NUMERICAL COMPUTATION OF PERTURBATION SOLUTIONS OF NONAUTONOMOUS SYSTEMS

by

Jeng-Sheng Huang

Dissertation submitted to the Graduate Faculty of the

Virginia Polytechnic Institute & State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering Mechanics

APPROVED:

Dr. L. Meirovitch, Chairman

Dr. D. Frederick, Head

Dr. R. P. McNitt

Dr. J. E. Kaiser

Dr. C. B. Ling

May, 1977

Blacksburg, Virginia

ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation to his advisor, Dr. L. Meirovitch, whose guiding influence has contributed immeasurably to the author's development in research. The author also wishes to express his thanks to the other members of his committee for their advice and criticisms: Dr. D. Frederick, Dr. R. P. McNitt, Dr. C. B. Ling and Dr. J. E. Kaiser.

Finally, the author dedicates this thesis to his wife, ,
for her love and encouragement during his academic pursuits.

TABLE OF CONTENTS

1. Introduction

1.1 Literature Survey and Background

The motion of a dynamical system can be described by a set of n second-order Lagrange's equations or a set of 2n first-order Hamilton's equations. In general, Hamilton's equations represent a system of nonlinear nonautonomous differential equations of the form

$$\dot{\underset{\sim}{x}} = \underset{\sim}{X}(\underset{\sim}{x}, t) \tag{1.1}$$

where $\underset{\sim}{x}$ is the state vector or phase vector and $\underset{\sim}{X}$ is a vector of the same dimension as $\underset{\sim}{x}$. The first n components of $\underset{\sim}{x}$ represent generalized displacements and the remaining n components represent generalized velocities. The components of $\underset{\sim}{X}$ satisfy Lipschitz conditions in a given domain D.

Because a closed-form solution of Eq.(1.1) is difficult to obtain, quite often one seeks special solutions by perturbation and numerical methods. Perturbation techniques can be used to obtain analytic solutions of differential systems associated mainly with weakly nonlinear autonomous systems or weakly nonautonomous systems. A number of perturbation techniques seeking periodic solutions of nonautonomous systems of the type (1.1) are described in Refs.1-3. One of the most widely used ones is Lindstedt's method, which seeks periodic solutions of nonlinear systems in which the nonlinear terms may affect the frequency of the periodic solutions. The frequency possesses a certain degree of arbitrariness which is removed by forcing the solution to be periodic. Another method concerned with the existence of periodic solutions of a quasi-harmonic system was developed by Krylov, Bogoliubov and Mitropolsky (KBM). The KBM method also builds into the solution a certain degree of arbitrariness, enabling us to produce a periodic solution by removing the arbitrariness. Although the approach is substantially different from that of Lindstedt's method, the basic idea behind the KBM method is essentially the same. One of the most important perturbation techniques for the determination of periodic solutions of nonlinear differential equations containing a small parameter is the method of averaging. The method attempts to determine under what conditions one can perform a time varying change of variables which has the effect of reducing a nonautonomous differential system to an autonomous one.

Although perturbation methods have many advantages, they are restricted to weakly nonlinear and weakly nonautonomous systems. For this reason, in more general cases iteration methods or methods of successive approximations are often used to solve Eq.(1.1) (see for example, Refs. 4-6). One of the methods generally used is the method based on Taylor's series. The method develops the Taylor's series expansion of solutions about the ordinary point $t = t_o$. The development of the expansion requires the values of the solutions and their derivatives at $t = t_o$. The solutions converge in the interior of a well-defined circle. Another method is the method of successive approximations. The method of successive approximations consists of forming by successive iteration a sequence of functions tending to converge uniformly to the solution in every finite interval. The method is convergent or divergent depending on the choice of starting iterative values. If the starting values are close to the exact solutions, then the convergence of the method is fast. Note, however, that the error estimation is difficult to compute.

From reviewing the perturbation methods and the numerical iteration methods, we find that each method has some limitations in solving a general nonautonomous system. For this reason, Cesari (Ref. 7) studied the solution of Eq.(1.1) by Galerkin's approximations, which is a method often applied to cases in which an exact solution is not known to exist. He proved that even for a very low order of Galerkin's approximation one may be able to obtain an upper estimate for the difference between the actual and the approximate solutions. Cesari's process reduces the problem to the study of a finite system of transcendental equations, known as a determining system, in a finite dimensional Euclidean space. Urabe (Ref. 8) used Galerkin's procedure for nonlinear periodic systems. He proved that the existence of a Galerkin's approximation of a sufficiently high order always implies the existence of an exact solution of Eq.(1.1) lying in the interior of the domain D. More recently, Urabe and Reiter (Ref. 9) have shown that high-order Galerkin's approximations can be obtained in solving the determining equations by Newton's method in conjunction with a computer program. For systems of order 15-20, the Galerkin's approximation is sufficiently refined and the corresponding error bounds between the actual and the approximate solutions, to be determined, proves to be particularly small. Although Newton's method is the most widely known method for solving nonlinear algebraic equations, it is relatively complicated as it necessitates the Jacobian matrix of $X_i(x, t)$ namely $\left[\dfrac{\partial X_i}{\partial x_j}\right]$. Therefore, Brown (Ref. 10) modified the Newton's method by replacing the Jacobian matrix by the first difference quotient approximation. Brown's method is derivative-free and second-order convergence has been proven.

The procedure for obtaining the variational equation from Eq.(1.1) is described in Ref. 1. We shall be interested in the perturbation equations about periodic solutions.

1.2 Description of the Present Work

In this study, we apply the higher-order Galerkin's approximation to the nonautonomous periodic system (1.1), which describes the motion of a dynamical system, to obtain the periodic solution of the unperturbed motion. By using a higher-order Galerkin's approximation, we reduce the nonlinear periodic system to a set of nonlinear determining algebraic equations. Then, we apply Brown's method in conjunction with a computer program to obtain coefficients of Galerkin's approximation from the determining equations. Furthermore, we derive the differential equations of the perturbed motion in the neighborhood of approximate periodic solutions for the unperturbed motion. The differential system is a set of nonlinear nonhomogeneous differential equations. The system contains extraneous force functions $\varepsilon_i(t)$ due to the use of approximate periodic solutions instead of the actual solutions. The force functions $\varepsilon_i(t)$ may be estimated by a trigonometric polynomial of higher-order terms. The corresponding error bound of the forces $\varepsilon_i(t)$ determined proves to be small.

In general, since the perturbation functions are small, we can expand the perturbation functions into a uniformly convergent series. Then, as we introduce the convergent series into the nonlinear nonhomogeneous differential system of perturbed motion, the system reduces to a linear nonhomogeneous differential system. Methods for solving linear nonhomogeneous differential systems are presented in Refs. 1-2 and 9-11. First, we can obtain the fundamental solution of the homogeneous system corresponding to the linear nonhomogeneous system by integration. Second, we introduce the fundamental solution matrix into the linear nonhomogeneous system to form a set of integral equations. Finally, we obtain the solutions of the series of perturbation functions by solving integral equations numerically.

The method is illustrated by means of a specific example, namely, the van der Pol equation with a harmonic forcing term. A computer program is developed to obtain the approximate periodic solution and the perturbation solutions of perturbed motion. The error bound between the actual and the approximate solution, to be also determined, proves to be small. The computations have been carried out through the use of IBM 370/158 computer at Virginia Polytechnic Institute and State University.

2. Theoretical Formulation

Let us assume that, following discretization, the dynamical system can be represented by n degrees of freedom, so that its motion is described by the 2n first-order Hamilton's equations

$$\dot{x}_i = X_i(x_1, x_2, \ldots, x_{2n}, t) \quad , \quad i = 1,2,\ldots,2n \tag{2.1}$$

where X_i are generally nonlinear functions of the variables x_i $(i = 1,2,\ldots,2n)$ and of the time t. The first n components of x_i represent generalized displacements and the remaining n components represent generalized velocities.

Next, let us consider a special solution of Eqs.(2.1) and denote it by $\phi_i(t)$. Recognizing that $\phi_i(t)$ must satisfy Eqs.(2.1), we can write

$$\dot{\phi}_i(t) = X_i(\phi_1, \phi_2, \ldots, \phi_{2n}, t) \quad , \quad i = 1,2,\ldots,2n \tag{2.2}$$

A case of particular interest is that in which the solution is periodic, $\phi_i(t + T) = \phi_i(t)$. We shall refer to $\phi_i(t)$ as the unperturbed motion and explore the behavior of the system in the vicinity of the periodic solutions $\phi_i(t)$ $(i = 1,2,\ldots,2n)$. To this end, the general perturbed motion can be written in the form

$$x_i(t) = \phi_i(t) + y_i(t) \quad , \quad i = 1,2,\ldots,2n \tag{2.3}$$

where $y_i(t)$ are small perturbations about the solutions $\phi_i(t)$ $(i = 1, 2,\ldots,2n)$. Introducing Eqs.(2.3) into Eqs.(2.1), we obtain

$$\dot{\phi}_i(t) + \dot{y}_i(t) = X_i(\phi_1 + y_1, \phi_2 + y_2, \ldots, \phi_{2n} + y_{2n}, t) \qquad i = 1,2,\ldots,2n \tag{2.4}$$

so that, considering Eqs.(2.2), we can reduce Eqs.(2.4) to

$$\dot{y}_i(t) = X_i(\phi_1+y_1,\phi_2+y_2,\ldots,\phi_{2n}+y_{2n},t) - X_i(\phi_1,\phi_2,\ldots,\phi_{2n},t)$$

$$i = 1,2,\ldots,2n \qquad (2.5)$$

which are referred to as the differential equations of the perturbed motion. Equations (2.5) can be expressed in a different form. Let us expand the first term on the right side of Eqs.(2.5) in a Taylor's series about solutions $\phi_i(t)$ and obtain

$$X_i(\phi_1+y_1,\phi_2+y_2,\ldots,\phi_{2n}+y_{2n},t) = X_i(\phi_1,\phi_2,\ldots,\phi_{2n},t) + \sum_{j=1}^{2n} \left.\frac{\partial X_i}{\partial x_j}\right|_{\underset{\sim}{x}=\underset{\sim}{\phi}} y_j$$

$$+ 0_i(\underset{\sim}{y}^2) \quad , \quad i = 1,2,\ldots,2n \qquad (2.6)$$

where $\underset{\sim}{x}$, $\underset{\sim}{\phi}$, and $\underset{\sim}{y}$ are 2n-dimensional vectors associated with x_i, ϕ_i, and y_i ($i = 1,2,\ldots,2n$), respectively and $0_i(\underset{\sim}{y}^2)$ represent series consisting of nonlinear terms of degree equal to or larger than two in $y_i(t)$ ($i = 1,2,\ldots,2n$). Introducing the notation

$$a_{ij}(t) = \left.\frac{\partial X_i}{\partial x_j}\right|_{\underset{\sim}{x}=\underset{\sim}{\phi}} \quad , \quad i,\ j = 1,2,\ldots,2n \qquad (2.7)$$

and recalling Eqs.(2.6), Eqs.(2.5) can be rewritten as

$$\dot{y}_i(t) = \sum_{j=1}^{2n} a_{ij}(t)y_j(t) + 0_i(\underset{\sim}{y}^2) \quad , \quad i = 1,2,\ldots,2n \qquad (2.8)$$

where in general the coefficients $a_{ij}(t)$ are periodic, $a_{ij}(t+T) = a_{ij}(t)$. Note that Eqs.(2.8) are nonlinear due to the nonlinear terms $0_i(\underset{\sim}{y}^2)$.

A case of particular interest is that in which the perturbations $y_i(t)$ are sufficiently small to permit second-order terms in $y_i(t)$ to be ignored. In this case, Eqs.(2.8) can be approximated by

$$\dot{y}_i(t) = \sum_{j=1}^{2n} a_{ij}(t)y_j(t) \quad , \quad i = 1,2,\ldots,2n \tag{2.9}$$

which represent the first approximation equations, and are called variational equations (see Ref. 1).

Let us consider the case in which closed-form solutions $\phi_i(t)$ of Eqs.(2.1) are difficult to obtain, so that we shall be interested in an approximate solution. Because Eqs.(2.1) represent a nonlinear nonautonomous system, we can use Galerkin's approximations to obtain a numerical approximation for a periodic solution. Applications of Galerkin's procedure for nonlinear periodic differential systems are described in Refs. 7-9.

