3}{G'}\Big(\frac{H'}{G'}\Big)^2(\epsilon_1{}^2\cos^2 h - \epsilon_2{}^2)\Big\}\Big\{v - l + e\sin v - \tfrac{1}{2}\sin(2v + 2g) - \frac{e}{2}\sin(v + 2g) - \frac{e}{6}\sin(3v + 2g)\Big\} - \Big\{\frac{3\epsilon_1{}^2}{G'}\Big(\frac{H'}{G'}\Big)\sin h\cos h\Big\}\Big\{\tfrac{1}{2}\cos(2v + 2g)$$

$$+ \frac{e}{2}\cos(v + 2g) + \frac{e}{6}\cos(3v + 2g)\Big\}\Big] + \frac{n'R^2}{10}\Big(\frac{L'}{G'}\Big)^4\Big[\Big\{\epsilon_1{}^2 + \epsilon_2{}^2 - \tfrac{3}{2}\epsilon_1{}^2\Big[\sin^2 h + \Big(\frac{H'}{G'}\Big)^2\cos^2 h\Big] - \tfrac{3}{2}\epsilon_2{}^2\Big[1 - \Big(\frac{H'}{G'}\Big)^2\Big]\Big\}\Big\{\frac{\sin v}{eL'}\Big\}\Big(3 + 3e\cos v - e^2\sin^2 v\Big)$$

$$+ \Big\{\tfrac{3}{2}\epsilon_1{}^2\Big[\Big(\frac{H'}{G'}\Big)^2\cos^2 h - \sin^2 h\Big] + \tfrac{3}{2}\epsilon_2{}^2\Big[1 - \Big(\frac{H'}{G'}\Big)^2\Big]\Big\}\Big\{\Big[\frac{\sin v(2 + e\cos v)}{eL'}\Big]\Big[\cos(2v + 2g) + \frac{e}{2}\cos(v + 2g) + \frac{e}{2}\cos(3v + 2h)\Big] + \Big[\tfrac{1}{2}\sin(v + 2g)$$

$$+ \tfrac{1}{6}\sin(3v + 2g)\Big]\Big[\frac{1}{eL'}\Big(\frac{G'}{L'}\Big)^2\Big]\Big\} - \Big\{3\epsilon_1{}^2\Big(\frac{H'}{G'}\Big)\sin h\cos h\Big\}\Big\{\Big[\sin(2v + 2g) + \frac{e}{2}\sin(v + 2g) + \frac{e}{2}\sin(3v + 2g)\Big]\Big[\frac{\sin v}{eL'}(2 + e\cos v)\Big]$$

$$- \Big[\frac{1}{eL'}\Big(\frac{G'}{L'}\Big)^2\Big]\Big[\tfrac{1}{2}\cos(v + 2g) + \tfrac{1}{6}\cos(3v + 2g)\Big]\Big\}\Bigg]\Bigg]$$

$$h = h' + \frac{3n'R^2}{100'}\Big(\frac{L'}{G'}\Big)^3\Big(\frac{H'}{G'}\Big)\Bigg[\Bigg[\Big(\epsilon_1{}^2\cos^2 h - \epsilon_2{}^2\Big)\Big\{v - l + e\sin v - \tfrac{1}{2}\sin(2v + 2g) - \frac{e}{2}\sin(v + 2g) - \frac{e}{6}\sin(3v + 2g)\Big\} - \Big\{\epsilon_1{}^2\Big(\frac{G'}{H'}\Big)\sin h\cos h\Big\}\Big\{\tfrac{1}{2}\cos(2v + 2g)$$

$$+ \frac{e}{2}\cos(v + 2g) + \frac{e}{6}\cos(3v + 2g)\Big\}\Bigg]\Bigg] \qquad (11)$$

Equations (11) give the complete relationships between the old and new coordinates. The primed coordinates are by construction independent of the position in orbit, that is, they have no short-period variations. These equations cannot, in general, be solved for the unprimed in terms of the primed elements because the equations are highly nonlinear, and in practice for a first-order solution, all elements on the right-hand sides of the above are generally considered as primed elements.

