APPLICATIONS OF TRILINEAR COORDINATES TO SOME PROBLEMS IN PLANE ELASTICITY

by

Ramanath Williams

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APPROVED:

H. F. Brinson, Chairman

C. W. Smith

R. P. McNitt

D. Frederick, Department Head

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Blacksburg, Virginia

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LIST OF SYMBOLS

A_i	points referring to vertices of a triangle
a_i	length of sides of triangle
C_{ni}	$\cos n\pi\zeta_i$
D	flexural rigidity, $D = Eh^3/12(1 - \nu^2)$
E	modulus of elasticity
e	dilatation
H_i	altitudes of a triangle, $H_i = \lambda/m_i$
h	plate thickness
i,j,k	dummy indices
M_i	triaxial bending moments
M_{in}	direct (normal) bending moments
ℓ	$\cos\theta$
m	$\sin\theta$
ℓ_i	$\cos\alpha_i$
m_i	$\sin\alpha_i$
in, is	i dummy index (i=1,2,3), suffix 'in' represents perpendicular to the ith side of the triangle of reference. Suffix 'is' represents parallel to the ith side of the triangle of reference.
q	transverse load intensity
R	radius of circumcircle
S_{ni}	$\sin n\pi\zeta_i$
(u_{in}, v_{is})	Cartesian displacements corresponding to (x_i, y_i) coordinates

w	transverse plate displacements
(x,y), (x_i,y_i)	rectangular Cartesian coordinates
(x_1,x_2,x_3)	triaxial coordinates
$(\alpha_1,\alpha_2,\alpha_3)$	angles referring to the triangle of reference
β	$\nu/m_1m_2m_3$
γ	$(1-\nu)/2$
Δ	area of the triangle of reference
$(\varepsilon_{in}, \varepsilon_{is}, \gamma_i)$	Cartesian strain components corresponding to (x_i,y_i) coordinates
$(\varepsilon_1, \varepsilon_2, \varepsilon_3)$	triaxial strain components
$\{\varepsilon_t\}$	$\{\varepsilon_{1n}\ \varepsilon_{2n}\ \varepsilon_{3n}\}$, direct strain components - triaxial system
$\{\varepsilon_c\}$	$\{\varepsilon_x\ \varepsilon_y\ \gamma_{xy}\}$, Cartesian strain components corresponding to (x_i,y_i) coordinates
ζ_i	areal coordinates, $\zeta_i = \eta_i/H_i$
(η_1,η_2,η_3)	trilinear coordinates
Θ	first stress invariant
λ	$2Rm_1m_2m_3$
ν	Poisson's ratio
ρ	radius of in-circle $\rho = 4R \sin \alpha_1/2 \,.\, \sin \alpha_2/2 \,.\, \sin \alpha_3/2$
$\{\sigma_t\}$	$\{\sigma_1\ \sigma_2\ \sigma_3\}$, triaxial stress components
$\{\sigma_c\}$	$\{\sigma_x\ \sigma_y\ \tau_{xy}\}$, Cartesian stress components corresponding to (x_i,y_i) coordinates
Φ	stress function
χ	second stress invariant

ω	frequency
$\lfloor\ \rfloor$	row matrix
$\{\ \}$	column matrix
$[\ ]^T$	transpose matrix
$[\ ]^{-1}$	inverse matrix

Introduction

In plane elasticity, it is necessary and sufficient to have only two independent coordinates to study elastic deformation. Because of the sufficiency of two independent coordinates, the most commonly employed coordinate systems for analyzing plane problems in elasticity have been Cartesian, polar, and other less familiar coordinate systems, each with its own peculiarities. However, in solving boundary value problems in plane elasticity, it was found that hitherto well established coordinate systems are not necessarily the best - in the sense that they are cumbersome - when confronted with triangular boundaries. In principle, the most suitable coordinate system for triangular boundaries would be one which conforms naturally to the boundaries of a triangle. In this respect, it is felt that it would be worthwhile to investigate the usefulness of trilinear coordinates to boundary value problems in two dimensional elasticity.

For reasons unknown, the subject of trilinear coordinates is omitted from modern textbooks on analytic geometry, but has been fully developed in older British treatises. Among these are the works of Whitworth[(1)] (1866), Ferrers [(2)] (1866), Askwith [(3)] (1918), Loney [(4)] (1923), and a concise treatment by Smith [(5)] (1919).

Just as any other coordinate system, the trilinear coordinates are described with respect to a frame of reference: it is called the triangle of reference. The trilinear coordinate of a point P, in the plane, is denoted by (η_1, η_2, η_3); in which η_1, η_2, and η_3 are perpendicular distances from the sides of the triangle of reference to the point P. Unlike other familiar coordinate systems, the coordinates which are to be utilized do not have an origin; that is, there is no ordered triad (0, 0, 0). However any finite point in the plane can be located with respect to the triangle of reference.

The trilinear coordinates, also referred to as homogeneous point coordinates in projective geometry, do not lend themselves readily to the direct investigation of purely metric properties. Formulae on which these investigations depend (e.g. for the distance from one point to another, for the angle made by one line with another, for the equation of a conic, and for many other formulae and equations in analytic geometry) can be derived or may be found in the classical texts on analytic geometry.

Trilinear coordinates have many applications in mineralogy and geochemistry [(6),(7)], but as the analytic geometry of these coordinates is not generally well known, the data derived by this system of coordinates are treated graphically rather than analytically. The first use of trilinear coordinates to problems in plane elasticity seems to have come about via the finite element stress analysis through the independent investigations of Argyris [(8)] and Zienkiewicz [(9)] with their respective collaborators. Argyris in his writings calls homogeneous point coordinates natural coordinates, and Zienkiewicz refers to them as areal coordinates. Both these writers, and their respective followers, have written numerous articles in the successful application of these coordinates to the finite element stress analysis

(Displacement Method). No investigation has been carried out so far to study the suitability of this coordinate system to other forms of stress analysis. The aim of the present series of investigations is to provide the necessary theoretical basis for all forms of stress analysis and ultimately establish this system of coordinates as a suitable alternative to two dimensional boundary value problems involving triangular boundaries.

Having discussed the reasons for the present investigation, a brief discussion will now be given on the content of this particular discourse with the aim of giving the reader a clear outline of the work.

In Section 1 trilinear coordinates are defined and the commonly accepted sign convention for this coordinate system is presented. In Section 2 a coordinate system termed triaxial is introduced. This system of coordinates also comes under the general category of homogeneous point coordinates. The reason for introducing these coordinates in the analysis is mainly to simplify the transition from the rectangular Cartesian coordinates to trilinear coordinates. This section also contains the derivations of a few relevant differential operator relationships between the Cartesian, triaxial, and trilinear variables.

Using the notion of triaxial coordinates, the triaxial stress system is defined in Section 3. The concept of a triaxial stress (and strain) state is not new, but it has been used mainly in experimental stress analysis. It is important to realize, that the triaxial stresses are not the same as the direct (or actual) stresses corresponding to the coordinate axes. In engineering design, the direct stresses corresponding to the coordinates are of importance, therefore in Section 3, the relationships between the triaxial and direct stress components are derived.

The stress-strain relationships for the triaxial coordinate system are developed in Section 4. Attention is however limited to an isotropic material. Section 5 is devoted to developing the relationships that exist between the stress components in the Cartesian and triaxial coordinate system. Expressions for the principal stresses and the strain energy of deformation, for the triaxial system are given in Section 6.

In Section 7, the equilibrium equations and equations governing bending of thin plates are deduced from their counterpart in rectangular Cartesian coordinates. It is also shown in Section 7, that the equilibrium equations could also be derived without restoring to the equations in Cartesian coordinates.

In Sections 8 and 9, respectively, the methods of integration and the construction of functional relations for the trilinear variables are investigated.

Section 10 is devoted to the application of the theory developed in this paper. Solutions to two examples are found. One of these examples is the bending of a simply supported equilateral triangular plate subjected to a uniform loading. Solution to this problem was originally obtained by Woinowsky-Krieger[10] in terms of Cartesian variables. Here, the solution to this problem is found in terms of trilinear variables. To illustrate the application of the theory to an eigenvalue problem, free transverse vibration of a simply supported triangular plate is considered, and the natural frequencies for all the symmetrical modes are found.

1. DEFINITION OF TRILINEAR COORDINATES

The trilinear coordinates η_i of a point P are the perpendicular distances of P from the straight lines forming the triangle of reference. We shall denote the vertices of the triangle of reference by A_i and the sides opposite to them by a_i, and the angles formed by the triangle at the vertices A_i will be denoted by α_i.

Sign Convention.

A coordinate η_i, which is measured from a_i, is said to be positive if it lies on the same side of a_i as the point A_i. If the side a_i lies between the points P and A_i, then η_i is negative. If the point P lies within the triangle of reference, as shown in Figure 1, then all of its coordinates are positive. In Figure 1, the point P' lies outside the triangle of reference, and η_1 has a negative value, but η_2 and η_3 are positive. No point in a plane has all its coordinates negative.

Fundamental Identity of Trilinear Coordinates.

If the area of the triangle of reference is denoted by Δ and the trilinear coordinates of any point by η_i, then

$$a_1\eta_1 + a_2\eta_2 + a_3\eta_3 = 2\Delta. \tag{1-1}$$

The above statement will now be verified for the point P shown in Figure 1 in which the area of the triangle A_2PA_3 is $\eta_1 a_1/2$, the area of the triangle A_3PA_1 is $\eta_2 a_2/2$, and the area of the triangle A_1PA_2 is $\eta_3 a_3/2$. Thus the total area Δ of the triangle $A_1A_2A_3$ is

$$\frac{a_1\eta_1}{2} + \frac{a_2\eta_2}{2} + \frac{a_3\eta_3}{2} = \Delta$$

from which equation (1-1) follows. Equation (1-1) can equally be verified for any other point in the plane by carefully adapting the sign convention mentioned earlier.

Now, if R is the radius of the circumcircle with center at 0, see Figure 2, then by sine rule,

$$2R \sin \alpha_i = a_i$$

also from analytical geometry, we know that

$$4R\Delta = a_1 a_2 a_3$$

Now denoting,

$$\sin \alpha_i = m_i \quad \text{and} \quad \cos \alpha_i = \ell_i$$

we have,

$$a_i = 2Rm_i \quad \text{and} \quad \Delta = 2R^2 m_1 m_2 m_3 \ . \tag{1-2}$$

Substituting equation (1-2) in equation (1-1) shows,

$$m_1\eta_1 + m_2\eta_2 + m_3\eta_3 = \lambda \tag{1-3}$$

where,

$$\lambda = 2R\, m_1 m_2 m_3 \tag{1-4}$$

and this quantity is a constant for a given triangle of reference. Dividing equation (1-3) by λ gives

$$\frac{m_1 \eta_1}{\lambda} + \frac{m_2 \eta_2}{\lambda} + \frac{m_3 \eta_3}{\lambda} = 1 \tag{1-5}$$

The quantities,

$$\frac{\lambda}{m_1} = 2Rm_2m_3, \quad \frac{\lambda}{m_2} = 2Rm_1m_3, \quad \frac{\lambda}{m_3} = 2Rm_1m_2$$

are called the altitudes of a triangle, and they are the lengths A_iX_i in Figure 2. Now denoting,

$$\frac{\lambda}{m_i} = H_i \qquad \text{and} \qquad \frac{\eta_i}{H_i} = \zeta_i \tag{1-6}$$

equation (1-3) becomes

$$\zeta_1 + \zeta_2 + \zeta_3 = 1 \tag{1-7}$$

The quantities ζ_i are called Areal or triangular coordinates, also referred to as Bary-centric coordinates of Möbius.

2. RELATIONSHIP BETWEEN CARTESIAN AND TRILINEAR COORDINATES.

In this section, we introduce a new coordinate system (x_1,x_2,x_3) with an origin (0,0,0) located at the in-center I of the triangle of reference. An illustration of these coordinates in relation to the triangle of reference is given in Figure 3. The coordinate system (x_1,x_2,x_3) will henceforth be called Triaxial coordinates, and these can be readily related to the Cartesian coordinates by usual coordinate transformations. Also by introducing these coordinates into the analysis, it becomes relatively easy to transform differential equations, which are in terms of the Cartesian coordinate variables (x,y), into trilinear variables (η_1,η_2,η_3).

In the subsequent analysis, we shall use simultaneously triaxial and trilinear coordinates. In deriving and stating equations of elasticity, we shall use triaxial coordinates (x_1,x_2,x_3) because they are more convenient and compact. When it comes to solving a boundary value problem with respect to a triangle, emphasis will be placed on the trilinear system (η_1,η_2,η_3) because it is more convenient to specify the boundary conditions.

In discussing derivatives of a function with respect to a coordinate system, one must distinguish between directional and partial derivatives. In our analysis, we shall use triaxial variables (x_1,x_2,x_3) for the specification of directional derivative of a function. For example, the directional derivatives of a function F will be denoted symbolically by $\partial F/\partial x_i$, $\partial^2 F/\partial x_i^2$ etc., and these derivatives have physical meanings, such as rate of change of F with respect to x_i, curvature of F with respect to x_i etc.

Although the directional derivatives with respect to x_i variables have physical interpretations, these are not convenient in finding solutions to physical problems. For this reason, prior to solving a boundary value problem, we shall express the directional derivatives in terms of the partial derivatives with respect to the trilinear coordinate variables (η_1, η_2, η_3).

It should be carefully noted that whenever $\partial/\partial x_i$ appears it represents a directional derivative and that whenever $\partial/\partial \eta_i$ appears it represents a partial derivative. Also it should be emphasized that each component of x_i refers to not only a triaxial coordinate but also a Cartesian coordinate.

In Figure 3, three orthogonal Cartesian coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) are shown. These coordinate variables are related to one another as follows.

$$\left.\begin{aligned}
\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \begin{bmatrix} -\ell_3 & m_3 \\ -m_3 & -\ell_3 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad : \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -\ell_3 & -m_3 \\ m_3 & -\ell_3 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\
\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} &= \begin{bmatrix} -\ell_2 & -m_2 \\ m_2 & -\ell_2 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad : \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -\ell_2 & m_2 \\ -m_2 & -\ell_2 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \\
\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} &= \begin{bmatrix} -\ell_1 & m_1 \\ -m_1 & -\ell_1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad : \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -\ell_1 & -m_1 \\ m_1 & -\ell_1 \end{bmatrix}\begin{bmatrix} x_3 \\ y_3 \end{bmatrix}
\end{aligned}\right\} \quad (2\text{-}1)$$

Further,

$$\boxed{\begin{aligned} m_1x_1 + m_2x_2 + m_3x_3 &= 0 \\ m_1y_1 + m_2y_2 + m_3y_3 &= 0 \end{aligned}} \tag{2-2}$$

the trilinear variables η_i are related to x_i and y_i variables by

$$\begin{aligned} \eta_1 &= \rho - x_1 \\ \eta_2 &= \rho - x_2 = \rho + x_1\ell_3 - y_1m_3 \\ \eta_3 &= \rho - x_3 = \rho + x_1\ell_2 + y_1m_2 \end{aligned} \tag{2-3}$$

In which ρ is the radius of in-circle, $\rho = 4R \sin \alpha_1/2 \sin \alpha_2/2 \sin \alpha_3/2$.

Equation (2-3) can be written in matrix form as,

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & \ell_3 & -m_3 \\ 1 & \ell_2 & m_2 \end{bmatrix} \begin{bmatrix} \rho \\ x_1 \\ y_1 \end{bmatrix} \tag{2-4}$$

Equation (2-4) can be inverted to give

$$\begin{bmatrix} \rho \\ x_1 \\ y_1 \end{bmatrix} = \frac{1}{(m_1 + m_2 + m_3)} \begin{bmatrix} m_1 & m_2 & m_3 \\ -(m_2 + m_3) & m_2 & m_3 \\ (\ell_2 - \ell_3) & -(1+\ell_2) & (1+\ell_3) \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \tag{2-5}$$

Using equations (2-1), (2-2) and (2-4), we have,

$$\left.\begin{aligned}
\frac{\partial}{\partial x_1} &= \left[-\frac{\partial}{\partial \eta_1} + \ell_3\frac{\partial}{\partial \eta_2} + \ell_2\frac{\partial}{\partial \eta_3}\right] \\
\frac{\partial}{\partial x_2} &= -\left[\ell_3\frac{\partial}{\partial x_1} - m_3\frac{\partial}{\partial y_1}\right] = \left[\ell_3\frac{\partial}{\partial \eta_1} - \frac{\partial}{\partial \eta_2} + \ell_1\frac{\partial}{\partial \eta_3}\right] \\
\frac{\partial}{\partial x_3} &= -\left[\ell_2\frac{\partial}{\partial x_1} + m_2\frac{\partial}{\partial y_1}\right] = \left[\ell_2\frac{\partial}{\partial \eta_1} + \ell_1\frac{\partial}{\partial \eta_2} - \frac{\partial}{\partial \eta_3}\right]
\end{aligned}\right\} \tag{2-6}$$

In matrix form the above equation can be written as,

$$\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -1 & \ell_3 & \ell_2 \\ \ell_3 & -1 & \ell_1 \\ \ell_2 & \ell_1 & -1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \eta_1} \\ \frac{\partial}{\partial \eta_2} \\ \frac{\partial}{\partial \eta_3} \end{bmatrix} \tag{2-7}$$

The above equation cannot be inverted, and it should be noticed.

$$\boxed{m_1\frac{\partial}{\partial x_1} + m_2\frac{\partial}{\partial x_2} + m_3\frac{\partial}{\partial x_3} = 0} \tag{2-8}$$

Equations developed so far in this section are utilized to construct the following further relations amoungst the differential operators, which are presented without further proof.

$$\left.\begin{aligned}
\frac{\partial}{\partial y_1} &= \frac{1}{m_1}\left[\ell_2\frac{\partial}{\partial x_2} - \ell_3\frac{\partial}{\partial x_3}\right] = \left[-m_3\frac{\partial}{\partial \eta_2} + m_2\frac{\partial}{\partial \eta_3}\right] \\
\frac{\partial}{\partial y_2} &= -\left[m_3\frac{\partial}{\partial x_1} + \ell_3\frac{\partial}{\partial y_1}\right] = \left[-m_1\frac{\partial}{\partial \eta_3} + m_3\frac{\partial}{\partial \eta_1}\right] \\
\frac{\partial}{\partial y_3} &= \left[m_2\frac{\partial}{\partial x_1} - \ell_2\frac{\partial}{\partial y_1}\right] = \left[-m_2\frac{\partial}{\partial \eta_1} + m_1\frac{\partial}{\partial \eta_2}\right]
\end{aligned}\right\} \tag{2-9}$$

$$\boxed{m_1 \frac{\partial}{\partial y_1} + m_2 \frac{\partial}{\partial y_2} + m_3 \frac{\partial}{\partial y_3} = 0} \tag{2-10}$$

$$\frac{\partial^2}{\partial x_1^2} = \left[\frac{\partial^2}{\partial \eta_1^2} + \ell_3^2 \frac{\partial^2}{\partial \eta_2^2} + \ell_2^2 \frac{\partial^2}{\partial \eta_3^2} - 2\ell_3 \frac{\partial^2}{\partial \eta_1 \partial \eta_2} - 2\ell_2 \frac{\partial^2}{\partial \eta_1 \partial \eta_3} + 2\ell_3\ell_2 \frac{\partial^2}{\partial \eta_2 \partial \eta_3}\right]$$

$$\frac{\partial^2}{\partial x_2^2} = \left[\frac{\partial^2}{\partial \eta_2^2} + \ell_3^2 \frac{\partial^2}{\partial \eta_1^2} + \ell_1^2 \frac{\partial^2}{\partial \eta_3^2} - 2\ell_3 \frac{\partial^2}{\partial \eta_1 \partial \eta_2} - 2\ell_1 \frac{\partial^2}{\partial \eta_2 \partial \eta_3} + 2\ell_3\ell_1 \frac{\partial^2}{\partial \eta_1 \partial \eta_3}\right]$$

$$\frac{\partial^2}{\partial x_3^2} = \left[\frac{\partial^2}{\partial \eta_3^2} + \ell_2^2 \frac{\partial^2}{\partial \eta_1^2} + \ell_1^2 \frac{\partial^2}{\partial \eta_2^2} - 2\ell_1 \frac{\partial^2}{\partial \eta_2 \partial \eta_3} - 2\ell_2 \frac{\partial^2}{\partial \eta_1 \partial \eta_3} + 2\ell_1\ell_2 \frac{\partial^2}{\partial \eta_1 \partial \eta_2}\right]$$

(2-11)

$$\left.\begin{aligned}
\frac{\partial^2}{\partial y_1^2} &= \left[m_3^2 \frac{\partial^2}{\partial \eta_2^2} + m_2^2 \frac{\partial^2}{\partial \eta_3^2} - 2m_2m_3 \frac{\partial^2}{\partial \eta_2 \partial \eta_3}\right] \\
\frac{\partial^2}{\partial y_2^2} &= \left[m_1^2 \frac{\partial^2}{\partial \eta_3^2} + m_3^2 \frac{\partial^2}{\partial \eta_1^2} - 2m_1m_3 \frac{\partial^2}{\partial \eta_1 \partial \eta_3}\right] \\
\frac{\partial^2}{\partial y_3^2} &= \left[m_2^2 \frac{\partial^2}{\partial \eta_1^2} + m_1^2 \frac{\partial^2}{\partial \eta_2^2} - 2m_1m_2 \frac{\partial^2}{\partial \eta_1 \partial \eta_2}\right]
\end{aligned}\right\} \tag{2-12}$$

$$\frac{\partial^2}{\partial x_1 \partial y_1} = \left[m_3 \frac{\partial^2}{\partial \eta_1 \partial \eta_2} - m_2 \frac{\partial^2}{\partial \eta_1 \partial \eta_3} - \ell_3 m_3 \frac{\partial^2}{\partial \eta_2^2} + \ell_2 m_2 \frac{\partial^2}{\partial \eta_3^2} + \right.$$

$$\left. (\ell_3 m_2 - \ell_2 m_3) \frac{\partial^2}{\partial \eta_2 \partial \eta_3}\right] \tag{2-13}$$

The Laplace's operator ∇^2

$$\nabla^2 = \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right] = \left[\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2}\right] = \left[\frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial y_3^2}\right]$$

$$= \left[\frac{\partial^2}{\partial \eta_1^2} + \frac{\partial^2}{\partial \eta_2^2} + \frac{\partial^2}{\partial \eta_3^2} - 2\ \ell_1 \frac{\partial^2}{\partial \eta_2 \partial \eta_3} + \ell_2 \frac{\partial^2}{\partial \eta_1 \partial \eta_3} + \ell_3 \frac{\partial^2}{\partial \eta_1 \partial \eta_2}\right] \quad (2\text{-}14)$$

Also,

$$\nabla^2 = \frac{1}{m_1 m_2 m_3}\left[\ell_1 m_1 \frac{\partial^2}{\partial x_1^2} + \ell_2 m_2 \frac{\partial^2}{\partial x_2^2} + \ell_3 m_3 \frac{\partial^2}{\partial x_3^2}\right] \quad (2\text{-}15)$$

and,

$$\nabla^2 = \frac{1}{m_1 m_2 m_3}\left[\ell_1 m_1 \frac{\partial^2}{\partial y_1^2} + \ell_2 m_2 \frac{\partial^2}{\partial y_2^2} + \ell_3 m_3 \frac{\partial^2}{\partial y_3^2}\right] \quad (2\text{-}16)$$

From equation (2-7),

$$\left[m_1 \frac{\partial}{\partial x_1} + m_2 \frac{\partial}{\partial x_2} + m_3 \frac{\partial}{\partial x_3}\right] = 0$$

therefore,

$$\left(m_1 \frac{\partial}{\partial x_1} + m_2 \frac{\partial}{\partial x_2} + m_3 \frac{\partial}{\partial x_3}\right) \cdot \left(\ell_1 \frac{\partial}{\partial x_1} + \ell_2 \frac{\partial}{\partial x_2} + \ell_3 \frac{\partial}{\partial x_3}\right) = 0$$

that is, by using equation (2-15),

$$\left.\begin{aligned} m_1 m_2 m_3 \nabla^2 &= \left[\ell_1 m_1 \frac{\partial^2}{\partial x_1^2} + \ell_2 m_2 \frac{\partial^2}{\partial x_2^2} + \ell_3 m_3 \frac{\partial^2}{\partial x_3^2}\right] \\ &= -\left[m_3 \frac{\partial^2}{\partial x_1 \partial x_2} + m_2 \frac{\partial^2}{\partial x_1 \partial x_3} + m_1 \frac{\partial^2}{\partial x_2 \partial x_3}\right] \end{aligned}\right\} \quad (2\text{-}17)$$

3. TRIAXIAL STRESS SYSTEM

In this section, using the concept of a triaxial coordinate system (x_1, x_2, x_3) shown in Figure 3, a stress system (σ_1, σ_2, σ_3) will be introduced. The three stresses σ_1, σ_2, and σ_3 are independent stress quantities corresponding to the triaxial system (x_1, x_2, x_3), in the same sense that (σ_x, σ_y, τ_{xy}) are three independent stress components in the familiar Cartesian coordinates (x,y).

