

ASYMPTOTIC SOLUTIONS OF A CIRCULAR
PLATE PROBLEM

by

Donald C. Tolefson

Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

DOCTOR OF PHILOSOPHY

in

ENGINEERING MECHANICS

APPROVED:

Chairman, R. T. Davis

C. W. Smith

R. P. McNitt

L. McFadden

G. W. Swift

D. H. Pletta, Head of Department

July, 1967

Blacksburg, Virginia

TABLE OF CONTENTS

<u>Chapter</u>		<u>Page</u>
	LIST OF ILLUSTRATIONS.....	6
	NOMENCLATURE.....	7
I	INTRODUCTION AND REVIEW OF THE LITERATURE.....	9
II	FORMULATION OF THE PROBLEM.....	13
	A. Governing Equations.....	13
	1. Small finite deformations.....	17
	2. Large finite deformations.....	19
	B. Boundary Conditions.....	20
III	INFINITESIMAL DEFORMATIONS.....	22
	A. Determination of the Parameters.....	22
	B. Perturbation Expansions.....	23
	1. Boundary conditions.....	26
	2. Solutions.....	26
	C. Stresses, displacements and other physical quantities.....	32
IV	SMALL FINITE DEFORMATIONS.....	35
	A. Determination of the Parameters.....	36
	B. Membrane Expansions.....	37
	1. Membrane expansions for ν less than $1/3$	42
	2. Membrane expansions for ν greater than $1/3$	46
	3. Membrane expansions for ν equal to $1/3$	48
	C. Boundary-Layer Expansions.....	50
	1. Boundary-layer expansions for the vicinity of the outer edge of the plate.....	50
	a. Boundary-layer coordinate system and equations.....	50
	b. Boundary Conditions.....	55
	c. Solutions to the boundary-layer equations...	56
	2. Boundary-layer expansions for the vicinity of the inner edge of the plate.....	58
	a. Boundary-layer coordinate system and equations.....	58

<u>Chapter</u>	<u>Page</u>
b. Boundary conditions.....	61
c. Solutions to the boundary-layer equations.....	61
D. Matching.....	63
1. Poisson's ratio equal to $1/3$	65
a. Matching conditions for the variable F.....	65
b. Matching conditions for the radial displacement U.....	69
c. Solutions for the constants of integration.....	74
2. Poisson's ratio less than $1/3$	76
a. Matching conditions for the variable F.....	76
b. Matching conditions for the radial displacement U.....	81
3. Poisson's ratio greater than $1/3$	87
a. Matching conditions for the variable F.....	87
b. Matching conditions for the radial displacement U.....	88
E. Summary.....	89
1. Poisson's ratio less than $1/3$	90
a. Membrane expansions.....	90
b. Boundary-layer expansions.....	90
2. Poisson's ratio greater than $1/3$	92
a. Membrane expansions.....	92
b. Boundary-layer expansions.....	92
3. Poisson's ratio equal to $1/3$	92
a. Membrane expansions.....	93
b. Boundary-layer expansions.....	93
c. Composite Expansions.....	94
d. Stresses and displacements.....	97
V LARGE FINITE DEFORMATIONS.....	101
A. Determination of the Parameters.....	101
B. Membrane Expansions.....	102
C. Boundary-Layer Expansions.....	103

<u>Chapter</u>	<u>Page</u>
1. Boundary-layer expansions for the vicinity of the outer edge of the plate.....	103
a. Boundary-layer coordinate system and equations.....	104
b. Boundary conditions.....	104
c. Solutions to the boundary-layer equations.....	105
2. Boundary-layer expansions for the vicinity of the inner edge of the plate.....	106
a. Boundary-layer coordinate system and equations.....	106
b. Boundary conditions.....	107
c. Solutions to the boundary-layer equations..	107
D. Matching.....	108
1. Matching conditions for the variable ψ	108
2. Matching conditions for the variable β	109
3. Matching conditions for the radial displacement	111
E. Summary for the Numerical Integration Process.....	114
1. Membrane expansions.....	114
2. Boundary-layer expansions.....	115
a. Boundary-layer expansions for the vicinity of the outer edge of the plate.....	115
b. Boundary-layer expansions for the vicinity of the inner edge of the plate.....	116
VI DISCUSSION OF RESULTS AND CONCLUSIONS.....	117
VII ACKNOWLEDGMENTS.....	128
VIII BIBLIOGRAPHY.....	129
IX VITA.....	130
APPENDIX.....	A.1
A.1 Evaluation of the Constants of Integration for the Case when Poisson's Ratio is less than 1/3.....	A.1

