RESPONSE OF PERIODIC STRUCTURES

by

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CHAPTER I

RESPONSE OF PERIODIC STRUCTURES BY THE METHOD OF Z-TRANSFORMS

1. INTRODUCTION

A periodic structure is a structure consisting of identical substructures connected to each other in identical manner. Many physical systems in natural state possess properties that are periodic in space. A typical example is the single crystal in which identically arranged atoms form an infinite or semi-infinite lattice. Many other physical systems are built in the form of periodic structures by design, the object being to reduce cost or to save time. For example, a segment of an aircraft fuselage is often made of identical bays connected by identical circumferential frames. Monorail tracks or pontoon bridges can be regarded as examples of periodic structures. Continuous systems with periodic properties can often be treated as periodic systems as will be shown in a later chapter.

Periodic structures are called one, two or three dimensional, if the connections are made along one, two or three spatial directions. A building with identical apartments is an example of a three dimensional periodic structure. It should be noted that, although the complete system may be one dimensional, the substructure can be two or three dimensional.

There is no single approach to the mathematical analysis of periodic structures. Indeed, the approach depends very much on the nature of the substructure. In particular, a scalar approach is suitable for the case in which the substructure represents a single-degree-of-freedom system, whereas the matrix approach is recommended for the case in which the substructure represents a multi-degree-of-freedom system. The scalar analysis of periodic structures is in a relatively advanced stage of development. Much progress has been made in the matrix analysis, but many questions remain still to be answered. In the following we present a survey of selected works on the subject.

A classical example of a simple periodic structure is the simple harmonic oscillator. Using such a mathematical model, Brillouin[1] has studied the propagation of harmonic waves in discrete crystal lattices. The same problem is discussed in the text by Morse and Ingard[2], who present solutions to both sinusoidal wave motion and transient motion in infinite arrays, where in the latter case the dispersive character of the array for wave motion is pointed out. In a related work, Weinstock[3] derives the response of a semi-infinite lattice to an excitation in the form of a uniform velocity imparted to the end particle. He proposes a method whereby the infinite set of ordinary differential equations is replaced by a single differential equation. Addressing himself to the same problem as that of Ref. 3, Goodman[4] presents a solution in a slightly more general form, obtained by a procedure he refers to as the "response function method". In an attempt to gain some insight into the behavior of elastic and viscoelastic composite materials, Nayfeh[5] investigates the same mathematical model as that of Refs. 1-4, but in addition he considers viscous damping both in series and in parallel with the springs. Exact and approximate solutions are obtained by integral transform techniques.

References 1-5 are concerned with periodic structures whose basic unit is a single-degree-of-freedom system. They all adopt essentially a scalar approach, where the dependent variable is the displacement. The same problem can be formulated in terms of a two dimensional "spatial state vector" whose components are the displacement and the force at every mass point. If the time is eliminated from the formulation, either by assuming harmonic motion or by means of the Laplace transform, then the original differential equations reduce to difference equations. As a result, the relation between the state at one particle and the state at an adjacent particle can be written in terms of a 2 x 2 transfer matrix. This gives rise to a formulation that has come to be known as "matrix difference equation". Lin and McDaniel[6] investigate the matrix difference equation associated with a periodic Bernoulli-Euler beam on many elastic supports, where the beam is of finite length. They point out various numerical difficulties which may be encountered in using a conventional transfer matrix approach involving the multiplication of a chain of transfer matrices, and propose a "complementary approach" to circumvent these numerical difficulties. A generalization of the matrix difference technique is provided by Mead[7], who considers the problem of harmonic wave propagation in arbitrary one dimensional periodic arrays with multiple coupling. He shows that the frequencies at which wave motion can occur for any given propagation constant can be obtained by solving an eigenvalue problem. An approach similar to that of Ref. 7 but more efficient from a computational point of view, is presented by Denke, Eide and Pickard[8]. They properly identify the objective of an efficient analysis of a periodic structure, namely to render the computing effort independent of the number of substructures. Consistent with this idea, they give the solution of the difference equations in terms of the eigenvalues and eigenvectors of a matrix derived from the substructure mechanical impedance matrix.

Although concerned with the same type of mathematical models as those of Refs. 7 and 8 , the approach to the mathematical solution proposed here is distinctly different. The approach can accommodate not only end excitations but also external forces at the boundaries interconnecting the various substructures. Moreover, the method is not restricted to infinite or semi-infinite periodic structures. Indeed, it can handle finite periodic structures as well.

As a consequence of using the Z-transform method to solve the matrix difference equation, it is only necessary to obtain the eigenvalues of the transfer matrix for a typical substructure. There is no need to calculate the corresponding eigenvectors, as is demanded by the method proposed in Ref. 8. It should be pointed out that eigenvalues can be obtained with significantly higher accuracy than eigenvectors, especially when the matrix is a general complex matrix. Finally, the procedure is very well suited for further investigation of periodic structures. This will be amply demonstrated in the subsequent chapters of this dissertation.

2. DERIVATION OF THE MATRIX DIFFERENCE EQUATION

Let us consider a discrete system consisting of a set of subsystems i, and denote by $\underset{\sim}{u}_{i,L}$ and $\underset{\sim}{u}_{i,R}$ the displacement vectors on the left and on the right side of subsystem i, as shown in Fig. 1.1. Similarly, $\underset{\sim}{p}_{i,L} + \underset{\sim}{f}_{i,L}$ and $\underset{\sim}{p}_{i,R} + \underset{\sim}{f}_{i,R}$ are force vectors, where $\underset{\sim}{p}$ denotes internal forces and $\underset{\sim}{f}$ external forces.

$$\underset{\sim}{u}_i = \begin{bmatrix} \underset{\sim}{u}_{i,L} \\ \hline \underset{\sim}{u}_{i,R} \end{bmatrix} , \qquad \underset{\sim}{p}_i = \begin{bmatrix} \underset{\sim}{p}_{i,L} \\ \hline \underset{\sim}{p}_{i,R} \end{bmatrix} , \qquad \underset{\sim}{f}_i = \begin{bmatrix} \underset{\sim}{f}_{i,L} \\ \hline \underset{\sim}{f}_{i,R} \end{bmatrix} \tag{1.1}$$

The equations of motion for the subsystem i can be written in the matrix form

$$M_i \ddot{\underset{\sim}{u}}_i + C_i \dot{\underset{\sim}{u}}_i + K_i \underset{\sim}{u}_i = \underset{\sim}{p}_i + \underset{\sim}{f}_i \tag{1.2}$$

where

$$M_i = \left[\begin{array}{c|c} m_{i,LL} & m_{i,LR} \\ \hline m_{i,RL} & m_{i,RR} \end{array}\right] , \qquad K_i = \left[\begin{array}{c|c} k_{i,LL} & k_{i,LR} \\ \hline k_{i,RL} & k_{i,RR} \end{array}\right]$$

$$C_i = \left[\begin{array}{c|c} c_{i,LL} & c_{i,LR} \\ \hline c_{i,RL} & c_{i,RR} \end{array}\right] \tag{1.3}$$

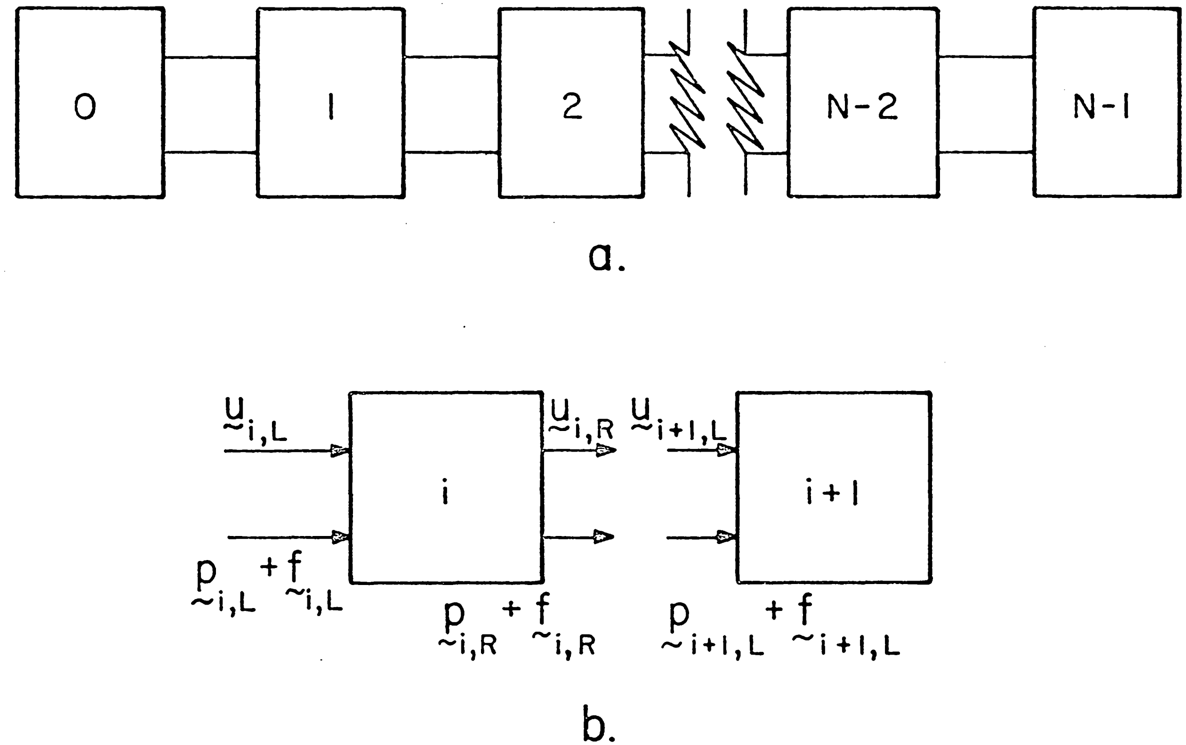


FIGURE 1.1 a. The Periodic Structure

b. The Substructure

are the mass matrix, the stiffness matrix and the damping matrix, respectively.

Next, let us consider the case in which the forces and the displacements are harmonic, oscillating with frequency ω . Assuming that the phase angles are included in the amplitudes and eliminating the time dependence, Eq.(1.2) can be written in the compact form

$$Z_i \underset{\sim}{u}_i = \underset{\sim}{p}_i + \underset{\sim}{f}_i \tag{1.4}$$

where

$$Z_i = -\omega^2 M_i + j\omega C_i + K_i \quad , \qquad j = \sqrt{-1} \tag{1.5}$$

is the impedance matrix, and $\underset{\sim}{u}_i$, $\underset{\sim}{p}_i$ and $\underset{\sim}{f}_i$ represent complex amplitude vectors.

Equation (1.4) describes the motion of a single subsystem. Next, we wish to obtain an equation relating the motion of subsystem i to that of subsystem $i + 1$. To this end, let us write Eq.(1.4) in the more explicit form

$$\begin{aligned} z_{i,LL} \underset{\sim}{u}_{i,L} + z_{i,LR} \underset{\sim}{u}_{i,R} &= \underset{\sim}{p}_{i,L} + \underset{\sim}{f}_{i,L} \\ z_{i,RL} \underset{\sim}{u}_{i,L} + z_{i,RR} \underset{\sim}{u}_{i,R} &= \underset{\sim}{p}_{i,R} + \underset{\sim}{f}_{i,R} \end{aligned} \tag{1.6}$$

where $z_{i,LL}, \ldots$ are corresponding submatrices of Z_i . But the displacements and the forces internal to the system are related by

$$\underset{\sim}{u}_{i,R} = \underset{\sim}{u}_{i+1,L} \quad , \quad \underset{\sim}{p}_{i,R} = - \underset{\sim}{p}_{i+1,L} \tag{1.7}$$

so, that Eqs.(1.6) can be rewritten as

$$\begin{aligned} z_{i,LR}\, \underset{\sim}{u}_{i+1,L} &= - z_{i,LL}\, \underset{\sim}{u}_{i,L} + \underset{\sim}{p}_{i,L} + \underset{\sim}{f}_{i,L} \\ z_{i,RR}\, \underset{\sim}{u}_{i+1,L} + \underset{\sim}{p}_{i+1,L} &= - z_{i,RL}\, \underset{\sim}{u}_{i,L} + \underset{\sim}{f}_{i,R} \end{aligned} \tag{1.8}$$

Next, it will prove convenient to introduce a "spatial state vector" defined as

$$\underset{\sim}{x}_{i,L} = \left[\begin{array}{c} \underset{\sim}{u}_{i,L} \\ \hline \underset{\sim}{p}_{i,L} + \underset{\sim}{f}_{i,L} \end{array}\right] \tag{1.9}$$

which permits us to write Eqs.(1.8) in the form

$$\left[\begin{array}{c|c} z_{i,LR} & 0 \\ \hline z_{i,RR} & 1 \end{array}\right] \underset{\sim}{x}_{i+1,L} = \left[\begin{array}{c|c} - z_{i,LL} & 1 \\ \hline - z_{i,RL} & 0 \end{array}\right] \underset{\sim}{x}_{i,L} + \left[\begin{array}{c} \underset{\sim}{0} \\ \hline \underset{\sim}{f}_{i,R} + \underset{\sim}{f}_{i+1,L} \end{array}\right] \tag{1.10}$$

where 0 and 1 denote the null matrix and the unit matrix, respectively and $\underset{\sim}{0}$ denotes the null vector. Equation (1.10) represents the <u>matrix difference equation</u> which can be written in the more convenient form

$$\underset{\sim}{x}_{i+1,L} = A_i \underset{\sim}{x}_{i,L} + \underset{\sim}{f}_i^* \tag{1.11}$$

where

$$A_i = \left[\begin{array}{c|c} z_{i,LR} & 0 \\ \hline z_{i,RR} & 1 \end{array}\right]^{-1} \left[\begin{array}{c|c} -z_{i,LL} & 1 \\ \hline -z_{i,RL} & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} -z_{i,LR}^{-1} z_{i,LL} & z_{i,LR}^{-1} \\ \hline z_{i,RR} z_{i,LR}^{-1} z_{i,LL} - z_{i,RL} & -z_{i,RR} z_{i,LR}^{-1} \end{array}\right] \tag{1.12}$$

Moreover,

$$\underset{\sim}{f}_i^* = \left[\begin{array}{c} \underset{\sim}{0} \\ \hline \underset{\sim}{f}_{i,R} + \underset{\sim}{f}_{i+1,L} \end{array}\right] \tag{1.13}$$

is recognized as the net external force acting between the subsystems i and $i + 1$. Note that the matrix A_i is commonly known as the transfer matrix.

Matrix A_i depends in general on the index i . However, if the system is periodic in space, i.e. if the subsytems are identical, then $A_i = A =$ constant and a more formal solution of Eq.(1.11) is possible.

3. THE Z - TRANSFORM

The matrix difference equation (1.11) may be solved by a variety of techniques, one of which is the Z- transform method. By using the Z- transform, the solutions to such difference equations become essentially algebraic in nature. Just as the Laplace transformation transforms a function of a continuous variable into an integral of a complex variable, the Z- transformation transforms a real sequence into a complex series.

If the sequence of numbers $l(k)$ is defined for positive integers k only, then the one-sided Z- transform of $l(k)$, is defined as

$$Z\{l(k)\} = l(z) = \sum_{k=0}^{\infty} l(k)\, z^{-k} \tag{1.14}$$

Note the similarity with the one-sided Laplace transform. Indeed, the Z- transform is the discrete counterpart of the Laplace transform. From the definition (1.14) it is possible to derive a theory of the Z- transform completely similar to the theory of the Laplace transform (Ref. 9).

From Eq.(1.14) it follows that the one-sided Z- transform of $\underset{\sim}{x}_{i,L}$ is given by

$$Z\{\underset{\sim}{x}_{i,L}\} = \underset{\sim}{x}_L(z) = \sum_{i=0}^{\infty} \underset{\sim}{x}_{i,L}\, z^{-i} \tag{1.15}$$

Similarly,

$$Z\{ \underset{\sim}{f}_i^* \} = \underset{\sim}{f}^*(z) \tag{1.16}$$

Moreover, it can be easily verified that

$$Z\{ \underset{\sim}{x}_{i+1,L} \} = z\, \underset{\sim}{x}_L(z) - z\, \underset{\sim}{x}_{0,L} \tag{1.17}$$

where $\underset{\sim}{x}_{0,L}$ is the state vector on the left side of the substructure 0 . Taking the Z- transform on both sides of Eq.(1.11) and rearranging, we obtain the transformed response

$$\underset{\sim}{x}_L(z) = z\,(z1 - A)^{-1}\, \underset{\sim}{x}_{0,L} + (\, z1 - A\,)^{-1} \underset{\sim}{f}^*(z) \tag{1.18}$$

The actual response $\underset{\sim}{x}_i$ can be determined from Eq.(1.18), provided the inverse Z- transform of $\underset{\sim}{x}_L(z)$ can be obtained. This in turn, requires that $\underset{\sim}{x}_{0,L}$ and $\underset{\sim}{f}^*(z)$ are known.

Next, let us introduce

$$Z^{-1}\{\, z\,(\, z1 - A\,)^{-1} \} = \Phi_i \tag{1.19}$$

from which it follows that

$$Z^{-1}\{\, (z1 - A)^{-1} \} = \Phi_{i-1} \tag{1.20}$$

Considering Eqs.(1.19) and (1.20) and using the convolution theorem for Z- transforms (Ref. 10), we obtain the response

$$\underset{\sim}{x}_{i,L} = \Phi_i \underset{\sim}{x}_{0,L} + \sum_{k=0}^{i-1} \Phi_{i-k-1} \underset{\sim}{f}_k^* \tag{1.21}$$

Although the upper summation limit in Eq.(1.15) suggests an infinite number of substructures, Eq.(1.21) is equally valid for a finite number of substructures provided we introduce the appropriate boundary conditions.

The matrix Φ_i is known as the fundamental matrix for the system, so that the response problem reduces to the determination of the fundamental matrix. Using Eq.(1.11) recursively by letting i = 0,1,2,... it can be shown that $\Phi_i = A^i$.

As i becomes large, however, accuracy is lost in evaluating A^i , so that a method whose accuracy does not depend on i appears highly desirable. Such a method is described in next section.

4. THE LEVERRIER ALGORITHM

There are several methods to evaluate A^i (Ref. 14) . In the case of periodic structures, the Leverrier algorithm seems to be the most suitable, both from a numerical and a theoretical point of view.

In using Eq.(1.19) to produce the fundamental matrix, it is necessary to compute the inverse of the matrix ($z1$ - A) which is a matrix polynomial in z. The problem becomes progressively more difficult as the order of the subsystem increases, so that an efficient method for obtaining the inverse of ($z1$ - A) for higher order systems is highly desirable. Such a method is the Leverrier algorithm (Refs. 10 and 14) according to which the inverse of the matrix ($z1$ - A) can be written in the form

$$(z1 - A)^{-1} = \frac{1}{\det(z1 - A)} \sum_{l=0}^{2n-1} z^{2n-l-1} H_l \tag{1.22}$$

where

$$\det(z1 - A) = z^{2n} + \theta_1 z^{2n-1} + \theta_2 z^{2n-2} + \dots + \theta_{2n} \tag{1.23}$$

is the characteristic polynomial of the matrix A and

$$\begin{aligned} H_0 &= 1 \\ H_l &= A\,H_{l-1} + \theta_l 1 \quad , \qquad l = 1, 2, \dots , 2n-1 \end{aligned} \tag{1.24}$$

are constant square matrices of order 2n . Moreover, the coefficients θ_l of the characteristic polynomial are given by

$$\theta_l = -\frac{1}{l} \operatorname{tr} A H_{l-1} \quad , \qquad l = 1, 2, \ldots , 2n \tag{1.25}$$

Using Eqs.(1.23) and (1.24) in conjunction with the Cayley-Hamilton theorem, it can be easily verified that

$$H_{2n} = A H_{2n-1} + \theta_{2n} 1 = 0 \tag{1.26}$$

which can be used to obtain an error estimate. Note that n is the number of actual degrees of freedom of the subsystem.

Next, let us introduce the notation

$$X(z) = (z1 - A)^{-1} = \frac{1}{\det(z1 - A)} \sum_{l=0}^{2n-1} z^{2n-l-1} H_l \tag{1.27}$$

and consider a partial fractions expansion of the matrix $X(z)$. The matrix has poles at the zeros of the characteristic polynomial of A . We shall denote these zeros by z_j ($j = 1, 2, \ldots , 2n$), so that, assuming that these zeros are all distinct, the partial fractions expansion of $X(z)$ can be written in the form

$$X(z) = \sum_{j=1}^{2n} \frac{1}{z - z_j} X_j \tag{1.28}$$

where the constant matrices X_j have the expressions

$$X_j = \lim_{z \to z_j} \{ (z-z_j)\, X(z) \} = \frac{\sum_{l=0}^{2n-1} z_j^{2n-l-1} H_l}{\sum_{l=0}^{2n-1} (2n-l)\theta_l z_j^{2n-l-1}} \tag{1.29}$$

where $\theta_0 = 1$ and $j = 1, 2, \ldots, 2n$. Note that the denominator in Eq.(1.29) is the derivative of the polynomial (1.23) with respect to z, taken at $z = z_j$. Hence, using Eq.(1.19), the fundamental matrix becomes

$$\Phi_i = Z^{-1}\{ z\, X(z) \} = Z^{-1} \left(\sum_{j=1}^{2n} \frac{z}{z-z_j} X_j \right) \tag{1.30}$$

But, from the tables of inverse Z- transforms[9],it can be verified that

$$\Phi_i = \sum_{j=1}^{2n} z_j^i X_j \tag{1.31}$$

so that, inserting Eq.(1.31) into Eq.(21), we obtain the general response

$$\underset{\sim}{x}_i = \sum_{j=1}^{2n} X_j \left(z_j^i \underset{\sim}{x}_0 + \sum_{k=0}^{i-1} z_j^{i-k-1} \underset{\sim}{f}_k^* \right) \tag{1.32}$$

where it is understood that all "state vectors" are to be taken at the left side of the substructure.

The above method requires the determination of the roots of the characteristic polynomial (1.23), which are really the eigenvalues of the matrix A. If the order of matrix A is relatively small,we might solve the characteristic equation directly with the aid of the reduction procedure developed in appendix A.If the order of matrix A is relatively high or if the substructure is symmetric,it will prove to be more accurate to calculate the eigenvalues of A directly from A.

By using Leverrier's algorithm it is not necessary to obtain the eigenvectors of the matrix A, as demanded by the method proposed in Ref. 8. Moreover, it should be pointed out that eigenvalues can be obtained with significantly higher accuracy than eigenvectors, so that the method proposed here is likely to be more accurate.

The matrix A is a transfer matrix and it possesses several useful properties to be derived in next section. These properties can be used to reduce the computational effort significantly. It should be pointed out that the matrix A can be constructed in a number of other ways, without making explicit use of the matrices M, C and K. The method chosen here, however, has the advantage that it permits the use of the symmetry of M, C and K to demonstrate various properties of A in a convenient manner.

5. PROPERTIES OF THE TRANSFER MATRIX - THE EIGENVALUE PROBLEM

In this section we shall derive some properties of the transfer matrix A. These will help us to reduce the computational effort and permit us to introduce some general statements about the response problem.

From Eq.(1.12) we conclude that A can be written in the form of the matrix product

$$A = \left[\begin{array}{c|c} z_{LR}^{-1} & 0 \\ \hline -z_{RR}z_{LR}^{-1} & 1 \end{array}\right] \left[\begin{array}{c|c} -z_{LL} & 1 \\ \hline -z_{RL} & 0 \end{array}\right] \tag{1.33}$$

where the index i has been dropped because the submatrices in question are the same for every substructure. But the determinant of a product of matrices is equal to the product of the determinants of the matrices, i.e.,

$$\det A = \det \left[\begin{array}{c|c} z_{LR}^{-1} & 0 \\ \hline -z_{RR}z_{LR}^{-1} & 1 \end{array}\right] \det \left[\begin{array}{c|c} -z_{LL} & 1 \\ \hline -z_{RL} & 0 \end{array}\right] \tag{1.34}$$

or

$$\det A = \det z_{LR}^{-1} \det z_{RL} \tag{1.35}$$

Finally, since $z_{RL} = z_{LR}^{T}$ we obtain

$$\det A = \det z_{LR}^{-1} \det z_{LR}^{T} = 1 \tag{1.36}$$

Similarly,

$$\det A^{-1} = 1 \tag{1.37}$$

Next, let us consider the eigenvalue problem associated with A. Recalling Eq.(1.12), we can write the problem in the form

$$\det(\lambda 1 - A) = \det\left[\lambda 1 - \left[\begin{array}{c|c} z_{LR} & 0 \\ \hline z_{RR} & 1 \end{array}\right]^{-1} \left[\begin{array}{c|c} -z_{LL} & 1 \\ \hline -z_{RL} & 0 \end{array}\right]\right] = 0$$

or

$$\det(\lambda 1 - A) = \det\left[\begin{array}{c|c} \lambda z_{LR} + z_{LL} & -1 \\ \hline \lambda z_{RR} + z_{RL} & \lambda 1 \end{array}\right] = 0 \tag{1.38}$$

Because the value of a determinant does not change if we add one row multiplied by a constant to another row, we can multiply the top half of the determinant by λ and add to the bottom half to obtain

$$\det\left[\begin{array}{c|c} \lambda z_{LR} + z_{LL} & -1 \\ \hline \lambda z_{LR} + \lambda(z_{LL} + z_{RR}) + z_{RL} & 0 \end{array}\right]$$

$$= \det\{ \lambda^2 z_{LR} + \lambda(z_{LL} + z_{RR}) + z_{RL} \} = 0 \tag{1.39}$$

Now, let us consider the eigenvalue problem associated with A^{-1}, and denote the eigenvalues of A^{-1} by $\bar{\lambda}$, and write

$$\det(\bar{\lambda}1 - A^{-1}) = \det \left[\bar{\lambda}1 - \left[\begin{array}{c|c} -z_{LL} & 1 \\ \hline -z_{RL} & 0 \end{array}\right]^{-1} \left[\begin{array}{c|c} z_{LR} & 0 \\ \hline z_{RR} & 1 \end{array}\right] \right]$$

$$= \det \left[\begin{array}{c|c} -\bar{\lambda} z_{LL} - z_{LR} & \bar{\lambda}1 \\ \hline -\bar{\lambda} z_{RL} - z_{RR} & -1 \end{array}\right]$$

$$= \det\{ -\bar{\lambda}^2 z_{RL} - \bar{\lambda} (z_{LL} + z_{RR}) - z_{LR}\} = 0 \tag{1.40}$$

Because $z_{RL} = z_{LR}^T$, $z_{LL} = z_{LL}^T$, $z_{RR} = z_{RR}^T$, we conclude that λ and $\bar{\lambda}$ satisfy the same eigenvalue problem. But $\bar{\lambda} = \frac{1}{\lambda}$, so that the eigenvalues of A (or of A^{-1}) occur in reciprocal pairs λ and $1/\lambda$. This fact has significant implications as far as the computation of the eigenvalues of A is concerned, because we need to calculate only half of the eigenvalues.

Next, let us consider Eq.(1.23). Letting $z = \lambda$, we can write the characteristic equation associated with a typical substructure in the form

$$\lambda^{2n} + \theta_1 \lambda^{2n-1} + \theta_2 \lambda^{2n-2} + \ldots + \theta_{2n} = 0 \qquad (1.41)$$

Dividing Eq.(1.41) through by λ^{2n} , we obtain

$$\theta_{2n} \bar{\lambda}^{2n} + \theta_{2n-1} \bar{\lambda}^{2n-1} + \theta_{2n-2} \bar{\lambda}^{2n-2} + \ldots + 1 = 0 \qquad (1.42)$$

Because λ and $\bar{\lambda}$ satisfy the same characteristic equation, we conclude that

$$\begin{aligned} \theta_{2n} &= 1 \\ \theta_{2n-l} &= \theta_l \quad , \qquad l = 1, 2, \ldots, 2n-1 \end{aligned} \qquad (1.43)$$

From Eqs.(1.43), we conclude that we need to calculate only one half of the coefficients θ_l . Hence, for the case in which the substructure has n actual degrees of freedom, the characteristic equation can be obtained by merely calculating n coefficients θ_l .

Finally, let us consider

$$A^{-1} = \left[\begin{array}{c|c} -z_{LL} & 1 \\ \hline -z_{RL} & 0 \end{array}\right]^{-1} \left[\begin{array}{c|c} z_{LR} & 0 \\ \hline z_{RR} & 1 \end{array}\right]$$

or

$$A^{-1} = \left[\begin{array}{c|c} 0 & -z_{RL}^{-1} \\ \hline 1 & -z_{LL}z_{RL}^{-1} \end{array}\right] \left[\begin{array}{c|c} z_{LR} & 0 \\ \hline z_{RR} & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} -z_{RL}^{-1}\, z_{RR} & -z_{RL}^{-1} \\ \hline z_{LR} - z_{LL}z_{RL}^{-1}z_{RR} & -z_{LL}z_{RL}^{-1} \end{array}\right] \tag{1.44}$$

Next, let us partition matrix A as follows

$$A = \left[\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right] \tag{1.45}$$

where the n x n submatrices a_{11}, ... are given by Eq.(1.12).

Then, comparing Eq.(1.44) and Eq.(1.12) and recalling that $z_{LL}^T = z_{LL}$, $z_{RR}^T = z_{RR}$ and $z_{LR}^T = z_{RL}$ we obtain

$$A^{-1} = \left[\begin{array}{c|c} a_{22}^T & -a_{12}^T \\ \hline -a_{21}^T & a_{11}^T \end{array}\right] \tag{1.46}$$

We conclude that the inversion process of matrix A is a simple procedure, involving no loss of accuracy.

Next, let us introduce the matrix operator T in the form

$$T(B) = \left[\begin{array}{c|c} b_{22}^T & -b_{12}^T \\ \hline -b_{21}^T & b_{11}^T \end{array}\right] \tag{1.47}$$

where b_{11}, ... are submatrices of the matrix B. Clearly, T exhibits the following properties

$$\begin{aligned} &T(a_1 A_1 + a_2 A_2) = a_1 T(A_1) + a_2 T(A_2) \\ &T(A_1 A_2) = T(A_2) T(A_1) \end{aligned} \tag{1.48}$$

From Eq.(1.31) we have

$$A^i = \sum_{j=1}^{2n} X_j z_j^i = \sum_{j=1}^{n} X_j z_j^i + \sum_{j=n+1}^{2n} X_j z_j^i \tag{1.49}$$

Let us arrange the eigenvalues z_j as follows

$$\left.\begin{aligned} &z_j \text{ with } |z_1|<|z_2|< \ldots <|z_n|< 1 \\ &z_{n+j} = \frac{1}{z_j} \end{aligned}\right\} , \ j = 1, 2, \ldots , n \tag{1.50}$$

then, we can write Eq.(1.49) as

$$A^i = \sum_{j=1}^{n} X_j z_j^i + \sum_{j=1}^{n} X_{n+j} z_j^{-i} \tag{1.51}$$

On the other hand, from Eqs.(1.46) and (1.47) we have

$$A^i = T(A^{-i}) = T\left(\sum_{j=1}^{2n} X_j z_j^{-i} \right) = \sum_{j=1}^{2n} T(X_j) z_j^{-i}$$

or

$$A^i = \sum_{j=1}^{n} T(X_j) z_j^{-i} + \sum_{j=1}^{n} T(X_{n+j}) z_j^i \tag{1.52}$$

From Eqs.(1.51) and (1.52) it follows that

$$X_{n+j} = T(X_j) \quad , \quad j = 1, 2, \ldots , n \tag{1.53}$$

Substituting Eq.(1.53) into Eq.(1.49) we obtain

$$A^i = \sum_{j=1}^{n} \{ X_j z_j^i + T(X_j) z_j^{-i} \} \tag{1.54}$$

Equation (1.54) allows us to use Eq.(1.47) to calculate X_{j+n} (j=1,2, ..., n), which reduces the computational effort considerably.

Finally, a similar computational saving can be obtained for the matrices H_l (l=0,1,...,2n-1). Indeed, examining Eq.(1.24) and using simple substitution, it can be verified that

$$H_l = \sum_{k=0}^{l} \theta_{l-k} A^k \tag{1.55}$$

so that Eqs.(1.46) and (1.48) yield

$$T(A H_l) = \sum_{k=0}^{l} \theta_{l-k} A^{-(k+1)} \tag{1.56}$$

On the other hand, from Eqs.(1.26) and (1.24) it can also be shown that

$$H_{2n-(l+1)} = - \sum_{k=0}^{l} \theta_{2n-(l-k)} A^{-(k+1)} \tag{1.57}$$

and, in view of relations (1.43), we obtain

$$H_{2n-(l+1)} = - \sum_{k=0}^{l} \theta_{l-k} A^{-(k+1)} \tag{1.58}$$

Comparing Eqs.(1.56) and (1.58),we finally arrive at

$$H_{2n-(l+1)} = -T(A H_l) \tag{1.59}$$

Hence, it is only necessary to calculate one half of the matrices H_l ($l=0,1,\ldots,2n-1$), as the balance can be obtained by the simple process (1.47).