At the present time, the short-period variations are of little practical importance because they would be unobservable in the motion of a lunar satellite as seen from the earth. In fact, most operational orbit determination schemes for earth satellites do not consider the short-period variations due to the earth's oblateness. Of more importance, are the secular and long-period terms. These terms could be derived by using von Zeipel's method in much the same manner as was done for the short-period terms. However, an alternate procedure yields the same results with less effort.

First, the new set of equations of motion for the primed coordinates is

$$\frac{dL'}{dt} = \frac{\partial F'}{\partial l'} \qquad \frac{dl'}{dt} = -\frac{\partial F'}{\partial L'}$$

$$\frac{dG'}{dt} = \frac{\partial F'}{\partial g'} \qquad \frac{dg'}{dt} = -\frac{\partial F'}{\partial G'}$$

$$\frac{dH'}{dt} = \frac{\partial F'}{\partial h'} \qquad \frac{dh'}{dt} = -\frac{\partial F'}{\partial H'}$$

where F' is given by equation (8-a) as

$$F' = F_0 + F_{1S} = \frac{\mu^2}{2L'^2} + \left(\frac{\mu}{L'G'}\right)^3 S_1 - \nu H'$$

where S_1 is a function of h', H', and G'. Since F' is independent of l' and g', it is seen that L' and G' do not have long-period or secular variations. This result might have been anticipated since L' and G' are measures of the total energy and angular momentum, respectively. If S_1 is expanded according to its definitions and the Hamiltonian is substituted into the equations of motion, the result is

$$\frac{dH'}{dt} = \frac{\alpha_2}{2}\left(\frac{\mu}{L'G'}\right)^3\left[1 - \left(\frac{H'}{G'}\right)^2\right]\sin 2h'$$

$$\frac{dl'}{dt} = n' + \frac{3}{L'}\left(\frac{\mu}{L'G'}\right)^3\left\{\alpha_1 + \frac{\alpha_2}{2}\left[\sin^2 h' + \cos^2 h'\left(\frac{H'}{G'}\right)^2\right] + \frac{\alpha_3}{2}\left[1 - \left(\frac{H'}{G'}\right)^2\right]\right\}$$

$$\frac{dg'}{dt} = \frac{3}{G'}\left(\frac{\mu}{L'G'}\right)^2\left\{\alpha_1 + \frac{\alpha_2}{2}\left[\sin 2h' + \left(\frac{H'}{G'}\right)^2\cos^2 h'\right] + \frac{\alpha_3}{2}\left[1 - \left(\frac{H'}{G'}\right)^2\right]\right\}$$
$$+ \frac{1}{G'}\left(\frac{\mu}{L'G'}\right)^3\left[\alpha_2 \cos^2 h' - \alpha_3\right]\left[\left(\frac{H'}{G'}\right)^2\right]$$

$$\frac{dh'}{dt} = \nu - \frac{1}{H'}\left(\frac{\mu}{L'G'}\right)^3\left[\alpha_2 \cos^2 h' - \alpha_3\right]\left(\frac{H'}{G'}\right)^2$$

Considering the last equation and recalling the orders of magnitude of the various terms it is seen that the first term ν is at least 10 times greater than the second term. Hence h' increases approximately linearly with time. As a first approximation to the integrals of these equations we assume that h' varies as νt and the other parameters on the right-hand side are constants. The integrals are then

$$\Delta H' = -\frac{\alpha_2}{4\nu}\left(\frac{\mu}{L'G'}\right)^3 \cos 2h' \sin^2 i'$$

$$\Delta l' = n't + \frac{3}{L'}\left(\frac{\mu}{L'G'}\right)^3\left[\alpha_1 + \frac{\alpha_2}{4}(1 + \cos^2 i') + \frac{\alpha_3}{2}\sin^2 i'\right]t$$

$$- \frac{3}{L'}\left(\frac{\mu}{L'G'}\right)^3\left(\frac{\alpha_2}{8\nu}\right)\sin^2 i' \sin 2h'$$

$$\Delta g' = \frac{3}{G'}\left(\frac{\mu}{L'G'}\right)^3\left[\alpha_1 + \frac{\alpha_2}{4}(1 + \cos^2 i') + \frac{\alpha_3}{2}\sin^2 i' + \left(\frac{\alpha_2}{6} - \frac{\alpha_3}{3}\right)\cos^2 i'\right]t$$

