ASPECTS OF RISK PROGRAMMING

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INTRODUCTION

1.1 LINEAR PROGRAMMING

1.1.1 BASIC CONCEPTS AND ASSUMPTIONS

"Programming, or program planning, may be defined as the construction of the schedule of actions by means of which an economy, organization, or other complex of activities may move from one defined state to another, or from a defined state toward some specifically defined objective" (15). Since we will here be concerned with the linear programming of a firm under competition, it will be of interest to first discuss some concepts and assumptions peculiar to linear programming analysis.

The three basic concepts (6) to be considered are resources, products, and production processes. Resources are all physical and intangible things used by the firm. Products, or outputs, consist of all of the results of the productive effort of the firm. Each resource and product is readily definable, homogeneous, divisible, and measurable by an appropriate unit of measure. A production process is a physical event or series of events which the firm conducts in order to transform resources into products. In a production process the only variation allowed is a variation of overall scale, called the intensity of the process, which is measured in terms of the unit level of the production process which is in turn defined to be some instance of the process (one acre of a crop, etc.). Let us consider the preparation of a seedbed for corn: "Each added acre plowed . . . requires an equal input and can be considered to add an equal increment to output. Corn planting, corn cultivation, and corn harvesting involve linear relationships between input and output over all technical units . . ." (13). Thus a production process is a specialized and restricted form of a linear homogeneous production function.

A firm can be defined by the collection of processes and by the collection of resources available to it. The operation of the firm can be defined as the simultaneous operation of some linear combination of unit levels of the various production processes. The production function for the firm can be constructed by considering all possible linear combinations of unit levels of processes.

"We may now state the basic assumptions of linear programming. They are:

1. The productive opportunities of an economy or economic unit are defined by the resources and the productive processes available to it. The quantities of at least some of the resources are finite and so is the number of productive processes available.
2. Any productive process may be used at any positive level consistent with the supply of resources available. The consumption of resources and the output of products is proportional to the level at which the process is used.
3. Several productive processes may be used simultaneously, if the supply of resources is adequate. If this is done, the consumption of each resource is the sum of the

consumptions of the individual processes used, and the output of products is the sum of the outputs of the individual processes.

"Within this framework, the productive problem becomes the problem of choosing which productive processes to use and the level at which to use each of them. It will be useful to formulate this problem algebraically" (6).

1.1.2 MATHEMATICAL FORMULATION

We may now algebraically formulate the linear programming problem of a competitive firm in which processes transform resources into final products.

Let us consider a firm with n processes where each process is uniquely defined by the resources needed and the products produced by the operation at the unit level. We will divide the resources into two groups: scarce resources, those that are in limited supply, and non-scarce resources, those that can be bought without limit in the open market. Denote the amounts of scarce resources necessary for the operation of the h-th process at the unit level by the vector

$$\underline{t}'_h = (t_{1h}\, t_{2h} \ldots t_{mh}),$$

the amounts of non-scarce resources needed by the vector

$$(\underline{t}^o_h)' = (t^o_{1h}\, t^o_{2h} \ldots t^o_{m^o h}),$$

and the amount of the products produced by the vector

$$\underline{u}'_h = (u_{1h}\, u_{2h} \ldots u_{kh}).$$

The subscripts m, m^o, and k denote respectively the number of scarce resources, the number of non-scarce resources, and the number of products involved in the whole production program of the firm. Thus some vectors corresponding to some processes will have zero elements if they do not involve all resources and products available to the firm. In this manner any process can be completely described by the vector

$$(\underline{t}_h' \underline{t}_h^{o\prime} \underline{u}_h')$$

The whole firm is described by the collection of the vectors corresponding to all available processes. The collection of the vectors can be combined into a matrix which describes the entire firm. This matrix, which is called the technology matrix, comes in three parts:

$$T = (\underline{t}_1 \underline{t}_2 \ldots \underline{t}_n)$$

which indicates the scarce resources needed for all purposes,

$$T^o = (\underline{t}_1^o \underline{t}_2^o \ldots \underline{t}_n^o),$$

which indicates the non-scarce resources needed for all processes, and

$$U = (\underline{u}_1 \underline{u}_2 \ldots \underline{u}_n),$$

which indicates the products produced by all processes. The complete technology matrix is then

$$\begin{pmatrix} T \\ T^o \\ U \end{pmatrix}.$$

Let us denote by x_h the number of unit levels at which the h-th process is operated. This is the intensity of the process. It follows that the vector

$$\underline{x}' = (x_1 x_2 \ldots x_n)$$

defines the production program of the firm. All of the elements of the $\underline{x}$ vector must be non-negative.

Since the firm is assumed to be operating under competition, all prices of resources and products are given. We can thus obtain the net revenue due to the operation of any process at the unit level. If we denote by p_i the price of the i-th scarce resource, by p_i^o the price of the i-th non-scarce resource and by q_j the price of the j-th product, the net revenue of the h-th process at the unit level is then

$$s_h = \sum_{j=1}^{k} q_j u_{jh} - \sum_{i=1}^{m} p_i t_{ih} - \sum_{i=1}^{m^o} p_i^o t_{ih}^o, h = 1,2,\ldots n.$$

The net revenues of all processes at the unit level can be summarised by the vector

$$\underline{s}' = (s_1 s_2 \ldots s_n)$$

which is called the unit level profit vector. Further, the net revenue due to any production program defined by a particular vector $\underline{x}$ is

$$r = \sum_{h=1}^{n} s_h x_h = \underline{s}'\underline{x} \tag{1.1}$$

The amount of the i-th scarce resource used by this production program is

$$t_{i.} = x_1 t_{i1} + x_2 t_{i2} + \ldots = x_n t_{in} \qquad i = 1,2,\ldots m,$$

similarly the amount of the i-th non-scarce resource used is

$$t_{i.}^o = x_1 t_{i1}^o + x_2 t_{i2}^o + \ldots + x_n t_{in}^o, \quad i = 1,2,\ldots, m^o,$$

and the amount of the j-th product produced is

$$u_{j.} = x_1 u_{j1} + x_2 u_{j2} + \ldots + x_n u_{jn}, \quad j = 1,2,\ldots,k.$$

These quantities can be summarized for all resources by the vectors

$$\underline{t}_{.}' = (t_{1.} t_{2.} \ldots t_{m.}) = T\underline{x}$$

$$\underline{t}^{o\prime} = (t^o_{1.} t^o_{2.} \ldots t^o_{m^o_.}) = T^o\underline{x}$$

$$\underline{u}_{.}' = (u_{1.} u_{2.} \ldots u_{k.}) = U\underline{x},$$

which represents the total amounts of scarce resources, non-scarce resources, and products, respectively, involved in a production program defined by $\underline{x}$.

The problem facing the firm is to find some combination of unit levels of processes that maximizes net revenue but which at the same time does not use more than the available supply of certain scarce resources. Denote by v_i, $i=1,2,\ldots,m$, the total available amount of the i-th scarce resource. The vector

$$\underline{v}' = (v_1 v_2 \ldots v_m)$$

then expresses the total amount of all scarce resources. The limitation imposed by the presence of the i-th scarce resource can then be expressed:

$$t_{i.} \leq v_i \qquad i = 1,2,\ldots,m, \tag{1.2}$$

or considering all scarce resources

$$\underline{t}_{.} = T\underline{x} \leq \underline{v}. \tag{1.2a}$$

The problem facing the firm can now be summarized

$$\text{to maximize } r = \underline{s}'\underline{x} \tag{1.3}$$

$$\text{subject to restriction } T\underline{x} \leq \underline{v}$$

$$\text{and } \underline{x} \geq 0.$$

The above maximization problem is a special case of what may be called "the" linear programming problem which is simply stated:

to maximize a linear function of non-negative variables subject to some linear restrictions, equalities or inequalities. We have considered the special problem of maximizing the net revenue of a firm since we intend to use that model.

1.1.3 THE SOLUTION OF THE MAXIMIZATION PROBLEM

Although the above maximization problem can be solved by several methods, it is usually solved by an iterative procedure which utilizes the "simplex criterion" developed by Dantzig and others. Explanations of the simplex criterion are available (Charnes, Henderson, and Cooper, 1953, Part II, pp. 41-62 and Dorfman, 1951, pp. 23-52), and need not be discussed here. It will suffice to give a brief outline of the procedure used in solving the linear programming problem.

It can be shown that the optimizing solution for a programming problem is one of a limited set of solutions (programs) which have the property that a number of the processes (elements of the vector $\underline{x}$) are carried on at zero levels. Associated with any solution of this set is a "simplex" vector with elements corresponding to the zero-level processes. Each element of this vector indicates how the net revenue is affected by adding to the program one unit level of one of these zero-level processes without violating any of the restrictions. If the simplex indicates that the addition of a unit of any zero-level process decreases net revenue, an optimum solution has been obtained. This simplex vector is used in an iterative procedure with proceeds from solution to solution until the optimum is obtained.

1.1.4 THE DUAL CONCEPT AND PRICE IMPUTATION

The linear programming solution has an additional feature: it allows the imputed price evaluation of the scarce resources. The imputed price of the i-th factor indicates the amount by which net revenue can be increased by adding to the supply of that factor one unit. It follows that resources that are not fully used will have zero imputed prices.

Another feature of these imputed prices is that the sum

$$\sum_{i=1}^{m} p_i^* v_i = \underline{p}^{*\prime} \underline{v}, \tag{1.4}$$

where p* is the vector of the imputed prices, will be equal to the net revenue r. Thus, the total imputed cost of the limited resources of an optimum production program is exactly equal to its net revenue (7). This phenomenon is associated with the fact that in solving the linear programming problem as stated in Section 1.1.2 we have simultaneously solved another problem which can be stated:

$$\begin{aligned} &\text{to minimize} \quad \underline{v}'\underline{p}^* \\ &\text{subject to the restrictions } \underline{p}^* \geq o \\ &\text{and } T'\underline{p}^* \geq \underline{s} \end{aligned} \tag{1.5}$$

This problem, known as the dual of the original problem, is to minimize the imputed cost of limited resources with the restrictions that all imputed prices must be non-negative, and that the total imputed cost of a unit level of each process must be greater than, or equal to, the net revenue of that process.