6. ILLUSTRATIVE EXAMPLES

Let us consider the periodic system in axial vibration shown in Fig. 1.2. The substructures can be identified as single-degree-of-freedom damped systems. It is easy to verify that the mass matrix, damping matrix, and stiffness matrix for a typical substructure have the form

$$M = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = c\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad K = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.60}$$

so that the impedance matrix is simply

$$Z = \begin{bmatrix} -\omega^2 m + j\,\omega c + k & -(j\,\omega c + k) \\ -(j\,\omega c + k) & -\omega^2 m + j\,\omega c + k \end{bmatrix} \tag{1.61}$$

using Eq.(1.12), we can write the transfer matrix

$$A = \begin{bmatrix} \alpha & -1/\beta \\ \beta(1-\alpha^2) & \alpha \end{bmatrix} \tag{1.62}$$

where

$$\alpha = \frac{-\omega^2 m + j\,\omega c + k}{j\,\omega c + k}, \quad \beta = j\omega c + k \tag{1.63}$$

Next, we shall use the Leverrier algorithm to obtain the fundamental

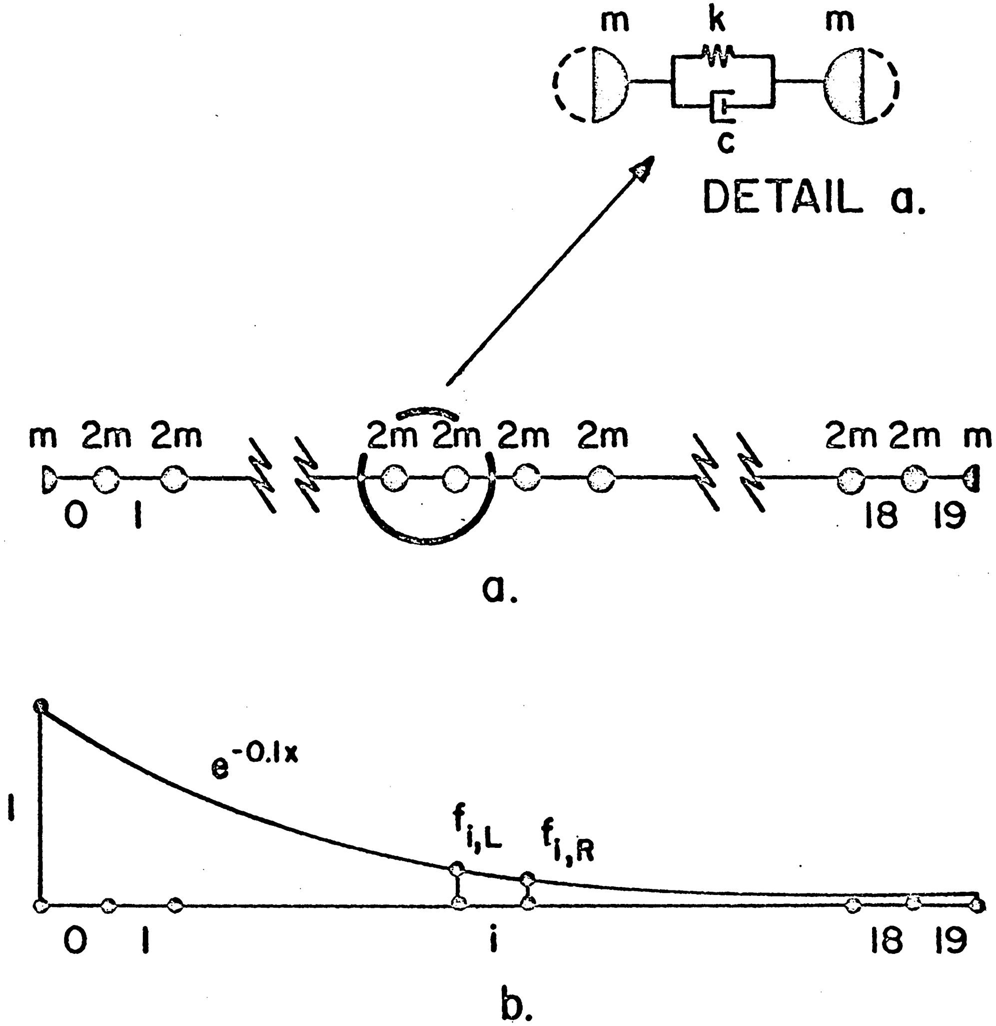


FIGURE 1.2 a. The Periodic Structure

b. The External Force Distribution

matrix Φ_i . To this end, we use Eqs.(1.24) and (1.25) and obtain

$$\theta_1 = -\operatorname{tr} A = -2\alpha$$

$$H_1 = A - 2\alpha 1 = \begin{bmatrix} -\alpha & -1/\beta \\ \beta(1-\alpha^2) & -\alpha \end{bmatrix} \tag{1.64}$$

Hence, Eq.(1.27) has the form

$$X(z) = \frac{1}{z^2 - 2\alpha z + 1} \begin{bmatrix} z-\alpha & -1/\beta \\ \beta(1-\alpha^2) & z-\alpha \end{bmatrix} \tag{1.65}$$

The characteristic equation is simply

$$z^2 - 2\alpha z + 1 = 0 \tag{1.66}$$

having the roots

$$z_1 = \alpha + (\alpha^2 - 1)^{1/2} \quad , \quad z_2 = \alpha - (\alpha^2 - 1)^{1/2} \tag{1.67}$$

and it is easy to verify that $z_1 = 1/z_2$. Using Eq.(1.29), we can write

$$X_1 = \frac{1}{z_1 - z_2} \begin{bmatrix} z_1-\alpha & -1/\beta \\ \beta(1-\alpha^2) & z_1-\alpha \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -\dfrac{1}{\beta(\alpha^2-1)^{1/2}} \\ -\beta(\alpha^2-1)^{1/2} & 1 \end{bmatrix} \tag{1.68a}$$

$$X_2 = \frac{1}{z_2 - z_1}\begin{bmatrix} z_2-\alpha & -1/\beta \\ \beta(1-\alpha^2) & z_2-\alpha \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \frac{1}{\beta(\alpha^2-1)^{1/2}} \\ \beta(\alpha^2-1)^{1/2} & 1 \end{bmatrix} \quad (1.68b)$$

Inserting Eqs.(1.68) into Eq.(1.31), we obtain the fundamental matrix

$$\Phi_i = \frac{1}{2}\begin{bmatrix} z_1^i + z_2^i & -\frac{z_1^i - z_2^i}{\beta(\alpha^2-1)^{1/2}} \\ -\beta(\alpha^2-1)^{1/2}(z_1^i - z_2^i) & z_1^i + z_2^i \end{bmatrix} \quad (1.69)$$

Hence, using Eq.(1.21), we obtain a general solution for the response in the form

$$u_i = \frac{1}{2}(z_1^i + z_2^i)\, u_0 - \frac{z_1^i - z_2^i}{2\beta(\alpha^2-1)^{1/2}}(p_0 + f_0)$$
$$- \frac{1}{2\beta(\alpha^2-1)^{1/2}} \sum_{k=0}^{i-1} (z_1^{i-k-1} - z_2^{i-k-1})\, f_k^*$$
$$(1.70)$$
$$p_i = -\frac{\beta(\alpha^2-1)^{1/2}}{2}(z_1^i - z_2^i)\, u_0 + \frac{1}{2}(z_1^i + z_2^i)(p_0 + f_0)$$
$$+ \frac{1}{2}\sum_{k=0}^{i-1}(z_1^{i-k-1} + z_2^{i-k-1})\, f_k^* - f_{i,L}$$

At this point, it appears desirable to make the example more specific. Hence, let us assume that the system consists of 20 substructures and consider two cases, one in which both ends are free and the other in which the left end is free and the right end is fixed. From Fig. 1.2 we conclude that the external load has the form

$$f_i^* = e^{-0.1(i+1)} \quad , \quad i=0,1,2,\ldots,19 \tag{1.71}$$

Moreover, we choose the following values for the system parameters

$$\omega = m = c = k = 1 \tag{1.72}$$

From Eqs.(1.63), it follows that

$$\alpha = \frac{1}{2}(1 + j) \quad , \quad \beta = 1 + j \tag{1.73}$$

so that, from Eqs.(1.67), we obtain

$$z_1 = 0.74293 + 1.52908\,j \quad , \quad z_2 = 0.25707 - 0.52908\,j \tag{1.74}$$

Considering first the free-free case, we can write

$$p_0 + f_0 = 0 + e^0 = 1 \quad , \quad p_{20} = 0 \tag{1.75}$$

Letting $i = 0$ in the second of Eqs.(1.70), we can solve for u_0 to obtain

$$u_0 = \frac{1}{\beta(\alpha^2-1)^{1/2}(z_1^{20} - z_2^{20})} \{ (z_1^{20} + z_2^{20}) + \sum_{k=0}^{19} (z_1^{19-k} + z_2^{19-k})\, e^{-0.1(k+1)} - 2\, e^{-2} \} \tag{1.76}$$

The system response is obtained by inserting Eqs.(1.73-76) into Eqs.(1.70). The actual evaluation is carried out most conveniently by a digital computer. The results are displayed in Fig. 1.3.

The response for the free-fixed case is also given by Eqs.(1.70), but the end conditions are different. Indeed, in this case

$$p_0 + f_0 = 1 \quad , \quad u_{20} = 0 \tag{1.77}$$

so that, from the first of Eqs.(1.70), we obtain

$$u_0 = \frac{1}{\beta(\alpha^2-1)^{1/2}\,(z_1^{20} + z_2^{20})} \{ z_1^{20} - z_2^{20} + \sum_{k=0}^{19} (z_1^{19-k} - z_2^{19-k})\, e^{-0.1(k+1)} \} \tag{1.78}$$

Hence, using Eqs.(1.70) in conjunction with the end conditions (1.77-78), we obtain the response plotted in Fig. 1.4.

Comparing the results obtained for the two cases, we observe that

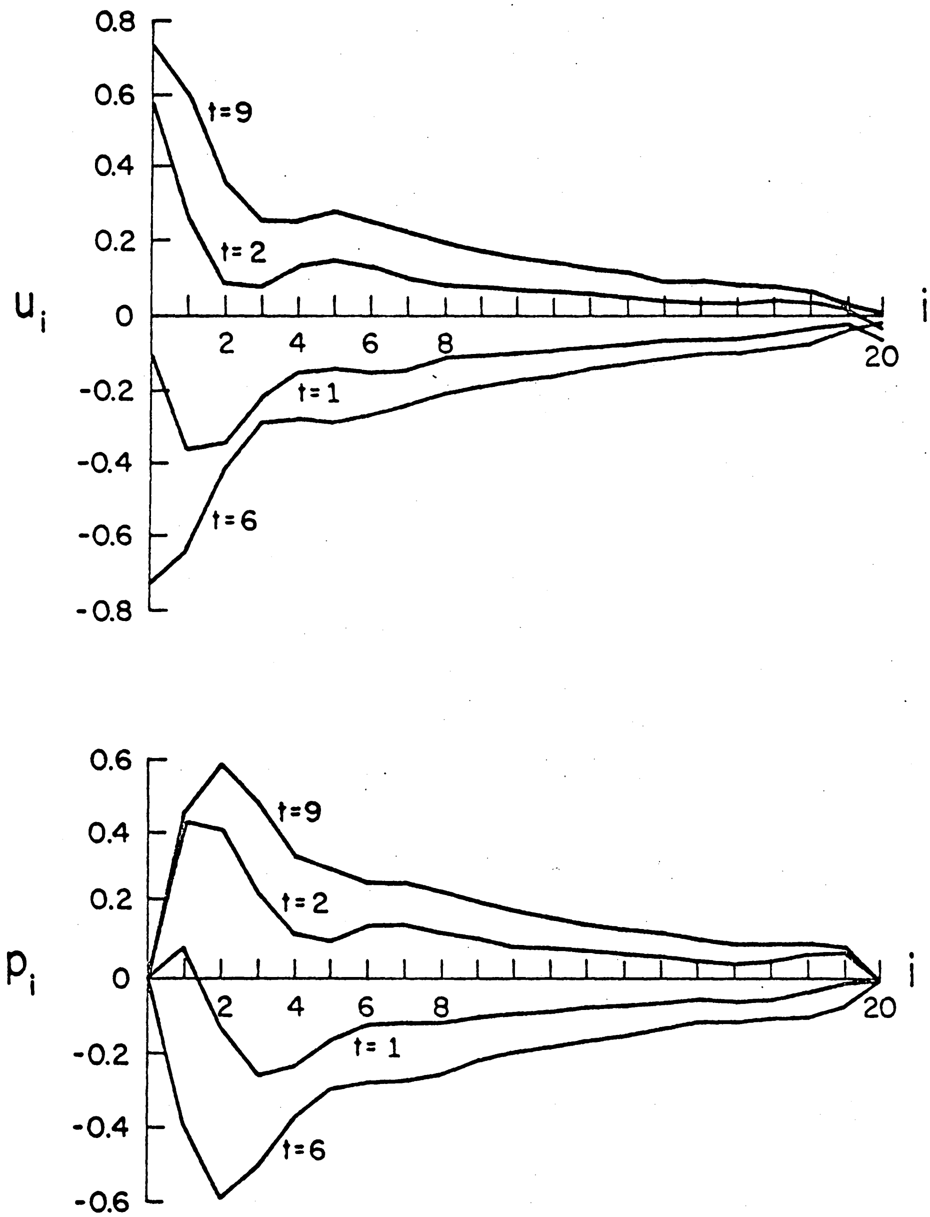


FIGURE 1.3 The Response in the Free-Free Case

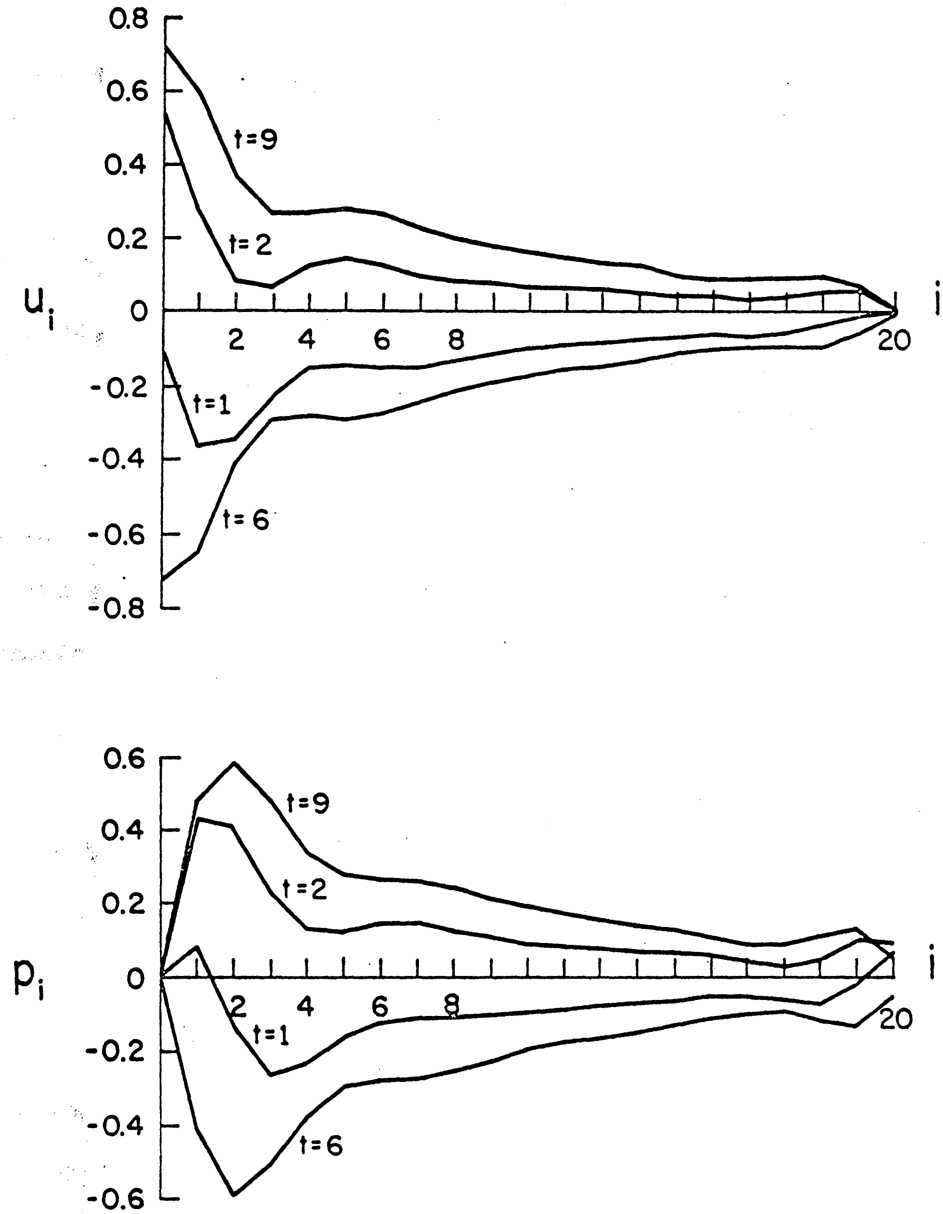


FIGURE 1.4 The Response in the Free-Fixed Case

the response in the left part of the system is not affected very much by the end conditions at the right end, which, of course, can be attributed to the relative large number of substructures together with a considerable amount of damping and the nature of the load. This fact does permit us to draw the conclusion, however, that many damped periodic structures with a large number of substructures can be regarded as being semi-infinite. In this case, significant computational saving can be obtained as will be shown in a later section.

Let us now consider a more realistic example, as given by the periodic truss in Fig. 1.5. The trusses consist of steel pipes with following dimensions for a horizontal pipe

length	=	24 in
weight per foot	=	1.68 lb
cross sectional area	=	0.494 in^2
Young's modulus	=	30×10^6 psi

Furthermore, we assume the damping matrix C to be proportional to the stiffness matrix K such that the impedance matrix Z in Eq.(1.5) can be written as

$$Z = -\omega^2 M + (1 + j\omega c) K \tag{1.79}$$

where

$$\omega = 100 \text{ rad/sec} \quad , \quad c = 0.01 \tag{1.80}$$

are the chosen values for the load frequency and damping factor, respectively. The mass matrix M and the stiffness matrix K are derived by the usual finite-element procedures. The symmetric matrix M can be represented as follows

$$M = (m_{ij}) \quad , \quad i,j = 1,2,\ldots,2n \tag{1.81}$$

with

$$m_{ij} = m_{ji} \quad , \quad i,j = 1,2,\ldots,2n$$

In the present example we have

$$m_{11} = m_{22} = m_{55} = m_{66} = -m_{13} = m_{24} = m_{57} = -m_{68} = 0.024595$$
$$m_{16} = m_{25} = m_{27} = m_{38} = -m_{18} = m_{45} = m_{47} = -m_{36} = 0.012297$$
$$m_{33} = m_{44} = m_{77} = m_{88} = 0.041986$$
$$m_{37} = m_{48} = 0.017391$$
$$m_{12} = m_{14} = m_{15} = m_{17} = m_{23} = m_{26} = m_{28} = m_{34} = m_{35}$$
$$= m_{46} = m_{56} = m_{58} = m_{67} = m_{78} = 0$$

Similarly, the symmetric matrix K can be written as

$$K = (k_{ij}) \quad , \quad i,j = 1,2,\ldots,2n \tag{1.82}$$

with

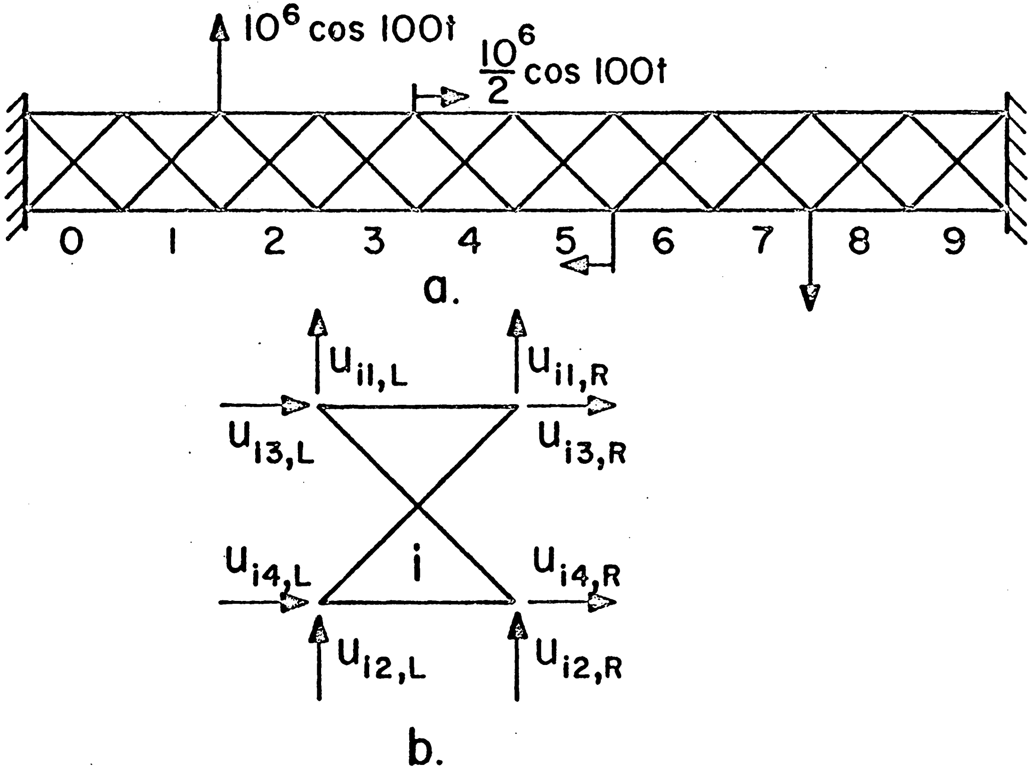


FIGURE 1.5 a. The Periodic Truss

b. The Substructure

$$k_{ij} = k_{ji} \quad , \quad i,j = 1,2,\ldots,2n$$

In the present example we obtain

$$\begin{aligned}
&k_{11} = k_{22} = k_{55} = k_{66} = k_{18} = - k_{13} = - k_{16} = k_{24} = k_{36} = \\
&k_{57} = - k_{25} = - k_{27} = - k_{38} = - k_{45} = - k_{47} = - k_{68} = 0.21832 \times 10^6 \\
&k_{33} = k_{44} = k_{77} = k_{88} = 0.83582 \times 10^6 \\
&k_{37} = k_{48} = - 0.61750 \times 10^6 \\
&k_{12} = k_{14} = k_{15} = k_{17} = k_{23} = k_{26} = k_{28} = k_{34} \\
&= k_{35} = k_{46} = k_{56} = k_{58} = k_{67} = k_{78} = 0
\end{aligned}$$

Note that in the present case, the substructure has four actual degrees of freedom, i.e. $2n = 8$.

Next, the matrices Z and A are computed, using a double precision Fortran computer program. Note that, due to the damping, A is a general complex matrix.

We now apply the Leverrier algorithm which yields the matrices H_l ($l = 1,2,\ldots,2n-1$) and the coefficients θ_l ($l = 0,1,\ldots,2n-1$). The property Eq.(1.59) can easily be incorporated into the computer program. The values for θ_l ($l=0,1,\ldots,8$) are

$$\begin{aligned}
&\theta_0 = \theta_8 = 1 \quad , \quad \theta_1 = \theta_7 = - 3.99887 - j\, 0.00113 \\
&\theta_2 = \theta_6 = 4.00451 - j\, 0.00450 \ , \ \theta_3 = \theta_7 = 3.96228 + j\, 0.03764 \qquad (1.83) \\
&\qquad\qquad \theta_4 = - 9.93582 - j\, 0.06401
\end{aligned}$$

Next, we solve for the eigenvalues of matrix A either directly or from the reduced characteristic equation

$$
\begin{aligned}
z_1 &= \frac{1}{z_5} = -\,0.97712 + j\,0.05139 \\
z_2 &= \frac{1}{z_6} = 0.92480 - j\,0.22601 \\
z_3 &= \frac{1}{z_7} = 0.79118 + j\,0.03551 \\
z_4 &= \frac{1}{z_8} = 0.98392 - j\,0.03685
\end{aligned}
\qquad (1.84)
$$

We are now in a position to compute the matrices $X_j (j=1,2,...,2n)$ where we make use of Eq.(1.53).

Finally, the unknown end vector $\underset{\sim}{x}_0$ is computed taking into account that in the present problem both the left and the right boundaries are fixed. The results are displayed in Fig. 1.6.

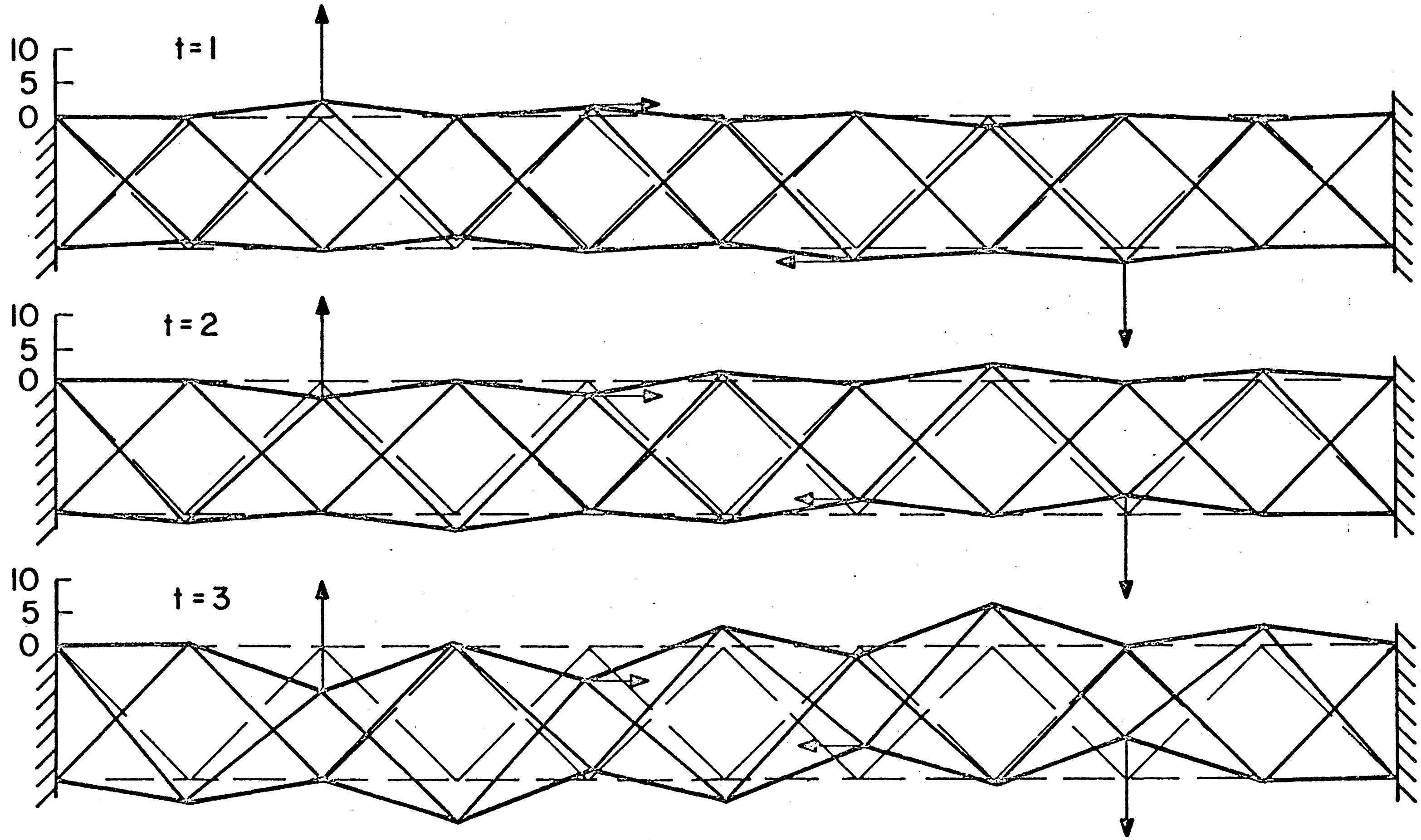


FIGURE 1.6   The Response

7. SUMMARY AND CONCLUSIONS

This chapter presents an efficient method to obtain the response of a periodic structure. A "state vector" $\tilde{x}_i$ is defined, containing both the displacement vector and the internal force vector at the left end of each station i. Equation (1.32) then yields $\tilde{x}_i$ in terms of the transition matrix Φ_i and the left-end state vector $\tilde{x}_0$. The solution Eq.(1.32) is very general in nature. It allows for arbitrary substructures, not necessarily symmetric. Also, it can accommodate harmonic loads at every station and different types of boundary conditions at both ends of the structure. Finally, the procedure is capable of handling damped periodic structures as well.

To obtain the response, it is necessary to compute $\Phi_i = A^i$. This is done by the Leverrier algorithm, which renders the evaluation of A^i independent of i. Therefore, the computation effort to analyse the entire structure reduces to that of a single substructure. Clearly, the time consuming part in this algorithm is the computation of the matrices H_l $(l=0,1,\ldots,2n-1)$ from Eqs.(1.24). However, only half of these matrices need to be computed by using relation (1.59). A similar saving is obtained for the matrices X_j $(j=1,2,\ldots,2n)$ by using Eq.(1.53). Also, it should be pointed out that the Leverrier algorithm does not require the eigenvectors of matrix A, at least not explicitly. In general, A represents a general complex matrix and its eigenvalues can be obtained with significantly higher accuracy than its eigenvectors. The eigenvalues are shown to be reciprocal and can be obtained from matrix A directly or from the reduced characteristic equation (A21) as derived in appendix A.

In principle, the solution Eq.(1.32) is valid for any number of substructures. However, due to the limited precision, inherent to any digital computer, the results become necessarily inaccurate when the number of substructures becomes too large. For the truss problem in the previous section, we can expect accurate results for trusses consisting of as many as a hundred substructures. Although the procedure is equally valid for a small number of subsystems, its main purpose is to investigate relatively large structures. In the next chapter we shall develop ways to adapt Eq.(1.32) to theoretically infinite or semi-infinite structures. Also, it will be demonstrated that more computational savings can be accomplished when the substructure is symmetric with respect to a central axis.

8. APPENDIX A : REDUCTION OF THE ORDER OF THE CHARACTERISTIC EQUATION

Let us write the characteristic polynomial, Eq.(1.41), in the form

$$\sum_{l=0}^{2n} \theta_l \lambda^{2n-l} = 0 \tag{A1}$$

where we recall that $\theta_0 = \theta_{2n} = 1$. Dividing Eq.(1.41) through by λ^n and making use of Eqs.(1.43), we obtain

$$\sum_{l=1}^{n} \theta_{n-l} (\lambda^l + \lambda^{-l}) + \theta_n = 0 \tag{A2}$$

Introducing the notation

$$\lambda_0 = 1 \quad , \quad \lambda_l = \lambda^l + \lambda^{-l} \quad , \quad l = 1, 2, \ldots , n \tag{A3}$$

Equation (A2) can be written in the form of the vector product

$$\underset{\sim}{\theta}^T \underset{\sim}{\lambda} = 0 \tag{A4}$$

where

$$\underset{\sim}{\theta} = (\theta_n \ \ \theta_{n-1} \ \ \ldots \ \ \theta_0)^T \quad , \quad \underset{\sim}{\lambda} = (\lambda_0 \ \ \lambda_1 \ \ \ldots \ \ \lambda_n)^T \tag{A5}$$

Next, let

$$\gamma = \lambda + 1/\lambda \tag{A6}$$

from which it follows that

$$\gamma^k = \sum_{l=0}^{k} \binom{k}{l} \lambda^{k-2l} \tag{A7}$$

where

$$\binom{k}{l} = \frac{k!}{l!\,(k-l)!} \tag{A8}$$

At this point, it will prove convenient to introduce a matrix $B = (b_{kl})$ $(k,l = 0,1,\ldots,n)$ defined as follows

$$b_{kl} = \binom{k}{\frac{k-l}{2}} \quad \text{if } k \text{ and } l \text{ are both even or both odd and } k \geqslant l$$

$$b_{kl} = 0 \qquad \text{otherwise}$$

Then, Eq.(A7) can be written in the form

$$\underset{\sim}{\gamma} = B\,\underset{\sim}{\lambda} \tag{A9}$$

where $\underset{\sim}{\gamma} = (\gamma^0\ \gamma^1\ \ldots\ \gamma^n)^T$. Introducing Eq.(A9) into Eq.(A4), we obtain simply

$$\underset{\sim}{\theta}^T B^{-1} \underset{\sim}{\gamma} = 0 \tag{A10}$$

From the definition of B, we conclude that the matrix is a unit lower triangular matrix, so that $\det B = 1$. It follows that $B^{-1} = \text{Adj } B$ is also a unit lower triangular matrix. When the notation

$$D = \text{Adj } B \tag{A11}$$

is introduced, where $D = (d_{kl})$, Eq. (A10) becomes

$$\underset{\sim}{\theta}^T D \underset{\sim}{\gamma} = 0 \tag{A12}$$

Moreover, it will prove convenient to introduce the vector

$$\underset{\sim}{\delta} = D^T \underset{\sim}{\theta} \tag{A13}$$

where $\underset{\sim}{\delta} = (\delta_0 \ \ \delta_1 \ \ \ldots \ \delta_n)^T$. Equation (A13) has the index form

$$\delta_k = \sum_{l=0}^{n} d_{lk} \, \theta_{n-l} \tag{A14}$$

and because $d_{lk} = 0$ for $l<k$, we have

$$\delta_k = \sum_{l=k}^{n} d_{lk} \, \theta_{n-l} \tag{A15}$$

On the other hand, for $l>k$ it can be easily verified that

$$d_{lk} = \det \begin{bmatrix} b_{k+1,k} & 1 & \cdots & 0 & 0 \\ b_{k+2,k} & b_{k+2,k+1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{l-1,k} & b_{l-1,k+1} & \cdots & b_{l-1,l-2} & 1 \\ b_{lk} & b_{l,k+1} & \cdots & b_{l,l-2} & b_{l,l-1} \end{bmatrix} \quad \text{(A16)}$$

From this determinant it follows that $d_{lk} = 0$ if $l + k$ is an odd number and

$$d_{lk} = \det \begin{bmatrix} b_{k+2,k} & 1 & \cdots & 0 & 0 \\ b_{k+1,k} & b_{k+4,k+2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{l-2,k} & b_{l-2,k+2} & \cdots & b_{l-2,l-4} & 1 \\ b_{lk} & b_{l,k+2} & \cdots & b_{l,l-4} & b_{l,l-2} \end{bmatrix} \quad \text{(A17)}$$

if $l + k$ is an even number. This permits us to construct the following recurrence formula

$$d_{lk} = - \sum_{s=k,k+2,\ldots}^{l-2} b_{ls}\, d_{sk} \quad , \qquad d_{kk} = 1 \quad \text{(A18)}$$

in which it is recalled that

$$b_{1s} = \binom{1}{\frac{1-s}{2}} \tag{A19}$$

In view of the above, Eq.(A15) reduces to

$$\delta_k = \sum_{1=k,k+2,\ldots}^{n} d_{1k}\, \theta_{n-1} \tag{A20}$$

Finally, considering Eqs.(A12) and (A13), the reduced characteristic equation can be written in the form

$$\sum_{k=0}^{n} \delta_k\, \gamma^{k} = o \tag{A21}$$

where δ_k is given by Eq.(A19).

