ANALYSIS OF A FOLDED - PLATE CONCRETE ROOF CONTINUOUS WITH OVERHANG

by

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I. INTRODUCTION

A. Background

During recent years, folded plate construction in reinforced concrete has found increasing application for roofs of industrial and commercial buildings, auditoriums and hangers, as well as for the sides and bottoms of elevated bunkers. Such construction is particularly well-suited to fairly long spans and possesses some of the attributes of circular thin shell construction with the advantage of somewhat simpler fabrication and forming.

It has been recognized that in folded plate, long span construction, the shape of the plates is a dominant factor. The folded plate roof structure has come into wide usage because of its low cost of construction for long span, high load-carrying capacity, rigidity, and pleasing aesthetic quality. Selection of reinforced concrete for the shell offers a high degree of fire resistivity, ease of molding to the desired alignment and profile, great degree of permanence, and low construction and maintenance costs.

B. Historical Development

The folded plate or hipped plate structure is a prismodial shell formed by a series of adjoining thin plane slabs, rigidly connected along their common edges, and usually closed off at their end supports by integral diaphragms, as shown in Fig. 1.

Along the longitudinal edges, the plates are assumed to be connected by joints at the folded edges, which do not slide and which are considered capable of transferring edge shears between the adjoining plate elements. The uniform load on the plates is transformed into a line load (P_c, in Fig. 1 for example) acting along the common edge. Basically, the load is carried by the

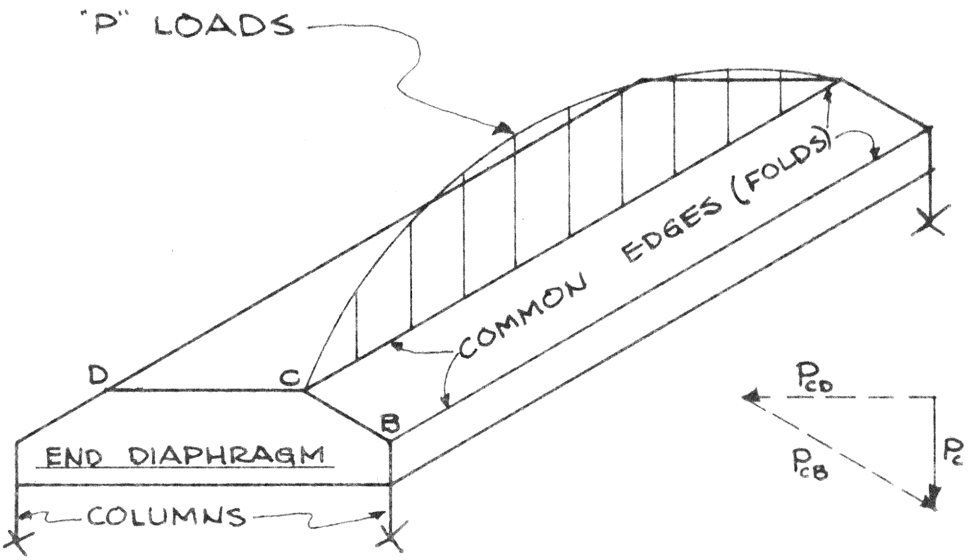


Fig. 1 Typical Simply Supported Folded Plate Structure

structure in two ways: first there is the slab action in the transverse direction, carrying the surface load P_c to the folded edges as shown in Fig. 1 by P_{cb} and P_{cd}, acting in the plane of the two adjoining plates BC and CD, along edge C , and second, there is the beam action in the longitudinal direction, by which the loads P_{cb} and P_{cd} on the folded edges are carried to the end supports. Shear stresses, V, are developed in maintaining equal longitudinal strains along the common edges. The strain condition along each folded edge is used to determine the magnitude and distribution of the edge shear stresses.

Early theories[4] neglected the transverse moments transmitted between adjacent plates due to the ridigity of their common joint.

In a refinement in the folded plate theory, E. Gruber published a paper[7] in 1932 which considered the effect of rigidity of the joints, the connecting moments acting along the common edges of the plates, and the relative displacements between joints. As a first approximation, the folded plate roof was assumed to be hinged along the joints. Then, using this assumed hinged structure as a basic system, Gruber developed his solution in the form of simultaneous differential equations of the fourth order which could be solved by a rapidly converging series. For a folded plate roof of $r + 1$ plates (r being the number of joints) the total number of unknowns is $7r + 2$. For a roof of five plates, there would be 30 unknowns in 30 simultaneous equations. In his solution, Gruber showed that the maximum longitudinal stresses on a cross-section, and the maximum deflections for a roof computed with hinged plates were about twice as great as those computed for the roof with rigidly connected plates. He concluded that the influence of the rigid connections should not be neglected as it had been according to previous practice.

Later, the theory was further developed and expanded in many respects by H. Craemer[3] and Gruber[7]. The European literature on the subject, mostly in German, is fairly extensive. All of the treatments of the theory by German authors were developed from elasticity equations which were mathematically involved.

In January 1947, Winter and Pei published a paper[15] on folded plate roof construction in which the algebraic solution was transformed into a stress distribution method having the advantage of numerical simplicity over other procedures. However, here the same simplifying assumption of neglecting the effect of the relative deflections of the joints was made.

In January 1958, Howard Simpson published[12] a solution of simply supported folded plate roofs featuring an iterative procedure which utilized moment distribution and stress relaxation operations to arrive at plate edge moments and stresses in terms of relative joint displacements and plate rotations.

In September 1958, David Yitzhaki published a book[16] on prismatic and cylindrical shell roof structures which featured the utilization of the method of superposition of particular loadings in the solution of the example problems contained therein.

Yitzhaki, in his treatment of prismatic roofs, developed a table of equations of load, shear, moment, slope, and deflection for simply supported, continuous, and cantilevered beams based on the assumption that "particular loadings similar to an elastic line (produced by the structure) can be used, in view of the close proximity in the form of elastic lines of beams with the same support conditions under the action of different types of load". These equations are based on three basic particular loadings - ordinary loading, uniformly distributed loading, and concentrated loading at mid-span - and cover four types of single-span beams: (1) both ends simply supported; (2) both ends clamped; (3) one end clamped, the other simply supported; (4) one end clamped, the other free.

For each of the four beam conditions described above, Yitzhaki tabulated the ordinates (at intervals of $\Delta X=0.1L$) of the elastic curves of the deflected beams produced by ordinary load, uniform load, and mid-span concentrated load, respectively. Arbitrarily choosing the ordinary load curve as the standard in each of the four support conditions, he found that nowhere did the deviations exceed 2% for uniform load, and 5.1% for mid-span concentrated load. The restrictions governing the analysis were that (1) the particular loadings be similar to the elastic curves of the plates structure and (2) it was assumed that the superposition of several particular loadings was practicable only if they were similarly distributed longitudinally along the plate structure. Yitzhaki did not consider the cross sectional geometry of the folded plates.

Then, in 1960, there appeared a paper[10] by A. L. Parme and John A. Sbarounis of the Portland Cement Association which dealt with the direct solution of folded plate roofs (simply supported). The procedure involved establishing two equations at each fold in terms of unknown transverse moments and longitudinal stresses.

These equations were solved simultaneously to determine the magnitude of moments and stresses at each point.

C. Thesis Objective

All of the available literature as discussed above, with the exception of Yitzhaki[16], deals only with the single-span, simply-supported folded plate structure. Yitzhaki presents a method of solution based on the elastic lines produced by simply supported and continuous beams. No particular attention is given to the geometry of the structure; values of stress and moment are obtained at selected points by applying the ratio of the ordinates of the elastic lines noted above. However, the problem still remains to relate the geometry of the folded plates to a number of points along the entire length of the plate edges, in an attempt to find the true condition of stress at these points, and the contribution of each stress condition to the overall effect.

In this thesis there is presented a suggested rational method of analysis for the designs of cantilevered and continuous reinforced concrete folded-plate structures which should serve as a supplement to the existing literature.

The method used is essentially the method presented by Simpson[12], but extended to apply to the cantilevered continuous beam. The method is illustrated by the solution of a problem in Section X.

Although cantilevered spans and continuous spans have been successfully built, their designs have been individually developed or extrapolated from computations based on the literature for simply-supported structures. At best, the extrapolations are only approximations and it is not always certain that these extrapolations are conservative and safe.

There is a need for a reliable rational method for the design and analysis of cantilevered and continuous folded plate structures. There are at least two locations* where studies are under way in this direction.

* Portland Cement Association Laboratories, Skokie, Illinois; University of Kentucky, Lexington, Kentucky

II. FOLDED PLATE THEORY

A. Description

The folded plate shell is a space structure formed by flat plates, monolithically joined along their common edges (folds). In folded plate roofs, the plates are usually rectangular or triangular. The discussion in this thesis will be limited to rectangular plates. The plates are usually very thin in relation to the free span of the structure (1/100 to 1/200); are generally less than twelve feet wide and up to 60 feet or more in length. The high strength-to-weight ratio of the folded plate structure depends mainly on the transfer of the load into the planes of the plates. A component of the load normal to the plate is carried by the plate acting as a one-way slab in the transverse direction. This component of load is determined by the slope of the roof and the plate width and it generally controls the plate thickness. The bending of the slabs in the longitudinal direction (longitudinal "slab" action) is usually ignored except at the ends, since the plate length-to-width ratio is usually very large.

B. Discussion of Plate Action

Several illustrations of plate action are presented showing the forces acting on a unit transverse strip in the interior of the plate structure.

Figure 2 shows the position of a unit transverse strip. The strip used in the computation of transverse moments and deflections is not shown.

Figure 3 shows the effect of the forces and moments acting on the strip in the transverse direction. The analysis of these forces and moments is made by treating the strip as a continuous beam, one foot wide by "t" inches thick, supported at the fold lines on flexible supports, which is the analysis of transverse "slab" action.

Figure 4 shows the effect of the forces and moments acting on the transverse strip. These forces and moments are analyzed by treating each plate element as a beam "t" inches wide by "h" inches deep, which is an analysis of longitudinal "beam" action.

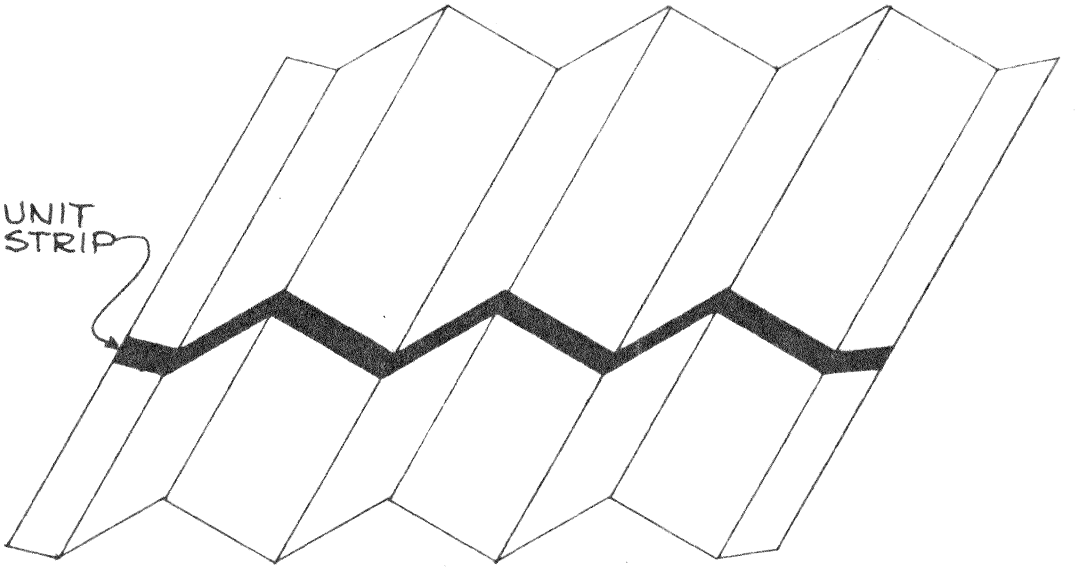


Fig. 2 Illustration of The General Position of The Unit Strip

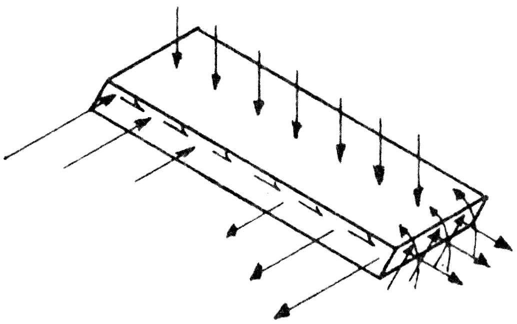


Fig. 3 Forces and Moments Acting on The Transverse Edges of The Unit Strip, Showing "Slab" Action

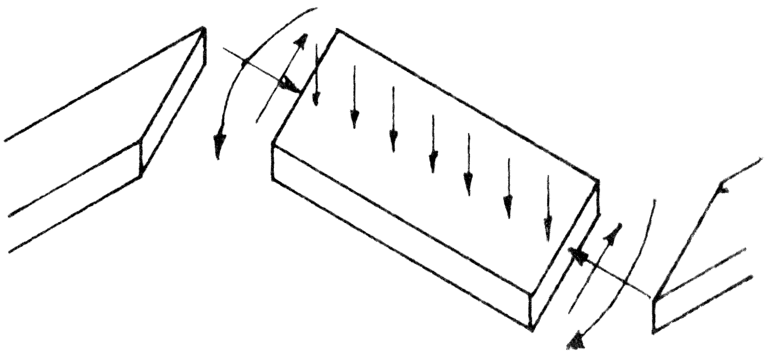


Fig. 4 Forces And Moments Acting on The Transverse Edges of The Unit Strip, Showing "Beam" Action

C. Reinforced Concrete Versus Prestressing

The use of prestressing in folded plates has not been considered in this anlysis, mainly because of a desire to keep the example problem uncomplicated and to concentrate on and validate the flexural theories involved in the cantilever condition. However, the analysis developed here can be used with some modification for prestressed folded plates.

Investigators in the folded plate field have inferred that prestressing of edge beams and/or the plates in a folded plate structure is often advisable. Yitzhaki[16], and Brough and Stephens[1] have concluded that simply reinforced concrete folded plates have larger deflections than prestressed folded plates, which in turn result in increased fold line displacements and increased transverse bending moments.

D. <u>Cracking in Reinforced Concrete; Elastic Versus Plastic Design</u>

In the majority of the theoretical investigations dealing with statically indeterminate systems such as plates and shells, a homogeneous material obeying Hooke's Law has been assumed. However, due to the presence of reinforcing rods and, in the higher stages of loading, the formation of cracks, reinforced concrete cannot even be assumedhomogeneous. (But even with reinforcement, it may have essentially a linear load pattern). Moreover, near failure, neither of the two materials follows Hooke's Law. Herman Craemer, in his treatise[3] on prismatic shells, attempts to bridge the gap between theory and practice in this area. He states:

"First of all, a distinction should be made between safety against cracks and safety against failure. Cracks will form if the tensile stresses occurring in the uncracked system exceed the tensile strength of the concrete. Since the percentage of reinforcing steel is small, the system is very nearly homogeneous; moreover, the behavior of the concrete in the early stages of loading coincides fairly well with the Hooke's Law assumption (in which stress is proportional to strain) and that of the steel does so completely. The safety against cracks, therefore, can be judged with the same accuracy as in ordinary reinforcing concrete design."

"To determine safety against failure, cracks should be taken into account; moreover, to be correct, Hooke's Law should be abandoned in the failure stage. In addition to the redistribution of stresses immediately after cracks have formed, there is another stress redistribution due to yielding of the steel and a certain amount of ductility of the concrete before failure. In a material with ideal plastic properties, this last redistribution is always favorable as has been proven in the (current) theory of plasticity. Therefore, a system design for safe stresses is safe even if elastic deformations are not geometrically compatible, and test results on cross-reinforced concrete slabs have proved that the plasticity theory, in certain cases, may well be applied to reinforced concrete."