$$+ \frac{1}{G'}\left(\frac{\mu}{L'G'}\right)^3 \frac{\alpha_2}{4\nu}\sin 2h' \cos^2 i' - \frac{3\alpha_2}{8G'\nu}\left(\frac{\mu}{L'G'}\right)^3 \sin^2 i' \sin 2h'$$

$$\Delta h = \nu t - \left[\frac{1}{H'}\left(\frac{\mu}{L'G'}\right)^3\left[\frac{\alpha_2}{2} - \alpha_3\right]\cos^2 i'\right]t - \frac{1}{H'}\left(\frac{\mu}{L'G'}\right)^3 \frac{\alpha_2}{4\nu}\sin 2h' \cos^2 i'$$

where Δ represents the change in the variable since zero time which is taken at any convenient periselenian passage, also the inclination has been introduced in the form $\cos i' = \frac{H'}{G'}$. It should be noted that these equations could have been integrated in a somewhat more accurate form by assuming L', G', and H' constant in the $\frac{dh'}{dt}$ expression as before but not assuming that h' varies linearly with time. The resulting expression for h' could have then been substituted into the remaining equations and they could have been integrated exactly. However, this procedure leads to terms which are not in the standard forms of secular or periodic terms. Hence, at the price of some accuracy we maintain a better physical insight into the results.

Introducing the more conventional

$$J = \frac{3}{10}\epsilon_2^2 \quad \text{and} \quad K = \frac{3}{10}\epsilon_1^2$$

the equation becomes

$$\left.\begin{aligned}
\Delta H' &= \left\{\frac{1}{4}\left(\frac{n'}{\nu}\right)(1 - e'^2)^{-3/2}\, n'R^2K \sin^2 i'\right\} \cos 2h' \\[2ex]
\Delta l' &= n't\left\{1 + (1 - e'^2)^{-3/2}\left(\frac{R}{a'}\right)^2\left[-\frac{J}{2}(2 - 3\sin^2 i') + \frac{K}{4}(1 - 3\cos^2 i')\right]\right\} \\
&\quad - \left\{\frac{3}{8}(1 - e'^2)^{-3/2}\left(\frac{R}{a'}\right)^2\left(\frac{n'}{\nu}\right)K \sin^2 i'\right\} \sin 2h' \\[2ex]
\Delta g' &= \left\{(1 - e'^2)^{-2}\left(\frac{R}{a'}\right)^2\left[\frac{1}{2}J(4 - 5\sin^2 i') + \frac{1}{4}K(1 - 5\cos^2 i')\right]\right\} n't \\
&\quad + \left\{\frac{K}{8}\left(\frac{n'}{\nu}\right)(1 - e'^2)^{-2}\left(\frac{R}{a'}\right)^2(5\cos^2 i' - 3)\right\} \sin 2h' \\[2ex]
\Delta h' &= \left\{+\left(\frac{\nu}{n'}\right) - (1 - e'^2)^{-2}\left(\frac{R}{a'}\right)^2 \cos i'\left(J - \frac{K}{2}\right)\right\} n't \\
&\quad + \left\{\frac{K}{4}(1 - e'^2)^{-2}\left(\frac{n'}{\nu}\right)\left(\frac{R}{a'}\right)^2 \cos i'\right\} \sin 2h'
\end{aligned}\right\} \tag{12}$$

The coefficients of $(n't)$ in braces when multiplied by 2π represent the secular change in each variable per orbital period. The coefficients of $\begin{pmatrix} \sin 2h \\ \cos 2h \end{pmatrix}$ in braces represent the amplitudes of the long-period variations.

IX. CONCLUDING REMARKS

Equations (11) and (12) represent the complete first-order solution to the motion of an artificial satellite of the moon under the moon's noncentral gravitational field. In performing the mathematical manipulations required to derive these equations there are numerous opportunities to make errors. However, in an effort to reduce the number of errors to a minimum, hopefully to zero, the analysis was carried through a number of times until successive results were identical. In addition, if e_1^2 is set equal to zero the short-period terms reduce to the same forms that Brouwer (2) gives for the earth satellite's motion and the secular terms also reduce to well-known relationships.

