COMPRESSIVE BUCKLING

OF A CLAMPED CIRCULAR PLATE

ON

AN ELASTIC FOUNDATION NOT IN ATTACHMENT

by

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NOMENCLATURE

A, C = constants in the deflection equation

D = flexural stiffness of plate

= $Et^3/12(1-\nu^2)$

E = Young's modulus of plate material

N = critical compressive load per unit length along the circumferential edge of plate

w_1 = plate deflection in the part of attachment with foundation

w_2 = plate deflection in the part not in attachment with foundation

r = radial distance from center of the plate to any point in consideration

ρ = radial distance from center of the plate to the point of separation of two parts in attachment and not in attachment

∇^4 = $\nabla^2\,\nabla^2$

= $\left(\frac{d^2}{dr^2} + \frac{1}{r}\,\frac{d}{dr}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\,\frac{d}{dr}\right)$

a = radius of plate

k = foundation modulus

t = plate thickness

ν = Poisson's ratio

$(Mr)_1$ = moment in the plate in the part of attachment with foundation at a distance r from the center of plate

$(Mr)_2$ = moment in the plate in the part not in attachment with foundation at a distance r from the center of plate

INTRODUCTION

The buckling problem of a thin circular plate, compressed by uniformly distributed loads in the plane of the plate and normal to the edge, has been studied by many authors; however, the effect of an elastic foundation not attached to the plate has not as yet been considered. The present thesis treats such a problem.

In 1891, G. H. Bryan[1] first attacked the stability problem of a plate under thrusts in its own plane. The buckling loads were found from the differential equation of equilibrium, which in turn was solved in terms of Bessel functions. Following the same process, many similar problems such as the circular annular plate under compressive loads in its plane was solved by A. Dannik, A. Nadai[2], A. Lokchine[3] and E. Meissner[4]. All of the above investigators obtained the exact solutions by using Bessel functions. In 1949, Grigolguk, E. I.[5] developed an approximate method for solving the stability problem of a circular annular plate. His results, compared with the exact solutions of the previous investigators, differed only by 2% to 4% at the most. One advantage of the approximate method is to reduce the computations connected with the evaluation of the Bessel functions. In the present investigation, a trial and error method was used in finding roots of the characteristic equations which were expressed in terms of the Bessel functions.

In 1958, P. Seide[6] published a paper on the compressive buckling of a long simply supported plate on an elastic foundation. His results showed that the lack of attachment of the plate to the foundation gives a drastic reduction in the buckling load.

The purposes of this thesis are: (1) to find the relation between the buckling load and the foundation modulus when the plate is not in attachment with the foundation and (2) to compare the results with that of the same plate attached to the foundation.

To perform the first objective, the author studied the second mode shape since the first mode is the well known case of full attachment of the plate with the foundation which the author also discussed in the later part of the next chapter. The governing differential equations were established from the classical small deflection theory. These equations were solved in terms of Bessel functions. The final characteristic equations contained three variables which are: the buckling load, N; the foundation modulus, k, and the distance, ρ , from the center of the plate to the line of separation of the two regions of action of the deformed plate. The present investigation is concerned with the relative values of N and k.

In solving these equations, an assumed value of k is required first and, by trial and error method, one set of roots which satisfy the assumed mode shape can be obtained. With many values of k and its corresponding values of N, a curve was plotted which showed the variation of the buckling load with the foundation modulus.

In the case of attachment, the problem is quite simple. There is only one characteristic equation to be solved which contains the buckling load, N, and the foundation modulus, k.

The results show that for the case of attachment, the relation between the buckling load and the foundation modulus is quite irregular; while for the case of unattachment, the N - k relation looks like an exponential curve.

THEORETICAL ANALYSIS

1. Assumptions

First of all, the author made the following assumptions:

(a) The plate is clamped along its edge.

(b) The forces acted on the plate are the uniform compressive load along its edge and the reactions of the foundation upon the bottom of the plate.

(c) The foundation is of the Winkle type: that is, the pressure at any point in attachment with the plate is proportional only to the deflection applied at that point.

(d) The plate will buckle in two cases as indicated in Fig. I and Fig. II.

