BITOPOLOGICAL SPACES

by

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Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

Mathematics

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August, 1970

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ACKNOWLEDGEMENTS

The author is indebted to Dr. C. W. Patty for his patience and assistance during the preparation of this thesis. She also wishes to thank the graduate faculty of the Virginia Polytechnic Institute for their advice and help during the course of her graduate study.

To her husband, Thurmon, the author is grateful for his unfailing encouragement and patience.

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CHAPTER I

INTRODUCTION

Definition 1.1.1. A bitopological space $(X,\mathcal{T},\mathcal{U})$ is a set X together with topologies $\mathcal{T}$ and $\mathcal{U}$.

The formal study of bitopological spaces was initiated by J. C. Kelly ([7]).

Definition 1.1.2. A quasi-pseudo-metric on a set X is a non-negative, real-valued function τ on $X \times X$ having the following properties:

(i) $\tau(x,x) = 0$ for each $x \in X$,

(ii) $\tau(x,z) \leq \tau(x,y) + \tau(y,z)$ for all $x,y,z \in X$.

If (i),(ii), and

(iii) $x = y$ if $\tau(x,y) = 0$

hold, then τ is a quasi-metric. If (i),(ii), and

(iv) $\tau(x,y) = \tau(y,x)$ for all $x,y \in X$

hold, then τ is a pseudo-metric.

Pseudo-metrics have been studied rather thoroughly (as in [6]), and Kelly proposed to consider problems concerning quasi-metrics.

If τ is a quasi-pseudo-metric on X, $\varepsilon > 0$, and $x \in X$, then $S_\tau(x,\varepsilon) = \{y : \tau(x,y) < \varepsilon\}$. The topology $\mathcal{T}$ determined by τ is the topology whose base is $\{S_\tau(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0\}$. If μ is the quasi-pseudo-metric on X defined by $\mu(x,y) = \tau(y,x)$, then we say that μ is the conjugate of τ, and we let $\mathcal{U}$ denote the topology determined by μ. One may then study the bitopological space $(X,\mathcal{T},\mathcal{U})$ (also written (X,τ,μ)).

Kelly's results "show in general that the existence of quasi-metrics is related to the existence of real-valued functions which are semi-continuous relative to two topologies in much the same way as the existence of metrics is related to the existence of real-valued functions which are continuous relative to the original topology of the space in question."

Definition 1.1.3. Let X be a set. A collection $\mathcal{C}$ of subsets of $X \times X$ is a quasi-uniformity for X provided that $\mathcal{C}$ has the following properties:

(i) If $C \in \mathcal{C}$, then $\Delta \subset C$, where $\Delta = \{(x,x) : x \in X\}$.

(ii) If $C, D \in \mathcal{C}$, then $C \cap D \in \mathcal{C}$.

(iii) If $C \in \mathcal{C}$, and $C \subset D \subset X$, then $D \in \mathcal{C}$.

(iv) If $C \in \mathcal{C}$, there exists a $D \in \mathcal{C}$ such that $D[D] \subset C$. (In general, $M[N] = \{(x,z) :$ for some $y \in X$, $(x,y) \in N$ and $(y,z) \in M\}$.)

If $\mathcal{C}$ is a quasi-uniformity for X, let $\mathcal{C}^{-1} = \{C^{-1} : C \in \mathcal{C}\}$. It is easy to see that $\mathcal{C}^{-1}$ is also a quasi-uniformity for X.

If X is a set, and $\mathcal{C}$ is a quasi-uniformity for X, then the topology generated by $\mathcal{C}$ will be denoted by $\mathcal{T}(\mathcal{C})$. That is, $\mathcal{T}(\mathcal{C}) = \{A \subset X :$ if $a \in A$, then there is a $C \in \mathcal{C}$ such that $C(a) \subset A\}$, where $C(a) = \{x \in X : (a,x) \in C\}$.

Definition 1.1.4. A bitopological space $(X, \mathcal{T}, \mathcal{U})$ is pairwise-uniform provided there exists a quasi-uniformity $\mathcal{C}$ such that $\mathcal{T} = \mathcal{T}(\mathcal{C})$ and $\mathcal{U} = \mathcal{T}(\mathcal{C}^{-1})$.

P. Fletcher ([3] and [4]) and E. P. Lane ([8]) continued Kelly's work on bitopological spaces. They independently proposed the same definition for a "pairwise-completely regular" bitopological space and proved the following theorem: "A bitopological space is pairwise-uniform if and only if it is pairwise-completely regular."

Definition 1.1.5. A bitopological space $(X,\mathcal{T},\mathcal{U})$ is quasi-pseudo-metrizable if there exist conjugate quasi-pseudo-metrics τ and μ such that $\mathcal{T}$ and $\mathcal{U}$ are determined by τ and μ, respectively.

C. W. Patty ([10]), Lane ([8]), Fletcher ([4]), and Kelly ([7]) have considered a number of quasi-pseudo-metrization theorems. All of these mathematicians have studied bitopological spaces in terms of "pairwise-separation" properties.

A number of difficulties with these axioms have been observed, however. Kelly proved the desired generalization of Tietze's Extension Theorem (Theorem 18.29 of [1]) for bitopological spaces (Theorem 2.9 of [7]), but Lane ([8]) found that his theorem was in error. Quasi-pseudo-metric spaces need not be pairwise-Hausdorff (Theorem 6.1.2(a)). Fletcher mentioned (in [3]) that he could find no concept of compactness which related meaningfully to bitopological spaces.

M. C. Datta ([2]) has proposed definitions for "quasi-Hausdorff" and "semi-compact" bitopological spaces. These two types of spaces, together with his "quasi-continuous" functions, have very desirable relationships. C. W. Patty suggested verbally some other "quasi-separation" axioms. The initial purpose of this thesis was to investigate these new quasi-separation properties, in the hope that they would relate satisfactorily to semi-compact spaces and quasi-continuous functions.

Up through completely regular spaces, pairwise-separation properties are stronger than quasi-separation properties. We cannot show that a quasi-completely regular space is pairwise-uniform. We do not think that quasi-pseudo-metric spaces need be quasi-normal (cf. the remarks following Theorem 6.1.2). Pairwise-separation axioms seem to relate more satisfactorily to semi-compactness and semi-paracompactness (e.g., Theorem 5.2.13 and Theorem 5.2.14).

It appears that comparing the topologies of bitopological spaces "separately," as pairwise-separation axioms do, yields more information than comparing them "jointly," as quasi-separation axioms do. It seems that the consideration of pairwise-separated spaces, semi-compact spaces, and functions that we shall call "bicontinuous" is very suitable for bitopological spaces.

Definition 1.1.6. Let $(X,\mathcal{T},\mathcal{U})$ be a bitopological space. The upper bound topology of $\mathcal{T}$ and $\mathcal{U}$, denoted by $\mathcal{T} \vee \mathcal{U}$, is the topology whose subbase is the collection $\mathcal{T} \cup \mathcal{U}$.

Let $f : X \to R$ and $g : X \to R$ be real-valued functions. We can define a new function $h = f \vee g$ by $h(x) = \sup\{f(x),g(x)\}$ for each $x \in X$.

Definition 1.1.7. A bitopological space $(X,\mathcal{T},\mathcal{U})$ is separable, first countable, or second countable if both $(X,\mathcal{T})$ and $(X,\mathcal{U})$ have the property in question.

If $(X,\mathcal{T},\mathcal{U})$ is first countable (second countable), then $(X,\mathcal{T} \vee \mathcal{U})$ is first countable (second countable). The space of Example 2.1.3 is a separable bitopological space such that $(X,\mathcal{T} \vee \mathcal{U})$ is not separable.

If $(X,\mathcal{T} \vee \mathcal{U})$ is separable, then $(X,\mathcal{T},\mathcal{U})$ is separable. The space of Example 6.1.8 has the property that $(X,\mathcal{T} \vee \mathcal{U})$ is first countable, but neither $(X,\mathcal{T})$ nor $(X,\mathcal{U})$ is first countable.

We shall follow only a few other conventions. Let R denote the set of real numbers, and let I denote the interval [0,1] with its usual topology. The phrase "bitopological space" will often be shortened to "space."

All definitions of terms of point-set topology, with the exception of those stated above, are due to Greever ([5]).

Finally, we shall refer to items using ordered triples. "Theorem 3.2.1" will denote the first theorem in the second section of Chapter III.

CHAPTER II

CONTINUITY

In this chapter, our principal concern will be with bicontinuous and quasi-continuous functions.

1. Preliminaries.

We shall need a number of developments due to Datta ([2]).

Definition 2.1.1. (Datta) A subset A of $(X,\mathcal{T},\mathcal{U})$ is quasi-open if for each $x \in A$, there exists a $\mathcal{T}$-open or $\mathcal{U}$-open set W such that $x \in W \subset A$. A set is quasi-closed if it is the complement of a quasi-open set.

Theorem 2.1.2. (Datta) A set is quasi-open if and only if it is the union of a $\mathcal{T}$-open set and a $\mathcal{U}$-open set.

Proof. Let A be quasi-open. Let $\alpha = \{x \in A :$ there exists a $\mathcal{T}$-open set T_x with $x \in T_x \subset A\}$, and let $\beta = \{x \in A :$ there exists a $\mathcal{U}$-open set U_x with $x \in U_x \subset A\}$. Clearly, $T = \bigcup\{T_x : x \in \alpha\}$ is a $\mathcal{T}$-open set, $U = \bigcup\{U_x : x \in \beta\}$ is a $\mathcal{U}$-open set, and $A = T \cup U$.

Of course, every $\mathcal{T}$-open ($\mathcal{U}$-open) set is quasi-open. The arbitrary union of a collection of quasi-open sets is quasi-open. However, the intersection of a finite collection of quasi-open sets need not be quasi-open.

Example 2.1.3. (Datta) Let $X = R$. Let $\mathcal{T}$ be the topology whose base is $\{[a,b) : a,b \in R$ and $a < b\}$, and let $\mathcal{U}$ be the topology whose base is $\{(c,d] : c,d \in R$ and $c < d\}$.

Let $a,b,c \in R$ such that $a < b < c$. The sets $(a,b]$ and $[b,c)$ are quasi-open, but $(a,b] \cap [b,c) = \{b\}$ is not quasi-open.

One can make dual statements about quasi-closed sets. A set is quasi-closed if and only if it is the intersection of a $\mathcal{T}$-closed set and a $\mathcal{U}$-closed set. Every $\mathcal{T}$-closed ($\mathcal{U}$-closed) set is quasi-closed. The intersection of an arbitrary collection of quasi-closed sets is quasi-closed, but the union of a finite collection of quasi-closed sets need not be quasi-closed.

Definition 2.1.4. (Datta) A subset A of $(X,\mathcal{T},\mathcal{U})$ is semi-open if A is open in $(X,\mathcal{T} \vee \mathcal{U})$.

Recall that $\mathcal{T} \cup \mathcal{U}$ is a sub-base for $\mathcal{T} \vee \mathcal{U}$. Each $\mathcal{T}$-open, $\mathcal{U}$-open, or quasi-open set is semi-open. The difficulty with quasi-open sets is that they are not closed under finite intersections, a fact which has bearing on the upper bound topology.

Theorem 2.1.5. The following conditions are equivalent:

(i) The topology $\mathcal{T} \vee \mathcal{U}$ is precisely the collection of quasi-open sets.

(ii) The intersection of a finite collection of quasi-open sets is quasi-open.

(iii) The intersection of any $\mathcal{T}$-open set and any $\mathcal{U}$-open set is quasi-open.

(iv) The union of a finite collection of quasi-closed sets is quasi-closed.

(v) The union of any $\mathcal{T}$-closed set and any $\mathcal{U}$-closed set is quasi-closed.

(vi) The collection $\mathcal{T} \cup \mathcal{U}$ is a base for $\mathcal{T} \vee \mathcal{U}$.

Proof. The proof is routine.

The conditions of Theorem 2.1.5 are very strong. However, we shall often need one of these conditions to prove a satisfactory theorem involving quasi-separation properties. We pause to give some examples of spaces satisfying the conditions of Theorem 2.1.5.

Example 2.1.6. (a). If $\mathcal{T}$ is the discrete topology on any space X, then $(X,\mathcal{T},\mathcal{U})$ has the properties of Theorem 2.1.5.

(b). Let X = R. Let $\mathcal{T}$ be the usual topology for R. Let $\mathcal{U}$ be the cofinite topology for R; i.e., $\mathcal{U}$-open sets are empty or complements of finite sets. The space $(X,\mathcal{T},\mathcal{U})$ has the properties of Theorem 2.1.5.

(c). Let X = R. Let $\mathcal{T} = \mathcal{L}$, where $\mathcal{L}$ is the lower limit topology for R. (Recall that $\{[a,b) : a,b \in R\}$ is a base for $\mathcal{L}$.) Let $\mathcal{U}$ be the usual topology for R. The space $(X,\mathcal{T},\mathcal{U})$ has the properties of Theorem 2.1.5.

(d). If $(X,\mathcal{T},\mathcal{U})$ is a space in which $\mathcal{U} \subset \mathcal{T}$, then $(X,\mathcal{T},\mathcal{U})$ has the properties of Theorem 2.1.5. Observe that this is the case in (a) - (c).

(e). Let X = R. Let τ be the quasi-pseudo-metric defined by

$$\tau(x,y) = \begin{cases} 0, \text{ if } y = x \\ |x-y|, \text{ if } x \geq 0 \\ |x+y|, \text{ if } x < 0 \text{ and } y \neq x \end{cases} ;$$

and let

$$\mu(x,y) = \tau(y,x) = \begin{cases} 0, \text{ if } y = x \\ |x-y|, \text{ if } y \geq 0 \\ |x+y|, \text{ if } y < 0 \text{ and } x \neq y \end{cases} .$$

One can verify that $\mathcal{T} \not\subset \mathcal{U}$ and $\mathcal{U} \not\subset \mathcal{T}$, but $\mathcal{T} \cup \mathcal{U}$ is a base for a topology.

Definition 2.1.7. (Datta) If A is a subset of $(X,\mathcal{T},\mathcal{U})$, then the quasi-closure of A, denoted by $\overline{A}$, is the set $\overline{A} = \mathcal{T}\text{-cl}(A) \cap \mathcal{U}\text{-cl}(A)$.

Theorem 2.1.8. (Datta) If A is a subset of $(X,\mathcal{T},\mathcal{U})$, then $\overline{A}$ is the smallest quasi-closed set containing A.

Proof. Since $\overline{A}$ is the intersection of two quasi-closed sets, $\overline{A}$ is quasi-closed. Let B be any quasi-closed set containing A, and let $x \in X \sim B$. Since $X \sim B$ is quasi-open, there is a $\mathcal{T}$-open or $\mathcal{U}$-open set W with $x \in W \subset X \sim B \subset X \sim A$. Hence, $x \notin \mathcal{T}\text{-cl}(A)$ or $x \notin \mathcal{U}\text{-cl}(A)$, which implies that $x \notin \overline{A}$. Thus, $X \sim B \subset X \sim \overline{A}$, and so $B \supset \overline{A}$.

Obviously, $x \in \overline{A}$ if and only if every quasi-open set about x contains some point of A. Moreover, we observe that $\overline{A \cup B} \supset \overline{A} \cup \overline{B}$, $\overline{\bigcap_\alpha A_\alpha} \subset \bigcap_\alpha \overline{A}_\alpha$, and $\overline{\bigcup_\alpha A_\alpha} \supset \bigcup_\alpha \overline{A}_\alpha$.