Although the state of stress is completely defined by the triaxial stress components (σ_1, σ_2, σ_3), one may not consider these quantities as the direct stress components corresponding to the triaxial coordinate system. It should be remembered that the stress state (σ_1, σ_2, σ_3) is introduced merely for the purpose of analytical convenience in the formulation of the problem, and it has no direct significance in terms of strength related stresses. In engineering design, the normal and shear stress components corresponding to the coordinate axes are of importance. It is therefore essential to define the triaxial stress components in terms of the normal and shear stress components, that is, to derive the relationships between these stress components.

In Figure 4 the familiar normal and shear stress components (i.e., the Cartesian stress components) are shown in relation to the triangle of reference. With reference to this figure, normal stresses perpendicular to the boundary are denoted by (σ_{1n}, σ_{2n}, σ_{3n}), normal stresses parallel to the boundary are denoted by (σ_{1s}, σ_{2s}, σ_{3s}) and the shear stress components are denoted by (τ_1, τ_2, τ_3). It will now be shown, that the Cartesian stress components (σ_{in}, σ_{is}, τ_i), $i = 1, 2, 3$, are linear combinations of the triaxial stresses. For this purpose, consider the stress state in a triangular element, subjected to a triaxial stress field (σ_1, σ_2, σ_3) as shown in Figure 5.

As the primary interest of this section is to define the triaxial stresses, it will be assumed here that the element shown in Figure 5 is in static equilibrium. With this assumption one may conclude that any element within this triangle must also be in equilibrium. For example the square element of side δ inside the triangle of Figure 5 is in equilibrium.

The relationships between the triaxial stress components and the Cartesian stress components (σ_{1n}, σ_{1s}, τ_1) will now be found by considering the forces acting on the square element. In Figure 6(a) the forces acting on the square element are illustrated. With reference to this figure, forces acting on sides 1$\leftrightarrow$2 and 2$\leftrightarrow$3 are specified in terms of the usual Cartesian stress components, but the forces on sides 3 $\leftrightarrow$ 4 and 4$\leftrightarrow$1 are specified in terms of the triaxial stress components. In Figure 6(b), the forces due to the triaxial stress components shown in Figure 6(a) are resolved into a parallel and a perpendicular component with respect to the x_1-axis.

Since by assumption the square element is in static equilibrium, one may readily deduce from Figure 6(b), the following equality between the triaxial stresses (σ_1, σ_2, σ_3) and the Cartesian stress components (σ_{1n}, σ_{1s}, τ_1).

$$\left.\begin{aligned} \sigma_{1n} &= \sigma_1 + \sigma_2\ell_3^2 + \sigma_3\ell_2^2 \\ \sigma_{1s} &= \sigma_2 m_3^2 + \sigma_3 m_2^2 \\ \tau_1 &= -\sigma_2\ell_3 m_3 + \sigma_3\ell_2 m_2 \end{aligned}\right\} \quad (3\text{-}1)$$

The stresses σ_{1n} and σ_{1s} are the direct stresses acting parallel and perpendicular to the coordinate axis x_1.

A similar procedure can now be followed to derive the stress system (σ_{2n}, σ_{2s}, τ_2) and (σ_{3n}, σ_{3s}, τ_3). In the former, the square element is rotated clockwise through an angle α_3, and in the latter, anti-clockwise by α_2. Hence,

$$\left.\begin{aligned} \sigma_{2n} &= \sigma_2 + \sigma_3\ell_1^2 + \sigma_1\ell_3^2 \\ \sigma_{2s} &= \sigma_3 m_1^2 + \sigma_1 m_3^2 \\ \tau_2 &= -\sigma_3\ell_1 m_1 + \sigma_1\ell_3 m_3 \end{aligned}\right\} \quad (3\text{-}2)$$

and,

$$\left.\begin{aligned} \sigma_{3n} &= \sigma_3 + \sigma_2\ell_1^2 + \sigma_1\ell_2^2 \\ \sigma_{3s} &= \sigma_2 m_1^2 + \sigma_1 m_2^2 \\ \tau_3 &= -\sigma_1\ell_2 m_2 + \sigma_2\ell_1 m_1 \end{aligned}\right\} \quad (3\text{-}3)$$

The relationships between the direct stress components and the triaxial stresses can now be summarized in matrix form as,

$$\begin{bmatrix} \sigma_{1n} \\ \sigma_{2n} \\ \sigma_{3n} \end{bmatrix} = \begin{bmatrix} 1 & \ell_3^2 & \ell_2^2 \\ \ell_3^2 & 1 & \ell_1^2 \\ \ell_2^2 & \ell_1^2 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \qquad (3\text{-}4)$$

$$\begin{bmatrix} \sigma_{1s} \\ \sigma_{2s} \\ \sigma_{3s} \end{bmatrix} = \begin{bmatrix} 0 & m_3^2 & m_2^2 \\ m_3^2 & 0 & m_1^2 \\ m_2^2 & m_1^2 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \qquad (3\text{-}5)$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} 0 & -\ell_3 m_3 & \ell_2 m_2 \\ \ell_3 m_3 & 0 & -\ell_1 m_1 \\ -\ell_2 m_2 & \ell_1 m_1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \qquad (3\text{-}6)$$

Equation (3-4) relates the direct stress components σ_{in} to the triaxial stress components σ_i. This equation can be inverted, to give,

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \frac{1}{2m_1^2 m_2^2 m_3^2} \begin{bmatrix} 1 - \ell_1^4 & \ell_1^2\ell_2^2 - \ell_3^2 & \ell_3^2\ell_1^2 - \ell_2^2 \\ \ell_1^2\ell_2^2 - \ell_3^2 & 1 - \ell_2^4 & \ell_2^2\ell_3^2 - \ell_1^2 \\ \ell_3^2\ell_1^2 - \ell_2^2 & \ell_2^2\ell_3^2 - \ell_1^2 & 1 - \ell_3^4 \end{bmatrix} \begin{bmatrix} \sigma_{1n} \\ \sigma_{2n} \\ \sigma_{3n} \end{bmatrix} \qquad (3\text{-}7)$$

Equation (3-5) can also be inverted to give

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \frac{1}{2m_1^2 m_2^2 m_3^2} \begin{bmatrix} -m_1^4 & m_1^2 m_2^2 & m_3^2 m_1^2 \\ m_1^2 m_2^2 & -m_2^4 & m_2^2 m_3^2 \\ m_3^2 m_1^2 & m_2^2 m_3^2 & -m_3^4 \end{bmatrix} \begin{bmatrix} \sigma_{1s} \\ \sigma_{2s} \\ \sigma_{3s} \end{bmatrix} \tag{3-8}$$

It should be noted that equation (3-6) cannot be inverted, because the determinant of the matrix

$$\begin{bmatrix} 0 & -\ell_3 m_3 & \ell_2 m_2 \\ \ell_3 m_3 & 0 & -\ell_1 m_1 \\ -\ell_2 m_2 & \ell_1 m_1 & 0 \end{bmatrix}$$

is zero, and there exists the relationship,

$$\boxed{\ell_1 m_1 \tau_1 + \ell_2 m_2 \tau_2 + \ell_3 m_3 \tau_3 = 0} \tag{3-9}$$

Using the trigonometrical identities

$$\left.\begin{aligned} m_i &= m_j \ell_k + \ell_j m_k \\ m_j m_k &= \ell_i + \ell_j \ell_k \end{aligned}\right\} \quad \begin{aligned} &i,\ j,\ k = 1,\ 2,\ 3 \\ &i \neq j \neq k \end{aligned} \tag{3-10}$$

it can be shown,

$$\left.\begin{aligned} &\ell_1 m_1\, \sigma_{1n} + \ell_2 m_2\, \sigma_{2n} + \ell_3 m_3\, \sigma_{3n} = m_1 m_2 m_3 (\sigma_1 + \sigma_2 + \sigma_3) \\ &\text{and} \\ &\ell_1 m_1\, \sigma_{1s} + \ell_2 m_2\, \sigma_{2s} + \ell_3 m_3\, \sigma_{3s} = m_1 m_2 m_3 (\sigma_1 + \sigma_2 + \sigma_3) \end{aligned}\right\} \tag{3-11}$$

It will be shown in Section 6, (equation (6-6)), that

$$(\sigma_1 + \sigma_2 + \sigma_3) = \Theta$$, the first stress invarient.

4. STRESS-STRAIN RELATIONS IN TERMS OF TRIAXIAL STRESS AND STRAIN COMPONENTS

The stress-strain relationships, to be derived in this section, are for an isotropic material having modulus of elasticity E, and Poisson's ratio ν. The direct in plane strains, acting along the triaxial coordinate axis x_1, x_2 and x_3 will be denoted by ε_{1n}, ε_{2n} and ε_{3n} respectively. In the case of plane stress (i.e.: the out of plane stress $\sigma_z = 0$), the in-plane direct strains ε_{in} are defined to be (11)

$$\varepsilon_{in} = \frac{1}{E}\left(\sigma_{in} - \nu\,\sigma_{is}\right) \qquad i = 1,\ 2,\ 3 \tag{4-1}$$

The quantities σ_{in} and σ_{is} ($i = 1, 2, 3$) are direct in-plane stresses. The orientation of these direct in-plane stresses and strains, in relation to the triangle of reference, is illustrated in Figure 4. The direct stresses σ_{in} and σ_{is} are related to the triaxial stresses (σ_1, σ_2, σ_3) and the relationships are given in Section 3.
From equation (4-1) the direct strain ε_{in} is given by

$$\varepsilon_{1n} = \frac{1}{E}\left(\sigma_{1n} - \nu\,\sigma_{1s}\right)$$

substituting for the direct stresses σ_{1n} and σ_{1s}, from Equations (3-4) and (3-5) into the above equation gives,

$$\varepsilon_{1n} = \frac{1}{E}\left[\sigma_1 + \sigma_2(\ell_3^2 - \nu m_3^2) + \sigma_3(\ell_2^2 - \nu m_2^2)\right] \tag{4-2}$$

similarly,

$$\varepsilon_{2n} = \frac{1}{E}\left[\sigma_2 + \sigma_1(\ell_3^2 - \nu m_3^2) + \sigma_3(\ell_1^2 - \nu m_1^2)\right] \tag{4-3}$$

$$\varepsilon_{3n} = \frac{1}{E}\left[\sigma_3 + \sigma_1(\ell_2^2 - \nu m_2^2) + \sigma_2(\ell_1^2 - \nu m_1^2)\right] \tag{4-4}$$

Equations (4-2), (4-3) and (4-4) can be conveniently represented in matrix form as,

$$\{\varepsilon_t\} = \left[T_{\varepsilon\sigma t}\right]\{\sigma_t\} \tag{4-5}$$

where,

$$\left[T_{\varepsilon\sigma t}\right] = \frac{1}{E}\begin{bmatrix} 1 & \ell_3^2 - \nu m_3^2 & \ell_2^2 - \nu m_2^2 \\ \ell_3^2 - \nu m_3^2 & 1 & \ell_1^2 - \nu m_1^2 \\ \ell_2^2 - \nu m_2^2 & \ell_1^2 - \nu m_1^2 & 1 \end{bmatrix} \tag{4-6}$$

$$\{\sigma_t\} = \{\sigma_1\ \sigma_2\ \sigma_3\}$$

and

$$\{\varepsilon_t\} = \{\varepsilon_{1n}\ \varepsilon_{2n}\ \varepsilon_{3n}\}$$

Matrix $\left[T_{\varepsilon\sigma t}\right]$ can be written as,

$$\left[T_{\varepsilon\sigma t}\right] = \left[R\right]\left[T_{\varepsilon\sigma}\right]\left[R\right]^T \tag{4-7}$$

where

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_3^2 & m_3^2 & -\ell_3 m_3 \\ \ell_2^2 & m_2 & \ell_2 m_2 \end{bmatrix} \quad (4\text{-}8)$$

and

$$[T_{\varepsilon\sigma}] = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \quad (4\text{-}9)$$

Therefore, substituting (4-7) in Equation (4-5) gives,

$$\{\varepsilon_t\} = [R]\,[T_{\varepsilon\sigma}]\,[R]^T\,\{\sigma_t\} \quad (4\text{-}10)$$

<u>Note</u>: Equation (4-7) was established using the following trigonometric relations.

Since, $m_1 = \sin\alpha_1$ and $\pi = \alpha_1 + \alpha_2 + \alpha_3$,

$$\begin{aligned} m_1 &= \sin\alpha_1 = \sin(\pi - (\alpha_2 + \alpha_3)) = \sin(\alpha_2 + \alpha_3) \\ &= \sin\alpha_2 \cos\alpha_3 + \sin\alpha_3 \cos\alpha_2 \\ &= m_2\,\ell_3 + m_3\,\ell_2 \end{aligned} \quad (4\text{-}11)$$

Similarly,

$$\begin{aligned} \ell_1 &= \cos\alpha_1 = -\cos(\alpha_2 + \alpha_3) = -(\cos\alpha_2 \cos\alpha_3 - \\ &\quad \sin\alpha_2 \sin\alpha_3) \\ &= -(\ell_2\,\ell_3 - m_2\,m_3) \end{aligned} \quad (4\text{-}12)$$

Equation (4-5), in its expanded form is given by

$$\begin{bmatrix} \varepsilon_{1n} \\ \varepsilon_{2n} \\ \varepsilon_{3n} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & \ell_3^2 - \nu m_3^2 & \ell_2^2 - \nu m_2^2 \\ \ell_3^2 - \nu m_3^2 & 1 & \ell_1^2 - \nu m_1^2 \\ \ell_2^2 - \nu m_2^2 & \ell_1^2 - \nu m_1^2 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \qquad (4\text{-}13)$$

The above relation can be inverted to give,

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = A \begin{bmatrix} m_1^2(\ell_1^2+\gamma m_1^2) & m_1 m_2(\ell_1\ell_2-\gamma m_1 m_2) & m_1 m_3(\ell_1\ell_3-\gamma m_1 m_3) \\ m_1 m_2(\ell_1\ell_2-\gamma m_1 m_2) & m_2(\ell_2^2+\gamma m_2^2) & m_2 m_3(\ell_2\ell_3-\gamma m_2 m_3) \\ m_1 m_3(\ell_1\ell_3-\gamma m_1 m_3) & m_2 m_3(\ell_2\ell_3-\gamma m_2 m_3) & m_3^2(\ell_3^2+\gamma m_3^2) \end{bmatrix} \begin{bmatrix} \varepsilon_{1n} \\ \varepsilon_{2n} \\ \varepsilon_{3n} \end{bmatrix} \qquad (4\text{-}14)$$

where,

$$\gamma = \frac{(1 - \nu)}{2} \quad \text{and} \quad A = \frac{E}{(1 - \nu^2)} \cdot \frac{1}{m_1^2 m_2^2 m_3^2} \qquad (4\text{-}15)$$

Equations (4-13) and (4-14) are matrix relationships between the tri-axial stress components and direct strain components. By using equation (3-7), in conjunction with equation (4-14), we can relate the direct stress components to direct strain components.

That is,

$$\begin{bmatrix} \sigma_{1n} \\ \sigma_{2n} \\ \sigma_{3n} \end{bmatrix} = \frac{E}{(1 - \nu^2)} \begin{bmatrix} 1 - \beta m_1 \ell_2 \ell_3 & \beta \ell_2 m_2 & \beta m_3 \ell_3 \\ \beta m_1 \ell_1 & 1 - \beta m_2 \ell_1 \ell_3 & \beta m_3 \ell_3 \\ \beta m_1 \ell_1 & \beta m_2 \ell_2 & 1 - \beta m_3 \ell_1 \ell_2 \end{bmatrix} \begin{bmatrix} \varepsilon_{1n} \\ \varepsilon_{2n} \\ \varepsilon_{3n} \end{bmatrix} \tag{4-16}$$

Where,

$$\beta = \frac{\nu}{m_1 m_2 m_3} \tag{4-17}$$

Equation (4-16) can be inverted to give,

$$\begin{bmatrix} \varepsilon_{1n} \\ \varepsilon_{2n} \\ \varepsilon_{3n} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 + \beta m_1 \ell_2 \ell_3 & -\beta m_2 \ell_2 & -\beta m_3 \ell_3 \\ -\beta m_1 \ell_1 & 1 + \beta m_2 \ell_1 \ell_3 & -\beta m_3 \ell_3 \\ -\beta m_1 \ell_1 & -\beta m_2 \ell_2 & 1 + \beta m_3 \ell_2 \ell_1 \end{bmatrix} \begin{bmatrix} \sigma_{1n} \\ \sigma_{2n} \\ \sigma_{3n} \end{bmatrix} \tag{4-18}$$

Note: Equations (4-16) and (4-18) are matrix relationships between the direct strain components and direct stress components. It should be noticed that these matrices are symmetrical only if the triangle of reference is equilateral.

5. RELATIONSHIP BETWEEN STRESS (STRAIN) COMPONENTS IN TRIAXIAL COORDINATES AND THE STRESS (STRAIN) COMPONENTS IN CARTESIAN COORDINATES

In this section, the stress components in the triaxial coordinates will be related to the stress components in Cartesian coordinates. In the process of deriving this relationship between the stress components, the relationships between the strain components in these two coordinate system will be established. The relative orientation of these two coordinate systems in the plane will be defined by an angle θ, as shown in Figure 7. The reason for introducing this angle will be apparent in Section 6, where it is shown that the relative orientation of these two coordinate systems is irrelevent in computing principal stresses and strain energy of deformation.