1.2 NON-LINEAR PROGRAMMING

1.2.1 GENERAL THEORY

Thus far we have considered the solutions of problems which are stated in terms of the maximization of a linear function of non-negative variables subject to linear restrictions. Non-linearities may arise in two distinct but not mutually exclusive ways:

1. The function to be maximized, the maximand, is non-linear, and

2. The restrictions are non-linear.

In general, very little work has been done on cases including non-linear restrictions. We will here discuss only a special case of a non-linear maximand, viz. a quadratic maximand. The programming problem for this case becomes:

$$\begin{aligned} &\text{to maximize} \quad y = \underline{d}'\underline{x}-\underline{x}'F\underline{x}, \\ &\text{subject to} \quad T\underline{x} \leq \underline{v}, \qquad\qquad (1.6) \\ &\text{and} \quad \underline{x} \geq 0. \end{aligned}$$

1.2.2 SOLVING THE QUADRATIC PROGRAMMING PROBLEM.

The quadratic programming problem may be solved by the use of a theorem developed by Kuhn and Tucker (16) and an iterative solution procedure developed by Hildreth (14).

The Kuhn-Tucker theorem states that the problem:

$$\begin{aligned} &\text{to maximize} \quad y = \underline{d}'\underline{x} - \underline{x}'F\underline{x}, \qquad\qquad (1.7) \\ &\text{subject to} \quad T\underline{x} \leq \underline{v}, \end{aligned}$$

where F is positive definite, is equivalent to the minimax problem:

$$\left.\begin{array}{l}\text{minimize with respect to } \underline{x} \\ \text{maximize with respect to } \underline{u}\end{array}\right) \phi(\underline{x},\underline{u}) = \underline{x}'F\underline{x}-\underline{d}'\underline{x}-\underline{u}'(\underline{v}-T\underline{x}), \quad (1.8)$$

subject to $\underline{u} \geq 0$,

where $\underline{u}$ is a vector consisting of as many elements as T has rows.

This problem differs in one respect from the programming problem stated in Section 1.2.1, viz., it does not include the restriction $\underline{x} \geq 0$. However, define

$$T^* = \begin{pmatrix} T \\ -I \end{pmatrix}, \qquad (1.9)$$

and

$$\underline{v}^* = \begin{pmatrix} \underline{v} \\ \underline{0} \end{pmatrix},$$

When -I is a negative identity matrix, and $\underline{0}$ is a vector of zeros. Then

$$T^*\underline{x} \leq \underline{v}^* \qquad (1.10)$$

is equivalent to

$$T\,\underline{x} \leq \underline{v}, \qquad (1.10a)$$

and $\underline{x} \geq 0.$

The above problem may now be made into a programming problem by the substitution of T^* and $\underline{v}^*$ for T and $\underline{v}$.

The first step in the solution procedure is to accomplish the unrestricted minimization of $\phi\,(\underline{x},\underline{u})$ with respect to $\underline{x}$. This can be accomplished by the usual procedure of setting the derivative equal to zero:

$$\frac{\partial\phi}{\partial\underline{x}} = 2F\underline{x}-d+T'u=0, \qquad (1.11)$$

whence

$$\underline{x} = \tfrac{1}{2}F^{-1}(\underline{d}-T'\underline{u}).$$

This is substituted into the original function, which becomes

$$\phi^*(\underline{u}) = -\tfrac{1}{4}(\underline{d}'F^{-1}\underline{d}) + \tfrac{1}{2}(d'F^{-1}T'\underline{u})-\tfrac{1}{4}(\underline{u}'TF^{-1}T'\underline{u})-\underline{u}'\underline{v}. \qquad (1.12)$$

It is desired to maximize $\phi^*\,(\underline{u})$ with respect to $\underline{u}$, subject to the restriction that $\underline{u} \geq 0$. Define

$$C = TF^{-1}T',$$

$$\underline{b} = \underline{v}-\tfrac{1}{2}TF^{-1}\underline{d},$$

and

$$k = \tfrac{1}{2}\underline{d}'F^{-1}\underline{d}.$$

We can now write

$$\theta(\underline{u}) = -2\phi^*(\underline{u}) = \tfrac{1}{2}\underline{u}'C\underline{u}+2\underline{b}'\underline{u}-k, \qquad (1.13)$$

whence

$$\text{Maximum } \phi^*(\underline{u}) = \text{minimum } \theta(\underline{u}).$$

The procedure of estimating the vector $\underline{u}$ which accomplishes this minimization is an iterative one[1]. After choosing an initial value of $\underline{u}$, say $\underline{u}^o$, we hold all except the first of the components constant at the level given by $\underline{u}^o$, and find that non-negative value of u_1 which minimizes $\Theta(\underline{u})$. Call this value u_1^1. Next, holding constant the elements u_1^1, u_3^o, ..., u_p^o, find the non-negative value of u_2 which minimizes $\Theta(\underline{u})$. Continuing in this manner, a new vector, $\underline{u}^1$, is constructed. The same procedure is used to construct in turn $\underline{u}^2$, $\underline{u}^3$, ..., until the desired degree of stability is obtained.

Let u_k, $k = 1, 2, \ldots$, be a component of u. The minimum of $\Theta(\underline{u})$ with respect to u_k will be attained either where $u_k = 0$, or where $\frac{\partial\Theta}{\partial u_k} = 0$. If the latter equation yields a non-negative value for u_k, then this is the minimizing value, otherwise $u_k = 0$ is the minimizing value.

The derivative

$$\frac{\partial\Theta}{\partial\underline{u}} = C\underline{u} + 2\underline{b}. \tag{1.14}$$

Define w_k^q as the value of the k-th coordinate of the q-th iteration that is obtained by setting $\partial\Theta/\partial u_k = 0$. Thus

$$w_k^q = -\sum_{i=1}^{k-1} \frac{c_{ki}}{c_{kk}} u_i^q - \sum_{i=k+1}^{p} \frac{c_{ki}}{c_{kk}} u_i^{q-1} - 2\,\frac{b_k}{c_{kk}}, \tag{1.15}$$

where c_{ki} are elements of C and b_1 are the elements of $\underline{b}$. The value of the k-th coordinate of $\underline{u}$ at the qth iteration is then obtained by taking

$$u_k^q = \text{maximum}\ (w_k^q, 0),$$

---

[1] The following two paragraphs follow closely the wording in Hildreth's paper ((14) p. 605).

or in other words $u_k^q = w_k^q$ if the latter is non-negative, zero if it is negative.

The iteration stops when two succeeding iterations are identical. The desired value of the factor $\underline{x}$ can then be found from the formula

$$\underline{x} = \tfrac{1}{2}\Sigma^{-1}\underline{s} - \tfrac{1}{2}\Sigma^{-1}T'\underline{u}, \qquad (1.16)$$

which was developed above.

The above iterative procedure has been developed for the case where the restriction matrix, T, is of full row rank. However, the restriction matrix, T-, of the programming problem must contain more rows than columns because of the added negative identity matrix and is, therefore, of less than full row rank. There is no reason to believe that the Hildreth procedure will not work for the case where the restriction matrix is not of full row rank. This has not yet been vigorously proved, but Freund (8) has shown that a solution for a programming problem obtained by Hildreth's method was optimal.

1.3 RISK PROGRAMMING

1.3.1 DESCRIPTION OF A RISK SITUATION

Under a no-risk situation the entrepreneur is faced with a given, known set of prices and production conditions that will prevail during a production period. These given conditions, in turn, specify uniquely the result that will be obtained by deciding to follow a certain program during the production period. Thus the entrepreneur can base his decisions on these known consequences. Under a risk condition, however, this does not hold; the entrepreneur is faced with a number of possible outcomes as a result of a decision to conduct a certain program. In order to make such a decision, then, it is of importance to be able to characterize the set of possible outcomes that are associated with a certain decision.

Probability statements may be used to characterize uncertain events. It can be said that those who propose to use probability for this purpose define it as some measure of the degree of belief that a particular outcome will occur, rather than some objective measure of relative frequency of the occurrence of a particular outcome. The possibility for inferring degrees of belief from observed actions of individuals has been developed and discussed by Savage (17). Some objections to the use of probability are discussed by Savage (17) and Arrow (2).

In describing a risk situation for programming purposes, all the components of the program; that is, prices p_i, p_i^o, and q_j, technical coefficients t_{ih} and t_{ih}^o, and quantities produced u_{ih} may be considered random variables. Tintner (20a) gives a procedure for handling this general use, but his results have, as yet, limited applications. We will consider here that only p_i, p_i^o, q_i, t_{ih}, t_{ih}^o, and u_{ih} are random variables. Furthermore, since $S_h = f(p_h, p_h^o, q_h, t_{ih}, t_{ih}^o, u_{ih})$, and since these quantities do not directly enter into any other part of the linear programming problem, we will simply state that the s_h are random variables.

If the s_h are normally distributed, then the vector $\underline{s}$ can be considered a multivariate normal with mean $\underline{u}$ and variance-covariance matrix Σ. The net revenue due to a program $\underline{x}$, is then, a univariate normal with mean $\mu_r = \underline{\mu}'\underline{x}$ and variance $\sigma^2 = \underline{x}'\Sigma\underline{x}$.

1.3.2 THE CHOICE AMONG ALTERNATIVE RISKY OUTCOMES

The choice among risky outcomes is equivalent to determining the "best" probability distribution. We may evaluate a probability distribution by means of the expected utility. The expected utility of the set of outcomes of a particular decision is, by definition, a measure of the relative value of the particular decision. Thus, if the expected utility due to all possible decisions is evaluated, the best decision can be picked by maximizing the expected utility.

The expected utility of a probability distribution of incomes is defined

$$E(y) = \int_r y(r)f(r)dr \qquad (1.17)$$

where $f(r)$ denotes the probability distribution of income and $y(r)$ denotes the relationship between outcome and utility. This relationship, called the utility function of income, gives a subjective value, a utility, for any given amount of income.

The simplest function relating utility and income or money is the linear function $y = r$. It can be shown that this utility function implies that maximizing expected utility is identical with maximizing expected income without regard to the variability of income, since

$$E(y) = \int_r rf(r)dr = E(r). \qquad (1.18)$$

However, experience shows that this is not an accurate picture of actual behavior.