Some interesting comparisons can be made between lunar and earth satellite motions. First, for earth satellites no long-period terms appear in the first-order analysis whereas for lunar satellites they play a fundamental role. Secondly, one of the interesting problems in earth satellite analyses is the "critical inclination" problem. This problem arises because the secular component of Δg vanishes if $\sin^2 i = \frac{4}{5}$. The motion near the critical inclination has been investigated by a number of authors and it has been shown that the motion of the apsidal line is similar to the motion of a simple pendulum. One might anticipate that the inclusion of the additional term in the lunar satellite potential might produce a secular term in Δg which does not vanish at $i = \sin^{-1}\left(\frac{4}{5}\right)^{1/2}$; however, it is seen from the equation that this is incorrect. Hence, the second-order analysis of the motion of the apsidal line might produce some interesting results.

To give some indications of the magnitudes of the various disturbances involved we consider a lunar satellite with the following characteristics:

$$a = 1222 \text{ miles} \quad e = 0.1 \quad i = 30^\circ$$

These conditions give L, G, and H and we let l, g, and h be such that we get the maximum perturbation. To give a better physical feeling for the numbers involved, changes in L, G, and H have been converted into changes in a, e, and i by using the proper relationships among the variables. Substituting the above characteristics into equations (11) and (12) along with the appropriate values for the constants involved leads to the following approximate amplitudes and rates:

	Short-period amplitudes	Secular variation per orbital period	Long-period amplitudes
l°	0.1	1.0	0.4
g°	.1	1.0	.5
h°	.1	.07	.04
a, miles	4.0	0	0
e	.002	0	0
i°	.04	0	.4

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XII. APPENDIX A

Solution of the Two-Body Problem by the Hamilton-Jacobi Equation

In this appendix the solution of the two-body problem will be demonstrated by the Hamilton-Jacobi technique and the relations between the canonical constants and the Lagrangian constants will be derived. In the usual spherical coordinates (r, θ, ϕ), the kinetic energy per unit mass is

$$T = \frac{1}{2}\left[\dot{r}^2 + r^2\cos^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2\right]$$

The momenta associated with each coordinate is given by

$$p_i = \frac{\partial T}{\partial \dot{q}_i}$$

or

$$p_r = \dot{r}, \quad p_\phi = r^2\cos^2\theta\dot{\phi}, \text{ and } \quad p_\theta = r^2\dot{\theta}$$

In terms of these momenta

$$T = \frac{1}{2}\left[p_r{}^2 + \frac{p_\phi{}^2}{r^2\cos^2\theta} + \frac{p_\theta{}^2}{r^2}\right]$$

The potential for this system is $-\frac{\mu}{r}$ and hence the Hamiltonian of the system is

$$H(q_i,p_i) = \frac{1}{2}\left[p_r{}^2 + \frac{p_\phi{}^2}{r^2\cos^2\theta} + \frac{p_\theta{}^2}{r^2}\right] - \frac{\mu}{r}$$

where μ is the product of the gravitational constant and the total mass of the two bodies. The Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2\cos^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2\right] - \frac{\mu}{r} = 0$$

This equation does not contain t or ϕ explicitly and can be separated by the substitution

$$S = -\alpha_1 t + \alpha_3\phi + S_r(r) + S_\theta(\theta)$$

into two equations

$$\left(\frac{dS_r}{dr}\right)^2 = 2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}$$

$$\left(\frac{dS_\theta}{d\theta}\right)^2 = \alpha_2^2 - \frac{\alpha_3^2}{\cos^2\theta}$$

Thus

$$S = -\alpha_1 t + \alpha_3\phi + \int_{r_p}^{r} \sqrt{2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2}}\, dr + \int_0^\theta \sqrt{\alpha_2^2 - \frac{\alpha_3^2}{\cos^2\theta}}\, d\theta$$

where r_p is the smallest root of

$$2\alpha_1 + \frac{2\mu}{r} - \frac{\alpha_2^2}{r^2} = 0$$

The integration will not be made at this point for reasons that will be clear later.