2. Convergence.

We shall need only a minimal theory of convergence.

Definition 2.2.1. Let $(X,\mathcal{T},\mathcal{U})$ be a space. A sequence (x_n) in X quasi-converges to a point $x \in X$ if for each quasi-open set W about x, there exists a positive integer N such that $x_n \in W$ for all $n \geq N$.

We remark that if (x_n) is quasi-convergent, then (x_n) is $\mathcal{T}$-convergent and $\mathcal{U}$-convergent.

Let A be a subset of $(X,\mathcal{T},\mathcal{U})$. If there exists a sequence (x_n) in A that quasi-converges to x, then $x \in \overline{A}$. If $(X,\mathcal{T},\mathcal{U})$ is first countable and $x \in \overline{A}$, then there is a sequence in A that quasi-converges to x.

Definition 2.2.2. (Kelly) A sequence (x_n) in a quasi-pseudo-metric space (X,τ,μ) is a τ-Cauchy sequence if for each $\varepsilon > 0$, there is an integer N such that $\tau(x_m,x_n) < \varepsilon$ whenever $m,n \geq N$. A subset A of X is τ-complete if every τ-Cauchy sequence in A has a τ-limit in A.

We state the following theorem to indicate a direction in which one might pursue the study of convergence. Certainly, nets and filters are other possibilities.

Theorem 2.2.3. (Kelly) Let (X, τ, μ) be a quasi-pseudo-metric space. If X is μ-complete, then X is of the second category in itself with respect to the topology $\mathcal{T}$, where $\mathcal{T}$ is determined by τ.

Proof. This is Theorem 2.11 of [7].

3. Bicontinuous and Quasi-continuous Functions.

We shall now establish for bitopological spaces a few analogues of standard results on continuity.

Definition 2.3.1. A function $f : (X,\mathcal{T}_1,\mathcal{U}_1) \to (Y,\mathcal{T}_2,\mathcal{U}_2)$ is bicontinuous if both $f : (X,\mathcal{T}_1) \to (Y,\mathcal{T}_2)$ and $f : (X,\mathcal{U}_1) \to (Y,\mathcal{U}_2)$ are continuous.

(Datta) The function is quasi-continuous if the inverse image of every quasi-open set is quasi-open.

Theorem 2.3.2. Every bicontinuous function is quasi-continuous.

Proof. The result is obvious.

Theorem 2.3.3. If $f : (X,\mathcal{T}_1,\mathcal{U}_1) \to (Y,\mathcal{T}_2,\mathcal{U}_2)$, then the following conditions are equivalent:

(i) The function f is quasi-continuous.

(ii) The inverse image of every quasi-closed set is quasi-closed.

(iii) For each $x \in X$, the inverse of each quasi-open set about $f(x)$ is a quasi-open set about x.

(iv) For each $x \in X$ and each quasi-open set W about $f(x)$, there is a quasi-open set Z about x such that $f(Z) \subset W$.

(v) For each subset A of X, $f(\bar{A}) \subset \overline{f(A)}$.

(vi) For each subset B of Y, $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are easy. We shall show that (i) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (v). We repeat Datta's proof ([2]). Let $x \in \overline{A} = \mathcal{T}_1\text{-cl}(A) \cap \mathcal{U}_1\text{-cl}(A)$. Let V be any $\mathcal{T}_2$-open neighborhood of f(x). By (i), $f^{-1}(V)$ is a quasi-open set in X such that $x \in f^{-1}(V)$; so there is a $\mathcal{T}_1$-open or $\mathcal{U}_1$-open neighborhood W of x such that $W \subset f^{-1}(V)$. Since $x \in \overline{A}$, we have $W \cap A \neq \emptyset$. Hence, $f^{-1}(V) \cap A \neq \emptyset$, which implies that $V \cap f(A) \neq \emptyset$. Thus, $f(x) \in \mathcal{T}_2\text{-cl}(f(A))$. A similar argument shows that $f(x) \in \mathcal{U}_2\text{-cl}(f(A))$. Therefore, $f(x) \in \overline{f(A)}$. We conclude that $f(\overline{A}) \subset \overline{f(A)}$.

(v) $\Rightarrow$ (vi). If B is any subset of Y, then $f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} \subset \overline{B}$. Hence, $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.

(vi) $\Rightarrow$ (ii). If F is any quasi-closed subset of Y, then $f^{-1}(F) = f^{-1}(\overline{F}) \supset \overline{f^{-1}(F)}$ by (vi). Since $\overline{f^{-1}(F)}$ is the smallest quasi-closed subset of X which contains $f^{-1}(F)$, we infer that $f^{-1}(F)$ is quasi-closed in X.

Theorem 2.3.4. (a). If $f : (X,\mathcal{T}_1,\mathcal{U}_1) \to (Y,\mathcal{T}_2,\mathcal{U}_2)$ is bicontinuous, and (x_n) is $\mathcal{T}_1$-convergent ($\mathcal{U}_1$-convergent) to x in X, then $(f(x_n))$ is $\mathcal{T}_2$-convergent ($\mathcal{U}_2$-convergent) to f(x) in Y.

(b). If f is quasi-continuous, and (x_n) is quasi-convergent to x in X, then $(f(x_n))$ is quasi-convergent to f(x) in Y.

Proof. Both proofs are trivial.

Obviously, the restriction of a bicontinuous (quasi-continuous) function is bicontinuous (quasi-continuous).

Definition 2.3.5. A function $f : (X,\mathcal{T}_1,\mathcal{U}_1) \to (Y,\mathcal{T}_2,\mathcal{U}_2)$ is bi-open if $f : (X,\mathcal{T}_1) \to (Y,\mathcal{T}_2)$ and $f : (X,\mathcal{U}_1) \to (Y,\mathcal{U}_2)$ are open functions.

We say f is <u>quasi-open</u> if the image of each quasi-open set is quasi-open.

The definitions for "bi-closed" and "quasi-closed" functions can be inferred.

As Datta has pointed out ([2]), if $f : (X,\mathcal{T}_1,\mathcal{U}_1) \to (Y,\mathcal{T}_2,\mathcal{U}_2)$ is quasi-continuous, then $f : (X,\mathcal{T}_1 \vee \mathcal{U}_1) \to (Y,\mathcal{T}_2 \vee \mathcal{U}_2)$ is continuous. The converse of this assertion is false, as the following example from [2] shows.

<u>Example 2.3.6.</u> Let X = R. Let τ be the quasi-pseudo-metric defined by $\tau(x,y) = \begin{cases} \min\{1, |x-y|\}, & \text{if } x \le y \\ 1, & \text{if } x > y \end{cases}$, and let $\mu(x,y) = \tau(y,x)$.

Let $\mathcal{T}$ and $\mathcal{U}$ be the topologies determined by τ and μ, respectively.

Let Y = R. Let ρ be the quasi-pseudo-metric defined by

$$\rho(x,y) = \begin{cases} 0, & \text{if } y = x \\ 1, & \text{if } y \neq x \text{ and } x \text{ is rational} \\ \dfrac{|x-y|}{1+|x-y|}, & \text{if } y \neq x \text{ and } x \text{ is irrational} \end{cases}$$

; and define η by $\eta(x,y) = \rho(y,x)$. Let $\mathcal{R}$ and $\mathcal{E}$ be the topologies for Y determined by ρ and η, respectively.

The identity map $i : (X,\mathcal{T} \vee \mathcal{U}) \to (Y,\mathcal{R} \vee \mathcal{E})$ is continuous since $\mathcal{T} \vee \mathcal{U}$ is the discrete topology. However, $i : (X,\mathcal{T},\mathcal{U}) \to (Y,\mathcal{R},\mathcal{E})$ is not quasi-continuous. The set $\{1\}$ is $\mathcal{R}$- (quasi-) open in Y, but $\{1\}$ is not quasi-open in X.

4. <u>Product spaces</u>.

In this section, we shall only define the product of a collection of bitopological spaces and obtain simple results about the projection maps and about upper bound topologies. We shall consider particular product spaces as we proceed.

Definition 2.4.1. (Datta) Let $\{(X_\lambda,\mathcal{T}_\lambda,\mathcal{U}_\lambda)\}_{\lambda\in\Lambda}$ be a collection of bitopological spaces. Let X be the product set $X = \prod_{\lambda\in\Lambda} X_\lambda$. Let $\mathcal{T}$ be the product topology on X generated by $\{\mathcal{T}_\lambda\}_{\lambda\in\Lambda}$, and let $\mathcal{U}$ be the topology on X generated by the collection $\{\mathcal{U}_\lambda\}_{\lambda\in\Lambda}$. We shall call $(X,\mathcal{T},\mathcal{U})$ the <u>product</u> of the spaces in the collection $\{(X_\lambda,\mathcal{T}_\lambda,\mathcal{U}_\lambda)\}_{\lambda\in\Lambda}$.

Theorem 2.4.2. Let $(X,\mathcal{T},\mathcal{U})$ be the product of the spaces in $\{(X_\lambda,\mathcal{T}_\lambda,\mathcal{U}_\lambda)\}_{\lambda\in\Lambda}$.

(a). For each $\lambda \in \Lambda$, the natural projection $\pi_\lambda : (X,\mathcal{T},\mathcal{U}) \to (X_\lambda,\mathcal{T}_\lambda,\mathcal{U}_\lambda)$ is bicontinuous and bi-open.

(b). (Datta) For each $\lambda \in \Lambda$, the natural projection $\pi_\lambda : (X,\mathcal{T},\mathcal{U}) \to (X_\lambda,\mathcal{T}_\lambda,\mathcal{U}_\lambda)$ is quasi-continuous and quasi-open.

Proof. (a). Fix $\lambda \in \Lambda$. If T is a $\mathcal{T}_\lambda$-open subset of X_λ, then $\pi_\lambda^{-1}(T)$ is $\mathcal{T}$-open by definition of $\mathcal{T}$. Hence, $\pi_\lambda : (X,\mathcal{T}) \to (X_\lambda,\mathcal{T}_\lambda)$ is continuous. Similarly, $\pi_\lambda : (X,\mathcal{U}) \to (X_\lambda,\mathcal{U}_\lambda)$ is continuous. Thus, π_λ is bicontinuous.

If T is a basic $\mathcal{T}$-open set, then $T = \bigcap_{j=1}^{n} \pi_{\lambda_j}^{-1}(T_j)$, where each T_j is open in $(X_{\lambda_j},\mathcal{T}_{\lambda_j})$. Since $\pi_\lambda(T) = \begin{cases} T_j, & \text{if } \lambda_j = \lambda \text{ for some } j \\ X_\lambda, & \text{if } \lambda_j \neq \lambda \text{ for all } j \end{cases}$, it follows that $\pi_\lambda : (X,\mathcal{T}) \to (X_\lambda,\mathcal{T}_\lambda)$ is open. Similarly, $\pi_\lambda : (X,\mathcal{U}) \to (X_\lambda,\mathcal{U}_\lambda)$ is open. We conclude that π_λ is bi-open.

(b). If W is quasi-open in $(X_\lambda,\mathcal{T}_\lambda,\mathcal{U}_\lambda)$, then $W = T \cup U$, where T is $\mathcal{T}_\lambda$-open and U is $\mathcal{U}_\lambda$-open. By (a), $\pi_\lambda^{-1}(W) = \pi_\lambda^{-1}(T) \cup \pi_\lambda^{-1}(U)$ is quasi-open. Thus, π_λ is quasi-continuous.

A similar argument shows that π_λ is quasi-open.

Our next result may be helpful in obtaining examples of bitopological spaces satisfying certain conditions. We shall observe it often to be the case that in order for a bitopological space to have a certain pairwise-separation property, it is necessary that the upper bound topology have the corresponding topological separation property.

Theorem 2.4.3. Let $(X,\mathcal{T},\mathcal{U})$ be the product of the spaces $\{(X_\lambda,\mathcal{T}_\lambda,\mathcal{U}_\lambda)\}_{\lambda\in\Lambda}$. The space $(X,\mathcal{T}\vee\mathcal{U})$ is the same as the product $(X,\mathcal{P})$ of the spaces $\{(X_\lambda,\mathcal{T}_\lambda\vee\mathcal{U}_\lambda)\}_{\lambda\in\Lambda}$.

Proof. We shall let $\pi_\lambda : X \to X_\lambda$ be the projection of X onto X_λ, regardless of what topologies X and X_λ have.

Let S be a $\mathcal{T}\vee\mathcal{U}$-open subset of X, and let $x \in S$. Since S is $\mathcal{T}\vee\mathcal{U}$-open, S is of the form $\cup\{T_\alpha \cap U_\alpha : \alpha\in\mathcal{Q}\}$, where each T_α is $\mathcal{T}$-open, and each U_α is $\mathcal{U}$-open. Hence, for some $\lambda \in \Lambda$, $x \in T_\lambda \cap U_\lambda \subset S$. Since $x \in T_\lambda$, there are a finite subset Γ of Λ and a collection $\{T_\gamma : \gamma \in \Gamma\}$, where each $T_\gamma \subset X_\gamma$ is $\mathcal{T}_\gamma$-open, such that $x \in \cap\{\pi_\gamma^{-1}(T_\gamma) : \gamma\in\Gamma\} \subset T_\lambda$. Since $x \in U_\lambda$, there are a finite subset Ω of Λ and a collection $\{U_\gamma : \gamma\in\Omega\}$, where each $U_\gamma \subset X_\gamma$ is $\mathcal{U}_\gamma$-open, such that $x \in \cap\{\pi_\gamma^{-1}(U_\gamma) : \gamma\in\Omega\} \subset U_\lambda$. Hence, $x \in (\cap\{\pi_\gamma^{-1}(T_\gamma) : \gamma\in\Gamma\}) \cap (\cap\{\pi_\gamma^{-1}(U_\gamma) : \gamma\in\Omega\}) = (\cap\{\pi_\gamma^{-1}(T_\gamma) \cap \pi_\gamma^{-1}(U_\gamma) : \gamma\in\Gamma\cap\Omega\}) \cap (\cap\{\pi_\gamma^{-1}(T_\gamma) : \gamma\in\Gamma\sim\Omega\}) \cap (\cap\{\pi_\gamma^{-1}(U_\gamma) : \gamma\in\Omega\sim\Gamma\}) \subset S$. If $\gamma \in \Gamma\cap\Omega$, then $\pi_\gamma^{-1}(T_\gamma) \cap \pi_\gamma^{-1}(U_\gamma) = \pi_\gamma^{-1}(T_\gamma \cap U_\gamma)$ is the inverse image of a $\mathcal{T}_\gamma\vee\mathcal{U}_\gamma$-open set and so is $\mathcal{P}$-open. If $\gamma \in \Gamma \sim \Omega$, then $\pi_\gamma^{-1}(T_\gamma)$ is the inverse image of a $\mathcal{T}_\gamma \vee \mathcal{U}_\gamma$-open set and so is $\mathcal{P}$-open. Similarly, if $\gamma \in \Omega \sim \Gamma$, then $\pi_\gamma^{-1}(U_\gamma)$ is $\mathcal{P}$-open. We have shown that there is a $\mathcal{P}$-open set P such that $x \in P \subset S$. Thus, $\mathcal{T}\vee\mathcal{U} \subset \mathcal{P}$.

An argument of similar kind can be used to show that $\mathcal{P} \subset \mathcal{T} \vee \mathcal{U}$. Therefore, $\mathcal{P} = \mathcal{T} \vee \mathcal{U}$.