If the strain components in the Cartesian coordinates are denoted by (ε_x ε_y γ_{xy}), then, the direct triaxial strains (ε_{1n} ε_{2n} ε_{3n}) can be related to these by a resolution of strain components.(11)

For example, the direct strain ε_{1n} corresponding to the x_1-axis is,

$$\varepsilon_{1n} = \varepsilon_x \ell^2 + \varepsilon_y m^2 + \gamma_{xy} \ell m \tag{5-1}$$

where

$$\ell = \cos\theta \quad \text{and} \quad m = \sin\theta$$

The direct strain corresponding to the x_2-axis is,

$$\varepsilon_{2n} = \varepsilon_x \cos^2(\alpha_1 + \alpha_2 + \theta) + \varepsilon_y \sin^2(\alpha_1 + \alpha_2 + \theta)$$

$$+ \gamma_{xy} \cos(\alpha_1 + \alpha_2 + \theta) \sin(\alpha_1 + \alpha_2 + \theta) \qquad (5\text{-}2)$$

This equation can be simplified using the following trigonometric relations. Since,

$$\pi = \alpha_1 + \alpha_2 + \alpha_3$$

it follows that,

$$\sin(\alpha_1 + \alpha_2 + \theta) = \sin(\pi - (\alpha_3 - \theta)) = \sin(\alpha_3 - \theta)$$

$$= \sin\alpha_3 \cos\theta - \cos\alpha_3 \sin\theta$$

$$= m_3\ell - \ell_3 m \qquad (5\text{-}3)$$

$$\cos(\alpha_1 + \alpha_2 + \theta) = \cos(\pi - (\alpha_3 - \theta)) = -\cos(\alpha_3 - \theta)$$

$$= -(\cos\alpha_3 \cos\theta + \sin\alpha_3 \sin\theta)$$

$$= -(\ell_3\ell + m_3 m) \qquad (5\text{-}4)$$

From Equations (5-3) and (5-4) it follows that,

$$\cos\,(\alpha_1 + \alpha_2 + \theta)\,\sin\,(\alpha_1 + \alpha_2 + \theta)$$

$$= -\,(m_3\ell - \ell_3 m)\,(\ell_3\ell + m_3 m)$$

$$= \ell m\,(\ell_3^2 - m_3^2) - \ell_3 m_3\,(\ell^2 - m^2) \tag{5-5}$$

Substituting (5-3), (5-4) and (5-5) into Equation (5-2) gives,

$$\varepsilon_{2n} = \varepsilon_x\,(\ell_3\ell + m_3 m)^2 + \varepsilon_y\,(\ell m_3 - m\ell_3)^2$$

$$+ \gamma_{xy}\left[\ell_m\,(\ell_3^2 - m_3^2) - \ell_3 m_3\,(\ell^2 - m^2)\right] \tag{5-6}$$

Similarly the direct strain corresponding to the x_3-axis is given by,

$$\varepsilon_{3n} = \varepsilon_x \cos^2\,(\pi + \alpha_2 + \theta) + \varepsilon_y \sin^2\,(\pi + \alpha_2 + \theta)$$

$$+ \gamma_{xy}\left[\cos\,(\pi + \alpha_2 + \theta)\,\sin\,(\pi + \alpha_2 + \theta)\right]$$

Once again, using the trigonometric relations, the above equation can be written as,

$$\varepsilon_{3n} = \varepsilon_x\,(\ell\ell_2 - mm_2)^2 + \varepsilon_y\,(m\ell_2 + \ell m_2)^2$$

$$+ \gamma_{xy}\left[\ell m\,(\ell_2^2 - m_2^2) + \ell_2 m_2\,(\ell^2 - m^2)\right] \tag{5-7}$$

The direct strains given by Equations (5-1), (5-5) and (5-7) can be conveniently represented in the following matrix form,

$$\{\varepsilon_t\} = \left[B_{\varepsilon tc}\right]\,\{\varepsilon_c\} \tag{5-8}$$

where

$$\left[B_{\varepsilon tc}\right] = \begin{bmatrix} \ell^2 & m^2 & \ell m \\ (\ell\ell_3 + mm_3)^2 & (\ell m_3 - m\ell_3)^2 & \ell m(\ell_3^2 - m_3^2) - \ell_3 m_3(\ell^2 - m^2) \\ (\ell\ell_2 - mm_2)^2 & (m\ell_2 + \ell m_2)^2 & \ell m(\ell_2^2 - m_2^2) + \ell_2 m_2(\ell^2 - m^2) \end{bmatrix} \tag{5-9}$$

$$\{\varepsilon_c\} = \{\varepsilon_x \; \varepsilon_y \; \gamma_{xy}\}$$

and

$$\{\varepsilon_t\} = \{\varepsilon_{1n} \; \varepsilon_{2n} \; \varepsilon_{3n}\}$$

The matrix $\left[B_{\varepsilon tc}\right]$ can be factorized in the following form,

$$\left[B_{\varepsilon tc}\right] = \left[R\right]\left[\theta\right] \tag{5-10}$$

where, the matrix $\left[R\right]$ is defined by Equation (4-8),

$$\left[\theta\right] = \begin{bmatrix} \ell^2 & m^2 & \ell m \\ m^2 & \ell^2 & -\ell m \\ -2\ell m & 2\ell m & (\ell^2 - m^2) \end{bmatrix} \tag{5-11}$$

and,

$$\ell = \cos\theta, \; m = \sin\theta$$

Substituting (5-10) into Equation (5-8) gives,

$$\{\varepsilon_t\} = \left[R\right] \left[\theta\right] \{\varepsilon_c\} \tag{5-12}$$

Using Equation (4-5), Equation (5-12) becomes,

$$\{\sigma_t\} = \left[T_{\varepsilon\sigma t}\right]^{-1} \left[R\right] \left[\theta\right] \{\varepsilon_c\} \tag{5-13}$$

The stress-strain relation in the Cartesian coordinates is given by,

$$\{\varepsilon_c\} = \left[T_{\varepsilon\sigma}\right] \{\sigma_c\} \tag{5-14}$$

Where, matrix $\left[T_{\varepsilon\sigma}\right]$ is defined by Equation (4-9).

Substituting (5-14) into Equation (5-13) gives,

$$\{\sigma_t\} = \left[T_{\varepsilon\sigma t}\right]^{-1} \left[R\right] \left[\theta\right] \left[T_{\varepsilon\sigma}\right] \{\sigma_c\} \tag{5-15}$$

From this Equation it follows that,

$$\{\sigma_c\} = \left[T_{\varepsilon\sigma}\right]^{-1} \left[\theta\right]^{-1} \left[R\right]^{-1} \left[T_{\varepsilon\sigma t}\right] \{\sigma_t\} \tag{5-16}$$

Substituting (4-7) in Equation (5-16) gives,

$$\{\sigma_c\} = \left[T_{\varepsilon\sigma}\right]^{-1} \left[\theta\right]^{-1} \left[T_{\varepsilon\sigma}\right] \left[R\right]^{T} \{\sigma_t\} \tag{5-17}$$

Using Equation (4-9) and (5-11), it can be shown that,

$$\left[T_{\varepsilon\sigma}\right]^{-1} \left[\theta\right]^{-1} \left[T_{\varepsilon\sigma}\right] = \left[\theta\right]^{T} \tag{5-18}$$

Substituting (5-18) in Equation (5-17) gives,

$$\{\sigma_c\} = \left[\theta\right]^{T} \left[R\right]^{T} \{\sigma_t\} \tag{5-19}$$

Therefore, expanding Equation (5-19) yields,

$$\left.\begin{aligned}\sigma_x &= \sigma_1\ \ell^2 + \sigma_2(\ell\ell_3 + mm_3)^2 + \sigma_3(\ell\ell_2 - mm_2)^2 \\ \sigma_y &= \sigma_1\ m^2 + \sigma_2(m\ell_3 - \ell m_3)^2 + \sigma_3(m\ell_2 + \ell m_2)^2 \\ \tau_{xy} &= \sigma_1\ \ell m + \sigma_2\ \left[\ell m(\ell_3^2 - m_3^2) - \ell_3 m_3(\ell^2 - m^2)\right] \\ &\qquad + \sigma_3\ \left[\ell m(\ell_2^2 - m_2^2) + \ell_2 m_2(\ell^2 - m^2)\right]\end{aligned}\right\} \quad (5\text{-}20)$$

Using Equation (5-20), it will be shown in Section 6 that the principal stresses and the strain energy of deformation are independent of the arbitrary angle θ. After considering Section 6, and without loss of generality, it would be advantageous to take the angle θ equal to zero. The implication of the angle θ equal to zero is that the x axis of the Cartesian coordinates is coaxial with the x_1 axis of the triaxial coordinates. In the particular case when θ equal to zero,

$$\ell = \cos 0 = 1 \text{ and } m = \sin 0 = 0$$

and therefore,

$$\left[\theta\right]^T = \left[I\right] \qquad \text{(identity matrix).}$$

Hence (5-19) becomes,

$$\{\sigma_c\} = \left[R\right]^T \{\sigma_t\} \qquad (5\text{-}21)$$

expanding of this equation yields,

$$\left.\begin{aligned} \sigma_x &= \sigma_1 + \sigma_2 \ \ell_3^2 + \sigma_3 \ \ell_2^2 \\ \sigma_y &= \sigma_2 \ m_3^2 + \sigma_3 \ m_2^2 \\ \tau_{xy} &= - \sigma_2 \ \ell_3 m_3 + \sigma_3 \ \ell_2 m_2 \end{aligned}\right\} \quad (5\text{-}22)$$

which is the same as equation (3-1).

6. PRINCIPAL STRESSES AND STRAIN ENERGY OF DEFORMATION

The principal stresses, in terms of the triaxial stress components, can be readily obtained from the corresponding expressions in the Cartesian system by using the results of Section 5.

The principal stresses σ_{p1} and σ_{p2} and the maximum shearing stress τ_{max} are related to the Cartesian stress components $(\sigma_x, \sigma_y, \tau_{xy})$ by the following expressions.[11]

$$\left.\begin{aligned} \sigma_{p1} &= \frac{\Theta + \chi}{2} \\ \sigma_{p2} &= \frac{\Theta - \chi}{2} \\ \tau_{max} &= \frac{\chi}{2} \end{aligned}\right\} \qquad (6\text{-}1)$$

where,

$$\Theta = \sigma_x + \sigma_y \qquad (6\text{-}2)$$

and,

$$\chi = 2\left[\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2\right]^{1/2} \qquad (6\text{-}3)$$

The stress combinations Θ and χ are known as stress invariants of plane elasticity. In brief, a quantity is said to be invariant if it is unchanged when referred to a new set of rectangular axes obtained by rotation of the original axes.

The stress invariants Θ and χ can be readily obtained in terms of the triaxial stresses by using the relationship between the Cartesian and triaxial stresses given by Equation (5-20). Therefore the first invariant, Θ, is given by,

$$\Theta = \sigma_x + \sigma_y = \sigma_1 \ (\ell^2 + m^2) + \sigma_2 \left[(\ell\ell_3 + mm_3)^2 + (m\ell_3 - \ell m_3)^2\right]$$
$$+ \sigma_3 \left[(\ell\ell_2 - mm_2)^2 + (m\ell_2 + \ell m_2)^2\right] \qquad (6\text{-}4)$$

where,

$$\ell = \cos\theta, \ m = \sin\theta$$

$$\ell_2 = \cos\alpha_2, \ m_2 = \sin\alpha_2$$

and,

$$\ell_3 = \cos\alpha_3, \ m_3 = \sin\alpha_3$$

Now,

$$\ell^2 + m^2 = 1$$

$$\left.\begin{aligned}
(\ell\ell_3 + mm_3)^2 &= (\cos\theta\cos\alpha_3 + \sin\theta\sin\alpha_3)^2 = \cos^2(\theta - \alpha_3) \\
(m\ell_3 - \ell m_3)^2 &= (\sin\theta\cos\alpha_3 - \cos\theta\sin\alpha_3)^2 = \sin^2(\theta - \alpha_3) \\
(\ell\ell_2 - mm_2)^2 &= (\cos\theta\cos\alpha_2 - \sin\theta\sin\alpha_2)^2 = \cos^2(\theta + \alpha_2) \\
(m\ell_2 + \ell m_2)^2 &= (\sin\theta\cos\alpha_2 + \cos\theta\sin\alpha_2)^2 = \sin^2(\theta + \alpha_2)
\end{aligned}\right\} \qquad (6\text{-}5)$$

Hence, substituting (6-5) into Equation (6-4) gives,

$$\Theta = \sigma_x + \sigma_y = (\sigma_1 + \sigma_2 + \sigma_3) \qquad (6\text{-}6)$$

The second invariant, χ, is obtained by substituting (5-20) into Equation (6-3) and using similar trigonometric relations as previously. Therefore

$$\chi = 2\left[\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2\right]^{1/2}$$

$$= 2\left[\left[\frac{\sigma_1 + \sigma_2 + \sigma_3}{2}\right]^2 - (\sigma_1\sigma_2 m_3^2 + \sigma_1\sigma_3 m_2^2 + \sigma_2\sigma_3 m_1^2)\right]^{1/2} \qquad (6\text{-}7)$$

Substituting Equations (6-6) and (6-7) into Equation (6-1) gives the following expressions for the principal stresses σ_{p1} and σ_{p2} and the maximum shear stress τ_{max}

$$\left.\begin{aligned}
\sigma_{p1} &= \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{2} + \left[\left[\frac{\sigma_1 + \sigma_2 + \sigma_3}{2}\right]^2 - (\sigma_1\, \sigma_2\, m_3^2 + \sigma_1\, \sigma_3\, m_2^2 + \sigma_2\, \sigma_3\, m_1^2)\right]^{1/2} \\
\sigma_{p2} &= \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{2} - \left[\left[\frac{\sigma_1 + \sigma_2 + \sigma_3}{2}\right]^2 - (\sigma_1\, \sigma_2\, m_3^2 + \sigma_1\, \sigma_3\, m_2^2 + \sigma_2\, \sigma_3\, m_1^2)\right]^{1/2} \\
\tau_{max} &= \left[\left[\frac{\sigma_1 + \sigma_2 + \sigma_3}{2}\right]^2 - (\sigma_1\, \sigma_2\, m_3^2 + \sigma_1\, \sigma_3\, m_2^2 + \sigma_2\, \sigma_3\, m_1^2)\right]^{1/2}
\end{aligned}\right\} \qquad (6\text{-}8)$$

These relationships show that the principal stresses and the maximum shear stress are independent of the arbitrary angle θ, which was introduced in Section 5. These relationships therefore imply that the stress quantities σ_{p1}, σ_{p2} and τ_{max} are invariants and independent of the relative orientation of the triaxial and Cartesian coordinate systems.

The strain energy of deformation in terms of the triaxial stress components can be obtained by considering the analogous expression for the strain energy in terms of the Cartesian stress components.

In the case of plane stress, the strain energy density V_o, is given by,[11]

$$V_o = \frac{1}{2} \lfloor \sigma_c \rfloor \left[T_{\varepsilon\sigma} \right] \{\sigma_c\} \tag{6-9}$$

This expression is in terms of the Cartesian stress components. Therefore, substituting Equation (5-19) into (6-9), the energy in triaxial stress components is given by,

$$V_o = \frac{1}{2} \lfloor \sigma_t \rfloor \left[R \right] \left[\theta \right] \left[T_{\varepsilon\sigma} \right] \left[\theta \right]^T \left[R \right]^T \{\sigma_t\}$$

Substituting Equation (5-18) for $\left[\theta\right]^T$ only, in the above, gives

$$V_o = \frac{1}{2} \lfloor \sigma_t \rfloor \left[R \right] \left[T_{\varepsilon\sigma} \right] \left[R \right]^T \{\sigma_t\}$$

Substituting Equation (4-7) into the right hand side of this equation gives,

$$V_o = \frac{1}{2} \lfloor \sigma_t \rfloor \left[T_{\varepsilon\sigma_t} \right] \{\sigma_t\} \tag{6-10}$$

This gives the strain energy density in terms of triaxial stress components. Since Equation (6-10) is independent of the angle θ, it follows that the strain energy per unit volume V_o is an invariant and independent of the relative orientation of the triaxial and Cartesian coordinate systems.

7. SOME EQUATIONS OF PLANE ELASTICITY IN TRILINEAR COORDINATES.

Some of the governing equations of two dimensional elasticity, with plane stress assumptions, will now be derived in terms of the trilinear system by considering the analogous equations in the rectangular Cartesian coordinates.

7.1. Equilibrium equations.

The equilibrium equations corresponding to the trilinear system will now be obtained by making appropriate transformations to the corresponding set of equations in the rectangular Cartesian coordinates. A geometrical interpretation of these equations will be given later in this section.

In the case of absence of body force and inertia forces, the equations of equilibrium in the rectangular coordinates (x,y) are derived to be,(11)

$$\left.\begin{aligned} \frac{\partial\sigma_x}{\partial_x} + \frac{\partial\tau_{xy}}{\partial_y} &= 0 \\ \\ \frac{\partial\sigma_y}{\partial_y} + \frac{\partial\tau_{xy}}{\partial_x} &= 0 \end{aligned}\right\} \quad (7\text{-}1)$$

Substituting equation (5-22) in equation (7-1) yields,

$$\left.\begin{aligned}
&\frac{\partial\sigma_1}{\partial x} + \ell_3 \left[\ell_3 \frac{\partial}{\partial x} - m_3 \frac{\partial}{\partial y}\right] \sigma_2 + \ell_2 \left[\ell_2 \frac{\partial}{\partial x} + m_2 \frac{\partial}{\partial y}\right] \sigma_3 = 0 \\
&\text{and} \\
&-m_3 \left[\ell_3 \frac{\partial}{\partial x} - m_3 \frac{\partial}{\partial y}\right] \sigma_2 + m_2 \left[\ell_2 \frac{\partial}{\partial x} + m_2 \frac{\partial}{\partial y}\right] \sigma_3 = 0
\end{aligned}\right\} \quad (7\text{-}2)$$

Now writing $\partial/\partial x = \partial/\partial x_1$ and $\partial/\partial y = \partial/\partial y_1$ in the above two equations and then making use of the operator identities given by equation (2-5), we have the following two equations for the equilibrium conditions in the triaxial systems.

$$\left.\begin{aligned}
&\frac{\partial\sigma_1}{\partial x_1} - \ell_3 \frac{\partial\sigma_2}{\partial x_2} - \ell_2 \frac{\partial\sigma_3}{\partial x_3} = 0 \\
&m_3 \frac{\partial\sigma_2}{\partial x_2} - m_2 \frac{\partial\sigma_3}{\partial x_3} = 0
\end{aligned}\right\} \quad (7\text{-}3)$$

These first order partial differential equations can be more elegantly represented by using some trigonometrical identities. For example, multiplying the first equation (7-3) by m_3 and the second equation of (7-3) by ℓ_3 and adding, we have the following system of equations.

$$\left.\begin{aligned}
&m_3 \frac{\partial\sigma_1}{\partial x_1} - (\ell_2 m_3 + m_2 \ell_3) \frac{\partial\sigma_3}{\partial x_3} = 0 \\
&m_3 \frac{\partial\sigma_2}{\partial x_2} - m_2 \frac{\partial\sigma_3}{\partial x_3} = 0
\end{aligned}\right\} \quad (7\text{-}4)$$

Since $m_i = \sin \alpha_i \quad , \quad \ell_i = \cos \alpha_i$ and,

$\pi = (\alpha_1 + \alpha_2 + \alpha_3)$, it follows that,

$\sin \alpha_1 = \sin \alpha_2 \cos \alpha_3 + \sin \alpha_3 \cos \alpha_2$, that is

$$m_1 = m_2\ell_3 + \ell_2 m_3 \tag{7-5}$$

Thus using equation (7-5), the equilibrium equations (7-4) can be written as,

$$\boxed{\begin{aligned} m_3 \frac{\partial \sigma_1}{\partial x_1} - m_1 \frac{\partial \sigma_3}{\partial x_3} &= 0 \\ m_3 \frac{\partial \sigma_2}{\partial x_2} - m_2 \frac{\partial \sigma_3}{\partial x_3} &= 0 \end{aligned}} \tag{7-6}$$

Equation (7-6) is the equilibrium condition for the triaxial system, and it can be satisfied by selecting a stress function Φ, such that

$$\left.\begin{aligned} \sigma_1 &= m_1 \frac{\partial^2 \Phi}{\partial x_2 \partial x_3} \\ \sigma_2 &= m_2 \frac{\partial^2 \Phi}{\partial x_1 \partial x_3} \\ \sigma_3 &= m_3 \frac{\partial^2 \Phi}{\partial x_1 \partial x_2} \end{aligned}\right\} \tag{7-7}$$

<u>Geometrical interpretation of equilibrium conditions for an infinitesimal triangular element.</u>

Consider a triangular element of unit thickness, area Δ^* whose sides are perpendicular to the axes (x_1, x_2, x_3). The element in relation to the triaxial coordinate system is shown in Figure 8. The net resultant force F_1 due to the triaxial stress component σ_1 will now be found. With reference to Figure 8, the force F_1 due to the triaxial stress σ_1 in the x_1 direction is given by

$$F_1 = \int_{K_1}^{K_2} \frac{\partial \sigma_1}{\partial x_1} x_{1p} \, dx_1 \tag{7-8}$$

Using the sine rule it can be shown that

$$x_{1p} = \frac{m_1}{m_2 m_3} (x_1 - K_1) \tag{7-9}$$

The area of the shaded portion of the triangle, denoted by Δ, is given by,

$$\Delta = \frac{(x_1 - K_1)}{2} x_{1p} \tag{7-10}$$

Substituting (7-9) in (7-10) gives,

$$\Delta = \frac{m_1}{2m_2m_3} (x_1 - K_1)^2 \tag{7-11}$$

Therefore,

$$d\Delta = \frac{m_1}{m_2 m_3} (x_1 - K_1) \, dx_1$$

Using equation (7-9), the above equation can be written as,

$$d\Delta = x_{1p} . dx_1 \tag{7-12}$$

Substituting (7-12) into (7-8) and then changing the lower and upper limit of the integral to zero and Δ^*, respectively, gives,

$$F_1 = \int_0^{\Delta^*} \frac{\partial \sigma_1}{\partial x_1} \, d\Delta \tag{7-13}$$

By following the method used in finding the force F_1, it can be shown that the force F_2 due to the stress component σ_2, and the force F_3 due to the stress component σ_3 are as follows

$$\left.\begin{aligned} F_2 &= \int_0^{\Delta^*} \frac{\partial \sigma_2}{\partial x_2} \, d\Delta \\ F_3 &= \int_0^{\Delta^*} \frac{\partial \sigma_3}{\partial x_3} \, d\Delta \end{aligned}\right\} \tag{7-14}$$

For the equilibrium of the element, the net resultant force in any direction must be zero. For example, the equilibrium condition in the x_1 direction requires.

$$F_1 - \ell_3 F_2 - \ell_2 F_3 = 0$$

Similarly the equilibrium conditions in the x_2 and x_3 direction requires

$$-\ell_3 F_1 + F_2 - \ell_1 F_3 = 0$$

$$-\ell_2 F_1 - \ell_1 F_2 + F_3 = 0$$

Substituting for F_1, F_2 and F_3, from (7-13) and (7-14), into the last three equations yields,

$$\int_0^{\Delta^*} \left[\frac{\partial\sigma_1}{\partial x_1} - \ell_3 \frac{\partial\sigma_2}{\partial x_2} - \ell_2 \frac{\partial\sigma_3}{\partial x_3}\right] d\Delta = 0$$

$$\int_0^{\Delta^*} \left[-\ell_3 \frac{\partial\sigma_1}{\partial x_1} + \frac{\partial\sigma_2}{\partial x_2} - \ell_1 \frac{\partial\sigma_3}{\partial x_3}\right] d\Delta = 0$$

$$\int_0^{\Delta^*} \left[-\ell_2 \frac{\partial\sigma_1}{\partial x_1} - \ell_1 \frac{\partial\sigma_2}{\partial x_2} + \frac{\partial\sigma_3}{\partial x_3}\right] d\Delta = 0$$

The above equilibrium conditions must hold irrespective of the area of the triangle. Hence the integrands themselves must vanish in each of the above three equations. That is,

$$\left.\begin{aligned} \frac{\partial\sigma_1}{\partial x_1} - \ell_3 \frac{\partial\sigma_2}{\partial x_2} - \ell_2 \frac{\partial\sigma_3}{\partial x_3} &= 0 \\ -\ell_3 \frac{\partial\sigma_1}{\partial x_1} + \frac{\partial\sigma_2}{\partial x_2} - \ell_1 \frac{\partial\sigma_3}{\partial x_3} &= 0 \\ -\ell_2 \frac{\partial\sigma_1}{\partial x_1} - \ell_1 \frac{\partial\sigma_2}{\partial x_2} + \frac{\partial\sigma_3}{\partial x_3} &= 0 \end{aligned}\right\} \quad (7\text{-}15)$$

It should be stated at this point that, not all of the above three equations are independent, and one of these can be eliminated. For example, we shall now eliminate the first equation of (7-15). Multiply the first, second, and third equations of (7-15) by m_1, m_2 and m_3 respectively and add the second and third equation to the first. This yields the following set of equations,

$$\left.\begin{aligned} &(m_1-\ell_3m_2-\ell_2m_3)\frac{\partial\sigma_1}{\partial x_1} + (m_2-\ell_3m_1-\ell_1m_3)\frac{\partial\sigma_2}{\partial x_2} + (m_3-\ell_2m_1-\ell_1m_2)\frac{\partial\sigma_3}{\partial x_3} = 0 \\ &-\ell_3\frac{\partial\sigma_1}{\partial x_1} + \frac{\partial\sigma_2}{\partial x_2} - \ell_1\frac{\partial\sigma_3}{\partial x_3} = 0 \\ &-\ell_2\frac{\partial\sigma_1}{\partial x_1} - \ell_1\frac{\partial\sigma_2}{\partial x_2} + \frac{\partial\sigma_3}{\partial x_3} = 0 \end{aligned}\right\} \quad (7\text{-}16)$$

Because of trigonometrical relationships, the first equation of (7-15) is identically zero.