A second utility function is one whose first derivative is everywhere positive and second derivative is everywhere negative. This function exhibits the usual assumption of positive but decreasing marginal utility of money. Such a function may be regardld as the one that best describes the behavior of an entrepreneur with limited capital who must practice relatively conservative policies. This utility function may be represented by $y = 1-e^{-ar}$ where a is an arbitrary constant which indicates the entrepreneur's aversion to risk. This can be shown by the fact that the maximization of the expected utility under this utility function and the assumption that r is normally distributed can be accomplished if we choose that probability distribution with the maximum value of the function

$$\mu - \frac{a}{2}\sigma^2 \qquad (1.19)$$

Note that a large value for a indicates that the variance plays a relatively larger role in the decision making process.

Another utility function is one represented by $e^{br} - 1$, where b is another arbitrary constant. This would be the utility function for a natural gambler, one who prefers those outcomes with wide dispersions and greater possibilities of great gains and losses. If we use the above utility function, the corresponding function to be maximized is

$$\mu + \frac{b}{2}\sigma^2 \qquad (1.20)$$

The above function indicates that the gambler is willing to reduce income to increase variance.

1.3.3 METHOD OF SOLUTION

As was stated in Section 1.3.1, net revenue associated with a program $\underline{x}$ is assumed to be normally distributed with $\mu_r = \underline{\mu}'\underline{x}$ and $\sigma^2 = \underline{x}'\Sigma_s\underline{x}$, where Σ is the matrix of the variances of net revenues. Thus, using the maximization of expected utility according to (1.19) above, the utility function to be maximized will be

$$y^* = E(\underline{s})\underline{x} - \frac{a}{2}\underline{x}'\Sigma_s\underline{x}.$$

The programming problem resulting from this maximization of expected utility is

$$\begin{aligned} &\text{maximize } y = E(\underline{s})\underline{x} - \frac{a}{2}\underline{x}'\Sigma_s\underline{x}, \\ &\text{subject to } T\underline{x} \leq \underline{v}, \qquad (1.22) \\ &\text{and } \underline{x} \geq 0. \end{aligned}$$

The procedure to be used for solving this problem will be that derived by Hildreth for the use of the Kuhn and Tucker minimax problem (See Section 1.2.2) with alterations as made by Freund (8). It was noted that to use this method the two restrictions to the above problem must be combined:

$$\begin{pmatrix} T \\ -I \end{pmatrix} \underline{x} \leq \begin{pmatrix} \underline{v} \\ 0 \end{pmatrix}$$

which was denoted

$$T^* \underline{x} \leq \underline{v}^* \qquad (1.10)$$

Since we will only be concerned with the vector $\underline{v}^*$ and the matrix T^*, we will omit the asterisk. Also, we will denote the matrix

$$\frac{a}{2}\Sigma_s$$

by

$$\Sigma .$$

The minimax equivalent for the programming problem can thus be written:

$$\left.\begin{array}{l} \text{minimize with respect to } \underline{x} \\ \text{maximize with respect to } \underline{u} \end{array}\right\} \phi(\underline{x},\underline{u}) = \underline{x}'\Sigma\,\underline{x} - \underline{s}'\underline{x} - \underline{u}'(\underline{v} - T\underline{x}). \quad (1.23)$$

CHAPTER II

AN EXAMPLE

2.1 THE DESCRIPTION OF THE FIRM AND THE NO RISK PROGRAM

2.1.1 THE DESCRIPTION OF THE FIRM

We shall use a programming model for a representative sixty acre farm in the Blacksburg area of Southwestern Virginia.

The farm may be identified by the processes[1] available to it:

1. Forkedeer Fall Oats,
2. Vahart Wheat,
3. Wong Barley,
4. U. S. 13 Corn,
5. Irish Cobbler Potatoes,
6. Rutgers Tomatoes,
7. Kansas Common Alfalfa,

and by the scarce resources:

1. Land,
2. Production capital, period 1 (September 10 - January 9),
3. Production capital, period 2 (January 10 - June 9),
4. Production capital, period 3 (June 10 - September 9),
5. Managerial labor, period 1,
6. Managerial labor, period 2,
7. Managerial labor, period 3.

Land refers to that portion of cleared cropland that is suitable for cultivation. Production capital is the availability of cash for net expenses. Managerial labor refers to the farm operator's own labor. Certain non-managerial tasks can be performed by hired labor, which, it is assumed, can be hired without limit at a given wage rate.

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[1]For the sake of simplicity only harvestable crops were used.

TABLE 2-1

BASIC BUDGET FOR ONE ACRE OF FALL OATS

Item	Price Dollars	Unit	Quan-tity	Amount Dollars	Period
Income					
Yield	.77	bu.	60.0	46.20	3
Expenses					
Seed	1.85	bu.	2.25	4.16	1
Fertilizer 5-10-10	2.25	cwt.	5.0	11.25	1
Machine operation					
Land preparing and planting	.56	hrs.	1.96	1.10	1
Harvesting	.56	hrs.	1.04	.58	3
Total all expenses				17.09	
Net Revenue				29.11	
Operator's Labor					
Land preparation and planting		hrs.	2.37		1
Harvesting		hrs.	2.08		3

TABLE 2-2

BASIC BUDGET FOR ONE ACRE OF WHEAT

Item	Price Dollars	Unit	Quantity	Amount Dollars	Period
Income					
Yield	1.90	bu.	30.0	57.00	3
Expenses					
Seed	3.00	bu.	1.5	4.50	1
Fertilizer 5-10-10	2.25	cwt.	5.0	11.25	1
Machine operation					
Land preparation and planting	.56	hrs.	1.96	1.10	1
Harvesting	.56	hrs.	1.04	.58	3
Total all expenses				17.43	
Net Revenue				39.57	
Operator's Labor					
Land preparation and planting		hrs.	2.37		1
Harvesting		hrs.	2.08		3

TABLE 2-3

BASIC BUDGET FOR ONE ACRE OF BARLEY

Item	Price Dollars	Unit	Quantity	Amount Dollars	Period
Income					
Yield	.95	bu.	40.0	38.00	3
Expenses					
Seed	2.25	bu.	2.0	4.50	1
Fertilizer 5-10-10	2.25	cwt.	5.0	11.25	1
Machine operation					
Land preparation and planting	.56	hrs.	1.96	1.10	1
Harvesting	.56	hrs.	1.04	.58	3
Total all expenses				17.43	
Net Revenue				20.57	
Operator's Labor					
Land preparation and planting		hrs.	2.37		1
Harvesting		hrs.	2.08		3

TABLE 2-4

BASIC BUDGET FOR ONE ACRE OF CORN

Item	Price Dollars	Unit	Quantity	Amount Dollars	Period
Income					
Yield	1.60	bu.	80.0	128.00	1
Expenses					
Seed, hybrid	11.00	bu.	0.2	2.20	2
Fertilizer 10-10-10	3.00	cwt.	8.0	24.00	2
Tractor operation					
Land preparation and planting	.48	hrs.	2.83	1.36	2
Cultivating	.48	hrs.	2.22	1.06	3
Harvesting	.48	hrs.	1.78	.85	1
Total all expenses				29.47	
Net Revenue				98.53	
Operator's Labor					
Land preparation and planting		hrs.	3.03		2
Cultivating		hrs.	4.17		3
Harvesting		hrs.	2.63		1

TABLE 2-5

BASIC BUDGET FOR ONE ACRE OF IRISH POTATOES

Item	Price Dollars	Unit	Quantity	Amount Dollars	Period
Income					
Yield	1.05	bu.	250.0	262.50	3
Expenses					
Seed, Cobbler	4.45	cwt.	8.1	36.05	2
Fertilizer 6-8-6	2.15	cwt.	23.0	49.45	2
Dust, 5% D.D.T.	.17	lb.	60.0	10.20	2
Bags, new	.22	bag	140.0	30.80	3
Washing and Grading	.35	bag	140.0	49.00	3
Digging, contracted	9.00	acre	1.0	9.00	3
Picking up, contracted	.13	70 lb. bag	215.0	27.95	3
Hauling, contracted	.07	70 lb. bag	215.0	15.05	3
Tractor operation					
Land preparation and planting	.48	hrs.	4.0	1.92	2
Poisoning	.48	hrs.	1.0	.48	3
Total all expenses				229.90	
Net Revenue				32.60	
Operator's Labor					
Land preparation and planting		hrs.	4.0		2
Digging, supervising		hrs.	.8		3
Poisoning		hrs.	1.0		3

TABLE 2-6

BASIC BUDGET FOR ONE ACRE OF TOMATOES

Item	Price Dollars	Unit	Quan-tity	Amount Dollars	Period
Income					
Yield	.90	bu.	533.0	480.00	3
Expenses					
Plants	12.00	M	3.0	36.00	2
Fertilizer 1200 lbs. 8-20-16	6.00	cwt.	12.0	72.00	2
Machine operation					
Growing	.49	hrs.	7.9	3.87	2
Growing	.49	hrs.	7.9	3.87	3
Spraying and dusting	17.85	acre	1.0	17.85	3
Trucking	.20	mi.	63.0	12.60	3
Picking, contracted	6.00	ton	16.0	96.00	3
Containers & others	8.12	acre	1.0	8.12	3
Total all expenses				250.31	
Net Revenue				229.69	
Operator's Labor					
Growing		hrs.	18.2		2
Growing		hrs.	18.2		3
Harvesting		hrs.	20.0		3

TABLE 2-7

BASIC BUDGET FOR ONE ACRE OF ALFALFA

Item	Price Dollars	Unit	Quantity	Amount Dollars	Period
<u>Income</u>					
Yield	42.85	Ton	2.67	114.27	2
Yield	42.85	Ton	1.33	57.13	3
<u>Expenses</u>					
Seed (Pro-rated seeding cost over 4 yrs.)	13.00	acre	1.0	13.00	1
Top Dressing 0-10-20	2.20	cwt.	8.0	17.60	1
Machine expenses					
For establishing	.48	hrs.	1.18	.57	1
For maintenance and	.56	hrs.	3.14	2.35	2
harvesting	.56	hrs.	2.14	1.17	3
Hired labor	.80	hrs.	1.41	1.13	2
Hired labor	.80	hrs.	.71	.57	3
<u>Total all expenses</u>				36.39	
<u>Net Revenue</u>				135.01	
<u>Operator's Labor</u>					
Establishing		hrs.	1.41		1
Maintenance and Harvesting		hrs.	7.33		2
Maintenance and Harvesting		hrs.	3.67		3

The budgets for the model were constructed by Dr. R. G. Kline of the Agricultural Economics Department at the Virginia Polytechnic Institute and the author with the assistance of Mr. K. E. Loope and Dr. C. W. Allen, also of the Agricultural Economics Department.