CHAPTER III

SEPARATION AXIOMS

In this chapter, we shall consider analogues for the definitions of T_o-, T_1-, Hausdorff, and regular spaces.

1. Pairwise-T_o ($-T_1$) and Quasi-T_o ($-T_1$) Spaces.

We shall begin by studying pairwise-T_o and quasi-T_o spaces. We shall need only a few properties of such spaces. Some of the properties of pairwise-T_o spaces have been summarized by Murdeshwar and Naimpally ([9]).

Definition 3.1.1. (Murdeshwar and Naimpally) A space $(X,\mathcal{T},\mathcal{U})$ is pairwise-T_o if for each pair of distinct points in X, there is a $\mathcal{T}$- or $\mathcal{U}$-neighborhood of one point which does not contain the other.

A space $(X,\mathcal{T},\mathcal{U})$ is quasi-T_o if for each pair of distinct points in X, there is a quasi-open neighborhood of one point which does not contain the other.

Theorem 3.1.2. A space $(X,\mathcal{T},\mathcal{U})$ is pairwise-T_o if and only if $(X,\mathcal{T},\mathcal{U})$ is quasi-T_o. If $(X,\mathcal{T},\mathcal{U})$ is pairwise- (quasi-) T_o, then $(X,\mathcal{T}\vee\mathcal{U})$ is a T_o-space.

Proof. The results are obvious.

If $(X,\mathcal{T})$ or $(X,\mathcal{U})$ is a T_o-space, then $(X,\mathcal{T},\mathcal{U})$ is pairwise- (quasi-) T_o.

It is clear that every subspace of a pairwise- (quasi-) T_o space is pairwise- (quasi-) T_o. The desired product theorems also hold.

Theorem 3.1.3. (a). The product of a collection of pairwise-T_o

spaces is pairwise-T_o.

(b). The product of a collection of quasi-T_o spaces is quasi-T_o.

Proof. (a). Let $(X,\mathcal{T},\mathcal{U})$ be the product of the pairwise-T_o spaces $\{(X_\lambda,\mathcal{T}_\lambda,\mathcal{U}_\lambda)\}_{\lambda\in\Lambda}$. If (x_λ) and (y_λ) are distinct points of X, there is some $\alpha \in \Lambda$ such that $x_\alpha \neq y_\alpha$. Since $(X_\alpha,\mathcal{T}_\alpha,\mathcal{U}_\alpha)$ is pairwise-T_o, there is a $\mathcal{T}_\alpha$- or $\mathcal{U}_\alpha$-open set W containing, say, x_α which does not contain y_α. Since π_α is bicontinuous, $\pi_\alpha^{-1}(W)$ is a $\mathcal{T}$- or $\mathcal{U}$-open set containing (x_λ) which does not contain (y_λ). We conclude that $(X,\mathcal{T},\mathcal{U})$ is pairwise-T_o.

The proof of (b) is similar.

We now consider analogues of definitions of T_1-spaces. There have been two definitions proposed for pairwise-T_1 spaces, one due to Fletcher ([4]) and the other to Murdeshwar and Naimpally ([9]). Fletcher's is the stronger of the two.

In general, the weaker definition is sufficient for our purposes. However, we need Fletcher's definition for Theorem 3.3.5 and Corollary 5.2.6(a). We shall try to prove the best possible theorem in each case.

Definition 3.1.4. (Fletcher) A space $(X,\mathcal{T},\mathcal{U})$ is strongly pairwise-T_1 (or s-pairwise-T_1) if for each pair of distinct points x and y, there is a $\mathcal{T}$-open set containing x but not y and a $\mathcal{U}$-open set containing y but not x.

(Murdeshwar and Naimpally) A space $(X,\mathcal{T},\mathcal{U})$ is weakly pairwise-T_1 (or w-pairwise-T_1) if for each pair of distinct points x and y, there is a $\mathcal{T}$- or $\mathcal{U}$-neighborhood of x not containing y and a $\mathcal{T}$- or $\mathcal{U}$-neighborhood of y not containing x.

A space $(X,\mathcal{T},\mathcal{U})$ is <u>quasi</u>-T_1 if for each pair of distinct points x and y, there is a quasi-open set containing x but not y and a quasi-open set containing y but not x.

<u>Theorem 3.1.5</u>. Every s-pairwise-T_1 space is w-pairwise-T_1, and a space is w-pairwise-T_1 if and only if it is quasi-T_1. If $(X,\mathcal{T},\mathcal{U})$ is w-pairwise- (s-pairwise- or quasi-) T_1, then $(X,\mathcal{T} \vee \mathcal{U})$ is a T_1-space.

<u>Proof</u>. The results are obvious.

The space of Example 3.1.9 is w-pairwise-T_1 and not s-pairwise-T_1.

<u>Theorem 3.1.6</u>. (<u>a</u>). Every w-pairwise-T_1 space is pairwise-T_o.

(<u>b</u>). Every quasi-T_1 space is quasi-T_o.

<u>Proof</u>. The proofs are easy.

As is to be expected, the converse of Theorem 3.1.6 is not true.

<u>Example 3.1.7</u>. Let X = R. Let $\mathcal{T}$ be the topology on X whose base is $\{(a,\infty) : a \in R\}$; and let $\mathcal{U}$ be the topology on X whose base is $\{[b,\infty) : b \in R\}$. The space $(X,\mathcal{T},\mathcal{U})$ is pairwise-T_o and not w-pairwise-T_1. By Theorem 3.1.2 and Theorem 3.1.5, $(X,\mathcal{T},\mathcal{U})$ is also quasi-T_o and not quasi-T_1.

If $\mathcal{T}$ or $\mathcal{U}$ is a T_1-space, then $(X,\mathcal{T},\mathcal{U})$ is w-pairwise- (quasi-) T_1. A space is s-pairwise-T_1 if and only if $(X,\mathcal{T})$ and $(X,\mathcal{U})$ are T_1-spaces.

Every subspace of a w-pairwise- (s-pairwise- or quasi-) T_1 space is w-pairwise- (respectively, s-pairwise- or quasi-) T_1. The bi-closed (quasi-closed) image of a s-pairwise- (quasi-) T_1 space is s-pairwise- (quasi-) T_1. As is true for pairwise-T_o spaces, the product of any collection of w-pairwise- (s-pairwise- or quasi-) T_1 spaces is w-pairwise-

(respectively, s-pairwise- or quasi-) T_1.

The next theorem is an example of a desirable property of quasi-T_1 spaces that is not possessed by w-pairwise-T_1 spaces.

Theorem 3.1.8. (a). A space $(X,\mathcal{T},\mathcal{U})$ is s-pairwise-T_1 if and only if each $\{x\}$ in X is $\mathcal{T}$-closed and $\mathcal{U}$-closed.

(b). If every $\{x\}$ in $(X,\mathcal{T},\mathcal{U})$ is $\mathcal{T}$-closed or $\mathcal{U}$-closed, then $(X,\mathcal{T},\mathcal{U})$ is w-pairwise-T_1.

(c). A space $(X,\mathcal{T},\mathcal{U})$ is quasi-T_1 if and only if each $\{x\}$ in X is quasi-closed.

Proof. The proofs are routine.

The following example shows that the converse of Theorem 3.1.8(b) is indeed false.

Example 3.1.9. Let $X = \{a,b,c\}$, let $\mathcal{T} = \{\emptyset, \{a,b\},\{c\},X\}$, and let $\mathcal{U} = \{\emptyset,\{a,c\},\{b\},X\}$. Clearly, $(X,\mathcal{T},\mathcal{U})$ is w-pairwise-T_1, but $\{a\}$ is neither $\mathcal{T}$-closed nor $\mathcal{U}$-closed.

2. Pairwise-Hausdorff and Quasi-Hausdorff Spaces.

Definition 3.2.1. (Kelly) A space $(X,\mathcal{T},\mathcal{U})$ is pairwise-Hausdorff if for each pair of distinct points x and y, there are a $\mathcal{T}$-neighborhood T_1 of x and $\mathcal{U}$-neighborhood U_1 of y with $T_1 \cap U_1 = \emptyset$, and there are a $\mathcal{U}$-neighborhood U_2 of x and a $\mathcal{T}$-neighborhood T_2 of y with $U_2 \cap T_2 = \emptyset$.

A space $(X,\mathcal{T},\mathcal{U})$ is quasi-Hausdorff if any two distinct points are contained in disjoint quasi-open sets.

Theorem 3.2.2. Every pairwise-Hausdorff space is quasi-Hausdorff.

Proof. The result is obvious.

There are quasi-Hausdorff spaces that are not pairwise-Hausdorff, as the following example shows.

Example 3.2.3. Let $X = R$. Let $\tau(x,y) = \begin{cases} 0, & \text{if } y = x \\ |x-y|, & \text{if } x \geq 0 \\ |x+y|, & \text{if } x < 0 \text{ and } y \neq x \end{cases}$,

and let $\mathcal{T}$ be the topology determined by τ. Let $\mu(x,y) = \tau(y,x)$, i.e.,

$\mu(x,y) = \begin{cases} 0, & \text{if } x = y \\ |x-y|, & \text{if } y \geq 0 \\ |x+y|, & \text{if } y < 0 \text{ and } y \neq x \end{cases}$, and let $\mathcal{U}$ be the topology determined by μ.

It can be verified that $(X,\mathcal{T},\mathcal{U})$ is a quasi-Hausdorff space. By Theorem 3.2.4(b) and Theorem 3.1.5, the space $(X,\mathcal{T},\mathcal{U})$ is w-pairwise-T_1.

If $(X,\mathcal{T},\mathcal{U})$ is a pairwise-Hausdorff space, then, as Kelly ([7]) observed, $(X,\mathcal{T})$ and $(X,\mathcal{U})$ are T_1-spaces. Patty asserted (Example 4.2 of [10]) that in the space we have described, neither $(X,\mathcal{T})$ nor $(X,\mathcal{U})$ is a T_1-space. Hence, $(X,\mathcal{T},\mathcal{U})$ is not a pairwise-Hausdorff space.

As we mentioned above, if $(X,\mathcal{T},\mathcal{U})$ is a pairwise-Hausdorff space, then $(X,\mathcal{T})$ and $(X,\mathcal{U})$ are T_1-topological spaces.

Theorem 3.2.4. (a). Every pairwise-Hausdorff space is s-pairwise-(w-pairwise-) T_1.

(b). Every quasi-Hausdorff space is quasi-T_1.

Proof. The results are obvious.

The spaces of Example 3.2.3 and Example 3.2.5 are w-pairwise-T_1 spaces that are not pairwise-T_2. We give an example of a quasi-T_1 space that is not quasi-T_2.

Example 3.2.5. Let $X = R$. Let $\mathcal{T} = \{\emptyset\} \cup \{T \subset X : 1 \in T \text{ and } X \sim T$ is finite$\}$, and let $\mathcal{U} = \{\emptyset\} \cup \{U \subset X : 0 \in U \text{ and } X \sim U \text{ is finite}\}$.

One can easily verify that $(X,\mathcal{T},\mathcal{U})$ is quasi-T_1 and not quasi-Hausdorff. Furthermore, $(X,\mathcal{T},\mathcal{U})$ is w-pairwise-T_1 by Theorem 3.1.5 and not pairwise-Hausdorff by Theorem 3.2.2.

Theorem 3.2.6. If $(X,\mathcal{T})$ or $(X,\mathcal{U})$ is a Hausdorff space, then $(X,\mathcal{T},\mathcal{U})$ is quasi-Hausdorff. If $(X,\mathcal{T},\mathcal{U})$ is a quasi-Hausdorff (in particular, pairwise-Hausdorff) space, then $(X,\mathcal{T} \vee \mathcal{U})$ is a Hausdorff space.

Proof. The proofs are easy.

We include a result on quasi-Hausdorff spaces due to Datta ([2]).

Theorem 3.2.7. (Datta) Let $(X,\mathcal{T},\mathcal{U})$ be a quasi-Hausdorff space. If $x \neq y$, then there are quasi-open neighborhoods W of x and V of y such that $\overline{W} \cap V = \emptyset$.

Proof. Since $(X,\mathcal{T},\mathcal{U})$ is quasi-Hausdorff, there are disjoint, quasi-open sets W and V such that $x \in W$ and $y \in V$. Suppose $\overline{W} \cap V \neq \emptyset$, and let $z \in \overline{W} \cap V$. Since $z \in V$, and V is quasi-open, there is a $\mathcal{T}$-open or $\mathcal{U}$-open set Z such that $z \in Z \subset V$. Since $z \in \overline{W} = \mathcal{T}\text{-cl}(W) \cap \mathcal{U}\text{-cl}(W)$, we have $Z \cap W \neq \emptyset$. This implies that $W \cap V \neq \emptyset$, contradicting the fact that W and V are disjoint.

It is clear that every subspace of a pairwise- (quasi-) Hausdorff space is pairwise- (quasi-) Hausdorff. One can use a proof similar to that of Theorem 3.1.3 to establish that the product of a collection of pairwise- (quasi-) Hausdorff spaces is pairwise- (quasi-) Hausdorff.

3. Pairwise-Regular and Quasi-regular Spaces.

Definition 3.3.1. (Kelly) In a space $(X,\mathcal{T},\mathcal{U})$, we say that $\mathcal{T}$ is regular with respect to $\mathcal{U}$ if for each $\mathcal{T}$-closed set A and each x in $X \sim A$, there are a $\mathcal{T}$-open set T and a $\mathcal{U}$-open set U such that $A \subset U$, $x \in T$, and $T \cap U = \phi$. If $\mathcal{T}$ is regular with respect to $\mathcal{U}$, and $\mathcal{U}$ is regular with respect to $\mathcal{T}$, then $(X,\mathcal{T},\mathcal{U})$ is pairwise-regular.

A space $(X,\mathcal{T},\mathcal{U})$ is quasi-regular if for each quasi-closed set F and each $x \in X \sim F$, there are disjoint, quasi-open sets V and W such that $x \in V$ and $F \subset W$.

Theorem 3.3.2. If $(X,\mathcal{T},\mathcal{U})$ is pairwise-regular, then $(X,\mathcal{T},\mathcal{U})$ is quasi-regular.

Proof. Let F be a quasi-closed subset of $(X,\mathcal{T},\mathcal{U})$, and let $x \in X \sim F$. Since $F = \mathcal{T}\text{-cl}(F) \cap \mathcal{U}\text{-cl}(F)$, and $x \notin F$, we have, say, $x \notin \mathcal{T}\text{-cl}(F)$. Since $(X,\mathcal{T},\mathcal{U})$ is pairwise-regular, there are a $\mathcal{T}$-open set T and a $\mathcal{U}$-open set U such that $x \in T$, $\mathcal{T}\text{-cl}(F) \subset U$, and $T \cap U = \phi$. Then T and U are quasi-open sets such that $x \in T$, $F \subset U$, and $T \cap U = \phi$. Therefore, $(X,\mathcal{T},\mathcal{U})$ is quasi-regular.

We give an example to show that there is no reasonable way to reverse the implication of Theorem 3.3.2.

Example 3.3.3. Let $X = \{a,b,c\}$. Let $\mathcal{T} = \{\phi, \{a,b\}, \{c\}, X\}$, and let $\mathcal{U} = \{\phi, \{a\}, \{b,c\}, X\}$.

One can verify easily that $\mathcal{T}$ and $\mathcal{U}$ are regular topologies and that $(X,\mathcal{T},\mathcal{U})$ is quasi-regular. However, $\{a,b\}$ is a $\mathcal{T}$-closed set such that every $\mathcal{U}$-open set containing $\{a,b\}$ contains c. Hence, $(X,\mathcal{T},\mathcal{U})$ is not pairwise-regular.