Since $\quad m_i = \sin\alpha_i$, $\ell_i = \cos\alpha_i$, and $\alpha_1 + \alpha_2 + \alpha_3 = \pi$,

we have $\quad (m_1-\ell_3m_2-\ell_2m_3) = (m_2-\ell_3m_1-\ell_1m_3) = (m_3-\ell_2m_1-\ell_1m_2) = 0$

The remaining two equations of equation (7-15) can be written in numerous ways; but we shall now put these two equations in the same form as equation (7-6). Multiply the second equation by ℓ_1 and add the third equation to it. This yields,

$$(\ell_3\ell_1 + \ell_2)\frac{\partial\sigma_1}{\partial x_1} + (\ell_1^2 - 1)\frac{\partial\sigma_3}{\partial x_3} = 0$$

$$-\ell_2\frac{\partial\sigma_1}{\partial x_1} - \ell_1\frac{\partial\sigma_2}{\partial x_2} + \frac{\partial\sigma_3}{\partial x_3} = 0$$

Since $\ell_2 = -\ell_3\ell_1 + m_3m_1$ and $(\ell_1^2 - 1) = -m_1^2$, the above two equations are equivalent to,

$$m_3 \frac{\partial\sigma_1}{\partial x_1} - m_1 \frac{\partial\sigma_3}{\partial x_3} = 0$$

$$-\ell_2 \frac{\partial\sigma_1}{\partial x_1} - \ell_1 \frac{\partial\sigma_2}{\partial x_2} + \frac{\partial\sigma_3}{\partial x_3} = 0$$

Now multiply the first and second of above equations by ℓ_2 and m_3 respectively and add to the second of the resulting equation. This procedure yields the following equations,

$$m_3 \frac{\partial\sigma_1}{\partial x_1} - m_1 \frac{\partial\sigma_3}{\partial x_3} = 0$$

$$-\ell_1 m_3 \frac{\partial\sigma_2}{\partial x_2} + (m_3 - \ell_2 m_1) \frac{\partial\sigma_3}{\partial x_3} = 0$$

Since, $m_3 = \ell_2 m_1 + \ell_1 m_2$, the above two equations are equivalent to,

$$m_3 \frac{\partial\sigma_1}{\partial x_1} - m_1 \frac{\partial\sigma_3}{\partial x_3} = 0$$

$$m_3 \frac{\partial\sigma_2}{\partial x_2} - m_2 \frac{\partial\sigma_3}{\partial x_3} = 0$$

the above two equations are identical to equation (7-6). Now differentiating the first of the above equation with respect to x_1, and the second of the above equation with respect to x_2, and then adding these equations yields,

$$m_3 \frac{\partial^2\sigma_1}{\partial x_1^2} + m_3 \frac{\partial^2\sigma_2}{\partial x_2^2} - \frac{\partial}{\partial x_3}\left[m_1 \frac{\partial}{\partial x_1} + m_2 \frac{\partial}{\partial x_2}\right] \sigma_3 = 0$$

From equation (2-8), $-m_3 \frac{\partial}{\partial x_3} = \left[m_1 \frac{\partial}{\partial x_1} + m_2 \frac{\partial}{\partial x_2}\right]$, therefore the above equation can be written as,

$$\boxed{\frac{\partial^2\sigma_1}{\partial x_1^2} + \frac{\partial^2\sigma_2}{\partial x_2^2} + \frac{\partial^2\sigma_3}{\partial x_3^2} = 0} \tag{7-17}$$

The equation corresponding to equation (7-17) in the Cartesian coordinate system is the

$$\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial x^2} + 2\,\frac{\partial^2\tau_{xy}}{\partial x\partial y} = 0$$

7.2. Kinematics and compatibility equation.

This section is devoted to the analysis of some aspects of kinematics and developing the compatibility equation that is natural to the triaxial (and trilinear) coordinates. Discussions provided in this section are necessarily brief, since the method of formulation of the problem is essentially the same as that adopted in previous sections.

In Figure 9, three orthogonal Cartesian coordinate systems (x_1, y_1), (x_2, y_2) and (x_3, y_3) are shown in relation to the triangle of reference. Corresponding to these three coordinate systems, there exist the following displacement components.

(x_1, y_1) coordinate system: displacements (u_{1n}, v_{1s})
(x_2, y_2) coordinate system: displacements (u_{2n}, v_{2s})
(x_3, y_3) coordinate system: displacements (u_{3n}, v_{3s})

Not all of the above displacements are independent, and the relationships among these displacements can be given by the following equations.

$$\left.\begin{aligned}
\begin{bmatrix} u_{2n} \\ v_{2s} \end{bmatrix} &= \begin{bmatrix} -\ell_3 & m_3 \\ -m_3 & -\ell_3 \end{bmatrix} \begin{bmatrix} u_{1n} \\ v_{1s} \end{bmatrix} \quad ; \quad \begin{bmatrix} u_{1n} \\ v_{1s} \end{bmatrix} = \begin{bmatrix} -\ell_3 & -m_3 \\ m_3 & -\ell_3 \end{bmatrix} \begin{bmatrix} u_{2n} \\ v_{2s} \end{bmatrix} \\
\begin{bmatrix} u_{3n} \\ v_{3s} \end{bmatrix} &= \begin{bmatrix} -\ell_2 & -m_2 \\ m_2 & -\ell_2 \end{bmatrix} \begin{bmatrix} u_{1n} \\ v_{1s} \end{bmatrix} \quad : \quad \begin{bmatrix} u_{1n} \\ v_{1s} \end{bmatrix} = \begin{bmatrix} -\ell_2 & m_2 \\ -m_2 & -\ell_2 \end{bmatrix} \begin{bmatrix} u_{3n} \\ v_{3s} \end{bmatrix} \\
\begin{bmatrix} u_{3n} \\ v_{3s} \end{bmatrix} &= \begin{bmatrix} -\ell_1 & m_1 \\ -m_1 & -\ell_1 \end{bmatrix} \begin{bmatrix} u_{2n} \\ v_{2s} \end{bmatrix} \quad : \quad \begin{bmatrix} u_{2n} \\ v_{2s} \end{bmatrix} = \begin{bmatrix} -\ell_1 & -m_1 \\ m_1 & -\ell_1 \end{bmatrix} \begin{bmatrix} u_{3n} \\ v_{3s} \end{bmatrix}
\end{aligned}\right\} \quad (7\text{-}18)$$

Further,

$$\boxed{m_1 u_{1n} + m_2 u_{2n} + m_3 u_{3n} = 0} \quad (7\text{-}19)$$

$$\boxed{m_1 v_{1s} + m_2 v_{2s} + m_3 v_{3s} = 0} \quad (7\text{-}20)$$

Strain Components

Corresponding to the three Cartesian coordinates shown in Figure 4 there exist the following strain components

(x_1, y_1) coordinate system : strain components $(\varepsilon_{1n}, \varepsilon_{1s}, \gamma_1)$

(x_2, y_2) coordinate system : strain components $(\varepsilon_{2n}, \varepsilon_{2s}, \gamma_2)$

(x_3, y_3) coordinate system : strain components $(\varepsilon_{3n}, \varepsilon_{3s}, \gamma_3)$

The relationships among these strain components can be written in the following form,

$$\begin{bmatrix} \varepsilon_{2n} \\ \varepsilon_{2s} \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \ell_3^2 & m_3^2 & -\ell_3 m_3 \\ m_3^2 & \ell_3^2 & \ell_3 m_3 \\ \ell_3 m_3 & -\ell_3 m_3 & (\ell_3^2 - m_3^2) \end{bmatrix} \begin{bmatrix} \varepsilon_{1n} \\ \varepsilon_{1s} \\ \gamma_1 \end{bmatrix} \tag{7-21}$$

$$\begin{bmatrix} \varepsilon_{3n} \\ \varepsilon_{3s} \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} \ell_2^2 & m_2^2 & \ell_2 m_2 \\ m_2^2 & \ell_2^2 & -\ell_2 m_2 \\ -\ell_2 m_2 & \ell_2 m_2 & (\ell_2^2 - m_2^2) \end{bmatrix} \begin{bmatrix} \varepsilon_{1n} \\ \varepsilon_{1s} \\ \gamma_1 \end{bmatrix} \tag{7-22}$$

Triaxial strain components

Prior to developing the compatibility equation for the triaxial system, three strain components ε_1, ε_2, and ε_3 will be introduced, and these will be called the triaxial strain components which correspond to the triaxial coordinates x_1, x_2 and x_3. The triaxial strains ε_i are not the same as the direct triaxial strains ε_{in}. The triaxial strains ε_1, ε_2, and ε_3 have no direct physical meaning in the same sense that the triaxial stresses σ_1, σ_2 and σ_3, introduced in Section 3, had no physical meaning. The relationships between the Cartesian strain components (ε_{in}, ε_{is}, γ_i) and the triaxial strains (ε_1, ε_2, ε_3) will be written in the following form

$$\begin{bmatrix} \varepsilon_{1n} \\ \varepsilon_{2n} \\ \varepsilon_{3n} \end{bmatrix} = \begin{bmatrix} 1 & \ell_3^2 & \ell_2^2 \\ \ell_3^2 & 1 & \ell_1^2 \\ \ell_2^2 & \ell_1^2 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \qquad (7\text{-}23)$$

$$\begin{bmatrix} \varepsilon_{1s} \\ \varepsilon_{2s} \\ \varepsilon_{3s} \end{bmatrix} = \begin{bmatrix} 0 & m_3^2 & m_2^2 \\ m_3^2 & 0 & m_1^2 \\ m_2^2 & m_1^2 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \qquad (7\text{-}24)$$

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = 2\begin{bmatrix} 0 & -\ell_3 m_3 & \ell_2 m_2 \\ \ell_3 m_3 & 0 & -\ell_1 m_1 \\ -\ell_2 m_2 & \ell_1 m_1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \qquad (7\text{-}25)$$

Both equation (7-23) and (7-24) can be inverted; the results are

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \frac{1}{2\ m_1^2 m_2^2 m_3^2} \begin{bmatrix} 1-\ell_1^4 & \ell_1^2\ell_2^2-\ell_3^2 & \ell_3^2\ell_1^2-\ell_2^2 \\ \ell_1^2\ell_2^2-\ell_3^2 & 1-\ell_2^4 & \ell_2^2\ell_3^2-\ell_1^2 \\ \ell_3^2\ell_1^2-\ell_2^2 & \ell_2^2\ell_3^2-\ell_1^2 & 1-\ell_3^4 \end{bmatrix} \begin{bmatrix} \varepsilon_{1n} \\ \varepsilon_{2n} \\ \varepsilon_{3n} \end{bmatrix} \qquad (7\text{-}26)$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \frac{1}{2\ m_1^2 m_2^2 m_3^2} \begin{bmatrix} -m_1^4 & m_1^2 m_2^2 & m_3^2 m_1^2 \\ m_1^2 m_2^2 & -m_2^4 & m_2^2 m_3^2 \\ m_3^2 m_1^2 & m_2^2 m_3^2 & -m_3^4 \end{bmatrix} \begin{bmatrix} \varepsilon_{1s} \\ \varepsilon_{2s} \\ \varepsilon_{3s} \end{bmatrix} \qquad (7\text{-}27)$$

It should be noted that equation (7-25) cannot be inverted. Further, the following relations are of general interest

$$\left.\begin{aligned}
&\ell_1 m_1\ \gamma_1 + \ell_2 m_2\ \gamma_2 + \ell_3 m_3\ \gamma_3 = 0\\
&\ell_1 m_1\ \varepsilon_{1n} + \ell_2 m_2\ \varepsilon_{2n} + \ell_3 m_3\ \varepsilon_{3n} = m_1 m_2 m_3\ (\varepsilon_1+\varepsilon_2+\varepsilon_3)\\
&\ell_1 m_1\ \varepsilon_{1s} + \ell_3 m_3\ \varepsilon_{2s} + \ell_3 m_3\ \varepsilon_{3s} = m_1 m_2 m_3\ (\varepsilon_1+\varepsilon_2+\varepsilon_3)
\end{aligned}\right\} \quad (7\text{-}28)$$

It can be verified that the quantity

$$(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = \quad e, \text{ the first strain invarient.}$$

Relationship between direct stresses and triaxial strains

In the case of plane stress, the normal stress σ_{in} is related to the normal strains ε_{in} and ε_{is} by this equation[11]

$$\sigma_{in} = \frac{E}{1-\nu^2}\ (\varepsilon_{in} + \nu\varepsilon_{is}) \quad , \quad i = 1,2,3 \qquad (7\text{-}29)$$

The orientation of these stress and strain components, in relation to the triangle of reference, is illustrated in Figure 4.

Using equations (7-23) and (7-24), the stress-strain relationship given by equation (7-29) can be written as,

$$\begin{bmatrix}\sigma_{1n}\\ \sigma_{2n}\\ \sigma_{3n}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & (\ell_3^2+\nu m_3^2) & (\ell_2^2+\nu m_2^2)\\ (\ell_3^2+\nu m_3^2) & 1 & (\ell_1^2+\nu m_1^2)\\ (\ell_2^2+\nu m_2^2) & (\ell_1^2+\nu m_1^2) & 1\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix} \qquad (7\text{-}30)$$

The relationships between the triaxial stresses (σ_1, σ_2, σ_3) and the triaxial strains (ε_1, ε_2, ε_3) can be obtained by combining equations (4-13) and (7-23). The result is,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \frac{1}{E} \begin{bmatrix} (1+\nu) - \beta \ell_1 m_1 & -\beta \ell_1 m_1 & -\beta \ell_1 m_1 \\ -\beta \ell_2 m_2 & (1+\nu) - \beta \ell_2 m_2 & -\beta \ell_2 m_2 \\ -\beta \ell_3 m_3 & -\beta \ell_3 m_3 & (1+\nu) - \beta \ell_3 m_3 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \tag{7-31}$$

where, $\beta = \dfrac{\nu}{m_1 m_2 m_3}$

<u>Compatibility equation.</u>

The compatibility equation in Cartesian coordinates (x_i, y_i) is given by,[11]

$$\frac{\partial^2}{\partial y_i^2} \varepsilon_{in} + \frac{\partial^2}{\partial x_i^2} \varepsilon_{is} = \frac{\partial^2 \gamma_i}{\partial x_i \partial y_i} \tag{7-32}$$

Considering the case i = 1, we have from equations (7-23), (7-24) and (7-25)

$$\varepsilon_{1n} = \varepsilon_1 + \ell_3^2 \varepsilon_2 + \ell_2^2 \varepsilon_3$$

$$\varepsilon_{1s} = m_3^2 \varepsilon_2 + m_2^2 \varepsilon_3$$

$$\gamma_1 = -2\ell_3 m_3 \varepsilon_2 + 2\ell_2 m_2 \varepsilon_3$$

Substituting the above three equations in equation (7-31), for i = 1, gives,

$$\frac{\partial}{\partial y_1^2}(\varepsilon_1 + \ell_3^2\varepsilon_2 + \ell_2^2\varepsilon_3) + \frac{\partial^2}{\partial x_1^2}(m_3^2\varepsilon_2 + m_2^2\varepsilon_3) = \frac{2\,\partial^2}{\partial x_1 \partial y_1}(-\ell_3 m_3\varepsilon_2 + \ell_2 m_2 \varepsilon_3)$$

That is,

$$\frac{\partial^2}{y_1^2}\varepsilon_1 + \left[\ell_3\frac{\partial}{\partial y_1} + m_3\frac{\partial}{\partial x_1}\right]^2\varepsilon_2 + \left[\ell_2\frac{\partial}{\partial y_1} - m_2\frac{\partial}{\partial x_1}\right]^2\varepsilon_3 = 0 \qquad (7\text{-}33)$$

From equation (2-9),

$$\frac{\partial^2}{\partial y_2^2} = \left[\ell_3\frac{\partial}{\partial y_1} + m_3\frac{\partial}{\partial x_1}\right]^2 \quad \text{and} \quad \frac{\partial^2}{\partial y_3^2} = \left[\ell_2\frac{\partial}{\partial y_1} - m_2\frac{\partial}{\partial x_1}\right]^2$$

Therefore, equation (7-33) can be written as,

$$\boxed{\frac{\partial^2\varepsilon_1}{\partial y_1^2} + \frac{\partial^2\varepsilon_2}{\partial y_2^2} + \frac{\partial^2\varepsilon_3}{\partial y_3^2} = 0} \qquad (7\text{-}34)$$

Equation (7-34) is the compatibility equation for the triaxial system, and this equation can be written in a different form.

Since, $\nabla^2 = \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2}$, equation (7-33) can be written as,

$$\nabla^2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) - \left[\frac{\partial^2}{\partial x_1^2}\varepsilon_1 + \frac{\partial^2}{\partial x_2^2}\varepsilon_2 + \frac{\partial^2}{\partial x_3^2}\varepsilon_3\right] = 0 \qquad (7\text{-}35)$$

The quantity $(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = e$, is known as the first-strain invarient (also called the dilatation or volume expansion) and in terms of Cartesian strain components, $e = (\varepsilon_x + \varepsilon_y)$. In the case of absence of body force the dilatation e is a harmonic function [11], therefore

$$\nabla^2 e = \nabla^2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = 0$$

hence, equation (7-35) can be written as,

$$\boxed{\frac{\partial^2 \varepsilon_1}{\partial x_1^2} + \frac{\partial^2 \varepsilon_2}{\partial x_2^2} + \frac{\partial^2 \varepsilon_3}{\partial x_3^2} = 0} \qquad (7\text{-}36)$$

It is of general interest to note that the compatibility equation (7-36) is of the same form as the equilibrium equation (7-17).

The combined compatibility-equilibrium equation.

It was shown that the stress function Φ, defined by equation (7-7) satisfies the equilibrium equation. Substituting equation (7-7) in equation (7-31) will yield the following relationships between the triaxial strain components and the stress function

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \frac{1}{E} \begin{bmatrix} (1+\nu) - \beta\ell_1 m_1 & -\beta\ell_1 m_1 & -\beta\ell_1 m_1 \\ -\beta\ell_2 m_2 & (1+\nu) - \beta\ell_2 m_2 & -\beta\ell_2 m_2 \\ -\beta\ell_3 m_3 & -\beta\ell_3 m_3 & (1+\nu) - \beta\ell_3 m_3 \end{bmatrix} \begin{bmatrix} m_1 \; \Phi,_{23} \\ m_2 \; \Phi,_{13} \\ m_3 \; \Phi,_{12} \end{bmatrix}$$

where,

$$\Phi,_{ij} = \frac{\partial^2 \Phi}{\partial x_i \partial x_j}$$

Substituting the above expressions for strains ε_i into equation (7-36) yields,

$$\left[\frac{(1+\nu)}{E}\frac{\partial}{\partial x_1 \partial x_2 \partial x_3}\left[m_1 \frac{\partial}{\partial x_1} + m_2 \frac{\partial}{\partial x_2} + m_3 \frac{\partial}{\partial x_3}\right]\Phi \right.$$
$$\left. - \frac{\beta}{E}\left[\ell_1 m_1 \frac{\partial^2}{\partial x_1^2} + \ell_2 m_2 \frac{\partial^2}{\partial x_2^2} + \ell_3 m_3 \frac{\partial^2}{\partial x_3^2}\right]\left[m_1 \frac{\partial^2}{\partial x_2 \partial x_3} + m_2 \frac{\partial^2}{\partial x_1 \partial x_3} + m_3 \frac{\partial^2}{\partial x_1 \partial x_2}\right]\Phi\right] = 0$$

(7-37)

From equation (2-8), $\left[m_1 \frac{\partial}{\partial x_1} + m_2 \frac{\partial}{\partial x_2} + m_3 \frac{\partial}{\partial x_3}\right] = 0$

From equation (2-17),

$$m_1 m_2 m_3 \nabla^2 = \left[\ell_1 m_1 \frac{\partial^2}{\partial x_1^2} + \ell_2 m_2 \frac{\partial^2}{\partial x_2^2} + \ell_3 m_3 \frac{\partial^2}{\partial x_3^2}\right]$$
$$= -\left[m_1 \frac{\partial^2}{\partial x_2 \partial x_3} + m_2 \frac{\partial^2}{\partial x_1 \partial x_3} + m_3 \frac{\partial^2}{\partial x_1 \partial x_2}\right]$$

Therefore, using the above three equations, the governing differential equation given by (7-37) can be written as,

$$\boxed{\nabla^4 \Phi = 0} \qquad (7\text{-}38)$$

7.3. Governing equations for thin triangular plate.

We shall now express the equations governing bending of triangular plates in the triaxial stress and coordinate system. The equations to be derived are based on classical thin plate theory and Kirchhoff's hypotheses for the boundary conditions.