III. ASSUMPTIONS

The analysis of the reinforced concrete folded plate structure will be based on the following general assumptions:[12]

1. The material is considered homogeneous, uncracked, and elastic, and has the properties of the concrete.
2. Longitudinal edge joints are fully monolithic and continuous; there is no relative rotation or translation of two adjoining plates at their common boundary.
3. The principles of superposition hold whereby the structure may be analyzed separately for the effects of its reactions and various external loadings, and the results added. This is to say that to the center span is applied the roof and corrective loads. Next, a corrective load is applied to the cantilever, first assuming that the cantilevered plates are separated and free to deflect. Other corrective loads are then applied to the center span. Roof and corrective loads are then applied to the cantilever (with plates free to deflect also), and corrective loads are subsequently applied to the center span and to the cantilever, respectively (see Figs. 20 and 21). The results of these many analyses may be added algebraically. Moreover, in any one analysis, the results of the effects of "slab" action may be added to the results of the effects of "beam" action.
4. The structure is analyzed by first considering a transverse

strip subjected to "slab" action under components of normal load, and then considering the plates as longitudinal deep beams subjected to "beam" action under load components acting in the plane of the plates. Correlation is made between these actions by providing compatibility of deflections.

5. Individual plates possess negligible torsional resistance. Torsional stresses due to twisting of the plates is ignored.

If each plate is relatively long compared to its width (length-to-width ratio of three or more), the following additional assumptions can be made:

6. Longitudinal strain due to longitudinal beam action varies linearly across the width of each plate; plane sections remain plane after bending. The rate of change of strain with respect to width ordinarily differs from plate to plate however, from which it can be inferred that there is some relative displacement of the joints of a cross section.
7. Longitudinal slab action can be neglected. Loads applied to the plate surfaces are carried by slab bending to the longitudinal edges only, as in one-way slab system (known as "slab" action).
8. Because of the continuous nature of the structure, it is assumed that the elastic line produced by the deflected folded plate structure is similar to that of an ordinary

beam as discussed by Yitzhaki[16].

Slide rule accuracy is maintained when computing moments and stresses; figures are rounded to the nearest unit. In arriving at beam-deflection relationships, the coefficients K_2 and K_3 are rounded to thousandths or ten-thousandths.

IV. LIMITATIONS

The limitations governing the theory used in the solution of the example problem are as follows:

1. All plates are of rectangular shape.
2. The length of each plate is at least six times its width, making it possible to treat the plates as one-way slabs, to assume linear distribution of bending stresses, and to neglect the effects of torsional rigidity. This treatment was justified by the experimental results of tests[5] conducted by Gaafar.
3. Each plate is of uniform thickness.
4. The roof is analyzed as a homogeneous elastic structure and the moments and stresses obtained are used as the basis for determining the amount of reinforcing steel required.
5. All loads on the structure act symmetrically about the transverse axis. The structure itself is symmetrical about both axes. Where convenient, advantage is taken of this symmetry by using only one-half of the structure for computations.

V. INDEX OF NOTATIONS AND SIGN CONVENTION

a - length of the cantilever

A_n - cross-sectional area of plate "n"

C - distance from centroidal axis to extreme fiber

e - subscript referring to the extreme end of the cantilever

F.E.M. - fixed end moments (foot-pounds except where otherwise noted)

f'_c - allowable compressive strength developed by a 28-day concrete cylinder

f_s - tensile strength of reinforcing steel

h_n - transverse dimension, or width, or depth of plate "n"

I_n - moment of inertia of plate "n" about the centroidal X-X axis of the section normal to the plane of the plate

k_n - factors used in the determination of plate edge moments and logitudinal plate stresses

M - logitudinal or transverse moment (foot-pounds except where otherwise noted)

mp - subscript referring to the cantilever midpoint (a/2)

N - resultant of the shear forces acting along longitudinal edge of a plate

r_n - radius of gyration of plate "n" about the cross-sectional X-X axis of the section normal to the plane of the plate

s - subscript referring to the supports (diaphragms)

v - shear force per unit of length

w - total roof loading in pounds per foot of length

w_1, w_2 - corrective loading in pounds per foot of length

δ_n - final displacement of plate "n" in its own plane

α_{mn} - angle between plates "m" and "n"

ψ_n - angular rotation of plate "n" in radians

Δ_n - deflection of one end of plate "n" relative to the other (in the transverse direction)

σ_T, σ_B - longitudinal extreme fiber stress in the plates at the fold lines

τ - shearing stress acting longitudinally along the plate edges

Sign Convention:

- Stresses are considered positive when they are compressive.
- Moments are considered positive when they tend to rotate a joint (or fold line) clockwise.
- At center span, downward deflections are positive (Fig. 16). On the cantilever, upward deflections are positive (Step 12).
- The angular rotation (ψ) of a plate is considered positive when clockwise.

VI. DESCRIPTION OF THE EXAMPLE PROBLEM

A folded plate roof structure, as shown in Fig. 5, is to be analyzed under its own dead weight. Cross section dimensions are shown in Fig. 6.

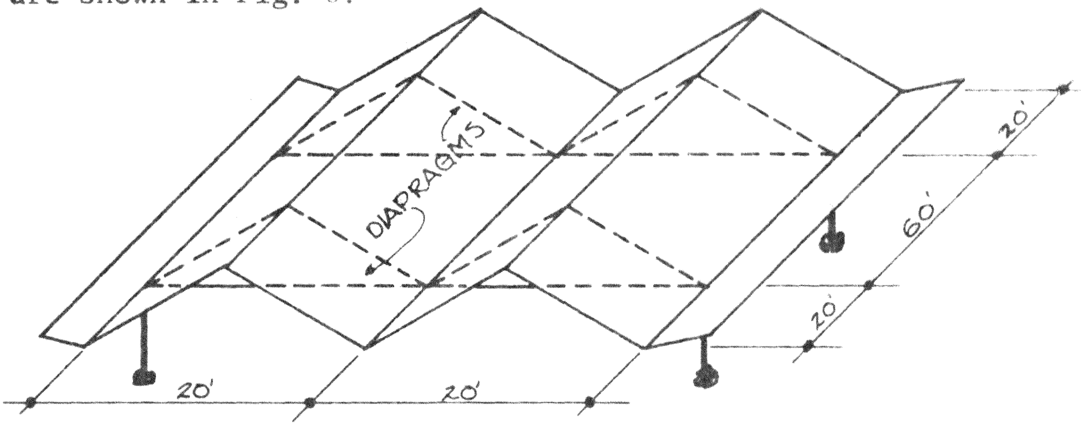


Fig. 5 Isometric View of The Structure

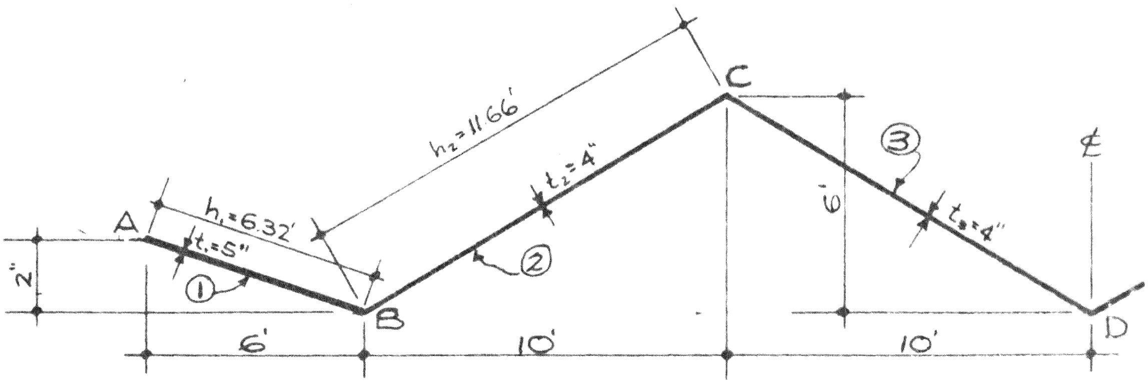


Fig. 6 End View of Half Section

Additional data needed for computations is as follows:

$$I_1 = \frac{bh^3}{12} = \frac{(5)(6.32 \times 12)^3}{12} = 183{,}000 \text{ IN}^4$$

$$I_2, I_3 = \frac{(4)(11.66 \times 12)^3}{12} = 916{,}000 \text{ IN}^4$$

$$A_1 = (5)(6.32 \times 12) = 380 \text{ IN}^2 \qquad \alpha_{12} = 49^\circ$$

$$A_2, A_3 = (4)(11.66 \times 12) = 560 \text{ IN}^2 \qquad \alpha_{23} = 62^\circ$$

VII. RELATIONSHIP BETWEEN PLATE DEFLECTIONS AND ROTATIONS

A. Rotation and Translation of Plate 2

The geometric relationship between the interdependent plate deflections and rotations must be established, using a transverse strip. Displacements of the plates in their planes (δ) are exaggerated.

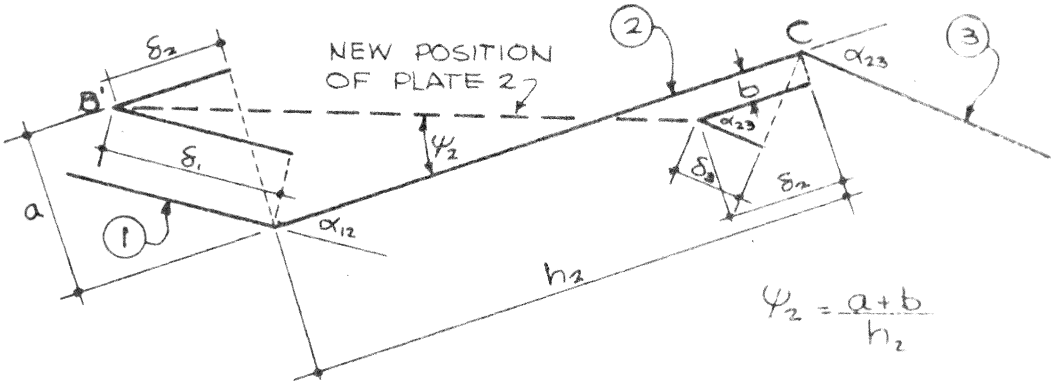


Fig. 7 Rotation And Translation of Plate 2

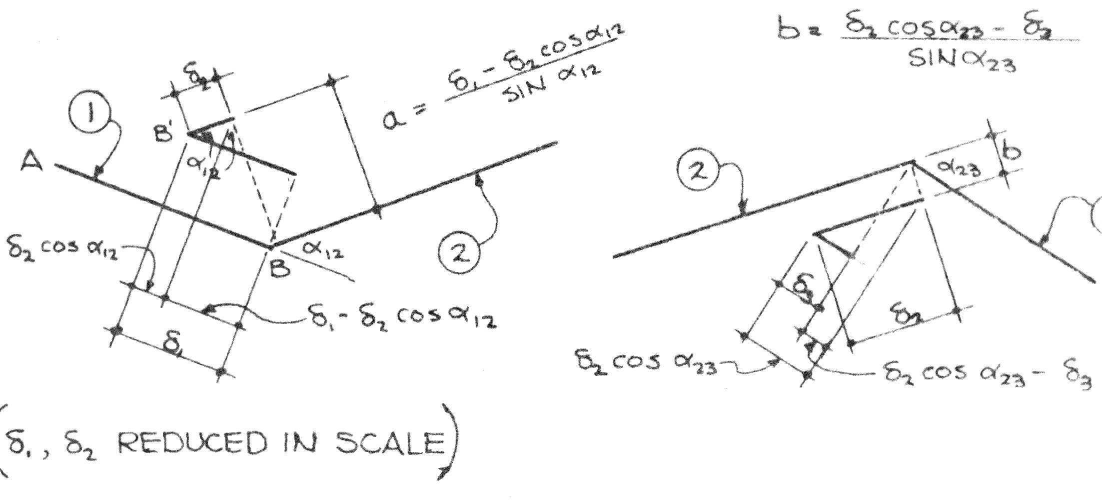


Fig. 8 Details of Joints B and C

From the preceding figure, we can say that

$$\psi_2 = \frac{a+b}{h_2} = \frac{1}{h_2}\left[\frac{\delta_1 - \delta_2\cos\alpha_{12}}{\sin\alpha_{12}} + \frac{\delta_2\cos\alpha_{23} - \delta_3}{\sin\alpha_{23}}\right] \quad \text{------ (1)}$$

$$\psi_2 = \frac{1}{h_2}\left[\frac{\delta_1}{\sin\alpha_{12}} - \delta_2\left(\cot\alpha_{12} - \cot\alpha_{23}\right) - \frac{\delta_3}{\sin\alpha_{23}}\right] \quad \text{------ (2)}$$

$$\psi_2 = -0.0971\,\delta_3 - 0.0280\,\delta_2 + 0.1130\,\delta_1 \quad \text{------ (3)}$$

And by similar construction for plate 3, it can be shown that

$$\psi_3 = \frac{1}{h_3}\left[\frac{\delta_4 - \delta_2}{\sin\alpha_{23}}\right] = 0.0972(\delta_4 - \delta_2) \quad \text{------ (4)}$$

The values of ψ_2 and ψ_3 are not known, and the "δ" values are not the final values. They could be the final values if multiplied by a factor of k_n.

B. Finite Rotation - One Plate At A Time

In order to obtain the factors of "k_n", it is necessary to induce finite rotations in the plates (actually a sidesway type of correction) which causes a known initial Fixed End Moment (FEM). One plate at a time is treated, working toward the centerline of the structure.

(1) Arbitrary rotation of plate 2 (see Fig. 9):

Assume a FEM = $+\frac{3EI_2}{h_2}(\psi_{20})$, to give a rotation of ψ_{20}

Take $\frac{EI_2}{h_2}(\psi_{20}) = 1$ ft-kip ------ (5)

The final moments and reactions are found in Fig. 10.