The basic budgets for processes 1-7 are presented in tables 2-1, 2-2, 2-3, 2-4, 2-5, 2-6, and 2-7, respectively.

2.1.2 THE TECHNOLOGY MATRIX

The derivation of the (scarce resources) technology matrix, T, from the budgets will be illustrated for process 1, Fall Oats. We will present the data for the other budgets without further explanation. From Table 2-1 we obtain the various amounts of the scarce resources needed for one acre of oats. The amount of scarce resources needed and the net revenue are given in Table 2-8.

TABLE 2-8

SCARCE RESOURCES NEEDED AND NET REVENUE

FOR ONE ACRE OF FALL OATS

Resources	Amounts
Land, acres	1.0
Production capital, dollars	
Period 1	16.51
Period 2	.00
Period 3	-45.62
Managerial labor, hours	
Period 1	2.37
Period 2	.0
Period 3	2.08
Net revenue, dollars	29.11

Note that Table 2-8 gives the net requirements of scarce resources. In the case of production capital for the third period, production capital is created by the sale of oats. Thus, the net requirements of capital for the third period becomes \$0.58 - \$46.20 = -\$45.62, the negative amount denoting the creation rather than the use of the resource.

For computational simplicity it is advantageous that unit levels of the various processes produce equal amounts of net revenue. For this reason, all processes are redefined such that a unit level produces \$100 net revenue. In the case of oats this is accomplished by raising 100/29.11 = 3.435 acres of oats. This redefinition causes all resources requirements to be multiplied by 3.435. Finally all production capital requirements and the unit level profits are expressed in hundreds of dollars. This scaling device makes the numbers in the technology matrix roughly of the same magnitude, which tends to make computations less subject to rounding errors. The resulting vector, $\underline{t}_1$, and the unit level profit, s_1, are presented in table 2-9.

TABLE 2-9

SCARCE RESOURCE VECTOR, $\underline{t}$, FOR PROCESS 1

FALL OATS

Resources	Amounts
Land, acres	3.435
Production capital, \$100	
Period 1	.567
Period 2	.000
Period 3	-1.567
Managerial labor, hours	
Period 1	8.140
Period 2	.000
Period 3	7.140
Unit level profit, \$100	1.000

The resulting technology matrix, T, of the entire firm, the unit level profit vector, s, and the resources limitations vector, v, are presented in Table 2-10.

The only item in Table 2-10 that has not been discussed is the construction of the resources limitation vector, v. This vector is somewhat arbitrary, particularly when one wishes to characterize a representative farm. In our case we have assumed a sixty acre farm with homogeneous fertility of the soil. The availability of production capital is based on an initial stock of $4500 on hand as of the beginning of Period 1, of which $1500 must be reserved for living expenses in each of the three periods. The amount of managerial labor available is limited to the number of good weather daylight hours in each period and is based on past weather records.

TABLE 2-10

THE TECHNOLOGY MATRIX FOR THE PROGRAMMING PROBLEM

Scarce Resources	INPUT VECTORS FOR PROCESSES AT UNIT LEVELS t_1 Oats	t_2 Barley	t_3 Wheat	t_4 Corn	t_5 Potatoes	t_6 Tomatoes	t_7 Alfalfa	Availability Vector $\underline{v}$
Land, acres	3.435	2.527	4.861	1.015	3.067	.435	.740	60
Production Capital, $100								
Period 1	.567	.426	.819	-1.291	.0	.0	.231	30
Period 2	.0	.0	.0	-1.011	2.994	.487	-1.243	15
Period 3	-1.567	-1.426	-1.819	-1.000	-3.994	-1.486	-1.330	0
Managerial Labor, hours								
Period 1	8.140	5.989	11.520	2.670	.0	.0	1.040	799
Period 2	.0	.0	.0	3.075	12.268	10.870	5.424	867
Period 3	7.140	5.256	10.110	4.230	5.521	13.660	2.716	783

Under no circumstances will the restriction on production capital in Period 1 or Period 3, and the restriction on labor in Period 1 become effective. This is true since these restrictions, for any process, are less stringent than the other restrictions. We can, therefore, regard the corresponding resources as being in unlimited supply and can omit these restrictions from the programming problem.

2.1.3 THE OPTIMUM NO RISK PROGRAM

Solution by the simplex technique (see Chapter I) resulted in the optimum program presented in Table 2-11. This is the No Risk Program since variability was not considered. In this program, note the heavy reliance on tomatoes, which are a highly unstable, or risky, crop. Also small grain crops, the usual backbone of a general purpose farm, are not raised. The above features, which have been noted in other linear programming problems, cause us to consider the effect of risk into a programming model.

TABLE 2-11

THE OPTIMUM NO RISK PROGRAM

Process	Unit Levels	Acres
1. Oats	.00	.00
2. Barley	.00	.00
3. Wheat	.00	.00
4. Corn	.00	.00
5. Potatoes	.00	.00
6. Tomatoes	46.65	20.29
7. Alfalfa	53.66	39.72
Net Revenue ($100)	100.31	

2.2 THE RISK PROGRAM

2.2.1 THE DESCRIPTION OF A RISK SITUATION: THE COMPUTATION OF VARIANCES OF UNIT LEVEL NET REVENUES

The variabilities of the unit level net revenues will be assumed to depend almost entirely on the output prices and quantities (see Section 1.3.1), while most of the inputs are fixed. However, certain inputs, such as picking costs for tomatoes and potatoes, do vary with the quantity produced. We will thus deal with adjusted gross revenues, which in this case will be the product of prices and yields of the various crops, adjusted for variable inputs, since the variability of adjusted gross revenues is equal to the variance of net revenues.

We are interested in obtaining the entrepreneur's estimate of the variability of prices and yields of individual crops as an indication of a probability distribution. This estimate is usually based on past behavior. Therefore variability estimates were obtained from data for 1948-1956. This was estimated in two stages: (1) the estimation of past prices in constant dollars, and (2) the estimation of past yields under constant cultivation practices.

Price estimates, with the exception of tomatoes, were obtained by using average Virginia prices (21) for each crop, deflating these prices by the United States Wholesale Price Index (1) and finally multiplying these deflated prices by a constant so as to obtain an average price equal to the price specified in the budget[1]. Records of past tomato prices were furnished by J. M. Johnson of the Department of Agricultural Economics at Virginia Polytechnic Institute.

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[1]This adjustment procedure assumes that variation in yield is proportioned to the mean.

The yields of each of the crops used were obtained from records of experiments conducted by the Virginia Agricultural Experiment Station at Blacksburg. Data on small grains were obtained from T. M. Starling (18 & 19), data on corn from C. F. Genter (10, 11 & 12), data on alfalfa from T. S. Smith and P. T. Gish, and the data for potatoes and tomatoes from P. H. Massey, Jr., and others. All data were obtained from experiments conducted in the Blacksburg area such that yield variability caused by different locations was eliminated. A linear time trend as an indicator of improvement in crop growing practices was computed for each crop by least squares regression. In no case did the linear trend prove to be significant, and consequently no adjustment for time trend was made. The yields were then adjusted by the ratio of the yield specified in the budget to the average yield obtained above so as to obtain the yield specified in the budget[1].

The product of estimated per acre yield and estimated price for each year was then used as the estimate of gross revenue for each crop. The gross revenue was then adjusted for variable inputs as was mentioned previously. Variances were computed from this data and multiplied by the appropriate constants in order to reduce them to variances of unit levels. The computations for the estimated gross revenue of process 1, oats, are reproduced in Table 2-12 to illustrate the procedure.

---

[1]This adjustment procedure assumes that variation in yield is proportioned to the mean.

TABLE 2-12

ESTIMATION OF GROSS REVENUE FROM ONE ACRE OF OATS

Year	Price $	Deflated Price $	Adj. to $.77 $	Yield Bu.	Estimated Yield Bu.	Adjusted[1] Gross Revenue $
1948	.87	.84	.87	56.4	49.86	43.38
1949	.70	.71	.74	83.3	73.64	54.49
1950	.81	.79	.82	49.5	43.76	35.88
1951	.79	.69	.72	47.3	41.81	30.10
1952	.91	.81	.84	50.8	44.91	37.72
1953	.82	.74	.77	94.5	83.54	64.33
1954	.80	.73	.76	90.4	79.91	60.73
1955	.71	.64	.66	67.0	59.23	39.09
1956	.76	.68	.71	71.6	63.29	44.94

[1]In the case of oats, wheat, barley, corn and alfalfa, gross revenue was not adjusted for variable inputs.

The estimated per acre adjusted gross revenues for the seven crops in the example are presented in Table 2-13. These adjusted gross revenues were used to estimate variances of per acre gross revenue (Table 2-14). Covariances were assumed to be zero. The per acre variances were reduced to unit level variances (Table 2-15).

TABLE 2-13

THE ESTIMATED PER ACRE ADJUSTED REVENUES

Year	Oats	Wheat	Barley	Corn	Potatoes	Tomatoes	Alfalfa
1948	43.38	66.09	56.87	137.94	167.59	253.30	165.38
1949	54.49	58.62	36.10	129.27	281.43	396.53	116.50
1950	35.88	52.41	25.84	123.85	144.53	207.31	128.73
1951	30.10	46.44	53.42	113.89	272.61	260.16	186.52
1952	37.72	32.76	42.06	118.25	338.60	542.78	207.60
1953	64.33	56.10	40.93	91.50	139.50	356.02	177.04
1954	60.73	70.31	28.87	146.08	110.65	297.92	145.92
1955	39.09	58.88	25.46	124.47	180.85	503.93	126.22
1956	44.94	70.52	32.39	123.93	454.58	612.98	240.20

TABLE 2-14

PER ACRE VARIANCES

Crop	Variance
Oats	137.92
Wheat	146.20
Barley	46.85
Corn	236.97
Potatoes	12,836.92
Tomatoes	20,493.02
Alfalfa	1,700.56

TABLE 2-15

UNIT LEVEL VARIANCES

Crop	Variance
Oats	1,627.32
Wheat	933.63
Barley	1,107.02
Corn	244.08
Potatoes	120,744.06
Tomatoes	3,873.18
Alfalfa	931.91

2.2.2 A COMPUTATION OF THE RISK PROGRAM

We shall derive an optimum risk program resulting from the maximization of the expected utility of the form

$$y = \mu - \frac{a}{2}\sigma^2$$

as discussed in Section 1.3.2. The data used will be the same as in the linear program together with the variances as derived in the previous section. The constant a will be given an arbitrary value 1/1500. Further discussion of this constant can be found in Section 1.3.2 and Section 3.1.1.