Equation (A21) yields n roots γ_1 , γ_2 , ... , γ_n . Introducing these roots into Eq.(A6), we can obtain the 2n eigenvalues of A, i.e., $\lambda_1, \lambda_2, \ldots \cdot \lambda_{2n}$.

To illustrate the procedure, let us consider the following characteristic equation

$$\begin{aligned} &z^6 - 10.08333\, z^5 + 36.12500\, z^4 - 55.58333\, z^3 \\ &+ 36.12500\, z^2 - 10.08333\, z + 1 = 0 \end{aligned} \tag{A22}$$

In this case $n = 3$, and from Eq.(A21) it is seen that we need to find δ_k for $k=0,1,2,3$. Therefore, using Eq.(A20) yields

$$\delta_0 = d_{00}\,\theta_3 + d_{20}\,\theta_1 \quad , \quad \delta_1 = d_{11}\,\theta_2 + d_{31}\,\theta_0$$
$$\delta_2 = d_{22}\,\theta_1 \quad , \quad \delta_3 = d_{33}\,\theta_0 \tag{A23}$$

Then, from Eqs.(A18) and (A19) it follows

$$d_{00} = 1 \quad , \quad d_{20} = -\,b_{20}\,d_{00} = -\,2$$
$$d_{31} = -\,b_{31}\,d_{11} = -\,3 \quad , \quad d_{22} = d_{33} = 1 \tag{A24}$$

Also, by inspecting Eq.(A22) we obtain

$$\theta_0 = 1 \quad , \quad \theta_1 = -\,10.08333$$
$$\theta_2 = 36.12500 \quad , \quad \theta_3 = -\,55.58333 \tag{A25}$$

Finally, introducing Eqs.(A24) and (A25) into Eq.(A23) yields

$$\delta_0 = -\,35.41667 \quad , \quad \delta_1 = 33.12500$$
$$\delta_2 = -\,10.08333 \quad , \quad \delta_3 = 1 \tag{A26}$$

Therefore, the reduced characteristic equation (A21) becomes

$$\gamma^3 - 10.08333\,\gamma^2 + 33.12500\gamma - 35.41667 = 0 \tag{A27}$$

which has the following roots

$$\gamma_1 = 2.50000 \quad , \quad \gamma_2 = 3.33333 \quad , \quad \gamma_3 = 4.25000 \tag{A28}$$

Introducing these roots into Eq.(A6), we finally obtain the following eigenvalues

$$\begin{aligned} &\lambda_1 = 1/4 \quad , \quad \lambda_2 = 1/3 \quad , \quad \lambda_3 = 1/2 \\ &\lambda_4 = 4 \quad , \quad \lambda_5 = 3 \quad , \quad \lambda_6 = 2 \end{aligned} \tag{A29}$$

CHAPTER II

RESPONSE OF INFINITE PERIODIC STRUCTURES -SYMMETRIC SUBSTRUCTURES

1. INTRODUCTION

In previous chapter we derived the response of a general periodic structure. It was pointed out, however, that numerical difficulties arise when the number of substructures becomes too large. In that case, the structure is theoretically infinite and Eq.(1.32) cannot be applied any longer. The object of the first part of this chapter is to adapt Eq.(1.32) to several important cases of infinitely long periodic systems. Before we investigate the reasons for the failure of Eq.(1.32) for large structures, let us derive some useful relationships. First, from Eq.(1.31) we obtain for $i = 1$

$$A = \sum_{j=1}^{2n} X_j z_j \tag{2.1}$$

This equation represents matrix A in spectral form. From standard texts[14]

$$X_j = \underset{\sim}{y}_j \underset{\sim}{r}_j^* \quad , \qquad j = 1,2,\ldots,2n \tag{2.2}$$

where

$$\underset{\sim}{r}_j^* \underset{\sim}{y}_k = \delta_{jk} \quad , \qquad j,k = 1,2,\ldots,2n \tag{2.3}$$

The vectors $\underset{\sim}{y}_k$ are the right eigenvectors of A and $\underset{\sim}{r}_k$ are the left ones. The asterisk indicates the complex conjugate transpose. From Eqs.(2.2) and (2.3) we have

$$X_j \, X_k = (\underset{\sim}{y}_j \, \underset{\sim}{r}_j^*) (\underset{\sim}{y}_k \, \underset{\sim}{r}_k^*) = \underset{\sim}{y}_j \, \underset{\sim}{r}_k^* \, \delta_{jk}$$

or

$$X_j \, X_k = X_j \, \delta_{jk} \tag{2.4}$$

Note that Eq.(2.4) implies that all matrices X_j are idempotent and singular. In fact, it turns out that, when z_j is a simple eigenvalue, X_j has rank one [15].

Next, let us partition matrix X_j as follows

$$X_j = \left[\begin{array}{c|c} X_{11j} & X_{12j} \\ \hline X_{21j} & X_{22j} \end{array} \right] \quad , \qquad j = 1, 2, \ldots , 2n \tag{2.5}$$

Then, from Eq.(2.4) we obtain

$$X_{11j} \, X_{11k} + X_{12j} \, X_{21k} = X_{11j} \, \delta_{jk} \tag{2.6}$$

$$X_{11j} \, X_{12k} + X_{12j} \, X_{22k} = X_{12j} \, \delta_{jk} \tag{2.7}$$

$$X_{21j} \, X_{11k} + X_{22j} \, X_{21k} = X_{21j} \, \delta_{jk} \tag{2.8}$$

$$X_{21j}\, X_{12k} \; + \; X_{22j}\, X_{22k} \; = \; X_{22j}\, \delta_{jk} \tag{2.9}$$

Equations (2.6-9) will prove very useful in the subsequent sections.

2. A SEMI-INFINITE STRUCTURE WITH A LEFT-END LOAD VECTOR

Let us consider a semi-infinite periodic structure subject to a single load vector at the left end, as depicted in Fig. 2.1. Since $\underset{\sim}{f}_k^* = \underset{\sim}{0}$ ($k = 0,1,\ldots$), Eq.(1.32) reduces to

$$\underset{\sim}{x}_i = \sum_{j=1}^{2n} X_j \, z_j^i \, \underset{\sim}{x}_0 \tag{2.10}$$

Considering the fact that the response at infinity approaches $\underset{\sim}{0}$, we must discard all the eigenvalues z_j ($j=n+1,\ldots,2n$) because eigenvalues with magnitude larger than one, would produce an unbounded response at infinity. Hence, Eq.(2.10) must be written as

$$\underset{\sim}{x}_i = \sum_{j=1}^{n} X_j \, z_j^i \, \underset{\sim}{x}_0 \quad . \tag{2.11}$$

This equation must be valid for $i = 0$, so that

$$\underset{\sim}{x}_0 = \sum_{j=1}^{n} X_j \, \underset{\sim}{x}_0 \tag{2.12}$$

or, after partitioning

$$\begin{aligned} \underset{\sim}{u}_0 &= \sum_{j=1}^{n} X_{11j} \, \underset{\sim}{u}_0 + \sum_{j=1}^{n} X_{12j} \, \underset{\sim}{f}_0 \\ \underset{\sim}{f}_0 &= \sum_{j=1}^{n} X_{21j} \, \underset{\sim}{u}_0 + \sum_{j=1}^{n} X_{22j} \, \underset{\sim}{f}_0 \end{aligned} \tag{2.13}$$

Hence,

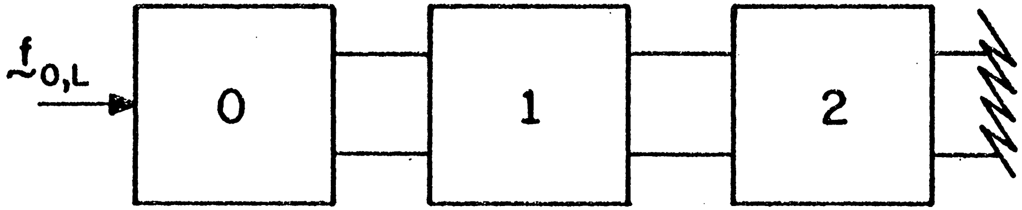


FIGURE 2.1 A Semi-Infinite Periodic Structure

Subject to a Left-End Load

$$\left(1 - \sum_{j=1}^{n} X_{11j}\right) \underset{\sim}{u}_0 = \sum_{j=1}^{n} X_{12j}\, \underset{\sim}{f}_0$$

$$\sum_{j=1}^{n} X_{21j}\, \underset{\sim}{u}_0 = \left(1 - \sum_{j=1}^{n} X_{22j}\right) \underset{\sim}{f}_0 \tag{2.14}$$

Since,

$$\sum_{j=1}^{2n} X_j = A^0 = 1 \tag{2.15}$$

we have

$$1 - \sum_{j=1}^{n} X_{11j} = \sum_{j=n+1}^{2n} X_{11j} \ , \quad 1 - \sum_{j=1}^{n} X_{22j} = \sum_{j=n+1}^{2n} X_{22j}$$

$$\sum_{j=1}^{n} X_{12j} = - \sum_{j=n+1}^{2n} X_{12j} \ , \quad \sum_{j=1}^{n} X_{21j} = - \sum_{j=n+1}^{2n} X_{21j} \tag{2.16}$$

and from Eqs.(2.14) and (2.16) we obtain

$$\underset{\sim}{u}_0 = \left(\sum_{j=n+1}^{2n} X_{11j} \right)^{-1} \left(\sum_{j=1}^{n} X_{12j} \right) \underset{\sim}{f}_0$$

$$\underset{\sim}{u}_0 = \left(\sum_{j=1}^{n} X_{21j} \right)^{-1} \left(\sum_{j=n+1}^{2n} X_{22j} \right) \underset{\sim}{f}_0 \tag{2.17}$$

It appears as if we obtained two different results for $\underset{\sim}{u}_0$. However, we shall now prove that these two results are the same. Indeed, after taking $\sum_{j=n+1}^{2n}$ and $\sum_{k=1}^{n}$, Eq.(2.6) becomes

$$(\sum_{j=n+1}^{2n} X_{11j})(\sum_{k=1}^{n} X_{11k}) + (\sum_{j=n+1}^{2n} X_{12j})(\sum_{k=1}^{n} X_{21k}) = \sum_{j=n+1}^{2n} \sum_{k=1}^{n} X_{11j}\delta_{jk} = 0$$

and using Eq.(2.16) we obtain

$$(\sum_{j=n+1}^{2n} X_{11j})^{-1} (\sum_{j=1}^{n} X_{12j}) = (\sum_{j=1}^{n} X_{11j}) (\sum_{j=1}^{n} X_{21j}) \qquad (2.18)$$

where we replaced the dummy variable k by j. Similarly, from Eq.(2.7) it follows that

$$(\sum_{j=1}^{n} X_{21j})^{-1} (\sum_{j=n+1}^{2n} X_{22j}) = (\sum_{j=1}^{n} X_{11j}) (\sum_{j=1}^{n} X_{21j})^{-1} \qquad (2.19)$$

From Eqs.(2.18) and (2.19) it is seen that the two equations (2.17) represent the same result

$$\underset{\sim}{u}_0 = (\sum_{j=1}^{n} X_{11j}) (\sum_{j=1}^{n} X_{21j})^{-1} \underset{\sim}{f}_0 \qquad (2.20)$$

Next, let us introduce Eq.(2.20) into Eq.(2.11) and write

$$\underset{\sim}{u}_i = \{(\sum_{j=1}^{n} X_{11j} z_j^i)(\sum_{j=1}^{n} X_{11j})(\sum_{j=1}^{n} X_{21j})^{-1} + \sum_{j=1}^{n} X_{12j} z_j^i\} \underset{\sim}{f}_0$$

$$(2.21)$$

$$\underset{\sim}{p}_i = \{(\sum_{j=1}^{n} X_{21j} z_j^i)(\sum_{j=1}^{n} X_{11j})(\sum_{j=1}^{n} X_{21j})^{-1} + \sum_{j=1}^{n} X_{22j} z_j^i\} \underset{\sim}{f}_0$$

Taking $\left(\sum_{j=1}^{n} X_{21j} \right)^{-1}$ outside the brackets and using Eqs.(2.6) and (2.8), we obtain

$$\underset{\sim}{u}_i = \left(\sum_{j=1}^{n} X_{11j} \, z_j^i \right) \left(\sum_{j=1}^{n} X_{21j} \right)^{-1} \underset{\sim}{f}_0 \tag{2.22a}$$

$$\underset{\sim}{p}_i = \left(\sum_{j=1}^{n} X_{21j} \, z_j^i \right) \left(\sum_{j=1}^{n} X_{21j} \right)^{-1} \underset{\sim}{f}_0 \tag{2.22b}$$

where we omit $\underset{\sim}{f}_{i,L}$ in (2.22b) since they are all $\underset{\sim}{0}$, except for $i = 0$, where we take $\underset{\sim}{f}_{0,L} = \underset{\sim}{f}_0 = \underset{\sim}{p}_0$.

Equations (2.22) represent the response of a semi-infinite structure subject to a left-end load vector. It should be pointed out that the solution is given in terms of known quantities and no need exists to solve for the unknown end vector $\underset{\sim}{x}_0$ as is the case in Ref. 8. Furthermore, it is seen that the response dies out with increasing i and that no numerical difficulties arise.

Let us now investigate some of the reasons why the general solution Eq.(1.32) becomes numerically unstable. To illustrate the procedure, let us assume the structure is free at both ends, i.e., $\underset{\sim}{p}_0 = \underset{\sim}{p}_N = \underset{\sim}{0}$. Furthermore, let us consider the case of a left end load vector only. To solve for the unknown displacement vector $\underset{\sim}{u}_0$, we write the lower half of Eq.(1.32) for $i = N$

$$\underset{\sim}{0} = \left(\sum_{j=1}^{2n} X_{21j} \, z_j^N \right) \underset{\sim}{u}_0 + \left(\sum_{j=1}^{2n} X_{22j} \, z_j^N \right) \underset{\sim}{f}_0 \tag{2.23}$$

from which it follows that

$$\underset{\sim}{u}_0 = - \left(\sum_{j=1}^{2n} X_{21j} \, z_j^N \right)^{-1} \left(\sum_{j=1}^{2n} X_{22j} \, z_j^N \right) \underset{\sim}{f}_0 \tag{2.24}$$

Therefore, it is necessary to compute the inverse of the matrix $\sum_{j=1}^{2n} X_{21j} \, z_j^N$. However, when the number of substructures N becomes large, it is seen that the powers z_j^N $(j=1,2,\ldots,n)$ become negligible when compared to their reciprocals z_{j+n}^N $(j=1,2,\ldots,n)$. In this case, the computer will invert the matrix $\sum_{j=n+1}^{2n} X_{21j} \, z_j^N$, which, in general, is still nonsingular. However, if N is such that even z_{2n}^N is lost, one usually ends up with a singular matrix $\sum_{j=n+1}^{2n-1} X_{21j} \, z_j^N$. This can be attributed to the fact that, for distinct eigenvalues, each of the matrices X_j $(j=1,2,\ldots,2n)$ are of rank one[15] . Because the eigenvalues of A are usually close to 1, it usually takes a relatively large number of substructures before the structure can be considered as infinite or semi-infinite.

3. AN INFINITE STRUCTURE WITH AN INTERIOR LOAD

Let us now consider the case of a structure which is infinite to both sides of an interior force vector, as indicated in Fig. 2.2. If we assume the load vector to be applied at the left end of the station 1, it follows from Eq.(1.32) for $i = 1$ that,

$$\underset{\sim}{x}_1 = A^1 \underset{\sim}{x}_0 + \underset{\sim}{f}^*_{1-1} \tag{2.25}$$

so that

$$\underset{\sim}{x}_0 = A^{-1} (\underset{\sim}{x}_1 - \underset{\sim}{f}^*_{1-1}) = A^{-1} \begin{bmatrix} \underset{\sim}{u}_1 \\ \text{----} \\ \underset{\sim}{p}_1 \end{bmatrix} \tag{2.26}$$

where we assumed that the total load at station 1 is applied at the left side. Note that there is no loss in generality in doing so. Substituting Eq.(2.26) into Eq.(1.32), we obtain

$$\underset{\sim}{x}_i = A^{i-1} \begin{bmatrix} \underset{\sim}{u}_1 \\ \text{----} \\ \underset{\sim}{p}_1 \end{bmatrix} , \qquad i = 0, 1, \dots , 1-1, 1 \tag{2.27}$$

Because the structure is semi-infinite to the left, we must discard the eigenvalues z_j $(j=1,2,\dots,n)$ and write Eq.(2.27) as follows

$$\underset{\sim}{u}_i = \sum_{j=n+1}^{2n} X_{11j}\, z_j^{i-1}\, \underset{\sim}{u}_1 + \sum_{j=n+1}^{2n} X_{12j}\, z_j^{i-1}\, \underset{\sim}{p}_1 \tag{2.28a}$$

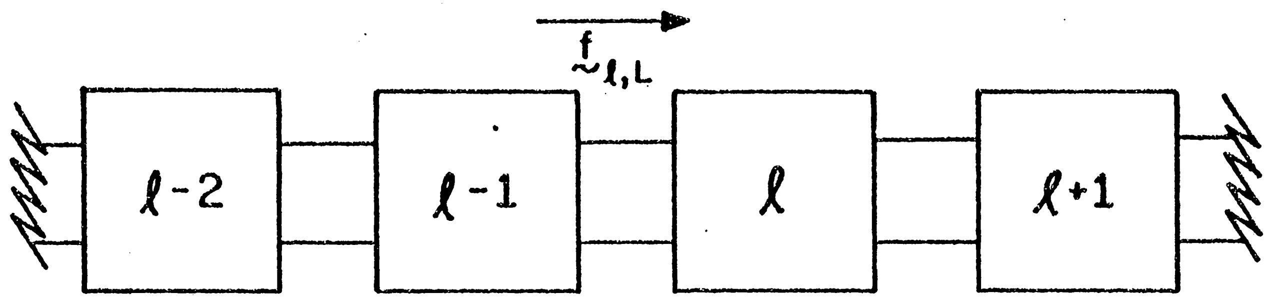


FIGURE 2.2 An Infinite Periodic Structure Subject to an Interior Load

$$\underset{\sim}{p}_i = \sum_{j=n+1}^{2n} X_{21j}\, z_j^{i-1}\, \underset{\sim}{u}_1 + \sum_{j=n+1}^{2n} X_{22j}\, z_j^{i-1}\, \underset{\sim}{p}_1 \tag{2.28b}$$

Again, to determine $\underset{\sim}{u}_1$ we can take either Eq.(2.28a) or (2.28b) for $i = 1$, which yields

$$\underset{\sim}{u}_1 = \Big(\sum_{j=n+1}^{2n} X_{11j}\Big)\Big(\sum_{j=n+1}^{2n} X_{21j}\Big)^{-1} \underset{\sim}{p}_1 \tag{2.29}$$

From Eq.(2.29) together with Eq.(2.28) we obtain

$$\underset{\sim}{u}_i = \Big\{\Big(\sum_{j=n+1}^{2n} X_{11j} z_j^{i-1}\Big)\Big(\sum_{j=n+1}^{2n} X_{11j}\Big) + \Big(\sum_{j=n+1}^{2n} X_{12j} z_j^{i-1}\Big)\Big(\sum_{j=n+1}^{2n} X_{21j}\Big) \times \Big(\sum_{j=n+1}^{2n} X_{21j}\Big)^{-1}\Big\}\underset{\sim}{p}_1$$

$$\underset{\sim}{p}_i = \Big\{\Big(\sum_{j=n+1}^{2n} X_{21j} z_j^{i-1}\Big)\Big(\sum_{j=n+1}^{2n} X_{11j}\Big) + \Big(\sum_{j=n+1}^{2n} X_{22j} z_j^{i-1}\Big)\Big(\sum_{j=n+1}^{2n} X_{21j}\Big) \times \Big(\sum_{j=n+1}^{2n} X_{21j}\Big)^{-1}\Big\}\underset{\sim}{p}_1$$

Then, using Eqs.(2.6) and (2.8), we obtain

$$\underset{\sim}{u}_i = \Big(\sum_{j=n+1}^{2n} X_{11j}\, z_j^{i-1}\Big)\Big(\sum_{j=n+1}^{2n} X_{21j}\Big)^{-1} \underset{\sim}{p}_1 \tag{2.30a}$$

$$, \quad i=0,1,\ldots,l$$

$$\underset{\sim}{p}_i = \Big(\sum_{j=n+1}^{2n} X_{21j}\, z_j^{i-1}\Big)\Big(\sum_{j=n+1}^{2n} X_{21j}\Big)^{-1} \underset{\sim}{p}_1 \tag{2.30b}$$

If the structure is also semi-infinite to the right, we can develop a completely similar response to that derived in section 2, where, in Eq.(2.22) we replace i by $i-l$ and $\underset{\sim}{f}_0$ by $\underset{\sim}{p}_l + \underset{\sim}{f}_{l-1}^{*\prime}$ with $\underset{\sim}{f}_{l-1}^{*\prime}$ being the lower half of $\underset{\sim}{f}_{l-1}^{*}$. Hence

$$\underset{\sim}{u}_i = \left(\sum_{j=1}^{n} X_{11j}\, z_j^{i-l}\right)\left(\sum_{j=1}^{n} X_{21j}\right)^{-1}\left(\underset{\sim}{p}_l + \underset{\sim}{f}_{l-1}^{*\prime}\right) \tag{2.31a}$$

$$i=l,l+1,\ldots$$

$$\underset{\sim}{p}_i = \left(\sum_{j=1}^{n} X_{21j}\, z_j^{i-l}\right)\left(\sum_{j=1}^{n} X_{21j}\right)^{-1}\left(\underset{\sim}{p}_l + \underset{\sim}{f}_{l-1}^{*\prime}\right) \tag{2.31b}$$

where, for $i = l$, we must replace $\underset{\sim}{p}_i$ by $\underset{\sim}{p}_l + \underset{\sim}{f}_{l-1}^{*\prime}$. Note, that Eqs.(2.31) reduce to Eqs.(2.22) for $l=0$, where $\underset{\sim}{f}_{-1}^{*\prime} = \underset{\sim}{f}_0$.

Finally, from Eqs.(2.30a) and (2.31a) for $i=l$, we have

$$\underset{\sim}{u}_l = \left(\sum_{j=n+1}^{2n} X_{11j}\right)\left(\sum_{j=n+1}^{2n} X_{21j}\right)^{-1}\underset{\sim}{p}_l = \left(\sum_{j=1}^{n} X_{11j}\right)\left(\sum_{j=1}^{n} X_{21j}\right)^{-1}\left(\underset{\sim}{p}_l+\underset{\sim}{f}_{l-1}^{*\prime}\right)$$

or

$$\left\{\left(\sum_{j=n+1}^{2n} X_{11j}\right)\left(\sum_{j=n+1}^{2n} X_{21j}\right)^{-1} - \left(\sum_{j=1}^{n} X_{11j}\right)\left(\sum_{j=1}^{n} X_{21j}\right)^{-1}\right\}\underset{\sim}{p}_l$$

$$= \left(\sum_{j=1}^{n} X_{11j}\right)\left(\sum_{j=1}^{n} X_{21j}\right)^{-1}\underset{\sim}{f}_{l-1}^{*\prime}$$

and, because of Eq.(2.16) we have

$$\sum_{j=n+1}^{2n} X_{21j} = \sum_{j=1}^{n} X_{21j}$$

and therefore,we obtain

$$\left(\sum_{j=n+1}^{2n} X_{11j} + \sum_{j=1}^{n} X_{11j}\right)\left(\sum_{j=n+1}^{2n} X_{21j}\right)^{-1} \underset{\sim}{p}_l = \left(\sum_{j=1}^{n} X_{11j}\right)\left(\sum_{j=1}^{n} X_{21j}\right)^{-1} \underset{\sim}{f}_{l-1}^{*'}$$

where

$$\sum_{j=n+1}^{2n} X_{11j} + \sum_{j=1}^{n} X_{11j} = \sum_{j=1}^{2n} X_{11j} = 1$$

Hence

$$\left(\sum_{j=n+1}^{2n} X_{21j}\right)^{-1} \underset{\sim}{p}_l = \left(\sum_{j=1}^{n} X_{11j}\right)\left(\sum_{j=1}^{n} X_{21j}\right)^{-1} \underset{\sim}{f}_{l-1}^{*'} \qquad (2.32)$$

Substituting Eq.(2.32) into Eq.(2.30) yields

$$\underset{\sim}{u}_i = \left(\sum_{j=n+1}^{2n} X_{11j}\, z_j^{i-l}\right)\left(\sum_{j=1}^{n} X_{11j}\right)\left(\sum_{j=1}^{n} X_{21j}\right)^{-1} \underset{\sim}{f}_{l-1}^{*'}$$

$$\underset{\sim}{p}_i = \left(\sum_{j=n+1}^{2n} X_{21j}\, z_j^{i-l}\right)\left(\sum_{j=1}^{n} X_{11j}\right)\left(\sum_{j=1}^{n} X_{21j}\right)^{-1} \underset{\sim}{f}_{l-1}^{*'}$$

or, using Eqs.(2.6) and (2.8), we obtain

$$\underset{\sim}{u}_i = -\sum_{j=n+1}^{2n} X_{12j}\, z_j^{i-l}\, \underset{\sim}{f}_{l-1}^{*'} \quad , \qquad i = 0,1, \dots ,l \qquad (2.33a)$$

$$\underset{\sim}{p}_i = - \sum_{j=n+1}^{2n} X_{22j} \, z_j^{i-l} \, \underset{\sim}{f}^{*'}_{l-1} \quad , \quad i = 0,1, \ldots ,l \tag{2.33b}$$

Furthermore, from Eq.(2.33b) it follows that

$$\underset{\sim}{p}_l = - \sum_{j=n+1}^{2n} X_{22j} \, \underset{\sim}{f}^{*'}_{l-1}$$

and

$$\underset{\sim}{p}_l + \underset{\sim}{f}^{*'}_{l-1} = \left(1 - \sum_{j=n+1}^{2n} X_{22j} \right) \underset{\sim}{f}^{*'}_{l-1} = \sum_{j=1}^{n} X_{22j} \, \underset{\sim}{f}^{*'}_{l-1} \tag{2.34}$$

Substituting Eq.(2.34) into Eqs.(2.31) and using similar arguments, we arrive at

$$\underset{\sim}{u}_i = \sum_{j=1}^{n} X_{12j} \, z_j^{i-l} \, \underset{\sim}{f}^{*'}_{l-1} \tag{2.35a}$$

$$, \quad i = l, \, l+1, \ldots$$

$$\underset{\sim}{p}_i = \sum_{j=1}^{n} X_{22j} \, z_j^{i-l} \, \underset{\sim}{f}^{*'}_{l-1} \tag{2.35b}$$

where for $i = l$, we should add $\underset{\sim}{f}^{*'}_{l-1}$ on the left hand side of Eq.(2.35b).

Finally, using Eq.(1.53), we can write Eqs.(2.33) as

$$\underset{\sim}{u}_i = \sum_{j=1}^{n} X^T_{12j} \, z_j^{l-i} \, \underset{\sim}{f}^{*'}_{l-1} \quad , \quad i = 0, 1, \ldots , l \tag{2.36a}$$

$$\underset{\sim}{p}_i = - \sum_{j=1}^{n} X_{11j}^T \, z_j^{l-i} \, \underset{\sim}{f}_{l-1}^{*'} \quad , \quad i = 0, 1, \dots , l \tag{2.36b}$$

Equations (2.35) and (2.36) represent the response of a structure subject to an interior load far enough from both ends, such that there exist no reflected waves. It is anticipated that Eqs.(2.35-36) yield the same results as Eqs.(2.22) for $\frac{1}{2} \underset{\sim}{f}_{l-1}^{*'}$.

Finally, let us consider a structure subject to several interior loads, each far enough from the boundaries so as not to cause any reflection. In Fig.2.3, we assume the loads to be applied between stations s and l. Then, for each load vector $\underset{\sim}{f}_k^{*'}$ (k = s, ... , l-1) we have a solution similar to Eqs.(2.35) and (2.36). Because the system is linear, we can superimpose these solutions to obtain

$$\underset{\sim}{u}_i = \sum_{k=s}^{i} \left(\sum_{j=1}^{n} X_{12j} \, z_j^{i-k-1} \, \underset{\sim}{f}_k^{*'} \right) + \sum_{k=i-1}^{l-1} \left(\sum_{j=1}^{n} X_{12j}^T \, z_j^{k+1-i} \, \underset{\sim}{f}_k^{*'} \right) \qquad i=0,1,\dots$$

$$\underset{\sim}{p}_i = \sum_{k=s}^{i} \left(\sum_{j=1}^{n} X_{22j} \, z_j^{i-k-1} \, \underset{\sim}{f}_k^{*'} \right) - \sum_{k=i-1}^{l-1} \left(\sum_{j=1}^{n} X_{11j}^T \, z_j^{k+1-i} \, \underset{\sim}{f}_k^{*'} \right)$$

or, finally

$$\underset{\sim}{u}_i = \sum_{j=1}^{n} \left(X_{12j} \sum_{k=s}^{i} z_j^{i-k-1} \, \underset{\sim}{f}_k^{*'} + X_{12j}^T \sum_{k=i-1}^{l-1} z_j^{k+1-i} \, \underset{\sim}{f}_k^{*'} \right) \tag{2.37a}$$

$$i = 0, 1, \dots$$

$$\underset{\sim}{p}_i = \sum_{j=1}^{n} \left(X_{22j} \sum_{k=s}^{i} z_j^{i-k-1} \, \underset{\sim}{f}_k^{*'} - X_{11j}^T \sum_{k=i-1}^{l-1} z_j^{k+1-i} \, \underset{\sim}{f}_k^{*'} \right) \tag{2.37b}$$

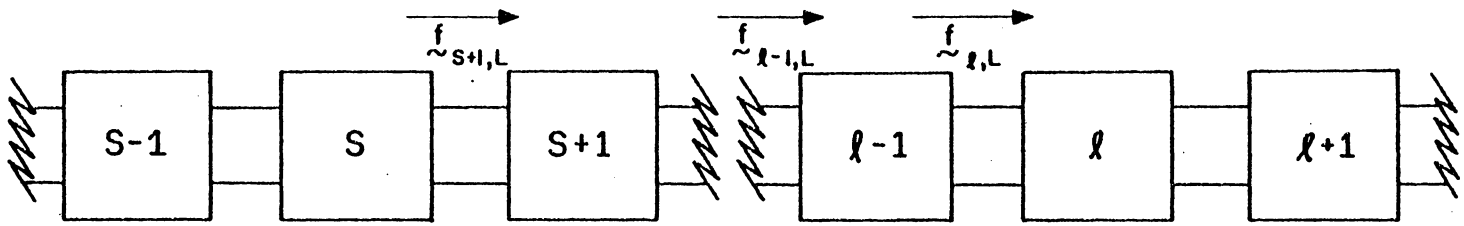


FIGURE 2.3 An Infinite Periodic Structure
Subject to Interior Loads

Equations (2.37) are valid for all types of boundary conditions and give the response of a periodic structure subject to loads between two stations far enough from the boundaries.

Finally, it should be pointed out that cases such as a semi-infinite structure subject to a load near the boundary can be handled in two steps by subdividing the structure into two parts as indicated in Fig.2.4. The left section is regarded as a finite structure, and the right section is an example of a semi-infinite structure subject to a left-end load $\underset{\sim}{p}_l + \underset{\sim}{f}^{*\prime}_{l-1}$.

As an example, let us assume that the system is free at the left end, i.e. $\underset{\sim}{p}_0 = 0$, $\underset{\sim}{f}_0 = 0$. Therefore, the equations can be written as follows

$$\underset{\sim}{u}_i = \sum_{j=1}^{2n} X_{11j}\, z_j^i\, \underset{\sim}{u}_0 \qquad (2.38a)$$

$$i \leq l$$

$$\underset{\sim}{p}_i = \sum_{j=1}^{2n} X_{21j}\, z_j^i\, \underset{\sim}{u}_0 \qquad (2.38b)$$

and, from Eqs.(2.31)

$$\underset{\sim}{u}_i = \left(\sum_{j=1}^{n} X_{11j}\, z_j^{i-l} \right)\left(\sum_{j=1}^{n} X_{21j} \right)^{-1} \left(\underset{\sim}{p}_l + \underset{\sim}{f}^{*\prime}_{l-1} \right) \qquad (2.38c)$$

$$i \leq l$$

$$\underset{\sim}{p}_i = \left(\sum_{j=1}^{n} X_{21j}\, z_j^{i-l} \right)\left(\sum_{j=1}^{n} X_{21j} \right)^{-1} \left(\underset{\sim}{p}_l + \underset{\sim}{f}^{*\prime}_{l-1} \right) \qquad (2.38d)$$

Again, in Eq.(2.38d) for $i = l$, we must add $\underset{\sim}{f}^{*\prime}_{l-1}$ to the left-hand side.