There is an equivalent definition for pairwise- (quasi-) regular spaces.

<u>Theorem 3.3.4</u>. (<u>a</u>). (Kelly) In a space $(X,\mathcal{T},\mathcal{U})$, $\mathcal{T}$ is regular with respect to $\mathcal{U}$ if and only if there is a $\mathcal{T}$-neighborhood base of $\mathcal{U}$-closed sets.

(<u>b</u>). A space $(X,\mathcal{T},\mathcal{U})$ is quasi-regular if and only if for each $x \in X$ and each quasi-open neighborhood W of x, there is a quasi-open set V such that $x \in V \subset \overline{V} \subset W$.

<u>Proof</u>. (<u>a</u>). This is a remark of Kelly ([7]).

(<u>b</u>). The proof is almost verbatim the same as the proof of the similar result in point-set topology.

We wish to compare these new separation axioms to those of the previous section. Here is one of the places mentioned earlier where Fletcher's stronger definition is required.

<u>Theorem 3.3.5</u>. (<u>a</u>). Each pairwise-regular, s-pairwise-T_1 space is pairwise-Hausdorff.

(<u>b</u>). Every quasi-regular, quasi-T_1 space is quasi-Hausdorff.

<u>Proof</u>. (<u>a</u>). If $(X,\mathcal{T},\mathcal{U})$ is s-pairwise-T_1, then every point is $\mathcal{T}$-closed and $\mathcal{U}$-closed. The result follows easily from this observation.

The proof of (b) follows from Theorem 3.1.8(c).

We shall give an example of a pairwise-regular, w-pairwise-T_1 space that is not pairwise-Hausdorff.

<u>Example 3.3.6</u>. Let $X = R$. Let τ be the quasi-pseudo-metric defined $\tau(x,y) = \begin{cases} 0, \text{ if } x \leq y \\ \min\{1, |x-y|\}, \text{ if } x > y \end{cases}$, and let $\mu(x,y) = \tau(y,x)$. Let $\mathcal{T}$ and

$\mathcal{U}$ be the topologies determined by τ and μ, respectively.

Datta asserted ([2]) that $(X,\mathcal{T},\mathcal{U})$ is a quasi-Hausdorff space that is not pairwise-Hausdorff. By Theorem 3.2.4(b) and Theorem 3.1.5, $(X,\mathcal{T},\mathcal{U})$ is a w-pairwise-T_1 space. It is easy to verify that $(X,\mathcal{T},\mathcal{U})$ is pairwise-regular. Thus, $(X,\mathcal{T},\mathcal{U})$ is a pairwise-regular, w-pairwise-T_1 space that is not pairwise-Hausdorff.

We obtain some information about the upper bound topology.

Theorem 3.3.7. If $(X,\mathcal{T},\mathcal{U})$ is quasi-regular (in particular, pairwise-regular), then $(X,\mathcal{T}\vee\mathcal{U})$ is regular.

Proof. The proof is easy.

Clearly, every subspace of a pairwise- (quasi-) regular space is pairwise- (quasi-) regular. Finally, the product of a collection of pairwise- (quasi-) regular spaces is pairwise- (quasi-) regular.

CHAPTER IV

COMPACTNESS PROPERTIES

In the first section of this chapter, we consider analogues of compactness, countable compactness, and the Lindelöf property. In the second section, we study briefly an analogue of paracompactness. The last section is devoted to subspaces, product spaces, and quasi-continuous functions of these new types of spaces.

1. Semi-compact and Semi-countably Compact Spaces.

Definition 4.1.1. (Datta) A space $(X,\mathcal{T},\mathcal{U})$ is semi-compact if $(X,\mathcal{T}\vee\mathcal{U})$ is compact.

The space is semi-countably compact if $(X,\mathcal{T}\vee\mathcal{U})$ is countably compact.

The space is semi-Lindelöf if $(X,\mathcal{T}\vee\mathcal{U})$ is a Lindelöf space.

Of course, every semi-compact space is semi-countably compact. Every semi-compact space is semi-Lindelöf. Every semi-Lindelöf, semi-countably compact space is semi-compact. Every second countable space is semi-Lindelöf. One may give examples to show that no other implications hold by using the standard examples of point-set topology and letting $\mathcal{T} = \mathcal{U}$.

Recall that a collection $\mathcal{D}$ of sets has the finite intersection property if every nonempty finite subcollection of $\mathcal{D}$ has nonempty intersection. It is well known that a topological space is compact if and only if every nonempty collection of closed sets having the finite intersection property has nonempty intersection.

Theorem 4.1.2. If $(X,\mathcal{T},\mathcal{U})$ is semi-compact, then every nonempty family of semi-closed (in particular, $\mathcal{T}$-closed, $\mathcal{U}$-closed, or quasi-closed) sets having the finite intersection property has nonempty intersection.

Proof. The theorem follows from the preceding remark.

We are primarily concerned here with $\mathcal{T}$-closed, $\mathcal{U}$-closed, and quasi-closed sets, rather than semi-closed sets. The converse of Theorem 4.1.2 in terms of quasi-closed sets is false. A "converse" may be obtained by including an extra hypothesis. The necessity of this added hypothesis is due to the fact that the union of two quasi-closed sets need not be quasi-closed.

Theorem 4.1.3. Let $(X,\mathcal{T},\mathcal{U})$ be a space in which $\mathcal{T} \cup \mathcal{U}$ is a base for a topology. If every nonempty family of quasi-closed sets with the finite intersection property has nonempty intersection, then $(X,\mathcal{T},\mathcal{U})$ is semi-compact.

Proof. Let $\mathcal{D} = \{D_\lambda\}_{\lambda \in \Lambda}$ be any nonempty family of semi-closed sets having the finite intersection property. Since $\mathcal{T} \cup \mathcal{U}$ is a base for a topology, each D_λ is quasi-closed. Hence the hypothesis implies that $\bigcap \mathcal{D} \neq \phi$. We conclude that $(X,\mathcal{T} \vee \mathcal{U})$ is compact.

Another possibility for a converse of Theorem 4.1.2 is this: "Let $(X,\mathcal{T},\mathcal{U})$ be a space in which $\mathcal{T} \cup \mathcal{U}$ is a base for a topology. If every nonempty family of $\mathcal{T}$-closed sets with the finite intersection property has nonempty intersection, and every nonempty family of $\mathcal{U}$-closed sets with the finite intersection property has nonempty intersection, then $(X,\mathcal{T},\mathcal{U})$ is semi-compact." However, this conjecture is false, as the following example shows.

Example 4.1.4. Let $X = R$. Let $\mathcal{T} = \{\phi,\{0\},X\sim\{0\},X\}$. Let $\mathcal{U}$ have the collection $\{\{x\} : x \in R \text{ and } x \neq 0\} \cup \{U : 0 \in U \text{ and } X\sim U \text{ is finite}\}$ as a base.

It is easily verified that $\mathcal{T} \cup \mathcal{U}$ is a base for a topology. The space $(X,\mathcal{T})$ is compact, so every nonempty family of $\mathcal{T}$-closed sets with the finite intersection property has nonempty intersection. A similar condition holds in $(X,\mathcal{U})$. However, $(X,\mathcal{T} \vee \mathcal{U})$ is the discrete topology on X. Hence, $(X,\mathcal{T},\mathcal{U})$ is not semi-compact.

If $(X,\mathcal{T},\mathcal{U})$ is semi-compact, then $(X,\mathcal{T})$ and $(X,\mathcal{U})$ are compact. Example 4.1.4 shows that the converse of this statement is false.

<u>Theorem 4.1.5</u>. If $(X,\mathcal{T},\mathcal{U})$ is semi-countably compact, then any nonempty countable collection of semi-closed (in particular, $\mathcal{T}$-closed, $\mathcal{U}$-closed, or quasi-closed) sets having the finite intersection property has nonempty intersection. If $\mathcal{T} \cup \mathcal{U}$ is a base for a topology, and if every nonempty countable family of quasi-closed sets with the finite intersection property has nonempty intersection, then $(X,\mathcal{T},\mathcal{U})$ is semi-compact.

<u>Proof</u>. The proofs are like those of Theorem 4.1.2 and Theorem 4.1.3.

We shall prove one other result about semi-countably compact spaces.

<u>Definition 4.1.6</u>. A <u>quasi-cluster point</u> of a sequence (x_n) in a space $(X,\mathcal{T},\mathcal{U})$ is a point $x \in X$ such that for each quasi-open set W about x and each positive integer N, there is a positive integer n with $n \geq N$ and $x_n \in W$.

Notice that every $\mathcal{T} \vee \mathcal{U}$-cluster point is a quasi-cluster point. Also, every quasi-cluster point is a $\mathcal{T}$-cluster point ($\mathcal{U}$-cluster point).

<u>Theorem 4.1.7</u>. If $(X,\mathcal{T},\mathcal{U})$ is semi-countably compact, then each

sequence in X has a $\mathcal{T}$-cluster point and a $\mathcal{U}$-cluster point. If $\mathcal{T} \cup \mathcal{U}$ is a base for a topology, and if each sequence in X has a quasi-cluster point, then $(X,\mathcal{T},\mathcal{U})$ is semi-countably compact.

Proof. Recall that a topological space is countably compact if and only if each sequence has a cluster point.

Suppose there is a sequence in X with no $\mathcal{T}$-cluster points. Then $(X,\mathcal{T})$ is not countably compact, which implies that $(X,\mathcal{T},\mathcal{U})$ is not semi-countably compact. Therefore, if $(X,\mathcal{T},\mathcal{U})$ is semi-countably compact, then each sequence in X has a $\mathcal{T}$-cluster point. The proof for $\mathcal{U}$-cluster points is similar.

Now assume that $\mathcal{T} \cup \mathcal{U}$ is a base for a topology and that each sequence in X has a quasi-cluster point. Since $\mathcal{T} \cup \mathcal{U}$ is a base for a topology, each semi-open set is quasi-open. Under this condition, every quasi-cluster point is a $\mathcal{T} \vee \mathcal{U}$-cluster point. Therefore, every sequence in $(X,\mathcal{T},\mathcal{U})$ has a $\mathcal{T} \vee \mathcal{U}$-cluster point. We conclude that $(X,\mathcal{T} \vee \mathcal{U})$ is countably compact.

We shall conclude this section with a category theorem.

Theorem 4.1.8. If $(X,\mathcal{T},\mathcal{U})$ is a semi-countably compact, quasi-regular (in particular, pairwise-regular) space, then $(X,\mathcal{T} \vee \mathcal{U})$ is of the second category in itself.

Proof. By Theorem 3.3.7, the space $(X,\mathcal{T} \vee \mathcal{U})$ is regular. Every countably compact, regular space is of the second category in itself (Theorem 29.25 in [1]).

2. Semi-paracompact Spaces.

Definition 4.2.1. A space $(X,\mathcal{T},\mathcal{U})$ is semi-paracompact if $(X,\mathcal{T} \vee \mathcal{U})$

is paracompact.

Every semi-compact space is semi-paracompact. We pause to give an example of a nontrivial semi-paracompact space.

<u>Example 4.2.2</u>. Let $X = R$. Let $\mathcal{T}$ be the topology for X whose base is $\{(-\infty,a) : a \in R\}$. Let $\mathcal{U}$ be the topology for X whose base is $\{(b,\infty) : b \in R\}$. The topology $\mathcal{T} \vee \mathcal{U}$ is the usual topology on R, which is paracompact. Hence, $(X,\mathcal{T},\mathcal{U})$ is semi-paracompact.

As we shall see (Corollary 6.1.7), there is a large class of semi-paracompact spaces, namely the quasi-pseudo-metrizable ones.

In order for $(X,\mathcal{T},\mathcal{U})$ to be semi-paracompact, it is not necessary that $(X,\mathcal{T})$ or $(X,\mathcal{U})$ be paracompact. We give an example.

<u>Example 4.2.3</u>. Let $X = R \times R$. Let $\mathcal{T}$ be the topology on X whose base is $\{[a,b) \times [c,d) : a,b,c,d \in R,\ a < b, \text{ and } c < d\}$, i.e., $\mathcal{T} = \mathcal{L} \times \mathcal{L}$. Let $\mathcal{U}$ be the topology on X whose base is $\{(a,b] \times (c,d] : a,b,c,d \in R,\ a < b, \text{ and } c < d\}$.

The spaces $(X,\mathcal{T})$ and $(X,\mathcal{U})$ are not paracompact (Example 1.9.19 of [5]). On the other hand, $(X,\mathcal{T} \vee \mathcal{U})$ is discrete, which implies that $(X,\mathcal{T},\mathcal{U})$ is semi-paracompact.

We conjecture that in order for a space $(X,\mathcal{T},\mathcal{U})$ to be semi-paracompact, it is not sufficient that $(X,\mathcal{T})$ and $(X,\mathcal{U})$ be paracompact. We have not yet found an example of such a space, however.

It seems, then, that the requirement that a bitopological space be semi-paracompact yields very little information about the two topologies involved.

Theorem 4.2.4. Every pairwise- (quasi-) regular, semi-Lindelöf space is semi-paracompact.

Proof. By Theorem 3.3.7, every pairwise- (quasi-) regular bitopological space is regular in the upper bound topology. By a theorem of point-set topology, every regular, Lindelöf topological space is paracompact.

We observe in passing that if $(X,\mathcal{T},\mathcal{U})$ is a semi-paracompact, pairwise- (quasi-) Hausdorff space, then $(X,\mathcal{T}\vee\mathcal{U})$ is normal. Moreover, if $(X,\mathcal{T},\mathcal{U})$ is a pairwise- (quasi-) regular, semi-Lindelöf space, then $(X,\mathcal{T}\vee\mathcal{U})$ is normal.

We conclude with some analogues of well-known equivalent definitions of paracompactness in regular, T_1-spaces.

Theorem 4.2.5. In a pairwise- (quasi-) regular, w-pairwise- (quasi-) T_1 space $(X,\mathcal{T},\mathcal{U})$, any two of the following conditions are equivalent:

(i) The space $(X,\mathcal{T},\mathcal{U})$ is semi-paracompact, i.e., any semi-open cover of X has a semi-open, locally finite refinement which covers X.

(ii) Every semi-open covering of X has a semi-closed, locally finite refinement which covers X.

(iii) Every semi-open covering of X has a locally finite refinement which covers X.

(iv) Every semi-open covering of X has a semi-open, σ- locally finite refinement which covers X.

Proof. If $(X,\mathcal{T},\mathcal{U})$ is a pairwise- (quasi-) regular, w-pairwise- (quasi-) T_1 space, then $(X,\mathcal{T}\vee\mathcal{U})$ is a regular T_1-space. The equivalences are well known in regular T_1-spaces ([1]).

3. Subspaces, Product Spaces, and Quasi-continuous Functions.

It follows from the results of point-set topology that every semi-closed subset of the semi-compact space $(X,\mathcal{T},\mathcal{U})$ is semi-compact. In particular, every $\mathcal{T}$-closed, $\mathcal{U}$-closed, or quasi-closed subset of $(X,\mathcal{T},\mathcal{U})$ is semi-compact.

Similarly, if $(X,\mathcal{T},\mathcal{U})$ is semi-paracompact, then every semi-closed (in particular, $\mathcal{T}$-closed, $\mathcal{U}$-closed, or quasi-closed) subset of $(X,\mathcal{T},\mathcal{U})$ is semi-paracompact.