To determine an approximate periodic solution $\underset{\sim}{\phi}^*(t)$ of Eqs.(2.1), we consider a Galerkin's approximation in the form of a trigonometric polynomial

$$\underset{\sim}{\phi}^*(t) = \underset{\sim}{c}_o + \sum_{k=1}^{m} (\underset{\sim}{c}_{2k-1}\sin kt + \underset{\sim}{c}_{2k}\cos kt) \tag{2.10}$$

with undetermined coefficients $\alpha = (\underset{\sim}{c}_o, \underset{\sim}{c}_1, \underset{\sim}{c}_2, \ldots, \underset{\sim}{c}_{2m})$, where $\underset{\sim}{c}_o, \underset{\sim}{c}_1, \underset{\sim}{c}_2, \ldots, \underset{\sim}{c}_{2m}$ are vectors of the same 2n-dimension as $\underset{\sim}{x}$ and $\underset{\sim}{X}(\underset{\sim}{x}, t)$. Introducing Eq.(2.10) into (2.1), we determine the coefficients from the following equations

$$\frac{d\underset{\sim}{\phi}^*(t)}{dt} = \frac{1}{T}\int_o^T \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(s),\ s\right] ds + \frac{2}{T}\sum_{k=1}^{m}\left\{\int_o^T \cos k(t-s)\ \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(s),\ s\right] ds\right\} \tag{2.11}$$

Equation (2.11) in conjunction with Eq.(2.10) yields

$$\begin{aligned}
\underset{\sim}{F}_o(\alpha) &= \frac{1}{T}\int_o^T \underset{\sim}{X}\left[\ \underset{\sim}{\phi}^*(s),\ s\ \right] ds &&= 0\\
\underset{\sim}{F}_{2k-1}(\alpha) &= \frac{2}{T}\int_o^T \underset{\sim}{X}\left[\ \underset{\sim}{\phi}^*(s),\ s\ \right] \sin ks\ ds + k\underset{\sim}{c}_{2k} &&= 0 \qquad (2.12)\\
\underset{\sim}{F}_{2k}(\alpha) &= \frac{2}{T}\int_o^T \underset{\sim}{X}\left[\ \underset{\sim}{\phi}^*(s),\ s\ \right] \cos ks\ ds - k\underset{\sim}{c}_{2k-1} &&= 0
\end{aligned}$$

$$k = 1,2,\ldots,2m$$

where $\alpha = (\ \underset{\sim}{c}_o,\ \underset{\sim}{c}_1,\ \underset{\sim}{c}_2,\ldots,\ \underset{\sim}{c}_{2m}\ )$ is a matrix of undetermined coefficients, which can be obtained by solving the set of equations (2.12).

When m is sufficiently large, a trigonometric polynomial $\underset{\sim}{\phi}^*(t)$ determined by the relations (2.12) should provide a reasonable approximation of the actual solution $\underset{\sim}{\phi}(t)$. A trigonometric polynomial $\underset{\sim}{\phi}^*(t)$ satisfying the relations (2.11) and (2.12) is known as a Galerkin's approximation of order m; Eqs.(2.12) are called the determining equations of the mth Galerkin's approximation.

Next, let us denote the difference between the approximate solution and the actual solution by $\underset{\sim}{\delta}(t)$, so that in vector form we have

$$\underset{\sim}{\delta}(t) = \underset{\sim}{\phi}^*(t) - \underset{\sim}{\phi}(t) \tag{2.13}$$

Moreover, letting $\underset{\sim}{y}^*(t)$ be the perturbation from the approximate solution instead of the actual solution, the perturbed motion can be written in the vector form

$$\underset{\sim}{x}(t) = \underset{\sim}{\phi}^*(t) + \underset{\sim}{y}^*(t) \tag{2.14}$$

Inserting Eq.(2.14) into Eqs.(2.1), we obtain an expression similar to Eqs.(2.4)

$$\dot{\phi}_i^*(t) + \dot{y}_i^*(t) = X_i(\phi_1^* + y_1^*, \phi_2^* + y_2^*, \ldots, \phi_{2n}^* + y_{2n}^*, t)$$

$$i = 1,2,\ldots,2n \tag{2.15}$$

Therefore, expanding $X_i(\underset{\sim}{\phi}^* + \underset{\sim}{y}^*, t)$ in a Taylor's series about the approximate solution $\underset{\sim}{\phi}^*(t)$, we can reduce Eqs.(2.15) to

$$\dot{y}_i^*(t) = X_i(\phi_1^*, \phi_2^*, \ldots, \phi_{2n}^*, t) - \dot{\phi}_i^*(t) + \sum_{j=1}^{2n} \left.\frac{\partial X_i}{\partial x_j}\right|_{\underset{\sim}{x}=\underset{\sim}{\phi}^*} y_j^* + O_i(\underset{\sim}{y}^{*2})$$

$$i = 1,2,\ldots,2n \tag{2.16}$$

Unlike the case in which the expansion was about the actual solution $\underset{\sim}{\phi}(t)$, however, the first two terms on the right side of Eqs.(2.16) do not cancel out, because $\underset{\sim}{\phi}^*(t)$ is only an approximate solution of Eqs.(2.1). It will prove of interest to examine these terms very closely.

Assuming that the difference between the actual and the approximate solutions is relatively small, we can write the expansion of $X_i(\underset{\sim}{\phi}^*, t)$ about the actual solution $\underset{\sim}{\phi}(t)$ as

$$X_i(\phi_1^*,\phi_2^*,\ldots,\phi_{2n}^*,\ t) \cong X_i(\phi_1,\phi_2,\ldots,\phi_{2n},\ t) + \sum_{j=1}^{2n} a_{ij}(t)\delta_j(t)$$

$$i = 1,2,\ldots,2n \qquad (2.17)$$

where $a_{ij}(t)$ are defined in Eqs.(2.7) as

$$a_{ij}(t) = \left.\frac{\partial X_i}{\partial x_j}\right|_{\underset{\sim}{x}=\underset{\sim}{\phi}} \quad , \quad i,\ j = 1,2,\ldots,2n \qquad (2.18)$$

and are the actual coefficients; they are generally not known. Considering Eqs.(2.13) and (2.18), as well as Eqs.(2.2), we obtain

$$X_i(\phi_1^*,\phi_2^*,\ldots,\phi_{2n}^*,\ t) - \dot{\phi}_i^*(t) \cong \sum_{j=1}^{2n} a_{ij}(t)\delta_j(t) - \dot{\delta}_i(t)$$

$$i = 1,2,\ldots,2n \qquad (2.19)$$

Moreover, introducing the notation

$$\varepsilon_i(t) = \sum_{j=1}^{2n} a_{ij}(t)\delta_j(t) - \dot{\delta}_i(t) \quad , \quad i = 1,2,\ldots,2n \qquad (2.20)$$

and

$$a_{ij}^*(t) = \left.\frac{\partial X_i}{\partial x_j}\right|_{\underset{\sim}{x}=\underset{\sim}{\phi}^*} \quad , \quad i,\ j = 1,2,\ldots,2n \qquad (2.21)$$

Equations (2.16) and (2.19) reduce to

$$\dot{y}_i^*(t) = \sum_{j=1}^{2n} a_{ij}^*(t)y_j^*(t) + \varepsilon_i(t) + 0_i(\underset{\sim}{y}^{*2}) \quad , \quad i = 1,2,\ldots,2n \qquad (2.22)$$

where

$$\varepsilon_i(t) \cong X_i(\phi_1^*, \phi_2^*, \ldots, \phi_{2n}^*, t) - \dot{\phi}_i^*(t) \quad , \quad i = 1,2,\ldots,2n \tag{2.23}$$

Let us write Eqs.(2.22) and (2.23) in the matrix form

$$\dot{\underset{\sim}{y}}^*(t) = A^*(t)\underset{\sim}{y}^*(t) + \underset{\sim}{\varepsilon}(t) + \underset{\sim}{O}(\underset{\sim}{y}^{*2}) \tag{2.24}$$

$$\underset{\sim}{\varepsilon}(t) \cong \underset{\sim}{X}(\underset{\sim}{\phi}^*, t) - \dot{\underset{\sim}{\phi}}^*(t) \tag{2.25}$$

where $A^*(t)$ is a matrix, whose elements are a_{ij}^*, i, j = 1,2,...,2n, that is periodic in t with period T, $A^*(t + T) = A^*(t)$. Hence, $\underset{\sim}{\varepsilon}(t)$ plays the role of an unknown extraneous force vector, introduced by the process of using the approximate solution $\underset{\sim}{\phi}^*(t)$, instead of the actual solution $\underset{\sim}{\phi}(t)$. Note that $\underset{\sim}{O}(\underset{\sim}{y}^{*2})$ represents a vector consisting of non-linear terms of degree equal to or larger than two in y_i^* (i = 1,2,..,2n).

Our interest is in estimating the unknown force vector $\underset{\sim}{\varepsilon}(t)$. To this end, let us consider the Fourier series

$$\underset{\sim}{X}[\underset{\sim}{\phi}^*(t), t] - \dot{\underset{\sim}{\phi}}^*(t) = \underset{\sim}{d}_o + \sum_{k=1}^{\infty} (\underset{\sim}{d}_{2k-1}\sin kt + \underset{\sim}{d}_{2k}\cos kt) \tag{2.26}$$

where $\underset{\sim}{d}_o$, $\underset{\sim}{d}_1$, $\underset{\sim}{d}_2$, ..., $\underset{\sim}{d}_{2m_o}$, ... are the Fourier coefficients of the function $\underset{\sim}{X}(\underset{\sim}{\phi}^*,t) - \dot{\underset{\sim}{\phi}}^*(t)$. By assumption, the function on the left side of Eq.(2.26) is continuous and periodic, so that it can be approximated by a trigonometric polynomial of the form

$$\underset{\sim}{d}_o + \sum_{k=1}^{m_o} (\underset{\sim}{d}_{2k-1}\sin kt + \underset{\sim}{d}_{2k}\cos kt) \tag{2.27}$$

with a large m_o. From Eq.(2.25), the force vector $\underset{\sim}{\varepsilon}(t)$ can be approximated equivalently to the trigonometric polynomial in Eq.(2.27) as

$$\underset{\sim}{\varepsilon}(t) = \underset{\sim}{d}_o + \sum_{k=1}^{m_o} (\underset{\sim}{d}_{2k-1}\sin kt + \underset{\sim}{d}_{2k}\cos kt) \tag{2.28}$$

Because $\underset{\sim}{d}_o, \underset{\sim}{d}_1, \underset{\sim}{d}_2, \ldots, \underset{\sim}{d}_{2m_o}$ are the Fourier coefficients of $\underset{\sim}{X}(\underset{\sim}{\phi}^*, t) - \dot{\underset{\sim}{\phi}}^*(t)$, we can use Galerkin procedure in Eqs.(2.25) and (2.28). Therefore, we can use formulas similar to Eq.(2.12) to express the coefficients $\underset{\sim}{d}_o, \underset{\sim}{d}_1, \underset{\sim}{d}_2, \ldots, \underset{\sim}{d}_{2m_o}$ as follow

$$
\begin{aligned}
\underset{\sim}{d}_o &= \frac{1}{T}\int_o^T \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(s), s\right] ds \\
\underset{\sim}{d}_{2k-1} &= \frac{2}{T}\int_o^T \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(s), s\right]\sin ks\, ds - k\underset{\sim}{c}_{2k} \\
\underset{\sim}{d}_{2k} &= \frac{2}{T}\int_o^T \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(s), s\right]\cos ks\, ds + k\underset{\sim}{c}_{2k-1} \\
\underset{\sim}{d}_{2p-1} &= \frac{2}{T}\int_o^T \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(s), s\right]\sin ps\, ds \\
\underset{\sim}{d}_{2p} &= \frac{2}{T}\int_o^T \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(s), s\right]\cos ps\, ds \\
& k = 1,2,\ldots,m \quad ; \quad p = m+1, m+2, \ldots, m_o
\end{aligned}
\tag{2.29}
$$

where $\beta = (\underset{\sim}{d}_o, \underset{\sim}{d}_1, \underset{\sim}{d}_2, \ldots, \underset{\sim}{d}_{2m_o})$ is an undetermined coefficient matrix. It will be obtained from Eqs.(2.29) by numerical integration methods, which will be discussed in a later section. Consequently, we can get a nonnegative constant R satisfying the following inequality

$$R \geq \left\| \underset{\sim}{d}_o + \sum_{k=1}^{m_o} (\underset{\sim}{d}_{2k-1}\sin kt + \underset{\sim}{d}_{2k}\cos kt) \right\| \tag{2.30}$$

where the symbol $\| \;\|$ denotes the norm. A value slightly greater than

$$\max_i \left\| \underset{\sim}{d}_o + \sum_{k=1}^{m_o} (\underset{\sim}{d}_{2k-1} \sin kt + \underset{\sim}{d}_{2k} \cos kt) \right\| \tag{2.31}$$

will yield a desired value R satisfying the following inequality

$$R \geq \left\| \underset{\sim}{X} \left[\underset{\sim}{\phi}^*(t),\; t\right] - \dot{\underset{\sim}{\phi}}^*(t) \right\| \geq \left\| \underset{\sim}{\varepsilon}(t) \right\| \tag{2.32}$$

Inequality (2.32) provides a bound for $\underset{\sim}{\varepsilon}(t)$ in terms of trigonometric polynomial approximations in Eq.(2.28).