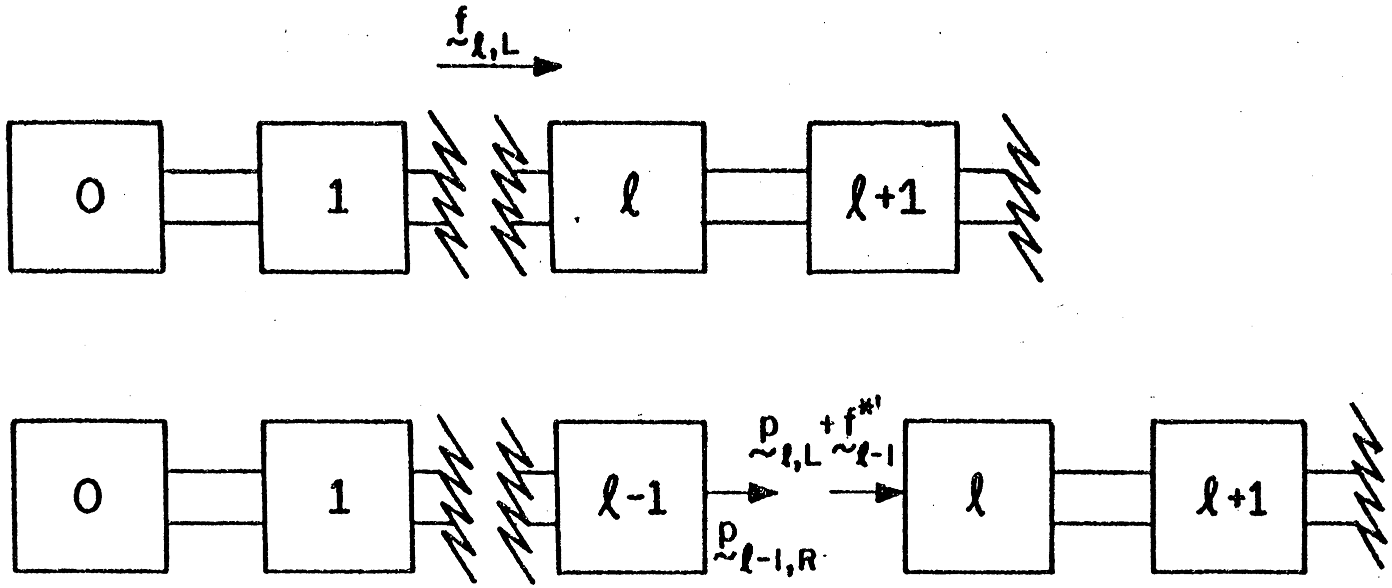


FIGURE 2.4 A Semi-Infinite Structure Subject to a Load near the Boundary

Also, in Eq.(2.38d) for $i = 1$, we again assume the total load to be applied at the left end of station 1 . To determine $\underset{\sim}{u}_0$, we write Eqs.(2.38a), (2.38b) and (2.38c) for $i = 1$,

$$\underset{\sim}{u}_1 = \sum_{j=1}^{2n} X_{11j}\, z_j^1\, \underset{\sim}{u}_0 \tag{2.39}$$

$$\underset{\sim}{p}_1 = \sum_{j=1}^{2n} X_{21j}\, z_j^1\, \underset{\sim}{u}_0 \tag{2.40}$$

$$\underset{\sim}{u}_1 = \left(\sum_{j=1}^{n} X_{11j} \right) \left(\sum_{j=1}^{n} X_{21j} \right)^{-1} \left(\underset{\sim}{p}_1 + \underset{\sim}{f}^{*\prime}_{1-1} \right) \tag{2.41}$$

We now substitute Eqs.(2.39) and (2.40) into (2.41) and obtain

$$\sum_{j=1}^{2n} X_{11j}\, z_j^1\, \underset{\sim}{u}_0 = \left(\sum_{j=1}^{n} X_{11j} \right)\left(\sum_{j=1}^{n} X_{21j} \right)^{-1} \left(\sum_{j=1}^{2n} X_{21j} z_j^1 \underset{\sim}{u}_0 + \underset{\sim}{f}^{*\prime}_{1-1} \right) \tag{2.42}$$

from which it follows that

$$\left\{ \sum_{j=1}^{2n} X_{11j}\, z_j^1 - \left(\sum_{j=1}^{n} X_{11j} \right)\left(\sum_{j=1}^{n} X_{21j} \right)^{-1} \left(\sum_{j=1}^{2n} X_{21j}\, z_j^1 \right) \right\} \underset{\sim}{u}_0$$

$$= \left(\sum_{j=1}^{n} X_{11j} \right)\left(\sum_{j=1}^{n} X_{21j} \right)^{-1} \underset{\sim}{f}^{*\prime}_{1-1} \tag{2.43}$$

From Eq.(2.6) we see that

$$- \left(\sum_{j=1}^{n} X_{11j} \right)\left(\sum_{j=1}^{n} X_{21j} \right)^{-1} = \left(\sum_{j=n+1}^{2n} X_{11j} \right)^{-1} \left(\sum_{j=n+1}^{2n} X_{12j} \right) \tag{2.44}$$

Substituting Eq.(2.44) into Eq.(2.43) and taking $\left(\sum_{j=n+1}^{2n} X_{11j} \right)^{-1}$ outside the square brackets yields

$$(\sum_{j=n+1}^{2n} X_{11j})^{-1}\{(\sum_{j=n+1}^{2n} X_{11j})(\sum_{j=1}^{2n} X_{11j}z_j^1)+(\sum_{j=n+1}^{2n} X_{12j})(\sum_{j=1}^{2n} X_{21j}z_j^1)\}\ \underset{\sim}{u}_0$$

$$= (\sum_{j=1}^{n} X_{11j})(\sum_{j=1}^{n} X_{21j})^{-1}\ \underset{\sim}{f}_{1-1}^{*'} \tag{2.45}$$

Again, applying Eq.(2.6) to the expression inside the square brackets, we obtain

$$(\sum_{j=n+1}^{2n} X_{11j})^{-1}\ (\sum_{j=1}^{n} X_{11j}z_j^1)\ \underset{\sim}{u}_0 = (\sum_{j=1}^{n} X_{11j})(\sum_{j=1}^{n} X_{21j})^{-1}\ \underset{\sim}{f}_{1-1}^{*'} \tag{2.46}$$

Finally, substituting Eq.(2.44) into Eqs.(2.46) and using Eq.(2.16),we obtain

$$\sum_{j=1}^{n} X_{11j}\ z_j^1\ \ \underset{\sim}{u}_0 = \sum_{j=1}^{n} X_{12j}\ \underset{\sim}{f}_{1-1}^{*'}$$

from which

$$\underset{\sim}{u}_0 = (\sum_{j=1}^{n} X_{11j}\ z_j^1)^{-1}\ (\sum_{j=1}^{n} X_{12j})\ \underset{\sim}{f}_{1-1}^{*'} \tag{2.47}$$

Equations (2.38a-d) together with Eq.(2.47) represent the response of a semi-infinite periodic structure with an interior load producing reflections at the left boundary.

A similar solution can be obtained for the case of a fixed left boundary. The solution to the problem of a semi-infinite structure subject to many interior loads affecting the left boundary can be found by superimposing the solutions to each load vector seperately.

We are now in a position to solve any type of periodic structure without any numerical difficulties. Indeed, when the structure is finite we use Eq.(1.32) directly. The term " finite " in the present context means that the number of substructures must be such that the unknown left-end vector $\underset{\sim}{x}_0$ can be computed without numerical difficulties, i.e., without encountering singular matrices. If, on the other hand, the structure is infinite, i.e., if the number of substructures is such that singular matrices appear when $\underset{\sim}{x}_0$ is calculated, we must distinguish between several types of loads. In general, for an infinite structure and starting from the left end, we can distinguish between following types of forces : (1) a left-end load not affecting the right boundary, (2) an interior load affecting the left end but not the right end, (3) an interior load affecting neither the left nor the right end, (4) a load affecting both ends, which is a finite structure, (5) an interior load affecting the right end but not the left end, (6) a right-end load not affecting the left end.

Each of these types can be handled by one of the previously developed procedures.

4. ILLUSTRATIVE EXAMPLES

Let us consider the periodic structure in axial vibration as shown in Fig.2.5, and let us assume the number of substructures N to be equal to forty . Furthermore, assume that the system is subject to a left-end load only. Such a system can be considered as a semi-infinite structure. The number of subsystems needed to constitute an infinite structure is primarily dependent on the relative magnitudes of z_{n+1} and z_{2n} and also on the magnitudes of the applied loads. The infinite model will be valid if the response at the right end is very close to 0 .

Considering Eqs.(2.22) together with Eqs.(1.68b) and (1.67),we obtain

$$u_i = \frac{z_2^i}{\beta(\alpha^2 - 1)^{1/2}} f_0 = z_2^i u_0 \qquad (2.48a)$$

$$i = 0, 1, 2, \ldots , 40$$

$$p_i = z_2^i f_0 \qquad (2.48b)$$

where for $i = 0$, we must replace p_0 by f_0 .

In the present case we have from Eqs.(1.73) and (1.74)

$$\begin{aligned} z_2 &= 0.25707 - 0.52908\, j \\ \alpha &= 0.5(1 + j) \quad , \quad \beta = 1 + j \end{aligned} \qquad (2.49)$$

Substituting these values into Eqs.(2.48) and assuming $f_0 = 1$, we

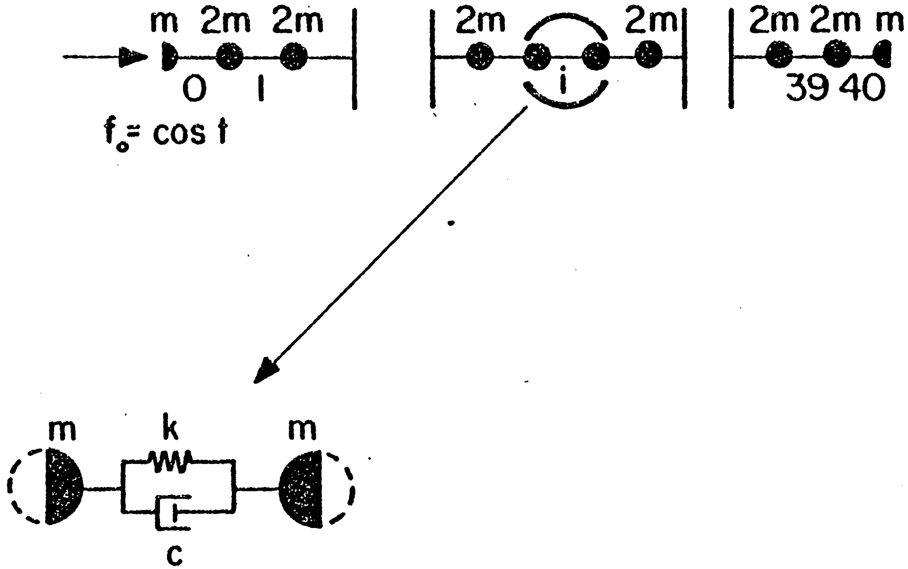


FIGURE 2.5   The Periodic Structure

obtain the response as displayed in Fig. 2.6. Note, that since $u_{40} = p_{40} = 0$, we can conclude that the infinite model is valid.

Next, let us consider the same system, but this time consisting of 90 substructures and subject to an interior load at station 40 . Such a structure can be considered as semi-infinite both to the left and to the right of the load. This time we use Eqs.(2.35), (2.36), (1.68b) and (1.67) and obtain

$$u_i = \frac{1}{2\beta(\alpha^2-1)^{1/2}} z_2^{i-l} f^{*'}_{l-1} \tag{2.50a}$$

$$i=l,l+1, \ldots ,N$$

$$p_i = \frac{1}{2} z_2^{i-l} f^{*'}_{l-1} \tag{2.50b}$$

$$u_i = \quad 2\beta(\alpha^2-1)^{1/2} \quad z_2^{l-i} f^{*'}_{l-1} \tag{2.50c}$$

$$i=0,1, \ldots ,l$$

$$p_i = -\frac{1}{2} z_2^{l-i} f^{*'}_{l-1} \tag{2.50d}$$

where for $i = l$, we should add $f^{*'}_{l-1}$ on the left side of Eq.(2.50b). Then, from Eqs.(2.49) and $f^{*'}_{l-1} = 1$ we obtain the response as plotted in Fig. 2.7. As could be expected, the displacement and reaction patterns are symmetric and anti-symmetric respectively.

Finally, let us consider the case of an interior load at station 3 of a semi-infinite structure. Note, that the structure should be slightly longer as in the case of a left-end load. Here we must use Eqs.(2.38) if we assume the structure to be free at the left end.

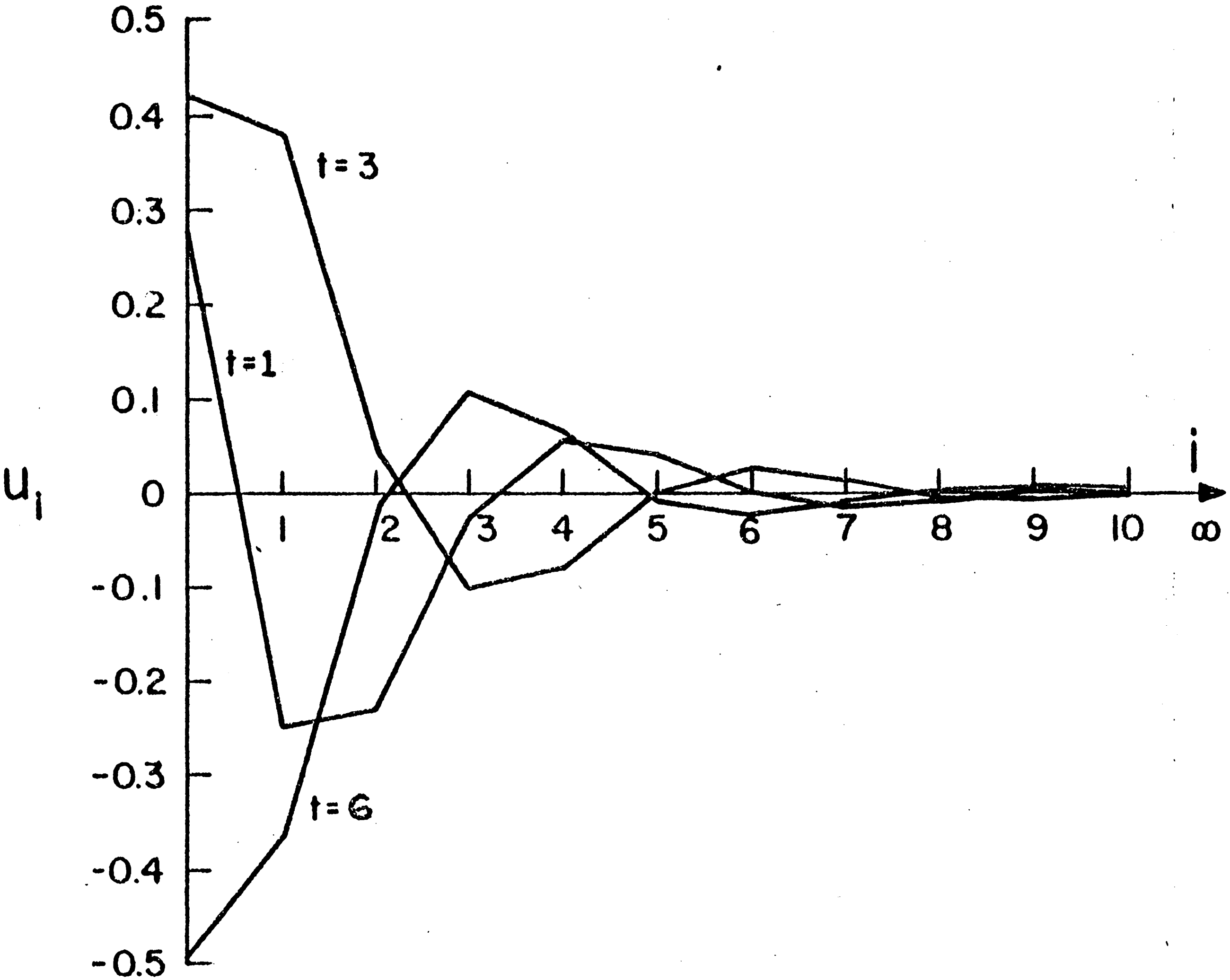


FIGURE 2.6.a   The Displacement Pattern

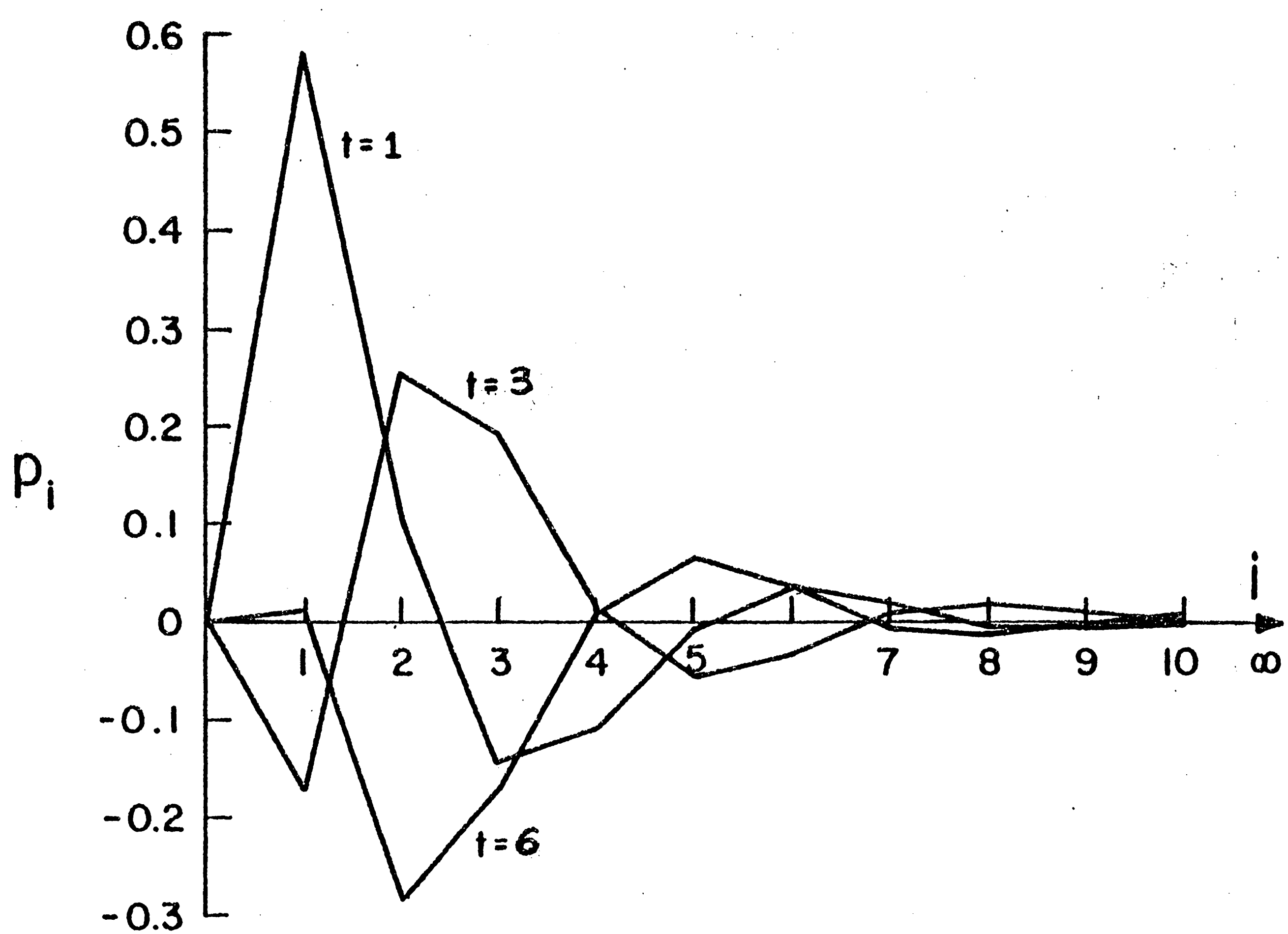


FIGURE 2.6.b The Reaction Pattern

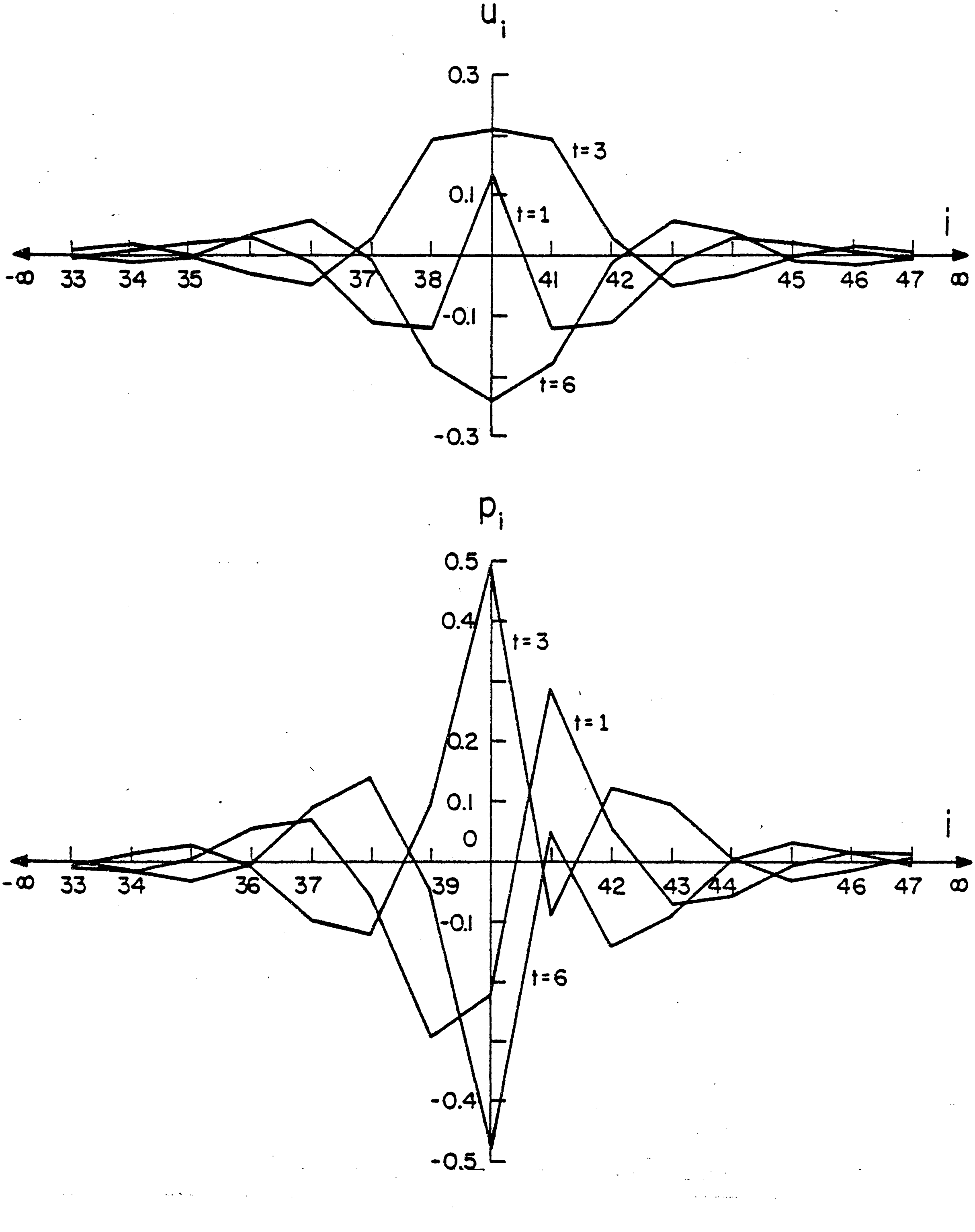


FIGURE 2.7 The Response

Hence,

$$u_i = \frac{1}{2}\,(z_1^i + z_2^i)\,u_0 \qquad (2.51a)$$

$$i \leqslant l$$

$$p_i = -\,\frac{\beta}{2}\,(\alpha^2 - 1)^{1/2}\,(z_1^i - z_2^i)\,u_0 \qquad (2.51b)$$

$$u_i = \frac{z_2^{i-l}}{\beta(\alpha^2 - 1)^{1/2}}\,(p_l + f^{*'}_{l-1}) \qquad (2.51c)$$

$$i \geqslant l$$

$$p_i = z_2^{i-l}\,(p_l + f^{*'}_{l-1}) \qquad (2.51d)$$

where, for $i = l$ we must add $f^{*'}_{l-1}$ to the left-hand side of Eq.(2.51d) . Furthermore, from Eq.(2.51c) it follows that

$$u_0 = \frac{z_2^{-l}}{\beta(\alpha^2 - 1)^{1/2}}\,f^{*'}_{l-1} \qquad (2.52)$$

It is easy to see that $l = 2$ and $f^{*'}_{l-1} = 1$. Substituting these values into Eqs.(2.51) and using Eqs.(2.49) and (1.74), we obtain the response as displayed in Fig. 2.8 .

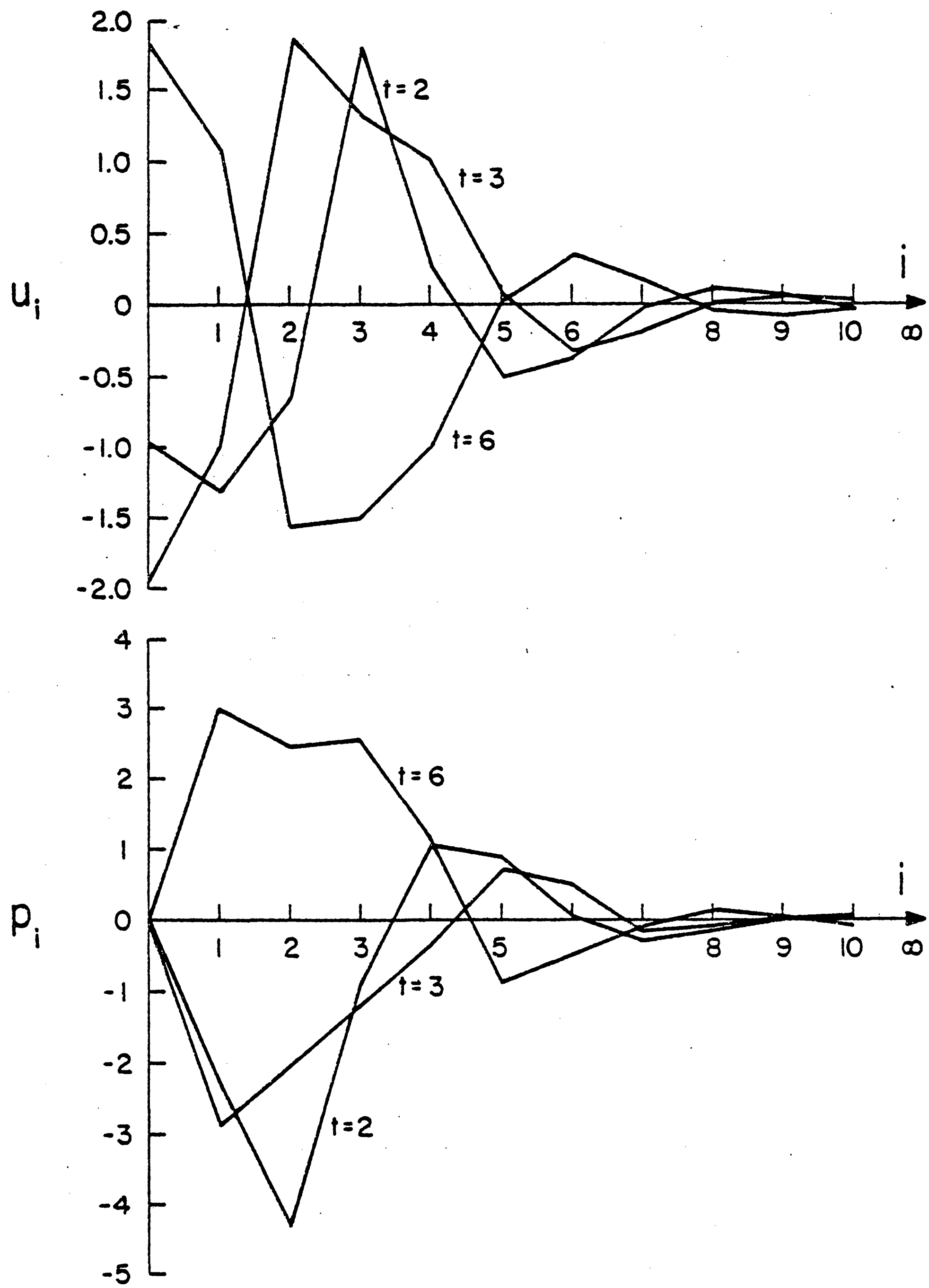


FIGURE 2.8 The Response

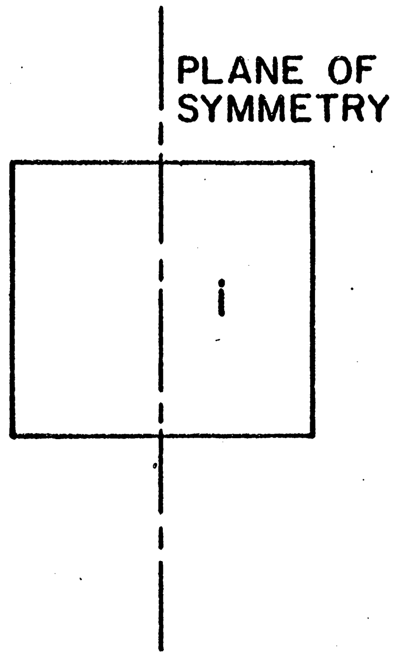


FIGURE 2.9 The Substructure

5. PERIODIC STRUCTURES WITH SYMMETRIC SUBSTRUCTURES

When the substructure has a plane of symmetry perpendicular to the i- axis, additional simplifications can be accomplished. Indeed, the impedance matrix for a substructure, such as depicted in Fig. 2.9 , can be written as follows

$$Z = \begin{bmatrix} z_{LL,ss} & z_{LL,sa} & z_{LR,ss} & z_{LR,sa} \\ z_{LL,as} & z_{LL,aa} & z_{LR,as} & z_{LR,aa} \\ & & & \\ z_{RL,ss} & z_{RL,sa} & z_{RR,ss} & z_{RR,sa} \\ z_{RL,as} & z_{RL,aa} & z_{RR,as} & z_{RR,aa} \end{bmatrix} \tag{2.53}$$

where the subscripts s and a , mean symmetric and anti-symmetric , respectively. Due to this symmetry and anti-symmetry,we must have

$$\begin{gathered} z_{LL,ss} = z_{RR,ss},\ z_{LL,aa} = z_{RR,aa},\ z_{LL,sa} = -z_{RR,sa},\ z_{LL,as} = -z_{RR,as} \\ z_{RL,ss} = z_{LR,ss},\ z_{Rl,aa} = z_{LR,aa},\ z_{RL,as} = -z_{LR,as},\ z_{RL,sa} = -z_{LR,sa} \end{gathered} \tag{2.54}$$

and since Z is still symmetric we also have

$$\begin{gathered} z_{LL,ss} = z^T_{LL,ss},\ z_{LL,aa} = z^T_{LL,aa},\ z_{RR,ss} = z^T_{RR,ss},\ z_{RR,aa} = z^T_{RR,aa} \\ z_{LL,sa} = z^T_{LL,as},\ z_{LR,ss} = z^T_{RL,ss},\ z_{LR,sa} = z^T_{RL,as},\ z_{LR,as} = z^T_{RL,sa} \end{gathered} \tag{2.55}$$

From Eqs.(2.53-55) then it follows

$$Z = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{12}^T & z_{22} & -z_{14}^T & z_{24} \\ z_{13} & -z_{14} & z_{11} & -z_{12} \\ z_{14}^T & z_{24} & -z_{12}^T & z_{22} \end{bmatrix} \tag{2.56}$$

where, to simplify the notation, we put

$$\begin{aligned} z_{LL,ss} &= z_{11}\ ,\ z_{LL,sa} = z_{12}\ ,\ z_{LR,ss} = z_{13} \\ z_{LR,sa} &= z_{14}\ ,\ z_{LL,aa} = z_{22}\ ,\ z_{LR,aa} = z_{24} \end{aligned} \tag{2.57}$$

Note that z_{11} , z_{22} , z_{13} , and z_{24} are square and symmetric, whereas z_{12} and z_{14} may be square or rectangular matrices depending of the dimensions of $\underset{\sim}{u}_{i,Ls}$ and $\underset{\sim}{u}_{i,La}$ where

$$\underset{\sim}{u}_i = \begin{bmatrix} \underset{\sim}{u}_{i,Ls} \\ \hline \underset{\sim}{u}_{i,La} \\ \hline \underset{\sim}{u}_{i,Rs} \\ \hline \underset{\sim}{u}_{i,Ra} \end{bmatrix} \tag{2.58}$$

is the displacement vector.

Next, let us consider Eq.(1.12) and derive the matrix A . It is not difficult to show that A has the following form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & -a_{14}^T & a_{24} \\ a_{31} & -a_{41}^T & a_{11} & -a_{21}^T \\ a_{41} & a_{42} & -a_{12}^T & a_{22}^T \end{bmatrix} \tag{2.59}$$

where a_{13} , a_{24} , a_{31} and a_{42} are square and symmetric and

$$\begin{aligned} &\xi = z_{24} + z_{14}^T(z_{13}^{-1} z_{14}) \quad , \quad a_{24} = \xi^{-1} \\ &a_{14} = -(z_{13}^{-1} z_{14}) \xi^{-1} \quad , \quad a_{13} = z_{13}^{-1} - (z_{13}^{-1} z_{14}) \xi^{-1}(z_{14}^T z_{13}^{-1}) \\ &a_{11} = -a_{13} z_{11} - a_{14} z_{12}^T \quad , \quad a_{12} = - a_{13} z_{12} - a_{14} z_{22} \\ &a_{21} = a_{14}^T z_{11} - a_{24} z_{12}^T \quad , \quad a_{22} = a_{14}^T z_{12} - a_{24} z_{22} \\ &a_{31} = - z_{11} a_{11} + z_{12} a_{21} - z_{13} \quad , \quad a_{41} = z_{12}^T a_{11} - z_{22} a_{21} - z_{14}^T \\ &a_{42} = z_{12}^T a_{12} - z_{22} a_{22} - z_{24} \end{aligned} \tag{2.60}$$

From Eq.(2.59) it is seen that only 10 out of 16 submatrices must be computed. In addition, the properties established for the general case are still valid. For example, from Eq.(1.46) it follows that

$$A^{-1} = \begin{bmatrix} a_{11} & -a_{12} & -a_{13} & a_{14} \\ -a_{21} & a_{22} & -a_{14}^T & -a_{24} \\ -a_{31} & -a_{41}^T & a_{11}^T & a_{21}^T \\ a_{41} & -a_{42} & a_{12}^T & a_{22}^T \end{bmatrix} \tag{2.61}$$

A similar computational saving can be obtained for the calculation of the matrices H_l $(l=0,1, \ldots ,2n-1)$. Indeed, because k symmetric substructures constitute one large symmetric substructure with A^k as the transfer matrix, we conclude that A^k has the same structure as A . Also, any linear combination of powers of A must have the same structure. Again, it should be pointed out, that all the properties of the general case are still valid, such as Eqs.(1.53) and (1.59) .