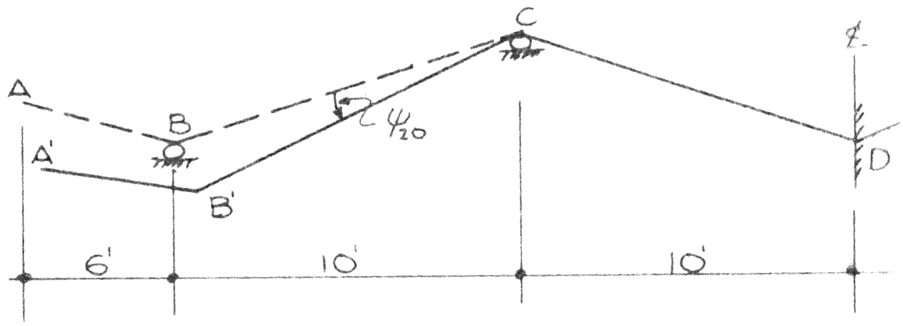


Fig. 9 Arbitrary Rotation of Plate 2

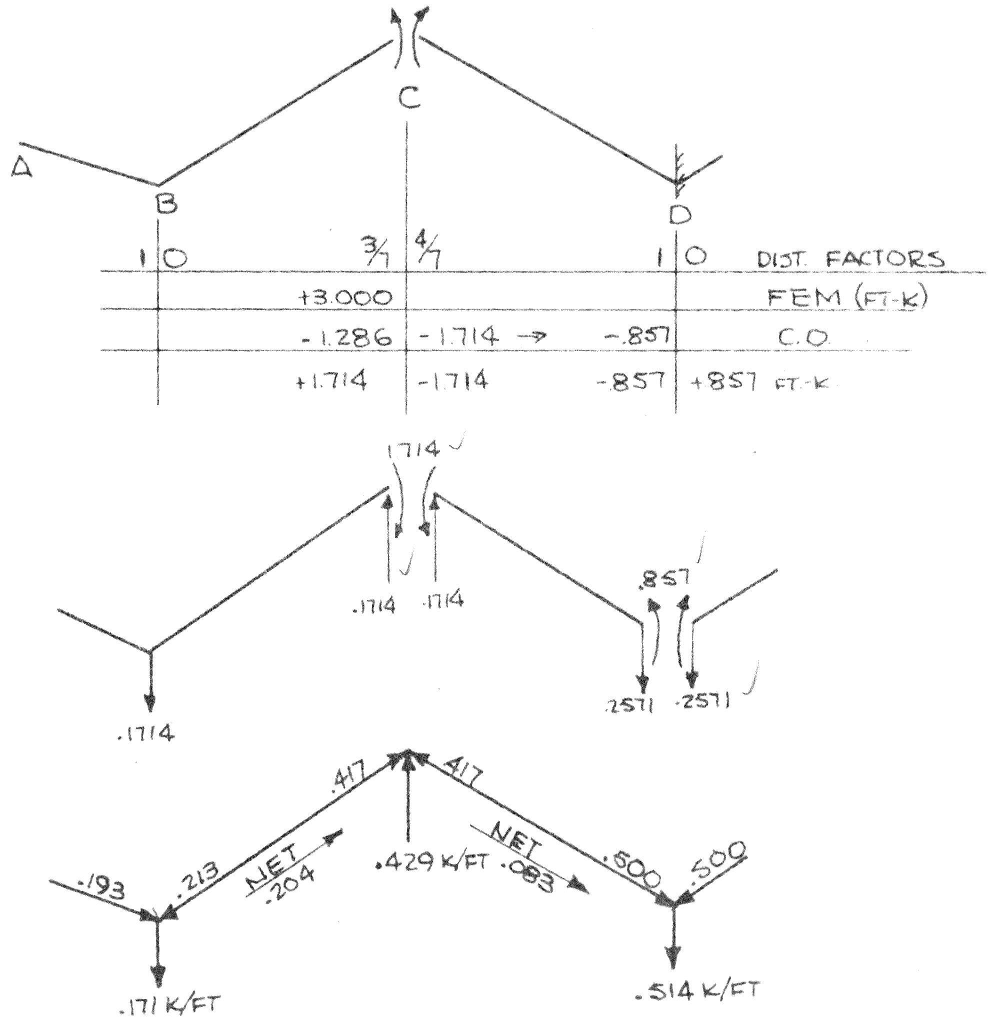


Fig. 10 Slab Action And Plate Loads Due to Arbitrary Rotation of Plate 2

(2) Arbitrary Rotation of Plate 3 (see Fig. 11):

Assume a FEM = $+\frac{6EI_3}{h_3}(\psi_{30})$, to give a rotation of ψ_{30}

Take $\frac{EI_3}{h_3}(\psi_{30}) = 1$ ft-kip .... (6)

And the final moments and reactions are found in Fig. 12.

Hence,

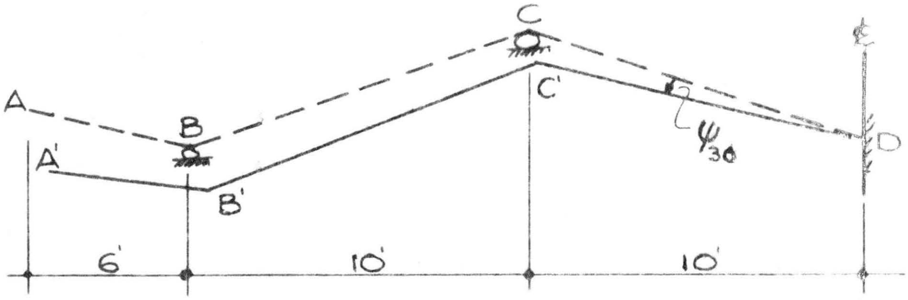
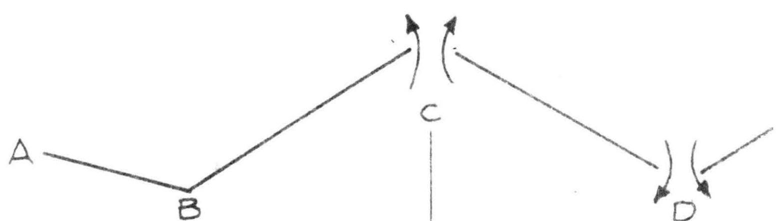
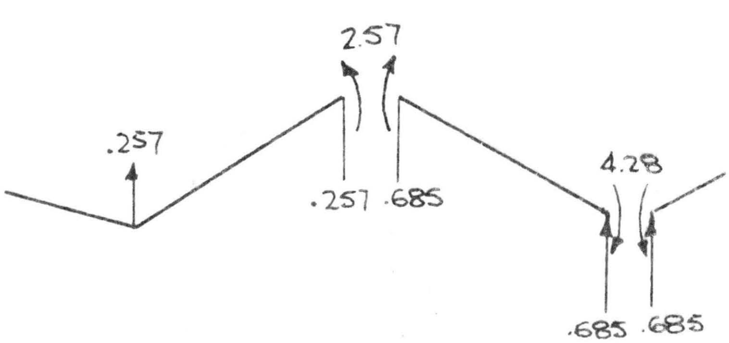


Fig. 11 Arbitrary Rotation of Plate 3



B		C		D		
1	0	3/7	4/7	1	0	DIST. FACTOR
			+6.00	+6.00		FEM (FT-K)
		-2.57	-3.43 →	-1.72		C.O
		-2.57	+2.57	+4.28	-4.28 FT-K	



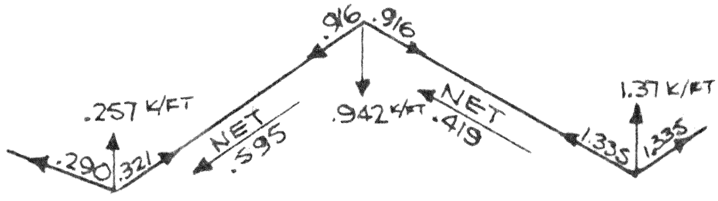


Fig. 12 Slab Action And Plate Loads Due to Arbitrary Rotation of Plate 3

Now, by definition,

$$\psi_2 = k_2 \psi_{20} \qquad (7)$$

$$\psi_3 = k_3 \psi_{30} \qquad (8)$$

And from Eqs. (5) and (6),

$$\psi_{20} = \psi_{30} = \frac{h_2}{I_2 E} = \frac{11.66}{(1/12)(1)(1/3)^3(1)(144)} = 26.2 \qquad (9)$$

where E = 1 kip/sq. in. (assumed).

The net transverse loads obtained in Figs. 10 and 12, from the arbitrary rotation of plates 2 and 3, respectively, are used later in the solution of the example problem. Those plate loads, called corrective loads, are assumed to vary sinusoidally and/or cosinusoidally, as the case may be, over the length of plate section under consideration.

VIII. DERIVATION OF FORMULAE FOR STRESS RELAXATION PROCEDURE

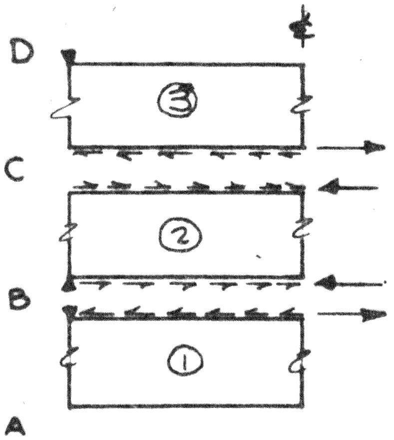
In order to maintain continuity at the plate edges, the longitudinal stresses at each fold, obtained from the "beam" action computations, must be modified and made compatible with strains. The general equation, derived according to Simpson[12], is as follows:

From the column formula, the stress at any point in a cross-section is given by

$$\sigma = \frac{N}{A} \pm \frac{Mc}{I} = \frac{r^2 N}{r^2 A} \pm \frac{N \frac{h}{2}\frac{h}{2}}{I}$$

From which

$$\sigma = \frac{N}{I}\left(r^2 \pm \frac{h^2}{4}\right) \text{ --------------- (10)}$$



$$N_C = -\frac{\sigma_{2B} A_2}{2} \text{ --------------- (11)}$$

$$N_B = +\frac{\sigma_{2B} A_2}{4} = +\frac{\sigma_{1B} A_1}{4} \text{ --------- (12)}$$

$$\sigma_{2B} = \sigma_{1B}\left(\frac{A_1}{A_2}\right); \quad \sigma_{1B} + \sigma_{2B} = \text{TOTAL } \Delta\sigma \text{ ---- (13)}$$

Therefore, the Stress Distribution Factors are:

$$K_1 = \frac{A_2}{A_1 + A_2} \qquad K_2 = \frac{A_1}{A_1 + A_2}$$

Fig. 13 Equations For Stress Distribution

IX. FORMULATION OF MOMENT AND DEFLECTION EQUATIONS

A. Plate Loads Expressed Sinusoidally

The principle problem associated with the analysis of folded plate structures is that of making the displacement computed from longitudinal behavior of the plates compatible at the folds. In many cases and for simply supported plate structures, reasonable values can be obtained by satisfying the requirements of compatible displacements at mid-span only.

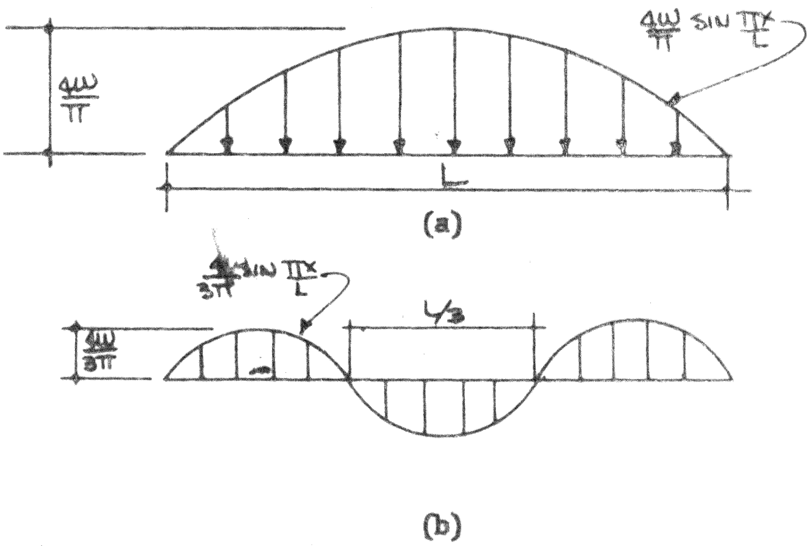
The uniform load can be expressed as the sum of partial sinusoidal loads, with each load corresponding to a term in a Fourier Series, given by the expression

$$w = \frac{4w}{\pi}\left(\sin\frac{x}{L} + \frac{1}{3}\sin\frac{3x}{L} + \frac{1}{5}\sin\frac{5x}{L} + \cdots + \cdots\right) \qquad (14)$$

or,

$$w = \sum_{n=1,3,5\ldots}^{\infty} \frac{4w}{n\pi}\left(\sin\frac{n\pi x}{L}\right) \qquad (15)$$

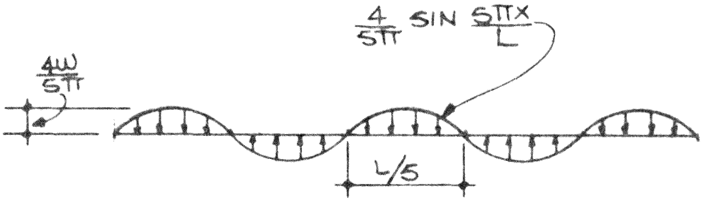
The loads are shown graphically in Fig. 14.



1st term

2nd term

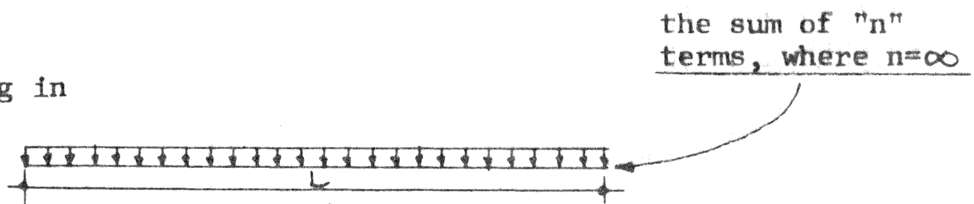
Fig. 14 (Continued)



3rd term

(c)

...resulting in



(d)

Fig. 14 Distribution of Loads as Given By The Fourier Services Expansion

It can be shown that the sum of the terms converges rather rapidly to equal a uniform load; the sum of three terms normally is sufficient. Usually, illustrated problems are solved using only the first term of the series[10,12] for obtaining a good approximation in the calculation of deflections and longitudinal stresses in the plates.

To represent the uniform load acting on the plates, the following pattern of loading is used (Fig. 15).

In solving a folded plate structure by an iterative procedure, the first computations are made for "slab" action when the fold lines are assumed to be fixed against transverse

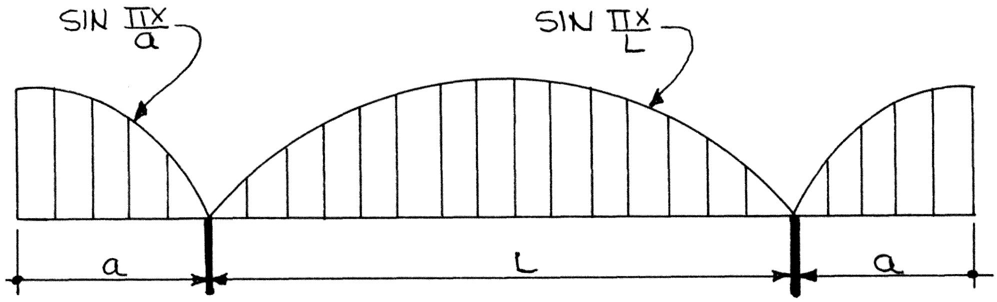


Fig. 15 Pattern of Sinusoidal Loading

rotation, hereinafter termed as CASE I(ψ_o=0), and the additional computations involve corrections due to displacements by an iterative procedure. For this thesis, the load is expressed as a sine series (Fig. 15) and only the first term is used. Justification for this is shown in Fig. 55 where values of plate stresses have been calculated using two terms (n=1 and n=3), and are compared.

In addition to the effects of CASE I loads, there are additional stresses in the structure caused when each plate is given a finite rotation in order to satisfy the requirement of displacement compatibility at the fold lines. In the case of very wide systems having many folded plates, the outer four or five plates will usually not substantially affect the interior plates, and may be treated independently of the interior plates. In the case of the example problem under consideration, each plate is given a finite rotation as developed in Section VII, and is designated as "CASE II(ψ_2)" (for plate 2), and "CASE III(ψ_3)" (for plate 3). In wider structures, this may be continued as many times as necessary to include all plates.

B. Development of Moments in Terms of Plate Edge Stresses

In order to obtain the constants k_2 and k_3 in Eqs. (7) and (8), it is first necessary to calculate beam deflections (Δ) in terms of the fold line stresses induced from the three cases of loading mentioned previously. The deflection formula, for substitution into subsequent equations, is now derived, (see Fig. 16).