In Eq.(2.24), the nonlinear terms $\underset{\sim}{O}(\underset{\sim}{y}^{*2})$ can be expressed in the following form

$$O_i(\underset{\sim}{y}^{*2}) = \sum_{h=2}^{\infty} \sum_{\substack{h_1+h_2+h_3+\ldots+h_{2n}=h \\ h_1,h_2,h_3,\ldots,h_{2n}\geq 0}} g_{i,h_1h_2\ldots h_{2n}} y_1^{*h_1} y_2^{*h_2} \ldots y_{2n}^{*h_{2n}} \tag{2.33}$$

By assumption, the perturbations $y_i^*(t)$ are small and Eq.(2.24) is a set of nonlinear, nonhomogeneous, first-order differential equations. Hence, the solutions $y_i^*(t)$ can be determined as the sum of a uniformly convergent series of the form

$$\underset{\sim}{y}^*(t) = \underset{\sim}{y}^{*(1)}(t) + \underset{\sim}{y}^{*(2)}(t) + \ldots + \underset{\sim}{y}^{*(\ell)}(t) + \ldots \tag{2.34}$$

First, let us introduce Eq.(2.34) into Eq.(2.33) and develop the power series for $O_i(\underset{\sim}{y}^{*2})$ into a new power series. Then, we can write Eq.(2.33) in the form

$$O_i(\underset{\sim}{y}^{*2}) = \sum_{h=2}^{\infty} \sum_{\substack{h_1+h_2+\ldots+h_k=h \\ h_1,h_2,\ldots,h_k\geq 0}} g_{i,h_1h_2\ldots h_k}\left(y^{*(\ell_1)}_{i_1}\right)^{h_1}\left(y^{*(\ell_2)}_{i_2}\right)^{h_2}\ldots\left(y^{*(\ell_k)}_{i_k}\right)^{h_k}$$

$$i = 1,2,\ldots,2n \qquad (2.35)$$

where $k \geq 1$, $\ell_1, \ell_2, \ldots, \ell_k \geq 1$, and $1 \leq i_1, i_2, \ldots, i_k \leq 2n$, are all integers, and where $g_{i,h_1h_2\ldots h_k}$ denotes a function of time. Let us denote by r the number $r = \ell_1h_1 + \ell_2h_2 + \ldots + \ell_kh_k$, or the weight of the term in the right side of Eqs.(2.35) and by $Q_i^{(\ell)}$ the finite sum of all terms of weight $r = \ell$ in the development of $O_i(\underset{\sim}{y}^{*2})$. Then, $\underset{\sim}{Q}(\underset{\sim}{y}^{*2})$ is formally given by the series

$$\underset{\sim}{Q}(\underset{\sim}{y}^{*2}) = \underset{\sim}{Q}^{(2)} + \underset{\sim}{Q}^{(3)} + \ldots + \underset{\sim}{Q}^{(\ell)} + \ldots \qquad (2.36)$$

Let us substitute Eqs.(2.34) and (2.36) into Eq.(2.24) assume that differentiation term by term is permissible, and obtain

$$\dot{\underset{\sim}{y}}^{*(1)} + \dot{\underset{\sim}{y}}^{*(2)} + \ldots + \dot{\underset{\sim}{y}}^{*(\ell)} + \ldots = A^*(t)(\underset{\sim}{y}^{*(1)} + \underset{\sim}{y}^{*(2)} + \ldots + \underset{\sim}{y}^{*(\ell)}$$
$$+ \ldots) + \underset{\sim}{\varepsilon}(t) + \underset{\sim}{Q}^{(2)} + \underset{\sim}{Q}^{(3)} + \ldots + \underset{\sim}{Q}^{(\ell)} + \ldots \qquad (2.37)$$

Since there are no terms of Eqs.(2.35) of weight $r < 2$, Eq.(2.37) can be written as the set of linear and nonhomogeneous differential system

$$\dot{\underset{\sim}{y}}^{*(1)} = A^*(t)\underset{\sim}{y}^{*(1)} + \underset{\sim}{\varepsilon}(t) \qquad (2.38)$$

and

$$\dot{\underset{\sim}{y}}^{*(\ell)} = A^*(t)\underset{\sim}{y}^{*(\ell)} + \underset{\sim}{Q}^{(\ell)} \quad , \quad \ell = 2, 3, \ldots \qquad (2.39)$$

where the vectors $\underset{\sim}{Q}^{(\ell)}$ (ℓ = 2, 3, ...) depend only on $\underset{\sim}{y}^{*(1)}$, $\underset{\sim}{y}^{*(2)}$, ..., $\underset{\sim}{y}^{*(\ell-1)}$, and their actual determination may be a tedious process; no general expression has been found for them, and they can only be determined in particular cases.

The method to solve the linear nonhomogeneous differential systems (2.38) and (2.39) is described in Refs. 1-4. Let us consider the corresponding homogeneous part of Eq.(2.38)

$$\dot{\underset{\sim}{y}}^{*(1)}(t) = A^{*}(t)\underset{\sim}{y}^{*(1)} \tag{2.40}$$

and let Y(t) be an arbitrary fundamental matrix of Eq.(2.40). Because det $Y(t) \neq 0$ in a given domain D, we can define the following expression for any solution $\underset{\sim}{y}^{*(1)}(t) = \underset{\sim}{u}(t)$ of Eq.(2.38) as

$$\underset{\sim}{u}(t) = Y(t)\ \underset{\sim}{z}(t) \tag{2.41}$$

or

$$\underset{\sim}{z}(t) = Y^{-1}(t)\ \underset{\sim}{u}(t) \tag{2.42}$$

where $\underset{\sim}{z}(t)$ is an unknown vector whose elements are functions of time. By introducing Eq.(2.41) into (2.38), we obtain

$$\dot{Y}(t)\underset{\sim}{z}(t) + Y(t)\dot{\underset{\sim}{z}}(t) = A^{*}(t)Y(t)\underset{\sim}{z}(t) + \underset{\sim}{\varepsilon}(t) \tag{2.43}$$

Because Y(t) is a fundamental matrix of Eq.(2.40), it satisfies

$$\dot{Y}(t) = A^{*}(t)Y(t) \tag{2.44}$$

so that Eq.(2.43) can be reduced to

$$\dot{\underset{\sim}{z}}(t) = Y^{-1}(t)\,\underset{\sim}{\varepsilon}(t) \tag{2.45}$$

Integrating Eq.(2.45), we obtain

$$\underset{\sim}{z}(t) = \int_{t_o}^{t} Y^{-1}(s)\underset{\sim}{\varepsilon}(s)\,ds + \underset{\sim}{C} \tag{2.46}$$

where $\underset{\sim}{C}$ is a constant vector, and t_o is an arbitrary initial value of time; $t_o = 0$ in the present case. Introducing Eq.(2.46) into (2.41), we thus have

$$\underset{\sim}{y}^{*(1)}(t) = Y(t)\underset{\sim}{C} + Y(t)\int_{o}^{t} Y^{-1}(s)\underset{\sim}{\varepsilon}(s)\,ds \tag{2.47}$$

The solution $\underset{\sim}{y}^{*(1)}(t)$ given by Eq.(2.47) is periodic in t of period T if and only if

$$\left[I - Y(T)\right]\underset{\sim}{C} = Y(T)\int_{o}^{T} Y^{-1}(s)\underset{\sim}{\varepsilon}(s)\,ds \tag{2.48}$$

where I is the unit matrix. Implicit is the assumption that $Y(0) = I$, so that $Y(t)$ is really the principal matrix. Because the solution is periodic, the Jacobian determinant is nonzero, $\det|I - Y(T)| \neq 0$. Hence, Eq.(2.48) implies

$$\underset{\sim}{C} = \left[I - Y(T)\right]^{-1} Y(T)\int_{o}^{T} Y^{-1}(s)\underset{\sim}{\varepsilon}(s)\,ds \tag{2.49}$$

By substituting Eq.(2.49) into (2.47), the solution $\underset{\sim}{y}^{*(1)}(t)$ becomes

$$\underset{\sim}{y}^{*(1)}(t) = \int_{o}^{T} H(t, s)\underset{\sim}{\varepsilon}(s)\,ds \tag{2.50}$$

where $H(t, s)$ is a continuous periodic matrix

$$H(t, s) = \begin{cases} Y(t)\left[I - Y(T)\right]^{-1} Y^{-1}(s) & 0 \le s \le t \le T \\ Y(t)\left[I - Y(T)\right]^{-1} Y(T)\, Y^{-1}(s) & 0 \le t \le s \le T \end{cases} \tag{2.51}$$

To solve Eqs.(2.39), we can use the same procedure as that for solving Eq.(2.38). The solutions of Eqs.(2.39) can be expressed in a form similar to that of Eq.(2.50), namely,

$$\underset{\sim}{y}^{*(\ell)}(t) = \int_0^T H(t, s)\underset{\sim}{Q}^{(\ell)}(s)\, ds \quad , \quad \ell = 2,3,4,\ldots \tag{2.52}$$

Finally, we solve Eqs.(2.50) and (2.52) by numerical integration. Then, the solutions for the perturbations $y_i^*(t)$ are obtained as the sum of the solutions $y_i^{*(\ell)}(t)$ (ℓ = 1, 2, 3, ...), Eq.(2.34).

To produce bounds for the difference between the actual and approximate solutions, let us rewrite Eqs.(2.20) in the matrix form

$$\dot{\underset{\sim}{\delta}}(t) = A(t)\,\underset{\sim}{\delta}(t) + \underset{\sim}{\varepsilon}(t) \tag{2.53}$$

where the matrix $A(t)$ consists of functions $a_{ij}(t)$ ($i, j = 1,2,\ldots,2n$) which are defined in Eqs.(2.7). Although the actual solution $\underset{\sim}{\phi}(t)$ is not known, we can express it by rewriting Eq.(2.13) as $\underset{\sim}{\phi}(t) = \underset{\sim}{\phi}^*(t) - \underset{\sim}{\delta}(t)$. Then we can obtain the matrix $A(t)$ by introducing this expression of $\underset{\sim}{\phi}(t)$ into Eqs.(2.7). Moreover, Urabe (Ref. 8) has shown that Eqs.(2.1) have an approximate solution $\underset{\sim}{x} = \underset{\sim}{\phi}^*(t)$ lying in domain D, and there is a continous periodic matrix $A(t)$ such that

$$\| A(t) - A^*(t) \| \le \frac{\gamma}{M_1} \quad \text{for} \quad \underset{\sim}{\delta}(t) = \underset{\sim}{\phi}^*(t) - \underset{\sim}{\phi}(t) \tag{2.54}$$

where γ is a small parameter, $0 < \gamma < 1$, and M_1 is a positive constant such that

$$M_1 \ge \left[T \cdot \max_{0 \le t \le T} \left\{ \int_0^T \sum_{k,\ell} H_{k\ell}^2(t, s)\, ds \right\} \right]^{\frac{1}{2}} \tag{2.55}$$

in which $H_{k\ell}(t, s)$ are the elements of the matrix $H(t, s)$. Then Eq.(2.53) can be rewritten as follows

$$\dot{\underset{\sim}{\delta}}(t) = A^*(t)\underset{\sim}{\delta}(t) + \left[A(t) - A^*(t) \right] \underset{\sim}{\delta}(t) + \underset{\sim}{\varepsilon}(t) \tag{2.56}$$