Another important simplification can be obtained for the computation of the eigenvalues of matrix A . The eigenvalue problem can be written as

$$A\,\underset{\sim}{x} = \lambda\,\underset{\sim}{x} \tag{2.62}$$

Because A and A^{-1} have reciprocal eigenvalues and the same eigenvectors we can write

$$A^{-1}\,\underset{\sim}{x} = \frac{1}{\lambda}\,\underset{\sim}{x} \tag{2.63}$$

Adding Eqs.(2.62) and (2.63) yields

$$(A + A^{-1}) \underset{\sim}{x} = (\lambda + \frac{1}{\lambda}) \underset{\sim}{x} = \mu \underset{\sim}{x} \tag{2.64}$$

where

$$\mu = \lambda + \frac{1}{\lambda} \tag{2.65}$$

and from Eqs.(2.59) and (2.61)

$$A + A^{-1} = 2 \begin{bmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & -a_{14}^T & 0 \\ 0 & -a_{41}^T & a_{11}^T & 0 \\ a_{41} & 0 & 0 & a_{22}^T \end{bmatrix} \tag{2.66}$$

Next, let us apply following elementary similarity transformation

$$k_{13} = k_{13}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.67}$$

from which it follows

$$k_{13}^{-1} (A + A^{-1}) k_{13} = 2 \begin{bmatrix} a_{11}^T & -a_{41}^T & 0 & 0 \\ -a_{14}^T & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{14} \\ 0 & 0 & a_{41} & a_{22}^T \end{bmatrix} \tag{2.68}$$

Because similarity transformations do not change the eigenvalues of the matrix,we conclude that the eigenvalues of A are also given by following characteristic equation

$$\det\begin{bmatrix} 2a_{11}^{T} - \mu 1 & -2a_{41}^{T} & 0 & 0 \\ -2a_{14}^{T} & 2a_{22} - \mu 1 & 0 & 0 \\ 0 & 0 & 2a_{11} - \mu 1 & 2a_{14} \\ 0 & 0 & 2a_{41} & 2a_{22}^{T} - \mu 1 \end{bmatrix} = 0 \qquad (2.69)$$

or

$$\det\begin{bmatrix} 2a_{11}^{T} - \mu 1 & -2a_{41}^{T} \\ -2a_{14}^{T} & 2a_{22} - \mu 1 \end{bmatrix} \times \det\begin{bmatrix} 2a_{11} - \mu 1 & 2a_{14} \\ 2a_{41} & 2a_{22}^{T} - \mu 1 \end{bmatrix} = 0 \qquad (2.70)$$

and because the determinant of a matrix is equal to the determinant of its transpose ,we can write

$$\det\begin{bmatrix} 2a_{11} - \mu 1 & -2a_{14} \\ -2a_{41} & 2a_{22}^{T} - \mu 1 \end{bmatrix} \times \det\begin{bmatrix} 2a_{11} - \mu 1 & 2a_{14} \\ 2a_{41} & 2a_{22}^{T} - \mu 1 \end{bmatrix} = 0 \qquad (2.71)$$

Finally, multiplying the first column of the first determinant by -1 and then the first row by -1,we obtain

$$\det\begin{bmatrix} 2a_{11} - \mu 1 & 2a_{14} \\ 2a_{41} & 2a_{22}^T - \mu 1 \end{bmatrix} \times \det\begin{bmatrix} 2a_{11} - \mu 1 & 2a_{14} \\ 2a_{41} & 2a_{22}^T - \mu 1 \end{bmatrix} = 0 \quad (2.72)$$

being the product of two identical determinants, so that the problem reduces to finding the eigenvalues $2\,\mu_k$ (k=1,2, ... n) of the matrix

$$\begin{bmatrix} a_{11} & a_{14} \\ a_{41} & a_{22}^T \end{bmatrix} \quad (2.73)$$

and then solving the equations

$$\lambda^2 - \mu_k \lambda + 1 = 0 \;, \quad k = 1,2, \dots ,n$$

for one root.

All these properties can be incorporated into a computer program and reduce the excecution time considerably. However, because of increasing compilation time the effort becomes only worthwhile when the number of degrees of freedom 2n of the substructure becomes fairly large.

CHAPTER III

RESPONSE OF ALMOST PERIODIC STRUCTURES

1. INTRODUCTION

In chapter I we developed a method to analyze strictly periodic structures. All the substructures were identical and had the same transfer matrix. An almost periodic structure is defined as a structure consisting of almost identical substructures. An example is given by the spinal column in which the geometrical dimensions of a typical vertebra vary slightly from those of another. These variations are not confined to geometry alone but apply also to other system parameters such as stiffness and mass distributions.

In this chapter we shall develop an algorithm capable of analyzing almost periodic structures using a perturbation technique. Finally, we shall apply the theory to an illustrative example and compare the results with the unperturbed solution.

2. PERTURBATION EQUATIONS FOR ALMOST PERIODIC STRUCTURES

Let us consider an almost periodic structure (Fig.3.1) and assume that its harmonic response is given by the vector difference equation

$$\underset{\sim}{x}_{i+1} = A_i \underset{\sim}{x}_i + \underset{\sim}{f}_i^* \tag{3.1}$$

where this time the matrix A_i is not constant but depends on i .

Because the structure is almost periodic, we can assume the transfer matrix to have the following form

$$A_i = A^{(0)} + \varepsilon A_i^{(1)} + \varepsilon A_i^{(2)} + \dots \tag{3.2}$$

in which $A^{(0)}$ is a constant transfer matrix and ε is a small parameter. We shall take $A^{(0)}$ as the transfer matrix corresponding to the left-end structure,

Next, in accordance with the basic perturbation technique (Ref. 16), we can write the solution of Eq.(3.1) in the form

$$\underset{\sim}{x}_i = \underset{\sim}{x}_i^{(0)} + \varepsilon \underset{\sim}{x}_i^{(1)} + \varepsilon^2 \underset{\sim}{x}_i^{(2)} + \dots \tag{3.3}$$

Introducing Eqs.(3.2) and (3.3) into Eq.(3.1), we can write

$$\begin{gathered}\underset{\sim}{x}_{i+1}^{(0)} + \varepsilon \underset{\sim}{x}_{i+1}^{(1)} + \varepsilon^2 \underset{\sim}{x}_{i+1}^{(2)} + \dots = \\ (A^{(0)} + \varepsilon A_i^{(1)} + \varepsilon^2 A_i^{(2)} + \dots)(\underset{\sim}{x}_i^{(0)} + \varepsilon \underset{\sim}{x}_i^{(1)} + \varepsilon^2 \underset{\sim}{x}_i^{(2)} + \dots) + \underset{\sim}{f}_i^*\end{gathered} \tag{3.4}$$

so that, equating coefficients of like powers in ε , we obtain the set of equations

$$\underset{\sim}{x}_{i+1}^{(l)} = A^{(0)} \underset{\sim}{x}_{i}^{(l)} + \underset{\sim}{f}_{i}^{(l)} \quad , \qquad l = 0,1,2, \ldots \tag{3.5}$$

where

$$\underset{\sim}{f}_{i}^{(0)} = \underset{\sim}{f}_{i}^{*} \quad , \quad \underset{\sim}{f}_{i}^{(l)} = \sum_{k=1}^{l} A_{i}^{(k)} \underset{\sim}{x}_{i}^{(l-k)} \quad , \quad l = 1,2, \ldots \tag{3.6}$$

Equations (3.5) represent difference equations associated with strictly periodic systems. Their solutions can be obtained sequentially.

3. RESPONSE OF ALMOST PERIODIC STRUCTURES

To obtain the response of almost periodic structures, we must solve Eqs.(3.5) sequentially. Indeed, letting $l=0$ in Eqs.(3.5), we obtain

$$\underset{\sim}{x}_{i+1} = A^{(0)} \underset{\sim}{x}_i^{(0)} + \underset{\sim}{f}_i^{(0)} \tag{3.7}$$

which is a matrix difference equation similar to Eq.(1.11). Hence, the solution of Eq.(3.7) is given by Eqs.(1.21) and (1.31), or

$$\underset{\sim}{x}_i^{(0)} = \sum_{j=1}^{2n} X_j \left(z_j^i \underset{\sim}{x}_0^{(0)} + \sum_{k=0}^{i-1} z_j^{i-k-1} \underset{\sim}{f}_k^{(0)} \right) \tag{3.8}$$

The remaining equations in (3.5) can be solved in an entirely similar manner, by letting $l=1,2,\ldots$ . . Indeed, for $l=1$, we obtain

$$\underset{\sim}{x}_{i+1}^{(1)} = A^{(0)} \underset{\sim}{x}_i^{(1)} + \underset{\sim}{f}_i^{(1)} \tag{3.9}$$

in which, from Eq.(3.6)

$$\underset{\sim}{f}_i^{(1)} = A_i^{(1)} \underset{\sim}{x}_i^{(0)} \tag{3.10}$$

The solution of Eq.(3.9) is

$$\underset{\sim}{x}_i^{(1)} = \sum_{j=1}^{2n} X_j \left(z_j^i \underset{\sim}{x}_0^{(1)} + \sum_{k=0}^{i-1} z_j^{i-k-1} \underset{\sim}{f}_k^{(1)} \right) \tag{3.11}$$

where $\underset{\sim}{f}_k^{(1)}$ is now a known quantity.

The higher-order approximations can be obtained in the same manner. However, in many problems the first-order approximation suffices, so that we can write the solution of Eq.(3.1) as follows

$$\underset{\sim}{x}_i \cong \underset{\sim}{x}_i^{(0)} + \varepsilon\, \underset{\sim}{x}_i^{(1)} \quad , \quad i = 0,1, \ldots \tag{3.12}$$

where $\underset{\sim}{x}_i^{(0)}$ and $\underset{\sim}{x}_i^{(1)}$ are given by Eqs.(3.9) and (3.11), respectively. Note that, for the first-order approximation, we only need $A^{(0)}$ and $A_i^{(1)}$ in Eq.(3.2).

Next, let us consider Eq.(3.2). Because the substructure parameters occur explicitly in the matrices M_i , C_i and K_i and the impedance matrix depends on these matrices, we may wish to derive the matrices $A^{(0)}$ and $A_i^{(1)}$ (1=1,2, ...) in terms of the impedance matrix. To this end, let some or all substructure parameters vary in an arbitrary fashion. The only restriction is that the variations be small. Then, any parameter p_i (such as mass, length, cross-sectionsl area, Young's modulus,etc.) can be written as

$$\underset{\sim}{p}_i = \underset{\sim}{p}^{(0)} + \varepsilon\, \underset{\sim}{p}_i^{(1)} \tag{3.13}$$

where $p^{(0)}$ is the left-end substructure parameter. Using expansions (3.13) it is easy to see that the impedance matrix Z_i can be written as

$$Z_i = Z^{(0)} + \varepsilon\, Z_i^{(1)} + \varepsilon^2\, Z_i^{(2)} + \ldots \tag{3.14}$$

where $Z^{(0)}$ is the left end substructure impedance matrix. Furthermore, the transfer matrix A_i can be written as

$$A_i = \left[\begin{array}{c|c} A_{i,LL} & A_{i,LR} \\ \hline A_{i,RL} & A_{i,RR} \end{array}\right] \tag{3.15}$$

It is now possible to substitute Eq.(3.14) into Eq.(1.12), where the submatrices z_{LL} , ... now depend on i, and obtain expressions for $A_{i,LL}$, ... , in terms of ε^0, ε^1, ε^2 , ... We shall restrict ourselves to the first-order approximation and write Eq.(3.14) as

$$Z_i \cong Z^{(0)} + \varepsilon Z_i^{(1)} \tag{3.16}$$

from which it follows that

$$z_{i,LR} \cong z_{LR}^{(0)} + \varepsilon z_{i,LR}^{(1)} = z_{LR}^{(0)} (1 + \varepsilon z_{LR}^{(0)-1} z_{i,LR}^{(1)})$$

so that

$$z_{i,LR}^{-1} \cong (1 + \varepsilon z_{LR}^{(0)-1} z_{i,LR}^{(1)})^{-1} z_{LR}^{(0)-1}$$

or

$$A_{i,LR} \cong z_{LR}^{(0)-1} - \varepsilon z_{LR}^{(0)-1} z_{i,LR}^{(1)} z_{LR}^{(0)-1} \tag{3.17}$$

Furthermore,

$$- z_{i,LR}^{-1}\, z_{i,LL} \;\tilde{\equiv}\; -\,(z_{LR}^{(0)-1} - \varepsilon z_{LR}^{(0)-1}\, z_{i,LR}^{(1)}\, z_{LR}^{(0)-1})(z_{LL}^{(0)} + \varepsilon z_{i,LL}^{(1)})$$

so that

$$A_{i,LL} \;\tilde{\equiv}\; - z_{LR}^{(0)-1}\, z_{LL}^{(0)} + \varepsilon(z_{LR}^{(0)-1}\, z_{i,LR}^{(1)}\, z_{LL}^{(0)-1} - z_{LR}^{(0)-1}\, z_{i,LL}^{(1)}) \quad (3.18)$$

Similarly

$$A_{i,RR} \;\tilde{\equiv}\; - z_{RR}^{(0)}\, z_{LR}^{(0)-1} + \varepsilon(z_{RR}^{(0)}\, z_{LR}^{(0)-1}\, z_{i,LR}^{(1)}\, z_{LR}^{(0)-1} - z_{i,RR}^{(1)} z_{LR}^{(0)-1}) \quad (3.19)$$

and

$$\begin{aligned} A_{i,RL} \;\tilde{\equiv}\; & z_{RR}^{(0)}\, z_{LR}^{(0)-1}\, z_{LL}^{(0)} - z_{RL}^{(0)} + \varepsilon\, (z_{i,RR}^{(1)}\, z_{LR}^{(0)-1}\, z_{LL}^{(0)} \\ & - z_{RR}^{(0)}\, z_{LR}^{(0)-1}\, z_{i,LR}^{(1)}\, z_{LR}^{(0)-1}\, z_{LL}^{(0)} + z_{RR}^{(0)}\, z_{LR}^{(0)-1}\, z_{i.LL}^{(1)} \\ & - z_{i,RL}^{(1)}\,) \end{aligned} \quad (3.20)$$

Introducing Eqs.(3.17-20) into Eq.(3.15) yields the desired transfer matrix A_i .

4. ILLUSTRATIVE EXAMPLE

Let us consider the almost periodic structure in axial vibration shown in Fig.3.2. The 20 substructures can be identified as single-degree-of-freedom damped systems. It is easy to verify that the mass matrix, damping matrix and stiffness matrix for a typical substructure have the form

$$M_i = m_i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_i = c_i \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad K_i = k_i \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.21}$$

Furthermore, for simplicity, we take M_i, C_i and K_i to be linear functions of the substructure index i

$$m_i = m_0 - \varepsilon\, m_0\, i \tag{3.22a}$$

$$c_i = c_0 - \varepsilon\, c_0\, i \tag{3.22b}$$

$$k_i = k_0 - \varepsilon\, k_0\, i \tag{3.22c}$$

where

$$\varepsilon = \frac{0.1}{19} \tag{3.23}$$

so that, the parameters of the right end substructure are 90 % of those of the left-end substructure.

From Eqs.(3.21-22) and Eq.(1.5) it follows that

$$Z^{(0)} = \begin{bmatrix} -\omega^2 m_0 + j\omega\, c_0 + k_0 & -(j\omega\, c_0 + k_0) \\ -(j\,\omega c_0 + k_0) & -\omega^2 m_0 + j\omega\, c_0 + k_0 \end{bmatrix} \tag{3.24}$$

Similarly

$$Z_i^{(1)} = -\, i\, Z^{(0)} \tag{3.25}$$

So that, from Eq.(3.15) and Eqs.(3.24-25) it follows that

$$A^{(0)} = \begin{bmatrix} \alpha_0 & -1/\beta_0 \\ \beta_0(1-\alpha_0^2) & \alpha_0 \end{bmatrix} \tag{3.26}$$

and

$$A_i^{(1)} = i \begin{bmatrix} 0 & -1/\beta_0 \\ -\beta_0(1-\alpha_0^2) & 0 \end{bmatrix} \tag{3.27}$$

where

$$\alpha_0 = \frac{-\omega^2 m_0 + j\omega\, c_0 + k_0}{j\omega\, c_0 + k_0} \quad , \quad \beta_0 = j\omega\, c_0 + k_0 \tag{3.28}$$

Note that, whenever $Z_i^{(1)} = -\,i\,Z^{(0)}$, we have

$$A_i^{(1)} = i \begin{bmatrix} 0 & A_{i,LR}^{(0)} \\ -A_{i,RL}^{(0)} & 0 \end{bmatrix} \tag{3.29}$$

From Eq.(3.8) we can solve for $\underset{\sim}{x}_i^{(0)}$ and then from Eq.(3.11) we solve for $\underset{\sim}{x}_i^{(1)}$. Moreover, Eqs.(3.10) and (3.27) yield

$$\underset{\sim}{f}_i^{(1)} = i \begin{bmatrix} 0 & -1/\beta_0 \\ -\beta_0(1-\alpha_0^2) & 0 \end{bmatrix} \underset{\sim}{x}_i^{(0)} \tag{3.30}$$

Let us now derive the general solution for the almost periodic structure in terms of the system parameters. The characteristic equation (1.41) becomes

$$z^2 - 2\,\alpha_0\,z + 1 = 0 \tag{3.31}$$

having roots

$$z_1 = \alpha_0 + (\alpha_0^2 - 1)^{1/2} \quad , \quad z_2 = \alpha_0 - (\alpha_0^2 - 1)^{1/2} \tag{3.32}$$

Furthermore, from Eq.(1.29) it follows that

$$X_1 = \frac{1}{2}\begin{bmatrix} 1 & -\dfrac{1}{\beta_0(\alpha_0^2 - 1)^{1/2}} \\ -\beta_0(\alpha_0^2 - 1)^{1/2} & 1 \end{bmatrix}$$

$$X_2 = \frac{1}{2}\begin{bmatrix} 1 & \dfrac{1}{\beta_0(\alpha_0^2 - 1)^{1/2}} \\ \beta_0(\alpha_0^2 - 1)^{1/2} & 1 \end{bmatrix} \tag{3.33}$$

Inserting Eqs.(3.34) into Eq.(1.31), we obtain the fundamental matrix

$$\Phi_i = \frac{1}{2}\begin{bmatrix} z_1^i + z_2^i & -\dfrac{z_1^i - z_2^i}{\beta_0(\alpha_0^2 - 1)^{1/2}} \\ -\beta_0(\alpha_0^2 - 1)^{1/2}(z_1^i - z_2^i) & z_1^i + z_2^i \end{bmatrix} \tag{3.34}$$

Hence, using Eq.(3.8), we obtain a general solution for the zero-th order response in the form

$$u_i^{(0)} = \frac{1}{2}(z_1^i + z_2^i)u_0^{(0)} - \frac{z_1^i - z_2^i}{2\beta_0(\alpha_0^2 - 1)^{1/2}}\left(p_0^{(0)} + f_0^{(0)} \right)$$

$$- \frac{1}{2\beta_0(\alpha_0^2 - 1)^{1/2}} \sum_{k=0}^{i-1} (z_1^{i-k-1} - z_2^{i-k-1}) f_k^{(0)} \tag{3.35}$$

$$p_i^{(0)} = -\frac{1}{2}\,\beta_0(\alpha_0^2 - 1)^{1/2}\,(z_1^i - z_2^i)\,u_0^{(0)} + \frac{1}{2}\,(z_1^i + z_2^i)(p_0^{(0)} + f_0^{(0)})$$

$$+ \frac{1}{2}\sum_{k=0}^{i-1}(z_1^{i-k-1} + z_2^{i-k-1})\,f_k^{(0)} - f_{i,L}^{(0)}$$

Similarly, from Eqs.(3.9) and (3.30) we obtain

$$u_i^{(1)} = \frac{1}{2}\,(z_1^i + z_2^i)\,u_0^{(1)} - \frac{1}{2\beta_0\,(\alpha_0^2 - 1)^{1/2}}\,(z_1^i - z_2^i\,)\,p_0^{(1)}$$

$$+ \frac{1}{2\beta_0^2\,(\alpha_0^2 - 1)^{1/2}}\sum_{k=0}^{i-1} k(z_1^{i-k-1} - z_2^{i-k-1})(p_k^{(0)} + f_{k,L}^{(0)}\,)$$

(3.36)

$$p_i^{(1)} = -\frac{1}{2}\,\beta_0(\alpha_0^2 - 1)^{1/2}\,(z_1^i - z_2^i\,)\,u_0^{(1)} + \frac{1}{2}\,(z_1^i + z_2^i)\,p_0^{(1)}$$

$$- \frac{1}{2}\,\beta_0(1-\alpha_0^2)\sum_{k=0}^{i-1} k(z_1^{i-k-1} + z_2^{i-k-1})\,u_k^{(0)}$$

Equations (3.35-36) can be solved sequentially once the boundary conditions, the external forces and the system parameters are specified. Finally, Eq.(3.23) together with Eq.(3.12) and Eqs.(3.35-36) yield the final solution.

Next, let us introduce the following values for the system parameters

$$\omega_0 = m_0 = c_0 = k_0 = 1 \tag{3.37}$$

Then, from Eq.(3.28), it follows that

$$\alpha_0 = \frac{1}{2}(1 + j) \quad , \quad \beta_0 = 1 + j \tag{3.38}$$

and from Eq.(3.32) we obtain the following eigenvalues

$$z_1 = 0.74293 + 1.52908\, j \ , \ z_2 = 0.25707 - 0.52908\, j \tag{3.39}$$

Next, from Fig.3.2 we conclude that the external load has the form

$$\underset{\sim}{f}_i^{(0)} = \underset{\sim}{f}_i^{*} = \begin{bmatrix} 0 \\ e^{-0.1(i+1)} \end{bmatrix} \tag{3.40}$$

Furthermore, let us assume that we have 20 substructures and that both ends of the system are free. Then, we have the following relations

$$p_0^{(0)} + f_0^{(0)} = 0 + e^0 = 1 \quad , \quad p_{20}^{(0)} = 0 \tag{3.41}$$

and

$$p_0^{(1)} = 0 \quad , \quad p_{20}^{(1)} = 0 \tag{3.42}$$

so that, from Eqs.(3.35) and (3.41) it follows that

$$u_0^{(0)} = \frac{1}{\beta_0(\alpha_0^2 - 1)^{1/2}\,(z_1^{20} - z_2^{20})} \{ z_1^{20} + z_2^{20}$$

$$+ \sum_{k=0}^{19} (z_1^{19-k} + z_2^{19-k})\, e^{-0.1(k+1)} - e^{-2.0} \} \qquad (3.43)$$

Moreover, from Eqs.(3.36) and (3.42), it also folljws that

$$u_0^{(1)} = \frac{(\alpha_0^2 - 1)^{1/2}}{z_1^{19} - z_2^{19}} \sum_{k=0}^{19} k\,(z_1^{19-k} + z_2^{19-k})\, u_k^{(0)} \qquad (3.44)$$

This concludes the illustrative example. Numerical results are displayed in Fig.3.3 where for comparison purposes we also indicated the unperturbed solution.

5. SUMMARY AND CONCLUSIONS

In this chapter we derived an efficient method for finding the response of an almost periodic structure subject to harmonic external forces. The algorithm represents a natural extension of the method for unperturbed structures developed in chapter I. As such, it retains all the computational advantages of the latter and reduces to it when $\varepsilon = 0$. Moreover, the procedure can accommodate variations in all the system parameters as long as the variations remain small. The method should find its application in many practical problems, such as the response of an aircraft fuselage or of a human spinal column.

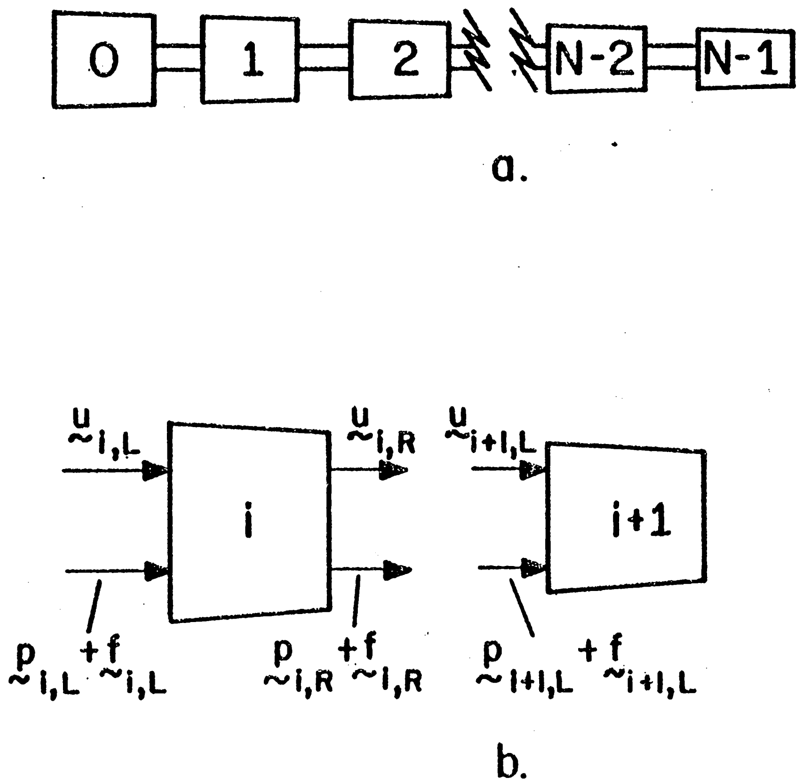


FIGURE 3.1 a. The Almost Periodic Structure

b. The Substructure

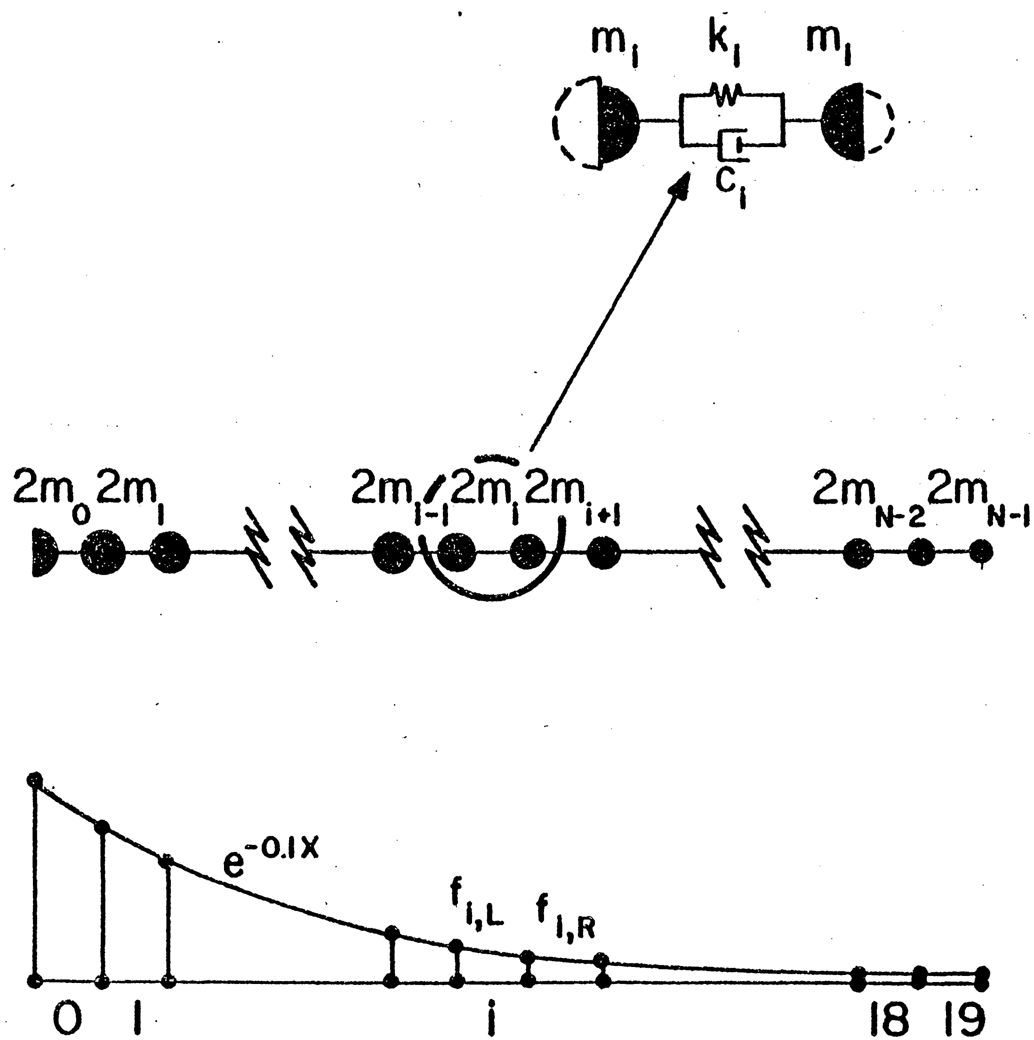


FIGURE 3.2 The Almost Periodic Structure

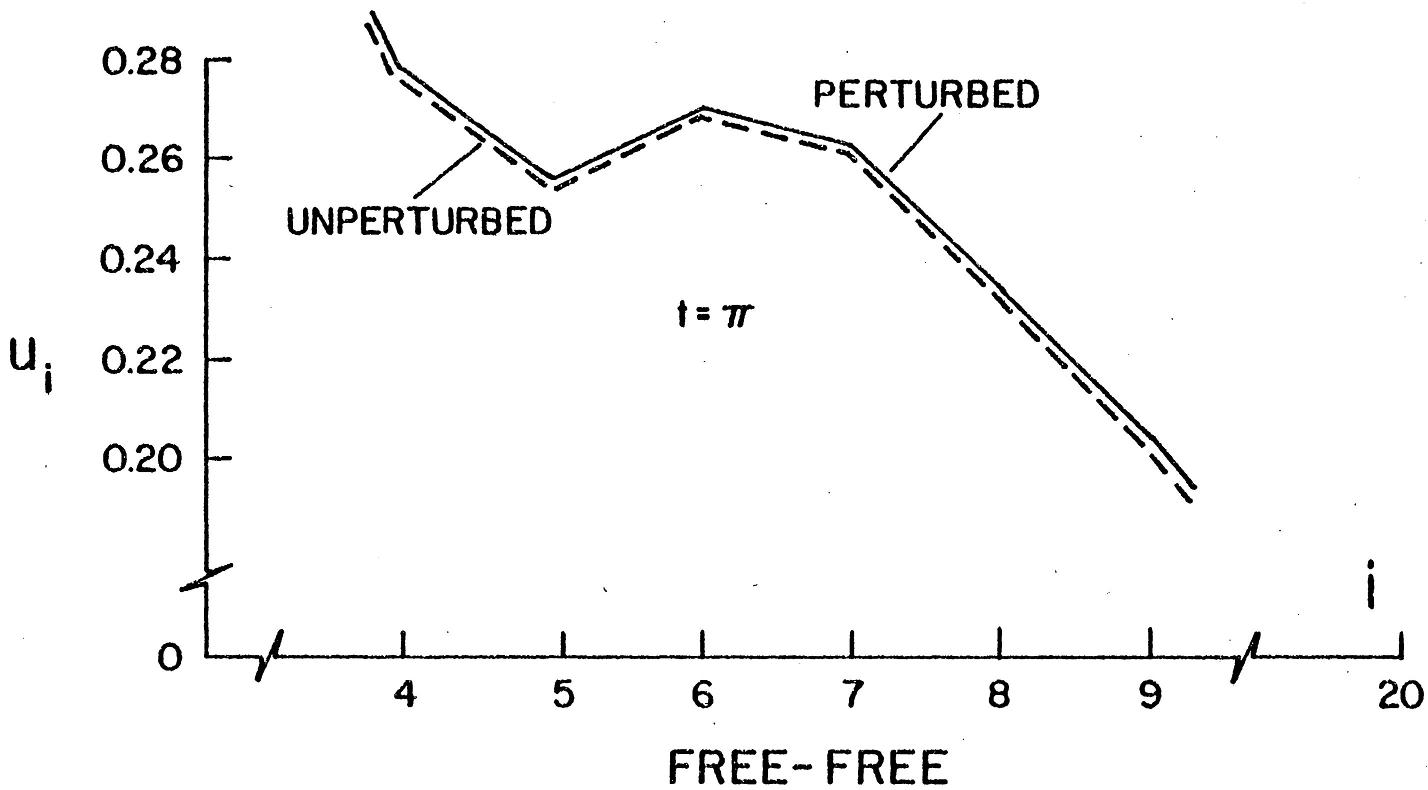


FIGURE 3.3 The Response

CHAPTER IV

RESPONSE OF PERIODIC STRUCTURES BY MODAL ANALYSIS

1. INTRODUCTION

In this chapter, an attempt is made to analyze a finite periodic structure using modal analysis. This approach not only enables us to determine the eigenvalues of the complete structure in a systematic way, but also to derive the response of the system to arbitrary excitations which is not possible with the fundamental theory of chapter I.

Nonharmonic loads have been studied to date only for single-degree-of-freedom semi-infinite structures, as for example in Ref.5. In Ref.8 an attempt is made to determine the " pseudo natural frequencies " of an infinite engine duct. Apparently, these natural frequencies were obtai by observing resonance peaks in the frequency-response curve using trial and error. This is a time-consuming process, so that a more direct method is highly desirable.

2. FORMULATION OF THE PROBLEM

Let us consider a finite periodic structure consisting of a set of subsystems i (i = 0, 1, ... , N-1) , subject to external excitations, not necessarily harmonic. In accordance with modal analysis, we first consider the case of undamped free vibration to determine the system natural frequencies and modes. To this end, let us assume harmonic motion and derive a matrix difference equation similar to Eq.(1.11)

$$\underset{\sim}{x}_{i+1} = A \underset{\sim}{x}_i \quad , \qquad i = 0, 1, \ldots , N-1 \tag{4.1}$$

where $\underset{\sim}{x}_i$ is a "state vector" given by

$$\underset{\sim}{x}_i = \left[\begin{array}{c} \underset{\sim}{u}_{i,L} \\ \hline \underset{\sim}{p}_{i,L} \end{array}\right] \tag{4.2}$$

in which the vector $\underset{\sim}{u}_{i,L}$ represents the amplitudes of the harmonic displacements at the left boundary of subsystem i, whereas $\underset{\sim}{p}_{i,L}$ is the amplitude of the corresponding internal force vector (Fig.4.1). The transfer matrix A is given by

$$A = \left[\begin{array}{c|c} - z_{LR}^{-1} z_{LL} & z_{LR}^{-1} \\ \hline z_{RR} z_{LR}^{-1} z_{LL} - z_{RL} & - z_{RR} z_{LR}^{-1} \end{array}\right] \tag{4.3}$$

where the matrices z_{LL} , ... are submatrices of the impedance matrix Z for the subsystem i , namely,

$$Z = -\omega^2 M + K \tag{4.4}$$

in which M and K represent the mass matrix and stiffness matrix, respectively.