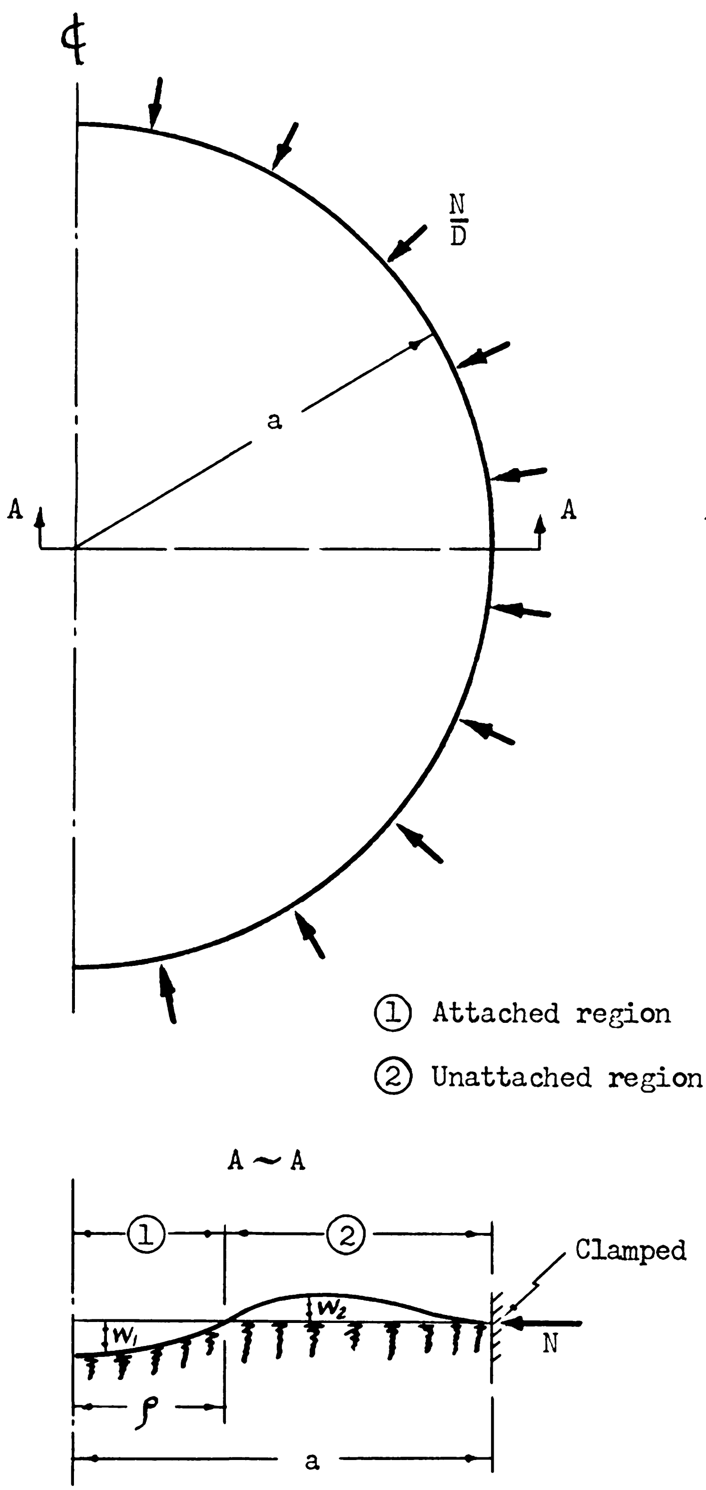


Fig. I

1st case of the deflection curve of the plate not in attachment with the foundation