Since $\underset{\sim}{\delta}(t)$ is a periodic vector of period T, the solution $\underset{\sim}{\delta}(t)$ of Eq.(2.56) can be expressed in the integral equation just as Eq.(2.50)

$$\underset{\sim}{\delta}(t) = \int_0^T H(t, s) \left\{ \left[A(s) - A^*(s) \right] \delta(s) + \varepsilon(s) \right\} ds \tag{2.57}$$

where $H(t,s)$ is the continuous periodic matrix defined by Eq.(2.51) by analogy with Eq.(2.40). To solve Eq.(2.57), let us consider the successive iteration process

$$\underset{\sim}{\delta}_{n+1}(t) = \int_0^T H(t, s) \left\{ \left[A(s) - A^*(s) \right] \underset{\sim}{\delta}_n(s) + \underset{\sim}{\varepsilon}(s) \right\} ds \tag{2.58}$$

Then by Eqs.(2.30), (2.54) and (2.55), Eq.(2.58) yields in the following inequality

$$\| \underset{\sim}{\delta} \|_n \le M_1 \left[\frac{\gamma}{M_1} \| \delta \|_n + R \right] = \gamma \| \underset{\sim}{\delta} \|_n + M_1 R \tag{2.59}$$

or

$$(1 - \gamma)\| \underset{\sim}{\delta} \|_n \leq M_1 R \tag{2.60}$$

which implies that

$$\| \underset{\sim}{\delta} \|_n \leq \frac{M_1 R}{1 - \gamma} \tag{2.61}$$

If the iterative process is convergent, then we have

$$\underset{\sim}{\delta}(t) = \lim_{n \to \infty} \underset{\sim}{\delta}_n(t) \tag{2.62}$$

Therefore inequality (2.61) can be written as

$$\| \underset{\sim}{\delta}(t) \| \leq \frac{M_1 R}{1 - \gamma} \tag{2.63}$$

Using inequalities (2.54) and (2.63), we obtain the estimated values of $\| \underset{\sim}{\delta}(t) \|$ and γ. Subsequently, the error bound of $\underset{\sim}{y}^*(t) - \underset{\sim}{y}(t)$ may be estimated by given bounds for $\underset{\sim}{\phi}^*(t) - \underset{\sim}{\phi}(t)$.

3. Numerical Techniques

3.1 Numerical Solution of the Determining System

Because the determining system (2.12) is a set of nonlinear algebraic equations, their solution can be obtained by using Brown's method. According to the method of approximate evaluation of Fourier coefficients (see Ref. 9), the determining equations (2.12) can be rewritten as follows

$$
\begin{aligned}
\underset{\sim}{F}_o(\alpha) &= \frac{1}{2N}\sum_{j=1}^{2N} \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(t_j),\ t_j\right] = 0 \\
\underset{\sim}{F}_{2k-1}(\alpha) &= \frac{1}{N}\sum_{j=1}^{2N} \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(t_j),\ t_j\right]\sin kt_j + k\underset{\sim}{c}_{2k} = 0 \\
\underset{\sim}{F}_{2k}(\alpha) &= \frac{1}{N}\sum_{j=1}^{2N} \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(t_j),\ t_j\right]\cos kt_j - k\underset{\sim}{c}_{2k-1} = 0
\end{aligned}
$$

$$k = 1,2,\ldots,m \tag{3.1}$$

where $\alpha = (\ \underset{\sim}{c}_o,\ \underset{\sim}{c}_1,\ \underset{\sim}{c}_2,\ldots,\ \underset{\sim}{c}_{2m-1},\ \underset{\sim}{c}_{2m}\ )$ is an unknown matrix, and $\underset{\sim}{\phi}^*(t)$ is defined by Eq.(2.10). Moreover

$$
\begin{aligned}
t_j &= \frac{2j-1}{4N}\,T \qquad , \quad j = 1,2,\ldots,2N \\
N &\geq m+1
\end{aligned}
\tag{3.2}
$$

In using Brown's method, Eqs.(3.1) are expanded simultaneously about an arbirary matrix α^n assumed to be close to the solution α, yielding

$$\underset{\sim}{F}_o(\alpha) = \underset{\sim}{F}_o(\alpha^n) + \frac{\underset{\sim}{F}_o(\alpha^n+h^n e) - \underset{\sim}{F}_o(\alpha^n)}{h^n}(\alpha - \alpha^n)$$

$$\underset{\sim}{F}_{2k-1}(\alpha) = \underset{\sim}{F}_{2k-1}(\alpha^n) + \frac{\underset{\sim}{F}_{2k-1}(\alpha^n+h^n e) - \underset{\sim}{F}_{2k-1}(\alpha^n)}{h^n}(\alpha - \alpha^n)$$

$$\underset{\sim}{F}_{2k}(\alpha) = \underset{\sim}{F}_{2k}(\alpha^n) + \frac{\underset{\sim}{F}_{2k}(\alpha^n+h^n e) - \underset{\sim}{F}_{2k}(\alpha^n)}{h^n}(\alpha - \alpha^n)$$

$$k = 1,2,\ldots,m \tag{3.3}$$

where e denotes the unit matrix consisting of j unit vectors and the scalar h^n is normally chosen such that $h^n = 0\left(\left\| F_k(\alpha^n) \right\|\right)$ (in detail see Ref. 10). With this choice, it can be proven that Brown's method yields a second-order convergence.

The starting values α^n for iteration can be usually found by solving the determining equations for small m. We substitute the starting values in a computer program and iterate untill the values α^n converge to the solutions α.

3.2 Computation of Forcing Function $\varepsilon(t)$

Using a formula similar to Eq.(3.1) to determine Fourier coeficients by numerical integration, Eqs.(2.29) can be formed approximately as follows

$$\underset{\sim}{d}_o = \frac{1}{2N}\sum_{j=1}^{2N} \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(t_j),\ t_j\right] \tag{3.4a}$$

$$\underset{\sim}{d}_{2k-1} = \frac{1}{N}\sum_{j=1}^{2N} \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(t_j),\ t_j\right]\sin kt_j + k\underset{\sim}{c}_{2k} \tag{3.4b}$$

$$\underset{\sim}{d}_{2k} = \frac{1}{N} \sum_{j=1}^{2N} \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(t_j),\ t_j\right]\cos kt_j - k\underset{\sim}{c}_{2k-1} \tag{3.4c}$$

$$\underset{\sim}{d}_{2p-1} = \frac{1}{N} \sum_{j=1}^{2N} \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(t_j),\ t_j\right]\sin pt_j \tag{3.4d}$$

$$\underset{\sim}{d}_{2p} = \frac{1}{N} \sum_{j=1}^{2N} \underset{\sim}{X}\left[\underset{\sim}{\phi}^*(t_j),\ t_j\right]\cos pt_j \tag{3.4e}$$

$$k = 1,2,\ldots,m \quad , \quad p = m+1, m+2, \ldots, m_o$$

where $N \geq m_o + 1$, and

$$t_j = \frac{2j - 1}{4N} T \quad , \quad j = 1,2,\ldots,2N \tag{3.5}$$

Substituting the solutions α obtained in Sec.(3.1) and the approximate solution $\underset{\sim}{\phi}^*(t)$ into Eqs.(3.4), the coefficients $\beta = (\ \underset{\sim}{d}_o,\ \underset{\sim}{d}_1,\ \ldots,\ \underset{\sim}{d}_{2m_o})$ are obtained by Simpson's integration at $t = t_1,\ t_2,\ \ldots,\ t_{2N}$. Then, we can obtain the forcing function $\underset{\sim}{\varepsilon}(t_j)$ for each time t_j ($j = 1,2,\ldots,2N$) by using the following expressions

$$\underset{\sim}{\varepsilon}(t_j) = \underset{\sim}{d}_o + \sum_{j=1}^{m_o} (\ \underset{\sim}{d}_{2k-1}\sin kt_j + \underset{\sim}{d}_{2k}\cos kt_j\ )$$

$$j = 1,2,\ldots,2N \tag{3.6}$$

Consequently, by means of Eq.(2.30), we can get a reasonable bounded value of R according to

$$R = |\ d_o\ | + \sum_{j=1}^{m_o} \sqrt{d_{2k-1}^2 + d_{2k}^2} \tag{3.7}$$

3.3 Computation of the Perturbation Function $\underset{\sim}{y}^*(t)$

Before we try to solve Eqs.(2.50) and (2.52), we need to find the fundamental matrix Y(t). It is convenient to write the fundamental matrix Y(t) in integral form as

$$Y(t) = I + \int_o^t A^*(s)\,ds + \int_o^t A^*(s) \int_o^s A^*(\tau)\,d\tau ds + \ldots \qquad (3.8)$$

where I is the identity matrix, and $A^*(t)$ is a Jacobian matrix whose elements $a^*_{ij}(t)$ are defined by Eqs.(2.7). In case the matrices $A^*(t)$ and $\int_o^t A^*(s)ds$ commute, we can write Eq.(3.8) in the following form

$$Y(t) = \exp(\int_o^t A^*(s)ds) \qquad (3.9)$$

To be able to carry out numerical integration, we divide the time interval into k small intervals. Thus Eq.(3.9) can be rewritten as

$$Y(t) = \exp(\int_o^{t_1} A^*(s)ds + \int_{t_1}^{t_2} A^*(s)ds + \ldots + \int_{t_{k-1}}^{t} A^*(s)ds)$$

or

$$Y(t) = \exp(\int_o^{t_1} A^*(s)ds)\cdot\exp(\int_{t_1}^{t_2} A^*(s)ds)\cdot \cdots \cdot\exp(\int_{t_{k-1}}^{t} A^*(s)ds)$$

$$(3.10)$$

Let us introduce a notation as follows

$$Y_{t_{k-1}}^{t_k} = \exp(\int_{t_{k-1}}^{t_k} A^*(s)ds) \qquad (3.11)$$

in which t_k and t_{k-1} denote the limits of the kth interval. Equation (3.10) can be rewritten as

$$Y(t) = Y_o^{t_1} \, Y_{t_1}^{t_2} \, \ldots \, Y_{t_{k-2}}^{t_{k-1}} \, Y_{t_{k-1}}^{t} \tag{3.12}$$

Equation (3.12) is based on the assumption that the time increment $\Delta t = t_k - t_{k-1}$ is a small quantity. The matrix $A^*(t)$ may be considered as a constant at instaneous time t_k ($k = 1, 2, \ldots, 2N$). For this reason, we can write Eq.(3.11) in the following approximate form

$$Y_{t_{k-1}}^{t_k} = I + A^*\big|_{t=t_k} \Delta t + \frac{1}{2} \left(A^*\big|_{t=t_k} \Delta t \right)^2 + \ldots$$

$$k = 1,2,\ldots,2N \tag{3.13}$$

Then the fundamental matrix $Y(t)$ can be obtained by using the property of Eq.(3.12) and the approximate form (3.13).