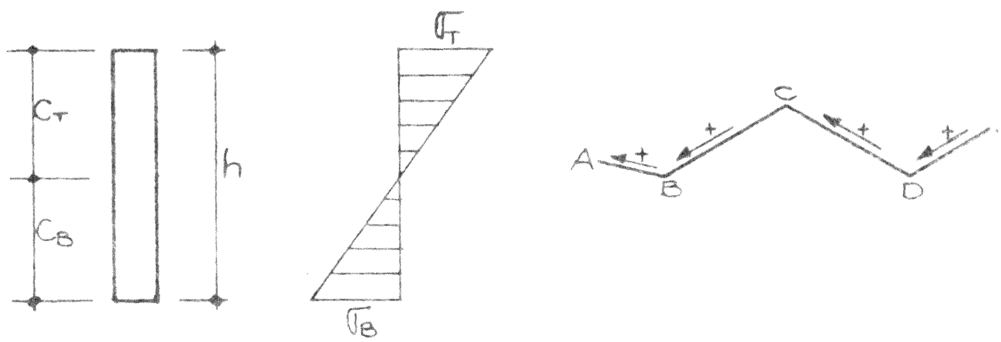


Fig. 16 Typical Plate Cross-section And Assumed Direction of Positive Deflections at The Centerline

From Strength of Materials,

$$\sigma = \frac{Mc}{I} \qquad \text{and} \qquad \frac{M}{EI} = \frac{\sigma}{Ec} = \frac{\sigma_T}{Ec_T} = \frac{\sigma_B}{Ec_B}$$

$$\therefore \quad M = \frac{\sigma_T - \sigma_B}{Eh} \qquad \text{(16)}$$

C. Formulation Of Corrective Loading Applied To The Cantilever

A particularly difficult problem associated with the solution of a continuous folded plate roof by the procedure outlined herein is that of applying corrective loads to the cantilevered spans when only the center span is loaded. Since the plates in the cantilevered span are first assumed to be separated (Assumption 3, page 13), they will deflect without bending as rotation occurs over the support, due to the load on the center span (Fig. 18-a). Arbitrary rotations must be applied to each plate in order to produce corrective loads sufficient to rejoin the plates at their fold lines.

An ideal corrective load, governed by the elastic lines of the deflected structure (discussion on pages 4 and 5), could be applied as a combination of bending load and shearing load, and would provide for rejoining the plates at all points along the separated edge. But, since corrective loads will cause bending, compatibility can be provided only at selected points along the edge. In this illustrative problem, an approximate solution will be obtained by selecting a form of corrective loading to provide compatibility at the center and ends of the cantilever span (Fig. 18-b).

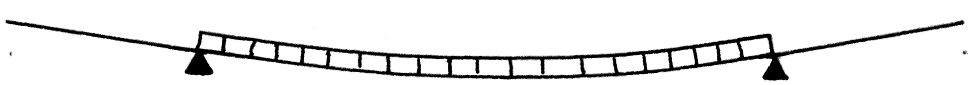


Fig. 18-a Cantilever Deflection Due to Center Span Loading

Fig. 18-b Corrective Loading Scheme on The Cantilever

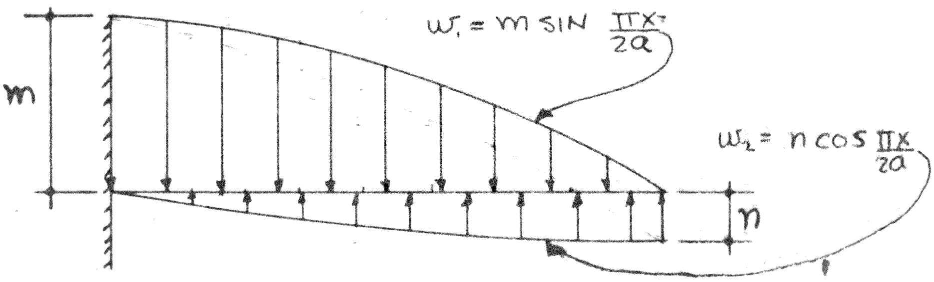


Fig. 19 Corrective Cantilever Loading to Provide Free End Deflection Equal to Twice the Midpoint Deflection

The corrective load shown in Fig. 19 consists of a combination of half-sine and half-cosine loading, and provides for a deflection at the free end equal to twice that at the midpoint when the other end of the cantilever is fixed.

The general equation for the loaded cantilever in Fig. 19 is:

$$EI\frac{d^4y}{dx^4} = -m \sin\frac{\pi x}{2a} + n \cos\frac{\pi x}{2a} \qquad (17)$$

Integrating Eq. (17) and solving for the constants,

$$EIy = -m\left(\frac{2a}{\pi}\right)^4\left[\sin\frac{\pi x}{2a} + \frac{\pi^3x^3}{48a^3} - \frac{\pi^3 x}{16a} - 1 + \frac{\pi^3}{24}\right]$$
$$+ n\left(\frac{2a}{\pi}\right)^4\left[\cos\frac{\pi x}{2a} + \frac{\pi^2x^2}{8a^2} + \frac{\pi x}{2a} - \frac{\pi^2 x}{4a} + \frac{\pi^2}{8} - \frac{\pi}{2}\right] \qquad (18)$$

At $x = 0$ $\quad EIy = \left(\frac{2a}{\pi}\right)^4 \left[-0.29\,m + 0.66\,n\right]$ _ _ _ _ _ _ _ _(19)

At $x = \frac{a}{2}$ $\quad EIy = \left(\frac{2a}{\pi}\right)^4 \left[-0.11\,m + 0.23\,n\right]$ _ _ _ _ _ _ _ _(20)

But, the correction at x=0 must be twice as large as the correction at $x=\frac{a}{2}$.

Therefore, $\left(-0.29\,m + 0.66\,n\right) = 2\left(-0.11\,m + 0.23\,n\right)$

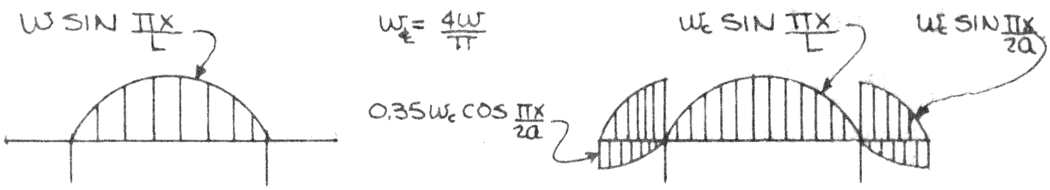
And $n = 0.35\,m$ _ _ _ _ _ _ _ _ _ _(21)

Thus, the cantilevered plate is corrected to meet at the support at the midpoint, and at the extreme end, if subjected to corrective sine and cosine loads acting in opposite directions and proportioned as shown by Eq. (21).

D. Derivation of Working Formulae.

It now remains to determine the equation of deflection of the plates in terms of the plate edge stresses using Eq. (16). To do this it will be necessary to set forth the sequences involved in the analysis.

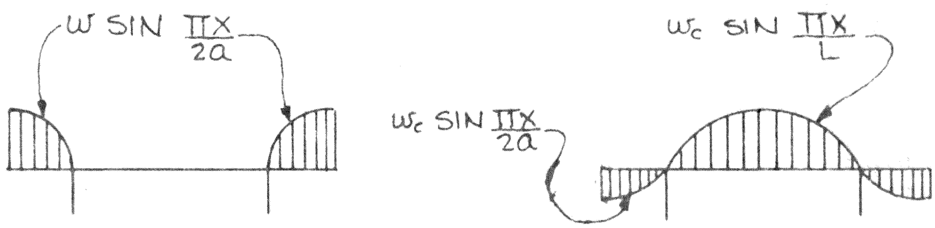
The analysis of the illustrative problem in Section X consists of two phases. Phase I (Steps 1 through 23) encompasses the effect of the uniform roof load acting on the center span of the structure, and the subsequent effect of this load and corrective loads on the cantilevered span and center section. Fig. 20(a) shows the application of the first term of the equivalent Fourier Series. For a previous discussion on the method of loading, refer to Assumption 3, page 13.



(a) Roof Loading (b) Corrective Loadings

Fig. 20 Roof and Corrective Loading Scheme-Phase 1

Phase 2 (Steps 24 through 33) considers the roof load (as the first term of the equivalent Fourier Series) on the cantilevered span, and the subsequent effect of this load and corrective loads on the cantilever and on the center section, as shown in Fig. 21.



(a) Roof Loading (b) Corrective Loadings

Fig. 21 Roof And Corrective Loading Scheme-Phase 2

Using the principal of superposition, the results from each of the phases described above are added algebraically to obtain the values of plate transverse moment and longitudinal stress at the centerline, supports, and cantilever ends.

For convenience in following the analysis of the problem, equations pertinent to the plate stresses and deflections are now derived and referenced to the steps in the illustrative problem in Section X:

From Eq. (16), $\frac{M}{EI} = \frac{\sigma_T - \sigma_B}{Eh}$

From Step 5, $\Delta_{\mathfrak{L}} = \frac{5}{384}\frac{wL^2}{EI}$; $M_{\mathfrak{L}} = \frac{wL^2}{8}$

And combined with Eq. (16) gives $\Delta_{\mathfrak{L}} = \frac{\sigma_T - \sigma_B}{Eh} \times \frac{5L^2}{48}$ ------ (22)

For Steps 6 and 8, for the sine load,

$\Delta_{\mathfrak{L}} = \frac{W_c L^4}{\pi^2 EI}$; $M_{\mathfrak{L}} = \frac{W_c L^2}{\pi^2}$

And combined with Eq. (16) gives $M_{\mathfrak{L}} = \frac{W_c L^2}{\pi^2}$ ------ (23)

For Steps 7 and 9, $\Delta_{\mathfrak{L}} = \frac{\sigma_T - \sigma_B}{Eh} \times \frac{L^2}{\pi^2}$ ---------- (24)

For Step 12, the cantilever deflection due to the center span uniform load is given by

$$\Delta = \frac{M_{\mathfrak{L}} L a}{3EI} \quad \text{----------} (25)$$

For Steps 13 and 15, for the corrective sine and cosine loads on the cantilever , integrating Eq. (17) and combining with Eq. (16) gives

$$M_{MP} = W_c \left(\frac{2a}{\pi}\right)^2 (0.0245) \quad \text{-------} (26)$$

For Steps 14 and 16, also for the corrective sine and cosine loads on the cantilever , integrating Eq. (17) and combining with Eq. (16) gives

$$\Delta_{mp} = \frac{\sigma_T - \sigma_B}{Eh}\left(\frac{2a}{\pi}\right)^2 (1.204) \qquad (27)$$

For Step 19, for corrective loads on the center span; at x=a=20', simplifying Eq. (18) gives

$$M = w_{net}\left(\frac{2a}{\pi}\right)^2 (0.22) \qquad (28)$$

For Step 20, the moment at the support is constant over the center span, and the centerline deflection is

$$\Delta_{℄} = \frac{w_c L^4}{384EI}$$

When combined with Eq. (16) gives $$\Delta_{℄} = \frac{\sigma_T - \sigma_B}{Eh} \times \frac{L^2}{3.81} \qquad (29)$$

For Step 25, for actual loads on the cantilever,

$$\Delta_x = \frac{w}{24EI}\left(\frac{a^4}{16} - 2a^4 + 3a^4\right) = \frac{17}{16}\frac{a^4 w}{EI} \; ; \; M_{MP} = \frac{wa^2}{4} = \frac{\sigma I}{h}$$

When combined with Eq. (16) gives

$$\Delta_{mp} = \frac{\sigma_T - \sigma_B}{Eh} \times \frac{17}{48} a^2 \qquad (30)$$

For Step 30, for the corrective loads applied the center span

$$M_S = w_c \frac{a^2}{2} = w_c \frac{L^2}{18} = \frac{\sigma I}{h} \; ; \; \Delta = \frac{wL^4}{144EI}$$

When combined with Eq. (16) gives $$\Delta_{℄} = \frac{\sigma_T - \sigma_B}{Eh} \times \frac{L^2}{8} \qquad (31)$$

X. SOLUTION OF THE ILLUSTRATIVE PROBLEM

Phase 1. Apply actual loads to the center span, and apply subsequent corrective loads to the center and cantilever spans.

Step 1. Support the fold lines and consider a transverse strip, as in a one-way slab (see Fig. 2). Apply loads and calculate moments and shears on the transverse strip, assuming continuity and no relative edge movements.

A	B		C		D	FOLD LINE
	0	1	3/7	4/7	1/2	D.F.
	-1194	+486	-486	+486	-486	FEM
		708 →	354			
			-152	-202 →	-101	
	-1194	+1194	-284	+284	-587	FINAL MOMENTS
	↑398	383↑	↑201	262↑	↑322	SHEARS (LBS)

Fig. 22 Calculation of Forces on A One Foot Transverse Strip

Step 2. For the transverse strip, convert computed edge shears due to roof load into dummy reactions in the directions of adjacent plates, to determine plate loads.

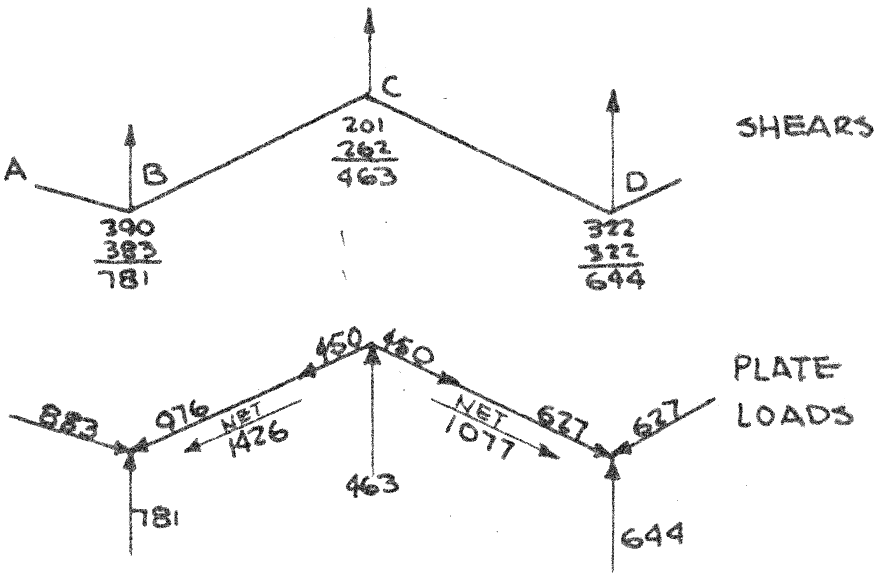


Fig. 23 Edge Shears And Plate Loads

Step 3. Using the plate loads, calculate the bending moments and edge stresses at the center of the center span.