Equations of bending of triangular plates.

The moment equilibrium equation in rectangular Cartesian co-ordinate system is given by,[12]

$$\frac{\partial^2 M_x}{\partial x^2} + 2\,\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \qquad (7\text{-}39)$$

in which q is the normal load intensity.

We shall now define the triaxial bending moments.

The bending moments M_x, M_{xy} and M_y are related to the Cartesian stress components by the following definite integrals[12]

$$M_x = \int_{-h/2}^{h/2} z\,\sigma_x\,dz \;:\; M_y = \int_{-h/2}^{h/2} z\,\sigma_y\,dz \;:\; M_{xy} = \int_{-h/2}^{h/2} z\,\tau_{xy}\,dz \qquad (7\text{-}40)$$

In equation (7-40), h is the thickness of the plate, and the variable z measured from the middle plane of the plate.

Substituting equation (5-22) in equation (7-40) yields the following equations,

$$\left.\begin{aligned} M_x &= M_1 + M_2\ell_3^2 + M_3\ell_2^2 \\ M_y &= M_2 m_3^2 + M_3 m_2^2 \\ M_{xy} &= -M_2\ell_3 m_3 + M_3\ell_2 m_2 \end{aligned}\right\} \qquad (7\text{-}41)$$

in which trilinear moments M_1, M_2 and M_3 are defined as,

$$M_1 = \int_{-h/2}^{h/2} z\,\sigma_1\,dz \quad : \quad M_2 = \int_{-h/2}^{h/2} z\,\sigma_2\,dz \quad : \quad M_3 = \int_{-h/2}^{h/2} z\,\sigma_3\,dz \tag{7-42}$$

Substituting equation (7-41) in equation (7-39) yields,

$$\frac{\partial^2}{\partial x^2} M_1 + \left[\ell_3^2 \frac{\partial^2}{\partial x^2} + m_3^2 \frac{\partial^2}{\partial y^2} - 2\ell_3 m_3 \frac{\partial^2}{\partial x \partial y} \right] M_2$$

$$+ \left[\ell_2^2 \frac{\partial^2}{\partial x^2} + m_2^2 \frac{\partial^2}{\partial y^2} + 2\ell_2 m_2 \frac{\partial^2}{\partial x \partial y} \right] M_3 = -q \tag{7-43}$$

Now writing $\partial/\partial x = \partial/\partial x_1$ and $\partial/\partial y = \partial/\partial y_1$ in equation (7-43), and using (2-6), the moment equilibrium condition in the triaxial system can be given as,

$$\boxed{\frac{\partial^2 M_1}{\partial x_1^2} + \frac{\partial^2 M_2}{\partial x_2^2} + \frac{\partial^2 M_3}{\partial x_3^2} = -q} \tag{7-44}$$

We shall now develop the relationship between the triaxial bending moments and the curvature of the bent plate. To develop the required relationships, we consider the isotropic stress-strain relationships, as given by equation (4-14). Denoting the transverse displacements of plate by w, we may now express the direct triaxial strains ε_{1n}, ε_{2n} and ε_{3n} in terms of w as follows,

$$\varepsilon_{1n} = -z \frac{\partial^2 w}{\partial x_1^2} \quad , \quad \varepsilon_{2n} = -z \frac{\partial^2 w}{\partial x_2^2} \quad , \quad \varepsilon_{3n} = -z \frac{\partial^2 w}{\partial x_3^2} \tag{7-45}$$

Substituting equation (7-45) in equation (4-14) and then multiplying the resulting equation by z and integrating over the thickness of the plate, we have the following relationship between the triaxial moments and the curvatures.

$$\left.\begin{aligned} M_1 &= -\frac{D}{m_2 m_3}\left[\ell_1 \nabla^2 w + (1-\nu)\frac{\partial^2 w}{\partial x_2 \partial x_3}\right] \\ M_2 &= -\frac{D}{m_1 m_3}\left[\ell_2 \nabla^2 w + (1-\nu)\frac{\partial^2 w}{\partial x_1 \partial x_3}\right] \\ M_3 &= -\frac{D}{m_1 m_2}\left[\ell_3 \nabla^2 w + (1-\nu)\frac{\partial^2 w}{\partial x_1 \partial x_2}\right] \end{aligned}\right\} \quad (7\text{-}46)$$

where,

$$D = \frac{Eh^3}{12(1-\nu^2)} \text{ is the flexural rigidity of plate.} \qquad (7\text{-}47)$$

Substituting equation (7-46) in equation (7-44) yields,

$$\frac{D}{m_1 m_2 m_3}\left[\ell_1 m_1 \frac{\partial^2}{\partial x_1^2} + \ell_2 m_2 \frac{\partial^2}{\partial x_2^2} + \ell_3 m_3 \frac{\partial^2}{\partial x_3^2}\right]\nabla^2 w$$

$$+ (1-\nu)\frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3}\left[m_1 \frac{\partial}{\partial x_1} + m_2 \frac{\partial}{\partial x_2} + m_3 \frac{\partial}{\partial x_3}\right] w = q \qquad (7\text{-}48)$$

From equations (2-8) and (2-15), respectively, we have

$$\left[m_1 \frac{\partial}{\partial x_1} + m_2 \frac{\partial}{\partial x_2} + m_3 \frac{\partial}{\partial x_3}\right] = 0$$

and,

$$\frac{1}{m_1 m_2 m_3}\left[\ell_1 m_1 \frac{\partial^2}{\partial x_1^2} + \ell_2 m_2 \frac{\partial^2}{\partial x_2^2} + \ell_3 m_3 \frac{\partial^2}{\partial x_3^2}\right] = \nabla^2$$

Therefore we can write equation (7-48) as

$$\boxed{\nabla^4 w = \frac{q}{D}} \tag{7-49}$$

Equation (7-49) is the well known differential equation governing the transverse bending of thin plates. In the case of polygonal plates, it is advantageous to replace the fourth order differential equation (7-49) developed for bending of a plate by two equations of the second order which represent the deflections of a membrane. This method of approaching problems in bending of plates by membrane analogy is known as the Marcus method [10]. The equations governing plate bending, by the Marcus method, can be developed for the triaxial system in the following manner.

Adding together all the three equations of (7-45), we have,

$$M_1 + M_2 + M_3 = \frac{-D}{m_1 m_2 m_3}\left\{(\ell_1 m_1 + \ell_2 m_2 + \ell_3 m_3)\nabla^2 w + (1-\nu)\left[m_1 \frac{\partial^2 w}{\partial x_2 \partial x_3} + m_2 \frac{\partial^2 w}{\partial x_1 \partial x_3} + m_3 \frac{\partial^2 w}{\partial x_1 \partial x_2}\right]\right\} \tag{7-50}$$

From equation (2-17), we have,

$$\nabla^2 w = -\frac{1}{m_1 m_2 m_3}\left[m_1 \frac{\partial^2 w}{\partial x_2 \partial x_3} + m_2 \frac{\partial^2 w}{\partial x_1 \partial x_3} + m_3 \frac{\partial^2 w}{\partial x_1 \partial x_3}\right]$$

Also, it is possible to show from trigonometrical properties, that

$$(\ell_1 m_1 + \ell_2 m_2 + \ell_3 m_3) = 2m_1 m_2 m_3 \tag{7-51}$$

Therefore using the last two identities, equation (7-50) can be written as,

$$M_1 + M_2 + M_3 = -D(1+\nu)\,\nabla^2 w \tag{7-52}$$

Now introducing a quantity M such that,

$$M = \frac{1}{(1+\nu)}(M_1 + M_2 + M_3) \tag{7-53}$$

we could write equation (7-52) as,

$$\boxed{\nabla^2 w = -\frac{M}{D}} \tag{7-54}$$

From equations (7-49) and (7-54) it follows that,

$$\boxed{\nabla^2 M = -q} \tag{7-55}$$

7.4. Boundary conditions for a triangular plate.

Based on the assumptions of Kirchhoff's theory of thin plates, we shall specify here the equations governing the following three types of edge conditions.

a) simply supported edge

b) clamped edge

c) free edge

a) Simply supported edge. If the side $\eta_i = 0$ is simply supported, then the deflection w and the normal (direct) bending moments M_{in} at the supported edge are zero. That is,

$$\left[w\right]_{\eta_i = 0} = 0 \quad \text{and} \quad \left[M_{in}\right]_{\eta_i = 0} = -D\left[\frac{\partial^2 w}{\partial x_i^2} + \nu \frac{\partial^2 w}{\partial y_i^2}\right]_{\eta_i = 0} = 0$$

observing that $\partial^2 w/\partial y_i^2$ must vanish together with w along the rectilinear edge $\eta_i = 0$, we find that the second of the above equation is equivalent to $(\partial^2 w/\partial x_i^2)_{\eta_i = 0} = 0$ or also $(\nabla^2 w)_{\eta_i = 0} = 0$. Therefore, the analytical expression for the boundary conditions in this case are

$$\left[w\right]_{\eta_i = 0} = 0 \quad \text{and} \quad \left[\nabla^2 w\right]_{\eta_i = 0} = 0 \tag{7-56}$$

b) Clamped edge. If the side $\eta_i = 0$ is clamped, then the deflection and the normal slope $\partial w/\partial x_i$ are zero at $\eta_i = 0$. That is,

$$\left.\begin{aligned} &\left[w\right]_{\eta_i = 0} = 0 \\ &\text{and} \\ &\left[\frac{\partial w}{\partial x_i}\right]_{\eta_i = 0} = \left[-\frac{\partial w}{\partial \eta_i} + \ell_k \frac{\partial w}{\partial \eta_j} + \ell_j \frac{\partial w}{\partial \eta_k}\right]_{\eta_i = 0} = 0 \end{aligned}\right\} \tag{7-57}$$

in which,

$$\{i,j,k = 1,2,3 : i \neq j \neq k\}$$

c) Free edge. If the side $\eta_i = 0$ is free, then the normal bending moment M_{in} and the edge reaction V_i are zero. That is,

$$\left[M_{in}\right]_{\eta_i = 0} = -D\left[\nu\nabla^2 w + (1-\nu)\frac{\partial^2 w}{\partial x_i^2}\right]_{\eta_i = 0} = 0 \qquad (7\text{-}58)$$

where the differential operators ∇^2 and $\partial^2/\partial x_i^2$ are given by

$$\left.\begin{aligned} \nabla^2 &= \left[\frac{\partial^2}{\partial \eta_1^2} + \frac{\partial^2}{\partial \eta_2^2} + \frac{\partial^2}{\partial \eta_3^2} - 2\left[\ell_1 \frac{\partial^2}{\partial \eta_2 \partial \eta_3} + \ell_2 \frac{\partial^2}{\partial \eta_1 \partial \eta_3} + \ell_3 \frac{\partial^2}{\partial \eta_1 \partial \eta_2}\right]\right] \\ &\text{and} \\ \frac{\partial^2}{\partial x_i^2} &= \left[\frac{\partial^2}{\partial \eta_i^2} + \ell_k^2 \frac{\partial^2}{\partial \eta_j^2} + \ell_j^2 \frac{\partial^2}{\partial \eta_k^2} - 2\ell_k \frac{\partial^2}{\partial \eta_i \partial \eta_j}\right. \\ &\qquad \left. - 2\ell_j \frac{\partial^2}{\partial \eta_i \partial \eta_k} + 2\ell_j \ell_k \frac{\partial^2}{\partial \eta_j \partial \eta_k}\right] \end{aligned}\right\} \qquad (7\text{-}59)$$

where,

$$\{i,j,k = 1,2,3 : i \neq j \neq k\}$$

The condition that the edge reaction V_i must vanish is specified by

$$V_i = \left[Q_{x_i} + \frac{\partial M_{x_i y_i}}{\partial y_i}\right]_{\eta_i = 0} = 0 \qquad (7\text{-}60)$$

where Q_{x_i} and $M_{x_i y_i}$ are respectively the shearing forces and twisting moments corresponding to the x_i direction. In terms of trilinear system, these quantities can be given as,

$$Q_{x_i} = -D\frac{\partial}{\partial x_i}\nabla^2 w = -D\left[-\frac{\partial}{\partial \eta_i} + \ell_k \frac{\partial}{\partial \eta_j} + \ell_j \frac{\partial}{\partial \eta_k}\right]\nabla^2 w \qquad (7\text{-}61)$$

and

$$M_{x_i y_i} = - D\,(1 - \nu)\,\frac{\partial^2 w}{\partial x_i \partial y_i} \tag{7-62}$$

Using the operator relationships of Section 2, equation (7-62) can be written as,

$$M_{x_i y_i} = - D\,(1-\nu)\left[m_k \frac{\partial^2 w}{\partial \eta_i \partial \eta_j} - m_j \frac{\partial^2 w}{\partial \eta_i \partial \eta_k} - \ell_k m_k \frac{\partial^2 w}{\partial \eta_j^2} + \ell_j m_j \frac{\partial^2 w}{\partial \eta_k^2} + (\ell_k m_j - \ell_j m_k)\frac{\partial^2 w}{\partial \eta_j \partial \eta_k}\right] \tag{7-63}$$

where,

$$\{i,j,k = 1,2,3 : i \neq j \neq k\}$$

Substituting (7-61) and (7-62) in equation (7-60) gives,

$$\left[\frac{\partial^2 w}{\partial x_i^3} + (2-\nu)\,\frac{\partial^2 w}{\partial x_i \partial y_i^2}\right]_{\eta_i = 0} = 0 \tag{7-64}$$

Therefore, the boundary conditions corresponding to the free edge, at $\eta_i = 0$, are specified by equations (7-58) and (7-64). The differential operators $\partial/\partial x_i$ and $\partial/\partial y_i$ are given by equations (2-7) and (2-9) respectively.

8. INTEGRATION OF A FUNCTION COMPOSED OF A TRILINEAR VARIABLE

In analyzing problems in elasticity by approximate methods, such as finite element method or methods based on energy theorems, there is a need to integrate some appropriate function with respect to the domain in question. Since the method of integration of a function composed of trilinear or areal coordinates is not obvious, an investigation is carried out here as regards the method of integration of these types of functions with respect to the triangle of reference.

Let the given function be F,

$$F = F(\zeta_1, \zeta_2, \zeta_3)$$

where, ζ_i $(i = 1,2,3)$ are the areal coordinates, and these quantities are related to the trilinear variables η_i, $(i = 1,2,3)$ according to equation (1-6). That is

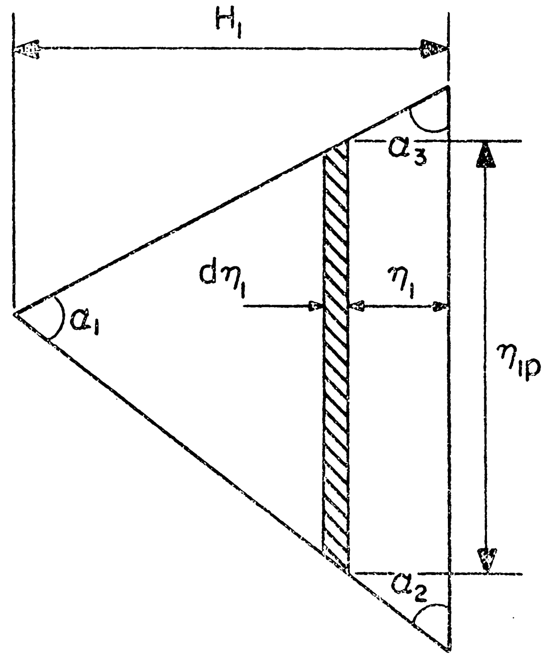
$$\zeta_i = \frac{m_i \eta_i}{\lambda}$$

Let us assume that we need to find

$$\int_\Delta F(\zeta_1, \zeta_2, \zeta_3)\, d\Delta$$

where, Δ is the area of the triangle of reference.

The infinitesimal element of area $d\Delta$ can be represented in more than one way. For example if $F = F(\zeta_1)$ then, with reference to the sketch below, $d\Delta$ can be represented as a function of ζ_1 (or η_1); details are as follows.



The area of the shaded area, $d\Delta$ is,

$$d\Delta = \eta_{1p}\, d\eta_1 \tag{8-1}$$

But using sine rule,

$$\eta_{1p} = 2R\, m_1 \frac{(H_1 - \eta_1)}{H_1} \tag{8-2}$$

In which R is the radius of circumcircle and H_1 is the altitude in the η_1 direction. From the analysis of Section 1, we have

$$H_1 = 2R\, m_2 m_3 \qquad \text{and} \qquad \frac{\eta_1}{H_1} = \zeta_1 \tag{8-3}$$

Substituting (8-3) in equation (8-2) yields

$$\eta_{1p} = \quad 2R\ m_1\ (1 - \zeta_1) \tag{8-4}$$

Therefore equation (8-1) can be written as

$$d\Delta = \eta_{1p}\ d\eta_1 = 2R\ m_1 H_1\ (1 - \zeta_1)\ \frac{d\eta_1}{H_1} = 4R^2\ m_1 m_2 m_3\ (1 - \zeta_1)\ d\zeta_1$$

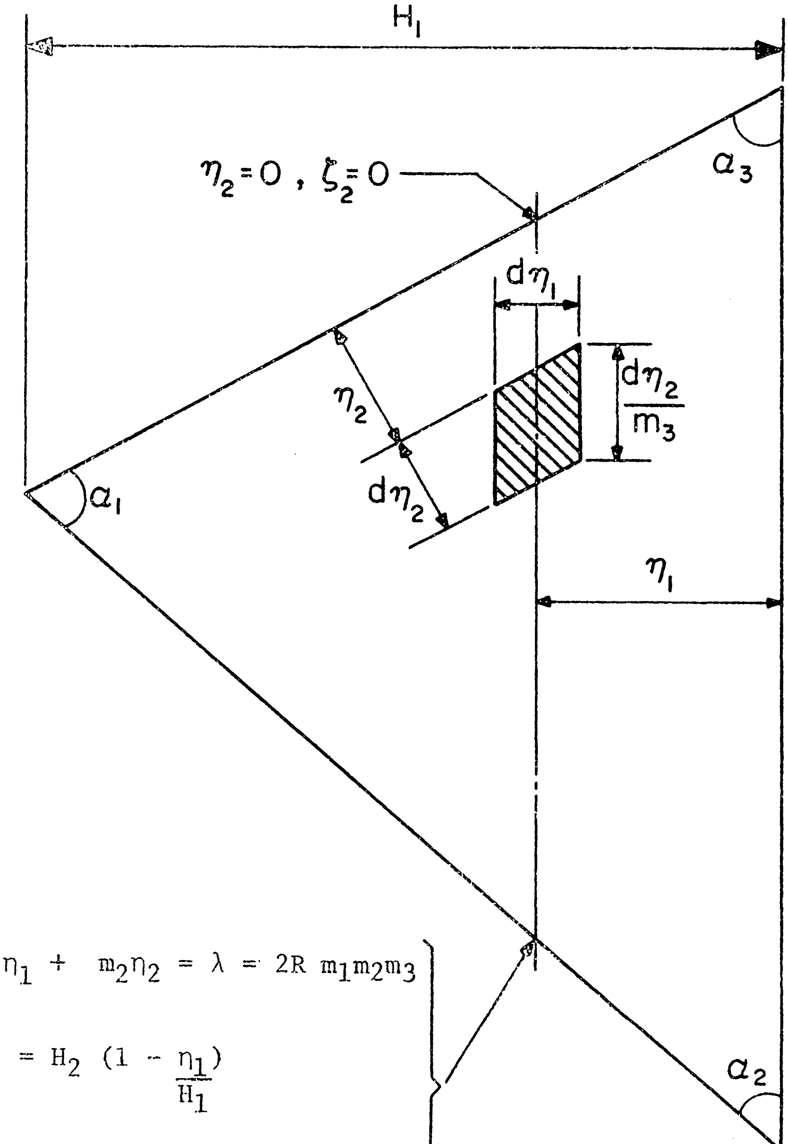
Using equation (1-2), the above equation can be written as

$$d\Delta = 2\Delta\ (1 - \zeta_1)\ d\zeta_1$$

Therefore,

$$\left.\begin{array}{l}
\text{if } F = F(\zeta_1),\text{ then } \displaystyle\int_\Delta F(\zeta_1)\ d\Delta = 2\Delta \int_0^1 (1 - \zeta_1)\ F(\zeta_1)\ d\zeta_1 \\[2ex]
\text{Similarly, it can be shown that} \\[2ex]
\text{if } F = F(\zeta_2),\text{ then } \displaystyle\int_\Delta F(\zeta_2)\ d\Delta = 2\Delta \int_0^1 (1 - \zeta_2)\ F(\zeta_2)\ d\zeta_2 \\[2ex]
\text{or, if } F = F(\zeta_3),\text{ then } \displaystyle\int_\Delta F(\zeta_3)\ d\Delta = 2\Delta \int_0^1 (1 - \zeta_3)\ F(\zeta_3)\ d\zeta_3
\end{array}\right\} \tag{8-5}$$

Suppose that the given function $F = F(\zeta_1, \zeta_2)$ and we need to evaluate the integral $\int_\Delta F(\zeta_1, \zeta_2)\ d\Delta$. In this case, we can express the differential $d\Delta$ as a function of ζ_1 and ζ_2, by making use of the properties of the triangle. Sketch below illustrates, an infinitesimal element of area $d\Delta$, in relation to the triangle of reference.



$$\left.\begin{aligned} \eta_3 = 0, \quad & m_1\eta_1 + m_2\eta_2 = \lambda = 2R\, m_1 m_2 m_3 \\ & \eta_2 = H_2 \left(1 - \frac{\eta_1}{H_1}\right) \\ & \zeta_2 = (1 - \zeta_1) \end{aligned}\right\}$$

The area of $d\Delta$, the shaded portion in the above sketch, is

$$d\Delta = \frac{d\eta_1 d\eta_2}{m_3} = \frac{H_1 H_2}{m_3}\, d\zeta_1 d\zeta_2 \tag{8-6}$$

From the analysis of Section 1, the altitudes H_i are given by

$$H_i = 2R\ m_j m_k$$

Hence, equation (8-6) can be written as

$$d\Delta = 4R^2\ m_1 m_2 m_3\ d\zeta_1\ d\zeta_2$$

Using equation (1-2), the above equation can be written as

$$d\Delta = 2\Delta\ d\zeta_1\ d\zeta_2 \tag{8-7}$$

Therefore,

$$\left.\begin{aligned}
&\text{if } F(\zeta_1,\zeta_2) \text{ then } \int_\Delta F(\zeta_1,\ \zeta_2)\ d\Delta = 2\Delta \int_0^1 \left[\int_0^{(1-\zeta_1)} F(\zeta_1,\zeta_2)\ d\zeta_2\right] d\zeta_1 \\
&\text{Similarly, it can be shown that} \\
&\text{if } F(\zeta_1,\zeta_3) \text{ then } \int_\Delta F(\zeta_1,\ \zeta_3)\ d\Delta = 2\Delta \int_0^1 \left[\int_0^{(1-\zeta_1)} F(\zeta_1,\zeta_2)\ d\zeta_3\right] d\zeta_1 \\
&\text{or, if } F(\zeta_2,\ \zeta_3) \text{ then } \int_\Delta F(\zeta_2,\ \zeta_3)\ d\Delta = 2\Delta \int_0^1 \left[\int_0^{(1-\zeta_2)} F(\zeta_2,\ \zeta_3)\ d\zeta_3\right] d\zeta_2
\end{aligned}\right\} \quad (8\text{-}8)$$

Suppose $F = F(\zeta_1,\ \zeta_2,\ \zeta_3)$, then we may eliminate one of the variables by using the identity, $\zeta_1 + \zeta_2 + \zeta_3 = 1$, and then equation (8-8) can be used to integrate F with respect to the area over the area of the triangle of reference.