Since the fundamental matrix $Y(t)$ is given, $Y(T)$ and $Y^{-1}(t)$ can be obtained by the same procedure used in obtaining $Y(t)$. Therefore, the integral kernel matrix $H(t, s)$ of Eqs.(2.50) and (2.52) will be obtained numerically by introducing the matrices $Y(t)$, $Y(T)$, and $Y^{-1}(t)$ into Eq.(2.51). Subsequently, Eqs.(2.50) and (2.52) can be written approximately as

$$\begin{aligned} \underset{\sim}{y}^{*(1)}(t_i) &= \sum_{j=1}^{2N} H(t_i, s_j) \, \underset{\sim}{\varepsilon}(s_j) \\ \underset{\sim}{y}^{*(\ell)}(t_i) &= \sum_{j=1}^{2N} H(t_i, s_j) \, \underset{\sim}{Q}^{(\ell)}(s_j) \quad , \quad \ell = 2,3, \ldots \end{aligned} \tag{3.14}$$

where

$$t_i = \frac{2i}{4N} T \quad , \quad i = 0,1,2,\ldots,2N$$

$$s_j = \frac{2j - 1}{4N} T \quad , \quad j = 1,2,\ldots\ldots,2N \tag{3.15}$$

The perturbation function $\underset{\sim}{y}^*(t)$ will be obtained by summing the numerical solutions of Eqs.(3.14) at time t_i ($i = 0,1,2,\ldots,2N$)

$$\underset{\sim}{y}^*(t_i) = \underset{\sim}{y}^{*(1)}(t_i) + \underset{\sim}{y}^{*(2)}(t_i) + \ldots + \underset{\sim}{y}^{*(\ell)}(t_i) + \ldots.$$

$$i = 0,1,2,\ldots,2N \tag{3.16}$$

Next, by determining the values of the matrix $H(t, s)$ at $t = t_o$, t_1, t_2, ..., t_{2N}, $s = s_1$, s_2, ..., s_{2N}, we can compute the value of

$$\int_o^T \sum_{k,\ell} H_{k\ell}^2(t, s)\, ds \tag{3.17}$$

by Simpson's numerical integration at $t = t_o$, t_1, t_2, ..., t_{2N}. Then we can compute the approximate values of inequality (2.55) by using the values of Eq.(3.17) as

$$M^* = \left[T \cdot \max_{i=0,1,2,\ldots,2N} \int_o^T \sum_{k,\ell} H_{k\ell}^2(t_i, s)\, ds \right]^{\frac{1}{2}} \tag{3.18}$$

After we have found the approximate value of M^*, we take a number slightly greater than the approximate value of M^*. Then, this gives a reasonable value of M_1 satisfying inequality (2.55).

Finally, we choose a value of γ such that $0 < \gamma < 1$. By substituting the values of M_1 and γ into inequalities (2.54) and (2.63), we compute the value of the norm $\|\underset{\sim}{\delta}(t)\|$. If inequalities (2.54) and (2.63) are satisfied, then the value of γ is a reasonable one, and the norm $\|\underset{\sim}{\delta}(t)\|$ is an estimated value of bounds between the perturbations $\underset{\sim}{y}(t)$ and $\underset{\sim}{y}^*(t)$.

4. Application to van der Pol Equation

In this section, we will apply our method to the van der Pol equation with a harmonic forcing term. Hence, let us consider

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = \epsilon E \sin \omega t \qquad (4.1)$$

or

$$P(t) = \ddot{x} - \epsilon(1 - x^2)\dot{x} + x - \epsilon E \sin \omega t = 0 \qquad (4.2)$$

Since a periodic solution of Eq.(4.2) is a solution with period $\frac{2\pi}{\omega}$ and we are interested in a periodic solution with period 2π, we can replace t by t/ω in Eq.(4.2) and define the notations

$$\frac{\epsilon}{\omega} = \epsilon_1 \quad , \quad \frac{E}{\omega} = E_1 \quad , \quad \frac{1 - \omega^2}{\epsilon\,\omega} = A_1 \qquad (4.3)$$

By substituting Eq.(4.3), Eq.(4.2) can be transformed into the following form

$$P(t) = \ddot{x} - \epsilon_1(1 - x^2)\dot{x} + (1 + \epsilon_1 A_1)x - \epsilon_1 E_1 \sin t = 0 \quad (4.4)$$

Equation (4.4) can be rewritten in the form of a first-order differential system as follows

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \epsilon_1(1 - x_1^2)x_2 - (1 + \epsilon_1 A_1)x_1 + \epsilon_1 E_1 \sin t \end{aligned} \qquad (4.5)$$

Now let $x_1 = x(t)$ be any periodic solution of Eq.(4.4). Then evidently $-x(t + \pi)$ is also a periodic solution. Therefore, the Fourier series of such a periodic solution must be of the form

$$x(t) \sim c_1 \sin t + c_2 \cos t + c_3 \sin 3t + c_4 \cos 3t + \ldots.$$

Taking this fact into consideration, we can assume the Kth Galerkin's approximations in the form of trigonometric polynomials

$$x_1(t) = \phi_1^*(t) = \sum_{k=1}^{K} \left[c_{2k-1} \sin(2k-1)t + c_{2k} \cos(2k-1)t \right] \qquad (4.6)$$

and

$$x_2(t) = \phi_2^*(t) = \sum_{k=1}^{K} (2k-1) \left[c_{2k-1} \cos(2k-1)t - c_{2k} \sin(2k-1)t \right] \qquad (4.7)$$

Taking a derivative with respect to t in Eq.(4.7), we obtain

$$\dot{x}_2(t) = \dot{\phi}_2^*(t) = -\sum_{k=1}^{K} (2k-1)^2 \left[c_{2k-1} \sin(2k-1)t + c_{2k} \cos(2k-1)t \right] \qquad (4.8)$$

Introducing Eqs.(4.6), (4.7) and (4.8) into Eq.(4.4), Eq.(4.4) becomes

$$P(t) = \sum_{k=1}^{3K-1} \left[F_{2k-1}(\alpha) \sin(2k-1)t + F_{2k}(\alpha) \cos(2k-1)t \right] \qquad (4.9)$$

where $F_k(\alpha)$ are nonlinear algebraic equations, Eqs.(3.1), in the undetermined coefficients $\alpha = (c_1, c_2, \ldots, c_{2K})$. We can write $F_k(\alpha)$ in the form

$$F_k(c_1, c_2,\ldots,c_{2k}) = Q_k(c_1,c_2,\ldots,c_{2k}) + \sum_{j=1}^{2K} R_{kj}c_j + \delta_k$$

$$k = 1,2,\ldots,2K \qquad (4.10)$$

where Q_k is the nonlinear part of $F_k(\alpha)$, which, in turn, can be written in the form

$$Q_k = \sum_{j=1}^{2K} G_{kj}c_j \qquad (4.11)$$

and

$$\delta_1 = - \epsilon_1 E_1$$
$$\delta_k = 0 \quad , \quad k = 2,3,\ldots,2K \qquad (4.12a)$$

$$R_{2j-1,2j-1} = R_{2j,2j} = 1 + \epsilon_1 A_1 - (2j - 1)^2$$
$$R_{2j,2j-1} = - R_{2j-1,2j} = - \epsilon_1 (2j - 1) \ , \quad j = 1,3,5,\ldots \qquad (4.12b)$$

and the remaining elements of the matrix $[R_{kj}]$ are equal to zero, as $R_{kj} = 0$ for all other values of the indices. To determine the matrix $[G_{kj}]$, whose elements are quadratic polynomials in the c_k's, we rewrite Eqs.(4.6) and (4.7) in the forms of the complex functions

$$\phi_1^*(t) = \sum_{k=1}^{K} \left[s_k e^{i(2k-1)t} + \bar{s}_k e^{-i(2k-1)t} \right] \qquad (4.13)$$

$$\phi_2^*(t) = \sum_{k=1}^{K} i(2k-1) \left[s_k e^{i(2k-1)t} - \bar{s}_k e^{-i(2k-1)t} \right] \qquad (4.14)$$

where

$$s_k = \frac{1}{2}(c_{2k} - ic_{2k-1}) \quad , \quad k = 1,2,\ldots,2K \tag{4.15}$$

and $\bar{s}_k$ are complex conjugates of s_k. Therefore, the term $x_1^2 x_2$ of Eq.(4.5) can be written as

$$\phi_1^{*2}\phi_2^{*} = \sum_{k=1}^{3K-1}\left[z_k e^{i(2k-1)t} + \bar{z}_k e^{-i(2k-1)t} \right] \tag{4.16}$$

where

$$z_k = \Sigma_1 + \Sigma_2 + \Sigma_3 \qquad (k = 1,2,\ldots,3K-1) \tag{4.17}$$

in which

$$\Sigma_1 = \sum_{j=1}^{k-1} s_j \left[\sum_{r=1}^{k-j} i(2r-1)s_r s_{k-j-r+1} + \sum_{r=k-j+1}^{K} i(2r-1)s_r \bar{s}_{r-k+j} - \sum_{r=1}^{K-k+j} i(2r-1)\bar{s}_r s_{k-j+r} \right] \tag{4.18a}$$

$$\Sigma_2 = \sum_{j=k+1}^{K} s_j \left[-\sum_{r=1}^{j-k} i(2r-1)\bar{s}_r \bar{s}_{j-k-r-1} - \sum_{r=j-k+1}^{K} i(2r-1)\bar{s}_r s_{r-j+k} + \sum_{r=1}^{K-j+k} i(2r-1)s_r \bar{s}_{j-k+r} \right] \tag{4.18b}$$

$$\Sigma_3 = \sum_{j=1}^{K} \bar{s}_j \left[\sum_{r=1}^{j+k-1} i(2r-1)s_r s_{j+k-r} + \sum_{r=j+k}^{K} i(2r-1)s_r \bar{s}_{r-j-k+1} - \sum_{r=1}^{K-j-k+1} i(2r-1)\bar{s}_r s_{j+k+r-1} \right] \tag{4.18c}$$

Interchanging the order of summation and defining $p = k+r+j-1$, we can write

$$z_k = \sum_{p=1}^{K} W_{kp} s_p \quad , \quad k = 1,2,\ldots,3K-1 \tag{4.19}$$

so that if p < k, we obtain

$$\begin{aligned} W_{kp} = \sum_{r=1}^{k-p} i(2r-1) s_r s_{k-p-r+1} &+ \sum_{r=k-p+1}^{K} i(2r-1) s_r \bar{s}_{r-k+p} \\ &- \sum_{r=1}^{K-k+p} i(2r-1) \bar{s}_r s_{k-p+r} + \sum_{r=1}^{K-k+p} i(2p-1) \bar{s}_r \bar{s}_{r+k-p} \end{aligned} \tag{4.20}$$

Combining the last three terms on the right side of Eq.(4.20), we also obtain

$$W_{kp} = \sum_{r=1}^{k-p} i(2r-1) s_r s_{k-p-r+1} + \sum_{r=k-p+1}^{K} i(2k-1) s_r \bar{s}_{r+p-k} \quad (p < k) \tag{4.21}$$

Similarly, we can obtain for the cases of k = p and k < p by substitution and transposition

$$W_{kk} = \sum_{r=1}^{K} i(2k-1) |s_r|^2 \qquad (k = p) \tag{4.22}$$

$$W_{kp} = \sum_{r=1}^{p-k} i(2r-1) \bar{s}_r \bar{s}_{p-k-r+1} + \sum_{r=p-k+1}^{K} i(2k-1) \bar{s}_r s_{r+k-p} \quad (k<p) \tag{4.23}$$

Since each W_{kp} is a sum of exactly K complex terms, we may write it as

$$W_{kp} = U_{kp} + iV_{kp} \tag{4.24}$$

where the elements U_{kp} and V_{kp} are real quantities. The elements of the matrix $[G_{kj}]$ may be obtained from the following expressions

$$G_{2k-1,2p-1} = G_{2k,2p} = \epsilon_1 U_{kp}$$

$$G_{2k,2p-1} = - G_{2k-1,2p} = \epsilon_1 V_{kp}$$

$$k = 1,2,\ldots,3K-1 \; ; \quad p = 1,2,\ldots,K \tag{4.25}$$

Substituting Eqs.(4.11), (4.12) and (4.25) into Eq.(4.10) and applying Brown's method (see Ref.11), we obtain the coefficients c_1, $c_2, \ldots,$ c_{2K}. Brown's method is a derivative-free analogue of Newton's method. The iteration steps can be expressed as follows

$$\underset{\sim}{c}^{n+1} = \underset{\sim}{c}^{n} - J^{-1}(\underset{\sim}{c}^{n})\cdot F(\underset{\sim}{c}^{n}) \tag{4.26}$$

where n denotes iterative numbers, and $J(\underset{\sim}{c})$ is the Jacobian matrix, given by

$$J(c_1,c_2,\ldots,c_{2K}) = \left[\frac{\partial F_k}{\partial c_j} \right] \tag{4.27}$$

In Brown's method, the partial derivatives of Jacobian matix are replaced by the first difference quotient approximations

$$\frac{F_k(\underset{\sim}{c}^{n}+h^{n}\underset{\sim}{e}_j) - F_k(\underset{\sim}{c}^{n})}{h^{n}} \tag{4.28}$$

where $\underset{\sim}{e}_j$ denotes the jth unit vector of the unit matrix e, and the scalar value h^n is normally chosen such that $h^n = O(\|F_k(c^n)\|)$. Therefore, Eq.(4.26) can be solved by a successive substitution iteration for $n = 0, 1, 2, \ldots,$ beginning with the initial guess $c_1^o, c_2^o, \ldots, c_{2K}^o$.