Semi-compact sets have special properties in pairwise- (quasi-) Hausdorff spaces.

Theorem 4.3.1. Any semi-compact subset of a quasi-Hausdorff (in particular, pairwise-Hausdorff) space $(X,\mathcal{T},\mathcal{U})$ is semi-closed.

Proof. If $(X,\mathcal{T},\mathcal{U})$ is a quasi-Hausdorff space, then $(X,\mathcal{T}\vee\mathcal{U})$ is a Hausdorff space. Any compact subset of $(X,\mathcal{T}\vee\mathcal{U})$ is $\mathcal{T}\vee\mathcal{U}$-closed.

Theorem 4.3.2. In any quasi-Hausdorff (in particular, pairwise-Hausdorff) space, disjoint semi-compact sets are contained in disjoint semi-open sets.

Proof. If $(X,\mathcal{T},\mathcal{U})$ is a quasi-Hausdorff space, then $(X,\mathcal{T}\vee\mathcal{U})$ is a Hausdorff space. Compact subsets of $(X,\mathcal{T}\vee\mathcal{U})$ are contained in disjoint $\mathcal{T}\vee\mathcal{U}$-open sets.

We shall consider the products of semi-compact and semi-paracompact spaces. The result for compact spaces was proved by Datta ([2]). We shall give a slightly different proof.

Theorem 4.3.3. (Datta) If $(X,\mathcal{T},\mathcal{U})$ is the product of a collection $\{(X_\lambda,\mathcal{T}_\lambda,\mathcal{U}_\lambda)\}_{\lambda\in\Lambda}$ of semi-compact spaces, then $(X,\mathcal{T},\mathcal{U})$ is semi-compact.

Proof. By Theorem 2.4.3, the space $(X,\mathcal{T}\vee\mathcal{U})$ is the product of the collection $\{(X_\lambda,\mathcal{T}_\lambda\vee\mathcal{U}_\lambda)\}_{\lambda\in\Lambda}$. By hypothesis, each $(X_\lambda,\mathcal{T}_\lambda\vee\mathcal{U}_\lambda)$ is a compact space. The Tychonoff Theorem implies that $(X,\mathcal{T}\vee\mathcal{U})$ is compact.

Example 4.3.4. Let $X = R$. Let $\mathcal{T}$ be the topology on R whose base is $\{[a,\infty) : a \in R\}$, and let $\mathcal{U}$ be the topology on R whose base is $\{(-\infty,b) : b \in R\}$. Then $\mathcal{T} \vee \mathcal{U} = \mathcal{L}$, and $(X,\mathcal{T},\mathcal{U})$ is semi-paracompact (Example 1.9.17 of [5]). However, $(X,\mathcal{T},\mathcal{U}) \times (X,\mathcal{T},\mathcal{U})$ is not semi-paracompact (Example 1.9.19 of [5]).

Finally, we consider quasi-continuous images of semi-compact and semi-paracompact spaces. One of the desired results has been obtained by Datta ([2]).

Theorem 4.3.5. (a). (Datta) The quasi-continuous image of a semi-compact space is semi-compact.

(b). The bicontinuous image of a semi-compact space is semi-compact.

Proof. (a). Let $f : (X,\mathcal{T}_1,\mathcal{U}_1) \to (Y,\mathcal{T}_2,\mathcal{U}_2)$ be a quasi-continuous function from the semi-compact space $(X,\mathcal{T}_1,\mathcal{U}_1)$ onto $(Y,\mathcal{T}_2,\mathcal{U}_2)$. We shall show that $(Y,\mathcal{T}_2,\mathcal{U}_2)$ is compact.

Let $\mathcal{G} = \{G_\lambda\}_{\lambda\in\Lambda}$ be a semi-open cover of Y. For each $\lambda \in \Lambda$, there are a collection $\{T_\lambda^\alpha : \alpha \in \mathcal{A}_\lambda\}$ of $\mathcal{T}$-open sets and a collection $\{U_\lambda^\alpha : \alpha \in \mathcal{A}_\lambda\}$ such that $G_\lambda = \cup\{T_\lambda^\alpha \cap U_\lambda^\alpha : \alpha \in \mathcal{A}_\lambda\}$. Since $f^{-1}(G_\lambda) = \cup\{f^{-1}(T_\lambda^\alpha) \cap f^{-1}(U_\lambda^\alpha) : \alpha \in \mathcal{A}_\lambda\}$ and f is quasi-continuous, each $f^{-1}(G_\lambda)$ is $\mathcal{T}_1 \vee \mathcal{U}_1$-open. Thus, $\{f^{-1}(G_\lambda) : \lambda \in \Lambda\}$ is a $\mathcal{T}_1 \vee \mathcal{U}_1$-open cover of X. Since $(X,\mathcal{T}_1,\mathcal{U}_1)$ is semi-compact, there is a finite subcover $\{f^{-1}(G_1), f^{-1}(G_2),\ldots,f^{-1}(G_n)\}$ of

$(X,\mathcal{T}_1,\mathcal{U}_1)$. The collection $\{G_1,\ldots,G_n\}$ is a subset of $\mathcal{G}$ which covers Y.

(b). Every bicontinuous function is quasi-continuous, so the result follows from (a).

The continuous image of a paracompact topological space need not be paracompact, so one could offer a trivial example to show that the quasi-continuous (bicontinuous) image of a semi-paracompact is not semi-paracompact. There is a theorem of point-set topology which states that the image of a paracompact Hausdorff space under a continuous closed map is paracompact. A desirable generalization for bitopological spaces would be this: "The bicontinuous (quasi-continuous), bi-closed (quasi-closed) image of a semi-paracompact, pairwise-Hausdorff space is semi-paracompact." We have not been able to prove this conjecture because it appears that a bi-closed (quasi-closed) function may not be closed in the upper bound topology.

CHAPTER V

MORE SEPARATION AXIOMS

In this chapter, we shall complete the study of pairwise- and quasi-separation axioms begun in Chapter III. We shall consider analogues of the definitions of completely regular, normal, completely normal, and perfectly normal spaces.

1. Pairwise-Completely Regular and Quasi-completely Regular Spaces.

As we mentioned in Chapter I, results on pairwise-completely regular spaces were obtained independently by Fletcher ([3] and [4]) and Lane ([7]). The result which interests us most is Theorem 5.1.8 below.

Definition 5.1.1. (Fletcher and Lane) Let $(X,\mathcal{T},\mathcal{U})$ be a bitopological space. We say that $\mathcal{T}$ is completely regular with respect to $\mathcal{U}$ if for each $\mathcal{T}$-closed set F and each $x \in X \sim F$, there is a $\mathcal{T}$-upper semi-continuous, $\mathcal{U}$-lower semi-continuous function $f : X \to I$ such that $f(x) = 0$ and $f(F) = \{1\}$. If $\mathcal{T}$ is completely regular with respect to $\mathcal{U}$, and $\mathcal{U}$ is completely regular with respect to $\mathcal{T}$, then $(X,\mathcal{T},\mathcal{U})$ is pairwise-completely regular.

Let $\mathcal{T}^*$ be the topology on $[0,1]$ for which a base is $\{[0,a) : 0 < a \leq 1\}$; and let $\mathcal{U}^*$ be the topology on $[0,1]$ for which a base is $\{(b,1] : 0 \leq b < 1\}$. A space $(X,\mathcal{T},\mathcal{U})$ is quasi-completely regular provided that for each quasi-closed set F and each $x \in X \sim F$, there is a quasi-continuous function $f : (X,\mathcal{T},\mathcal{U}) \to ([0,1], \mathcal{T}^*, \mathcal{U}^*)$ such that $f(x) = 0$ and $f(F) = \{1\}$.

Theorem 5.1.2. Every pairwise-completely regular space is quasi-completely regular.

Proof. Let C be a quasi-closed set, and let $x \in X \sim C$. Since C is quasi-closed, there are a $\mathcal{T}$-closed set A and a $\mathcal{U}$-closed set B such that $C = A \cap B$. Then, say, $x \notin A$. There is a $\mathcal{T}$-upper semi-continuous, $\mathcal{U}$-lower semi-continuous function $f : X \to I$ such that $f(x) = 0$ and $f(A) = \{1\}$. Then $f : (X,\mathcal{T},\mathcal{U}) \to ([0,1], \mathcal{T}^*, \mathcal{U}^*)$ is a quasi-continuous function such that $f(x) = 0$ and $f(C) \subset f(A) = \{1\}$.

The converse of Theorem 5.1.2 is false. Although it anticipates future results, we include this example here.

Example 5.1.3. It can be verified that the space $(X,\mathcal{T},\mathcal{U})$ of Example 3.3.3 is quasi-normal (Definition 5.2.1) and quasi-T_1. By Corollary 5.2.6(b), $(X,\mathcal{T},\mathcal{U})$ is quasi-completely regular. However, $(X,\mathcal{T},\mathcal{U})$ is not pairwise-regular, and so (by Theorem 5.1.5) $(X,\mathcal{T},\mathcal{U})$ is not pairwise-completely regular.

Clearly, every subspace of a pairwise- (quasi-) completely regular space is pairwise- (quasi-) completely regular. We conjecture, on the other hand, that the product of quasi-regular spaces is not quasi-regular, whereas pairwise-regular spaces do have this desirable property.

Theorem 5.1.4. The product of pairwise-completely regular spaces is pairwise-completely regular.

Proof. Let $(X,\mathcal{T},\mathcal{U})$ be the product of the pairwise-completely regular spaces $\{(X_\lambda,\mathcal{T}_\lambda,\mathcal{U}_\lambda)\}_{\lambda \in \Lambda}$. Let C be a $\mathcal{T}$-closed subset of X, and let $x = (x_\lambda) \in X \sim C$. Since $X \sim C$ is $\mathcal{T}$-open, there is a finite subset Γ of Λ and a collection $\{T_\lambda\}_{\lambda \in \Gamma}$, where $T_\lambda \in \mathcal{T}_\lambda$, such that $x \in \bigcap_{\lambda \in \Gamma} \pi_\lambda^{-1}(T_\lambda) \subset X \sim C$.

For each $\lambda \in \Gamma$, there is a $\mathcal{T}_\lambda$-upper semi-continuous, $\mathcal{U}_\lambda$-lower semi-continuous function $f_\lambda : (X_\lambda, \mathcal{T}_\lambda, \mathcal{U}_\lambda) \to I$ such that $f_\lambda(x_\lambda) = 0$ and $f_\lambda(X_\lambda \sim T_\lambda) = \{1\}$. For each $\lambda \in \Gamma$, let $g_\lambda : (X, \mathcal{T}, \mathcal{U}) \to I$ be defined by $g_\lambda(x) = f_\lambda(\pi_\lambda(x))$. Since each π_λ is bicontinuous, each g_λ is $\mathcal{T}$-upper semi-continuous and $\mathcal{U}$-lower semi-continuous. Moreover, $g_\lambda(x) = 0$ and $g_\lambda(X \sim \pi_\lambda^{-1}(T_\lambda)) = \{1\}$.

Define $g : (X, \mathcal{T}, \mathcal{U}) \to I$ by $g(x) = \sup\{g_\lambda(x) : \lambda \in \Gamma\}$ for each $x \in X$. Then g is $\mathcal{T}$-upper semi-continuous and $\mathcal{U}$-lower semi-continuous, $g(x) = 0$, and $g(C) \subset g(X \sim \bigcap_{\lambda\in\Gamma} \pi_\lambda^{-1}(T_\lambda)) = g(\bigcup_{\lambda\in\Gamma} [X \sim \pi_\lambda^{-1}(T_\lambda)]) = \{1\}$. Thus, $\mathcal{T}$ is completely regular with respect to $\mathcal{U}$. A similar argument shows that $\mathcal{U}$ is completely regular with respect to $\mathcal{T}$. We conclude that $(X, \mathcal{T}, \mathcal{U})$ is pairwise-completely-regular.

We cannot prove the analogous theorem for quasi-completely regular spaces because the finite intersection of quasi-open sets need not be quasi-open. This fact implies that a function defined like g in Theorem 5.1.4 may not be quasi-continuous.

It is easy to establish the place of pairwise- (quasi-) completely regular spaces in the hierarchy.

<u>Theorem 5.1.5</u>. Every pairwise- (quasi-) completely regular space is pairwise- (quasi-) regular.

<u>Proof</u>. The proofs are easy.

We give an example to show that the converse of Theorem 5.1.4 is false.

<u>Example 5.1.6</u>. Let $X = R$. Let $\mathcal{T}$ be the usual topology for X, and let $\mathcal{U}$ be the cofinite topology on X.

One can verify that $(X,\mathcal{T},\mathcal{U})$ is pairwise-regular and not pairwise-Hausdorff. If $x,y \in X$, then there is a $\mathcal{T}$-open set containing x and not y. By Theorem 4 of [4], $(X,\mathcal{T},\mathcal{U})$ is not pairwise-uniform. By Theorem 5.1.8, $(X,\mathcal{T},\mathcal{U})$ is not pairwise-completely regular. We observe in passing that $(X,\mathcal{T} \vee \mathcal{U})$ is completely regular, but $(X,\mathcal{T},\mathcal{U})$ is not pairwise-completely regular (and not quasi-completely regular by Theorem 5.1.7(b)).

<u>Theorem 5.1.7</u>. (<u>a</u>). If $(X,\mathcal{T},\mathcal{U})$ is pairwise-completely regular, then $(X,\mathcal{T} \vee \mathcal{U})$ is completely regular.

(<u>b</u>). If $(X,\mathcal{T},\mathcal{U})$ is quasi-completely regular and $\mathcal{T} \cup \mathcal{U}$ is a base for a topology, then $(X,\mathcal{T} \vee \mathcal{U})$ is completely regular.

(<u>c</u>). If $(X,\mathcal{T} \vee \mathcal{U})$ is completely regular and $\mathcal{T} \cup \mathcal{U}$ is a base for a topology, then $(X,\mathcal{T},\mathcal{U})$ is quasi-completely regular.

<u>Proof</u>. (<u>a</u>). Let C be a $\mathcal{T} \vee \mathcal{U}$-closed subset of X, and let $x \in X \sim C$. Then $C = \bigcap_{\lambda \in \Lambda} (E_\lambda \cup F_\lambda)$, where each E_λ is $\mathcal{T}$-closed, and each F_λ is $\mathcal{U}$-closed. Since $x \in X \sim C$, there is some $\lambda \in \Lambda$ such that $x \notin E_\lambda \cup F_\lambda$. Hence, there is a $\mathcal{T}$-upper semi-continuous and $\mathcal{U}$-lower semi-continuous function $g : (X,\mathcal{T},\mathcal{U}) \to I$ such that $g(x) = 0$ and $g(E_\lambda) = \{1\}$; and there is a $\mathcal{U}$-upper semi-continuous and $\mathcal{T}$-lower semi-continuous function $h : (X,\mathcal{T},\mathcal{U}) \to I$ such that $h(x) = 0$ and $h(F_\lambda) = \{1\}$. The function $f : (X,\mathcal{T},\mathcal{U}) \to I$ which is defined by $f = g \vee h$ is a $\mathcal{T} \vee \mathcal{U}$-continuous function such that $f(x) = 0$ and $f(C) \subset f(E_\lambda \cup F_\lambda) = \{1\}$. We conclude that $(X,\mathcal{T} \vee \mathcal{U})$ is completely regular.