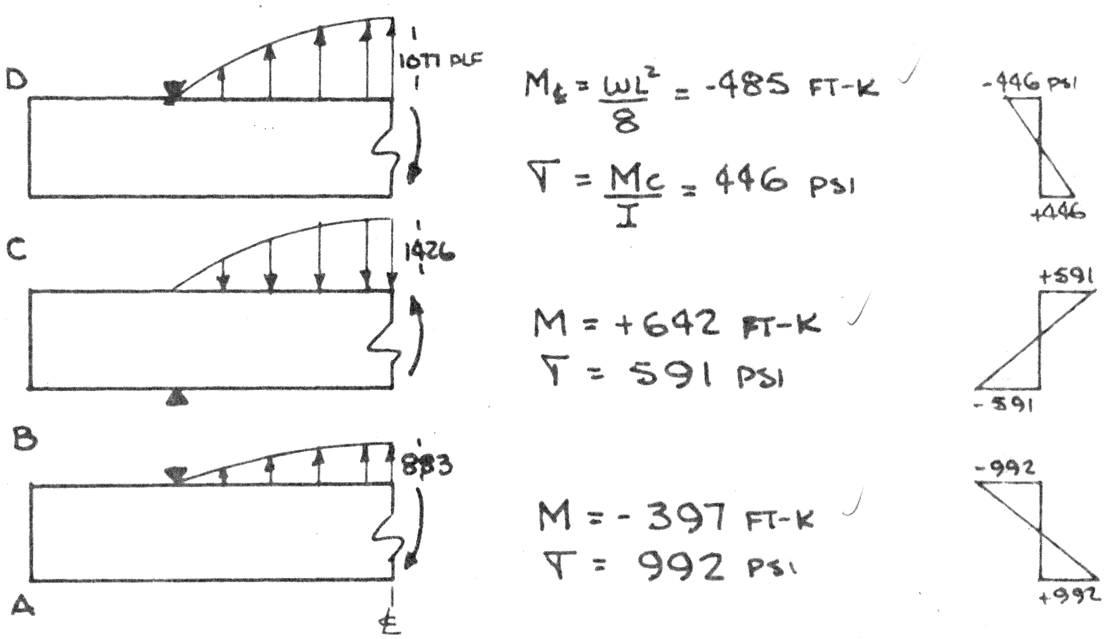
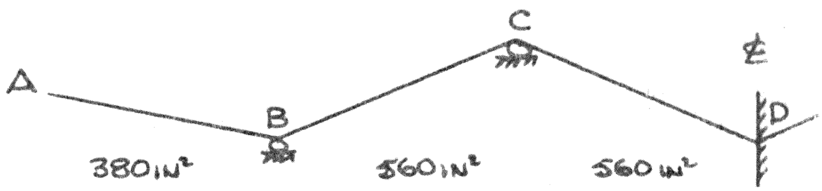


Fig. 24 Bending Moments And Edge Stresses At The Centerline

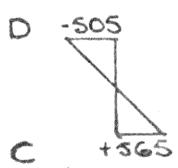
Step 4. Modify edge stresses for compatible continuity along the fold lines, according to the stress relaxation procedure in Fig. 13. Calculate the final stresses after about three cycles.



.0	1	.596	.404	1/2	1/2	0	1 D.F.
	-1/2		-1/2		-1/2		COF
	+992	-992	-591	+591	+446	-446	
		+239	-162	-72	+73		
	-120		+36	+81		-37	
		+21	-15	-41	+40		
	-10		+21	+8		-20	
		+13	-8	-4	+4		
	-7		+2	+4		-2	
		+1	-1	-2	+2		
	+855	-718	-718	+565	+565	-505	TOTAL (PSI)

Fig. 25 Corrected Longitudinal Stresses at Centerline. Phase 1, CASE I (ψ=0)

Step 5. Calculate free plate deflections, using Eq. (22) and values from Step 4. Final stress are independent of E, provided E is constant throughout.



$$\Delta_3 = \frac{\sigma_T - \sigma_B}{Eh} \times \frac{5L^2}{48} = -34.4$$

$$\Delta_4 = -\Delta_3 = 34.4$$

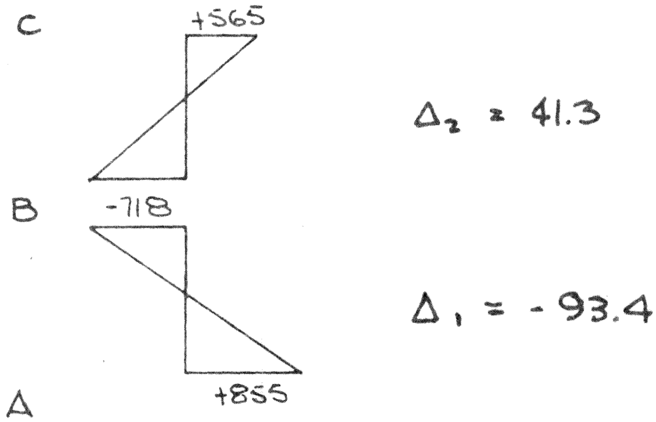


Fig. 26 Deflections at Centerline.
Phase 1, CASE I(ψ=0)

Step 6. Calculate plate moments and fold stresses (from sine loads) at the centerline due to an assumed value of transverse rotation of Plate 2. Refer to Eq. (23).

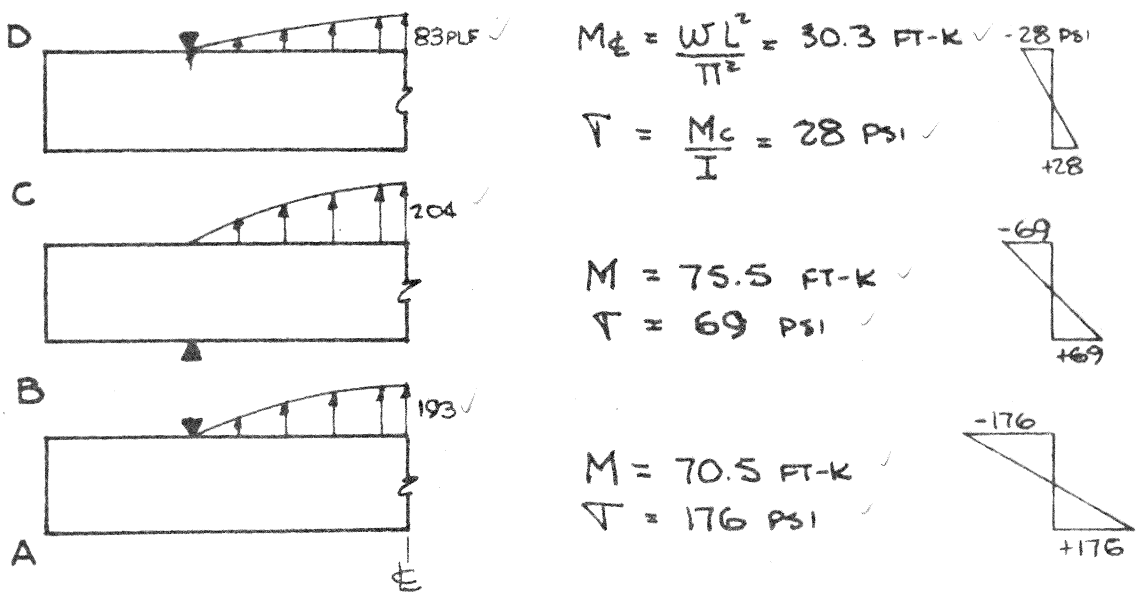
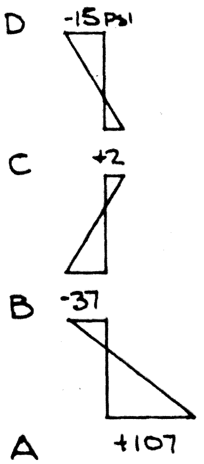


Fig. 27 Longitudinal Stresses Due
to Rotation of Plate 2.

Step 7. **Adjust edge stresses for continuity (as in Step 4), and calculate plate deflections (as in Step 5) caused by the rotation of Plate 2. Refer to Eq. (24).**

Stresses after distribution



$$\Delta_3 = \frac{\sigma_T - \sigma_B}{E h} \times \left(\frac{L}{\pi}\right)^2 = -0.53$$

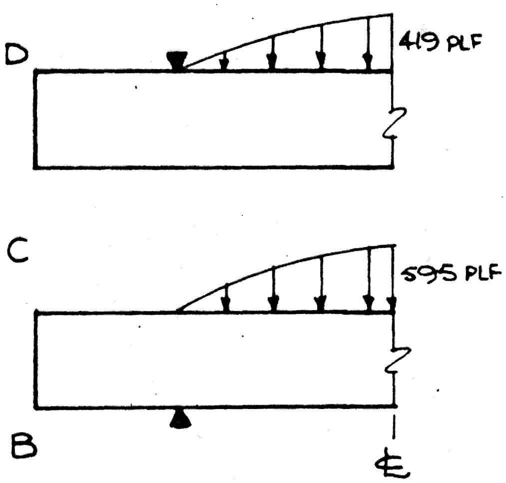
$$\Delta_4 = -\Delta_3 = 0.53$$

$$\Delta_2 = 1.22$$

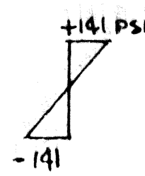
$$\Delta_1 = -8.32$$

Fig. 28 Deflections at Centerline Due to Plate 2 Rotation. Phase 1, CASE II (ψ_2)

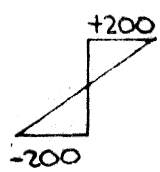
Step 8. Calculate plate moments and fold stresses due to rotation of Plate 3 (as in Step 6).

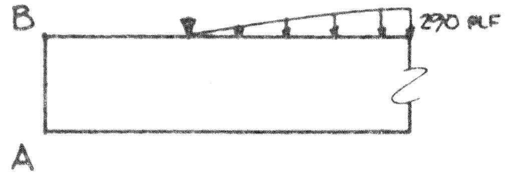


$M_{℄} = 152.7$ FT-K
$\tau = 141$ PSI



$M_{℄} = 217$ FT-K
$\tau = 200$ PSI





$M_{\mathbb{E}} = 105.9$ FT-K
$\sigma = 265$ PSI

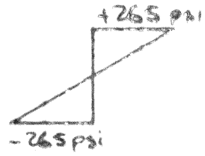
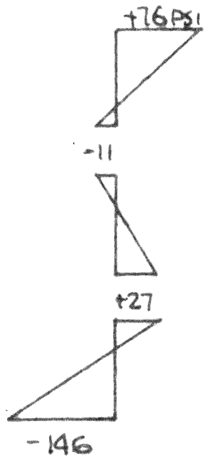


Fig. 29 Longitudinal Stresses Due to Rotation of Plate 3

Step 9. Adjust edge stresses for continuity (as in Step 4) and calculate plate deflections (Step 5) caused by the rotation of Plate 3.

Stresses after distribution



$$\Delta_3 = \frac{\sigma_T - \sigma_B}{Eh} \times \frac{L^2}{\pi^2} = 2.72$$

$$\Delta_4 = -\Delta_3 = -2.72$$

$$\Delta_2 = -1.19$$

$$\Delta_1 = 10.0$$

Fig. 30 Deflections at Centerline Due Plate 3 Rotation. Phase 1, CASE III (ψ_3).

Step 10. Referring to Section VII, use Eqs. (3) and (4) to find k_2 and k_3. Proceed by adding the CASE I solution to k_3 times CASE III solution plus k_2 times CASE II solution.

$$\psi_2 = 26.2\,K_2 = -.0971\,\delta_3 - 0.0280\,\delta_2 + 0.1130\,\delta_1$$

$$\begin{aligned} = &-0.0971\left(-34.4 - 0.53K_2 + 2.72K_3\right) \\ &-0.0280\left(+41.3 + 1.22K_2 - 1.19K_3\right) \\ &+0.1130\left(-93.4 - 8.32K_2 + 10.0K_3\right) \end{aligned}$$

$$\psi_3 = 26.2\,K_3 = 0.0971\left(\delta_4 - \delta_2\right)$$

$$\begin{aligned} = &+0.0971\left(+34.4 + 0.53K_2 - 2.72K_3\right) \\ &-0.0971\left(+41.3 + 1.22K_2 - 1.19K_3\right) \end{aligned}$$

From which

$$K_2 = -0.309$$
$$K_3 = -0.025$$

Step 11. Final moments and stresses can now be calculated by adding the appropriate values of specific Steps. From Figs. 10 and 12, right-hand values at each joint are used.

(A) Transverse Moments at Centerline (ft-lbs)

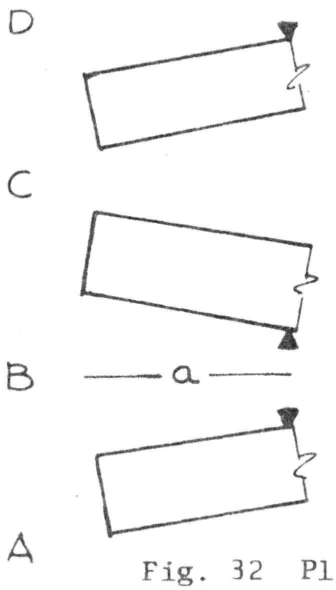
JOINT	CASE I +	$K_2 \times$ CASE II +	$K_3 \times$ CASE III =	TOTAL
D	+587	(-.309)(857)	(-.025)(-4280)	= +429
C	+284	(-.309)(-1719)	(-.025)(2570)	= +751
B	+1194	–	–	= +1194

(B) Longitudinal Stresses at Centerline (psi)

D	-505	(-.309)(-15)	(.025)(76)	= -501
C	+565	(-.309)(2)	(-.025)(-11)	= +564
B	-718	(-.309)(-37)	(-.025)(27)	= -708
A	+855	(-.309)(107)	(-.025)(-146)	= +826

Fig. 31 Summary of Transverse Moments And Longitudinal Stresses at Centerline, As Caused By Roof And Corrective Loads on Center Span.

Step 12. The deflection of weightless cantilevered plates due to the center span load will now be calculated, based on the moments calculated in Step 3. Refer to Eq. (25), and get deflections at midpoint.



$M_{CL} = -485$ FT-K $\quad \Delta_{3_{MP}} = \frac{1}{2} \cdot \frac{M_{CL} L a}{3EI} = -15.25$

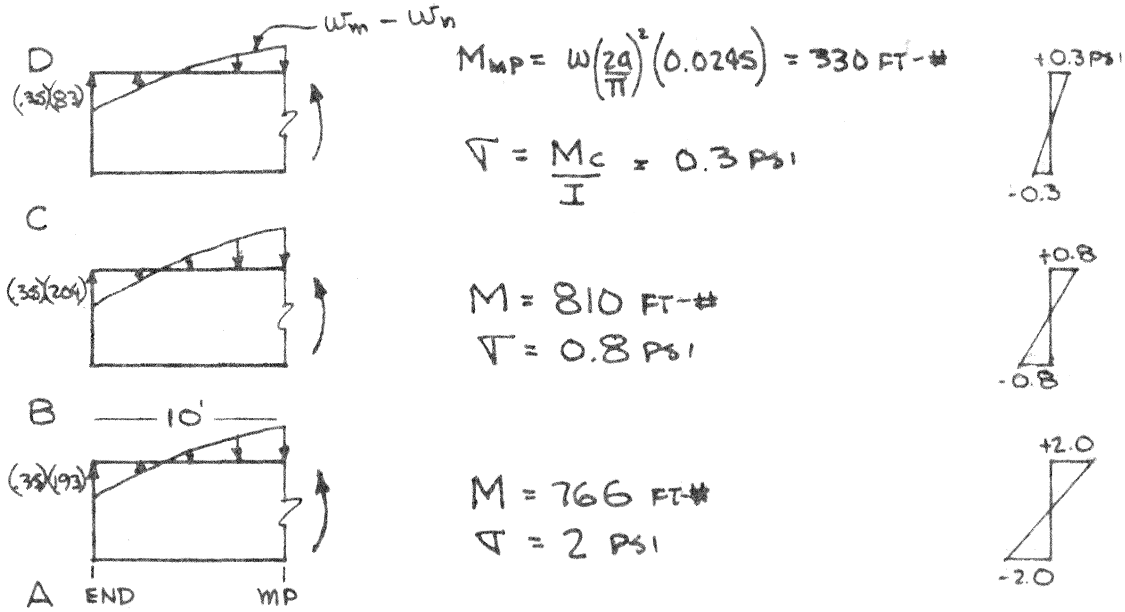
$\Delta_4 = -\Delta_3 = 15.25$

$M_{CL} = +642$ FT-K $\quad \Delta_2 = +20.2$

$M_{CL} = -397$ FT-K $\quad \Delta_1 = -62.5$

Fig. 32 Plate Deflections at Midpoint of The Cantilever, Due to Center Span Loading. Phase 1.