In Ref.7 it is proved that even for a very low order of Galerkin's approximation one may be able to obtain an approximate solution close to the actual solution. Hence, we can use Galerkin's approximations with K = 1 to estimate the first two values of the starting values c_1^o, c_2^o and let the remaining values $c_3^o, c_4^o, \ldots, c_{2K}^o$ equal to zero. The approximate solution $\underset{\sim}{\phi}^*(t)$ of Galerkin's approximation with K = 1 can be expressed as

$$\begin{aligned} \phi_1^*(t) &= c_1 \sin t + c_2 \cos t \\ \phi_2^*(t) &= c_1 \cos t - c_2 \sin t \end{aligned} \tag{4.29}$$

Substituting Eqs.(4.29) into Eq.(4.5) and using the form of Eq.(2.12), we obtain the following determining equations

$$\begin{aligned} F_1(c_1,c_2) &= \frac{1}{\pi} \int_o^{2\pi} X_2\left[\phi_1^*(s),\phi_2^*(s),s\right] \sin s \, ds + c_1 = 0 \\ F_2(c_1,c_2) &= \frac{1}{\pi} \int_o^{2\pi} X_2\left[\phi_1^*(s),\phi_2^*(s),s\right] \cos s \, ds - c_2 = 0 \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} X_2\left[\phi_1^*(t),\phi_2^*(t),t\right] = \frac{\epsilon_1}{4} \Big\{ &\left[4E_1 - 4c_2 + 4\left(\tfrac{1}{\epsilon_1} + A_1\right)c_1 + c_2^3 + c_1^2 c_2\right] \sin t \\ &+ \left[4c_1 - 4\left(\tfrac{1}{\epsilon_1} + A_1\right)c_2 - c_1^3 - c_1 c_2^2\right] \cos t + \left[c_2^3 - 3c_1^2 c_2\right] \sin 3t \\ &+ \left[c_1^3 - 3c_1^2 c_2\right] \cos 3t \Big\} \end{aligned} \tag{4.31}$$

Considering the following orthogonal properties of trigonometric functions

$$
\begin{aligned}
\int_0^{2\pi} \sin ms \sin ns \, ds &= \pi \qquad \text{for } m = n \\
&= 0 \qquad\quad m \neq n \\[1em]
\int_0^{2\pi} \cos ms \cos ns \, ds &= \pi \qquad \text{for } m = n \\
&= 0 \qquad\quad m \neq n \\[1em]
\int_0^{2\pi} \sin ms \cos ns \, ds &= 0 \qquad \text{for all } m,\ n
\end{aligned}
\tag{4.32}
$$

and performing integration on Eq.(4.30), we obtain the determining equations as follows

$$
\begin{aligned}
F_1(c_1,c_2) &= 4E_1 - 4c_2 - 4A_1c_1 + c_2^3 + c_1^2c_2 = 0 \\
F_2(c_1,c_2) &= 4c_1 - 4A_1c_2 - c_1^3 - c_1c_2^2 = 0
\end{aligned}
\tag{4.33}
$$

Let us assume values for the parameters ϵ, ω, E as follows

$$\epsilon = 0.1 \qquad \omega = 0.9 \qquad E = 3 \tag{4.34}$$

Then from Eq.(4.3), we have

$$\epsilon_1 = 0.11111\ , \quad E_1 = 3.33333\ , \quad A_1 = 2.11111 \tag{4.35}$$

Using Brown's method in conjuction with a computer program to solve Eqs.(4.33), we get the values

$$c_1 = 1.52471827 \qquad c_2 = 0.287548855 \tag{4.36}$$

If we let K = 2 in the Galerkin's approximation, Eqs.(4.6) and (4.7), the approximate solution $\phi^*_\ell(t)$ has the form

$$\phi_1^*(t) = c_1 \sin t + c_2 \cos t + c_3 \sin 3t + c_4 \cos 3t \qquad (4.37a)$$

and

$$\phi_2^*(t) = c_1 \cos t - c_2 \sin t + 3c_3 \cos 3t - 3c_4 \sin 3t \qquad (4.37b)$$

Using the same procedure as in the K = 1 case, the determining equations can be written as

$$F_1(c) = 4E_1 - 4c_2 - 4A_1c_1 + c_2^3 + c_1^2c_2 - c_4(c_1^2 - c_2^2) + 2c_2(c_3^2 + c_4^2)$$
$$+ 2c_1c_2c_3 = 0$$

$$F_2(c) = 4c_1 - 4A_1c_2 - c_1^3 - c_1c_2^2 + c_3(c_1^2 - c_2^2) - 2c_1(c_3^2 + c_4^2)$$
$$+ 2c_1c_2c_4 = 0 \qquad (4.38)$$

$$F_3(c) = -12c_4 + 4(\frac{8}{\epsilon_1} - A_1)c_3 + c_2^3 + 3c_4^3 - 3c_1^2c_2 + 3c_3^2c_4 + 6c_4(c_1^2 + c_2^2)$$
$$= 0$$

$$F_4(c) = 12c_3 + 4(\frac{8}{\epsilon_1} - A_1)c_4 + c_1^3 - 3c_3^3 - 3c_1^2c_2 - 3c_3c_4^2 - 6c_3(c_1^2 + c_2^2)$$
$$= 0$$

Using Brown's method, we arrive at the following solutions of Eqs.(4.38)

$$c_1 = 1.52911470 \qquad c_2 = 0.28671426$$
$$c_3 = 0.007212549 \qquad c_4 = -0.011375427 \qquad (4.39)$$

Thus, we take the following values

$$c_1 = 1.52911470 \quad c_2 = 0.28671426 \quad c_3 = 0.007212549$$
$$c_4 = -\ 0.011375427 \quad c_5 = c_6 = \dots = c_{29} = c_{30} = 0 \tag{4.40}$$

as the starting values. Then, we solve Eq.(4.10) by Brown's method with the starting values given by Eq.(4.40). After 30 iterations or after attainment of the convergence criterion 0.1×10^{-7}, the result of $\phi_1^*(t)$ by numerical computation is

$$\begin{aligned}
\phi_1^*(t) = {} & 0.152910541\times10^{1} \sin t + 0.286717997\times10^{0} \cos t \\
& + 0.720979824\times10^{-2} \sin 3t - 0.113708891\times10^{-1} \cos 3t \\
& - 0.110846828\times10^{-3} \sin 5t - 0.154798571\times10^{-3} \cos 5t \\
& - 0.293784239\times10^{-5} \sin 7t + 0.713603039\times10^{-6} \cos 7t \\
& - 0.725957484\times10^{-8} \sin 9t + 0.499475004\times10^{-7} \cos 9t \\
& - 0.749134287\times10^{-9} \sin 11t + 0.435594666\times10^{-9} \cos 11t \\
& + 0.119493425\times10^{-10} \sin 13t - 0.931094218\times10^{-11} \cos 13t \\
& - 0.740676508\times10^{-13} \sin 15t - 0.257734623\times10^{-12} \cos 15t \\
& - 0.476655138\times10^{-14} \sin 17t - 0.488788128\times10^{-15} \cos 17t \\
& - 0.401380855\times10^{-16} \sin 19t + 0.763192553\times10^{-16} \cos 19t \\
& + 0.101100246\times10^{-17} \sin 21t + 0.118897803\times10^{-17} \cos 21t \\
& + 0.269135883\times10^{-19} \sin 23t - 0.900105043\times10^{-20} \cos 23t \\
& + 0.306279653\times10^{-22} \sin 25t - 0.517109789\times10^{-21} \cos 25t \\
& - 0.857861135\times10^{-23} \sin 27t - 0.404969950\times10^{-23} \cos 27t \\
& - 0.127726525\times10^{-24} \sin 29t + 0.118479765\times10^{-24} \cos 29t
\end{aligned} \tag{4.41}$$

Furthermore, introducing the above results of $\phi_1^*(t)$ into Eq.(3.4) and setting $m_o = 50$ and $N = 75$, the force function $\varepsilon_2(t)$ yields

$$
\begin{aligned}
\varepsilon_2(t) = & - 0.266633967\times10^{-8} - 0.131692315\times10^{-6} \sin t \\
& - 0.423356799\times10^{-6} \cos t + 0.147737847\times10^{-10} \sin 2t \\
& - 0.533424111\times10^{-8} \cos 2t - 0.665691967\times10^{-7} \sin 3t \\
& + 0.185536134\times10^{-6} \cos 3t + 0.295831336\times10^{-10} \sin 4t \\
& - 0.533891885\times10^{-8} \cos 4t + 0.117747246\times10^{-8} \sin 5t \\
& - 0.338806089\times10^{-8} \cos 5t + 0.444656303\times10^{-10} \sin 6t \\
& - 0.534672920\times10^{-8} \cos 6t + 0.235462408\times10^{-9} \sin 7t \\
& - 0.538712167\times10^{-8} \cos 7t + 0.594577089\times10^{-10} \sin 8t \\
& - 0.535769093\times10^{-8} \cos 8t + 0.678841238\times10^{-10} \sin 9t \\
& - 0.537321344\times10^{-8} \cos 9t + 0.745976691\times10^{-10} \sin 10t \\
& - 0.537183059\times10^{-8} \cos 10t + 0.819338117\times10^{-10} \sin 11t \\
& - 0.538019297\times10^{-8} \cos 11t + 0.899231627\times10^{-10} \sin 12t \\
& - 0.538918421\times10^{-8} \cos 12t + 0.976635192\times10^{-10} \sin 13t \\
& - 0.539907555\times10^{-8} \cos 13t + 0.105473483\times10^{-9} \sin 14t \\
& - 0.540979369\times10^{-8} \cos 14t + 0.113346149\times10^{-9} \sin 15t \\
& - 0.542133589\times10^{-8} \cos 15t + 0.121290577\times10^{-9} \sin 16t \\
& - 0.543371191\times10^{-8} \cos 16t + 0.129311307\times10^{-9} \sin 17t \\
& - 0.544692923\times10^{-8} \cos 17t + 0.137414303\times10^{-9} \sin 18t \\
& - 0.546099872\times10^{-8} \cos 18t + 0.145606497\times10^{-9} \sin 19t \\
& - 0.547592724\times10^{-8} \cos 19t + 0.153892081\times10^{-9} \sin 20t \\
& - 0.549172281\times10^{-8} \cos 20t + 0.162276775\times10^{-9} \sin 21t \\
& - 0.550839896\times10^{-8} \cos 21t + 0.170767395\times10^{-9} \sin 22t \\
& - 0.552596509\times10^{-8} \cos 22t + 0.179369942\times10^{-9} \sin 23t \\
& - 0.554443185\times10^{-8} \cos 23t + 0.188089809\times10^{-9} \sin 24t
\end{aligned}
$$

$$- 0.556381271\times10^{-8}\cos 24t + 0.196933816\times10^{-9}\sin 25t$$
$$- 0.558411984\times10^{-8}\cos 25t + 0.205909357\times10^{-9}\sin 26t$$
$$- 0.560536639\times10^{-8}\cos 26t + 0.215023967\times10^{-9}\sin 27t$$
$$- 0.562756651\times10^{-8}\cos 27t + 0.224282697\times10^{-9}\sin 28t$$
$$- 0.565073584\times10^{-8}\cos 28t + 0.233694049\times10^{-9}\sin 29t$$
$$- 0.567488961\times10^{-8}\cos 29t + 0.243264935\times10^{-9}\sin 30t$$
$$- 0.570004420\times10^{-8}\cos 30t + \ldots. \tag{4.42}$$