For example (13), if $F = \zeta_1^i \zeta_2^j \zeta_3^k$, i,j,k are integers, then

$$\int_{\Delta} \zeta_1^i \zeta_2^j \zeta_3^k \; d\Delta = 2\Delta \frac{i!\, j!\, k!}{(i+j+k+2)!} \tag{8-9}$$

Details of the above examples can be found on page 84, Reference 13.

9. SOME FUNCTIONAL RELATIONS BETWEEN THE TRILINEAR VARIABLES

This section contains some functional relations between the trilinear variables. The relationships developed here are useful, in investigating the possible form of the solution to a boundary value problem involving triangular boundary.

From equation (1-7) we have,

$$\zeta_1 + \zeta_2 + \zeta_3 = 1$$

Multiplying this equation by $n\pi$, n an integer, we have

$$n\pi\ \zeta_1 + n\pi\ \zeta_2 + n\pi\ \zeta_3 = n\pi \tag{9-1}$$

and

$$\sin\ (n\pi\ \zeta_1 + n\pi\ \zeta_2 + n\pi\ \zeta_3) = 0$$

Expanding the above equation shows,

$$S_{n1}\ S_{n2}\ S_{n3} = \left[S_{n1}\ C_{n2}\ C_{n3} + S_{n2}\ C_{n1}\ C_{n3} + S_{n3}\ C_{n1}\ C_{n2}\right] \tag{9-2}$$

where,

$$S_{ni} = \sin\ n\pi\ \zeta_i \text{ and } C_{ni} = \cos\ n\pi\ \zeta_i \tag{9-3}$$

From equation (9-1),

$$\cos n\pi\ \zeta_1 = \cos n\pi\ (1 - \zeta_2 - \zeta_3) = \cos n\pi\ \ \cos n\pi\ (\zeta_2 + \zeta_3)$$

$$= (-1)^n \left[\cos n\pi\ \zeta_2 \cos n\pi\ \zeta_3 - \sin n\pi\ \zeta_2 \sin n\pi\ \zeta_3\right]$$

That is,

$$\left.\begin{aligned} C_{n2}\ C_{n3} &= (-1)^n\ C_{n1} + S_{n2}\ S_{n3} \\ &\text{similarly,} \\ C_{n1}\ C_{n3} &= (-1)^n\ C_{n2} + S_{n1}\ S_{n3} \\ C_{n1}\ C_{n2} &= (-1)^n\ C_{n3} + S_{n1}\ S_{n2} \end{aligned}\right\} \quad (9\text{-}4)$$

Using the relations (9-4), equation (9-2) can be written as,

$$S_{n1}\ S_{n2}\ S_{n3} = \frac{(-1)^{n+1}}{2}\left[S_{n1}\ C_{n1} + S_{n2}\ C_{n2} + S_{n2}\ C_{n3}\right] \quad (9\text{-}5)$$

Using (9-3), the above equation can be written as,

$$\sin n\pi\ \zeta_1 \sin n\pi\ \zeta_2 \sin n\pi\ \zeta_3 = \frac{(-1)^{n+1}}{4}\Big[\sin 2n\pi\ \zeta_1$$

$$+ \sin 2n\pi\ \zeta_2 + \sin 2n\pi\ \zeta_3\Big] \quad (9\text{-}6)$$

Using equation (9-1), numerous numbers of functional relations between the trilinear variables can be developed. Some of these relations are given below.

$$\left[C_{n1}\ C_{n2}\ C_{n3} + (-1)^{n+1}\right] = \left[S_{n1}\ S_{n2}\ C_{n3} + S_{n1}\ C_{n2}\ S_{n3} + S_{n3}\ S_{n2}\ C_{n1}\right] \tag{9-7}$$

$$= \frac{1}{2}\left[S_{n1}^{\ 2} + S_{n2}^{\ 2} + S_{n3}^{\ 2}\right] \tag{9-8}$$

Also,

$$\left[\cos 2n\pi\ \zeta_1 + \cos 2n\pi\ \zeta_2 + \cos 2n\pi\ \zeta_3\right] =$$

$$= 3 - 4\left[\cos n\pi\ \zeta_1\ \cos n\pi\ \zeta_2\ \cos n\pi\ \zeta_3 + (-1)^{n+1}\right] \tag{9-9}$$

Trigonometrical summation

From equation (512), page 96, Reference 14, we have

$$\frac{\pi}{2}(1 - 2\theta) = \sum_{n=1}^{\infty} \frac{\sin 2n\pi\theta}{n} \quad , 0 < \theta < 1 \tag{9-10}$$

Therefore,

$$\frac{\pi}{2}\left[(1-2\ \zeta_1) + (1-2\ \zeta_2) + (1-2\ \zeta_3)\right] = \frac{\pi}{2} \quad , \left[\text{since, } \zeta_1+\zeta_2+\zeta_3 = 1\right]$$

$$= \sum_{n=1}^{\infty} \frac{(\sin 2n\pi\zeta_1 + \sin 2n\pi\zeta_2 + \sin 2n\pi\zeta_3)}{n},$$

$(0 < \zeta_i < 1)$ and $i = 1,2,3.$

Using relation (9-6), the above equation becomes,

$$\frac{\pi}{2} = 4 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin n\pi\ \zeta_1 \sin n\pi\ \zeta_2 \sin n\pi\ \zeta_3}{n}$$

or

$$1 = \frac{8}{\pi} \sum_{1=n}^{\infty} (-1)^{n+1} \frac{\sin n\pi\ \zeta_1 \sin n\pi\ \zeta_2 \sin n\pi\ \zeta_3}{n} \qquad (9\text{-}11)$$

for $0 < \zeta_i < 1;\ i = 1,2,3$

10. APPLICATIONS

Applications of trilinear coordinates to problems in plane elasticity will be illustrated by considering two problems associated with a thin triangular plate. We will consider here however, only an equilateral triangular plate with all its edges simply supported. The problems investigated are:

1) Bending subjected to a uniform normal load
2) Free vibration of a triangular plate

Treating each of these problems separately, the details of the analysis are as follows.

10.1. Bending of a simply supported equilateral triangular plate.

The solution to this problem, in terms of Cartesian coordinates (x,y), was first obtained by Woinowsky-Krieger; details of his analysis can be found in page 313 of Reference 10. We will now proceed to obtain the solution to this problem based on the theory developed in the previous sections.

The governing differential equation to this problem is given by,

$$\nabla^4 w = \frac{q}{D} \tag{10-1}$$

in which q is the uniform normal load intensity. If h is the plate thickness, then the flexural rigidity D is given by,

$$D = \frac{E\ h^3}{12\ (1-\nu^2)} \tag{10-2}$$

In the case of simply supported polygonal plates, it is convenient to replace equation (10-1) by the following two differential equations of the second order. (Details of this procedure are given in Section 7.)

$$\nabla^2 M = -q \tag{10-3}$$

$$\nabla^2 w = -\frac{M}{D} \tag{10-4}$$

where,

$$M = \frac{1}{(1+\nu)} (M_1 + M_2 + M_3) \tag{10-5}$$

Since the plate is simply supported, it follows from Section (7.4) that the boundary conditions are given by,

$$\left.\begin{aligned} \left[w\right]_{\eta_i = 0} &= 0 \\ \left[\nabla^2 w\right]_{\eta_i = 0} &= 0 \end{aligned}\right\} (i = 1,2,3) \tag{10-6}$$

In the case of the equilateral triangle, $\ell_i = \cos \alpha_i = \cos \pi/3 = 1/2$, therefore, Laplace's operator ∇^2, given by equation (2-14), can be written as

$$\nabla^2 = \left[\frac{\partial^2}{\partial\eta_1^2} + \frac{\partial^2}{\partial\eta_2^2} + \frac{\partial^2}{\partial\eta_3^2} - \left[\frac{\partial^2}{\partial\eta_2\partial\eta_3} + \frac{\partial^2}{\partial\eta_1\partial\eta_3} + \frac{\partial^2}{\partial\eta_1\partial\eta_2}\right]\right] \tag{10-7}$$

Comparing equation (10-4) and the second equation of (10-6), we conclude that

$$\left[M\right]_{\eta_i = 0} = 0 \qquad (i = 1,2,3) \tag{10-8}$$

A tentative solution to (10-3), which is subjected to the boundary condition (10-8), will be taken as,

$$M = A\ \eta_1\ \eta_2\ \eta_3 \tag{10-9}$$

where, A is a constant.

Substituting equation (10-9) in (10-3) yields,

$$\nabla^2(A\eta_1\eta_2\eta_3) = \left[\frac{\partial^2}{\partial\eta_1^2} + \frac{\partial^2}{\partial\eta_2^2} + \frac{\partial^2}{\partial\eta_3^2} - \left[\frac{\partial^2}{\partial\eta_2\partial\eta_3} + \frac{\partial^2}{\partial\eta_1\partial\eta_3} + \frac{\partial^2}{\partial\eta_2\partial\eta_3}\right]\right](A\eta_1\eta_2\eta_3)$$

$$= -A\ (\eta_1 + \eta_2 + \eta_3) = -q \tag{10-10}$$

In the case of the equilateral triangle, $m_i = \sin \alpha_i = \sin \pi/3 = \sqrt{3}/2$. Therefore, the fundamental identity of the trilinear coordinates, given by equation (1-3), can be written as,

$$(\eta_1 + \eta_2 + \eta_3) = \frac{2\lambda}{\sqrt{3}} = \frac{2}{\sqrt{3}} (2R\, m_1 m_2 m_3) = \frac{3}{2} R \tag{10-11}$$

where, R is the radius of the circumcircle.

Substituting (10-11) into equation (10-10) shows that $A = 2q/3R$. Therefore, from equations (10-4) and (10-9), we have

$$\nabla^2 w = -\frac{2q}{3RD} \eta_1\eta_2\eta_3 \tag{10-12}$$

The solution to the above equation is subjected to the first equation of (10-6); that is, $[w]_{\eta_i = 0} = 0$, $(i = 1,2,3)$

If we assume the solution to equation (10-12) as,

$$w = B(\eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3)\,\eta_1\eta_2\eta_3 \quad , \quad (B, \text{ a constant}) \tag{10-13}$$

then the required condition that the displacement must vanish at the boundary is satisfied. Substituting (10-13) in equation (10-12) yields,

$$B\left[\frac{\partial^2}{\partial\eta_1^2} + \frac{\partial^2}{\partial\eta_2^2} + \frac{\partial^2}{\partial\eta_3^2} - \left[\frac{\partial^2}{\partial\eta_2\partial\eta_3} + \frac{\partial^2}{\partial\eta_1\partial\eta_3} + \frac{\partial^2}{\partial\eta_1\partial\eta_2}\right]\right]\left[\eta_1^2\eta_2^2\eta_3 + \eta_1^2\eta_3^2\eta_2 + \eta_2^2\eta_3^2\eta_1\right]$$

$$= -12B\,\eta_1\,\eta_2\,\eta_3 = -\frac{2q}{3RD}\,\eta_1\,\eta_2\,\eta_3 \tag{10-14}$$

Therefore,

$$B = \frac{q}{18RD}$$

and hence the transverse displacement is given by,

$$w = \frac{q}{18RD} (\eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3)\ \eta_1\eta_2\eta_3 \tag{10-15}$$

It is of interest to note that the expression $(\eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3)$ is the equation of the circumcircle, while $\eta_1\eta_2\eta_3$ is the equation of the triangle.

Having found the solution to the problem in terms of trilinear variables, we will now express the solution (10-15) in terms of the Cartesian variables (x,y) and then compare this result with the solution provided by Woinowsky-Krieger.

The relationships between the Cartesian and trilinear variables are given by equation (2-4): In the case of the equilateral triangle, these relationships become

$$\left.\begin{aligned} \eta_1 &= \rho - x \\ \eta_2 &= \rho + \frac{x}{2} - \frac{\sqrt{3}\,y}{2} \\ \eta_3 &= \rho + \frac{x}{2} + \frac{\sqrt{3}\,y}{2} \end{aligned}\right\} \tag{10-16}$$

in which ρ is the radius of incircle; for the equilateral triangle,

$$\rho = 4R \sin\frac{\alpha_1}{2} \sin\frac{\alpha_2}{2} \sin\frac{\alpha_3}{2} = 4R \sin^3\frac{\pi}{6} = \frac{R}{2} \tag{10-17}$$

Substituting (10-16) and (10-17) in equation (10-15) gives,

$$w = \frac{q}{18RD}\left[\frac{3}{16}\left[R^2 - (x^2+y^2)\right]\left[\frac{R^3}{2} - x^3 - \frac{3R}{2}(x^2+y^2) + 3y^2x\right]\right] \qquad (10\text{-}18)$$

We could replace R in the above equation by the altitude H of the triangle. For an equilateral triangle all the three altitudes H_i are of the same magnitude, say H; but from Section 1, H = 3R/2. Therefore equation (10-18) takes the form,

$$W = \frac{q}{64\ DH}\left[\left[\frac{4}{9}H^2 - x^2 - y^2\right]\left[\frac{4}{27}H^3 - x^3 - H(x^2 + y^2) + 3y^2x\right]\right] \qquad (10\text{-}19)$$

It may appear that there is a conflict in sign between the above result and the solution given by Woinowsky-Krieger (equation (201), page 313, Reference 10). The reason for this disagreement in sign is due to the adapted conventions regarding the orientation of the triangle with respect to the reference Cartesian coordinates. If we substitute, x = -x and y = -y, in equation (10-19), we have the same form of solucion as that obtained by Woinowsky-Krieger.

10.2. Free vibration of a simply supported equilateral triangular plate.

The governing equation corresponding to the free vibration of a plate, of uniform thickness and of mass density μ, is given by

$$D\nabla^4 w + \mu\frac{\partial^2 w}{\partial t^2} = 0 \qquad (10\text{-}20)$$

Solution to the above equation is subjected to the simply supported edge conditions.

$$\left.\begin{aligned} \left[w\right]_{\eta_i = 0} &= 0 \\ \left[\nabla^2 w\right]_{\eta_i = 0} &= 0 \end{aligned}\right\} \quad (i = 1,2,3) \tag{10-21}$$

Solution to the above problem will now be found by utilizing the analysis of Section 9.

From equation (2-14), the Laplacian operator ∇^2 for the trilinear coordinate system is

$$\nabla^2 = \left[\frac{\partial^2}{\partial\eta_1^2} + \frac{\partial^2}{\partial\eta_2^2} + \frac{\partial^2}{\partial\eta_3^2} - 2\left[\ell_1 \frac{\partial^2}{\partial\eta_2\partial\eta_3} + \ell_2 \frac{\partial^2}{\partial\eta_1\partial\eta_3} + \ell_3 \frac{\partial^2}{\partial\eta_2\partial\eta_3}\right]\right]$$

Using the relationships between the trilinear and aereal coordinates, given by equation (1-6), we could write ∇^2 in terms of the non-dimensional variables ζ_i (areal coordinates). That is,

$$\begin{aligned} \nabla^2 = \frac{1}{\lambda^2} \Bigg[& m_1^2 \frac{\partial^2}{\partial\zeta_1^2} + m_2^2 \frac{\partial^2}{\partial\zeta_2^2} + m_3^2 \frac{\partial^2}{\partial\zeta_3^2} \\ & - 2\left[\ell_1 m_2 m_3 \frac{\partial^2}{\partial\zeta_2\partial\zeta_3} + \ell_2 m_1 m_3 \frac{\partial^2}{\partial\zeta_1\partial\zeta_3} + \ell_3 m_1 m_2 \frac{\partial^2}{\partial\zeta_1\partial\zeta_2}\right]\Bigg] \end{aligned} \tag{10-22}$$

where,

$$m_i = \sin\alpha_i \quad , \quad \ell_i = \cos\alpha_i \qquad (i = 1,2,3)$$

and $\quad \lambda = 2R\, m_1 m_2 m_3 \quad$, R is the radius of circumcircle

In the case of the equilateral triangle, $m_i = \sqrt{3}/2$, $\ell_i = 1/2$ and $\lambda = R3\sqrt{3}/4$. In this case equation (10-22) reduces to,

$$\nabla^2 = \frac{4}{9R^2}\left[\frac{\partial^2}{\partial\zeta_1^2} + \frac{\partial^2}{\partial\zeta_2^2} + \frac{\partial^2}{\partial\zeta_3^2} - \left[\frac{\partial^2}{\partial\zeta_2\partial\zeta_3} + \frac{\partial^2}{\partial\zeta_1\partial\zeta_3} + \frac{\partial^2}{\partial\zeta_1\partial\zeta_2}\right]\right] \tag{10-23}$$

Since $\zeta_i = 0$ implies and implied by $\eta_i = 0$, we can write equation (10-24) as,

$$\left.\begin{aligned} \left[w\right]_{\zeta_i = 0} &= 0 \\ \left[\nabla^2 w\right]_{\zeta_i = 0} &= 0 \end{aligned}\right\} (i = 1,2,3) \tag{10-24}$$

We will now intuitively assume that the solution to equation (10-20) is of the form,

$$w = \sum_{n=1}^{\infty} B_n \left[S_{n1}\ S_{n2}\ S_{n3}\right] \cos \omega_n t \tag{10-25}$$

where, B_n represents the amplitude of the n^{th} mode, and ω_n represents the natural frequency of n^{th} mode, and this quantity has to be determined. The term S_{ni}, occuring in equation (10-25), are defined by equation (9-3); that is, $S_{ni} = \sin n\pi\, \zeta_i$, (i = 1,2,3).

Equation (10-25) satisfies the first equation of (10-24); we will now verify that the expression (10-25) also satisfies the second equation of (10-24).

Operating equation (10-25) with ∇^2, yields

$$\nabla^2 w = \frac{-4\pi^2}{9R^2} \sum_{n=1}^{\infty} n^2 B_n \Big[3\, S_{n1}\, S_{n2}\, S_{n3}$$

$$+ (S_{n1}\, C_{n2}\, C_{n3} + S_{n2}\, C_{n1}\, C_{n3} + S_{n3}\, C_{n1}\, C_{n2}) \Big] \cos \omega_n t \qquad (10\text{-}26)$$

where,

$C_{ni} = \cos n\pi\, \zeta_i$, and from equation (9-2)

$S_{n1}\, S_{n2}\, S_{n3} = (S_{n1}\, C_{n2}\, C_{n3} + S_{n2}\, C_{n1}\, C_{n3} + S_{n3}\, C_{n1}\, C_{n2})$

Therefore, using the last relationship, equation (10-26) can be written as,

$$\nabla^2 w = -\frac{16\pi^2}{9\, R^2} \sum_{n=1}^{\infty} n^2 B_n \Big[S_{n1}\, S_{n2}\, S_{n3} \Big] \cos \omega_n t \qquad (10\text{-}27)$$

Since S_{ni} vanishes at the i^{th} edge of the triangle, it follows from equation (10-27) second equation of (10-24) is also satisfied.