Examining Eq.(4.1) and using simple substitution, it can be verified that

$$\underset{\sim}{x}_i = \Phi_i \underset{\sim}{x}_0 \tag{4.5}$$

where $\Phi_i = A^i$ is known as the fundamental matrix of the system. Actually, Φ_i is not calculated by raising A to the power i, because of numerical difficulties as explained in chapter I. In fact, Φ_i is computed by the Leverrier algorithm as follows

$$\Phi_i = \sum_{j=1}^{2n} X_j z_j^i \tag{4.6}$$

in which the matrices X_j $(j = 1, 2, \ldots, 2n)$ are given by Eq.(1.29). Note that i only appears as a power of the eigenvalues $z_j (j=1,2,\ldots,2n)$ of the matrix A, so that the evaluation of Φ_i does not involve time consuming matrix multiplications.

To determine the frequency equation of the system, we let i=N in Eq.(4.5) and write

$$\underset{\sim}{x}_N = \left[\begin{array}{c|c} \Phi_{N11} & \Phi_{N12} \\ \hline \Phi_{N21} & \Phi_{N22} \end{array}\right] \underset{\sim}{x}_0 \tag{4.7}$$

Equations (4.5) and (4.7) will be used to determine both the eigenvalues and associated eigenvectors of the complete system, taking full advantage of the system periodicity.

3. THE EIGENVALUE PROBLEM

To obtain the frequency equation of the complete system, we use a technique similar to Holzer's method for torsional vibration[13]. To this end, let us write Eq.(4.7) in the more explicit form

$$\underset{\sim}{u}_N = \Phi_{N11}\,\underset{\sim}{u}_0 + \Phi_{N12}\,\underset{\sim}{p}_0 \tag{4.7a}$$

$$\underset{\sim}{p}_N = \Phi_{N21}\,\underset{\sim}{u}_0 + \Phi_{N22}\,\underset{\sim}{p}_0 \tag{4.7b}$$

where the subscript L in Eq.(4.2) has been dropped for simplicity.

At this point, we wish to consider various possible boundary conditions. When the left boundary is free and the right boundary is fixed, we have

$$\underset{\sim}{p}_0 = \underset{\sim}{0} \quad , \qquad \underset{\sim}{u}_N = \underset{\sim}{0} \tag{4.8}$$

Inserting Eqs.(4.8) into Eq.(4.7a), we obtain

$$\underset{\sim}{0} = \Phi_{N11}\,\underset{\sim}{u}_0 \tag{4.9}$$

Equation (4.9) represents a set of homogeneous algebraic equations, which has a nontrivial solution if and only if the determinant of the coefficients is equal to zero. Hence, we must have

$$\det\ \Phi_{N11} = 0 \tag{4.10}$$

which is recognized as the frequency equation for the free-fixed case. In a similar manner, for the fixed-free case, we obtain the frequency equation

$$\det \Phi_{N22} = 0 \tag{4.11}$$

for the free-free case the equation

$$\det \Phi_{N21} = 0 \tag{4.12}$$

and for the fixed-fixed case the equation

$$\det \Phi_{N12} = 0 \tag{4.13}$$

Note that Eqs.(4.10) and (4.11) are of order $N \times n$ in $\lambda = \omega^2$, where $2n$ is the number of degrees of freedom of the substructure. Equations (4.12) and (4.13) are of order $(N + 1) \times n$ and $(N - 1) \times n$, respectively. Note that the submatrices Φ_{Nrs} $(r,s = 1, 2)$ are evaluated by using Eq.(1.6), taking full advantage of the system periodicity.

It remains for us to obtain the roots of $\det \Phi_{Nrs} = 0$ $(r,s = 1, 2)$. However, it should be pointed out that $\det \Phi_{Nrs}$ $(r,s = 1,2)$ is not known explicitly in terms of $\lambda = \omega^2$, so that an algorithm which does not require explicit knowledge of the characteristic determinant is required. Such an algorithm is given in Ref.11 and will be referred to as Muller's method.

Let us denote the characteristic determinant associated with any of Eqs.(4.9-13) by $D(\lambda)$ and consider the solution of $D(\lambda) = 0$ by Muller's method. The method is iterative in nature and requires only one evaluation of $D(\lambda)$ per iteration, but no derivatives of $D(\lambda)$. To start the algorithm, three values of $D(\lambda)$ must be computed. Then, Lagrange's interpolation formula yields a quadratic passing through the three computed points. The zeros of this quadratic are calculated and a new $D(\lambda)$ is obtained for one of these roots. This new $D(\lambda)$ together with the two previously computed are used to repeat the procedure. Eventually it will converge to a root of $D(\lambda) = 0$. Following Ref.11 , we can write the algorithm as follows ,

i. Calculate $D(\lambda)$ for three arbitrary values of λ , e.g.

$$\lambda_1 = 2 \quad , \quad \lambda_2 = 1 \quad , \quad \lambda_3 = 0 \tag{4.14}$$

Note that by taking $\lambda_3 = 0$, we usually iterate to the fundamental frequency.

ii. Calculate

$$h_3 = \lambda_3 - \lambda_2 \quad , \quad h_2 = \lambda_2 - \lambda_1 \quad , \quad l_3 = \frac{h_3}{h_2} \tag{4.15}$$

iii. Start following scheme with $i = 3$

$$\delta_i = 1 + l_i \tag{4.16}$$

$$g_i = D(\lambda_{i-2})\, l_i^2 - D(\lambda_{i-1})\, \delta_i^2 + D(\lambda_i)(l_i + \delta_i) \tag{4.17}$$

$$l_{i+1} = \frac{-2\, D(\lambda_i)\, \delta_i}{g_i \pm \{g_i^2 - 4D(\lambda_i)\delta_i l_i \left[D(\lambda_{i-2}) l_i - D(\lambda_{i-1})\delta_i + D(\lambda_i)\right]\}^{1/2}} \tag{4.18}$$

where the sign in Eq.(4.18) is to be chosen so that the denominator has the greater magnitude. As a consequence, λ_{i+1} will be the root closest to λ_i , and is obtained from

$$\begin{aligned} h_{i+1} &= l_{i+1}\, h_i \\ \lambda_{i+1} &= \lambda_i + h_{i+1} \end{aligned} \tag{4.19}$$

where λ_{i+1} is the new approximation of the root λ_i .

The process is continued until a desired accuracy is obtained. The rate of convergence is very high and may be increased if an estimate of the eigenvalue is available[12]. Previously calculated roots may be removed from $D(\lambda)$ by dividing it by $\sum_{k=1}^{p} (\lambda - \lambda_k)$, where λ_k $(k=1,2,\ldots,p)$ are the first p roots. When the submatrix K_{LR} in Eq.(1.14) is singular, we must take $\lambda_3 \neq 0$ in Eq.(4.14). This is also the case when the structure has rigid-body degrees of freedom. Furthermore, it can happen that the square root in the denominator of Eq.(4.18) becomes imaginary. Geometrically, this means that the Lagrange curve does not intersect the λ- axis. In this case we must recompute $D(\lambda)$ for another value of λ.

Muller's algorithm permits us to compute the natural frequencies for the complete structure. There remains the problem of determining the associated natural modes. To this end, we assume that the vector $\underset{\sim}{x}_0$ corresponding to a given eigenvalue λ_k is known. Denoting this vector by $\underset{\sim}{x}_{0,k}$ and using Eq.(4.5), we can write

$$\underset{\sim}{x}_{i,k} = \Phi_i(\lambda_k)\, \underset{\sim}{x}_{0,k} \quad , \quad i = 1, 2, \ldots , N \tag{4.20}$$

where $\underset{\sim}{x}_{i,k}$ is the i^{th} state vector in the k^{th} mode. Equation (4.20) defines the k^{th} mode fully, in the sense that it gives not only the displacement pattern $\underset{\sim}{u}_{i,k}$ but also the internal forces $\underset{\sim}{p}_{i,k}$.

The above procedure is based on the knowledge of the vector $\underset{\sim}{x}_{0,k}$. This vector is not readily available but it can be determined with relative ease. This can be best demonstrated by way of a specific example, such as the free-fixed case. Substituting $\lambda = \lambda_k$ into Eq.(4.9) , we can write

$$\Phi_{N11}(\lambda_k)\, \underset{\sim}{u}_{0,k} = \underset{\sim}{0} \tag{4.21}$$

which represents a set of n homogeneous equations. Assuming that λ_k is a simple eigenvalue, the matrix $\Phi_{N11}(\lambda_k)$ has rank n-1 , which means that only n-1 of the set of equations (4.21) are independent. Taking arbitrarily one of the components of $\underset{\sim}{u}_{0,k}$ to be equal to unity and retaining only n-1 of the set (4.21), we can solve for the balance of the n-1 components of $\underset{\sim}{u}_{0,k}$. Recalling that $\underset{\sim}{p}_{0,k} = \underset{\sim}{0}$, this defines the vector

$\underset{\sim}{x}_{0,k}$ uniquely. Because $\underset{\sim}{p}_{0,k} = 0$, Eq.(4.20) can be separated into two equation , one for the displacement mode and the other for the internal - force mode. Indeed, it is easy to verify that the two equations are

$$\underset{\sim}{u}_{i,k} = \Phi_{i11}(\lambda_k)\, \underset{\sim}{u}_{0,k} \quad , \quad \underset{\sim}{p}_{i,k} = \Phi_{i21}(\lambda_k)\, \underset{\sim}{u}_{0,k} \quad , \; i=1,2,\ldots,N \tag{4.22}$$

The same approach can be used for other types of boundary conditions.

4. THE RESPONSE PROBLEM

The equations of motion for the complete periodic structure are given by

$$M \ddot{\underset{\sim}{q}} + K \underset{\sim}{q} = \underset{\sim}{Q} \tag{4.23}$$

where

$$\underset{\sim}{q} = (\underset{\sim}{u}_0^T , \underset{\sim}{u}_1^T , \ldots , \underset{\sim}{u}_N^T)^T \quad , \quad \underset{\sim}{Q} = (\underset{\sim}{f}_0^T , \underset{\sim}{f}_1^T , \ldots , \underset{\sim}{f}_N^T)^T \tag{4.24}$$

and $\underset{\sim}{f}_0 , \underset{\sim}{f}_1 , \ldots , \underset{\sim}{f}_N$ are arbitrary external excitations acting between the substructures. Furthermore, M and K represent the mass and stiffness matrices of the complete structure, respectively. Then, following the usual procedure in modal analysis (Ref. 13), we can write

$$\underset{\sim}{q} = \overline{U} \underset{\sim}{\eta} \tag{4.25}$$

where $\overline{U}$ represents the modal matrix, whose columns are the eigenvectors of the system. The eigenvectors are orthogonal with respect to the mass matrix M and can be normalized so as to satisfy

$$\overline{U}^T M \overline{U} = 1 \quad , \quad \overline{U}^T K \overline{U} = \Lambda \tag{4.26}$$

where 1 is the identity matrix and Λ is the diagonal matrix of the eigenvalues $\lambda_k = \omega_k^2$.

Introducing Eq.(4.25) into Eq.(4.23), multiplying on the left by $\bar{U}^T$ and considering Eqs.(4.26), we obtain

$$\ddot{\underset{\sim}{\eta}} + \Lambda \underset{\sim}{\eta} = \bar{U}^T \underset{\sim}{Q} = \underset{\sim}{N} \tag{4.27}$$

which represents a set of independent equations for the generalized coordinates η_r . The solution of these equations can be written in the general form

$$\eta_r = \frac{1}{\omega_r} \int_0^t N_r(\tau) \sin \omega_r(t-\tau)\, dt + \eta_r(0) \cos \omega_r(t)$$

$$+ \dot{\eta}_r(0) \frac{\sin \omega_r t}{\omega_r} \quad , \quad r = 1, 2, \ldots , n(N+1) \tag{4.28}$$

with

$$\underset{\sim}{\eta}(0) = \bar{U}^T M \underset{\sim}{q}(0) \quad , \quad \dot{\underset{\sim}{\eta}}(0) = \bar{U}^T M \underset{\sim}{q}(0) \tag{4.29}$$

The problem of normalizing the eigenvectors with respect to the mass matrix M is considered in Appendix A .

Finally, the response of the structure can be obtained from Eqs.(4.24) and (4.25) as follows

$$\underset{\sim}{u}_i = \sum_{r=1}^{n(N+1)} \bar{\underset{\sim}{u}}_{i,r} \;\; {}_r \quad , \quad i = 0, 1, \ldots , N \tag{4.30}$$

where $\bar{\underset{\sim}{u}}_{i,r}$ and η_r are given by Eqs.(A10) and (4.28),respectively.

Furthermore, it is expected that the series in Eq.(4.30) can be truncated to a reasonable number of terms. Although the response was derived for a free-free system, the procedure is equally valid for other types of boundary conditions, provided some minor changes are made.

5. ILLUSTRATIVE EXAMPLES

Let us consider the periodic system in axial vibration shown in Fig.4.2. It is easy to verify that the mass matrix and stiffness matrix for a typical substructure have the following form

$$M = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad K = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{4.31}$$

From Eq.(1.52) it follows that

$$\Phi_i = \frac{1}{2}\begin{bmatrix} z_1^i + z_2^i & -\dfrac{z_1^i - z_2^i}{\beta(\alpha^2 - 1)^{1/2}} \\ -\beta(\alpha^2 - 1)^{1/2}\,(z_1^i - z_2^i) & z_1^i + z_2^i \end{bmatrix} \tag{4.32}$$

where

$$\alpha = 1 - \frac{m}{k}\,\omega^2 \quad , \quad \beta = k \tag{4.33}$$

and z_1 and z_2 are reciprocal eigenvalues of the matrix A , namely,

$$z_1 = \alpha + (\alpha^2 - 1)^{1/2} \quad , \quad z_2 = \alpha - (\alpha^2 - 1)^{1/2} \tag{4.34}$$

Next, let us choose following values for the system parameters

$$m = k = 1 \tag{4.35}$$

For the system shown in Fig 4.2 , the characteristic equation of the complete structure is obtained from Eqs.(4.10) and (4.32) in the form

$$z_1^{15} + z_2^{15} = 0 \tag{4.36}$$

Using Eqs.(4.32-35) in conjunction with Muller's method,we obtain the following natural frequencies

$$\begin{aligned} \omega_1 &= 0.07401 \quad , \quad & \omega_2 &= 0.22123 \\ \omega_3 &= 0.36603 \quad , \quad & \omega_4 &= 0.50681 \\ \omega_5 &= 0.64204 \quad , \quad & \omega_6 &= 0.77024 \end{aligned} \tag{4.37}$$

etc.

Note, that for substructures with a single degree of freedom, it is possible to obtain a closed-form solution. For example, in the free-fixed case we have

$$\omega_k^2 = \{ 1 - \cos (2k+1) \frac{\pi}{30} \} \quad , \quad k = 0, 1, \ldots , N-1 \tag{4.38}$$

as shown in Appendix B.

Next, let us compute the corresponding eigenvectors. From Eq.(4.21) it follows that $u_{0,k}$ (k= 1, 2, ... , 15) can be chosen arbitrarily, e.g. , equal to 1

$$u_{0,k} = 1 \quad , \quad k = 1, 2, \ldots , 15 \tag{4.39}$$

The rest of the eigenvector follows from Eq.(4.22)

$$u_{i,k} = \frac{1}{2}\left(z_1^i + z_2^i \right)_k \quad , \quad \begin{array}{l} i = 1, 2, \ldots , 14 \\ k = 1, 2, \ldots , 15 \end{array} \qquad (4.40)$$

where the bracketed quantity must be evaluated at $\omega^2 = \omega_k^2$, where ω_k's are given by Eqs.(4.37).

The next step consists of normalizing the eigenvectors with respect to the mass matrix using Eq.(A7) modified so as to accommodate the free-fixed case. The equation can be written as

$$d_k^2 = \sum_{l=1}^{N-1} \underset{\sim}{b}_{l,k}^T \underset{\sim}{u}_{l,k} \quad , \quad k = 1, 2, \ldots , nN \qquad (4.41)$$

where

$$\begin{aligned} \underset{\sim}{b}_{0,k}^T &= \underset{\sim}{u}_{0,k}^T \alpha + \underset{\sim}{u}_{1,k}^T \beta^T \\ \underset{\sim}{b}_{l,k}^T &= \underset{\sim}{u}_{l-1,k}^T \beta + \underset{\sim}{u}_{l,k}^T \gamma + \underset{\sim}{u}_{l+1,k}^T \beta^T \quad , \quad l=1,2,\ldots,N-2 \qquad (4.42) \\ \underset{\sim}{b}_{N-1,k}^T &= \underset{\sim}{u}_{N-2,k}^T \beta + \underset{\sim}{u}_{N-1,k}^T \delta \end{aligned}$$

Using Eqs.(4.31) we have

$$\alpha = 1 \ , \ \beta = 0 \ , \ \gamma = 2 \qquad (4.43)$$

so that, from Eqs.(4.41) and (4.42) we obtain

$$d_k = \sqrt{15} = 3.87298 \quad , \quad k = 1,2, \ldots , 15 \tag{4.44}$$

Finally, Eq.(A10) yields the normalized eigenvectors

$$\bar{u}_k = \frac{1}{d_k} u_k \quad , \quad k = 1,2, \ldots, 15 \tag{4.45}$$

The first three modes are displayed in Fig. 4.3. Note that the closed form solution is given by

$$\bar{u}_{i,k+1} = 0.25820 \cos\{ i(2k+1) \frac{\pi}{30} \}, \quad \begin{array}{l} i=1,2, \ldots ,14 \\ k=0,1, \ldots ,14 \end{array}$$

as shown in Appendix B.

Next, let us assume that the structure is subject to a left-end load of the form

$$f_0 = 0.1 \, t \tag{4.46}$$

and furthermore assume zero initial conditions. From Eq.(4.27) it follows that,

$$N_r = \bar{u}_{0,r} f_0 \quad , \quad r = 1, 2, \ldots , 15 \tag{4.47}$$

or, using Eqs.(4.39), (4.45) and (4.46), we have

$$N_r = 0.02582 \, t \quad , \quad r = 1, 2, \ldots , 15$$

Next, from Eq.(4.28) we obtain

$$\eta_r = \frac{0.02582}{\omega_r^2} \left(t - \frac{1}{\omega_r} \sin\omega_r t \right) \quad , \quad r = 1, 2, \ldots , 15 \qquad (4.48)$$

Finally, the response is given by Eq(4.30) for $i = 0,1, \ldots , N-1$. Using the first four modes, the response can be obtained within 1 % of the actual solution. Some of the results are

$$u_0 = 1.4289\ t - 16.443 \sin 0.0740\ t - 0.6157 \sin 0.22123\ t$$
$$- 0.1359 \sin 0.36603\ t - 0.0512 \sin 0.50681\ t$$

$$u_1 = 1.4022\ t - 16.352 \sin 0.07401\ t - 0.5856 \sin 0.22123\ t$$
$$- 0.1177 \sin 0.36603\ t - 0.0381 \sin 0.50681\ t$$

$$u_{10} = 0.5101\ t - 8.2212 \sin 0.07401\ t + 0.6157 \sin 0.22123\ t$$
$$- 0.0680 \sin 0.36603\ t - 0.0256 \sin 0.50681\ t$$

The time histories of several displacements are displayed in Fig. 4.4, while in Fig. 4.5 an overall picture of the motion is given for some time values.

Next, let us consider a more realistic example, as given by the truss of Fig. 4-6. The system parameters are

lenght L of a steel pipe	: 0.61 m
weight per unit lenght W	: 24.52 N m^{-1}
cross sectional area A	: 3.187 10^{-4} m^2
Young's modulus E	: 2.068324 10^{11} N m^{-2}
number of substructures N	: 10
degrees of freedom of substructure n	: 4

Furthermore, the system is subjected to a force in the form

$$f = 10^5 \, u\,(t) \tag{4.49}$$

where $u(t)$ is the unit step function applied at $t=0$. The force is applied in the center of the structure corresponding to $u_{51,L}$. Furthermore, we assume the initial conditions to be zero.

A Fortran computer program has been developed to analyze any type of periodic structure. The program is general in nature, so that it is applicable to structures with relatively large n and N . The results for the above example are displayed in Figs. 4.7-9.

6. SUMMARY AND CONCLUSIONS

This chapter presents an efficient method for deriving the response of periodic structures. The procedure is based on an algorithm for the harmonic response of a periodic structure, previously derived in chapter I. The characteristic equation of the entire structure is derived by a transfer matrix technique inspired by Holzer's method for torsional vibration[13]. The eigenvalues of the complete structure are obtained by Muller's method, taking full advantage of the system periodicity. The corresponding modes of vibration can then be computed in a straightforward manner. Finally, the response is obtained by modal analysis.

The advantage of this approach is that the computational effort is virtually independent of the number of substructures, so that the method becomes increasingly efficient as the number of substructures increases. The algorithm allows for arbitrary external loads, applied throughout the entire structure.

7. APPENDIX A : NORMALIZATION OF EIGENVECTORS

The eigenvectors as computed from Eq.(4.22) are orthogonal to each other with repect to the mass matrix M of the entire structure. However, they are in general, not normalized with respect to M. Because M is usually of very high order, it is desirable to develop a procedure involving matrices of order n only.

Let us represent the computed modal matrix of a free-free sytem by the matrix U, namely

$$U = \begin{bmatrix} \underset{\sim}{u}_{0,1} & \underset{\sim}{u}_{0,2} & \cdots & \underset{\sim}{u}_{0,n(N+1)} \\ \underset{\sim}{u}_{1,1} & \underset{\sim}{u}_{1,2} & \cdots & \underset{\sim}{u}_{1,n(N+1)} \\ \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{u}_{N-1,1} & \underset{\sim}{u}_{N-1,2} & \cdots & \underset{\sim}{u}_{N-1,n(N+1)} \\ \underset{\sim}{u}_{N,1} & \underset{\sim}{u}_{N,2} & \cdots & \underset{\sim}{u}_{N,n(N+1)} \end{bmatrix} \tag{A1}$$

Because the eigenvectors are determined upon a constant factor only, we can write

$$U = \bar{U}\, D \tag{A2}$$

where $\bar{U}$ represents a normalized modal matrix, and D represents a diagonal matrix with constant elements to be determined. Fron Eq.(4.26) and (A2) it follows that

$$U^T M U = D^2 \tag{A3}$$

which yields the matrix D in terms of the computed eigenvectors.

Next, let us consider the mass matrix M of a free-free system. It is not difficult to show that the general form is given by the following n(N+1) order matrix

$$M = \begin{bmatrix} \alpha & \beta & 0 & 0 & 0 & & \cdots & 0 \\ \beta^T & \gamma & \beta & 0 & 0 & & \cdots & 0 \\ 0 & \beta^T & \gamma & \beta & 0 & & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & \cdots & & & 0 & \beta^T & \gamma & \beta \\ & & \cdots & & & 0 & 0 & \beta^T & \delta \end{bmatrix} \tag{A4}$$

where

$$\alpha = M_{LL} \quad , \quad \beta = M_{LR} \quad , \quad \gamma = M_{LL} + M_{RR} \quad , \quad \delta = M_{RR} \tag{A5}$$

and M_{LL}, ... are nxn submatrices of the mass matrix of a single substructure.

Next, let us consider Eq.(A3) and write it as

$$\underset{\sim}{u}_k^T M \underset{\sim}{u}_k = d_k^2 \quad , \quad k = 1, 2, \ldots, n(N+1) \tag{A6}$$

where $\underset{\sim}{u}_k$ is the k^{th} eigenvector.

In view of Eqs.(A1), (A4) and (4.36) it is not difficult to show that

$$d_k^2 = \sum_{l=0}^{N} \underset{\sim}{b}_{l,k}^T \, \underset{\sim}{u}_{l,k} \quad , \qquad k = 1, 2, \ldots , n(N+1) \tag{A7}$$

where

$$\underset{\sim}{b}_{0,k}^T = \underset{\sim}{u}_{0,k}^T \alpha + \underset{\sim}{u}_{1,k}^T \beta^T$$

$$\underset{\sim}{b}_{l,k}^T = \underset{\sim}{u}_{l-1,k}^T \beta + \underset{\sim}{u}_{l,k}^T \gamma + \underset{\sim}{u}_{l+1,k}^T \beta^T \quad , \quad l=1,2, \ldots ,N-1 \tag{A8}$$

$$\underset{\sim}{b}_{N,k}^T = \underset{\sim}{u}_{N-1,k}^T \beta + \underset{\sim}{u}_{N,k}^T \delta$$

From Eq.(A2) it follows that

$$\overline{U} = U \, D^{-1} \tag{A9}$$

or

$$\overline{\underset{\sim}{u}}_k = \frac{1}{d_k} \underset{\sim}{u}_k \quad , \qquad k = 1, 2, \ldots , n(N+1) \tag{A10}$$

which yields the normalized modes for the system.

8. APPENDIX B : CLOSED FORM SOLUTION OF EXAMPLE 1

Because the transfer matrix A is a real matrix it must have real or complex conjugate roots. Let us consider Eq.(4.34) and assume that z_1 and z_2 are complex conjugates. Moreover, it is easy to show that in this case, z_1 and z_2 are of the form

$$z_1 = \cos\theta + j \sin\theta \quad , \quad z_2 = \cos\theta - j \sin\theta \tag{B1}$$

where

$$\cos\theta = \alpha \tag{B2}$$

Next, let us consider the general characteristic equation for the free-fixed case

$$\frac{1}{2} (z_1^N + z_2^N) = 0 \tag{B3}$$

and substitute Eq.(B1) into Eq.(B3), to obtain

$$\cos N \theta = 0 \tag{B4}$$

This equation has the following roots

$$\theta_k = (2k + 1) \frac{\pi}{2N} \quad , \quad k = 0, 1, \ldots , N-1 \tag{B5}$$

Equation (B5) yields N roots. Because Eq.(B3) has only N roots, it is not possible to have real values for z_1 and z_2 .

Finally, from Eqs.(B2) , (4.33) and (B5) it follows that

$$\lambda_k = \omega_k^2 = \frac{k}{m}\{ 1 - \cos(2k+1)\frac{\pi}{2N} \} \quad , \quad k=0,1, \dots ,N-1 \qquad \text{(B6)}$$

Next, from Eq.(4.22) it follows that

$$u_{i,k+1} = \cos\{ i(2k+1)\frac{\pi}{2N} \} \quad , \quad \begin{matrix} i=1,2, \dots , N-1 \\ k=0,1, \dots , N-1 \end{matrix} \qquad \text{(B7)}$$

with

$$u_{0,k+1} = 1 \quad , \quad k=0,1, \dots ,N-1$$

Furthermore, it is easy to show that

$$\bar{u}_{0,k+1} = \frac{1}{N^{1/2}}$$

$$\bar{u}_{i,k+1} = \frac{1}{N^{1/2}}\cos\{ i(2k+1)\frac{\pi}{2N} \} \; , \quad \begin{matrix} i=1,2, \dots ,N-1 \\ k=0,1, \dots ,N-1 \end{matrix} \qquad \text{(B8)}$$