The bounded constant R of the force function $\underset{\sim}{\varepsilon}(t)$ is

$$R = \left| \max\left[d_o + \sum_{k=1}^{50} (d_{2k-1}\sin kt + d_{2k}\cos kt)\right]\right|$$
$$\approx |d_o| + \sum_{k=1}^{50} \sqrt{d_{2k-1}^2 + d_{2k}^2} = 0.249841354\times10^{-6} \tag{4.43}$$

and from Eq.(4.5), we have

$$\varepsilon_1(t) = 0 \tag{4.44}$$

Note that the matrix $\overset{*}{A}(t)$ corresponding to Eq.(4.5) is given by

$$\overset{*}{A}(t) = \left[\frac{\partial X_i}{\partial x_j}\Big|_{\underset{\sim}{x}=\underset{\sim}{\phi}^*} \right] = \begin{bmatrix} 0 & 1 \\ -2\epsilon_1\phi_1^*\phi_2^*-(1+\epsilon_1 A_1) & \epsilon_1(1-\phi_1^{*2}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -(1+\epsilon_1 A_1)-2\epsilon_1 \sum_{k=1}^{2K-1} \Big[(r_k+\bar{r}_k)\cos 2kt & \epsilon_1 - \epsilon_1 \sum_{k=0}^{2K-1} \Big[(q_k+\bar{q}_k)\cos 2kt \\ + i(r_k-\bar{r}_k)\sin 2kt\Big] & + i(q_k-\bar{q}_k)\sin 2kt\Big] \end{bmatrix} \tag{4.45}$$

where

$$r_k = \sum_{j=1}^{k} i(2j-1)s_j s_{k-j+1} + \sum_{j=k+1}^{K} i(2j-1)s_j \bar{s}_{j-k} - \sum_{j=1}^{K-k} i(2j-1)\bar{s}_j s_{j+k}$$

$$k = 1,2,\ldots 2K-1 \qquad (4.46)$$

and

$$q_o = \sum_{j=1}^{K} s_j \bar{s}_j$$

$$(4.47)$$

$$q_k = \sum_{j=1}^{k} s_j s_{k-j+1} + 2\sum_{j=k+1}^{K} s_j \bar{s}_{j-k} \quad , \; k = 1,2,\ldots,2K-1$$

The computed values of $A^*_{21}(t)$ and $A^*_{22}(t)$ are

$$\begin{aligned}
A^*_{21}(t) = &- 0.123456784\times10^{1} \\
&- 0.101750615\times10^{0}\cos 2t - 0.248939824\times10^{0}\sin 2t \\
&+ 0.671814221\times10^{-2}\cos 4t - 0.644399889\times10^{-2}\sin 4t \\
&+ 0.234944817\times10^{-3}\cos 6t + 0.106331525\times10^{-3}\sin 6t \\
&- 0.279930963\times10^{-6}\cos 8t + 0.645334668\times10^{-5}\sin 8t \\
&- 0.143961824\times10^{-6}\cos 10t + 0.506639430\times10^{-7}\sin 10t \\
&- 0.228932385\times10^{-8}\cos 12t - 0.260935361\times10^{-8}\sin 12t \\
&+ 0.347358041\times10^{-10}\cos 14t - 0.685771538\times10^{-10}\sin 14t \\
&+ 0.166271058\times10^{-11}\cos 16t + 0.145121850\times10^{-12}\sin 16t \\
&+ 0.103674454\times10^{-13}\cos 18t + 0.341508826\times10^{-13}\sin 18t \\
&- 0.587942966\times10^{-15}\cos 20t + 0.472085578\times10^{-15}\sin 20t \\
&- 0.137481017\times10^{-16}\cos 22t - 0.773462669\times10^{-17}\sin 22t \\
&+ 0.426491105\times10^{-19}\cos 24t - 0.324550212\times10^{-18}\sin 24t \\
&+ 0.653476757\times10^{-20}\cos 26t - 0.167654703\times10^{-20}\sin 26t \\
&+ 0.817671613\times10^{-22}\cos 28t + 0.111454951\times10^{-21}\sin 28t
\end{aligned}$$

$$- 0.150167634\times10^{-23}\cos 30t + 0.238485647\times10^{-23}\sin 30t$$
$$- 0.150167634\times10^{-23}\cos 32t + 0.238485635\times10^{-23}\sin 32t$$
$$+ \ldots\ldots \qquad (4.48)$$

$$\begin{aligned}A^*_{22}(t) = &- 0.233656424\times10^{-1}\\
&+ 0.124469912\times10^{0} \cos 2t - 0.508753077\times10^{-1} \sin 2t\\
&+ 0.161099972\times10^{-2}\cos 4t + 0.167953555\times10^{-2} \sin 4t\\
&- 0.177219209\times10^{-4}\cos 6t + 0.391574695\times10^{-4} \sin 6t\\
&- 0.806668335\times10^{-6}\cos 8t - 0.349913703\times10^{-7} \sin 8t\\
&- 0.506639430\times10^{-8}\cos 10t - 0.143961824\times10^{-7}\sin 10t\\
&+ 0.217446135\times10^{-9}\cos 12t - 0.190776987\times10^{-9}\sin 12t\\
&+ 0.489836813\times10^{-11}\cos 14t + 0.248112887\times10^{-11}\sin 14t\\
&- 0.907011559\times10^{-14}\cos 16t + 0.103919411\times10^{-12}\sin 16t\\
&- 0.189727126\times10^{-14}\cos 18t + 0.575969187\times10^{-15}\sin 18t\\
&- 0.236042789\times10^{-16}\cos 20t - 0.293971483\times10^{-16}\sin 20t\\
&+ 0.351573940\times10^{-18}\cos 22t - 0.624913712\times10^{-18}\sin 22t\\
&+ 0.135229255\times10^{-19}\cos 24t + 0.177704627\times10^{-20}\sin 24t\\
&+ 0.644825782\times10^{-22}\cos 26t + 0.251337214\times10^{-21}\sin 26t\\
&- 0.398053398\times10^{-23}\cos 28t + 0.292025576\times10^{-23}\sin 28t\\
&- 0.794952156\times10^{-25}\cos 30t - 0.500558780\times10^{-25}\sin 30t\\
&+ \ldots\ldots \qquad (4.49)\end{aligned}$$

The nonlinear terms $\underset{\sim}{O}(\underset{\sim}{y}^{*2})$ of the variation equation (2.24) are

$$\begin{aligned}O_1(\underset{\sim}{y}^{*2}) &= 0\\
O_2(\underset{\sim}{y}^{*2}) &= -\,\epsilon_1(\, x_2 y_1^{*2} + 2x_1 y_1^* y_2^* + y_1^{*2} y_2^*\,)\end{aligned} \qquad (4.50)$$

Since the solutions of $\underset{\sim}{y}^*(t)$ are small, it is convenient to take only the first three terms of the uniformly convergent series (2.34) giving

$$\underset{\sim}{y}^*(t) = \underset{\sim}{y}^{*(1)} + \underset{\sim}{y}^{*(2)} + \underset{\sim}{y}^{*(3)} \tag{4.51}$$

Introducing Eqs.(4.51) into (4.50), we obtain

$$\begin{aligned} O_2(\underset{\sim}{y}^{*2}) &= -\epsilon_1\Big[x_2(y_1^{*(1)} + y_1^{*(2)} + y_1^{*(3)})^2 + 2x_1(y_1^{*(1)} + y_1^{*(2)} + y_1^{*(3)})\times \\ &\quad (y_2^{*(1)} + y_2^{*(2)} + y_2^{*(3)}) + (y_1^{*(1)} + y_1^{*(2)} + y_1^{*(3)})^2(y_2^{*(1)} + y_2^{*(2)} + y_2^{*(3)})\Big] \\ &= Q_2^{(2)} + Q_2^{(3)} + \ldots.. \end{aligned} \tag{4.52}$$

where

$$Q_2^{(2)} = -\epsilon_1(x_2 y_1^{*(1)^2} + 2x_1 y_1^{*(1)} y_2^{*(1)}) \tag{4.53}$$

$$\begin{aligned} Q_2^{(3)} = -\epsilon_1\Big[& 2x_2 y_1^{*(1)} y_1^{*(2)} + 2x_1(y_1^{*(1)} y_2^{*(2)} + y_1^{*(2)} y_2^{*(1)}) + \\ & y_2^{*(1)} y_1^{*(1)2} \Big] \end{aligned} \tag{4.54}$$

Because the functions $\underset{\sim}{\varepsilon}(t)$, $\underset{\sim}{Q}^{(2)}$ and $\underset{\sim}{Q}^{(3)}$ are given, we can solve Eqs.(2.50) and (2.52) by the numerical integration described in Sec.3.3. The solution $\underset{\sim}{y}^{*(1)}(t)$ is listed in Table 1. Furthermore, the solutions $\underset{\sim}{y}^{*(2)}(t)$ and $\underset{\sim}{y}^{*(3)}(t)$ are listed in Tables 2 and 3, respectively. The solution of perturbation $\underset{\sim}{y}^*(t)$ is obtained by summing the results of $\underset{\sim}{y}^{*(1)}(t)$, $\underset{\sim}{y}^{*(2)}(t)$ and $\underset{\sim}{y}^{*(3)}(t)$.

Furthermore, from Eq.(3.14), the constant M^* is given by numerical

Table 1. NUMERICAL RESULTS OF SOLUTIONS $y_1^{*(1)}(t)$ AND $y_2^{*(1)}(t)$

TIME	$y_1^{*(1)}(t)$	$y_2^{*(1)}(t)$
0.000000000D 00	0.221204397D-06	-0.734638069D-06
0.216661566D 00	0.141233070D-06	-0.689345539D-06
0.433323131D 00	0.579369712D-07	-0.954907001D-06
0.649984697D 00	-0.446725710D-07	-0.118493841D-05
0.866646262D 00	-0.158854395D-06	-0.131379594D-05
0.108330783D 01	-0.273250857D-06	-0.132507628D-05
0.129996939D 01	-0.376961948D-06	-0.122358776D-05
0.151663096D 01	-0.461372063D-06	-0.102589218D-05
0.173329252D 01	-0.520832110D-06	-0.777082972D-06
0.194995409D 01	-0.553659243D-06	-0.540635258D-06
0.216661566D 01	-0.562064844D-06	-0.357130687D-06
0.238327722D 01	-0.549791552D-06	-0.225941544D-06
0.259993879D 01	-0.519336904D-06	-0.124532863D-06
0.281660035D 01	-0.470992782D-06	-0.242583173D-07
0.303326192D 01	-0.403303965D-06	0.110196040D-06
0.324992348D 01	-0.313828252D-06	0.305643126D-06
0.346658505D 01	-0.200847648D-06	0.548962654D-06
0.368324661D 01	-0.663194874D-07	0.784178330D-06
0.389990818D 01	0.821602057D-07	0.948214047D-06
0.411656974D 01	0.232492326D-06	0.100012469D-05
0.433323131D 01	0.371213420D-06	0.922331209D-06
0.454989288D 01	0.486283977D-06	0.723423268D-06
0.476655444D 01	0.569122760D-06	0.449767389D-06
0.498321601D 01	0.616202442D-06	0.169328265D-06
0.519987757D 01	0.628914662D-06	-0.724654187D-07
0.541653914D 01	0.610878163D-06	-0.269055752D-06
0.563320070D 01	0.564973982D-06	-0.425596383D-06
0.584986227D 01	0.492696044D-06	-0.545684522D-06
0.606652383D 01	0.394801514D-06	-0.657354201D-06
0.628318540D 01	0.271041093D-06	-0.115630373D-05

Table 2. NUMERICAL RESULTS OF SOLUTIONS $y_1^{*(2)}(t)$ AND $y_2^{*(2)}(t)$

TIME	$y_1^{*(2)}(t)$	$y_2^{*(2)}(t)$
0.000000000D 00	0.143891990D-12	-0.137990954D-12
0.216661566D 00	0.110471866D-12	-0.169226449D-12
0.433323131D 00	0.679106146D-13	-0.197030914D-12
0.649984697D 00	0.180009335D-13	-0.235968973D-12
0.866646262D 00	-0.371829250D-13	-0.290289276D-12
0.108330783D 01	-0.950108791D-13	-0.346371115D-12
0.129996939D 01	-0.151991141D-12	-0.380207273D-12
0.151663096D 01	-0.203959023D-12	-0.372193661D-12
0.173329252D 01	-0.246795166D-12	-0.321494795D-12
0.194995409D 01	-0.277463716D-12	-0.246020899D-12
0.216661566D 01	-0.294626606D-12	-0.164924119D-12
0.238327722D 01	-0.298135156D-12	-0.866800448D-13
0.259993879D 01	-0.288041919D-12	-0.129446235D-13
0.281660035D 01	-0.264161733D-12	0.547077728D-13
0.303326192D 01	-0.226304344D-12	0.111668943D-12
0.324992348D 01	-0.174888087D-12	0.151668985D-12
0.346658505D 01	-0.111559952D-12	0.175365919D-12
0.368324661D 01	-0.392414130D-13	0.195579718D-12
0.389990818D 01	0.384059484D-13	0.225115710D-12
0.411656974D 01	0.117302671D-12	0.259013357D-12
0.433323131D 01	0.192788258D-12	0.272624442D-12
0.454989288D 01	0.259572821D-12	0.238679914D-12
0.476655444D 01	0.312369126D-12	0.151153808D-12
0.498321601D 01	0.347216776D-12	0.312101804D-13
0.519987757D 01	0.362480989D-12	-0.933006034D-13
0.541653914D 01	0.358403148D-12	-0.205672702D-12
0.563320070D 01	0.335832361D-12	-0.299156118D-12
0.584986227D 01	0.295456208D-12	-0.373268729D-12
0.606652383D 01	0.237679280D-12	-0.431997747D-12
0.628318540D 01	0.162985600D-12	-0.473038412D-12