Step 13. Calculate the plate midpoint moments and stresses due to the rotation of Plate 2 on the cantilever. Refer to Step 6 and Eq. (26).



$$M_{MP} = w\left(\frac{2a}{\pi}\right)^2(0.0245) = 330 \text{ FT-\#}$$

$$\sigma = \frac{Mc}{I} = 0.3 \text{ PSI}$$

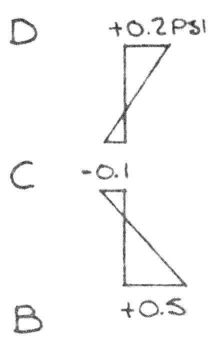
$$M = 810 \text{ FT-\#}$$
$$\sigma = 0.8 \text{ PSI}$$

$$M = 766 \text{ FT-\#}$$
$$\sigma = 2 \text{ PSI}$$

Fig. 33 Longitudinal Stresses at The Cantilever Midpoint, Due to Arbitrary Rotation of Cantilevered Plate 2.

Step 14. Adjust edge stresses for continuity, and calculate plate deflections. Refer to Steps 4 and 5, Eq. (27).

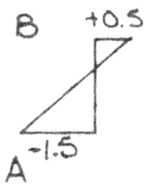
Stresses after distribution



$$\Delta_{3\,MP} = \frac{\sigma_T - \sigma_B}{Eh}\left(\frac{2a}{\pi}\right)^2(.204) = 0.005$$

$$\Delta_4 = -\Delta_3 = -0.005$$

$$\Delta_2 = -.010$$



$\Delta_1 = 0.055$

Fig. 34 Plate Deflections at The Cantilever Midpoint, Due to The Rotation of Plate 2. Phase 1, CASE II(ψ_2).

Step 15. Calculate the cantilevered plate midpoint moments and stresses due to the rotation of Plate 3 on the cantilever. Refer to Step 8 and Eq. (26).

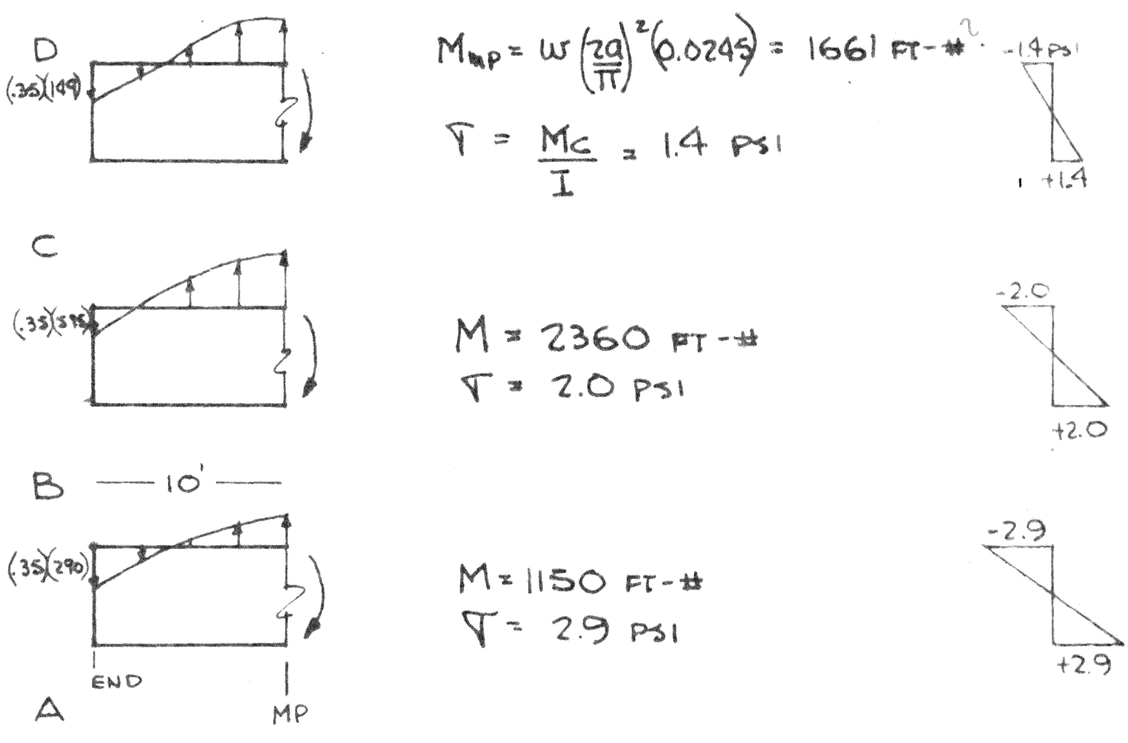
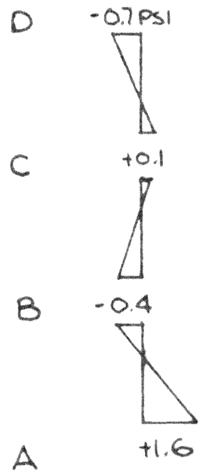


Fig. 35 Longitudinal Stresses at The Cantilever Midpoint, Due to Rotation of Plate 3 on The Cantilever. Phase 1, CASE III(ψ_3).

Step 16. Adjust plate edge stresses for continuity, and calculate plate deflections.

Stresses after distribution



$$\Delta_{3_{MP}} = \frac{\sigma_T - \sigma_B}{Eh} \times \left(\frac{2a}{\pi}\right)^2 (1.204) = -0.013$$

$$\Delta_4 = -\Delta_3 = 0.013$$

$$\Delta_2 = 0.008$$

$$\Delta_1 = -0.062$$

Fig. 36 Plate Deflections at The Cantilever Midpoint, Due to Rotation of Plate 3. Phase 1, CASE III(ψ_3)

Step 17. As in Step 10, use Eqs. (3) and (4) to find k_2 and k_3.

Then,

$$\begin{aligned}\psi_2 = 26.2\,K_2 = &-0.0971(-15.25 + 0.005K_2 - 0.013K_3)\\ &-0.0280(+20.2 - 0.010K_2 + 0.008K_3)\\ &+0.1130(-62.5 + 0.055K_2 - 0.062K_3)\end{aligned}$$

$$\begin{aligned}\psi_3 = 26.2\,K_3 = &\;0.0971(15.25 - 0.005K_2 + 0.013K_3)\\ &-0.0971(20.2 - 0.010K_2 + 0.008K_3)\end{aligned}$$

From which

$$K_2 = -0.235$$
$$K_3 = +0.0187$$

Step 18. Transverse moments and longitudinal stresses can now be calculated (as in Step 11) for points on the cantilever. These values result from corrective loads being applied to the cantilevered plates

(A) Transverse Moments at Extreme End (ft-lbs)

JOINT	CASE I	+ $K_2 \times$ CASE II	+ $K_3 \times$ CASE III	= TOTAL
D	-	(-.235)(857)	(-.0187)(4280)	= -121
C	-	(-.235)(-1714)	(-.0187)(2570)	= +355
B	-	-	-	

(B) Longitudinal Stresses Over Supports (psi)

D	-	<1.0
C	-	<1.0
B	-	<1.0
A	-	<1.0

Fig. 37 Summary of Transverse Moments And Longitudinal Stresses on The Cantilever, As Caused By Corrective Loads on The Cantilever.

The center span portion of the structure will now be corrected because of the additional loading imposed upon it, resulting from the correction of the cantilever.

Step 19. The deflections of the plates at the centerline are due to the net effect of the moments at the supports, caused in turn by the corrective loads on the cantilever. The plates will be assumed weightless and will be allowed to deflect freely. The moment is constant over the section. From Eq. (28), w_n is the net linear load obtained from Steps 13 and 15.

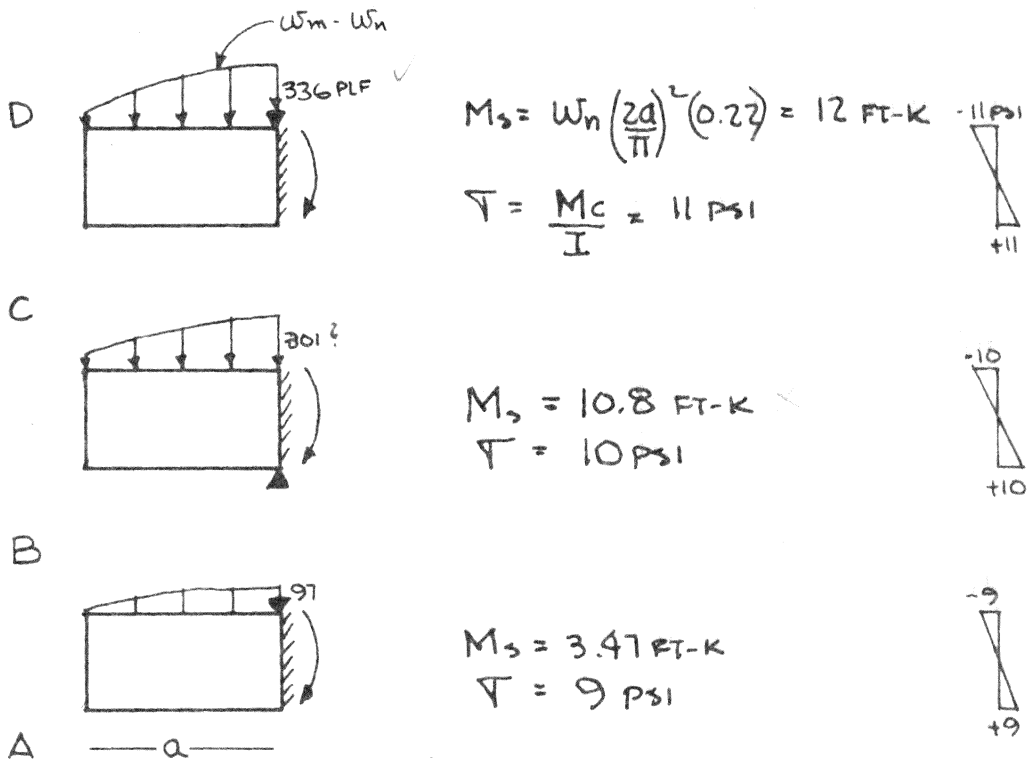
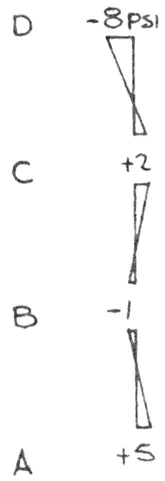


Fig. 38 Longitudinal Stresses at The Supports, Due to The Previous Corrective Loading of The Cantilever. Phase 1

Step 20. Adjust edge stresses for continuity and calculate plate deflections at the centerline. Refer to Eq. (29).

Stresses after distribution



$$\Delta_{3℄} = \frac{\sigma_T - \sigma_B}{Eh} \times \frac{L^2}{3.81} = -0.83$$

$$\Delta_4 = -\Delta_3 = 0.83$$

$$\Delta_2 = 0.24$$

$$\Delta_1 = -0.89$$

Fig. 29 Calculation of Deflection at The Centerline, Due to The Previous Corrective Loading of The Cantilever. Phase 1, CASE I(ψ_0).

Step 21. Apply the corrective sinusoidal loads, due to the rotations of Plates 2 and 3. As in Steps 6 and 7, for CASE II(ψ_2),

$\Delta_4 = 0.53$ $\qquad$ $\Delta_2 = 1.22$

$\Delta_3 = -0.53$ $\qquad$ $\Delta_1 = -8.32$

And similiarly, from Steps 8 and 9, for CASE III(ψ_3),

$\Delta_4 = -2.72$ $\qquad$ $\Delta_2 = -1.19$

$\Delta_3 = 2.72$ $\qquad$ $\Delta_1 = 10.0$

Fig. 39 Plate Deflections at The Centerline, Due to Rotation of Plates 2 And 3. Phase 2.

Step 22. As in Step 18, use Eqs. (3) and (4) to calculate the factors k_2 and k_3.

$$\psi_2 = 26.2K_2 = -0.0971(-0.83-0.53K_2+2.72K_3) -0.0280(+0.24+1.22K_2-1.19K_3) +0.1130(-0.089-8.32K_2+10.0K_3)$$

$$\psi_3 = 26.2K_3 = 0.0971(+0.83+0.53K_2-2.72K_3) -0.0971(+0.24+1.22K_2-1.19K_3)$$

From which

$$K_2 = -0.0234$$
$$K_3 = -0.0100$$

Step 23. Transverse moments and longitudinal stresses can now be calculated for the centerline.

(A) Transverse Moments at The Centerline (ft-lbs)

JOINT	CASE I +	$K_2 \times$ CASE II +	K_3 CASE III =	TOTAL
D	-	(-.0234)(857)	(-.0100)(-4280) =	+23
C	-	(-.0234)(-1714)	(-.0100)(2570) =	+14
B	-	-	-	

(B) Longitudinal Stress at Centerline.

D	-8	(-.0234)(-15)	(-.010)(-76)	=	-8 psi
B	2	(-.0234)(2)	(-.010)(11)	=	+2
C	-1	(-.0234)(-37)	(-.010)(-27)	=	-1
A	5	(-.0234)(107)	(-.010)(146)	=	+2

(C) Longitudinal Stresses Over The Supports (psi).

D	-8	-	-	=	-8
C	2	-	-	=	+2
B	-1	-	-	=	-1
A	5	-	-	=	+5

Fig. 41. Summary of Transverse Moments and Longitudinal Stresses, Due to Corrective Action on The Center Span. Phase 1.

With the application of actual loads to the center span, and subsequent application of corrective loads to the cantilever and center spans, respectively, Phase 1 has been completed.

Phase 2 deals with the application of actual loads to the cantilever and the subsequent corrective loads applied to the center and cantilever spans.

Step 24. Determine longitudinal stresses in the cantilevered plates, as caused by the actual loads acting thereon. The plates are assumed to be unconnected along their edges.

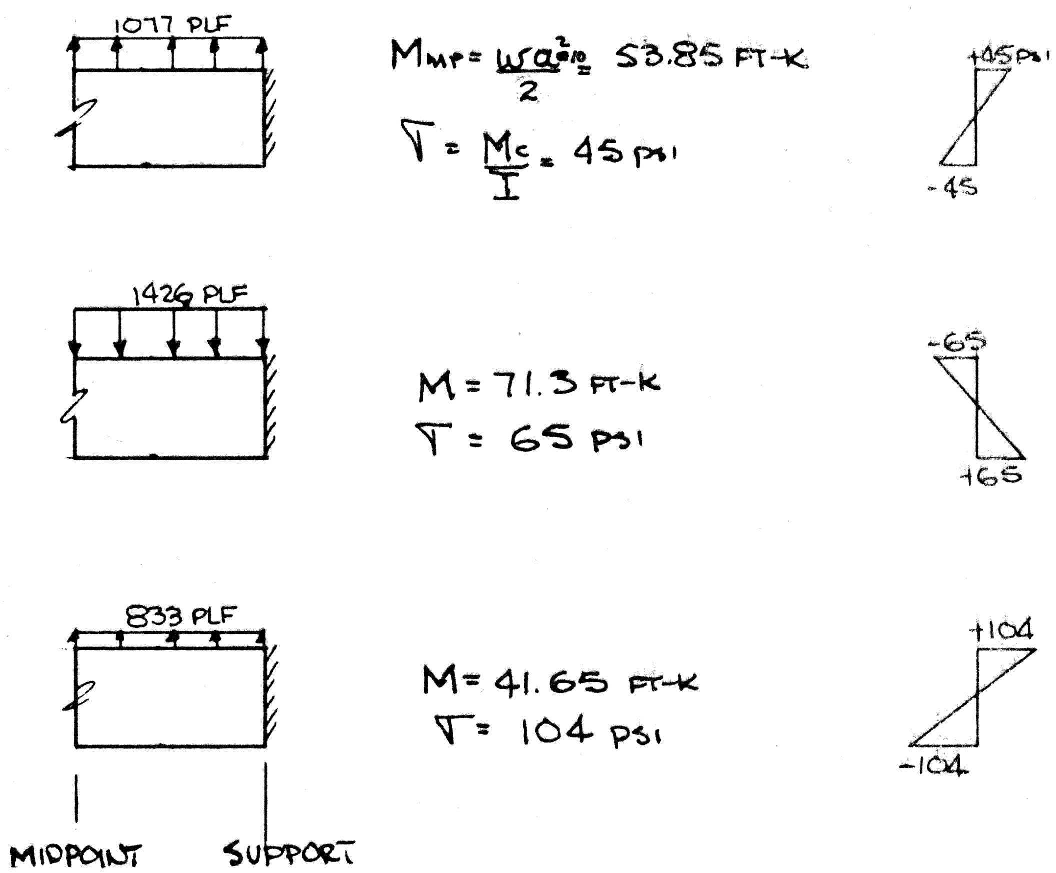
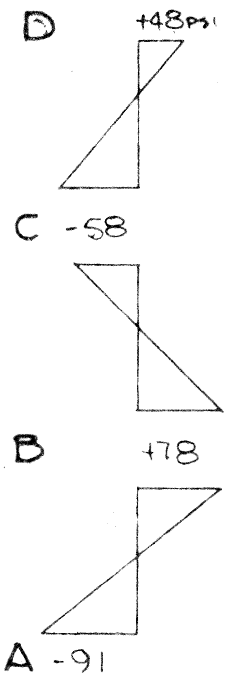


Fig. 42 Plate Free Edge Stresses on The Cantilever Midpoint Due to Actual Loading. Phase 2.