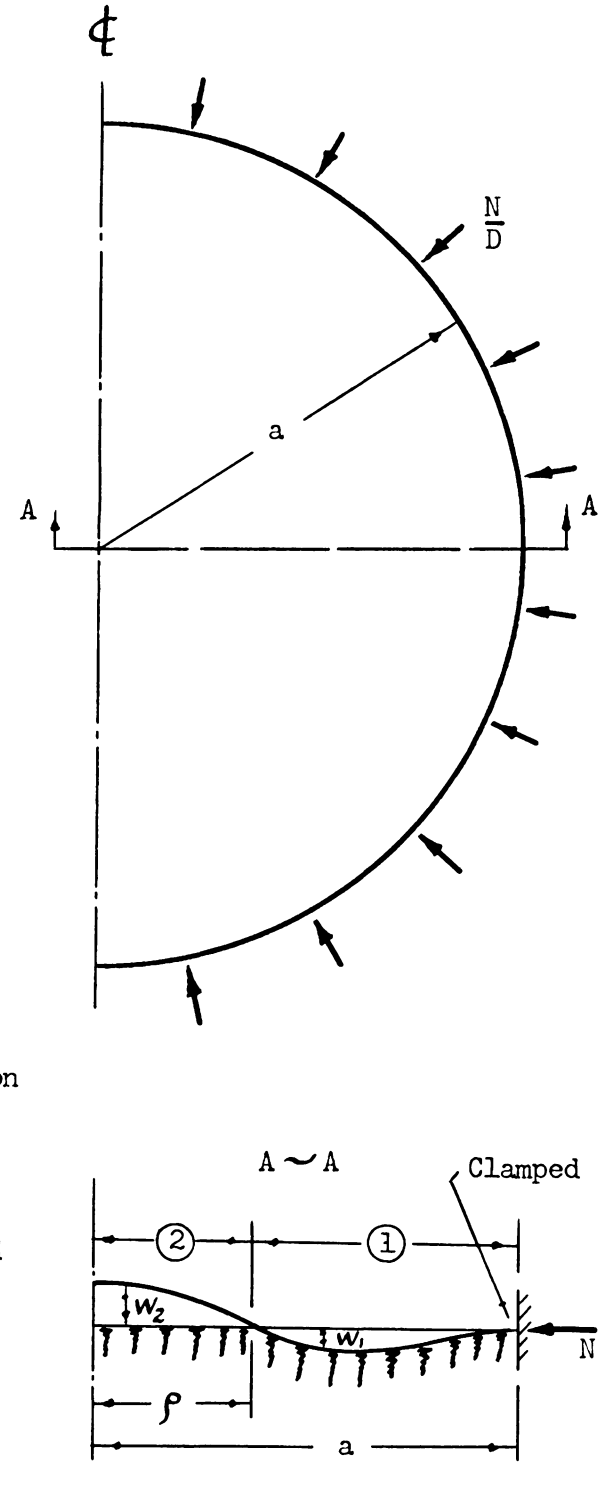


Fig. II

2nd case of the deflection curve of the plate not in attachment with the foundation

2. Differential Equations

The deflections of the plate are controlled by the following two differential equations based on classical small deflection theory:

$$(D\nabla^4 + N\nabla^2 + k)\, w_1 = 0 \qquad \text{attached region} \qquad (1)$$

$$(D\nabla^4 + N\nabla^2)\, w_2 = 0 \qquad \text{unattached region} \qquad (2)$$

in which w_1, w_2 are the deflections in the attached and unattached regions respectively.

The deflections of the two regions are connected by the following equations of continuity at $r = \rho$:

$$\left.\begin{aligned} w_1 &= w_2 = 0 \\ \left(\frac{dw_1}{dr}\right)_r &= \left(\frac{dw_2}{dr}\right)_r \\ (M_r)_1 &= (M_r)_2 \\ (Q_r)_1 &= (Q_r)_2 \end{aligned}\right\} \text{ or } \left.\begin{aligned} w_1 &= w_2 = 0 \\ \left(\frac{dw_1}{dr}\right)_{r=\rho} &= \left(\frac{dw_2}{dr}\right)_{r=\rho} \\ \left(\frac{d^2w_1}{dr^2} + \nu\frac{1}{r}\frac{dw_1}{dr}\right)_{r=\rho} &= \left(\frac{d^2w_2}{dr^2} + \nu\frac{1}{r}\frac{dw_2}{dr}\right)_{r=\rho} \\ \frac{d}{dr}\left(\frac{d^2w_1}{dr^2} + \frac{1}{r}\frac{dw_1}{dr}\right)_{r=\rho} &= \frac{d}{dr}\left(\frac{d^2w_2}{dr^2} + \frac{1}{r}\frac{dw_2}{dr}\right)_{r=\rho} \end{aligned}\right\} \quad (3)$$

The boundary conditions at $r = a$ are:

(a) for the first case of deflection of the plate (Fig. I)

$$\left.\begin{aligned} (w_2)_{r=a} &= 0 \\ \left(\frac{dw_2}{dr}\right)_{r=a} &= 0 \end{aligned}\right\} \qquad (4)$$

(b) for the second case of deflection of the plate (Fig. II)

$$\left.\begin{aligned} (w_1)_{r = a} &= 0 \\ \left(\frac{dw_1}{dr}\right)_{r = a} &= 0 \end{aligned}\right\} \qquad (4)'$$

In addition, at $r = 0$, the singularity properties require that the value of w_1 (Fig. I) or w_2 (Fig. II) be finite.

3. Solution of the Equations

(a) For the attached region, write equation (1) as

$$(\nabla^2 - m_1)(\nabla^2 - m_2)\, w_1(r) = 0, \text{ where } m_{1,2} = \frac{-N \pm \sqrt{N^2 - 4kD}}{2D}$$