The principle function affects a transformation of variables such that the new coordinates and new momenta are constants. If the

Hamilton-Jacobi equation is solved in the form

$$S = S(r,\ \theta,\ \phi,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ t)$$

then the relations between the new and old variables are given by

$$p_i = \frac{\partial s}{\partial q_i} \qquad\qquad \beta_i = \frac{\partial s}{\partial \alpha_i}$$

The second of these sets of equations gives

$$\beta_1 = \frac{\partial s}{\partial \alpha_1} = -\ t + \frac{\partial s_r}{\partial \alpha_1}$$

$$\beta_2 = \frac{\partial s}{\partial \alpha_2} = \frac{\partial s_r}{\partial \alpha_2} + \frac{\partial s_\theta}{\partial \alpha_2}$$

$$\beta_3 = \frac{\partial s}{\partial \alpha_3} = \phi + \frac{\partial s_\theta}{\partial \alpha_3}$$

To obtain explicit relations for the β's it is convenient to differentiate under the integral signs recalling that the method of finding r_p implies that r_p is a function of α_1 and α_2. Thus the first integral becomes

$$t + \beta_1 = \int_{r_p}^{r} \frac{r\,dr}{\sqrt{2\alpha_1 r^2 + 2\mu r - \alpha_2^2}} - r^{-1}\sqrt{2\alpha_1 r^2 + 2\mu r - \alpha_2^2}\,\frac{\partial r_p}{\partial \alpha_1}\Bigg|_{r_p}$$

$$= \int_{r_p}^{r} \frac{r\,dr}{\sqrt{2\alpha_1 r + 2\mu r - \alpha_2^2}}$$

by the fact that r_p is a root of the radical. At this point it is convenient to introduce the eccentric anomaly defined by

$$r = a(1 - e \cos E)$$

where

$$a = -\frac{\mu}{2\alpha_1} \quad \text{and} \quad a(1 - e^2) = \frac{\alpha_2^2}{\mu}$$

The integral then reduces to

$$t + \beta_1 = \sqrt{\frac{a^3}{\mu}} \int_0^E (1 - e \cos E)dE = \sqrt{\frac{a^3}{\mu}} (E - e \sin E)$$

that is, the familiar Kepler's equation. The relation between α_1, α_2, and β_1 and the three Lagrangian elements a, e, τ (time of periapse passage) is also demonstrated, namely

$$\alpha_1 = -\frac{\mu}{2a}, \quad \alpha_2 = \sqrt{\mu a(1 - e^2)}, \quad \beta_1 = -\tau$$

The third element β_3 is next evaluated.

$$\beta_3 = \phi - \alpha_3 \int_0^\theta \frac{\sec^2\theta \, d\theta}{\sqrt{\alpha_2^2 - \alpha_3^2\sec^2\theta}}$$

$$= \phi - \int_0^\theta \frac{\sec^2\theta \, d\theta}{\sqrt{\frac{\alpha_2^2 - \alpha_3^2}{\alpha_3^2} - \tan^2\theta}}$$

Thus $\alpha_2^2 \geq \alpha_3^2$ is required for real values of the integral. Define a new angle λ such that

$$\tan^2\lambda = \frac{\alpha_2^2 - \alpha_3^2}{\alpha_3^2}$$

then

$$\beta_3 = \phi - \sin^{-1}\left(\frac{\tan\theta}{\tan\lambda}\right)$$

or

$$\sin(\phi - \beta_3) = \tan\theta \cot\lambda$$

Thus when $\theta = 0$, $\phi = \beta_3$ and therefore $\beta_3 = \Omega$ is the longitude of the ascending node. Also $\theta_{max} = \lambda$ for $\left|\sin(\phi - \beta_3)\right| \leq 1$. Since the motion is planar θ_{max} is the inclination of the orbital plane to the fundamental plane, that is, $\theta_{max} = \lambda = i$ and we have related two more elements

$$\beta_3 = \Omega, \cos i = \sqrt{\frac{1}{1 + \tan^2 i}} = \frac{\alpha_3}{\alpha_2}$$