Step 25. Adjust plate edge stresses for continuity and calculate plate deflections at the midpoint of the cantilever. Refer to Eq. (30).

Stresses after distribution



$$\Delta_{3mp} = \frac{\sigma_T - \sigma_B}{Eh} \times \frac{17}{18} a^2 = 1.29$$

$$\Delta_4 = -\Delta_3 = -1.29$$

$$\Delta_2 = -1.86$$

$$\Delta_1 = 3.79$$

Fig. 43 Deflections of The Cantilevered Plates Free Edges, at The Midpoint. Phase 2, CASE I(ψ_o).

Step 26. Apply the corrective trigonometric loads to the cantilever, as in Steps 13 and 15. For CASE II (ψ_2).

$\Delta_{4_{MP}} = -0.005$ $\Delta_{2_{MP}} = -0.010$

$\Delta_{3_{MP}} = +0.005$ $\Delta_{1_{MP}} = +0.055$

And similiarly, from Step 16, for CASE III(ψ_3),

$\Delta_{4_{MP}} = +0.013$ $\Delta_{2_{MP}} = +0.008$

$\Delta_{3_{MP}} = -0.013$ $\Delta_{1_{MP}} = -0.062$

Fig. 44 Plate Deflections at The Cantilever Midpoint, Due to Rotation of Plates 2 and 3.

Step 27. As in Step 22, find k_2 and k_3.

$$\begin{aligned}\psi_2 = 26.2\,K_2 = &-0.0971\,(+1.29 + 0.005K_2 - 0.013K_3) \\ &- 0.0280\,(-1.86 - 0.010K_2 + 0.008K_3) \\ &+ 0.1130\,(+3.79 + 0.055K_2 - 0.062\,K_3)\end{aligned}$$

$$\psi_3 = 26.2K_3 = 0.0971\left(-1.29 - 0.005K_2 + 0.013K_3\right)$$
$$-0.0971\left(-1.86 - 0.010K_2 + 0.008K_3\right)$$

From which

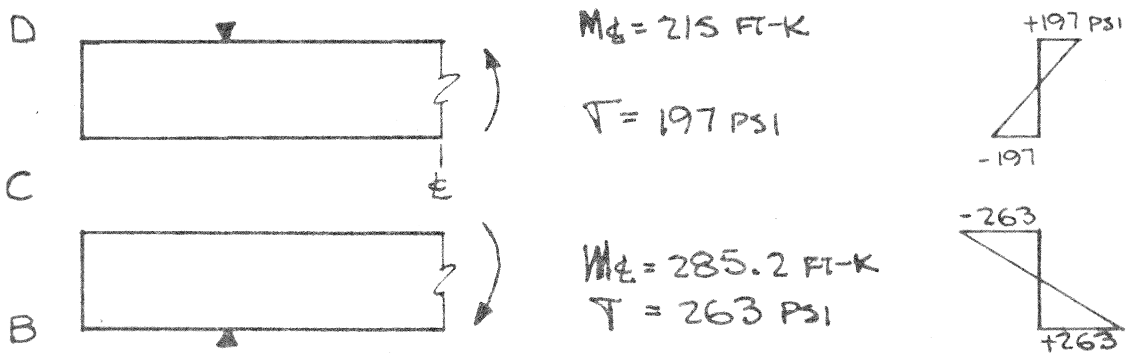
$$K_2 = +0.0128$$
$$K_3 = +0.0020$$

Step 28. Transverse moments can now be calculated at the extreme end of the cantilever, as caused by actual loads acting thereon.

JOINT	CASE I	+ $K_2 \times$ CASE II	+ $K_3 \times$ CASE III	= TOTAL
D	587	(.0128)(857)	(.002)(-4280)	= +611
C	284	(.0128)(-1714)	(.002)(2570)	= +116
B	1194	–	–	= +1194

Fig. 45 Summary of Transverse Moments at Extreme End, as Caused By Actual Loads on The Cantilever. Phase 2.

Step 29. Compute longitudinal stresses at the centerline, as caused by the cantilever loads, Refer to Step 24.



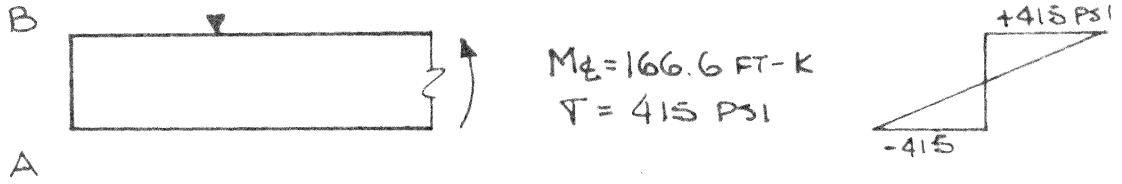
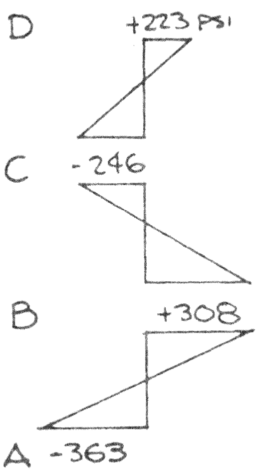


Fig. 46 Longitudinal Stresses at Center Span, Due to The Application of Actual Loads on The Cantilever. Phase 2.

Step 30. Adjust edge stresses for continuity and calculate plate deflections at the centerline. See Eq. (31).

Stresses after distribution



$$\Delta_{3℄} = \frac{\sigma_T - \sigma_B}{Eh} \times \frac{L^2}{8h} = 18.1$$

$$\Delta_4 = -\Delta_3 = -18.1$$

$$\Delta_2 = -21.4$$

$$\Delta_1 = 47.8$$

Fig. 47 Plate Deflections at The Centerline, Due to Actual Cantilever Loads. Phase 2, CASE I(ψ_0).

Step 31. Apply the corrective sinusoidal loads, due to the rotations of Plates 2 and 3. Refer to Step 21. For CASE II(ψ_2),

$$\Delta_4 = +0.53 \qquad \Delta_2 = +1.22$$
$$\Delta_3 = -0.53 \qquad \Delta_1 = -8.32$$

And similarily, for CASE III (ψ_3),

$$\Delta_4 = -2.72 \qquad \Delta_2 = -1.19$$

$$\Delta_3 = +2.72 \qquad \Delta_1 = +10.0$$

Fig. 48 Plate Deflections at The Centerline, Due to Rotation of Plates 2 And 3.

Step 32. As in Step 27, find k_2 and k_3.

$$\begin{aligned}\psi_2 = 26.2K_2 = &-0.0971(+18.1-0.53K_2+2.72K_3)\\ &-0.0280(-21.4+1.22K_2-1.19K_3)\\ &+0.1130(+47.8-8.32K_2+10.0K_3)\end{aligned}$$

$$\begin{aligned}\psi_3 = 26.2K_2 = &\ 0.0971(-18.1+0.53K_2-2.72K_3)\\ &-0.0971(-21.4+1.22K_2-1.19K_3)\end{aligned}$$

From which

$$K_2 = +0.156$$
$$K_3 = +0.012$$

Step 33. Transverse moments and longitudinal stresses can now be calculated for the centerline.

(A) Transverse Moments at The Centerline (ft-lbs)

JOINT	CASE I	+ $K_2 \times$ CASE II	+ $K_3 \times$ CASE III	= TOTAL
D	–	(.156)(857)	(012)(-4280)	= +82
C	–	(156)(-1714)	(012)(2570)	= -237
B	–	–	–	

(B) Longitudinal Stresses at The Centerline (psi)

D	223	(.156)(15)	(.012)(-76)	= +225
C	-246	(.156)(-2)	(.012)(11)	= -246
B	308	(.156)(37)	(.012)(-27)	= +312
A	-363	(.156)(-107)	(.012)(148)	= -379

(C) Longitudinal Stresses Over The Supports (psi)

D	223	–	–	= +223
C	-246	–	–	= -246
B	308	–	–	= +308
A	-363	–	–	= -363

Fig. 49 Summary of Transverse Moments and Longitudinal Stresses, Due to Corrective Action Applied to Center Span. Phase 2.

The final steps in this analysis were concerned with applying corrective loads to the cantilever (see Fig. 21-b) in order to adjust for the center span correction. However, it was found that the computations yielded the constants k_2 and k_3 to be of such small value (K_2=0.00013, K_3=0.00009) that the stresses and moments resulting from their application were much less than 10. This being the case, these steps have been emitted from this analysis.

Step 34. It now remains to tabulate the values of transverse moments and longitudinal stresses obtained from each of the preceding analyses, for n=1.

Joint	Step 11	Step 18	Step 23	Step 28	Step 33	Total
A. Transverse Moments At The Centerline (ft-lbs)						
D	+429	-	+23	-	+82	+534
C	+751	-	+14	-	-237	+528
B	+1194	-	-	-	-	+1194
B. Transverse Moments At Extreme Ends (ft-lbs)						
D	-	-121	-	+611	-	+490
C	-	+335	-	+116	-	+451
B	-	-	-	+1194	-	+1194
C. Longitudinal Stresses At The Centerline (psi)						
D	-501	-	-8	-	+225	-284
C	+564	-	+2	-	-246	+320
B	-708	-	-1	-	+312	-366
A	+826	-	+2	-	-379	+417
D. Longitudinal Stresses Over The Supports (psi)						
D	-	<1	-8	-	+223	+215
C	-	<1	+2	-	-246	-244
B	-	<1	-1	-	+308	+307
A	-	<1	+5	-	-363	-358

Fig. 50 Summary of Plate Transverse Moments And Longitudinal Stresses At Three Points in The Continuous Folded Plate Structure.

Step 35. The final calculation is to determine the approximate edge shear force in the plates at the centerline and at the supports. Refer to Section VIII.

A. Shear At The Centerline

Joint	Before And After Stress Distribution		Change In Stress	Calculation Of N
	STEP 3	STEP 4		
D	-446	-505	59	$N_C = \left(\frac{119-59}{2}\right)560 = 16.8K$
C	+446	+565	119	
C	+591	+565	26	
B	-591	-718	127	
B	-992	-718	274	$N_B = \left(\frac{274-137}{2}\right)380 = 26.0K$
A	+992	+855	137	

$$\text{CHECK: } N_C + N_B = \left(\frac{127-26}{2} + 26\right)560 = 42.8 \cong 16.8 + 20.0$$

Joint	STEP 29	STEP 30	Change In Stress	Calculation Of N
D	197	223	26	$N_C = \left(\frac{49-26}{2}\right)560 = 6.5K$
C	-197	-246	49	
C	-263	-246	17	
B	263	308	45	
B	415	308	107	$N_B = \left(\frac{107-52}{2}\right)380 = 10.6K$
A	-415	-363	52	

$$\text{CHECK: } N_C + N_B = \left(\frac{47-17}{2} + 17\right)560 = 17.1 \cong 6.5 + 10.6$$

Fig. 51 Calculation of Plate Edge Shear Forces

From the final values in Fig. 51, the approximate plate edge shear forces at the centerline may be obtained as shown in Fig. 52.

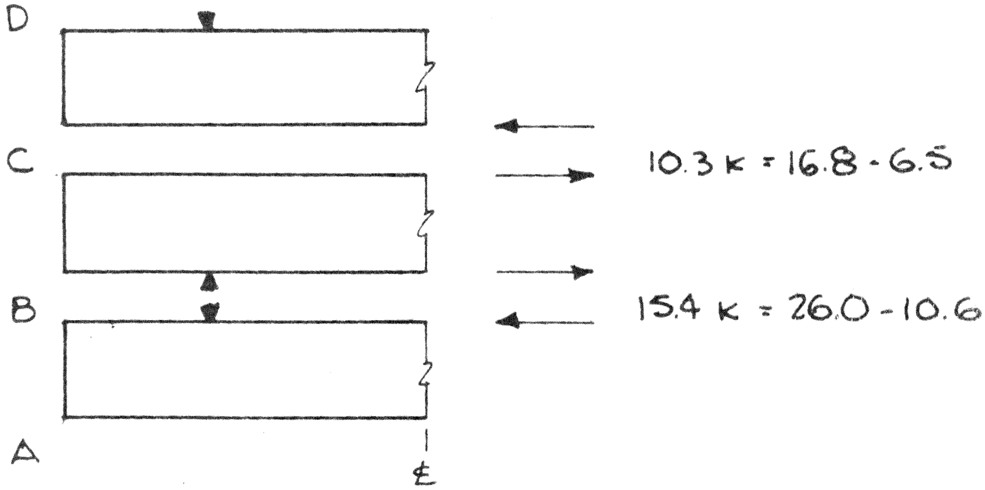


Fig. 52 Plate Edge Shear Forces at The Centerline

B. Shear Forces At The Supports

From Fig. 52, the approximate shear forces at the supports are seen to be as shown in Fig. 53.

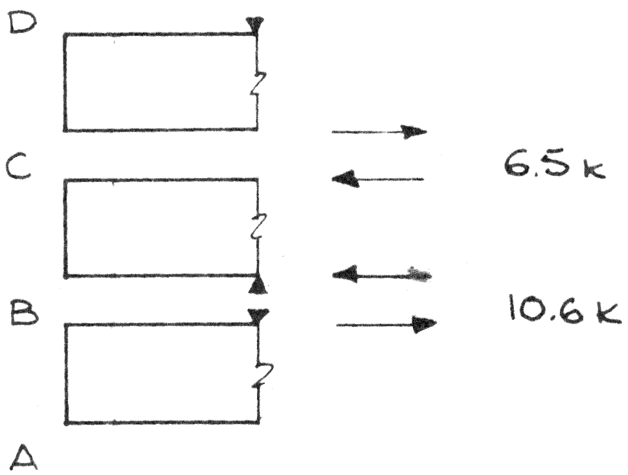
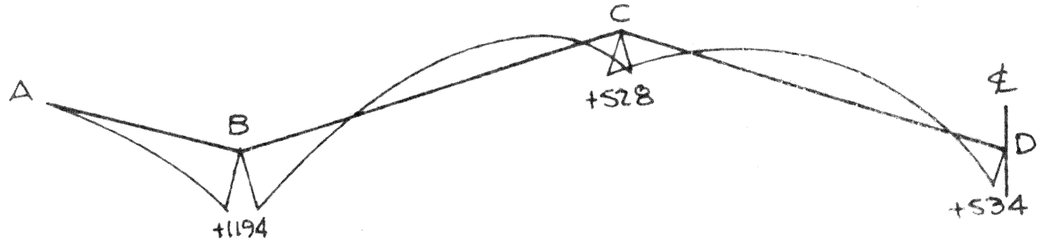


Fig. 53 Plate Edge Shear Forces at The Supports

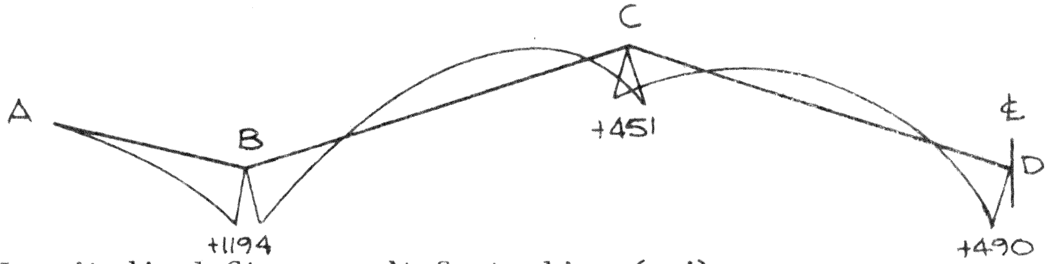
It should be noted that the values obtained in Figs. 52 and and 53 are approximate, as the accuracy involved in the stress relaxation procedures is reduced when used here.