Hence the complete solution of (1) is the sum of the solutions of the following two differential equations:

$$\frac{d^2w_1}{dr^2} + \frac{1}{r}\frac{dw_1}{dr} - m_1 w_1 = 0$$

$$\frac{d^2w_1}{dr^2} + \frac{1}{r}\frac{dw_1}{dr} - m_2 w_1 = 0$$

Assume $N^2 > 4kD$; then $m_{1,\,2}$ may be written as

$$m_1 = \left(\frac{-N + \sqrt{N^2 - 4kD}}{2D}\right) = -\left(\frac{N - \sqrt{N^2 - 4kD}}{2D}\right) = -p^2$$

$$m_2 = \left(\frac{-N - \sqrt{N^2 - 4kD}}{2D}\right) = -\left(\frac{N + \sqrt{N^2 - 4kD}}{2D}\right) = -q^2$$

where p, q are positive real numbers depending on the values of N and k. Substituting m_1, m_2 in the above two differential equations, we get

$$r^2\frac{d^2w_1}{dr^2} + r\frac{dw_1}{dr} + p^2 r^2 w_1 = 0$$

$$r^2\frac{d^2w_1}{dr^2} + r\frac{dw_1}{dr} + q^2 r^2 w_1 = 0$$

These are the typical Bessel equations and their solutions can be expressed as the sum of Bessel functions of the first kind and of the second kind of order zero. Hence the general solution of equation (1) may be written as

$$w_1 = A_1 J_o(pr) + A_2 J_o(qr) + A_3 Y_o(pr) + A_4 Y_o(qr) \qquad (5)$$

(b) For the unattached region, write equation (2) as

$$\nabla^2 (D\nabla^2 + N)\, w_2 = 0$$

Hence the complete solution of (2) is the sum of the solutions of the following two differential equations:

$$r^2 \frac{d^2 w_2}{dr^2} + r \frac{dw_2}{dr} = 0$$

$$r^2 \frac{d^2 w_2}{dr^2} + r \frac{dw_2}{dr} + \alpha^2 r^2 w_2 = 0 \qquad \text{where } \alpha^2 = \frac{N}{D} = p^2 + q^2$$

or $$w_2 = C_1 J_o(\alpha r) + C_2 Y_o(\alpha r) + C_3 \log r + C_4 \qquad (5)'$$

In (5) and (5)', A's and C's are arbitrary constants to be satisfied by the boundary conditions and the equations of continuity.

At $r = 0$, $Y_o\,(pr) \longrightarrow \infty$

$Y_o\,(qr) \longrightarrow \infty$

$Y_o\,(\alpha r) \longrightarrow \infty$

$\log r \longrightarrow \infty$

The singularity properties require the constants A_3, A_4 in equation (5) and C_2, C_3 in equation (5)' to vanish. Hence (5) and (5)' will be simplified as follows:

$$w_1 = A_1 J_o(pr) + A_2 J_o(qr) \qquad (6)$$

$$w_2 = C_1 J_o(\alpha r) + C_4 \qquad (6)'$$

Considering the first case of buckling, the equations of continuity at $r = \rho$ and the boundary conditions at $r = a$ yield the following seven algebraic equations with six constants, the A's and C's, as the unknowns. They are:

$$A_1 J_o(p\rho) + A_2 J_o(q\rho) = 0$$

$$A_1 J_o(p\rho) + A_2 J_o(q\rho) - C_1 J_o(\alpha\rho) - C_2 Y_o(\alpha\rho) - C_3 \log\rho - C_4 = 0$$

$$A_1 p J_1(p\rho) + A_2 q J_1(q\rho) - C_1 \alpha J_1(\alpha\rho) - C_2 \alpha Y_o(\alpha\rho) + C_3 \frac{1}{\rho} = 0$$

$$A_1\left[-p^2 J_o(p\rho) + (1-\nu)\frac{p}{\rho} J_1(p\rho)\right] + A_2\left[-q^2 J_o(q\rho) + (1-\nu)\frac{q}{\rho} J_1(q\rho)\right]$$
$$+ C_1\left[\alpha^2 J_o(\alpha\rho) - (1-\nu)\frac{\alpha}{\rho} J_1(\alpha\rho)\right] + C_2\left[\alpha^2 Y_o(\alpha\rho) - (1-\nu)\frac{\alpha}{\rho} Y_1(\alpha\rho)\right] + C_3(1-\nu)\frac{1}{\rho^2} = 0$$

$$A_1 p^3 J_1(p\rho) + A_2 q^3 J_1(q\rho) - C_1 \alpha^3 J_1(\alpha\rho) - C_2 \alpha^3 Y_1(\alpha\rho) = 0$$

$$C_1 J_o(\alpha a) + C_2 Y_o(\alpha a) + C_3 \log a + C_4 = 0$$

$$C_1 \alpha J_1(\alpha a) + C_2 \alpha Y_1(\alpha a) - C_3 \frac{1}{a} = 0$$

Elimination of the constants from the above seven equations yields two characteristic equations in terms of Bessel functions, which are:

$$\alpha h^2 J_o(p\rho) J_o(q\rho)\left[J_o(\alpha\rho) Y_1(\alpha\rho) - J_1(\alpha\rho) Y_o(\alpha\rho)\right]\left[\left(\log\frac{a}{\rho}\right)(\alpha a) Y_1(\alpha a) + Y_o(\alpha a) - Y_o(\alpha\rho)\right] + \left\{\left(\log\frac{a}{\rho}\right)(\alpha a)\left[J_1(\alpha a) Y_o(\alpha\rho) - J_o(\alpha\rho) Y_1(\alpha a)\right] + J_o(\alpha a) Y_o(\alpha\rho) - J_o(\alpha\rho) Y_o(\alpha a)\right\}\left\{p^2 J_o(q\rho)\left[p J_1(p\rho) Y_o(\alpha\rho) - \alpha J_o(p\rho) Y_1(\alpha\rho)\right] - q^2 J_o(p\rho)\left[q J_1(q\rho) Y_o(\alpha\rho) - \alpha J_o(q\rho) Y_1(\alpha\rho)\right]\right\} = 0 \qquad (7)$$

$$h^2 J_o(p\rho) J_o(q\rho)\left[\left(\log\frac{a}{\rho}\right)(\alpha a) Y_1(\alpha a) + Y_o(\alpha a) - Y_o(\alpha\rho)\right]\left\{a\left[J_1(\alpha a) Y_o(\alpha\rho) - J_o(\alpha\rho) Y_1(\alpha a)\right] + \rho\left[J_o(\alpha\rho) Y_1(\alpha\rho) - J_1(\alpha\rho) Y_o(\alpha\rho)\right]\right\} + \left\{\left(\log\frac{a}{\rho}\right)(\alpha a)\left[J_1(\alpha a) Y_o(\alpha\rho) - J_o(\alpha\rho) Y_1(\alpha a)\right] + J_o(\alpha a) Y_o(\alpha\rho) - J_o(\alpha\rho) Y_o(\alpha a)\right\}\left\{\rho\alpha Y_o(\alpha\rho)\left[p J_1(p\rho) J_o(q\rho) - q J_1(q\rho) J_o(p\rho)\right] + h^2 J_o(p\rho) J_o(q\rho)\left[\rho Y_o(\alpha\rho) - a Y_1(\alpha a)\right]\right\} = 0 \qquad (8)$$

where $h^2 = q^2 - p^2$

Similarly, for the second case of buckling, the following two characteristic equations are established:

$$(l_1l_2 - l_3l_4)\left[qJ_1(p\rho)J_o(q\rho)-pJ_o(p\rho)J_1(q\rho)\right] + l_1l_5p\left[J_o(q\rho)Y_1(q\rho) -J_1(q\rho)Y_o(q\rho)\right] + l_2l_5q\left[J_1(p\rho)Y_o(p\rho)-J_o(p\rho)Y_1(p\rho)\right] + l_3l_5\left[pJ_1(q\rho)Y_o(p\rho) -qJ_o(q\rho)Y_1(p\rho)\right] + l_4l_5\left[pJ_o(q\rho)Y_1(q\rho)-qJ_1(p\rho)Y_o(q\rho)\right] + l^2{}_5\left[pY_o(p\rho)Y_1(q\rho) -qY_o(q\rho)Y_1(p\rho)\right] = 0 \qquad (7)'$$

$$\alpha J_o(\alpha\rho)\left\{(l_1l_2-l_3l_4)\left[pJ_1(p\rho)J_o(q\rho)-qJ_o(p\rho)J_1(q\rho)\right] + l_1l_5q\left[J_1(q\rho)Y_1(q\rho) -J_1(q\rho)Y_o(q\rho)\right] + l_2l_5p\left[J_1(p\rho)Y_o(p\rho)- J_o(p\rho)Y_1(p\rho)\right] + l_3l_5\left[qJ_1(q\rho)Y_o(p\rho) -pJ_o(q\rho)Y_1(p\rho)\right] + l_4l_5\left[qJ_o(p\rho)Y_1(q\rho)- pJ_1(p\rho)Y_o(q\rho)\right] + l^2{}_5\left[qY_o(p\rho)Y_1(q\rho) -pY_o(q\rho)Y_1(p\rho)\right]\right\} + h^2J_1(\alpha\rho)\left\{(l_1l_2 - l_3l_4)\, J_o(p\rho)J_o(q\rho)- l_3l_5J_o(q\rho)Y_o(p\rho) - l_4l_5J_o(p\rho)Y_o(q\rho) - l^2{}_5Y_o(p\rho)Y_o(q\rho)\right\} = 0 \qquad (8)'$$