Table 3. NUMERICAL RESULTS OF SOLUTIONS $y_1^{*(3)}(t)$ AND $y_2^{*(3)}(t)$

TIME	$y_1^{*(3)}(t)$	$y_2^{*(3)}(t)$
0.000000000D 00	0.889965312D-19	-0.849089015D-19
0.216661566D 00	0.683512202D-19	-0.100768561D-18
0.433323131D 00	0.422272943D-19	-0.112951349D-18
0.649984697D 00	0.117762146D-19	-0.131436868D-18
0.866646262D 00	-0.217351951D-19	-0.160029544D-18
0.108330783D 01	-0.567530587D-19	-0.194209585D-18
0.129996939D 01	-0.913778706D-19	-0.220655651D-18
0.151663096D 01	-0.123319271D-18	-0.223100252D-18
0.173329252D 01	-0.150040355D-18	-0.194548426D-18
0.194995409D 01	-0.169334390D-18	-0.142968828D-18
0.216661566D 01	-0.179951606D-18	-0.821154203D-19
0.238327722D 01	-0.181567527D-18	-0.217843405D-19
0.259993879D 01	-0.174271461D-18	0.323379241D-19
0.281660035D 01	-0.158238115D-18	0.760729992D-19
0.303326192D 01	-0.133742499D-18	0.106288582D-18
0.324992348D 01	-0.101362443D-18	0.123112807D-18
0.346658505D 01	-0.621555382D-19	0.132990553D-18
0.368324661D 01	-0.176457853D-19	0.146009822D-18
0.389990818D 01	0.302337806D-19	0.167842508D-18
0.411656974D 01	0.791025218D-19	0.193899716D-18
0.433323131D 01	0.126111639D-18	0.207927878D-18
0.454989288D 01	0.167964192D-18	0.188523857D-18
0.476655444D 01	0.201134100D-18	0.126291508D-18
0.498321601D 01	0.222674006D-18	0.344877521D-19
0.519987757D 01	0.231188718D-18	-0.628569941D-19
0.541653914D 01	0.226881986D-18	-0.147345747D-18
0.563320070D 01	0.210745964D-18	-0.210076705D-18
0.584986227D 01	0.183820340D-18	-0.250657890D-18
0.606652383D 01	0.146839016D-18	-0.274905282D-18
0.628318540D 01	0.100300825D-18	-0.285565917D-18

computations as

$$M^* = 61.854675 \tag{4.55}$$

From Eq.(4.22), we have

$$A(\phi_1,\phi_2) - \mathring{A}(\phi_1^*,\phi_2^*) = \begin{bmatrix} 0 & 0 \\ 2\epsilon_1(\phi_1^*\phi_2^*-\phi_1\phi_2) & \epsilon_1(\phi_1^{*2}-\phi_1^2) \end{bmatrix} \tag{4.56}$$

From this the matrix norm is

$$\begin{aligned}
\| A(\phi_1,\phi_2) - \mathring{A}(\phi_1^*,\phi_2^*) \|^2 &\le 4\epsilon_1^2(\phi_1^*\phi_2^* - \phi_1\phi_2)^2 + \epsilon_1^2(\phi_1^{*2} - \phi_1^2)^2 \\
&= \epsilon_1^2 \left\{4\left[(\phi_1^* - \phi_1)\phi_2^* + \phi_1(\phi_2^* - \phi_2)\right]^2 + (\phi_1^* - \phi_1)^2(\phi_1^* + \phi_1)^2\right\} \\
&\le \epsilon_1^2 \left\{4\left[(\phi_1^* - \phi_1)^2 + (\phi_2^* - \phi_2)^2\right](\phi_1^2 + \phi_2^{*2}) + (\phi_1^* - \phi_1)^2(\phi_1^* + \phi_1)^2\right\} \\
&\le \epsilon_1^2 \left[(\phi_1^* - \phi_1)^2 + (\phi_2^* - \phi_2)^2\right]\left[4(\phi_1^2 + \phi_2^{*2}) + (\phi_1^* + \phi_1)^2\right]
\end{aligned} \tag{4.57}$$

and the norm $\|\delta\|$ is defined as

$$\|\delta\| = \sqrt{(\phi_1^* - \phi_1)^2 + (\phi_2^* - \phi_2)^2} \tag{4.58}$$

Therefore,

$$\begin{aligned}
\|A(\phi_1,\phi_2) - \mathring{A}(\phi_1^*,\phi_2^*)\| &\le |\epsilon_1| \|\delta\| \left(8\phi_1^{*2} + 4\phi_2^{*2} + 12\|\delta\| |\phi_1^*| + 5\|\delta\|^2\right)^{\frac{1}{2}} \\
&\le \frac{\gamma}{M_1} = \frac{\gamma}{62}
\end{aligned} \tag{4.59}$$

$$\|\delta\| \le \frac{M_1 R}{1 - \gamma} = \frac{62\times 2.5\times 10^{-7}}{1 - \gamma} \tag{4.60}$$

where

$$|\phi_1^*| = \sum_{k=1}^{K} \sqrt{c_{2k-1}^2 + c_{2k}^2} = 1.56941145$$

$$|\phi_2^*| = \sum_{k=1}^{K} k \sqrt{c_{2k-1}^2 + c_{2k}^2} = 1.58326548$$

Let us assume that

$$\|\delta\| < 10^{-7}$$

and

$$\|\delta\| \left(8\phi_1^{*2} + 4\phi_2^{*2} + 12 \|\delta\| |\phi_1^*| + 5\|\delta\|^2\right)^{\frac{1}{2}} \le \frac{9\gamma}{62}$$

$$0.545264509\times10^{-6} \le \frac{9\gamma}{62} \tag{4.62}$$

When we take the value

$$\gamma = 10^{-5} \tag{4.63}$$

which satisfies both the condition $0 \le \gamma \le 1$ and Eq.(4.62). Introducing Eq.(4.63) into Eq.(4.60), the value of $\|\delta\|$ can be obtained as

$$\|\delta\| < 1.550 \times10^{-5} \tag{4.64}$$

From Eq.(4.64), we know the error bound of $\underset{\sim}{\phi}^* - \underset{\sim}{\phi}$ is small. Therefore, the approximate solutions of a higher-order Galerkin's approximation are close to the actual solutions.

5. Summary and Conclusions

The purpose of this study is to provide a general procedure for investigating the behavior of dynamical systems in the neighborhood of periodic solutions which are known only approximately. An example of such a dynamical system is a helicopter in forward flight or hover. The procedure also estimates the effect of the extraneous forces introduced by the process of using approximate solutions instead of the actual periodic solutions.

The procedure is divided into two major parts: 1) the computation of an approximate solution of the unperturbed motion by means of Galerkin's approximations and Brown's method to nonautonomous system, and 2) the evaluation of the solutions of perturbations by solving a set of nonlinear nonhomogeneous differential equations. The perturbed motion occurs in the vicinity of the approximate periodic solution. Furthermore, the extraneous forcing terms resulting from the use of approximate periodic solutions are calculated by using a trigonometric polynomial. We also obtain the error bounds between the approximate and the actual solutions.

Comparing the results of very low order Galerkin's approximations (see Sec. 4) with $m = 1$ and $m = 2$, to a higher-order Galerkin's approximation with $m=15$, we find close agreement, with a difference of only a few percent. This indicates that Galerkin's approximation to the nonlinear nonautonomous system converges both rapidly and accurately. It appears that applying Brown's method to the nonlinear algebraic systems is considerably less complicated than Newton's method because the number of computation per iteration is reduced from $\bar{N}^2 + \bar{N}$ to $(\bar{N}^2 + 3\bar{N})/2$, where $\bar{N}$ denotes the order of the algebraic system. Moreover, Brown's method is a derivative-free method.

Finally, the solutions of perturbations depend on the values of the extraneous forces. The values of the extraneous forces depend on the differences between the approximate solutions and the actual solutions. In the present example, the values of the extraneous forces are small, so that the effect of these forces on the stability of the perturbed motion is relatively small. The perturbations are shown in Tables 1, 2, and 3. In general, the effect of introducing the extraneous forces on the stability of the perturbed motion depends on how large the difference between the approximate and the actual periodic solution is.

6. References

1. Meirovitch, L., Methods of Analytical Dynamics, McGraw-Hill Book Co., N.Y., 1970

2. Cesari, L., Asymptotic Behavior and Stability Problems in Ordinary Differential Equations, Springer-Verlag, N.Y., 1971

3. Hale, J. K., Ordinary Differential Equations, John Wiley & Sons, Inc., N.Y., 1969

4. Collatz, L., The Numerical Treatment of Differential Equations, Springer-Verlag, N.Y., 1966

5. Carnahan, B., Luther H. A., and Wilkes, J. O., Applied Numerical Methods, John Wiley & Sons, Inc., N.Y., 1969

6. Uhlmann, W., "Fehlerabschatzungen bei Anfangswertaufgahen gewohnlicher Differentialgleichungssysteme 1 Ordnung," ZAMM, Vol. 37, April 1957, pp. 88-99

7. Cesari, L., "Functional Analysis and Galerkin's Method", Michigan Math. J., Vol. 11, 1964, pp. 385-414

8. Urabe, M., "Galerkin's procedure for nonlinear periodic systems" Arch. Rat. Mech. Anal., 20, 1965, pp. 120-152

9. Urabe, M., and Reiter, A., "Numerical computation of nonlinear forced oscillations by Galerkin's procedure", J. Math. Anal. Appl., 14, 1966, pp. 107-140

10. Brown, K. M., "Computer oriented algorithms for solving systems of simultaneous nonlinear algebraic equations", Numerical Solution of Systems of Nonlinear Algebraic Equations, edited by Byrue, G., and Hall, C., Academic Press, N.Y., 1973, pp. 281-348

11. Trujillo, D. M., "The direct numerical integration of linear matrix differential equations using Pade approximations", Int. J. For Num. Methods in Eng., Vol. 9, 1975, pp. 259-270

12. Hahn, W., Stability of Motion, Springer-Verlag, N.Y., 1967

13. Urabe, M., Nonlinear Autonomous Oscillations, Academic Press, N.Y., 1967

14. Ceschine, F., and Kuntzmann, Numerical Solution of Initial Value Problems, Prentice-Hall Inc., Englewood Cliffs, N.J., 1966

15. Moser, J., "New aspects in the theory of stability of Hamiltonian systems", Communications on Pure and App. Math., Vol. 11, 1958, pp. 81-114

16. Birkhoff, G. D., "Stability and the equations of dynamics" Amer. J. Math., Vol. 49, 1927, pp. 1-38

The vita has been removed from the scanned document

NUMERICAL COMPUTATION OF PERTURBATION SOLUTIONS

OF NONAUTONOMOUS SYSTEMS

by

Jeng-Sheng Huang

(ABSTRACT)

A numerical investigation of 2n first-order Hamilton's equations, which describe the motion of a dynamical system, has been conducted using Galerkin's approximations and a derivative-free analogue of Newton's iteration method. Furthermore, the motion stability of a dynamical system in the neighborhood of the approximate periodic solutions due to the effect of the extraneous forces, introduced by the process of using the approximate solutions rather than the actual solutions, has been studied by solving the nonlinear nonhomogeneous differential systems of the perturbed motion. The perturbation solutions are obtained to determine the motion stability.

An example, using the van der Pol equation, illustrates the accuracy and error bounds between the approximate solutions and the actual solutions. Furthermore, the example also illustrates the motion stability of perturbation solutions. A computer program for numerical computions has been developed for solving the van der Pol equation with a harmonic forcing term.