XI. PRESENTATION OF RESULTS

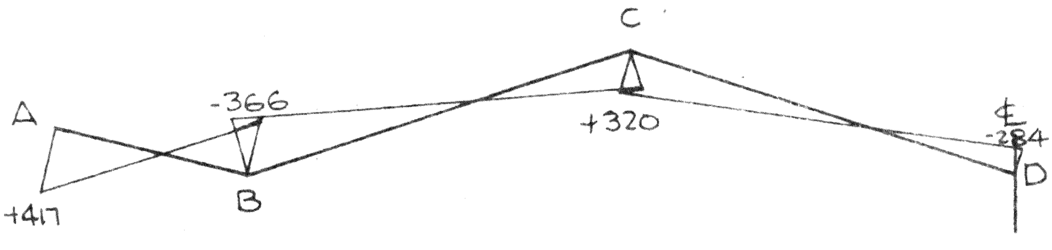
A. Transverse Moments At Centerline (ft-lbs)



B. Transverse Moments At Extreme End (ft-lbs)



C. Longitudinal Stresses At Centerline (psi)



D. Longitudinal Stresses At Supports (psi)

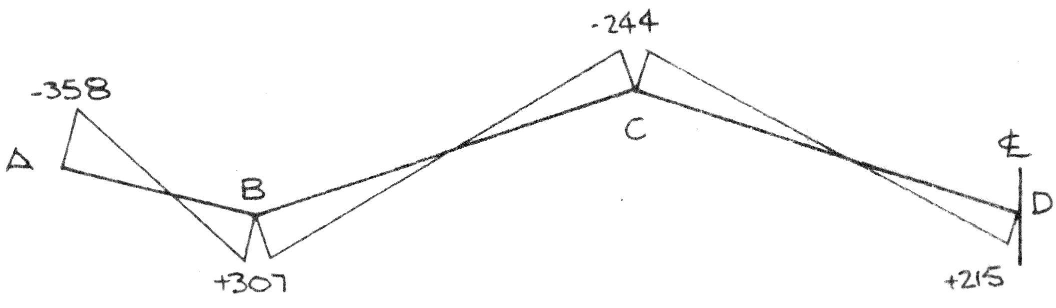


Fig. 54 Presentation of Results

XII. SUMMARY

A. General Discussion

This thesis presents a method for the analysis of continuous folded plate structures. Specifically, it deals with the problem of a folded plate roof simply supported over one central span, cantilevered out at each end, and carrying a uniform load throughout.

The method of analysis used is essentially an extension of that proposed by Simpson[12] in which the plate elements are considered separated from each other, acted upon by the applied load and edge moments of adjacent plates, and caused to deflect. Deflections are computed and adjusted as the plates are made to join together again. The resulting reactions and stresses are computed.

As mentioned at the beginning of Section IX, the uniform plate loadings have been represented by Fourier Series expressions. The values obtained in Fig. 50 have resulted from using only the first term of this series. As a check of the accuracy of using the first term only, values of longitudinal plate stress at the centerline have been calculated for $n=3$ and are compared in Fig. 55 with the total stress values obtained from Fig. 50-C.

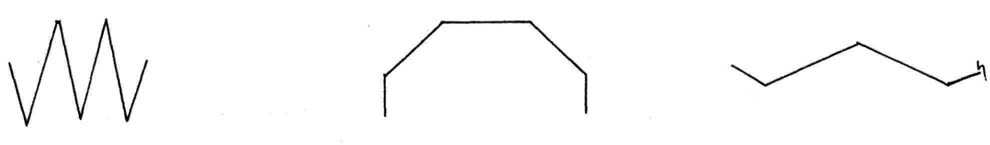
The illustrated problem was analyzed using only $n=1$. If greater refinement is desired, additional partial loading may be considered, as n=3, 5, 7, etc.

Joint	n=1	n=3
D	-284	+49
C	+320	-57
B	-366	+74
A	+417	-85

Fig. 55 Longitudinal Plate Stresses at The Centerline for n=1, 3

In applying the corrective loads to the plates, the use of the algebraic sign followed the pattern shown in Fig. 16. However, in Step 12, the sign of the moment takes precedent, giving plate deflections on the cantilever a sign opposite to that produced in the center span.

The analysis procedure which has been presented for the solution of the example problem would have to be modified for applications to folded plate structures whose configuration is not like that shown in Fig. 6. Examples of various other types of folded plates are shown in Fig. 56. It is apparent that the corrective action, applied in each case, would vary.



(a) (b) (c)

Fig. 56 Examples of Folded Plate Configurations

As a further refinement in treating the different folded plate roofs shown in Fig. 56, there are three ratios of plate geometry inherent in each plate configuration, which affect the analytical results. In Fig. 57 is shown a typical folded - plate roof configuration (as opposed to the so-called "hipped" plate shape shown in Fig. 56-b).

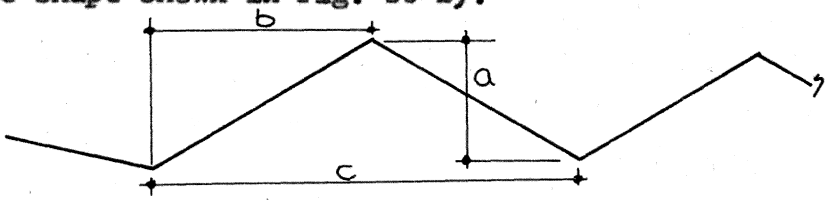


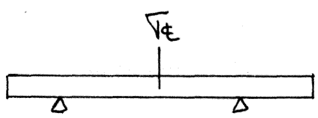
Fig. 57 Typical Folded Plate Roof Cross Section

In Fig. 57, the dimensions of pitch "a/b", the distance "c" between plate valleys or ridges, and the length of plates in the structure each combined in such a way as to affect results, both qualitatively and quantitatively. The selection of optimum values a, b, c and L to obtain the lowest possible plate moments and stressesfor a given plate loading can be found by experimentation; the selection of a flat versus steep roof is a compromise among a series of factors. The ratio of cantilever length to center span length also plays an important part in the selection and subsequent analysis of such a structure.

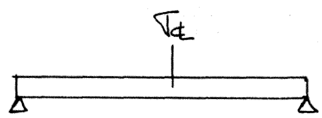
B. <u>Verification of Analysis Procedure</u>

In correspondence[17] one noted designer of folded plate structures suggested that the method of superposition was of questionable accuracy when applied to an analysis such as has been set forth for a cantilevered folded-plate roof. In the discussion which follows an attempt will be made to discover whether the procedure set forth herein produces results which are sufficiently accurate to justify use of the procedure.

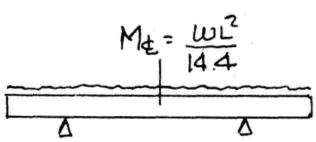
Among several theoretical analysis[2,18] of folded plate roof structures, there is a strong inference that the central portions of long folded plate structure acts as a beam. All deviate from beam action at the boundaries, with short structures having a larger percentage of deviation. The outside elements "spring" sideways (the plates twist). Assuming this to be true, a ratio based on the four beam conditions in Fig. 58 will be used to obtain longitudinal stresses at the and at the supports of the cantilevered plate structure.



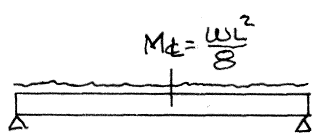
(a) Idealized Folded Plate Cantilevered Roof



(b) Idealized Simply Supported Folded Plate Roof



(c) Simply Supported Beam



(d) Simply Supported Beam

Fig. 58 Idealized Folded Plate System Versus Simply Supported Beam

From Fig. 58 it can be seen that (a) is to (c) as (b) is to (d) or, put into terms of moments and stresses,

$$\frac{\sigma_{\text{℄}a}}{\sigma_{\text{℄}b}} = \frac{M_{\text{℄}c}}{M_{\text{℄}d}} \; ; \quad \sigma_{\text{℄}a} = \sigma_{\text{℄}b} \times \frac{M_{\text{℄}c}}{M_{\text{℄}d}} = \frac{8}{14.4} \times \sigma_{\text{℄}b} \quad \text{---------} \quad (32)$$

By obtaining values of $\sigma_{\text{℄}b}$ from Fig. 31, and using Eq. (32), the following comparisons can be made with values of $\sigma_{\text{℄}B}$ from the last column of Fig. 50-c.

Joint	$\sigma_{\text{℄}a}$	$\sigma_{\text{℄}B}$	$\frac{a}{B}$
D	-279	-284	0.98
C	314	320	0.98
B	-393	-366	1.08
A	459	417	1.10

Fig. 59 Comparison of Longitudinal Stresses at The Centerline

Referring to Fig. 59, the values obtained from the author's analysis (Fig. 50) are used as the basis for comparison with the stresses calculated from the ratio of center-line moments (as shown in Fig. 58). The ratios in Fig. 59 say, in effect, that the results obtained by superposition range from 8% to 10% different than those based on these ratios of moment and stresses.

The question may be raised: Are the plates, under working loads, stressed within the allowable flexural stresses in concrete (based on the best estimates available)? To determine this, assume values of $f_c' = 3000$ psi, $f_c = .45\ f_c'$, and $f_s = 30{,}000$ psi.

Use Eq. (16) and calculate the amount of longitudinal tension steel needed near the edge for plate 2. Stresses are obtained from Fig. 50.

$$M = \frac{(-366-320)(916,000)}{(11.66)(12)} = 374 \text{ ft=kips}$$

$$As = \frac{(374)(12)}{(20)(.866)(138)} = 2.03 \text{ in } 2 \text{ or } 2 \# 9 \text{ bars}$$

From the above calculations, it is seen that the plates are more than able to accommodate the reinforcing steel required.

C. Corrective Loads Applied to the Cantilever

Corrective loads were applied to the cantilevered plates at least twice; first, after the application of real loads to the between-supports section, and second, after real loads were applied to the cantilevered section. In each case, as seen from the derivations in Section IX, the effect of the application of these corrective loadings was to reconnect (or re-join) edges of the plates at their extreme ends and at their midpoints.

In Steps 12 and 34, the cantilevered plates were assumed to be deflected up (for Step 12) and down (for Step 34) in a straight line, and trigonometric loadings were applied to produce theoretical compatability at the midpoint and at the extreme ends, as explained previously. Carrying this reasoning one step further, one may observe that since the plate loads and corrective loads are trigonometric on both cantilever and center span sections, one could possibly conclude with a reasonable degree of certainty that the deflection curve (of the structure under load) assumes a trigonometric (sinusoidal or cosinusoidal) shape. If, then, plate edge compatability can be obtained at four points (the centerline, over supports, the cantilever midpoint, and extreme end), plate edge compatability can be assumed to be such that values of moment and stress obtained are representative.

As stated earlier, the results due to the application of corrective loads to the cantilever (Step 34 on) were found to be negligible; values of k_2 and k_3 were found to equal 0.00013 and 0.00009, respectively. Because of the small values of k_2 and k_3, the cantilever would probably have to span a good deal longer than 20' in order for this sequence of steps to appreciably affect the final results.

XIII. CONCLUSIONS

With reference to the comments in Section XII dealing with the possibility of the deflection curve of the plates structure assuming a trigonometric shape, it should be stated that it is beyond the scope of this thesis to evolve a solution based upon this exact trigonometric shape. What has been done represents an attempt to improve upon the approximate solution suggested by Parme[18] and others[2,16] of comparing the plate structure with an ordinary continuous beam and using ratios as a comparison (as was done in Section XII-B). However, although there can be a case made for the method of ratios, this technique is affected only with the overall continuous length of the members; the solution developed here takes into account the cross-sectional geometry of the plates and attempts also to account for the change in the transverse and longitudinal behavior of the plates from centerline to support to cantilevered end. Hence, this solution is much more refined. The solution presented herein appears to be valid because it differs only slightly from comparison with the simple beam analysis. In support of this statement, the following is found[18] in ASCE Manual 31 (page 46): "the effect of continuity over the supports on stresses in ordinary beams. End restraint of the shell by continuity creates longitudinal stresses at the support whose magnitude (and sign) are approximately in the same ratio to the longitudinal stresses in the simply supported shell as the end moments in a continuous beam are to the positive (middle span) moment in the same beam, simply supported. In some cases, the values obtained by a rigorous satisfaction of the boundry conditions and those obtained by proportioning the interal forces on the basis of the ratio of end moments to the moment in a simply supported beam are practically identical. For instance, the longitudinal stresses at the center support of a continuous shell with two equal spans, computed (1) by means of the rigorous solution, and (2) by multiplying the stresses at mid-span of the simply supported shell by the ratio of the negative moment over the support of a continuous beam having two equal spans to the mid-span moment of a simply supported beam are as follows:[11]

Angle Ø*	Computed by Formulae	Computed by Ratio of Moments
40	1, 550 psi	1, 820 psi
30	7, 450	7, 670
20	16, 160	16, 220
10	2, 130	1, 940
0	-73, 000	-73, 500

"It is evident from the foregoing that the approximate procedure by ratio of moments is entirely satisfactory for long shells."[11]

* Ø is the angle measured from the vertical, of the radius line of the barrel shell

Several investigators** have worked on exact solutions for the continuous folded plate using theory of elasticity and other approaches, but, to date there are no published results.

The author is not satisfied that his analysis results in the best solution. In order to confirm the stresses and moments obtained herein, the following is suggested:

(1) Check future published solutions.

(2) Rework this problem to establish plate edge continuity at 8 or more points instead of at four points.

(3) Approach this problem from the point of view of using a generalized equation of deflections working the computations arithmetically using the theory of finite differences.

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ABSTRACT

ANALYSIS OF A FOLDED PLATE CONCRETE ROOF CONTINUOUS WITH OVERHANG

Concrete folded plate structures have gained increasing use and popularity in the United States during the past fifteen years. They have proved to be especially economical in the construction of longspan roof systems.

The folded plate shape has come into wide usage because of its low cost of construction for long spans, high loadcarrying capacity, and rigidity. Moreover, the folded plate structure has the structural advantage of thinshelled, curved surface structures, and is much easier and less expensive to construct than the latter.

With reference to folded plate structures continuous over a support, the problem arises of relating the geometry of the folded plates to a number of points along the entire length of the plate edges, in an attempt to find the true condition of stress at these points, and to find the contribution of each stress condition to the overall effect.

The investigation of this thesis consists of analyzing a reinforced folded plate concrete roof, continuous with overhang. The method of analysis used is that of successive iteration, i.e. balancing angle changes against internal moments and/or stresses. This method is similiar to one presented by Howard Simpson - "Design of Folded Plate Roofs"- in the ASCE Proceedings, Vol. 84, January, 1958, but extended to apply to the cantilevered folded plates.