where

$$l_1 = \left[J_1(pa)Y_o(pa) - J_o(pa)Y_1(pa)\right] p$$

$$l_2 = \left[J_o(qa)Y_1(qa) - J_1(qa)Y_o(qa)\right] q$$

$$l_3 = pJ_1(pa)Y_o(qa) - qJ_o(pa)Y_1(qa)$$

$$l_4 = pJ_o(qa)Y_1(pa) - qJ_1(qa)Y_o(pa)$$

$$l_5 = qJ_o(pa)J_1(qa) - pJ_1(pa)J_o(qa)$$

4. Study of the Case in Attachment

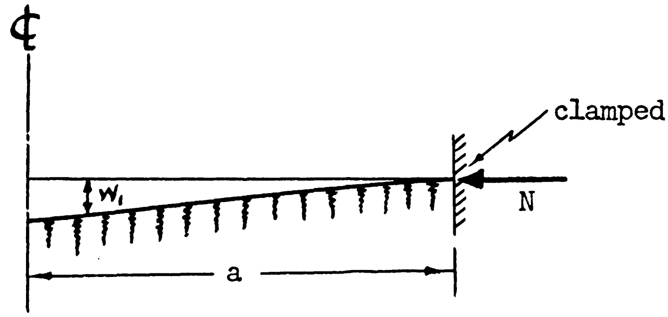


Fig. III

Deflection of the plate in
attachment with the foundation

In case the plate is in full attachment with the foundation, equation (1) will govern the whole region and the solution is expressed by equation (5) above. On applying the boundary conditions at the clamped edge and the singularity properties at the center of plate, the following characteristic equation is established:

$$qJ_1(qa)J_o(pa) - pJ_o(qa)J_1(pa) = 0 \qquad (9)$$

DATA AND RESULTS

There are two characteristic equations as expressed by (7), (8) or (7)', (8)' for the two buckling cases of the plate. These equations contain three variables: k/D, N/D and ρ. For any given value of k/D, there is a set of roots which satisfies these two equations and is, at the same time, consistent with the mode shape as shown in Fig. I or Fig. II. As the computational work in solving the characteristic equations is quite lengthy, the numerical calculations covered only the first case of buckling in the present thesis. In doing the calculations, the author wanted to use the existing tables[7] of the Bessel functions. Since the arguments of the Bessel functions are limited to 16.00 in these tables, the radius of the plate was taken equal to 3. Using trial and error method, the results for the indicated values of k/D and N/D are shown in Table I.

Table I

Foundation Modulus, k, and the Buckling Loads, N, for the Plate in Attachment and Not in Attachment with the Foundation

Foundation Modulus, k/D		15	25	35	45	65	85
Buckling Load, N/D	Unattached	11.71	12.09	12.44	13.42	16.92	25.10
	Attached	12.46	14.02	13.02	14.67	17.19	23.29

To find the mode shape, it was necessary to express the deflections, w_1 and w_2, in terms of an arbitrary constant by using the following relations:

$$A_1 = -A_2 \frac{J_o(q\rho)}{J_o(p\rho)}$$

$$C_1 = \frac{A_1 p^2 \left[pJ_1(p\rho)Y_o(\alpha\rho) - \alpha J_o(p\rho)Y_1(\alpha\rho) \right] + A_2 q^2 \left[qJ_1(q\rho)Y_o(\alpha\rho) - \alpha J_o(q\rho)Y_1(\alpha\rho) \right]}{\alpha^3 \left[J_1(\alpha\rho)Y_o(\alpha\rho) - J_o(\alpha\rho)Y_1(\alpha\rho) \right]}$$