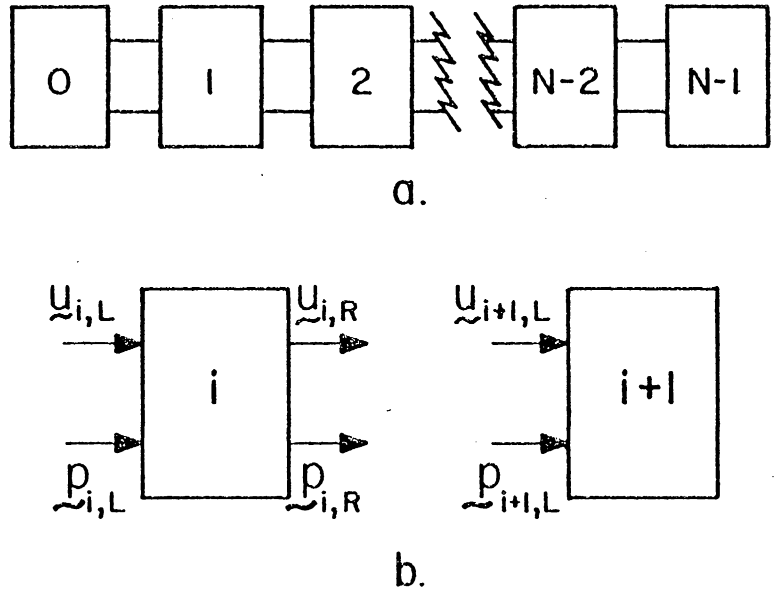


FIGURE 4.1 a. The Periodic Structure

b. The Substructure

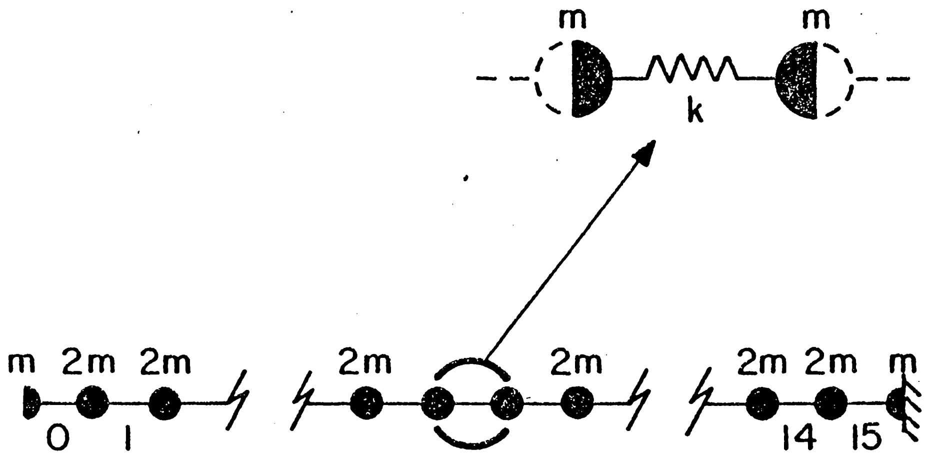


FIGURE 4.2 The Periodic Structure

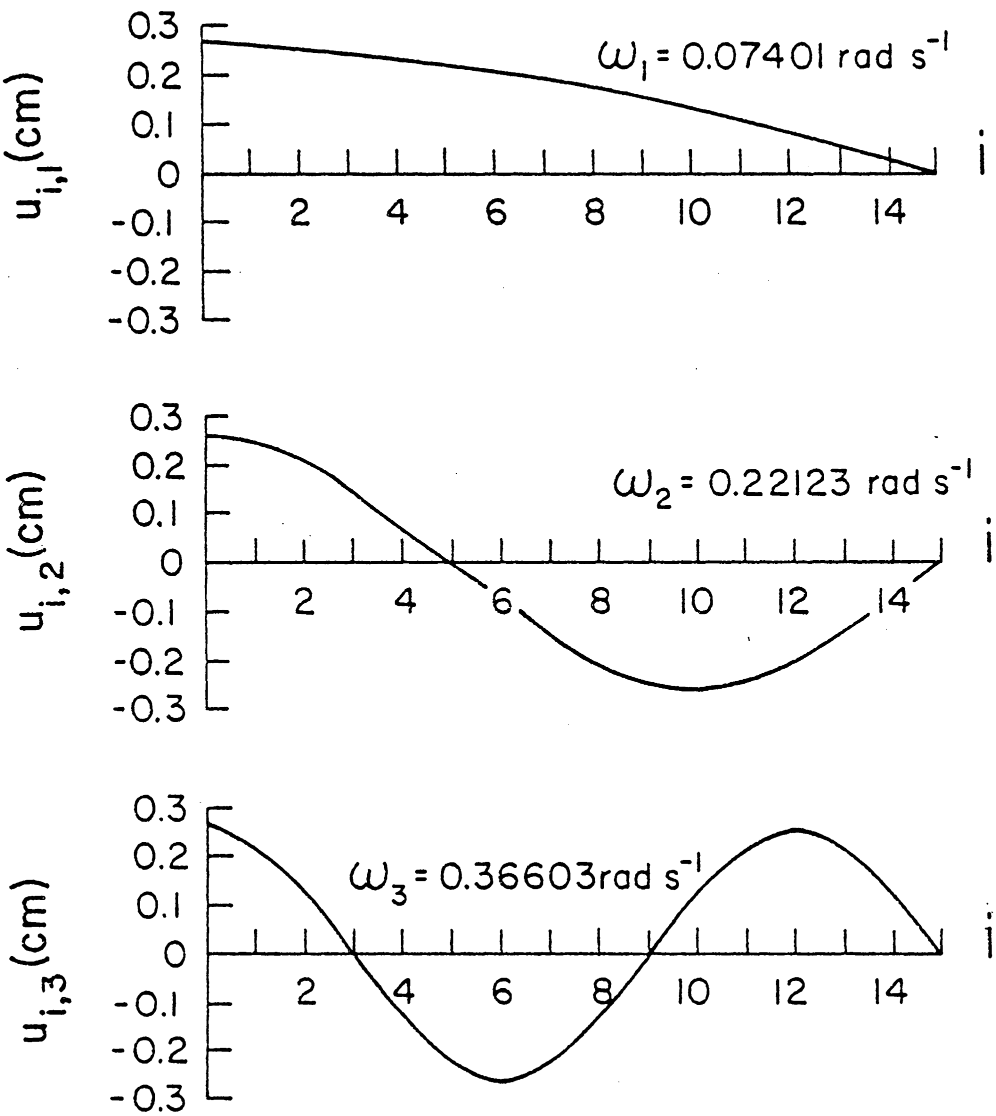


FIGURE 4.3 The Natural Frequencies and Modes of Vibration

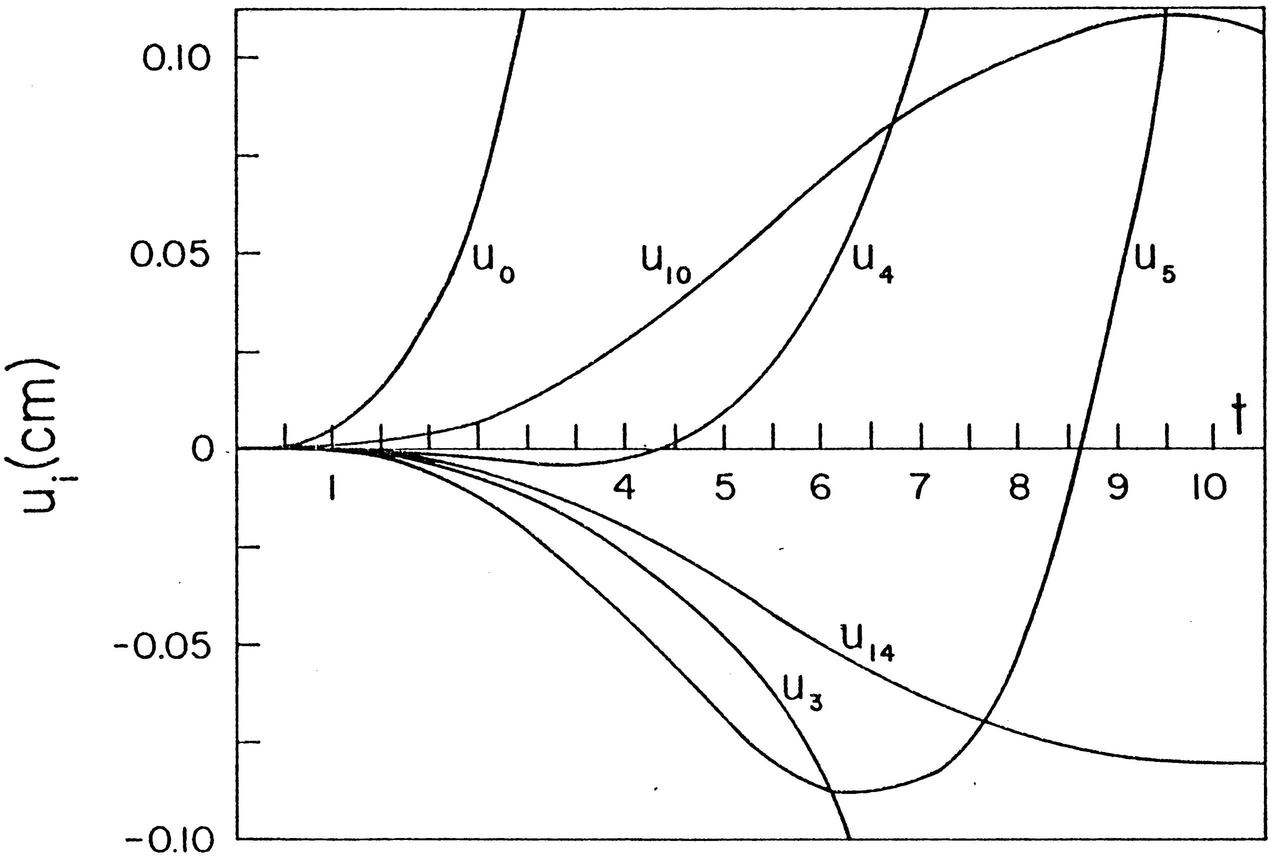


FIGURE 4.4 The Time Histories of the Displacements

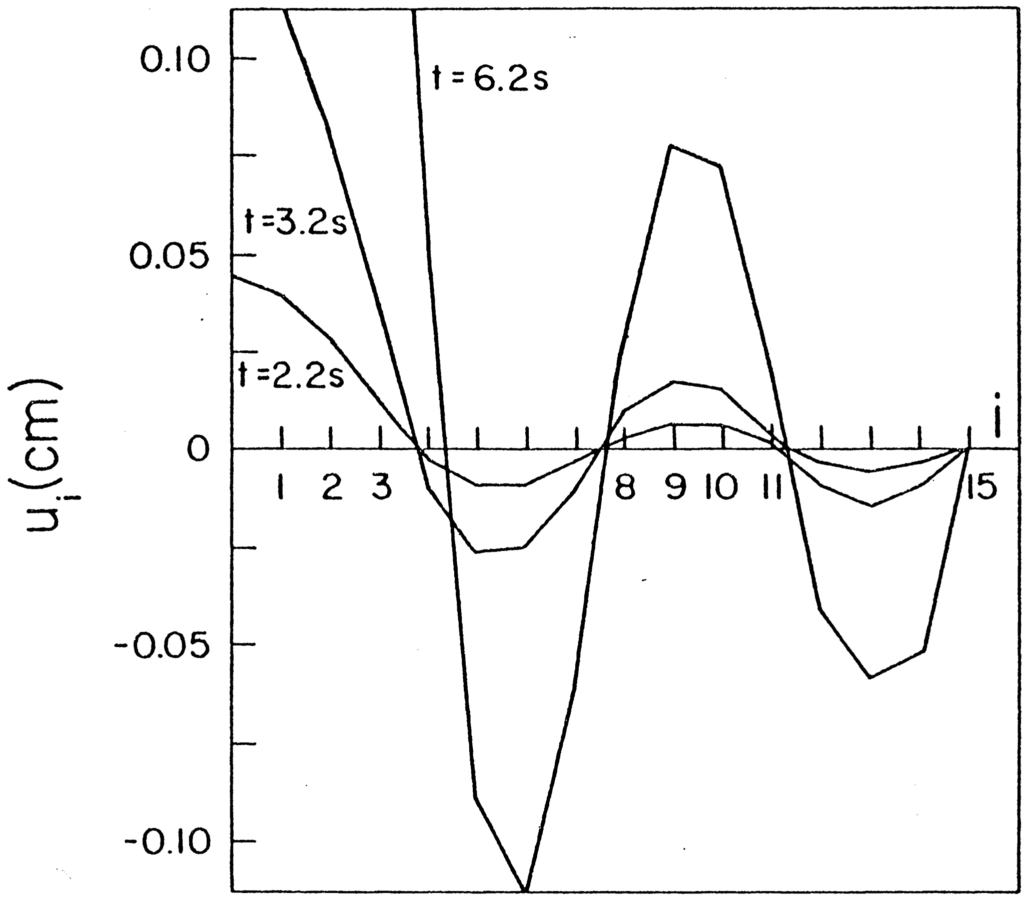


FIGURE 4.5 The Response for Several Time Values

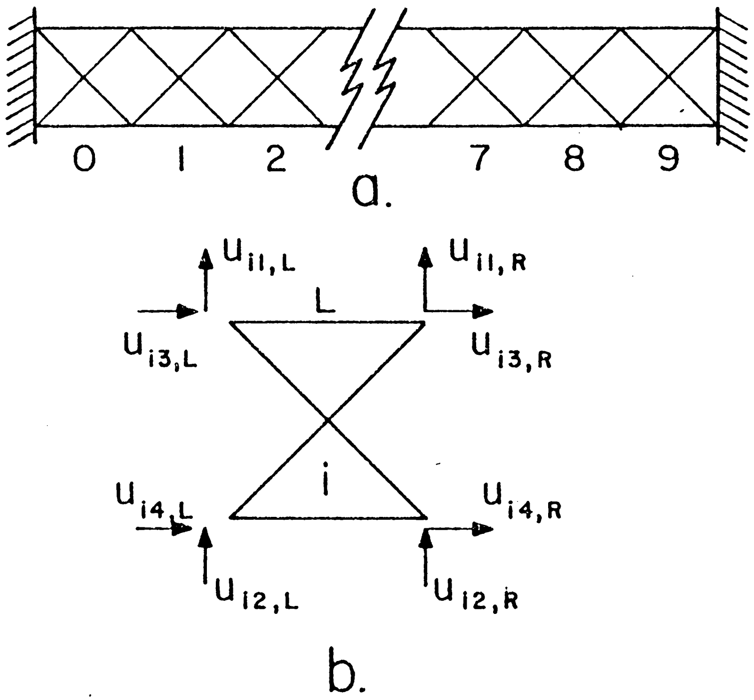


FIGURE 4.6 The Periodic Truss

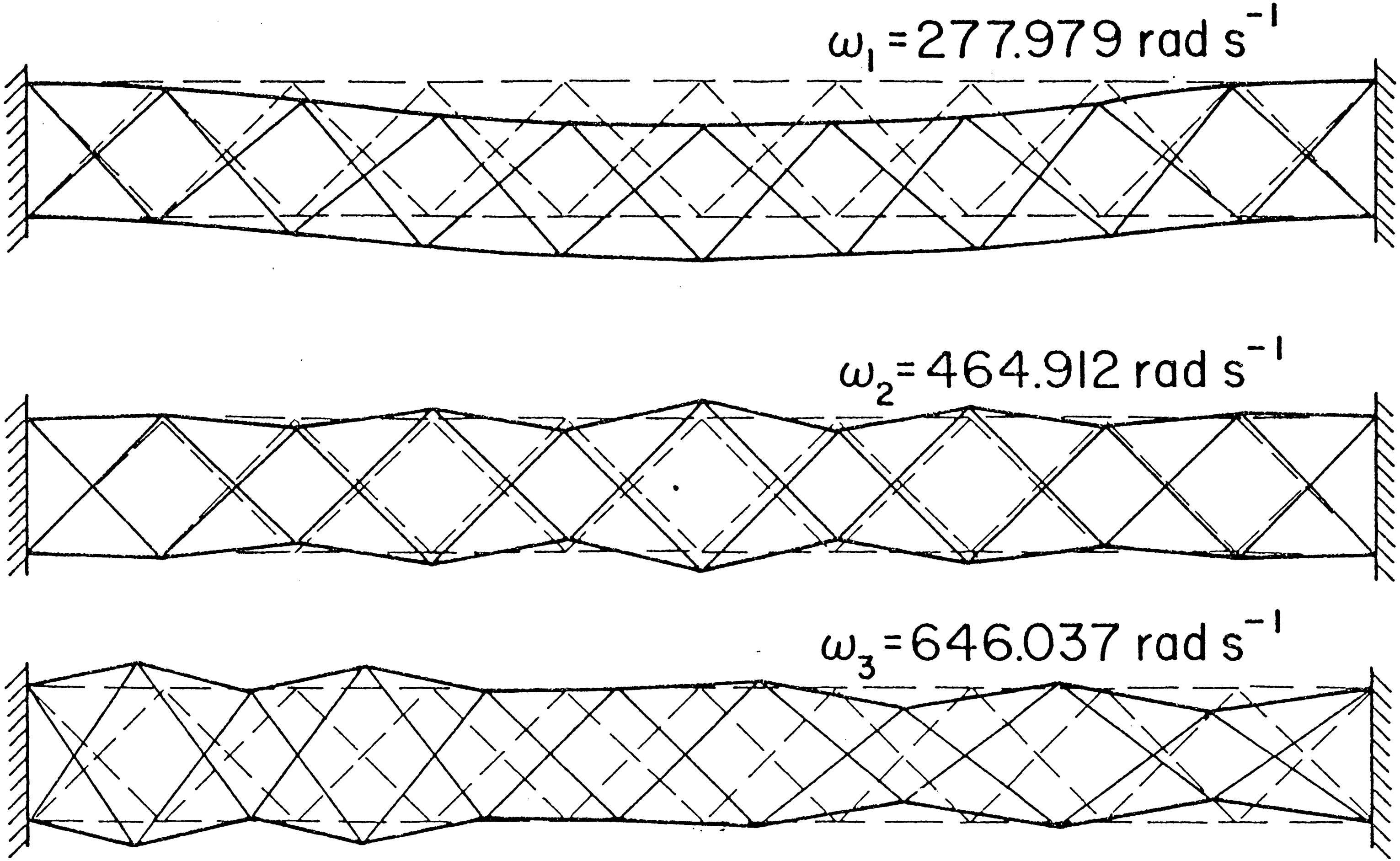


FIGURE 4.7 The Natural Frequencies and Modes of Vibration

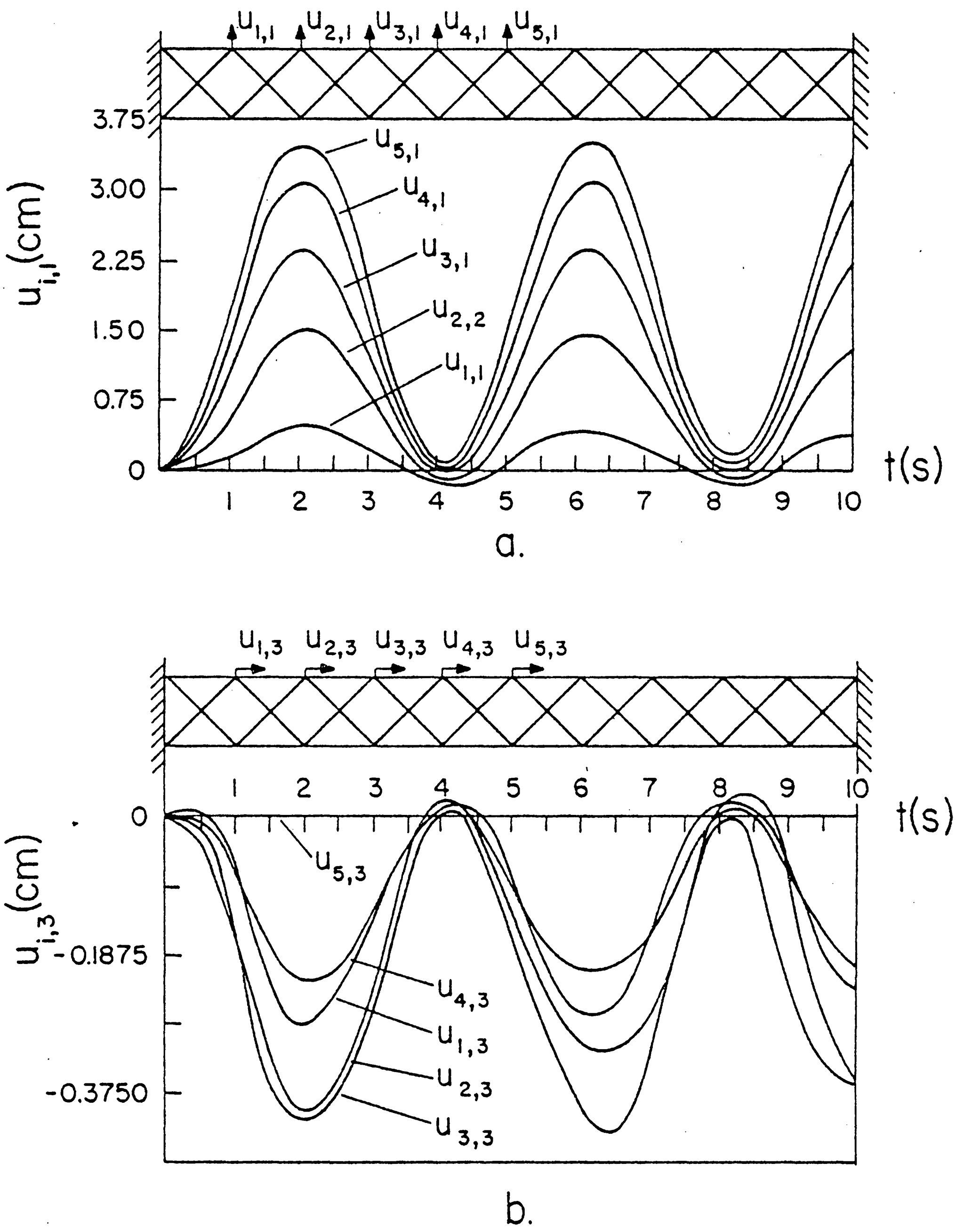


FIGURE 4.8 a. The Time Histories of the Vertical Displacements

b. The Time Histories of the Horizontal Displacements

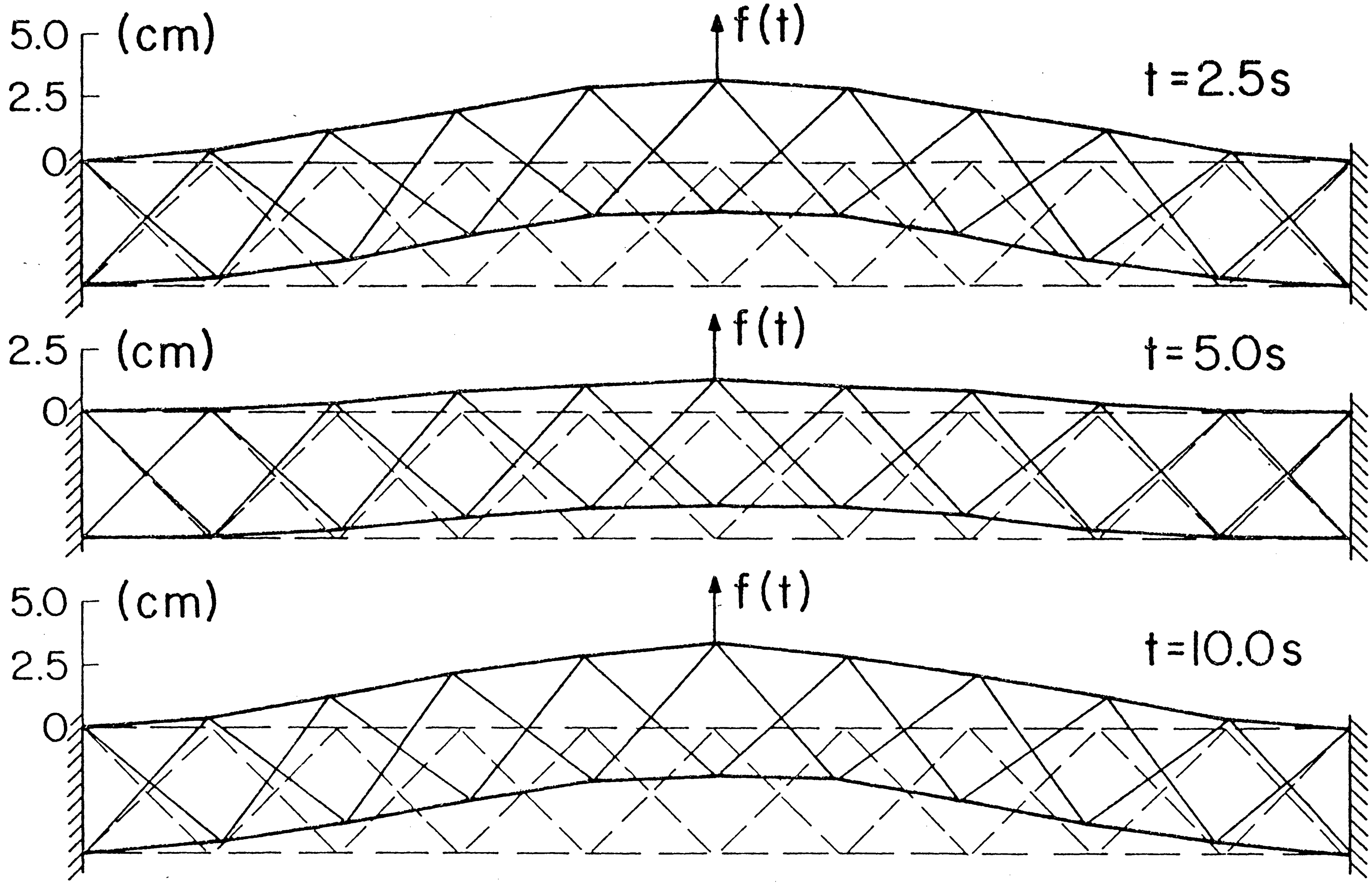


FIGURE 4.9 The Displacement Pattern for Several Time Values

CHAPTER V

SIMULATION OF CONTINUOUS SYSTEMS BY PERIODIC STRUCTURES

1. INTRODUCTION

The object of the present chapter is to analyze a certain class of continuous sytems as if they are periodic structures. Indeed, many continuous systems can be subdivided into identical subsystems having the same properties. A simple example is a homogeneous beam in axial vibration. The basic theory, as developed in chapter I, will be applied to this class of continuous systems. Because the computational effort of the fundamental algorithm does not depend on the number of substructures, it is seen that the accuracy of the solution can be increased without increasing the cost of the computations. Furthermore, all the advantages of the original method are still valid. For example, it is still possible to include damping and distributed harmonic loads.

Let us subdivide a continuous system into N identical substructures, each having n actual degrees of freedom. As an illustrative example let us consider a beam in axial vibration with $N = 100$ and $n = 1$. Obviously, the numerical accuracy of the response should increase with increasing N. However, it turns out that the procedure developed in chapter I fails when the length of the substructure Δx becomes relatively small compared to the other parameters in the problem. The reason for this becomes clear when we consider the elements of the impedance matrix. Indeed, for the beam in axial vibration, we obtain the following impedance matrix, using the finite-element approach

$$Z = \begin{bmatrix} -m\,\omega^2\,\frac{\Delta x}{3} + \frac{EA}{\Delta x} & -m\,\omega^2\,\frac{\Delta x}{6} - \frac{EA}{\Delta x} \\ \\ -m\,\omega^2\,\frac{\Delta x}{6} - \frac{EA}{\Delta x} & -m\,\omega^2\,\frac{\Delta x}{3} + \frac{EA}{\Delta x} \end{bmatrix} \tag{5.1}$$

where m is the mass density, ω is the load frequency, E is Young's modulus, A is the cross sectional area of the beam and $\Delta x = \frac{L}{N}$ is the length of the substructure in which L is the total length of the beam. In a practical problem, E is a relatively large number compared to the other parameters in the problem. Moreover, when Δx becomes smaller, the term $-m\,\omega^2\,\frac{\Delta x}{3}$ becomes negligible compared to the term $\frac{EA}{\Delta x}$. In fact, when the algorithm is performed by a digital computer, there is a large possibility that the mass term is lost. Therefore, the procedure as presented in chapter I, must be adapted in order to circumvent this problem. This will be done in the next sections of this chapter. Essentially, we shall find the limit of the solution as Δx approaches zero. In the last section we shall illustrate the method by solving a practical problem.

2. THE EIGENVALUES OF THE TRANSFER MATRIX

In the introduction, we discussed some of the numerical difficulties associated with the application of the basic algorithm for periodic structures to continuous structures. To circumvent these problems, we shall take the limit of Eq.(1.32) as the length of the substructure Δx, approaches zero, namely,

$$\lim_{\Delta x \to 0} \underset{\sim}{x}_i = \lim_{\Delta x \to 0} \sum_{j=1}^{2n} X_j \left(z_j^i \underset{\sim}{x}_0 + \sum_{k=0}^{i-1} z_j^{i-k-1} \underset{\sim}{f}_k^* \right) \tag{5.2}$$

From Eq.(5.2) it is seen that we shall need to find the limit of z_j^i as Δx approaches zero. To this end, let us consider the transfer matrix A as given by Eq.(1.12). Because A is a transfer matrix we must have that A approaches 1 when Δx approaches zero, so that A can be written as follows

$$A = \sum_{k=0}^{\infty} A_k \, \Delta x^k = 1 + \overline{A} \tag{5.3}$$

where

$$A_0 = 1 \quad , \quad \overline{A} = \sum_{k=1}^{\infty} A_k \, \Delta x^k \tag{5.4}$$

Note that the matrices A_k $(k = 0, 1, \ldots)$ are coefficient matrices easily determined from the matrix A. The eigenvalues z_j $(j = 1,2,\ldots,2n)$ of A are the roots of the characteristic equation

$$\det (z\,1 - A) = 0 \tag{5.5}$$

or, using Eq. (5.3)

$$\det \{(z-1)1 - \overline{A} \} = 0 \tag{5.6}$$

This implies that the eigenvalues of A have the following form

$$z_j = 1 + \sum_{k=1}^{\infty} z_{jk}\, \Delta x^k = 1 + \overline{z}_j \quad , \qquad j = 1, 2, \ldots , 2n \tag{5.7}$$

where

$$\overline{z}_j = \sum_{k=1}^{\infty} z_{jk}\, \Delta x^k \qquad , \qquad j = 1, 2, \ldots , 2n \tag{5.8}$$

are the eigenvalues of $\overline{A}$.

Because we wish to find the limit of Eq.(5.2) as Δx approaches zero, we need to find $\lim\limits_{\Delta x \to 0} z_j^i$. But, from Eq.(5.7) it follows that

$$\lim_{\Delta x \to 0} z_j^i = \lim_{\Delta x \to 0} (1 + \overline{z}_j)^i \tag{5.9}$$

or

$$\lim_{\Delta x \to 0} z_j^i = \{ \lim_{z_{j1}\Delta x \to 0} (1 + z_{j1}\, \Delta x + O(\Delta x^2))^{\frac{1}{z_{j1}\Delta x}} \}^{z_{j1}\, i\, \Delta x} \tag{5.10}$$

where $O(\Delta x^2)$ represents terms of order Δx^2. Note that Eq.(5.10) represents a standard limit given by

$$\lim_{\Delta x \to 0} z_j^i = e^{z_{j1} x} \tag{5.11}$$

where $x = i\,\Delta x$ has a finite value, being the x-coordinate of the left boundary of substructure i .

Next, we shall prove that the z_{j1} $(j=1,2, \ldots ,2n)$ are the eigenvalues of A_1 , where A_1 is the matrix corresponding to the linear term in Δx in Eq.(5.3). To this end, let us introduce the following theorem[17] ,

If α and β are two matrices for which

$$|\alpha_{ij}| < 1 \qquad , \qquad |\beta_{ij}| < 1 \tag{5.12}$$

and λ_1 is a simple eigenvalue of α, then, for sufficiently small ε , the corresponding eigenvalue $\lambda_1(\varepsilon)$ of $\alpha + \varepsilon\beta$ will be given by a convergent power series

$$\lambda_1(\varepsilon) = \lambda_1 + k_1\varepsilon + k_2\,\varepsilon^2 + \ldots \tag{5.13}$$

where

$$|\lambda_1(\varepsilon) - \lambda_1| = O(\varepsilon)$$

and the k_j can be determined.

Next, let us write $\overline{A}$ in the following form

$$\overline{A} = \sum_{k=1}^{\infty} A_k\,\Delta x^k = A_1\,\Delta x + \left(\sum_{k=2}^{\infty} A_k\,\Delta x^{k-1}\right)\Delta x \tag{5.14}$$

or

$$\bar{A} = \alpha + \varepsilon \beta \tag{5.15}$$

where

$$\alpha = A_1 \Delta x \quad , \quad \beta = \sum_{k=2}^{\infty} A_k \Delta x^{k-1} \quad , \quad \Delta x = \varepsilon \tag{5.16}$$

Because in the limit Δx approaches zero, we can take it small enough so that inequalities (5.12) are satisfied. Moreover, because $\lambda_1 = z_{j1} \Delta x$ $(j=1,2, \ldots ,2n)$, it follows from Eq.(5.16) that z_{j1} $(j=1,2, \ldots ,2n)$ are the eigenvalues of the matrix A_1 . Note that we assumed that the eigenvalues z_j $(j=1,2, \ldots ,2n)$ are simple. This completes the proof.

3. THE MATRICES X_j

To obtain the limiting response from Eq.(5.3), we also need to find the limit of X_j $(j = 1, 2, \ldots, 2n)$ as Δx approaches zero. Using simple substitution in Eq.(1.24) it is easy to verify that

$$H_l = \sum_{k=0}^{l} \theta_{l-k} \, A^k \tag{5.17}$$

Substituting Eq.(5.3) into Eq.(5.17), we obtain

$$H_l = \sum_{k=0}^{l} \theta_{l-k} \, (1 + \overline{A})^k \tag{5.18}$$

or

$$H_l = \sum_{k=0}^{l} \theta_{l-k} \sum_{s=0}^{k} \binom{k}{s} \overline{A}^{\,s} \tag{5.19}$$

where

$$\binom{k}{s} = \frac{k!}{s!(k-s)!} \tag{5.20}$$

Next, let us write Eq.(5.19) as a matrix polynomial in $\overline{A}$

$$H_l = \theta_l \, 1 + \theta_{l-1} \sum_{s=0}^{1} \binom{1}{s} \overline{A}^{\,s} + \theta_{l-2} \sum_{s=0}^{2} \binom{2}{s} \overline{A}^{\,s} + \ldots$$

$$\ldots + \theta_1 \sum_{s=0}^{l-1} \binom{l-1}{s} \overline{A}^{\,s} + \theta_0 \sum_{s=0}^{l} \binom{l}{s} \overline{A}^{\,s} \tag{5.21}$$

or

$$H_l = \{\, \theta_l \binom{0}{0} + \theta_{l-1} \binom{1}{0} + \theta_{l-2} \binom{2}{0} + \dots + \theta_0 \binom{l}{0} \,\} \, 1$$

$$+ \{\, \theta_{l-1} \binom{1}{1} + \theta_{l-2} \binom{2}{1} + \dots + \theta_0 \binom{l}{1} \,\} \, \bar{A}$$

$$+ \{\, \theta_{l-2} \binom{2}{2} + \theta_{l-3} \binom{3}{2} + \dots + \theta_0 \binom{l}{2} \,\} \, \bar{A}^2 \tag{5.22}$$

$$+ \dots$$

so that,

$$H_l = \sum_{s=0}^{l} \binom{s}{0} \theta_{l-s} + \sum_{s=1}^{l} \binom{s}{1} \theta_{l-s} \bar{A} + \sum_{s=2}^{l} \binom{s}{2} \theta_{l-s} \bar{A}^2$$

$$+ \dots + \sum_{s=l}^{l} \binom{s}{l} \theta_{l-l} \bar{A}^l \tag{5.23}$$

or finally,

$$H_l = \sum_{k=0}^{l} \{ \sum_{s=k}^{l} \binom{s}{k} \theta_{l-s} \} \bar{A}^k \tag{5.24}$$

Next, let us substitute Eq.(5.3) into Eq.(1.25) and write

$$\theta_l = - \frac{1}{l} \, \mathrm{tr} \, (1 + \bar{A}) \, H_{l-1} \tag{5.25}$$

or

$$\theta_l = - \frac{1}{l} \operatorname{tr} H_{l-1} + 0 \, (\Delta x) \tag{5.26}$$

and because

$$\operatorname{tr} H_{l-1} = (2n - l + 1) \, \theta_{l-1} \tag{5.27}$$

we have

$$\theta_l = - \frac{1}{l} (2n - l - 1) \, \theta_{l-1} + 0 \, (\Delta x) \tag{5.28}$$

or, by simple induction we obtain

$$\theta_l = (-1)^k \frac{(2n-l+1)(2n-l+2)\ldots(2n-l+k)}{l(l+1)\ldots(l-k+1)} \, \theta_{l-k} + 0(\Delta x) \tag{5.29}$$

For $k = l$ we have $\theta_{l-k} = \theta_0 = 1$, so that from Eq.(5.29) it follows that

$$\theta_l = (-1)^l \frac{(2n-l+1)\ldots 2n}{l(l-1)\ldots 2.1} + 0(\Delta x) \tag{5.30}$$

or

$$\theta_l = (-1)^l \binom{2n}{l} + 0(\Delta x) \tag{5.31}$$

From Eq.(5.27) and Eq.(5.31) it finally follows that

$$\operatorname{tr} H_l = (2n-l)\{ (-1)^l \binom{2n}{l} + 0(\Delta x) \} \tag{5.32}$$

or

$$\mathrm{tr}\ H_l = (-1)^l\ 2n \binom{2n-1}{l} + O(\Delta x) \tag{5.33}$$

Let us now consider the denominator in Eq.(1.29) and use Eq.(5.33) to obtain

$$\sum_{l=0}^{2n-1} z_j^{2n-l-1}\ \mathrm{tr}\ H_l = \sum_{l=0}^{2n-1} z_j^{2n-l-1} \{(-1)^l\ 2n \binom{2n-1}{l} + O(\Delta x)\} \tag{5.34}$$

or

$$\sum_{l=0}^{2n-1} z_j^{2n-l-1}\ \mathrm{tr}\ H_l = 2n \sum_{l=0}^{2n-1} (-1)^l \binom{2n-1}{l} z_j^{2n-l-1} + \text{higher order terms} \tag{5.35}$$

and by simplifying the right side of Eq.(5.35), we finally obtain

$$\sum_{l=0}^{2n-1} z_j^{2n-l-1}\ \mathrm{tr}\ H_l = 2n\ \bar{z}_j^{2n-1} + \text{higher order terms} \tag{5.36}$$

where we also used Eq.(5.7).