Substituting (10-25) in equation (10-20) yields,

$$D\, n^4\, \frac{256 \quad {}^4}{81\, R^4} = \mu\, \omega_n^{\,2}$$

or,

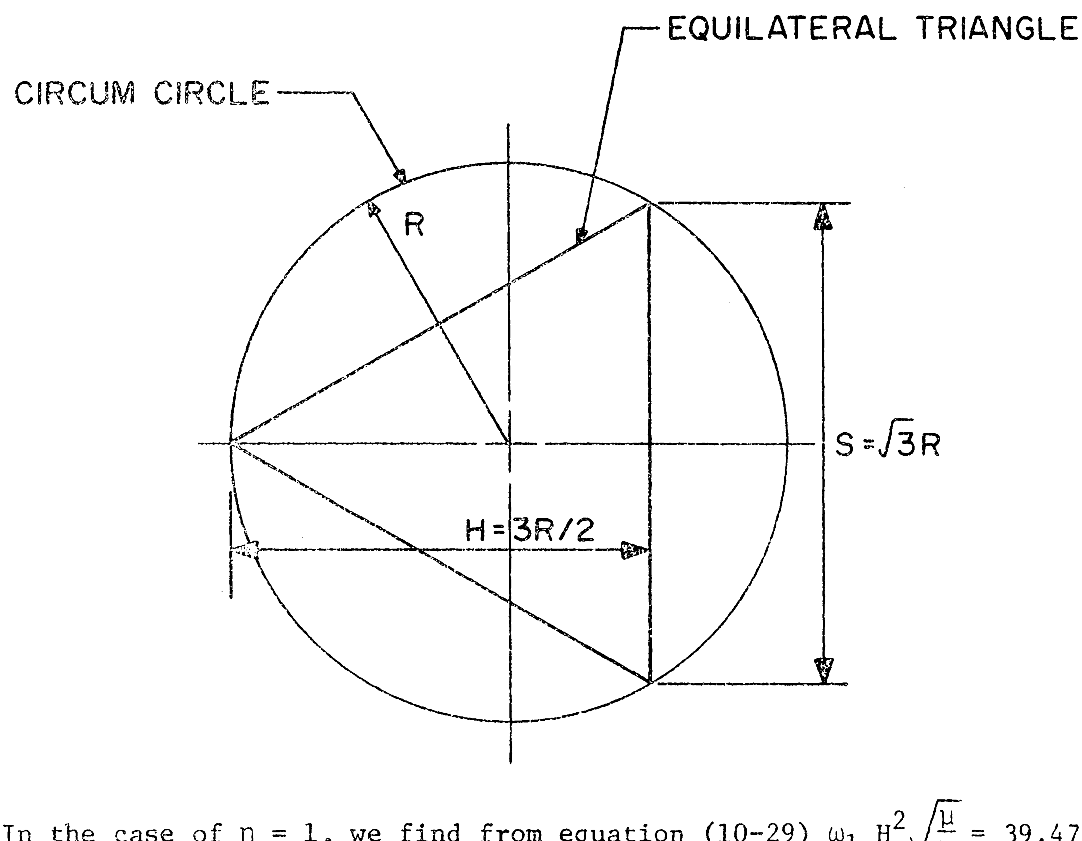
$$\boxed{\omega_n\, R^2 \sqrt{\frac{\mu}{D}} = n^2\, \pi^2\, \frac{16}{9}} \qquad (10\text{-}28)$$

With reference to the sketch below, the above equation for the natural frequency ω_n can also be written

$$\boxed{\omega_n H^2 \sqrt{\frac{\mu}{D}} = 4 n^2 \pi^2} \qquad (10\text{-}29)$$

or,

$$\boxed{\omega_n S^2 \sqrt{\frac{\mu}{D}} = n^2 \pi^2 \frac{16}{3}} \qquad (10\text{-}30)$$



In the case of $\eta = 1$, we find from equation (10-29) $\omega_1 H^2\sqrt{\frac{\mu}{D}} = 39.478$ and this result is in agreement with the established results for the fundamental frequency.

DISCUSSION

The concept of trilinear coordinates, also known as homogeneous point coordinates in the plane, are well known for their convenience in the study of problems in projective geometry. The present investigation was carried out in the belief that these coordinates could also be of great value in the study of boundary value problems involving triangular boundaries.

The analytical properties of trilinear coordinates -- as applied to the physical sciences -- are not yet fully established. Therefore, considerable analysis has yet to be performed to fully understand their complexity and to exploit their merits. In this respect, this discourse may be considered as a preliminary investigation on the applicability of trilinear coordinates to boundary value problems. Although this report considers only the suitability of trilinear coordinates to problems in plane elasticity, it is believed that in due time, and with further investigation, they could also be of use in the study of other physical sciences.

In the present investigation, it was found useful to introduce the concepts of triaxial stress, strain, moments and curvature. These quantities are very unfamiliar and do not have direct physical meaning; however they are found to possess certain qualities, that make it possible to express the governing equations in a simpler and more elegant form. This makes it possible to aim at a solution in terms of trilinear variables in a relatively easy manner. Although the triaxial quantities, i.e., triaxial stress, triaxial strain, etc., do not have physical meaning, they are linear combinations of known physical quantities, such as normal stress, normal strain, etc. The algebraic relationships between physical and non-physical quantities are explicitly given in this discourse, and therefore one can always obtain physical interpretation at any stage of the analysis.

To illustrate the application of the theory developed herein, two examples of simply supported equilateral triangular plates are considered. Solutions to both of these problems were previously known in terms of Cartesian coordinates. These two simple examples are included here merely for the purpose of illustrating the techniques that one must adopt in dealing with trilinear coordinates.

At present, progress is being made on the application of trilinear coordinates to some problem in plane elasticity, which are not of triangular configuration (i.e.: hexagon, rombus, etc.). It appears that the trilinear coordinates may also have some advantages over the conventional Cartesian coordinates in the presence of induced anisotropy such as laminated and semi-monoque structures. Application of trilinear coordinates to these problems is also under consideration. The results of the present series of investigations will be presented in a subsequent report.

REFERENCES

1. Whitworth, W. A., Trilinear Coordinates, and Other Methods of Modern Analytical Geometry of Two Dimensions. Deighton, Bell and Co., Cambridge (1866).

2. Ferrers, N. M., An Elementry Treatise on Trilinear Coordinates, the Method of Reciprocal Polars, and the Theory of Projections. Macmillan and Co., London (1866).

3. Askwith, E. H., Analytical Geometry of the Conic Sections. A & C. Black Ltd., London (1918).

4. Loney, S. L., The Elements of Coordinate Geometry, Vol. 2, Trilinear Coordinates. Macmillan and Co., London (1923).

5. Smith, C., An Elementry Treatise on Conic Sections. Macmillan and Co., London (1919).

6. Aiken, H., "Trilinear Coordinates," Journal of Applied Physics, Vol. 8, (July 1937).

7. Mertie, J. B., "Transformation of Trilinear and Quadriplanar Coordinates to and from Cartesian Coordinates," The American Mineralogist, Vol. 49 (July 1964).

8. Argyris, J. H., "Continua and Discontinua," Matrix Methods in Structural Mechanics, page 11-189, Proceedings of the Congress held at Wright-Patterson Air Force Base, Ohio (October 1965).

9. Bazeley, G. P., Cheung, Y. K., Irons, R. M., Zienkiewicz, O. C., "Triangular Elements in Plate Bending - Conforming and Non-conforming Solutions," Matrix Methods in Structural Mechanics, page 547-561, Proceedings of the congress held at Wright-Patterson Air Force Base, Ohio (October 1965).

10. Timoshenko, S., and Woinowsky-Krieger, S., Theory of Plates and Shells, Second Edition. McGraw-Hill Book Co., New York (1959).