The procedure which will be followed in the computation is identical to the procedures outlined in Section 1.3.3. It should be recalled that we need to find

$$C = T\Sigma^{-1}T', \tag{2.1}$$

and

$$2\underline{b} = 2\underline{v} - T\Sigma^{-1}\underline{s}, \tag{2.2}$$

where

$$\Sigma = \frac{a}{2}\Sigma_s \tag{2.3}$$

The data for this problem have been discussed before but will be presented again at this point to facilitate the presentation.

$$\underline{s}' = (100 \quad 100 \quad 100 \quad 100 \quad 100 \quad 100 \quad 100)$$

$$\Sigma = \begin{bmatrix} .542 & & & & & & \\ & .311 & & & & & \\ & & .369 & & & & \\ & & & .081 & & & \\ & & & & 40.248 & & \\ & & & & & 1.291 & \\ & & & & & & .311 \end{bmatrix}$$

$$\underline{v}' = (60 \quad 15 \quad 867 \quad 783 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$$

$$T = \begin{bmatrix} 3.435 & 2.527 & 4.861 & 1.015 & 3.067 & .435 & .740 \\ 0 & 0 & 0 & -1.011 & 2.994 & .487 & -1.243 \\ 0 & 0 & 0 & 3.075 & 12.268 & 10.870 & 5.424 \\ 7.140 & 5.256 & 10.11 & 4.230 & 5.521 & 13.660 & 2.716 \\ -1 & & & & & & \\ & -1 & & & & & \\ & & -1 & & & & \\ & & & -1 & & & \\ & & & & -1 & & \\ & & & & & -1 & \\ & & & & & & -1 \end{bmatrix}$$

From this we must now compute Σ^{-1}, C, $T\Sigma^{-1}$, and $2\underline{b} = 2\underline{v} - T\Sigma^{-1}\underline{s}$:

$$\Sigma^{-1} = \begin{bmatrix} 1.845 & & & & & & \\ & 3.215 & & & & & \\ & & 2.710 & & & & \\ & & & 12.346 & & & \\ & & & & .025 & & \\ & & & & & .775 & \\ & & & & & & 3.215 \end{bmatrix}$$

$$2\underline{b} = \begin{pmatrix} -4,169.370 \\ 1,627.500 \\ -4,664.100 \\ -11,329.200 \\ 184.350 \\ 321.300 \\ 270.990 \\ 1,229.100 \\ 2.485 \\ 77.454 \\ 321.900 \end{pmatrix}$$

$$
{}_{37}C = \begin{bmatrix}
121.107 & -15.180 & 55.884 & 285.315 & -6.332 & -8.119 & -13.173 & -12.474 & -.076 & -.337 & -2.382 \\
-15.180 & 17.943 & -54.906 & -57.867 & 0 & 0 & 0 & 12.426 & -.074 & -.377 & 4.002 \\
55.884 & -54.906 & 306.180 & 323.970 & 0 & 0 & 0 & -37.800 & -.305 & -8.418 & 17.460 \\
285.315 & -57.864 & 323.940 & 848.610 & -13.161 & -16.887 & -27.396 & -51.990 & -.137 & -10.578 & -8.742 \\
-6.332 & 0 & 0 & -13.161 & 1.844 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8.119 & 0 & 0 & -16.887 & 0 & 3.215 & 0 & 0 & 0 & 0 & 0 \\
-13.173 & 0 & 0 & -27.396 & 0 & 0 & 2.710 & 0 & 0 & 0 & 0 \\
-12.474 & 12.426 & -37.800 & -51.990 & 0 & 0 & 0 & 12.346 & 0 & 0 & 0 \\
-.076 & -.074 & -.305 & -.137 & 0 & 0 & 0 & 0 & .025 & 0 & 0 \\
-.337 & -.377 & -8.418 & -10.578 & 0 & 0 & 0 & 0 & 0 & .775 & 0 \\
-2.382 & 4.002 & 17.460 & -8.742 & 0 & 0 & 0 & 0 & 0 & 0 & 3.215
\end{bmatrix}
$$

The iterations for obtaining the vector $\underline{u}$ were performed using the IBM type 650 magnetic drum computer located at North Carolina State College in Raleigh, North Carolina. The program for use with the 650 was devised by R. J. Freund of the Department of Statistics at Virginia Polytechnic Institute. This program computes the iterations at a speed of approximately 100 times faster than an experienced operator on an automatic desk calculator. The iterations for this problem became stable after about 350 iterations. Without the use of the 650, the work as outlined in the results of this paper would have been practically impossible to obtain.

2.2.3 THE OPTIMUM RISK PROGRAM

The risk program and the no-risk program, together with other pertinent information are presented in Table 2-16. The figures in the table conform very well as to what was expected. The consideration of risk increased the importance of corn, the crop with the lowest unit level variance, and decreased the importance of tomatoes, a comparatively high risk crop.

The total revenue was, of course, decreased, but the expected utility of the form $\mu - \frac{a}{2}\sigma^2$ was increased and the standard deviation of the net revenue was substantially decreased. The risk program also requires less capital and labor than the no-risk program.

TABLE 2-16

COMPARING THE RISK AND THE NO RISK PROGRAM

Item	Risk Program	No Risk Program
Process Intensities		
Unit Levels		
Oats	.00	.00
Wheat	.00	.00
Barley	.00	.00
Corn	15.61	.00
Potatoes	.00	.00
Tomatoes	22.55	46.65
Alfalfa	46.58	53.66
Acres		
Oats	.00	.00
Wheat	.00	.00
Barley	.00	.00
Corn	15.80	.00
Potatoes	.00	.00
Tomatoes	9.80	20.29
Alfalfa	34.40	39.70
Expected Net Revenues, dollars	8,475.00	10,031.00
Expected Utility, arbitrary units	7,126	6,326
Standard Deviation of Net Revenue, dollars	2,012	3,334
Use of Resources		
Land	60.00	59.99
Production capital		
Period 1	-9.3925	12.3997
Period 2	-62.6980	-43.9923
Period 3	-106.4100	-135.336
Managerial Labor		
Period 1	90.14	55.81
Period 2	545.77	798.19
Period 3	500.57	783.00

CHAPTER III

FURTHER ASPECTS OF RISK PROGRAMMING

The example of the previous Chapter will be used to investigate additional phases of programming for the risk case. We shall discuss the "Risk Aversion" map in some detail and briefly touch on price, resource and variance maps as aspects of risk programming. Most of these have been perfected and used in the no-risk case, and extension of these aspects to risk programming should lead to useful results.

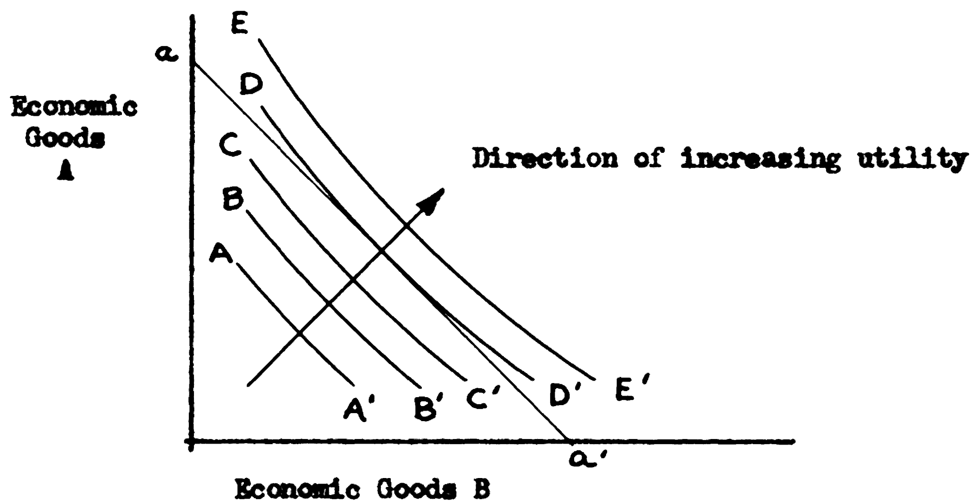
3.1 THE "RISK AVERSION" MAP

In Section 2.2, we did not justify the use of a particular value for the risk aversion constant. We shall present in this Section a method of risk programming which avoids this difficulty.

To a certain extent each combination of economic goods is a substitute for every other combination. Among the possible combinations, there are some which are preferable to others; but there are some which are equally attractive to us, so that we feel it is a matter of indifference which one we choose (3). This may be represented by an indifference curve, which is the loci of all points, combinations of goods, or of indifference. Another way of stating this condition is that an indifference curve is the loci of all points of equal utility.

There are other combinations of economic goods lying outside a given indifference curve which give rise to other indifference curves. A set of indifference curves combine to produce an indifference map showing indifference curves ranked from those of lowest utility to those of highest utility. There exists a unique relationship between a utility function and an indifference map. An idealized indifference map is illustrated by the set of curves AA', BB', CC', DD' and EE' in Figure 3.1.

Figure 3.1.