(<u>b</u>). The proof is similar to that for (a). The added hypothesis is necessary in order for each semi-closed set to be quasi-closed.

(<u>c</u>). The proof is easy.

Observe that if $(X,\mathcal{T})$ and $(X,\mathcal{U})$ are completely regular, then $(X,\mathcal{T},\mathcal{U})$ is quasi-completely regular, and $(X,\mathcal{T}\vee\mathcal{U})$ is completely regular.

We shall now state the theorem which to us most recommends the study of pairwise-separation properties. The proof is very hard.

Theorem 5.1.8. (Fletcher and Lane) A space is pairwise-completely regular if and only if it is pairwise-uniform.

Proof. See [4] or [8].

Corollary 5.1.9. Every pairwise-uniform space is quasi-completely regular.

Proof. The result follows from Theorem 5.1.8 and Theorem 5.1.2.

2. Pairwise-Normal and Quasi-normal Spaces.

Kelly's ([7]) concern with quasi-pseudo-metric spaces caused him to consider pairwise-normal spaces.

Definition 5.2.1. (Kelly) A space $(X,\mathcal{T},\mathcal{U})$ is pairwise-normal if for each $\mathcal{T}$-closed set A and each $\mathcal{U}$-closed set B with $A \cap B = \phi$, then there is a $\mathcal{T}$-open set T and a $\mathcal{U}$-open set U such that $A \subset U$, $B \subset T$, and $T \cap U = \phi$.

A space $(X,\mathcal{T},\mathcal{U})$ is quasi-normal if for each pair of disjoint, quasi-closed sets A and B, there exist disjoint quasi-open sets V and W such that $A \subset V$ and $B \subset W$.

Unexpectedly, there are pairwise-normal spaces that are not quasi-normal.

Example 5.2.2. Let $X = \{0,1,2,3\}$. Let $\mathcal{T}$ be the topology given by

$\mathcal{T} = \{\emptyset, \{0\}, X\}$, and let $\mathcal{U}$ be the topology on X given by $\mathcal{U} = \{\emptyset, \{0\}, \{0,1\}, \{0,1,2\}, \{0,2\}, \{0,2,3\}, X\}$.

The $\mathcal{T}$-closed sets are $\emptyset$, $\{1,2,3\}$, and X. The $\mathcal{U}$-closed sets are $\emptyset$, $\{1,2,3\}$, $\{2,3\}$, $\{3\}$, $\{1,3\}$, and $\{1\}$. Since there do not exist a nonempty $\mathcal{T}$-closed set A and a non-empty $\mathcal{U}$-closed set B such that $A \cap B = \emptyset$, the space $(X,\mathcal{T},\mathcal{U})$ is trivially pairwise-normal. On the other hand, $\{1\}$ and $\{3\}$ are disjoint quasi-closed sets which are not contained in disjoint quasi-open sets, and so $(X,\mathcal{T},\mathcal{U})$ is not quasi-normal.

There are quasi-normal spaces that are not pairwise-normal.

<u>Example 5.2.3</u>. Let X = R. Let $\mathcal{T}$ be the usual topology on R, and let $\mathcal{U}$ be the cofinite topology on R. Then $(X,\mathcal{T} \vee \mathcal{U})$ is R with the usual topology, which is a normal space. By Theorem 5.2.7, $(X,\mathcal{T},\mathcal{U})$ is quasi-normal. However, $(X,\mathcal{T},\mathcal{U})$ is not pairwise-normal. Consider the $\mathcal{T}$-closed set $A = [0,1]$ and the $\mathcal{U}$-closed set $B = \{2\}$. Any $\mathcal{T}$-open set containing B must contain an open interval, but there is no $\mathcal{U}$-open set containing A whose complement contains an open interval.

Next, we wish to compare pairwise- (quasi-) normal spaces to pairwise- (quasi-) completely regular spaces. As in the classical case, we shall need an equivalent definition for pairwise-normal spaces.

<u>Theorem 5.2.4</u>. (<u>a</u>). (Kelly) A space $(X,\mathcal{T},\mathcal{U})$ is pairwise-normal if and only if for each $\mathcal{U}$-closed set C and each $\mathcal{T}$-open set D such that $C \subset D$, there are a $\mathcal{T}$-open set G and a $\mathcal{U}$-closed set F such that $C \subset G \subset F \subset D$.

(<u>b</u>). A space $(X,\mathcal{T},\mathcal{U})$ is quasi-normal if and only if for any quasi-closed set and F and any quasi-open set G with $F \subset G$, there is a quasi-open set U such that $F \subset U \subset \overline{U} \subset G$.

Proof. (a). This is a remark of Kelly ([7]). The proof is much like that of the standard result for topological spaces.

(b). One need only mimic the standard proof to establish this theorem. We omit the details.

In order to relate pairwise- (quasi-) normal spaces to other bitopological spaces, we shall need analogues of Urysohn's Lemma (Theorem 1.10.8 of [5]).

Theorem 5.2.5. (a). (Kelly) If $(X,\mathcal{T},\mathcal{U})$ is pairwise-normal, then for each $\mathcal{U}$-closed set F and each $\mathcal{T}$-closed set H with $F \cap H = \emptyset$, there is a real-valued function $g : X \to I$ such that $g(F) = \{0\}$, $g(H) = \{1\}$, and g is $\mathcal{T}$-upper semi-continuous and $\mathcal{U}$-lower semi-continuous.

(b). If $(X,\mathcal{T},\mathcal{U})$ is a quasi-normal, then for each pair of disjoint, quasi-closed sets A and B, there is a quasi-continuous function $f : (X,\mathcal{T},\mathcal{U}) \to ([0,1],\mathcal{T}^*,\mathcal{U}^*)$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Proof. (a). This is Theorem 2.7 of [7]. The proof is like that of Urysohn's Lemma, and it depends on Theorem 5.2.4(a).

(b). To establish this result, one need only mimic the proof of Urysohn's Lemma, using Theorem 5.2.4(b).

Corollary 5.2.6. (a). (Fletcher) If $(X,\mathcal{T},\mathcal{U})$ is a pairwise-normal, s-pairwise-T_1 space, then $(X,\mathcal{T},\mathcal{U})$ is pairwise-completely regular.

(b). If $(X,\mathcal{T},\mathcal{U})$ is a quasi-normal, quasi-T_1 space, then $(X,\mathcal{T},\mathcal{U})$ is quasi-completely regular.

Proof. (a). This is Theorem 6 of [4]. Let C be a $\mathcal{T}$-closed subset of X, and let $x \in X \sim C$. If $y \in C$, then the fact that $(X,\mathcal{T},\mathcal{U})$ is s-pairwise-T_1 implies there is a $\mathcal{U}$-neighborhood of y not containing x. Hence, $\mathcal{U}\text{-cl}(\{x\}) \cap C = \emptyset$. By Theorem 5.2.5(a), there is a $\mathcal{T}$-upper semi-continuous, $\mathcal{U}$-lower semi-continuous function $g: X \to I$ such that $g(x) = 0$ and $g(C) = \{1\}$. Thus, $\mathcal{T}$ is completely regular with respect to $\mathcal{T}$. A similar argument shows that $\mathcal{U}$ is completely regular with respect to $\mathcal{T}$. We conclude that $(X,\mathcal{T},\mathcal{U})$ is pairwise-completely regular.

(b). By Theorem 3.1.8(b), every point in a quasi-T_1 space is quasi-closed. The result then follows immediately from Theorem 5.2.5(b).

Of course, there are pairwise- (quasi-) completely regular, pairwise-(quasi-) T_1 spaces that are not pairwise- (quasi-) normal. If $(X,\mathcal{T})$ is a non-normal Tychonoff space, then $(X,\mathcal{T},\mathcal{T})$ has these properties.

We shall consider the upper bound topology only briefly. The space of Example 5.2.2 is pairwise-normal, but $(X,\mathcal{T} \vee \mathcal{U})$ is not normal. We can obtain the following theorem.

Theorem 5.2.7. If $(X,\mathcal{T},\mathcal{U})$ is quasi-normal, and $\mathcal{T} \cup \mathcal{U}$ is a base for a topology, then $(X,\mathcal{T} \vee \mathcal{U})$ is normal. If $(X,\mathcal{T} \vee \mathcal{U})$ is normal, and $\mathcal{T} \cup \mathcal{U}$ is a base for a topology, then $(X,\mathcal{T},\mathcal{U})$ is quasi-normal.

Proof. Assume that $(X,\mathcal{T},\mathcal{U})$ is quasi-normal and that $\mathcal{T} \cup \mathcal{U}$ is a base for a topology. Let C_1 and C_2 be disjoint semi-closed subsets of X. Since $\mathcal{T} \cup \mathcal{U}$ is a base for a topology, C_1 and C_2 are quasi-closed. Since $(X,\mathcal{T},\mathcal{U})$ is quasi-normal, there are disjoint, quasi-open (and, hence, semi-open) sets U_1 and U_2 such that $C_1 \subset U_1$ and $C_2 \subset U_2$. Thus, $(X,\mathcal{T} \vee \mathcal{U})$ is normal.

Now assume that $(X,\mathcal{T} \vee \mathcal{U})$ is normal and that $\mathcal{T} \cup \mathcal{U}$ is a base for a topology. Let F_1 and F_2 be disjoint, quasi-closed (and, hence, semi-closed) subsets of X. Since $(X,\mathcal{T} \vee \mathcal{U})$ is normal, there are disjoint, semi-open sets U_1 and U_2 such that $F_1 \subset U_1$ and $F_2 \subset U_2$. The sets U_1 and U_2 are quasi-open because $\mathcal{T} \cup \mathcal{U}$ is a base for a topology. We conclude that $(X,\mathcal{T},\mathcal{U})$ is quasi-normal.

We pause to give an equivalent formulation, due to Lane ([8]), of the definition of pairwise-normal spaces.

Theorem 5.2.8. (Lane) In order for $(X,\mathcal{T},\mathcal{U})$ to be pairwise-normal, it is necessary and sufficient that for every pair of functions f and g defined on X such that f is $\mathcal{T}$-upper semi-continuous, g is $\mathcal{U}$-lower semi-continuous, and $g \leq f$, there exists a $\mathcal{T}$-upper semi-continuous and $\mathcal{U}$-lower semi-continuous function h on X such that $g \leq h \leq f$.

Proof. This is Theorem 2.5 of [8].

Much of Lane's paper is devoted to studying the possibility of extending certain functions. Kelly (Theorem 2.9 of [7]) proposed the following generalization of Tietze's Theorem: "Let $(X,\mathcal{T},\mathcal{U})$ be a pairwise-normal bitopological space. Let $A \subset X$ be $\mathcal{T}$-closed and $\mathcal{U}$-closed. Let f be a real-valued function defined on A which is $\mathcal{T}$-upper semi-continuous and $\mathcal{U}$-lower semi-continuous. Then there exists an extension F of f to the whole of X which is $\mathcal{T}$-upper semi-continuous and $\mathcal{U}$-lower semi-continuous." Lane showed that Kelly's theorem was in error by giving the following example.

Example 5.2.9. (Lane) Let X be any uncountable set. Let $\mathcal{T}$ be the discrete topology on X, and let $\mathcal{U}$ consist of the null set and complements of countable sets. It is easily verified that $(X,\mathcal{T},\mathcal{U})$ is pairwise-normal.

Let $A = \{x_n : n \in N\}$ be any countably infinite subset of X. Then A is $\mathcal{T}$-closed and $\mathcal{U}$-closed. However, if $f : A \to I$ is defined by $f(x_n) = n$, then f cannot be extended to a function on all of X that is $\mathcal{T}$-lower semi-continuous and $\mathcal{U}$-upper semi-continuous. (See pp. 242-3 of [8]).

Lane did, however, prove the following result on extending functions.

<u>Theorem 5.2.10</u>. (Lane) If A is a $\mathcal{T}$-closed and $\mathcal{U}$-closed subset of the pairwise-normal space $(X,\mathcal{T},\mathcal{U})$, then every bounded, $\mathcal{T}$-upper semi-continuous, $\mathcal{U}$-lower semi-continuous function $f : X \to R$ can be extended to a function $F : X \to R$ which is $\mathcal{T}$-upper semi-continuous and $\mathcal{U}$-lower semi-continuous.

<u>Proof</u>. This is Theorem 2.6 of [8].

Kelly obtained an additional property of pairwise-normal spaces.

<u>Theorem 5.2.11</u>. (Kelly) If $(X,\mathcal{T},\mathcal{U})$ is a second countable, pairwise-regular space, then $(X,\mathcal{T},\mathcal{U})$ is pairwise-normal.

<u>Proof</u>. This is Lemma 3.2 of [7].

We observe that if $(X,\mathcal{T},\mathcal{U})$ satisfies the hypotheses of Theorem 5.2.11, then $(X,\mathcal{T} \vee \mathcal{U})$ is a second countable, regular space and so is normal.

The quasi-normality property for bitopological spaces relates obliquely to the property of semi-paracompactness. Recall our remarks following Theorem 4.2.4.

<u>Theorem 5.2.12</u>. If $(X,\mathcal{T},\mathcal{U})$ is a semi-paracompact, quasi-Hausdorff space such that $\mathcal{T} \cup \mathcal{U}$ is a base for a topology, then $(X,\mathcal{T},\mathcal{U})$ is quasi-normal.

Proof. Our hypotheses imply that $(X,\mathcal{T}\vee\mathcal{U})$ is a paracompact, Hausdorff topological space, and so $(X,\mathcal{T}\vee\mathcal{U})$ is normal. Apply Theorem 5.2.7.

Semi-compactness relates better than semi-paracompactness to pairwise-normality.

Theorem 5.2.13. Every semi-compact, pairwise-Hausdorff space $(X,\mathcal{T},\mathcal{U})$ is pairwise-normal.

Proof. Let A and B be subsets of X such that $A \cap B = \emptyset$, A is $\mathcal{T}$-closed, and B is $\mathcal{U}$-closed.

Fix $x \in A$. Since $(X,\mathcal{T},\mathcal{U})$ is pairwise-Hausdorff, for each $y \in B$, there exist a $\mathcal{T}$-open set T_y and a $\mathcal{U}$-open set U_y such that $x \in T_y$, $y \in U_y$, and $T_y \cap U_y = \emptyset$. The collection $\{T_y : y \in B\} \cup \{X \sim B\}$ is a semi-open cover of X, and so there is a finite subcover $\{T_{y_1}, T_{y_2}, \ldots, T_{y_n}\}$ of B. Let $T_x = \bigcup_{m=1}^{n} T_{y_m}$, and let $U_x = \bigcap_{m=1}^{n} U_{y_m}$. Then T_x is a $\mathcal{T}$-open set, U_x is a $\mathcal{U}$-open set, $x \in U_x$, $B \subset T_x$, and $U_x \cap T_x = \emptyset$.

Now $\{U_x : x \in X\} \cup \{X \sim A\}$ is a semi-open cover of X, so there is a finite subcover $\{U_{x_1}, U_{x_2}, \ldots, U_{x_r}\}$ of A. Let $U = \bigcup_{s=1}^{r} U_{x_s}$, and let $T = \bigcap_{s=1}^{r} T_{x_s}$. Then U is $\mathcal{U}$-open, T is $\mathcal{T}$-open, $A \subset U$, $B \subset T$, and $U \cap T = \emptyset$. We conclude that $(X,\mathcal{T},\mathcal{U})$ is pairwise-normal.

Theorem 5.2.14. Every pairwise-regular, semi-Lindelöf space is pairwise-normal.