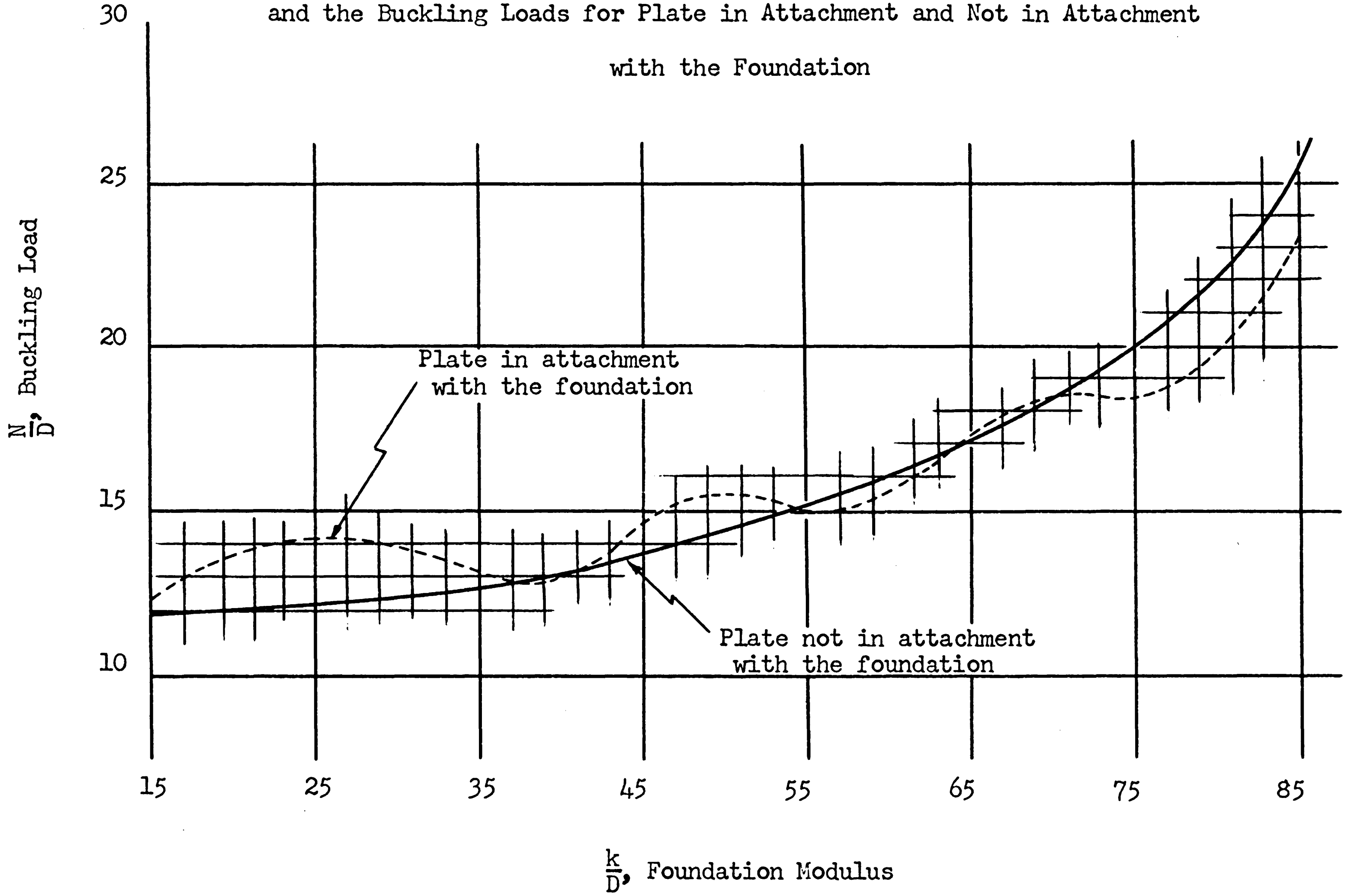
$$C_2 = \frac{A_1 p \rho J_1(p\rho) + A_2 q \rho J_1(q\rho) + C_1 \left[(\alpha a)J_1(\alpha a) - (\alpha\rho)J_1(\alpha\rho) \right]}{\alpha \left[\rho Y_1(\alpha\rho) - aY_1(\alpha a) \right]}$$

$$C_3 = C_1 \alpha a J_1(\alpha a) + C_2 \alpha a Y_1(\alpha a)$$

$$C_4 = - C_1 \left[J_o(\alpha a) + (\log a)(\alpha a) J_1(\alpha a) \right] - C_2 \left[Y_o(\alpha a) + (\log a)(\alpha a) Y_1(\alpha a) \right]$$

The relationship between the foundation modulus and the buckling loads, and the mode shape of the plate is shown in Fig. IV and Fig. V.

Fig. IV Curves Showing Relations Between Foundation Modulus and the Buckling Loads for Plate in Attachment and Not in Attachment with the Foundation