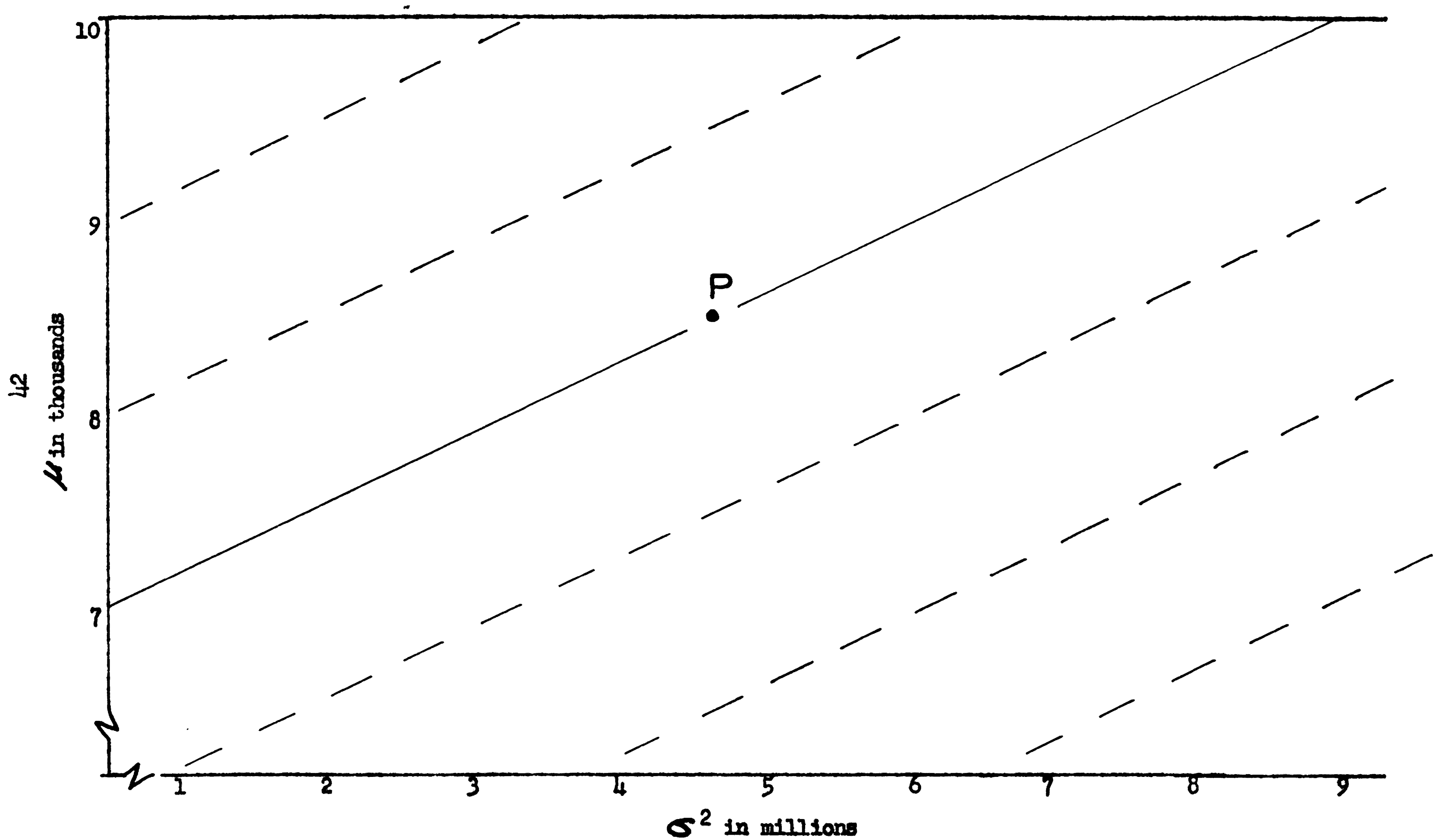


The problem facing an entrepreneur, or consumer, is to maximise utility subject to income limitations. An idealized income limitation is given by the line aa' in Figure 3.1; that is, linear combination of goods can be purchased by a given amount of income. It is intuitively obvious that the point of maximum utility occurs at the point where the income or opportunity line is tangent to an indifference curve. More vigorous arguments may be found in economic reference works, e. g., (3).

In risk conditions, where risky outcomes may be represented by the expected return (μ) and the standard deviation of this net return (σ), we may construct an indifference map for various combinations of μ and σ. The indifference map of these "economics goods" is somewhat the reverse of the illustration in Figure 3.1. since standard deviation, being the reverse of reliability, is an unfavorable good. The indifference map resulting from the utility function of income (r) discussed in Section 1.3.2, $u = \mu - \frac{a}{2}\sigma^2$, for $a = 1/1500$, is given in Figure 3.2. The curves represent loci of equal values of u at 4000, 5000, 6000, 7000, 8000 and 9000.

Figure 3.2

INDIFFERENCE MAP FOR a = 1/1500



In Section 2.2.3 the optimum risk program assuming the utility function $u = \mu - \frac{1}{2(1500)} \sigma^2$ was obtained, resulting in μ = \$8475, σ = \$2012, and u = 7126. It can be seen that this combination of μ and σ lies very close to the indifference curve for u $\sim$ 7000 in Figure 3.2. This point is marked by P. In Fact, according to the above theory, the point P should lie on the opportunity line of the entrepreneur which is tangent to the indifference curve for u = 7126.

Indifference maps similar to Figure 3.2. could be constructed for various values of the risk aversion constant. By recomputing optimum risk programs for each of these values, we can obtain several points on the opportunity line which are tangent to the indifference curves corresponding to utility functions with different values of "a". The different points of tangency may be joined together to form an opportunity curve.

This opportunity curve represents combinations of net revenue and the variance of net revenue which are available to the entrepreneur. An entrepreneur could choose a point on the curve which to him represents the best combination of net revenue and variance. In doing so, he will effectively be choosing his own risk aversion constant and corresponding optimum program.

Using the computational shortcut discussed in the appendix, optimum programs were developed for $a = \frac{1}{750}, \frac{1}{1000}, \ldots, \frac{1}{2500}$. The results of the computations are summarized in Table 3-1, and the corresponding opportunity curve in figure 3.3.

As was expected, the larger the value of the risk aversion constant "a", the more the optimum program depended on the comparatively low risk crop, corn. The smaller the value of "a", the closer the optimum

TABLE 3-1

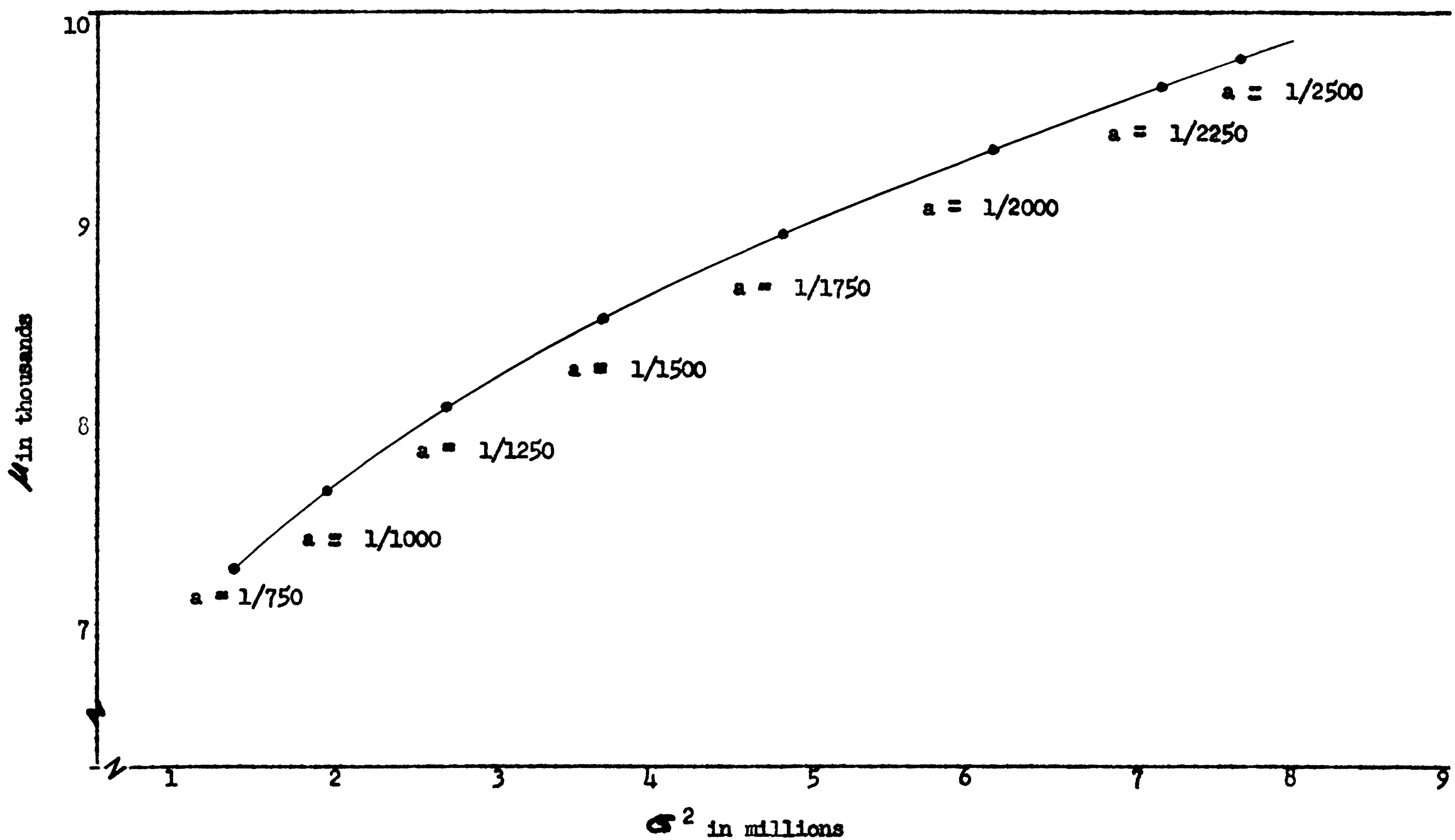
OPTIMUM PROGRAMS FOR VARIOUS RISK AVERSION CONSTANTS

Item	Various values of "a"								Optimum Program
Process Intensities* - Unit levels	1/750	1/1000	1/1250	1/1500	1/1750	1/2000	1/2250	1/2500	Linear
Corn	33.48	27.52	21.57	15.61	9.67	3.69	0	0	0
Tomatoes	11.97	15.49	19.02	22.55	26.08	29.61	32.76	35.21	46.65
Alfalfa	28.20	34.32	40.46	46.58	52.72	58.85	62.35	60.88	53.66
Acres									
Corn	33.98	27.93	21.80	15.80	9.81	3.70	0	0	0
Tomatoes	5.20	6.74	8.27	9.80	11.24	12.87	14.05	15.01	20.29
Alfalfa	20.86	25.40	29.93	34.40	39.01	43.50	46.03	45.05	39.70
Expected Net Revenue, Dollars	7365	7735	8105	8475	8847	9215	9511	9609	10031
Standard Deviation of Net Revenue, Dollars	1252	1487	1743	2013	2290	2574	2789	2873	3334

*Oats, Wheat, Barley and Potato crops never entered the program.