11. Timoshenko, S., Goodier, J. N., Theory of Elasticity, Second Edition. McGraw-Hill Book Co., New York (1951).

12. Mansfield, E. H., The Bending and Stretching of Plates. Macmillan and Co., New York (1964).

13. Holland, I., and Bell, K., Finite Element Method in Stress Analysis. Tapir, Technical University of Norway (1969).

14. Jolley, L. B. W., Summation of Series, Second Edition. Dover Publications, Inc., New York (1961).

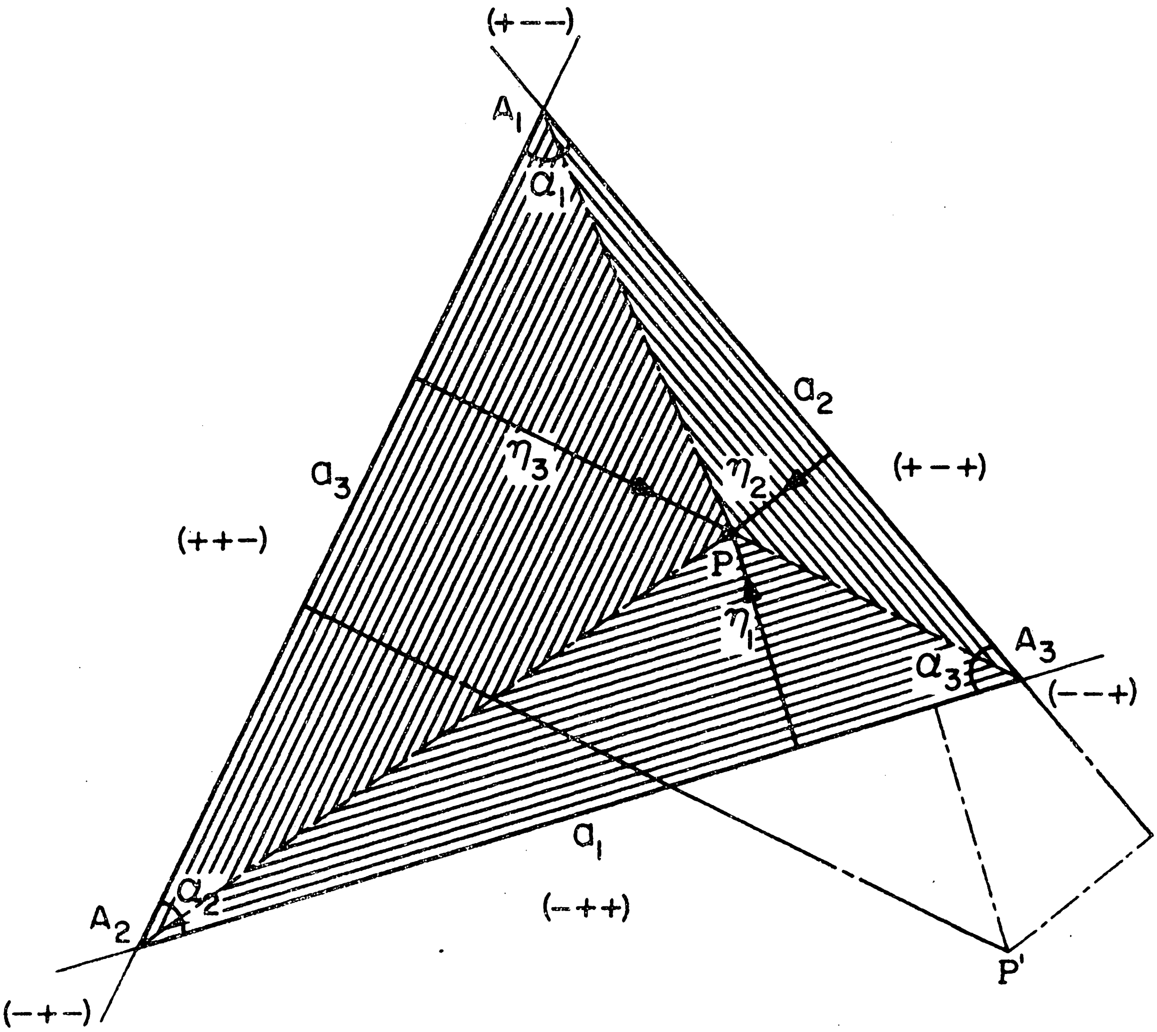


FIGURE I. TRILINEAR COORDINATES-SIGN CONVENTION

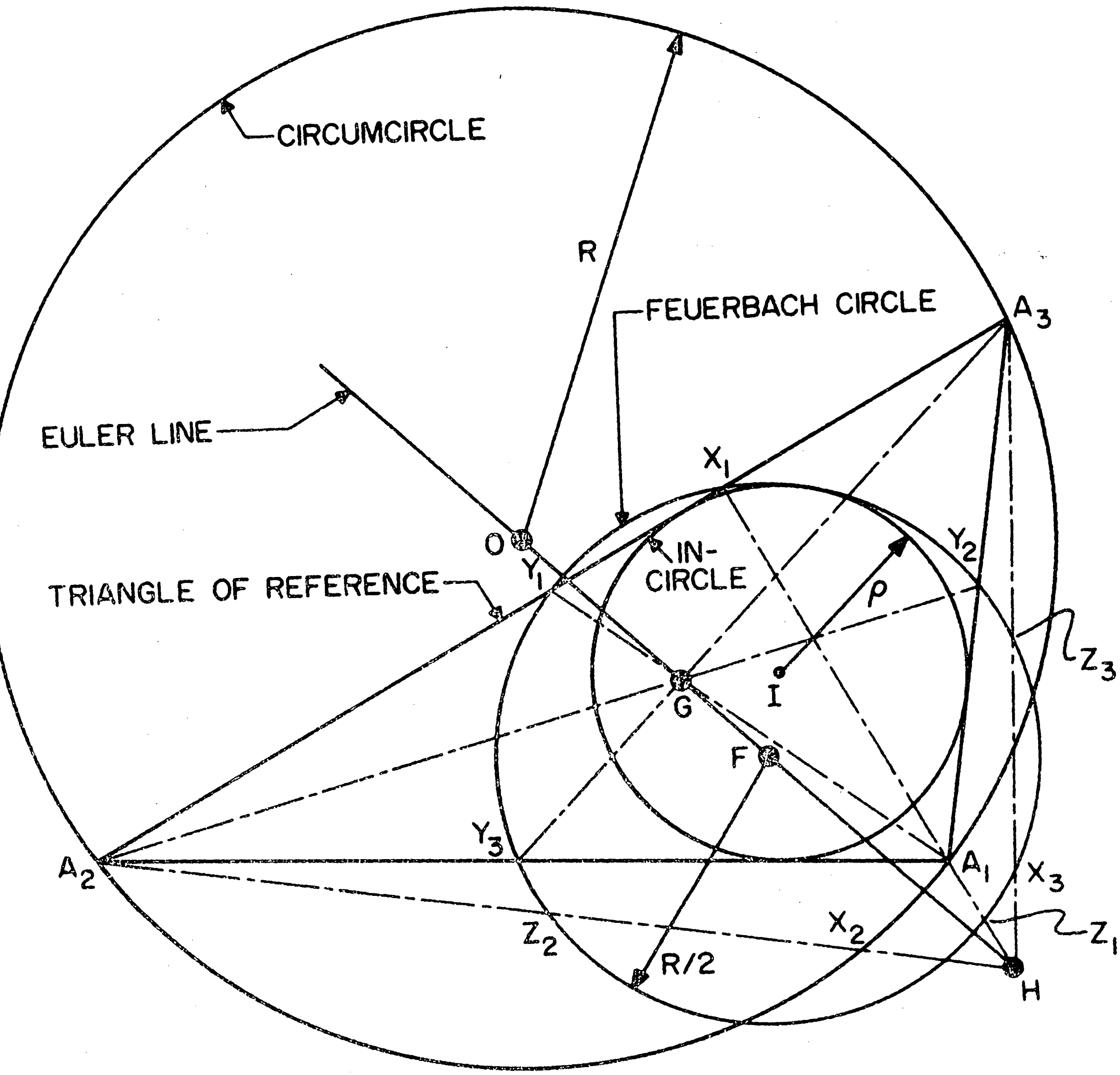


FIGURE 2. SOME RELEVANT POINTS AND CIRCLES ASSOCIATED WITH A TRIANGLE

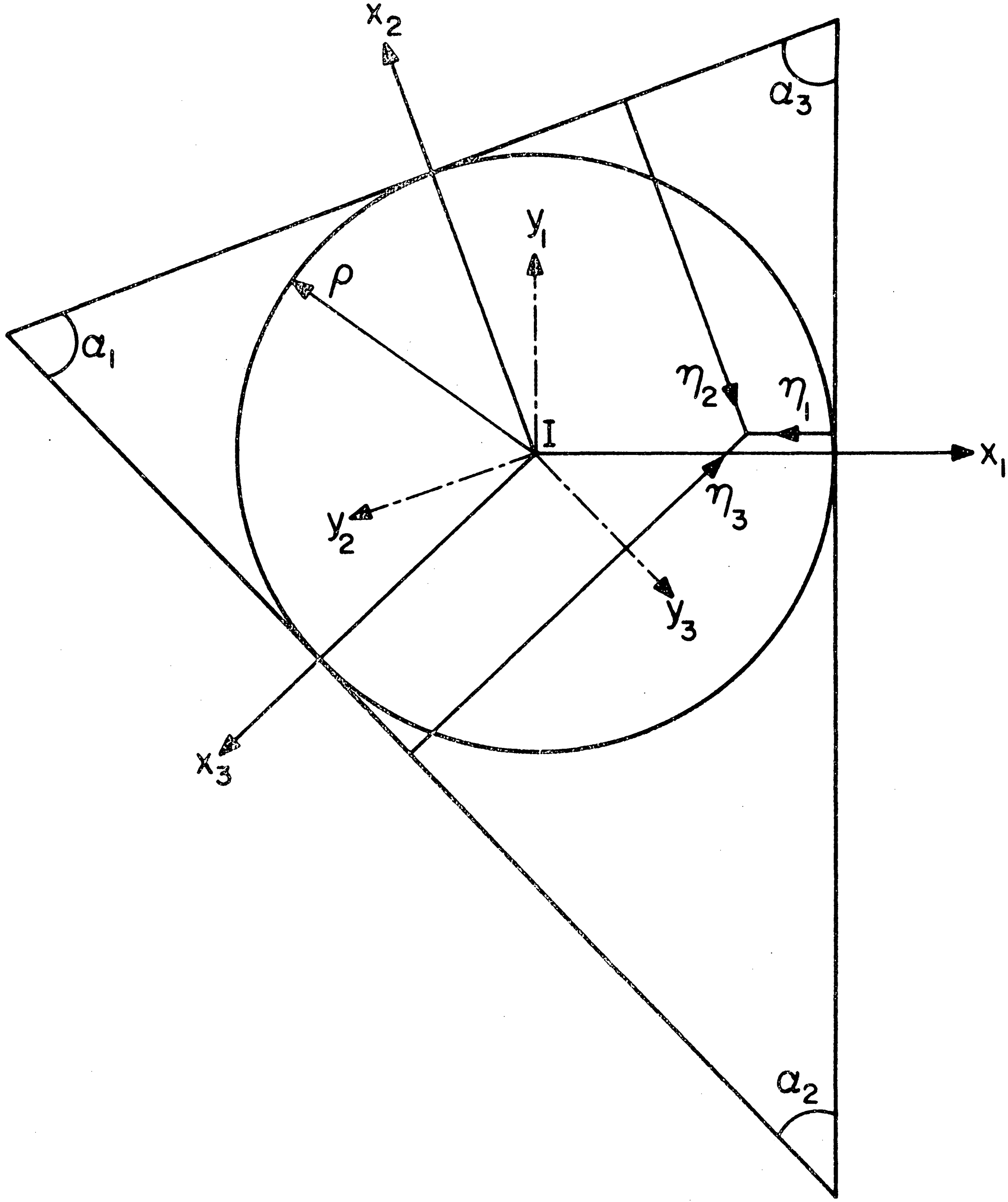


FIGURE 3. TRIAXIAL COORDINATES IN RELATION TO THE TRIANGLE OF REFERENCE

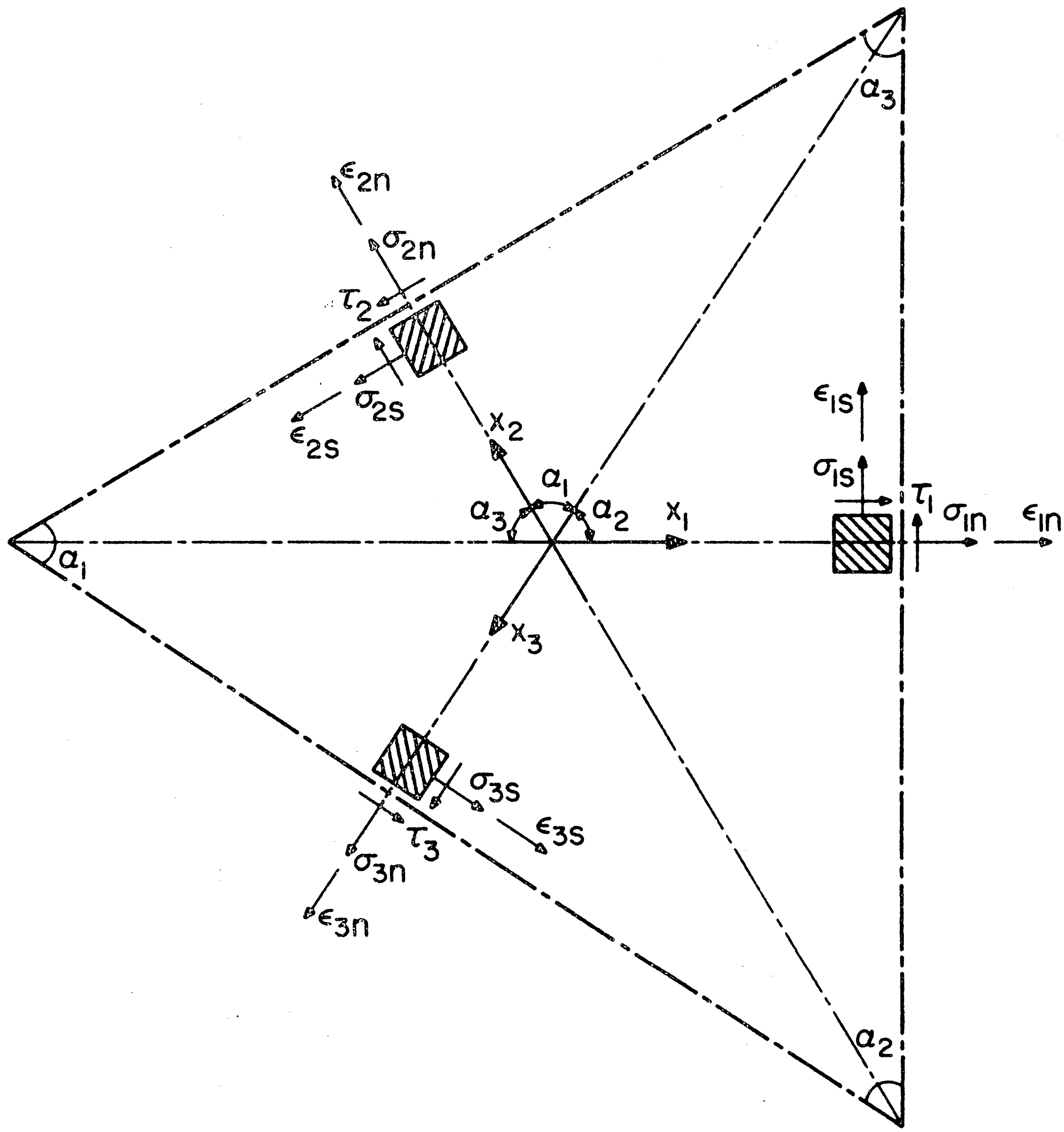


FIGURE 4. ORIENTATION OF DIRECT STRESS AND AND STRAIN COMPONENTS IN RELATION TO THE TRIANGLE OF REFERENCE

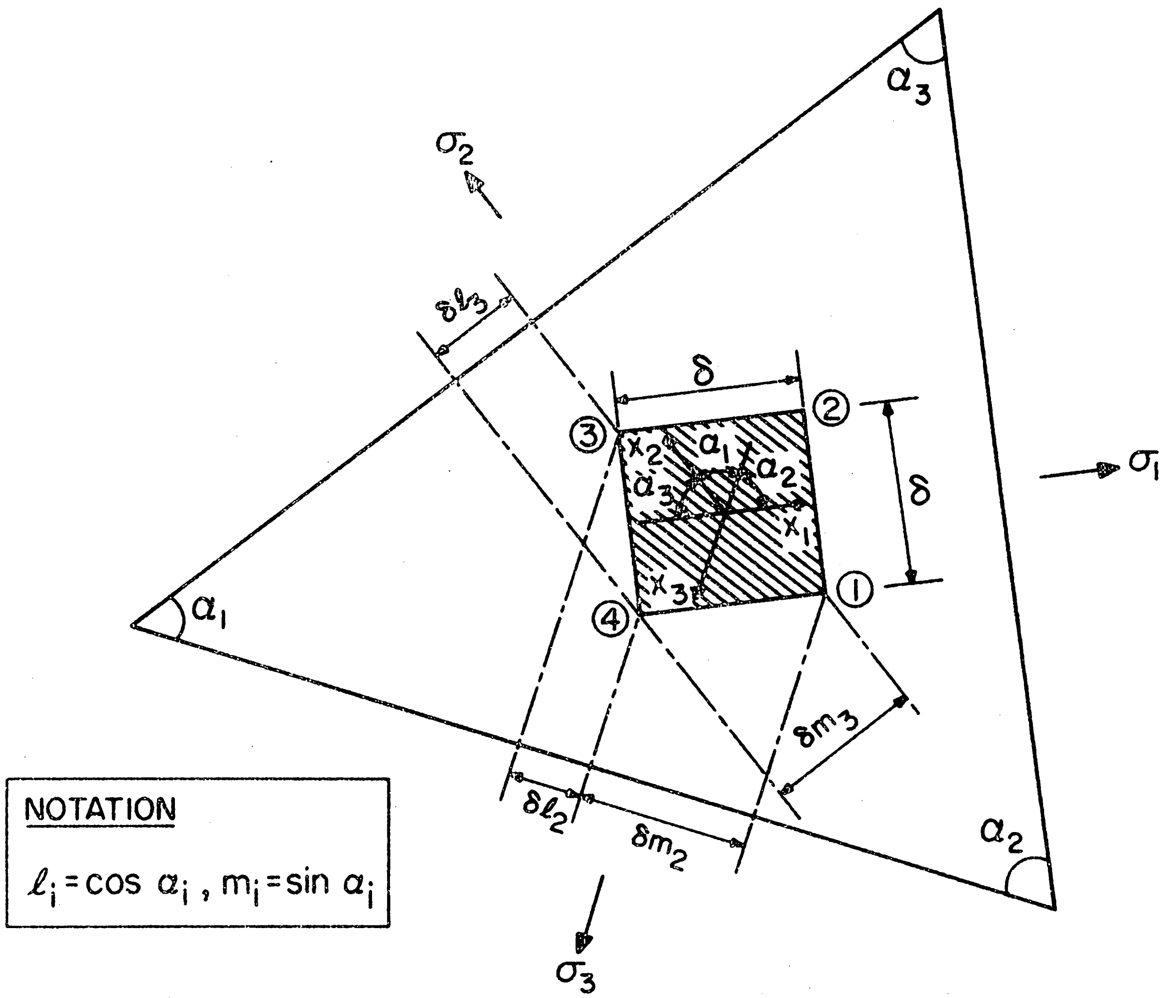


FIGURE 5. A TRIANGULAR ELEMENT SUBJECTED TO TRIAXIAL STRESS COMPONENTS ($\sigma_1, \sigma_2, \sigma_3$)

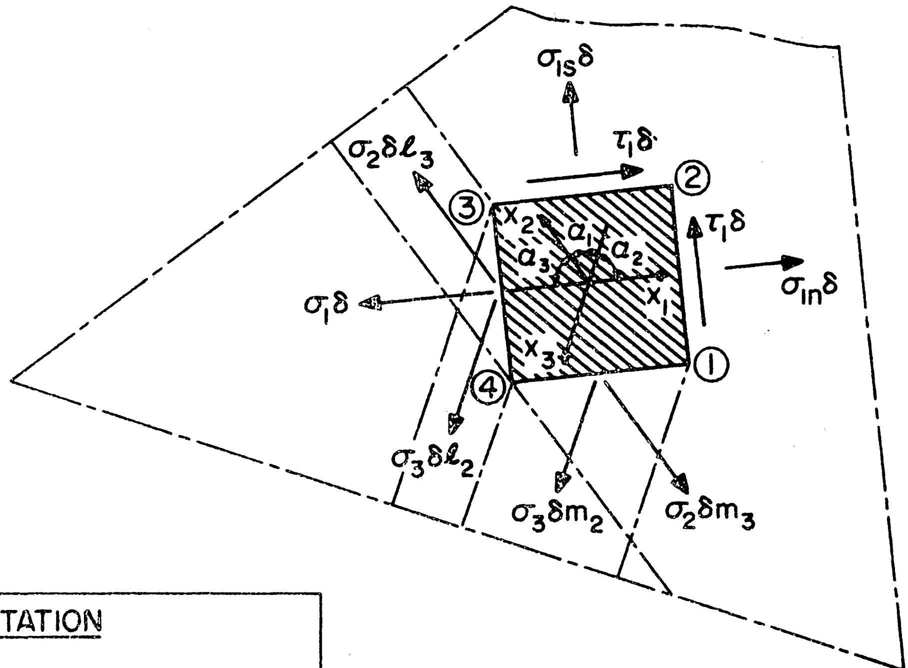


FIGURE 6a

NOTATION

$\ell_i = \cos\alpha_i$, $m_i = \sin\alpha_i$

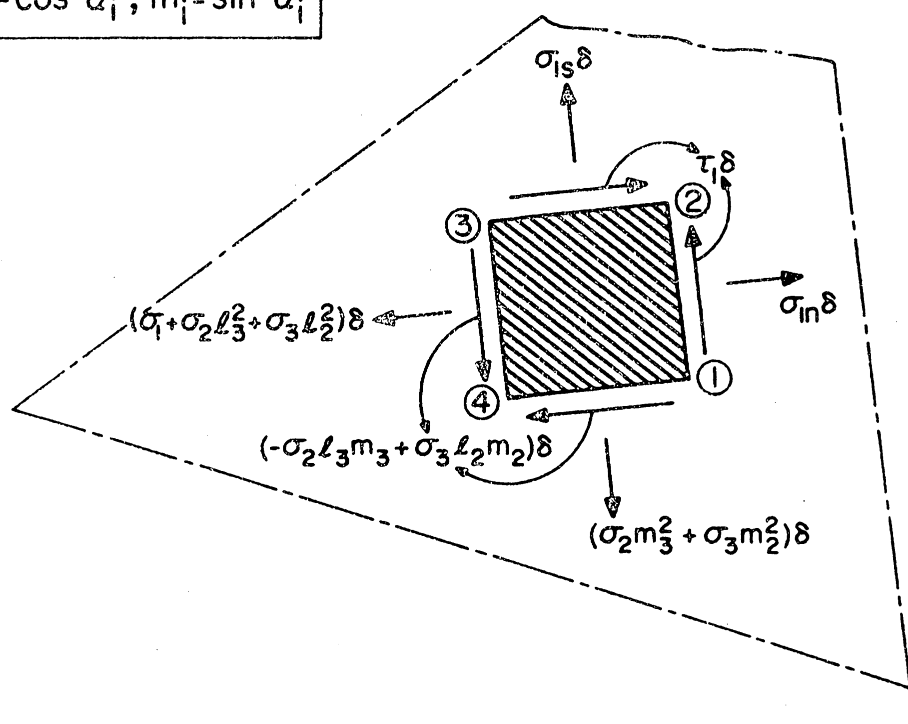


FIGURE 6b

FIGURE 6. RELATIONSHIPS BETWEEN TRIAXIAL STRESSES ($\sigma_1, \sigma_2, \sigma_3$) AND CARTESIAN STRESSES ($\sigma_{1n}, \sigma_{1s}, \tau_1$)

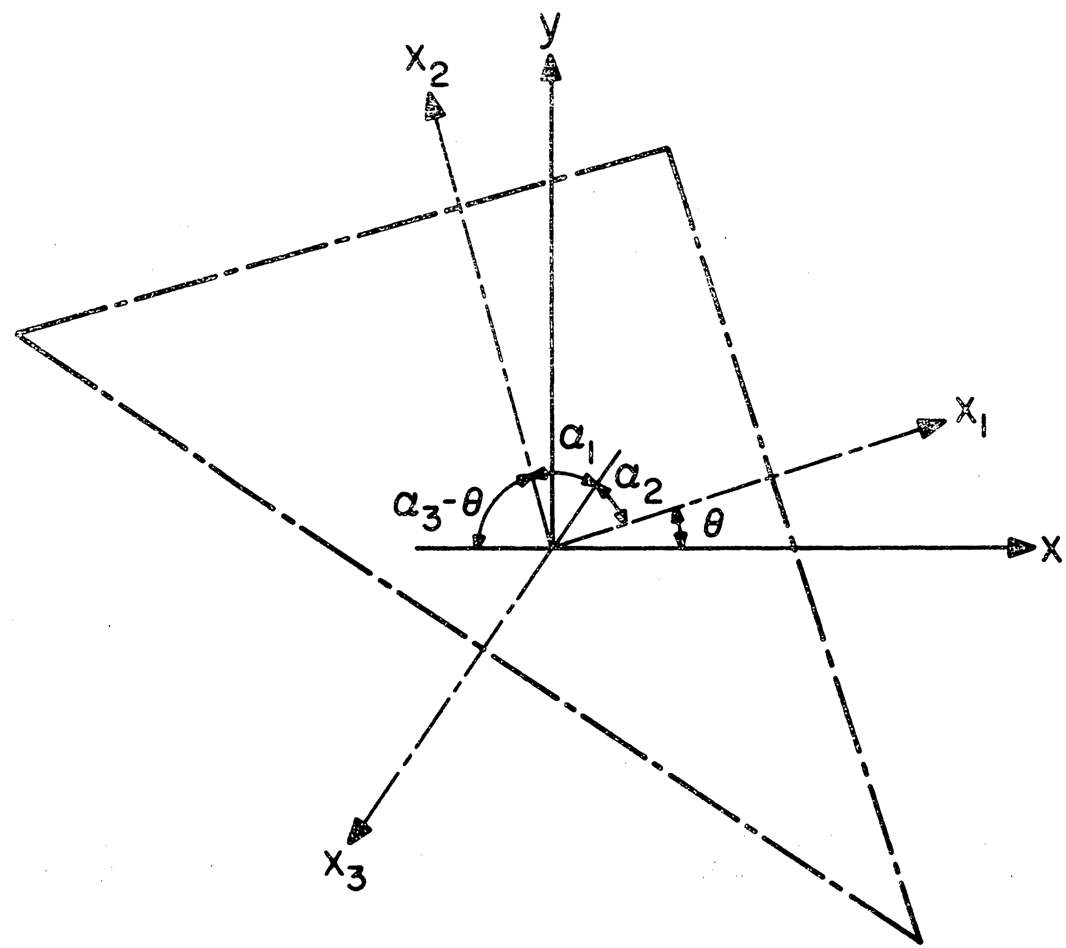


FIGURE 7. RELATIVE ORIENTATION OF TRIAXIAL COORDINATES WITH RESPECT TO A REFERENCE CARTESIAN COORDINATE

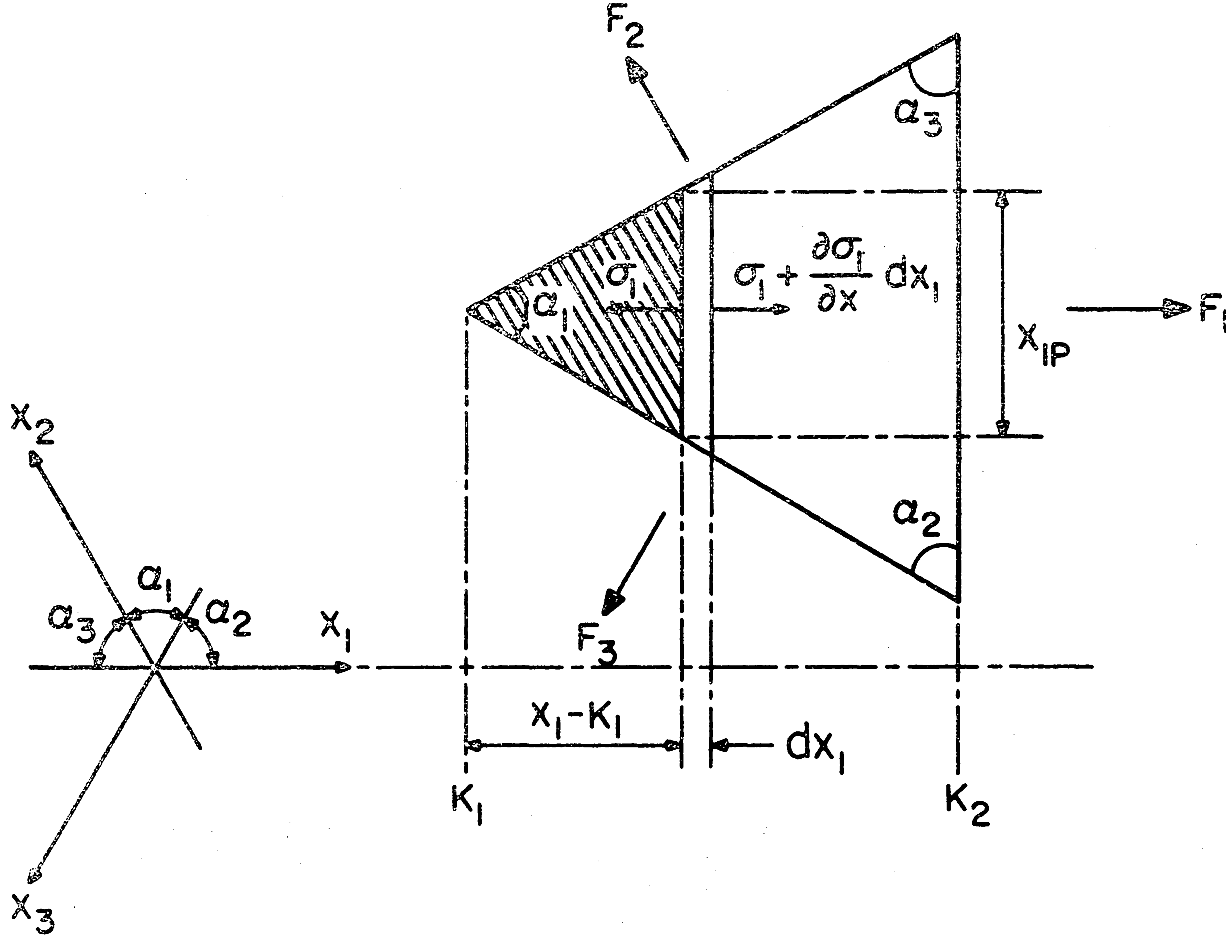


FIGURE 8. EQUILIBRIUM OF AN INFINITESIMAL TRIANGULAR ELEMENT SUBJECTED TO TRIAXIAL STRESS COMPONENTS

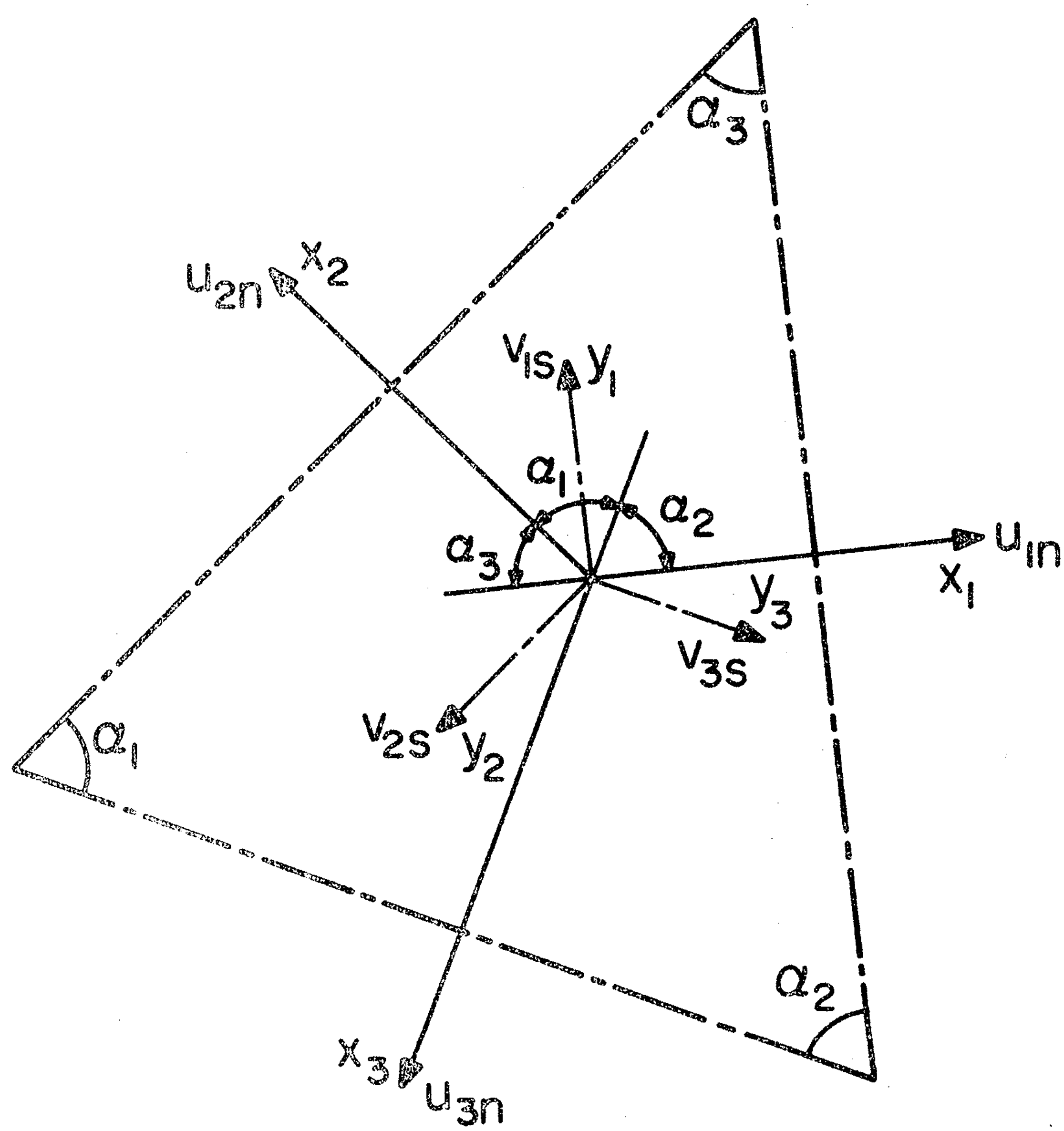


FIGURE 9. CARTESIAN DISPLACEMENTS IN RELATION TO THE TRIANGLE OF REFERENCE

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APPLICATIONS OF TRILINEAR COORDINATES TO

SOME PROBLEMS IN PLANE ELASTICITY

by

Ramanath Williams

(ABSTRACT)

This discourse considers some analytical aspects involved in the application of trilinear coordinates to boundary value problems in plane elasticity. Trilinear coordinates, also known as homogeneous point coordinates, are defined. The concepts of triaxial stress, strain, bending moments and curvature are introduced: utilizing these concepts, the stress-strain relationship, moment-curvature relationship and a few other basic equations of two dimensional elasticity are developed for an isotropic material. All these relationships are presented in matrix form - as an aid to finite element stress analysis. Governing equations corresponding to some two dimensional problems in elasticity are deduced for the trilinear system. An investigation was carried out on the method of integration of a function composed of trilinear variables. A few functional relations between the trilinear variables are also developed.

To illustrate the application of the theory, two examples on simply supported equilateral triangular plate are considered.