Finally, the integral for β_2 is evaluated

$$\beta_2 = \alpha_2 \int_0^\theta \frac{d\theta}{\sqrt{\alpha_2^2 - \alpha_3^2\sec^2\theta}} - \alpha_2 \int_{r_p}^r \frac{r^{-1}\,dr}{\sqrt{2\alpha_1 r^2 + 2\mu r - \alpha_2^2}}$$

$$- \frac{\partial r_p}{\partial \alpha_2}\left(\frac{\sqrt{2\alpha_1 r^2 + 2\mu r - \alpha_2^2}}{r}\right)\Bigg|_{r_p}$$

The last term is again zero, eliminating the α's in favor of $\cos i$ in the first integral and introducing the true anomaly v into the second integral by

$$r = \frac{a(1 - e^2)}{1 + e \cos v}$$

gives

$$\beta_2 = \sin^{-1}\left(\frac{\sin \theta}{\sin i}\right) - v$$

or

$$\sin(\beta_2 + v)\sin i = \sin \theta$$

Thus $\beta_2 = \omega$ is the argument of periapse and all of the canonical elements have been identified with the Lagrangian elements, namely

$$\alpha_1 = -\frac{\mu}{2a} \qquad \beta_1 = -\tau$$

$$\alpha_2 = \sqrt{\mu a(1 - e^2)} \qquad \beta_2 = \omega$$

$$\alpha_3 = \sqrt{\mu a(1 - e^2)}\cos i \qquad \beta_3 = \Omega$$

XIII. APPENDIX B

Derivation of the Perturbation Equations

There are essentially two methods of obtaining the perturbation equations. The earliest method, outlined by Brouwer (2), is due to Lagrange who used a standard variation of arbitrary constants technique resulting in Lagrangian brackets. A second method due to Hamilton utilizes the theory of contact transformations. The second approach will be given here because it demonstrates one of the advantages of canonical variables in simplifying the equations of motion.

A generating function $\phi(q_i, Q_j, t)$ defines a canonical transformation of variables if the new Hamiltonian is related to the old by

$$K = H + \frac{\partial\phi}{\partial t} \tag{B-1}$$

The relations between the new (P_i, Q_i) and old (p_i, q_i) coordinates are given by

$$p_i = \frac{\partial\phi}{\partial q_i} \qquad P_i = -\frac{\partial\phi}{\partial Q_i} \tag{B-2}$$

and the new equations of motion are

$$\dot{Q}_j = \frac{\partial K}{\partial P_j} \qquad \dot{P}_j = -\frac{\partial K}{\partial Q_j} \tag{B-3}$$

These relationships can be found in almost any book on classical mechanics Corben (3). An additional relation needed in what is to follow is

$$d\phi = p_i dq_i - P_i dQ_i$$

where the summation convention is invoked. We now turn to the derivation of the perturbation equations.

Consider a system with a Hamiltonian expressible as

$$H = H_0 + R \tag{B-5}$$

where R is called the disturbing function and H_0 is that part of the total Hamiltonian for which there exists a generating function which satisfies the Hamilton-Jacobi equation of appendix A. The equations of motion for equation (B-5) are from equation (B-3)

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{B-6}$$

Also we have solutions for H_0

$$q_i = q_i(\alpha_i, \beta_i, t) \qquad p_i = p_i(\alpha_i, \beta_i, t) \tag{B-7}$$

derivable from the function $S(q_i, \alpha_i, t)$ where S satisfies

$$\frac{\partial S}{dt} + H_0\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0 \tag{B-8}$$

and the relations between the coordinates are given by equation (B-2)

$$p_i = \frac{\partial S}{\partial q_i} \text{ and } \beta_i = \frac{\partial S}{\partial \alpha_i}$$

Now regard α_i and β_i as variables so that equation (B-7) satisfies equation (B-6). If now $Q_i = \alpha_i$ and $P_i = -\beta_i$ then

$$p_i dq_i - P_i dQ_i = p_i dq_i + \beta_i d\alpha_i = \frac{\partial S}{\partial q_i} dq_i + \frac{\partial S}{\partial \beta_i} d\alpha_i = dS$$

Thus by equation (B-4) S can be used to define a transformation in which the new Hamiltonian is

$$K = H + \frac{\partial S}{\partial t}$$

but by equation (B-8) $\frac{\partial S}{\partial t} = H_o$, therefore

$$K = H - H_o = R$$

The new equations of motion are given by equation (B-3) which in light of the definition of Q_1 and P_1 above become

$$\dot{\alpha}_1 = -\frac{\partial R}{\partial \beta_1} \quad \text{and} \quad \dot{\beta}_1 = \frac{\partial R}{\partial \alpha_1}$$

These are the perturbation equations for the variation of the arbitrary constants α_1 and β_1 derived in appendix A.