Proof. Let A and B be disjoint subsets of X such that A is $\mathcal{T}$-closed and B is $\mathcal{U}$-closed. Then $X \sim B$ is $\mathcal{U}$-open, and $A \subset X \sim B$; and $X \sim A$ is $\mathcal{T}$-open, and $B \subset X \sim A$. By Theorem 5.2.4(a), for each $a \in A$, (since

$\mathcal{T}$-cl$(\{a\}) \subset A$) there are a $\mathcal{U}$-open set U_a and a $\mathcal{T}$-closed set F_a such that $a \in U_a \subset F_a \subset X \sim B$. Similarly, for each $b \in B$, there are a $\mathcal{T}$-open set T_b and a $\mathcal{U}$-closed set E_b such that $b \in T_b \subset E_b \subset X \sim A$.

Let $\mathcal{A} = \{U_a : a \in A\}$, and let $\mathcal{B} = \{T_b : b \in B\}$. Since A and B are semi-closed subsets of a semi-Lindelöf space, there exist countable subcollections $\{U_1, U_2, \ldots\} \subset \mathcal{A}$ and $\{T_1, T_2, \ldots\} \subset \mathcal{B}$ such that $A \subset \bigcup_{n=1}^{\infty} U_n$ and $B \subset \bigcup_{n=1}^{\infty} T_n$.

Let $G_1 = U_1$, and for each $n > 1$, let $G_n = U_n \sim \bigcup_{j=1}^{n-1} E_j$. Let $H_1 = T_1$, and for each $n > 1$, let $H_n = T_n \sim \bigcup_{j=1}^{n-1} F_j$. Let $G = \bigcup_{n=1}^{\infty} G_n$, and let $H = \bigcup_{n=1}^{\infty} H_n$. Then G is $\mathcal{U}$-open, and H is $\mathcal{T}$-open. Moreover, $A \subset \bigcup_{n=1}^{\infty} G_n$ and $B \subset \bigcup_{n=1}^{\infty} H_n$. Finally, $G \cap H = \emptyset$. Therefore, $(X,\mathcal{T},\mathcal{U})$ is pairwise-normal.

Note that Theorem 5.2.11 is a corollary to Theorem 5.2.14. We remark that in the course of proving Theorem 5.2.13 and Theorem 5.2.14, we often required the finite intersection of open sets be open (or an equivalent condition). It seems probable that no analogue of either theorem exists for quasi-normal spaces. We have not, however, found the desired examples.

As is well known, not every subspace of a normal space need be normal. If $(X,\mathcal{T})$ is such a space, then $(X,\mathcal{T},\mathcal{T})$ is a pairwise- (quasi-) normal space having a subspace which is not pairwise- (quasi-) normal.

Similarly, the finite product of pairwise- (quasi-) normal spaces need not be pairwise- (quasi-) normal.

For functions on pairwise-normal spaces, we obtain analogues of the usual result.

<u>Theorem 5.2.15</u>. (<u>a</u>). Let $f : (X,\mathcal{T}_1,\mathcal{U}_1) \to (y,\mathcal{T}_2,\mathcal{U}_2)$ be a bi-closed, bi-continuous function from the pairwise-normal space $(X,\mathcal{T}_1,\mathcal{U}_1)$ onto $(y,\mathcal{T}_2,\mathcal{U}_2)$. Then $(X,\mathcal{T}_2,\mathcal{U}_2)$ is pairwise-normal.

(<u>b</u>). Let $f : (X,\mathcal{T}_1,\mathcal{U}_2) \to (y,\mathcal{T}_2,\mathcal{U}_2)$ be a quasi-closed, quasi-continuous function from the quasi-normal space $(X,\mathcal{T}_1,\mathcal{U}_1)$ onto $(y,\mathcal{T}_2,\mathcal{U}_2)$. Then $(X,\mathcal{T}_2,\mathcal{U}_2)$ is quasi-normal.

<u>Proof</u>. The proofs are almost exactly like the usual one (e.g., Theorem 1.10.30 of [5]).

3. <u>Pairwise- (Quasi-) Completely Normal and Pairwise-Perfectly Normal Spaces</u>.

We shall consider the spaces of this section only briefly. Patty's work ([10]) was aimed toward quasi-metrization results.

<u>Definition 5.3.1</u>. (Patty) A space $(X,\mathcal{T},\mathcal{U})$ is <u>pairwise-completely normal</u> provided that whenever A and B are subsets of X such that $\mathcal{T}\text{-cl}(A) \cap B = \emptyset$ and $A \cap \mathcal{U}\text{-cl}(B) = \emptyset$, then there are a $\mathcal{U}$-open set U and a $\mathcal{T}$-open set T such that $A \subset U$, $B \subset T$, and $U \cap T = \emptyset$.

A space $(X,\mathcal{T},\mathcal{U})$ is <u>quasi-completely normal</u> provided that whenever A and B are subsets of X such that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$, then there exist disjoint quasi-open sets U and V such that $A \subset U$ and $B \subset V$.

We are primarily interested in pairwise-completely regular spaces because of their applications to quasi-pseudo-metric spaces. We can find no relations between pairwise-completely normal spaces and quasi-completely normal spaces.

<u>Theorem 5.3.2</u>. (<u>a</u>). A space $(X,\mathcal{T},\mathcal{U})$ is pairwise-completely normal

if and only if every subspace of $(X,\mathcal{T},\mathcal{U})$ is pairwise-normal.

(b). If $(X,\mathcal{T},\mathcal{U})$ is quasi-completely normal, then every subspace of $(X,\mathcal{T},\mathcal{U})$ is quasi-normal.

Proof. In both cases, one can mimic the usual proof (e.g., Theorem 10.6 of [5]) with only minor changes.

Corollary 5.3.3. Every pairwise- (quasi-) completely normal space is pairwise- (quasi-) normal.

Proof. The result follows at once from Theorem 5.3.2.

We conjecture that the converse of Theorem 5.3.2(b) is not true becuase the finite intersection of quasi-open sets may not be quasi-open.

There are normal spaces that are not completely normal (e.g., Example 1.10.21 of [5]). Hence, there are pairwise-normal spaces that are not pairwise-completely normal.

Also, the finite product of completely normal spaces need not be completely normal (e.g., Example 1.10.25 of [5]). So the finite product of pairwise- (quasi-) completely normal spaces need not be pairwise-(quasi-) completely normal.

We shall include a definition of one other pairwise-separation axiom. We contend that defining a corresponding quasi-separation axiom would be fruitless in the light of the observed divergence of the two approaches.

Definition 5.3.4. (Patty) A space $(X,\mathcal{T},\mathcal{U})$ is pairwise-perfectly normal provided $(X,\mathcal{T},\mathcal{U})$ satisfies the following conditions:

(i) The space $(X,\mathcal{T},\mathcal{U})$ is pairwise-normal.

(ii) Each $\mathcal{T}$-closed set is the intersection of a countable collection of $\mathcal{U}$-open sets.

(iii) Each $\mathcal{U}$-closed set is the intersection of a countable collection of $\mathcal{T}$-open sets.

Theorem 5.3.5. (Patty) Every pairwise-perfectly normal space is pairwise-completely normal.

Proof. This is Theorem 2.5 of [10]. Let A and B be subsets of X such that $\mathcal{T}\text{-cl}(A) \cap B = \phi$ and $A \cap \mathcal{U}\text{-cl}(B) = \phi$. Then $X \sim \mathcal{U}\text{-cl}(B) = \bigcup_{j=1}^{\infty} M_j$, where each M_j is $\mathcal{T}$-closed; and $X \sim \mathcal{T}\text{-cl}(A) = \bigcup_{j=1}^{\infty} N_j$, where each N_j is $\mathcal{U}$-closed. Since $(X,\mathcal{T},\mathcal{U})$ is normal, for each j there is a $\mathcal{U}$-open set S_j and a $\mathcal{T}$-closed set S'_j such that $M_j \subset S_j \subset S'_j \subset X \sim \mathcal{U}\text{-cl}(B)$; and for each j there is a $\mathcal{T}$-open set V_j and a $\mathcal{U}$-closed set V'_j such that $N_j \subset V_j \subset V'_j \subset X \sim \mathcal{T}\text{-cl}(A)$. For each positive integer n, let $U_n = S_n \sim \bigcup_{j=1}^{n} V'_j$ and $T_n = V_n \sim \bigcup_{j=1}^{n} S'_j$. Finally, let $U = \bigcup_{n=1}^{\infty} U_n$ and $T = \bigcup_{n=1}^{\infty} T_n$. Then U is $\mathcal{U}$-open, T is $\mathcal{T}$-open, $A \subset U$, $B \subset T$, and $T \cap U = \phi$.

CHAPTER VI

QUASI-PSEUDO-METRIC SPACES

In this chapter, we first study some properties of quasi-pseudo-metric bitopological spaces. In the second section, we summarize the known quasi-pseudo-metrization theorems.

1. Properties of Quasi-pseudo-metric Spaces.

Some of the properties of quasi-pseudo-metric spaces have been studied, but the results are scattered. In this section, we shall try to collect them.

Kelly's work ([7]) included a study of pairwise-regular and pairwise-normal quasi-pseudo-metric spaces.

Theorem 6.1.1. Let (X, τ, μ) be a quasi-pseudo-metric space, and let $\mathcal{T}$ and $\mathcal{U}$ be the topologies determined by τ and μ, respectively.

(a). (Kelly) For a fixed point x in X, $\tau(x,y)$ is a $\mathcal{T}$-upper semi-continuous and $\mathcal{U}$-lower semi-continuous function of y; and for a fixed point y in X, $\tau(x,y)$ is a $\mathcal{T}$-lower semi-continuous and $\mathcal{U}$-upper semi-continuous function of X.

(b). Let $\mathcal{A}$ be the topology on R whose base is $\{(a,\infty) : a \in R\}$, and let $\mathcal{B}$ be the topology on R whose base is $\{(-\infty,b) : b \in R\}$. For a fixed point x in X, $\tau(x,y)$ is a quasi-continuous function from $(X,\mathcal{T},\mathcal{U})$ into $(R,\mathcal{A},\mathcal{B})$; and for a fixed point y in X, $\tau(x,y)$ is a quasi-continuous function from $(X,\mathcal{T},\mathcal{U})$ into $(R,\mathcal{A},\mathcal{B})$.

Proof. (a). This is Porposition 4.1 of [7]. The proof depends on showing that, for every x in X, the sets $\{y : \tau(x,y) < \varepsilon\}$ and

$\{y : \tau(x,y) \le \varepsilon\}$ are respectively $\mathcal{T}$-open and $\mathcal{U}$-closed.

(b). The result follows immediately from (a).

We use the observation in the proof of Theorem 6.1.1(a) to establish the following theorem.

Theorem 6.1.2. Let (X, τ, μ) be a quasi-pseudo-metric space, and let $\mathcal{T}$ and $\mathcal{U}$ be the topologies determined by τ and μ, respectively.

(a). (Kelly) The space $(X,\mathcal{T},\mathcal{U})$ is pairwise-regular and pairwise-normal. Moreover, $(X,\mathcal{T},\mathcal{U})$ is pairwise-Hausdorff if and only if τ and μ are quasi-metrics.

(b). The space $(X,\mathcal{T},\mathcal{U})$ is quasi-regular. If τ and μ are quasi-metrics, then (X, τ, μ) is quasi-Hausdorff.

Proof. (a). This is Proposition 4.2 of [7]. We include the proof because it is an interesting one.

Let $x \in X$, and let T be a $\mathcal{T}$-open set about x. Since T is $\mathcal{T}$-open, there is a basic open set $\{y : \tau(x,y) < \varepsilon\}$ such that $x \in \{y : \tau(x,y) < \varepsilon\} \subset T$. Let $V = \{y : \tau(x,y) < \varepsilon/2\}$, and let $F = \{y : \tau(x,y) \le \varepsilon/2\}$. By the remark in the proof of Theorem 6.1.1(a), the sets V and F are $\mathcal{T}$-open and $\mathcal{U}$-closed, respectively. Moreover, $x \in V \subset F \subset T$. Thus, $\mathcal{T}$ is regular with respect to $\mathcal{U}$ by Theorem 3.3.4(a). Similarly, $\mathcal{U}$ is regular with respect to $\mathcal{T}$. We conclude that $(X,\mathcal{T},\mathcal{U})$ is pairwise-regular.

Let A and B be disjoint subsets of X such that A is $\mathcal{T}$-closed and B is $\mathcal{U}$-closed.

For each x in X, let $\tau(x,A) = \inf\{\tau(x,a) : a \in A\}$ and $\mu(x,B) = \inf\{\mu(x,b) : b \in B\}$. Then $A = \{y : \tau(y,A) = 0\}$, and $B = \{y : \mu(y,B) = 0\}$. Let $U = \{x : \tau(x,A) < \mu(x,B)\}$ and $T = \{x : \mu(x,B) < \tau(x,A)\}$. Then $T \cap U = \emptyset$, $A \subset U$, and $B \subset T$. To establish that $(X,\mathcal{T},\mathcal{U})$ is pairwise-normal, we need to show that T and U are respectively $\mathcal{T}$-open and $\mathcal{U}$-open.

We shall show that T is $\mathcal{T}$-open. Suppose that $z \in T$ and that $\tau(z,A) - \mu(z,B) = \epsilon > 0$. If $x \in S_\tau(z,\epsilon/4)$, then $\mu(x,B) \le \mu(x,z) + \mu(z,B) = \tau(z,x) + \mu(z,B)$, and $\tau(z,A) \le \tau(z,x) + \tau(x,A)$. Hence

$$\tau(x,A) - \mu(x,B) \ge \tau(z,A) - \mu(z,B) - 2\tau(z,x) > \epsilon/2$$

and so $x \in T$. Thus, $x \in S_\tau(z,\epsilon/4) \subset T$.

Clearly, $(X,\mathcal{T},\mathcal{U})$ is pairwise-Hausdorff if and only if τ and μ are quasi-metrices.

(b). Our assertions follow from Theorem 3.3.2, Theorem 3.2.2, and part (a).

Observe that a subset A of (X,τ,μ) is quasi-closed if and only if $A = \{y : \tau(y,A) = 0 = \mu(y,A)\}$. We have been unable to modify the proof of Theorem 6.1.2 to show that a quasi-pseudo-metric space is quasi-normal. We conjecture that such is not the case.

The space of Example 3.2.3 is a quasi-pseudo-metric space that is quasi-Hausdorff, but neither τ nor μ is a quasi-metric.

We have repeatedly proved theorems in which we required that $\mathcal{T} \cup \mathcal{U}$ be a base for a topology. The quasi-pseudo-metric space of Example 2.1.6(e) has this useful property. However, the space of Example 2.1.3 does not. In Example 2.1.3, $\mathcal{T}$ is determined by the quasi-pseudo-metric τ defined by $\tau(x,y) = \begin{cases} |x-y|, \text{ if } x \le y \\ 1, \text{ if } x > y \end{cases}$; and $\mathcal{U}$ is determined by the conjugate μ of τ.

If τ and μ are quasi-metrics, then Theorem 6.1.2 implies that $(X,\mathcal{T},\mathcal{U})$ is pairwise-Hausdorff, s-pairwise-T_1 (Theorem 3.2.4(a)), pairwise-T_o (Theorem 3.1.6(a)), pairwise-normal, pairwise-completely regular (Corollary 5.2.6(a)), and pairwise-regular. In terms of quasi-separation axioms, $(X,\mathcal{T},\mathcal{U})$ is quasi-completely regular (Theorem 5.1.3) quasi-regular, quasi-Hausdorff, quasi-T_1 (Theorem 3.2.4(b)), and quasi-T_o (Theorem 3.1.6(a)).