Figure 3.3

OPPORTUNITY CURVE



program approached the linear program, which may be considered a risk program with a = $\frac{1}{\infty}$. It is believed, however, that a finite value of "a" will produce an optimum exactly equal to the linear program. As the value of "a" increases, indicating greater aversion to risk, the net revenue decreases and the standard deviation of net revenue decreases. It is interesting to note that the lower 2% limit ($\mu - t_{.02}\sigma$) of the net revenue in the linear program is less than the lower 2% limit of any of the risk programs.

3.2 PRICE MAPS

In practical applications of linear programming, it is of interest to investigate changes in the optimum program due to variations in prices of various products produced by the firm. Such investigations are usually graphically presented in the form of price maps which show the various optimum programs. Price maps can conceptually be derived for various combinations of all prices, but for the sake of simplicity we will concern ourselves only with price changes in one product.

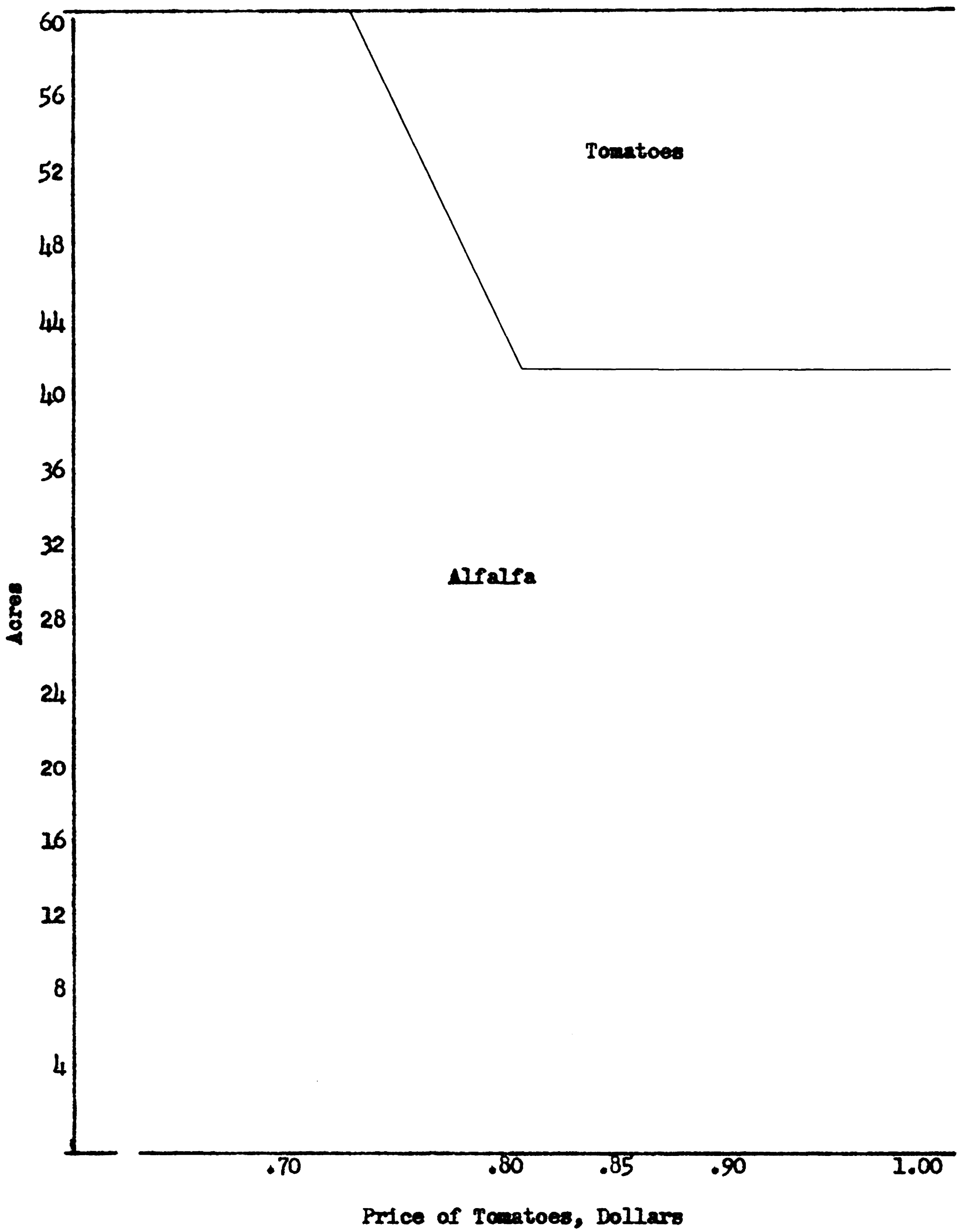
3.2.1. PRICE MAPS FOR A NO-RISK PROGRAM

The net effect of price changes on the linear programming model consists of changes in the elements of the vector $\underline{s}$. Computational short-cuts permit the computation of optimum programs for price changes without the complete recomputation for each price (8).

A price map, showing optimum programs for a range of tomato prices, is presented in Figure 3.4. Tomato prices of less than \$.70 per bushel indicate programs with no tomatoes; between \$.70 and \$.80 tomatoes replace alfalfa. Above \$.80 no additional tomatoes can be grown due to

Figure 3.4

PRICE MAP, NO-RISK PROGRAM



the third-period labor restriction.

3.2.2 PRICE MAP FOR THE RISK PROGRAM

Effects of changes in the price of tomatoes were used to determine the effect on the optimum program. These changes are easily incorporated in the risk program.

Consider equation (2.2) where

$$\underline{b} = \underline{v} - \tfrac{1}{2}T\Sigma^{-1}\underline{s}.$$

Price changes for products produced by a process can be incorporated into new values for the vector $\underline{s}$. It is then necessary to recompute $\underline{b}$ and change the last column of the multiplier matrix Z, (see appendix equation 5.2), and proceed with the Hildreth solution procedure.

Programs for a = 1/1500 for various prices of tomatoes are shown in Table 3-2. Using these data we may construct, by interpolation, a price map which shows the production programs as continuous functions of tomato prices. This price map is given in Figure 3.5.

The risk price map shows some similarities to the no-risk map. The primary difference consists of a "flattening" of the price response: a given price change has more effect on the no-risk program. Extrapolation indicates that tomatoes will enter the no-risk program at about \$.70 per bushel and the risk program at about \$.65 per bushel; at \$.80 per bushel the maximum tomato production is already attained with the no-risk program; in the case of the risk program, production of tomatoes is still increasing at a price of \$1.00 per bushel.

3.3 RESOURCE MAPS

The risk program is, again, easily adaptable to the comparison

TABLE 3-2

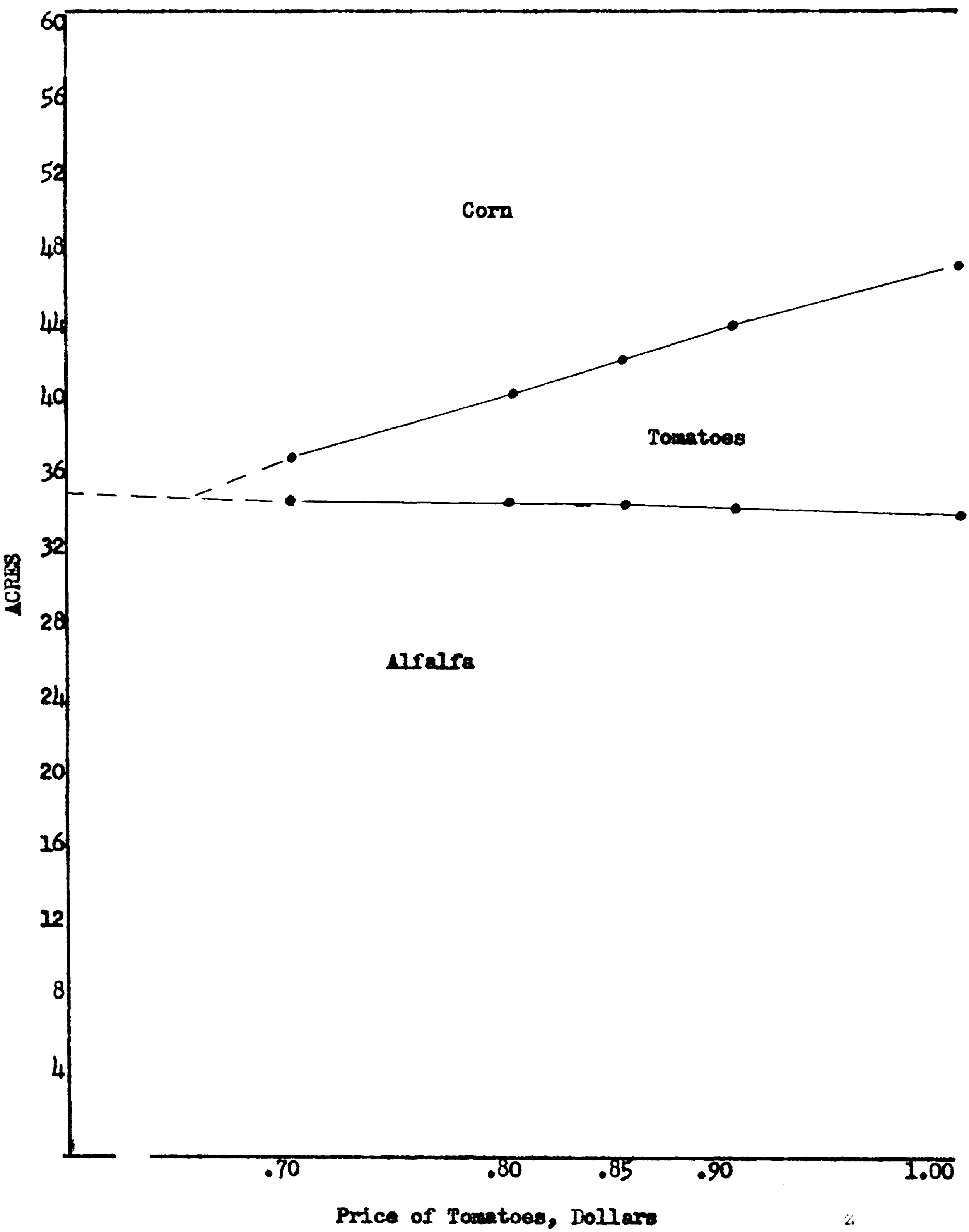
OPTIMUM RISK PROGRAMS FOR VARIOUS TOMATO PRICES (a = 1/1500)