In the expansion of the disturbing function terms of the form

$$A \cos j(v + \omega) \qquad j = 1, 2, \ldots$$

will appear. This term can be expanded in terms of the canonical elements in the form of an infinite series

$$\Sigma\, B_K \cos\left[K\left(\frac{1}{\mu}\right)\sqrt{(-2\alpha_1)^3}(t + \beta_1) + \omega\right]$$

Hence the $\frac{\partial R}{\partial \alpha_1}$ will introduce terms with $(t + \beta_1)$ as the coefficient. These terms give the appearance of being secular; however, this may not be the case. Thus we seek a transformation of variables which will remove these unwanted ambiguous terms. The new variable is

$$l = \left(\frac{1}{\mu}\right)\sqrt{(-2\alpha_1)^3}(t + \beta_1) = l(\alpha_1, \beta_1, t)$$

that is, the mean anomaly. Kepler's equation gives the relation between the mean anomaly and the position in orbit

$$l = E - e \sin E$$

We need to find the momenta conjugate to this new variable. Consider the generating function

$$\phi = -\alpha_1 t + \mu(-2\alpha_1)^{-1/2}B_1 + \alpha_2 B_2 + \alpha_3 B_3$$

The new Hamiltonian is given by equation (B-1)

$$F = R - \alpha_1$$

The relations between the coordinates are given by equation (B-2)

$$\beta_1 = -t + \mu(-2\alpha_1)^{-3/2}B_1 \qquad A_1 = \mu(2\alpha_1)^{-1/2}$$

$$\beta_2 = B_2 \qquad A_2 = \alpha_2$$

$$\beta_3 = B_3 \qquad A_3 = \alpha_3$$

The new coordinates are

$$B_1 = \left(\frac{1}{\mu}\right)\sqrt{(-2\alpha_1)^3}(t + \beta_1) = l$$

$$B_2 = \beta_2 = \omega = g$$

$$B_3 = \beta_3 = \Omega = h$$

$$A_1 = \mu(-2\alpha_1)^{-1/2} = \sqrt{\mu a} = L$$

$$A_2 = \alpha_2 = \sqrt{\mu a(1 - e^2)} = G$$

$$A_3 = \alpha_3 = \sqrt{\mu a(1 - e^2)}\cos i = H$$

The set L, G, H, l, g, h are the Delauney elements.

The equations of motion are given by equation (B-3)

$$\frac{dl}{dt} = - \frac{\partial F}{\partial L} \qquad \frac{dL}{dt} = \frac{\partial F}{\partial l}$$

$$\frac{dg}{dt} = - \frac{\partial F}{\partial G} \qquad \frac{dG}{dt} = \frac{\partial F}{\partial g}$$

$$\frac{dh}{dt} = - \frac{\partial F}{\partial H} \qquad \frac{dH}{dt} = \frac{\partial F}{\partial h}$$

with

$$F = \frac{\mu^2}{2L^2} + R$$

THE MOTION OF A LUNAR SATELLITE UNDER THE INFLUENCE OF THE MOON'S NONCENTRAL FORCE FIELD

By

Robert H. Tolson

ABSTRACT

For lunar satellites within a few hundred miles of the lunar surface the primary cause of disturbance from pure Keplerian motion is the disturbing force due to the moon's noncentral gravitational field which, unlike the earth's field, is unsymmetrical and rotating. This thesis presents a first-order approximation to the motion of lunar satellites under the forces resulting from this field. The results are a set of equations which give the short period, long period and secular variations of a slightly modified set of Delaunay elements. A typical set of orbital characteristics are utilized to obtain an estimate of the order of magnitude of the variations in the various elements.