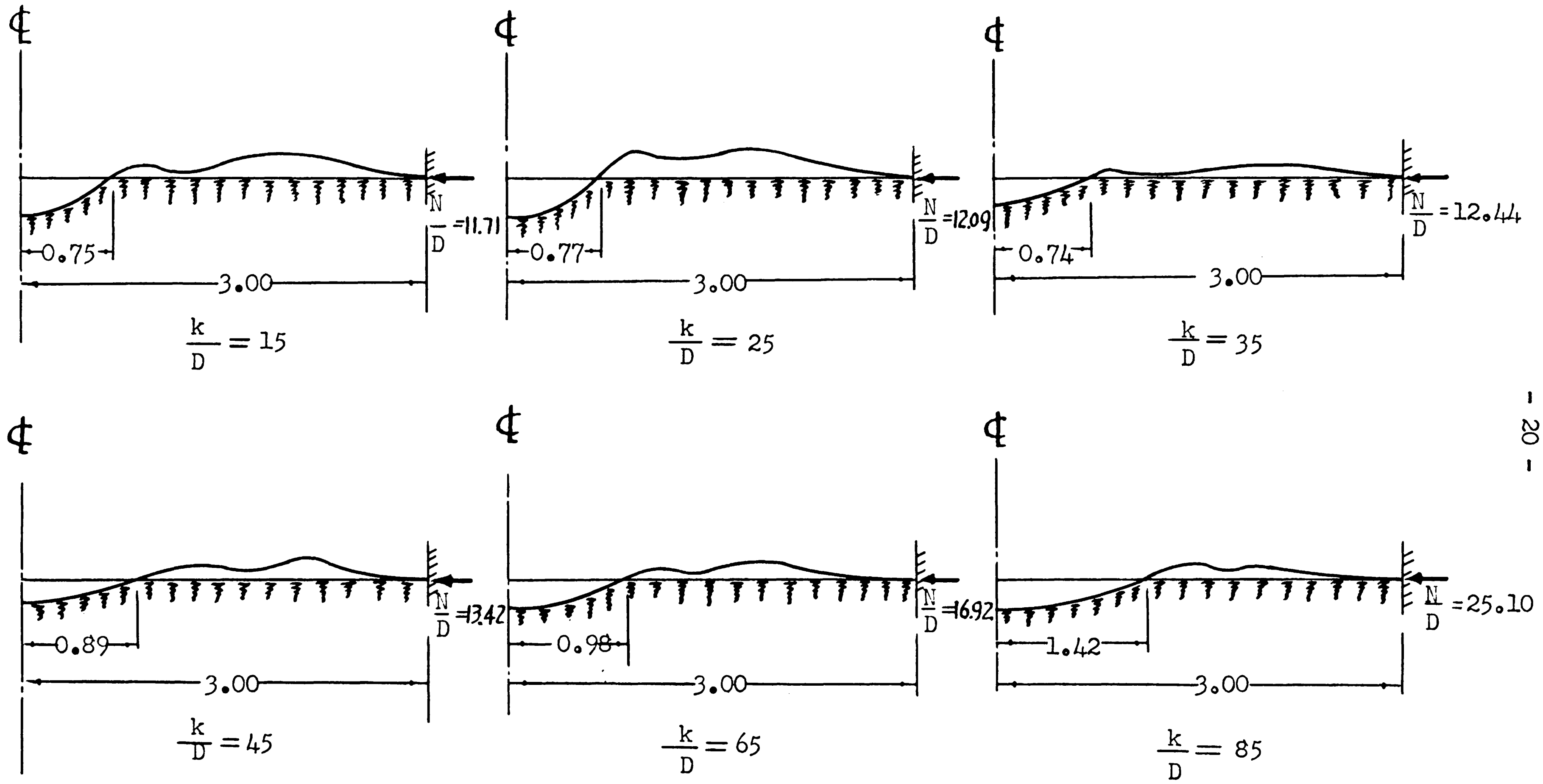


Fig. V Mode Shapes of the Plate

DISCUSSION AND CONCLUSION

In establishing the general solutions of equations (1) and (2), the author made the assumption that $N^2 - 4kD > 0$. As a result, all the characteristic equations become identities when $N^2 = 4kD$ or $p = q$. This indicates that for such a compressive force, no deflection occurs in the plate because for any value of ρ , where no deflection occurs, both characteristic equations are satisfied. Hence to make the plate buckle, it is necessary that N be larger than $\sqrt{4kD}$. In other words, the inequality, $N^2 - 4kD > 0$ must hold true in solving this problem.

By studying the results obtained in Chapter V, the author wishes to make the following conclusions:

(a) For the compressive buckling of a circular plate in the second mode shape when the plate is clamped on an elastic foundation not in attachment, the load and foundation modulus relation looks like an exponential curve; while for a long simply supported rectangular plate it is a logarithmic curve[6].

(b) If the plate is kept in attachment with the foundation, the load and foundation modulus relation has the same trend as that for the plate not in attachment with the foundation but it moves up and down behaving more like an ascending sine curve.

(c) Theoretically, the third mode shape or any mode larger than the third mode exists for certain values of N and k, but the problem is much more complicated. Hence modes higher than the second were not considered in this thesis.

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Abstract

A circular thin plate under radial compressive forces resting on an elastic foundation not in attachment was studied with regard to it's behavior in the 2nd mode shape.

Two regions of action are controlled by two differential equations of the fourth order which were solved in terms of the Bessel functions. The relations between the foundation modulus and the buckling load were found from two characteristic equations expressed in terms of Bessel functions.