Price of Tomatoes, Dollars Per Bushel

Item	0.70	0.80	0.85	0.90	1.00
Process Intensities					
Unit Levels					
Corn	22.32	18.98	17.30	15.61	12.28
Tomatoes	4.71	13.60	18.05	22.55	31.40
Alfalfa	47.87	47.23	46.91	46.59	45.95
Acres					
Corn	22.60	19.20	17.50	15.84	12.46
Tomatoes	2.05	5.91	7.85	9.81	13.60
Alfalfa	35.40	34.95	34.71	34.40	34.00
Net Revenue,					
Dollars	7490	7981	8226	8475	8963

Figure 3.5

PRICE MAP, RISK PROGRAM



programs when the availability of one or more scarce resources are changed. Procedures for doing this have already been developed for the no-risk case (8). In the case of risk programming if we are interested in changing the k-th resource (v_k), it can be seen that only one element, $\frac{2b_k}{c_{kk}}$, of the matrix Z (equation 5.2) is affected by such a change. From equation (5.6),

$$\frac{2b_k}{c_{kk}} = \frac{av_k - T\Sigma_s^{-1} s_k}{c_{kk}},$$ where v_k, s_k, and c_{kk} are elements of

$\underline{v}$, $\underline{s}$, and C. If we are interested in changing the availability of more than one resource, a direct extension may be applied.

The inadequacy of computational facilities prevented a complete investigation of resource maps.

3.4 VARIANCE MAPS

The variance, of course, does not enter into no-risk programming and, as such, variance maps have no parallel in no-risk programming. Computational shortcuts were investigated in connection with comparing programs when one or more variances were changed. No simplifying transformation were found.

CHAPTER IV

SUMMARY

Linear, or no-risk, programming has been used for finding the optimum combination of enterprises for a firm. This method of programming has a definite limitation in that provisions have not been made for considering the relative risk involved in various alternative programs. If we incorporate the risk feature into a programming problem by considering the theory of choice under risk as equivalent to the choice from among several alternative probability distributions, the expected utility hypothesis may be used. Under this hypothesis, the entrepreneur acts as if he maximizes expected utility which is defined:

$$E(y) = \int_r y(r)f(r)dr,$$

where y is the expected utility, y(r) is the utility of a particular amount of income, r, and f(r) is the probability density function. The resulting expected utility depends, then, on the form of the utility of income and on the distribution of income. As shown by Freund (8), if the utility function of income is assumed to be

$$y(r) = 1 - e^{-ar}$$

and income is assumed to be normally distributed, the maximization of expected utility becomes the equivalent of maximizing the linear combination of expected income (μ) and variance of income (σ^2)

$$E(y) = \mu - \frac{a}{2}\sigma^2,$$ where "a" represents the risk aversion constant. The maximization of expected utility can be used for a programming analysis where it results in a quadratic programming problem.

In the example presented in this thesis, experimental data based on varietal tests were used to obtain estimates of the variance of net revenue of the various crops considered. The results conformed very well as to what was expected. The consideration of risk increased the importance of corn, the crop with the lowest unit level variance, and decreased the importance of tomatoes, a comparatively high risk crop. In the risk program the total revenue was, of course, decreased, but the expected utility was increased and the standard deviation of the net revenue was substantially decreased. The risk program also requires less capital and labor than the no-risk program.

The risk aversion constant, a, plays an important part in the programming problem, since the wrong choice for the risk aversion constant would invalidate any results obtained. We may overcome this difficulty by constructing indifference maps for various values of the risk aversion constant, recomputing optimum risk programs for each of these values, and obtaining several points on the opportunity line which are tangent to the indifference curves corresponding to utility functions with different values of "a". The different points of tangency may then be joined together to form an opportunity curve. This opportunity curve represents combinations of net revenue and the variance of net revenue which are available to the entrepreneur. An entrepreneur could choose a point on the curve which to him represents the best combination of net revenue and variance. In doing so, he will effectively be choosing his own risk aversion constant and corresponding optimum program.

Computational shortcuts for arriving at optimum programs for various risk aversion constants were developed. A method of constructing resource maps using simplifying transformations was indicated. Price maps for the risk program were constructed incorporating computational shortcuts which were developed. Variance maps were also investigated.

CHAPTER V

APPENDIX

Using the Hildreth solution procedure, we must find

$$C = T\Sigma^{-1}T'$$

and

$$\underline{b} = \underline{v} - \tfrac{1}{2}T\Sigma^{-1}\underline{s},$$

where

$$\Sigma^{-1} = \frac{2}{a}\Sigma_s^{-1}$$

The iterations of the Hildreth procedure (Section 1.2.2)

$$w_k^q = -\sum_{i=1}^{k-1}\frac{c_{ki}}{c_{kk}}u_i^q - \sum_{i=k+1}^{p}\frac{c_{ki}}{c_{kk}}u_i^{q-1} - 2\frac{b_k}{c_{kk}}, \text{ where } c_{ij}$$

and b_i are elements of C and $\underline{b}$, can be simplified by defining a matrix of multipliers, Z, where

$$\begin{aligned} z_{ki} &= -\frac{c_{ki}}{c_{kk}},\ i \neq k,\ i \neq p+1 \\ z_{kk} &= 0 \\ z_{k,\,p+1} &= -\frac{2b_k}{c_{kk}}. \end{aligned} \tag{5.1}$$

Then

$$w_k^q = \sum_{i=1}^{k-1} z_{ki}u_i^q + \sum_{i=k+1}^{p} z_{ki}u_i^{q-1} + z_{k,p+1}.$$

Then

$$Z = \begin{vmatrix} 0 & -\frac{c_{12}}{c_{11}} & \cdots & -\frac{c_{1i}}{c_{11}} & \cdots & -\frac{2b_1}{c_{11}} \\ -\frac{c_{21}}{c_{22}} & 0 & \cdots & -\frac{c_{2i}}{c_{22}} & \cdots & -\frac{2b_2}{c_{22}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{c_{k1}}{c_{kk}} & -\frac{c_{k2}}{c_{kk}} & \cdots & -\frac{c_{ki}}{c_{kk}} & \cdots & -\frac{2b_k}{c_{kk}} \end{vmatrix} \tag{5.2}$$

It can be shown that all z_{ki}, $i \neq p+1$ and some $z_{k,\ p+1}$ are invariant through changes of a. Let us define

$$C^* = T\Sigma_s^{-1}T', \text{ where } \Sigma_s^{-1} = \frac{a}{2}\Sigma^{-1}. \tag{5.3}$$

Consider the general element for Z

$$\begin{aligned} z_{ki} &= \frac{-c_{ki}}{c_{kk}} \\ &= \frac{\frac{2}{-a}\,c^*_{ki}}{\frac{2}{a}\,c^*_{kk}} = \frac{-c^*_{ki}}{c^*_{kk}} = z^*_{ki} \end{aligned} \tag{5.4}$$

where c_{ki} are elements of C and c^*_{ki} are elements of C*.

Now

$$\begin{aligned} \underline{b} &= \underline{v} - \tfrac{1}{2}T\Sigma^{-1}\underline{s} \\ &= \underline{v} - \tfrac{1}{2}\,\tfrac{2}{a}\,T\Sigma_s^{-1}\underline{s} \\ &= \underline{v} - \tfrac{1}{a}T\Sigma_s^{-1}\underline{s}. \end{aligned} \tag{5.5}$$

Consider the element, $z_{k,p+1}$

$$\begin{aligned} -\frac{2b_k}{c_{kk}} &= -\frac{2(v_k - \frac{1}{a}T\Sigma_s^{-1}s_k}{\frac{2}{a}c^*_{kk}} \\ &= -\frac{av_k - T\Sigma_s^{-1}s_k}{c^*_{kk}}\,. \end{aligned} \tag{5.6}$$

Note that $z_{k,\ p+1} = z^*_{k,\ p+1}$ if $v_k = o$ which happens in the non-negativity restrictions.

In order to change the risk aversion constant, a, as can be seen from (5.4) and (5.6), it is only necessary to multiply the availability vector, $\underline{v}$, by the desired value of the risk aversion constant, insert this in expression (5.6), and proceed with the Hildreth procedure as outlined in Section 1.3.3.

The iterations for the various programs become more numerous as the value of the risk aversion constant decreased. There were about 680 iterations to find the optimum program for $a = 1/2500$ as compared to about 100 iterations for $a = 1/750$.

CHAPTER VI

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CHAPTER VII

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ASPECTS OF RISK PROGRAMMING

by

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May 1958

Blacksburg, Virginia

ABSTRACT

The purpose of this thesis was to investigate certain aspects of risk programming.

In the computational example, experimental data based on varietal tests were used to obtain estimates of the variances of net revenue of the various crops considered. As was expected, the consideration of risk increased the importance of corn, the crop with the lowest unit level variance, and decreased the importance of tomatoes, a comparatively high risk crop. In the risk program, the total revenue was, of course, decreased, but the expected utility was increased and the standard deviation of the net revenue was substantially decreased. The risk program also requires less capital and labor than the no-risk program.

An opportunity curve was formed by joining several points of tangency between the opportunity line and indifference curve corresponding to utility functions with different values of the risk aversion constant "a". This opportunity curve represents combinations of net revenue and the variance of net revenue which are available to the entrepreneur. An entrepreneur could choose a point on the curve which to him represents the best combination of net revenue and variance. In doing so, he will effectively be choosing his own risk aversion constant and corresponding optimum program. By this procedure, the difficulty of hypothesising an incorrect risk aversion constant can be avoided.

Computational shortcuts for arriving at optimum programs for various risk aversion constants were developed as were methods for varying the price of a process and the availability of a scarce resource.