Next, let us consider the numerator of Eq.(1.29) and use Eq.(5.24) to obtain

$$\sum_{l=0}^{2n-1} z_j^{2n-l-1} H_l = \sum_{l=0}^{2n-1} z_j^{2n-l-1} \sum_{k=0}^{l} \{ \sum_{s=k}^{l} \binom{s}{k} \theta_{l-s} \} \bar{A}^k \tag{5.37}$$

Again, we shall write the right side of Eq.(5.37) as a matrix polynomial in $\overline{A}$

$$\sum_{l=0}^{2n-1} z_j^{2n-1-l} H_l = z_j^{2n-1-0} H_0 + z_j^{2n-1-1} H_1 + \ldots$$

$$\ldots + z_j^{(2n-1)-(2n-1)} H_{2n-1} \qquad (5.38)$$

From Eq.(5.24) it then follows that

$$\sum_{l=0}^{2n-1} z_j^{2n-1-l} H_l = z_j^{2n-1} \sum_{k=0}^{0} \overline{A}^k \sum_{s=k}^{0} \binom{s}{k} \theta_{0-s}$$

$$+ z_j^{(2n-1)-1} \sum_{k=0}^{1} \overline{A}^k \sum_{s=k}^{1} \binom{s}{k} \theta_{1-s} + \ldots \qquad (5.39)$$

$$\ldots + z_j^{(2n-1)-(2n-1)} \sum_{k=0}^{2n-1} \overline{A}^k \sum_{s=k}^{2n-1} \binom{s}{k} \theta_{2n-1-s}$$

or

$$\sum_{l=0}^{2n-1} z_j^{2n-1-l} H_l = z_j^{2n-1} \{ \overline{A}^0 \sum_{s=0}^{0} \binom{s}{0} \theta_{0-s} \}$$

$$+ z_j^{(2n-1)-1} \{ \overline{A}^0 \sum_{s=0}^{1} \binom{s}{0} \theta_{1-s} + \overline{A}^1 \sum_{s=1}^{1} \binom{s}{1} \theta_{1-s} \} + \ldots$$

$$\ldots + z_j^{(2n-1)-(2n-1)} \{ \overline{A}^0 \sum_{s=0}^{2n-1} \binom{s}{0} \theta_{2n-1-s} \qquad (5.40)$$

$$+ \overline{A}^1 \sum_{s=1}^{2n-1} \binom{s}{1} \theta_{2n-1-s} + \ldots + \overline{A}^{2n-1} \sum_{s=2n-1}^{2n-1} \binom{s}{2n-1} \theta_{2n-1-s} \}$$

Now, we collect terms in $\bar{A}^0$, $\bar{A}^1$, ... to obtain

$$\sum_{l=0}^{2n-1} z_j^{2n-l-1} H_l = \bar{A}^0 \{ z_j^{2n-1} \sum_{s=0}^{0} \binom{s}{0} \theta_{0-s}$$

$$+ z_j^{(2n-1)-1} \sum_{s=0}^{1} \binom{s}{0} \theta_{1-s} + \ldots + z_j^{(2n-1)-(2n-1)} \sum_{s=0}^{2n-1} \binom{s}{0} \theta_{2n-1-s} \}$$

$$+ \bar{A}^1 \{ z_j^{(2n-1)-1} \sum_{s=1}^{1} \binom{s}{1} \theta_{1-s} + z_j^{(2n-1)-2} \sum_{s=1}^{2} \binom{s}{1} \theta_{2-s} + \ldots$$

$$\ldots + z_j^{(2n-1)-(2n-1)} \sum_{s=1}^{2n-1} \binom{s}{1} \theta_{2n-1-s} \} + \ldots$$

$$\ldots + \bar{A}^{2n-1} \{ z_j^{(2n-1)-(2n-1)} \sum_{s=2n-1}^{2n-1} \binom{s}{2n-1} \theta_{2n-1-s} \} \qquad (5.41)$$

so that

$$\sum_{l=0}^{2n-1} z_j^{2n-l-1} H_l = \sum_{l=0}^{2n-1} \{ \sum_{k=1}^{2n-1} z_j^{(2n-1)-k} \sum_{s=1}^{k} \binom{l}{s} \theta_{k-s} \} \bar{A}^l \qquad (5.42)$$

Using Eq.(5.31) in Eq.(5.42), we obtain

$$\sum_{l=0}^{2n-1} z_j^{2n-l-1} H_l = \sum_{l=0}^{2n-1} \{ \sum_{k=1}^{2n-1} z_j^{(2n-1)-k} \sum_{s=1}^{k} \binom{l}{s} (-1)^{k-s} \binom{2n}{k-s} \} \bar{A}^l$$

$$+ \text{higher order terms} \qquad (5.43)$$

Let us now take the coefficient of $\bar{A}^l$ in Eq.(5.43) seperately and perform the following manipulations

$$2n - 1 - k = t \tag{5.44}$$

so that, when $k = l$, we have $t = 2n - 1 - l$ and when $k = 2n - 1$, we have $t = 0$. Replacing t by k again, we obtain for the coefficient of $\bar{A}^{l}$ in Eq.(5.43) the following equivalent expression ;

$$\sum_{k=0}^{2n-1-l} z_j^k \sum_{s=l}^{2n-1-k} \binom{s}{l} \binom{2n}{2n-1-k-s} (-1)^{2n-1-k-s} \tag{5.45}$$

Similarly, for $2n-1-k-s = t$ we obtain

$$\sum_{k=0}^{2n-1-l} z_j^k \sum_{s=0}^{2n-1-l-k} \binom{2n-1-k-s}{l} \binom{2n}{s} (-1)^{s} \tag{5.46}$$

Now, in general, it can be shown that

$$\sum_{s=0}^{p-l} \binom{p-s}{l} \binom{a}{s} (-1)^{l} \equiv (-1)^{p-l} \binom{a-l-1}{p-l} \tag{5.47}$$

where the positive integers a,s,l and p must be such that $p-l$, $p-s$ and $a-l-l$ are non-negative. Using the identity Eq.(5.47) we can write the coefficient of $\bar{A}^{l}$ as follows

$$\sum_{k=0}^{2n-1-l} z_j^k (-1)^{2n-1-k-l} \binom{2n-1-l}{2n-1-k-l} \tag{5.48}$$

or

$$\sum_{k=0}^{2n-1-l} (-1)^{2n-1-l-k} \binom{2n-1-l}{2n-1-l-k} z_j^k \tag{5.49}$$

or

$$(z_j - 1)^{2n-1-l} \tag{5.50}$$

and using Eq.(5.7), we finally obtain

$$\bar{z}_j^{\,2n-1-l} \tag{5.51}$$

Substituting this coefficient into Eq.(5.43), we obtain

$$\sum_{l=0}^{2n-1} z_j^{2n-1-l} H_l = \sum_{l=0}^{2n-1} \bar{z}_j^{\,2n-1-l} \bar{A}^{l} + \text{higher order terms} \tag{5.52}$$

Next, using Eqs.(1.29), (5.36) and (5.52), we can write the matrices X_j ($j=1,2, \ldots ,2n$) as follows

$$X_j = \frac{\sum_{l=0}^{2n-1} \bar{z}_j^{\,2n-1-l} \bar{A}^{l} + \text{higher order terms}}{2n\, \bar{z}_j^{\,2n-1} + \text{higher order terms}} \tag{5.53}$$

We now take the limit of Eq.(5.53) as x approaches zero

$$\lim_{\Delta x \to 0} X_j = \lim_{\Delta x \to 0} \frac{\sum_{l=0}^{2n-1} \bar{z}_j^{\,2n-1-l} \bar{A}^{l}}{2n\, \bar{z}_j^{\,2n-1}} \tag{5.54}$$

or,

$$\lim_{\Delta x \to 0} X_j = \lim_{\Delta x \to 0} \frac{1}{2n} \sum_{l=0}^{2n-1} \bar{z}_j^{\,-l} \, \bar{A}^l \tag{5.55}$$

so that

$$\lim_{\Delta x \to 0} X_j = \lim_{\Delta x \to 0} \frac{1}{2n} \sum_{l=0}^{2n-1} \left(\frac{\bar{A}}{\bar{z}_j}\right)^l \tag{5.56}$$

Using Eqs.(5.8) and (5.4),we finally obtain

$$\lim_{\Delta x \to 0} X_j = \frac{1}{2n} \sum_{l=0}^{2n-1} \left(\frac{A_1}{z_{j1}}\right)^l \tag{5.57}$$

4. THE RESPONSE OF CONTINUOUS STRUCTURES

The response can now be obtained by substituting Eqs.(5.11) and (5.57) into Eq.(5.2). Hence, in the limit, as $\Delta x \to 0$

$$\underset{\sim}{x} = \sum_{j=1}^{2n} X_j \{ e^{z_{j1} x} \underset{\sim}{x}(0) + \int_0^x e^{z_{j1}(x-l)} \begin{bmatrix} \underset{\sim}{0} \\ \hline \underset{\sim}{f}(l) \end{bmatrix} dl \} \qquad (5.58)$$

where

$$X_j = \frac{1}{2n} \sum_{l=0}^{2n-1} \left(\frac{A_1}{z_{j1}}\right)^l \quad , \qquad j = 1, 2, \ldots , 2n \qquad (5.59)$$

and

$$\underset{\sim}{x} = \lim_{\Delta x \to 0} \underset{\sim}{x}_i \quad , \qquad \underset{\sim}{x}(0) = \lim_{\Delta x \to 0} \underset{\sim}{x}_0 \qquad (5.60)$$

and $\underset{\sim}{f}(l)$ is the amplitude of the distributed external force at $x = l$.

Because the eigenvalues of the matrix A occur in reciprocal pairs,we can arrange the z_j $(j=1,2,\ldots,2n)$ as follows

$$z_j \text{ with } |z_1|<\ldots<|z_n|<1$$

$$, \quad j = 1, 2, \ldots , n \qquad (5.61)$$

$$z_{n+j} = \frac{1}{z_j}$$

as we already demonstrated in Eq.(1.50).

Let us now consider the second half of Eqs.(5.61) and use Eq.(5.7) to obtain

$$1 + \bar{z}_{j+n} = \frac{1}{1 + \bar{z}_j} \quad , \qquad j = 1, 2, \ldots , n \tag{5.62}$$

In view of Eq.(5.8) and the fact that the right side of Eq.(5.62) can be expanded in a power series, we can write

$$1 + z_{j+n,1} \; \Delta x + O(\Delta x^2) = 1 - z_{j1} \, \Delta x + O(\Delta x^2) \; , \; j=1,2,\ldots,n \tag{5.63}$$

from which it follows that

$$z_{j+n,1} = - z_{j1} \quad , \qquad j = 1, 2, \ldots , n \tag{5.64}$$

so that only half of the eigenvalues of A_1 must be calculated.

Furthermore, transformation (1.53) is still valid, so that, from Eqs.(5.59) and (5.64) it follows that

$$\begin{aligned} X_j &= \frac{1}{2n} \sum_{l=0}^{2n-1} \left(\frac{A_1}{z_{j1}}\right)^l \\ & \qquad\qquad j=1,2, \ldots ,n \\ X_{j+n} &= T(X_j) = \frac{1}{2n} \sum_{l=0}^{2n-1} (-1)^l \left(\frac{A_1}{z_{j1}}\right)^l \end{aligned} \tag{5.65}$$

Next, let us define the following quantities

$$X_j^{(e)} = \frac{1}{n} \sum_{l=0,2,\ldots}^{2n-2} \left(\frac{A_1}{z_{jl}}\right)^l$$

$$j=1,2, \ldots ,n \qquad (5.66)$$

$$X_j^{(o)} = \frac{1}{n} \sum_{l=1,3,\ldots}^{2n-1} \left(\frac{A_1}{z_{jl}}\right)^l$$

in which (e) means even and (o) means odd. Then, from Eqs.(5.65) and (5.66) we can write

$$X_j = \frac{1}{2} \left(X_j^{(e)} + X_j^{(o)} \right)$$

$$j=1,2, \ldots ,n \qquad (5.67)$$

$$X_{j+n} = \frac{1}{2} \left(X_j^{(e)} - X_j^{(o)} \right)$$

Let us now return to Eq.(5.58) and write

$$\sum_{j=1}^{2n} X_j e^{z_{jl} x} = \sum_{j=1}^{n} X_j e^{z_{jl} x} + \sum_{j=n+1}^{2n} X_j e^{z_{jl} x} \qquad (5.68)$$

or

$$\sum_{j=1}^{2n} X_j e^{z_{jl} x} = \sum_{j=1}^{n} X_j e^{z_{jl} x} + \sum_{j=1}^{n} X_{j+n} e^{z_{j+n,l} x} \qquad (5.69)$$

Introducing Eqs.(5.64) and (5.67) into Eq.(5.69), we obtain

$$\sum_{j=1}^{2n} X_j e^{z_{jl} x} = \sum_{j=1}^{n} X_j^{(e)} \left(\frac{e^{z_{jl}x} + e^{-z_{jl}x}}{2}\right) + \sum_{j=1}^{n} X_j^{(o)} \left(\frac{e^{z_{jl}x} - e^{-z_{jl}x}}{2}\right) \qquad (5.70)$$

or

$$\Phi(x) = \sum_{j=1}^{2n} X_j \, e^{z_{j1} x} = \sum_{j=1}^{n} \{ X_j^{(e)} \cosh z_{j1}x + X_j^{(o)} \sinh z_{j1}x \} \quad (5.71)$$

Equation (5.71) represents the fundamental matrix $\Phi(x)$ of any continuous structure with homogeneous properties. The fundamental matrix essentially transfers the state at the left end of the structure to the state at position x .

Using Eq.(5.71), we can write Eq.(5.58) in the following form

$$\tilde{x} = \sum_{j=1}^{n} \{ (X_j^{(e)} \cosh z_{j1}x + X_j^{(o)} \sinh z_{j1}x) \, \tilde{x}(0) + \int_0^x (X_j^{(e)} \cosh z_{j1}(x-l) + X_j^{(o)} \sinh z_{j1}(x-l)) \begin{bmatrix} \tilde{0} \\ \tilde{f}(l) \end{bmatrix} dl \} \quad (5.72)$$

Equation (5.72) represents the response for a continuous structure obtained as the limiting case of a periodic structure. The matrices $X_j^{(e)}$ and $X_j^{(o)}$ (j=1,2, ... ,n) are given by Eqs.(5.66) while $\tilde{f}(l)$ represents the amplitude of the distributed external force at $x = l$. The scalars z_{j1} (j=1,2, ... ,n) represent half of the eigenvalues of the matrix A_1 , where A_1 is formed by the coefficients of Δx in the elements of the transfer matrix A. In general, it is necessary to evaluate the integral in Eq.(5.72) numerically.

Finally, it should be pointed out that Eq.(5.72) yields the complex amplitudes of the displacements and internal forces. To obtain the actual response we must evaluate

$$\mathrm{Re}(\underset{\sim}{x}\, e^{j\,\omega t}) = \mathrm{Re}(\underset{\sim}{x}) \cos \omega t - \mathrm{Im}(\underset{\sim}{x}) \sin \omega t \tag{5.73}$$

5. ILLUSTRATIVE EXAMPLE

Let us consider a viscously damped bar in flexural vibration, subjected to a continuous harmonic load as shown in Fig. 5.1. The transfer matrix is given by[13]

$$A = \begin{bmatrix} 1 + \frac{1}{12}(a\Delta x)^4 & \Delta x & -\frac{\Delta x^3}{6\ b} & \frac{\Delta x^2}{2\ b} \\ a^4\ \frac{\Delta x^3}{4} & 1 & -\frac{\Delta x^2}{2\ b} & \frac{\Delta x}{b} \\ -\frac{b}{2}\ a^4 \Delta x(2 + \frac{a^4 \Delta x^4}{12}) & -\frac{b}{2}\ a^4 \Delta x^2 & 1 + \frac{a^4 \Delta x^4}{12} & -\frac{a^4 \Delta x^3}{4} \\ \frac{b}{2}\ a^4 \Delta x^2 & 0 & -\Delta x & 1 \end{bmatrix} \tag{5.74}$$

where

$$\underset{\sim}{u}_{i,L} = \begin{bmatrix} w_{i,L} \\ \\ \psi_{i,L} \end{bmatrix} \quad , \quad \underset{\sim}{p}_{i,L} = \begin{bmatrix} Q_{i,L} \\ \\ M_{i,L} \end{bmatrix} \tag{5.75}$$

and $w_{i,L}$ represents the deflection at the left end of substructure i, while $\psi_{i,L}$ represents the corresponding angle of rotation. Furthermore, $Q_{i,L}$ and $M_{i,L}$ represent the shearing force and the bending moment, respectively. Finally, Δx is the length of the substructure whereas a and b are given by

$$a = \frac{m\omega^2}{b} \quad , \quad b = (1 + j\omega c)\ EI \tag{5.76}$$

in which m is the mass density and EI is the bending rigidity of the bar. The constant c represents a viscous damping coefficient and ω is the forcing frequency.

From Eq.(5.74) it follows that

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{b} \\ -ba^4 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{5.77}$$

which yields the simple characteristic equation

$$z_{j1}^4 - a^4 = 0 \tag{5.78}$$

so that, the eigenvalues of A_1 are given by

$$z_{11} = a \quad , \quad z_{21} = ja \quad , \quad z_{31} = -a \quad , \quad z_{41} = -ja \tag{5.79}$$

Note that the roots appear in pairs with opposite signs.

It is now easy to verify that

$$A_1^2 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{b} \\ 0 & 0 & -\frac{1}{b} & 0 \\ 0 & -ba^4 & 0 & 0 \\ ba^4 & 0 & 0 & 0 \end{bmatrix} \tag{5.80a}$$

and

$$A_1^3 = \begin{bmatrix} 0 & 0 & -\frac{1}{b} & 0 \\ a^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a^4 \\ 0 & ba^4 & 0 & 0 \end{bmatrix} \tag{5.80b}$$

Moreover, from Eqs.(5.66), we obtain

$$\chi_1^{(e)} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \frac{1}{ba^2} \\ 0 & 1 & -\frac{1}{ba^2} & 0 \\ 0 & -ba^2 & 1 & 0 \\ ba^2 & 0 & 0 & 1 \end{bmatrix} \tag{5.81a}$$

$$\chi_2^{(e)} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{ba^2} \\ 0 & 1 & \frac{1}{ba^2} & 0 \\ 0 & ba^2 & 1 & 0 \\ -ba^2 & 0 & 0 & 1 \end{bmatrix} \tag{5.81b}$$

$$\chi_1^{(o)} = \frac{1}{2}\begin{bmatrix} 0 & \frac{1}{a} & -\frac{1}{ba^3} & 0 \\ a & 0 & 0 & \frac{1}{ba} \\ -ba^3 & 0 & 0 & -a \\ 0 & ba & -\frac{1}{a} & 0 \end{bmatrix} \tag{5.81c}$$

$$X_2^{(o)} = \frac{1}{2}\begin{bmatrix} 0 & -\frac{j}{a} & -\frac{j}{ba^3} & 0 \\ ja & 0 & 0 & -\frac{j}{ba} \\ jba^3 & 0 & 0 & -ja \\ 0 & jba & \frac{j}{a} & 0 \end{bmatrix} \tag{5.81d}$$

Substituting Eqs.(5.79) and (5.81) into Eq.(5.71), we obtain the following matrix $\Phi(x) = \sum_{j=1}^{2n} X_j e^{z_j x}$

$$\Phi(x) = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\ -ba^4\Phi_{13} & \Phi_{11} & -\Phi_{14} & \frac{1}{b}\Phi_{12} \\ -ba^4\Phi_{12} & -b^2a^4\Phi_{14} & 11 & ba^4\Phi_{14} \\ b^2a^4\Phi_{14} & -b^2a^4\Phi_{14} & -\Phi_{12} & \Phi_{11} \end{bmatrix} \tag{5.82}$$

where

$$\begin{aligned} \Phi_{11} &= \frac{1}{2}(\cosh ax + \cos ax) \\ \Phi_{12} &= \frac{1}{2a}(\sinh ax + \sin ax) \\ \Phi_{13} &= \frac{-1}{2ba^3}(\sinh ax - \sin ax) \\ \Phi_{14} &= \frac{1}{2ba^2}(\cosh ax - \cos ax) \end{aligned} \tag{5.83}$$

in which we used, $\cosh jax = \cos ax$ and $\sinh jax = j \sin ax$. If a is real, it is possible to use the fundamental matrix $\Phi(x)$ directly as given by Eq.(5.82). If there is damping in the problem, then a is a complex number and we must use expressions such as

$$\cosh ax = \cosh \alpha x \cos \beta x + j \sinh \alpha x \sin \beta x \tag{5.84}$$

where α and β are the real and imaginary parts of a, respectively.

Let us now introduce the following values for the system parameters

$$m = \omega = c = EI = 1 \tag{5.85}$$

so that, from Eq.(5.76), we obtain

$$a = \frac{1}{2}(1 - j) \quad , \quad b = 1 + j \tag{5.86}$$

Furthermore, we assume the following external load vector

$$\underset{\sim}{f}(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{5.87}$$

Moreover, assume that the bar is clamped at the right end, and free at the left-end, so that

$$\underset{\sim}{p}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad , \quad \underset{\sim}{u}(L) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.88}$$

where L is the length of the beam.

Using Eqs.(5.82) and (5.72), we can write the general solution as follows

$$\underset{\sim}{u}(x) = \Phi_{LL}(x)\,\underset{\sim}{u}(0) + \Phi_{LR}(x)\{\,\underset{\sim}{p}(0) + \underset{\sim}{f}(0)\} + \int_0^x \Phi_{LR}(x-l)\underset{\sim}{f}(l)dl$$

$$\underset{\sim}{p}(x) = \Phi_{LR}(x)\,\underset{\sim}{u}(0) + \Phi_{RR}(x)\{\,\underset{\sim}{p}(0) + \underset{\sim}{f}(0)\} + \int_0^x \Phi_{RR}(x-l)\underset{\sim}{f}(l)dl \tag{5.89}$$

where Φ_{LL} , ... are submatrices of Φ .

The first of Eq.(5.89) and the second of Eq.(5.88) yield $\underset{\sim}{u}(0)$ in terms of known quantities. The final expression for the response in the present case is given by

$$\underset{\sim}{u}(x) = \Phi_{LL}(x)\,\underset{\sim}{u}(0) + \underset{\sim}{\psi}_1(x)$$

$$\underset{\sim}{p}(x) = \Phi_{LR}(x)\,\underset{\sim}{u}(0) + \underset{\sim}{\psi}_2(x) \tag{5.90}$$

where Φ_{LL} and Φ_{LR} follow from Eq.(5.72) and

$$\underset{\sim}{\psi}_1(x) = \begin{bmatrix} \frac{1}{2ba^4}(2 - \cosh ax - \cos ax) - \frac{1}{2ba^3}(\sinh ax - \sin ax) \\ -\frac{1}{2ba^3}(\sinh ax - \sin ax) - \frac{1}{2ba^2}(\cosh ax - \cos ax) \end{bmatrix} \tag{5.91}$$

$$\underset{\sim}{\psi}_2(x) = \begin{bmatrix} \frac{1}{2a}(\sinh ax + \sin ax) + \frac{1}{2}(\cosh ax + \cos ax) - 1 \\ \\ -\frac{1}{2a^2}(\cosh ax - \cos ax) - \frac{1}{2a}(\sinh ax + \sin ax) \end{bmatrix} \quad (5.92)$$

Furthermore

$$\underset{\sim}{u}(0) = -\Phi_{LL}^{-1}(L)\,\underset{\sim}{\psi}_1(L) \quad (5.93)$$

Note that, to obtain $\underset{\sim}{u}(0)$, we must introduce the total length L of the bar. In this example we assume $L = 10$.

The numerical results of this particular example are displayed in Figs. 5.2.

6. SUMMARY AND CONCLUSIONS

This chapter represents an adaptation to continuous systems of the algorithm to obtain the response of periodic structures subject to harmonic loads, as developed in chapter I. The continuous system is subdivided into a number of arbitrarily small but identical subsystems. Then, the fundamental theory for periodic structures is applied and the limit is taken as the length of the substructure approaches zero. The number of degrees of freedom of the substructure is arbitrary. The computational effort does not depend on the number of substructures, so that, the limiting process does not involve an increase in computation time. As can be seen from the example, it is sometimes possible to obtain a closed form solution depending on the order of the matrix A_1. The method can also accommodate damping but is restricted to continuous systems with homogeneous properties and subject to harmonic loads.

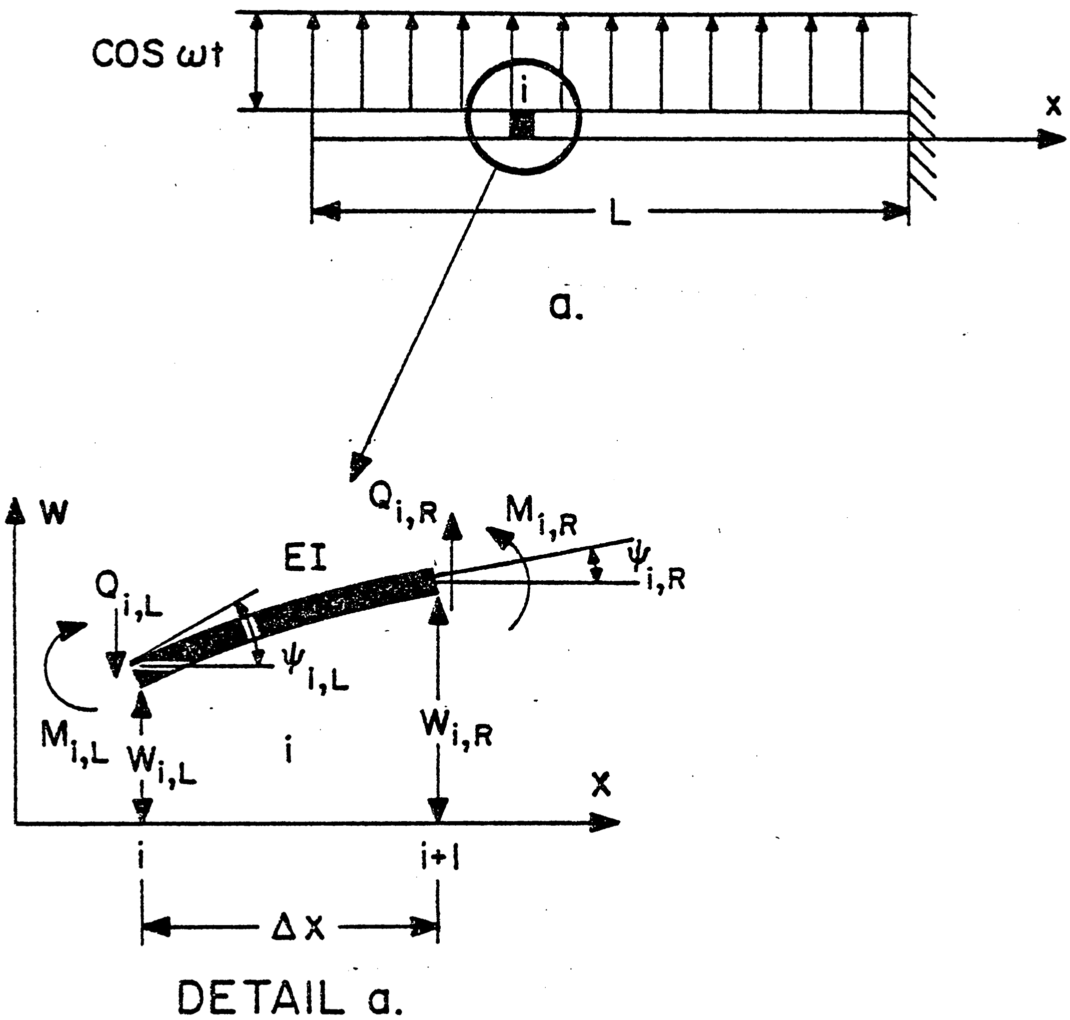
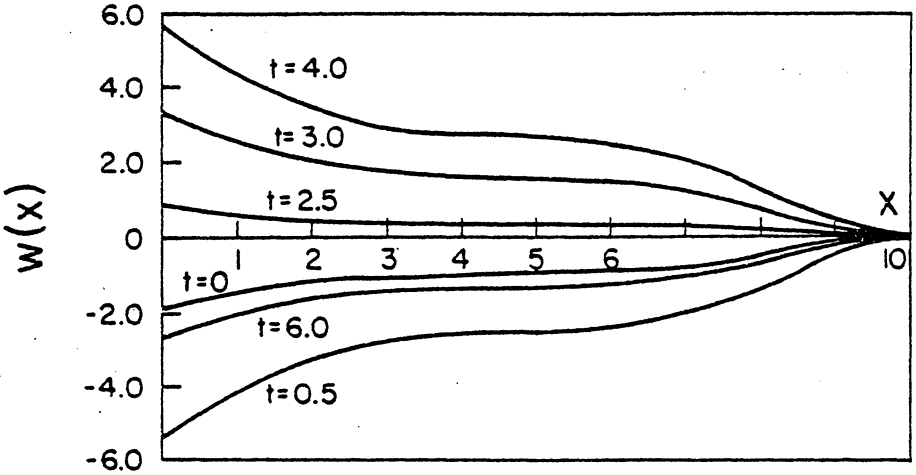
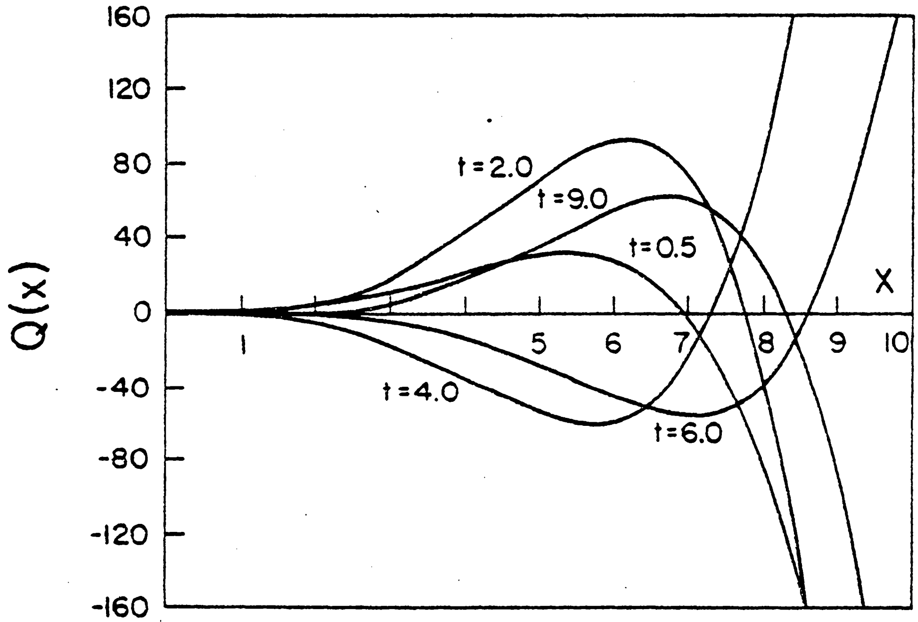


FIGURE 5.1 A Bar in Flexural Vibration


6.0
4.0
2.0
0
-2.0
-4.0
-6.0
w(x)
t=4.0
t=3.0
t=2.5
t=0
t=6.0
t=0.5
X
1
2
3
4
5
6
10


160
120
80
40
0
-40
-80
-120
-160
Q(x)
t=2.0
t=9.0
t=0.5
t=4.0
t=6.0
X
1
5
6
7
8
9
10

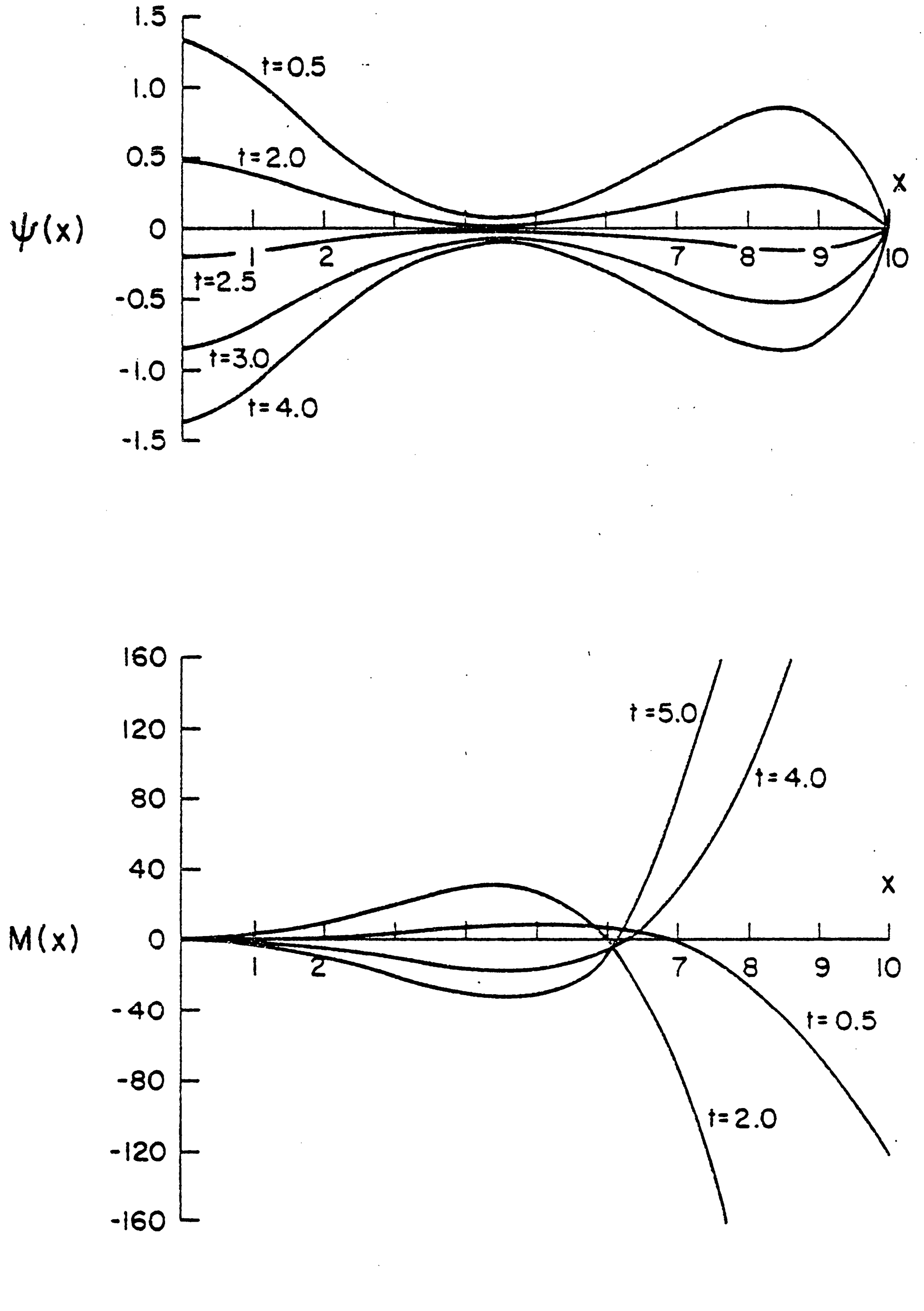


FIGURE 5.2 The Response

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RESPONSE OF PERIODIC STRUCTURES

by

Remi Carlos Engels

(ABSTRACT)

Periodic structures are defined as structures consisting of identical substructures connected to each other in identical manner. In chapter I, the response of periodic structures to harmonic excitation is described by a matrix difference equation. The solution of the matrix difference equation can be obtained by the Z-Transform method and it yields the response to both end conditions and external excitations. The method developed necessitates the eigenvalues of the transfer matrix for a tyoical substructure, so that the procedure is capable of analyzing a periodic structure with the same computational effort necessary to analyze a single substructure. Added advantage is derived from the fact that the method does not require the eigenvectors of the transfer matrix.

In chapter II, infinite periodic structures are considered for different types of loading. Furthermore, it is demonstrated that additional savings are possible when the substructure is symmetric.

Chapter III, considers the problem of almost periodic structures. If the system parameters differ slightly from one substructure to another, then the structure becomes almost periodic. An efficient method using a perturbation technique to derive the response of an almost periodic structure is presented. The procedure reduces the solution to a sequential application of the basic algorithm for periodic

structures as developed in chapter I.

In chapter IV, we consider the undamped response of a periodic structure using a modal analysis technique. The method allows for arbitrary loads and takes full advantage of the periodic properties of the structure.

In chapter V, an attempt is made to simulate certain continuous systems by periodic structures. The algorithm as developed in chapter I is now adapted to the treatment of continuous systems. The method is capable of deriving the response of damped and undamped systems subject to harmonic distributed loads. The length of the substructure can be made arbitrarily small without increasing the computational effort.