In view of the above observations, it seems to be more valuable to study quasi-pseudo-metric spaces in terms of the stronger pairwise-separation properties.

Patty has shown ([10]) that quasi-pseudo-metric spaces have additional pairwise-separation properties.

<u>Theorem 6.1.3</u>. (Patty) A quasi-pseudo-metric space (X,τ,μ) is pairwise-perfectly normal.

<u>Proof</u>. Let $\mathcal{T}$ and $\mathcal{U}$ be the topologies determined by τ and μ respectively. By Theorem 6.1.2(a), $(X,\mathcal{T},\mathcal{U})$ is normal.

Let A be a $\mathcal{T}$-closed set. For each positive integer n, let $U_n = \{x : \mu(A,x) < \frac{1}{n}\}$. Then $A = \{x : \tau(x,A) = 0\} = \{x : \mu(A,x) = 0\} = \bigcap_{n=1}^{\infty} U_n$. To see that U_n is $\mathcal{U}$-open, let $x \in U_n$ and $r_n = \frac{1}{n} - \mu(A,x)$. We assert that $x \in S_\mu(x,r_n) \subset U_n$. If $y \in S_\mu(x,r_n)$, then $\mu(A,y) \leq \mu(A,x) + \mu(x,y) < \mu(A,x) + \frac{1}{n} - \mu(A,x) = \frac{1}{n}$. Thus, each $\mathcal{T}$-closed set is the countable intersection of $\mathcal{U}$-open sets.

<u>Corollary 6.1.4</u>. Every quasi-pseudo-metric space is pairwise-completely normal.

<u>Proof</u>. The result follows from Theorem 6.1.3 and Theorem 5.3.5.

Of course, we have the following theorem.

<u>Theorem 6.1.5</u>. Every quasi-pseudo-metric space is first countable.

Proof. The proof is obvious.

In metric spaces, separability, second countability, and the Lindelöf property are equivalent. No analagous result can be obtained for quasi-metric spaces. We observed in Chapter IV that every second countable space $(X,\mathcal{T},\mathcal{U})$ is semi-Lindelöf. Of course, every second countable space is separable. The quasi-metric space of Example 2.1.3 (cf. the remark following Theorem 6.1.2) is separable, but it is neither second countable nor semi-Lindelöf. We conjecture that other examples can be given to show that the other implications need not hold.

We do obtain an interesting result on semi-paracompactness.

Theorem 6.1.6. If (X,τ,μ) is a quasi-pseudo-metric space, and $\mathcal{T}$ and $\mathcal{U}$ are the topologies determined by τ and μ, respectively, then $(X,\mathcal{T}\vee\mathcal{U})$ is pseudo-metrizable.

Proof. Define $d : X \times X \to R$ by $d(x,y) = \max\{\tau(x,y), \mu(x,y)\}$. It is easy to verify that d is a pseudo-metric. Let $\mathfrak{D}$ be the topology determined by d. We shall show that $\mathfrak{D} = \mathcal{T} \vee \mathcal{U}$.

Let $S_d(x,\varepsilon)$ be a basic open set in $\mathfrak{D}$. If $y \in S_d(x,\varepsilon)$, then $\max\{\tau(x,y), \tau(y,x)\} < \varepsilon$. Let $\varepsilon' = \min\{\varepsilon - \tau(x,y), \varepsilon - \tau(y,x)\}$, and consider $S_\tau(y,\varepsilon') \cap S_\mu(y,\varepsilon')$. If $z \in S_\tau(y,\varepsilon') \cap S_\mu(y,\varepsilon')$, then $\tau(x,z) \le \tau(x,y) + \tau(y,z) < \tau(x,y) + \varepsilon' \le \tau(x,y) + \varepsilon - \tau(x,y) = \varepsilon$, and by a similar argument, $\mu(x,z) = \tau(z,x) < \varepsilon$. Hence, $S_\tau(y,\varepsilon') \cap S_\mu(y,\varepsilon') \subset S_d(x,\varepsilon)$. We conclude that $\mathfrak{D} \subset \mathcal{P} \vee \mathcal{U}$.

Conversely, if B is a basic $\mathcal{P} \vee \mathcal{U}$-open set, then $B = T \cap U$, where T is $\mathcal{T}$-open, and U is $\mathcal{U}$-open. Let $x \in B = T \cap U$. Then there is a basic $\mathcal{T}$-open set $S_\tau(y,\varepsilon_1)$ such that $x \in S_\tau(y,\varepsilon_1) \subset T$; and there is a basic

$\mathcal{U}$-open set $S_\mu(y,\varepsilon_2)$ such that $x \in S_\mu(y,\varepsilon_2) \subset U$. Let $\varepsilon = \min\{\varepsilon_1 - \tau(x_1,x), \varepsilon_2 - \mu(x_2,x)\}$. We assert that $x \in S_d(x,\varepsilon) \subset B$. If $z \in S_d(x,\varepsilon)$, then $d(x,z) < \varepsilon$. Hence, $\tau(x_1,z) \leq \tau(x_1,x) + \tau(x,z) \leq \tau(x_1,x) + d(x,z) < \tau(x_1,x) + \varepsilon \leq \tau(x_1,x) + \varepsilon_1 - \tau(x_1,x) = \varepsilon_1$, which implies that $z \in T$. Thus $S_d(x,\varepsilon) \subset T$. A similar argument shows that $S_d(x,\varepsilon) \subset U$. Therefore, $x \in S_d(x,\varepsilon) \subset B$, and so $B \in \mathcal{D}$. We then have that $\mathcal{T} \vee \mathcal{U} \subset \mathcal{D}$.

Corollary 6.1.7. Every quasi-pseudo-metrizable space is semi-paracompact.

Proof. If $(X,\mathcal{T},\mathcal{U})$ is quasi-pseudo-metrizable, then $(X,\mathcal{T} \vee \mathcal{U})$ is pseudo-metrizable by Theorem 6.1.6. Every pseudo-metrizable space is paracompact (Corollary 5.35 of [6]).

The converse of Theorem 6.1.6 is not true, as the following example shows.

Example 6.1.8. Let $X = R$. Let $\mathcal{T}$ be the topology for R whose base is $\{\{0\}\} \cup \{T : X \sim T \text{ is finite}\}$. Let $\mathcal{U}$ be the topology for R whose base is $\{\{x\} : x \neq 0\} \cup \{U : 0 \in U \text{ and } X \sim U \text{ is finite}\}$.

The space $(X,\mathcal{T} \vee \mathcal{U})$ is metrizable since $(X,\mathcal{T} \vee \mathcal{U})$ is discrete. However, neither $(X,\mathcal{T})$ nor $(X,\mathcal{U})$ is first countable. Therefore, by Theorem 6.1.5, $(X,\mathcal{T},\mathcal{U})$ is not quasi-pseudo-metrizable.

Finally, the pseudo-metrizability of the upper bound topology yields an easy theorem about compactness.

Theorem 6.1.9. A quasi-metric space is semi-compact if and only if it is semi-countably compact.

Proof. If (X,τ,μ) is a quasi-metric space, and d is the pseudo-metric defined in the proof of Theorem 6.1.6, then it is immediate that d is a metric on X. In a metric space, compactness and countable compactness are equivalent properties.

2. Quasi-pseudo-metrization Theorems.

There are only a few quasi-pseudo-metrization theorems known. The first, proved by Kelly ([7]), is an analogue of a metrization theorem due to Urysohn.

Theorem 6.2.1. (Kelly) Every pairwise-regular, second countable bitopological space $(X,\mathcal{T},\mathcal{U})$ is quasi-pseudo-metrizable. If, in addition, $(X,\mathcal{T},\mathcal{U})$ is pairwise-regular, then $(X,\mathcal{T},\mathcal{U})$ is quasi-metrizable.

Proof. This is Theorem 2.8 of [7]. In the proof, Kelly uses the property of Theorem 5.2.11.

In the Urysohn-Tychonoff Embedding Theorem (Theorem 2.5.2 of [5]) it is shown that every second countable, regular T_1-space is homeomorphic to a subspace of the Hilbert cube. Kelly's proof uses some of the same techniques as the classical proof, but the definition of the quasi-pseudo-metric is, predictably, more complex.

Kelly also includes a lesser pseudo-metrization theorem.

Lemma 6.2.2. (Kelly) If $(X,\mathcal{T},\mathcal{U})$ is a quasi-pseudo metric space in which $\mathcal{U} \subset \mathcal{T}$, then $\mathcal{T}$ is pseudo-metrizable.

Proof. Since $\mathcal{U} \subset \mathcal{T}$, we have $\mathcal{T} = \mathcal{T} \vee \mathcal{U}$. By Theorem 6.1.6, $\mathcal{T} = \mathcal{T} \vee \mathcal{U}$ is pseudo-metrizable.

Theorem 6.2.3. (Kelly) Let (X,τ,μ) be a quasi-pseudo-metric space, and let $\mathcal{T}$ and $\mathcal{U}$ be the topologies determined by τ and μ, respectively.

If, for each x in X, the function $\tau(x,y)$ is $\mathcal{T}$-continuous in y, then $\mathcal{T}$ is regular. If, for each x in X the function $\tau(x,y)$ is $\mathcal{U}$-continuous in y, then $\mathcal{U}$ is pseudo-metrizable.

Proof. To prove the first assertion, we observe that the $\mathcal{T}$-continuity of $\tau(x,y)$ implies that $\{y : \tau(x,y) \le k\}$ is a $\mathcal{T}$-closed set for each $k > 0$.

If $\tau(x,y)$ is $\mathcal{U}$-continuous, then each set $S_\tau(x,\varepsilon)$ is $\mathcal{U}$-open. Hence, $\mathcal{T} \subset \mathcal{U}$. By Lemma 6.2.2, $\mathcal{U}$ is pseudo-metrizable.

Fletcher ([4]) obtained a necessary and sufficient condition for a bitopological space to be quasi-metrizable. A space $(X,\mathcal{T},\mathcal{U})$ is s-pairwise-T_o if for $x,y \in X$, either there is a $\mathcal{T}$-open set containing x but not y, or there is a $\mathcal{U}$-open set containing y but not x.

Theorem 6.2.4. (Fletcher) If $(X,\mathcal{T},\mathcal{U})$ is quasi-metrizable, then $(X,\mathcal{T},\mathcal{U})$ is s-pairwise-T_o and pairwise-uniform and is generated by a quasi-uniformity with a countable base. If $(X,\mathcal{T}(\mathcal{C}),\mathcal{T}(\mathcal{C}^{-1}))$ is a pairwise-uniform, s-pairwise-T_o space, and $\mathcal{C}$ has a countable base, then $(X,\mathcal{T}(\mathcal{C}),\mathcal{T}(\mathcal{C}^{-1}))$ is quasi-metrizable.

Proof. These results are Theorem 7 and Theorem 8 of [4]. The first is not too hard to prove, but the second requires much work.

Theorem 6.2.1 is a very desirable metrization theorem. We should be satisfied if we had a generalization of the Nagata-Smirnov Metrization Theorem (Theorem 2.5.7 of [5]). A possible generalization would be this: "We shall say that $(X,\mathcal{T},\mathcal{U})$ has a σ- locally finite base if both $(X,\mathcal{T})$ and $(X,\mathcal{U})$ have such bases. If $(X,\mathcal{T},\mathcal{U})$ is a pairwise-regular space with a σ- locally finite base, then $(X,\mathcal{T},\mathcal{U})$ is quasi-pseudo-metrizable."

Patty ([10]) gave a counterexample to disprove this conjecture.

Example 6.2.5. (Patty) Let $X = R$. Let $\mathcal{T}$ be the usual topology on X, and let $\mathcal{U}$ be the discrete topology on X.

The spaces $(X,\mathcal{T})$, $(X,\mathcal{U})$, and $(X,\mathcal{T} \vee \mathcal{U})$ have σ- locally finite bases, and $(X,\mathcal{T},\mathcal{U})$ is pairwise-regular. However, $(X,\mathcal{T},\mathcal{U})$ is not quasi-pseudo-metrizable. The set of rational numbers, which is $\mathcal{U}$-closed, is not the countable intersection of $\mathcal{T}$-open sets, and so our assertion follows from Theorem 6.1.3.

A generalization of the Nagata-Smirnov Theorem was proved by Lane. The proof is very hard.

Theorem 6.2.6. (Lane) Suppose that the space $(X,\mathcal{T},\mathcal{U})$ is pairwise-regular. If there is a sequence

$$\gamma_n = \{\Gamma_{n\alpha}\}_\alpha \quad (n = 1,2,\ldots)$$

of $\mathcal{T}$- and $\mathcal{U}$-locally finite $\mathcal{U}$-open families such that $\gamma = \bigcup_{n=1}^{\infty} \gamma_n$ is a basis for $\mathcal{U}$, and there is a sequence

$$\tau_n = \{T_{n\beta}\}_\beta \quad (n = 1,2,\ldots)$$

of $\mathcal{T}$- and $\mathcal{U}$-locally finite $\mathcal{T}$-open families such that $\tau = \bigcup_{n=1}^{\infty} \tau_n$ is a basis for $\mathcal{T}$, then $(X,\mathcal{T},\mathcal{U})$ is quasi-pseudo-metrizable. If, in addition, X is pairwise-Hausdorff, then $(X,\mathcal{T},\mathcal{U})$ is quasi-metrizable.

Proof. This is Theorem 3.1 of [8].

It is not known whether the condition of Theorem 6.2.6 is necessary.

BIBLIOGRAPHY

[1] Cullen, Helen F., _Introduction to General Topology_, D. C. Heath and Co., Boston, 1967.

[2] Datta, M. C., "Projective bitopological spaces," submitted for publication.

[3] Fletcher, Peter, "Pairwise uniform spaces," _Notices Amer. Math. Soc._ 83 (1965), 612 (abstract).

[4] ______________, "Pairwise uniform spaces," Ph. D. dissertation, Univ. of North Carolina, 1966.

[5] Greever, John, _Theory and Examples of Point-Set Topology_, Brooks/Cole Publ. Co., Belmont, Calif., 1967.

[6] Kelley, John L., _General Topology_, D. Van Nostrand Co., Princeton, N. J., 1955.

[7] Kelly, J. C., "Bitopological spaces," _Proc. London Math. Soc._ (3) 13 (1963), 71-89.

[8] Lane, E. P., "Bitopological spaces and quasi-uniform spaces," _Proc. London Math. Soc._ (3) 17 (1967) 241-256.

[9] Murdeshwar, M. G. and Naimpally, S. A., _Quasi-uniform Topological Spaces_, Noordhoff, 1966.

[10] Patty, C. W., "Bitopological spaces," _Duke Math. Jour._ 34 (1967), 387-392.

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BITOPOLOGICAL SPACES

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Abstract

A bitopological space $(X,\mathcal{T},\mathcal{U})$ is a set X with two topologies. The study of bitopological spaces was initiated by J. C. Kelly.

In this thesis, we study pairwise-separation axioms as defined by J. C. Kelly, C. W. Patty, and E. P. Lane. In addition, definitions for semi-compactness, semi-paracompactness, and bicontinuous functions are proposed and are related to the definitions of pairwise-separated spaces.

Finally, quasi-pseudo-metric spaces are defined, and a number of quasi-pseudo-metrization theorems are summarized.