Chapter

Page

A.2	Expressions for the Stresses and Displacements.....	A.6
a.	Direct stresses.....	A.6
b.	Bending stresses.....	A.9
c.	Displacements.....	A.10

LIST OF ILLUSTRATIONS

<u>Figure</u>		<u>Page</u>
(1)	Plate element in the undeformed and deformed position.....	15
(2)	Circular plate with central rigid inclusion.....	15
(3)	Maximum stress at the inner edge of the plate.....	120
(4)	Lateral displacement profile.....	121
(5)	Transverse displacement at the inner edge of the plate for $b/a = .2$	122
(6)	Transverse displacement at the inner edge of the plate for $b/a = .4$	123
(7)	Transverse displacement at the inner edge of the plate for $b/a = .6$	124
(8)	Transverse displacement at the inner edge of the plate for $b/a = .8$	125
(9)	Bending and direct stresses at the inner edge of the plate for $b/a = .2$	126
(10)	Bending and direct stresses at the inner edge of the plate for $b/a = .8$	127
(A.1)	Curves for determining corresponding values of b/a and γ_b	A.12
(A.2)	Curves for determining corresponding values of b/a and γ_a	A.13
(A.3)	Curves illustrating the tangency condition at $b/a = 1.0$	A.14
(A.4)	Maximum value of γ_a as a function of Poisson's ratio.....	A.15

NOMENCLATURE

\bar{r}	radial coordinate from the center of the plate
$\bar{\beta}$	rotation of the middle surface of the plate
$\bar{\psi}$	stress function
a	outer radius of the plate
b	inner radius of the plate
h	plate thickness
E	modulus of Elasticity
ν	Poisson's ratio
D	flexural rigidity of the plate ($Eh^3/12(1-\nu^2)$)
P	applied load
V	vertical stress resultant
H	Horizontal stress resultant
$B, \delta, \lambda, L, \epsilon, \tau$	parametric quantities
β, ψ, F, α	transformed dependent variables
γ	parameter related to F in the case of ν less than 1/3
$\bar{\delta}$	parameter related to F in the case of ν greater than 1/3
r, ρ	transformed independent variables
η	boundary-layer coordinate in the vicinity of the outer edge of the plate
ζ	boundary-layer coordinate in the vicinity of the inner edge of the plate
N_θ, N_r	circumferential and radial stress resultants
M_θ, M_r	circumferential and radial moment resultants
$\sigma_{\theta D}, \sigma_{rD}$	circumferential and radial direct stresses respectively
$\sigma_{\theta B}, \sigma_{rB}$	circumferential and radial bending stresses respectively

U, W radial and transverse displacements respectively
~ tilde, placed over a variable to indicate that
 the variable applies in the vicinity of the
 outer edge of the plate
= double bar, placed over a variable to indicate
 that the variable applies in the vicinity of
 the inner edge of the plate

Plus miscellaneous symbols defined in the text.

CHAPTER I

INTRODUCTION AND REVIEW OF THE LITERATURE

We consider the problem of the rotationally symmetric deformations of a thin circular plate with a central rigid inclusion of finite radius. The plate is deformed due to a load which acts normal to the plane of the rigid inclusion at its center, and the outer edge of the plate is considered to be clamped such that it undergoes no rotation or displacement. The plate is initially flat, of constant thickness and consists of a uniform elastic material. We assume that the strains in the plate are small but allow for the possibility of finite rotations and displacements.

With regard to the mathematical formulation of the problem we begin with the basic differential equations due to E. Reissner in the form as presented in (1). In a more recent paper (2) Reissner presents a new theory where certain terms in his original equations, as given in (1), are considered to be insignificant and are omitted. Herein we determine from an order of magnitude analysis that these terms are important if the deformations are large, and furthermore, it is observed that without these terms in the basic differential equations certain inconsistencies arise in the problem under consideration (see chapter V).

The first approximations to the Reissner equations (1) are derived by an order of magnitude analysis in the separate cases of small and large finite deformations. We find, as is well known, that the first approximations in the case of small finite deformations correspond to the von Karman equations (3).

It is the purpose of this dissertation to solve the von Karman equations for the limiting cases of deformation which exist within their range of validity and, in addition, to investigate the case of large finite deformations when the deformations are exceedingly large.

Solutions to the von Karman equations in the form of perturbation expansions in powers of a small parameter are obtained in the limiting cases of infinitesimal and increasingly large deformations. With regard to the case of infinitesimal deformations the classical linearized bending theory is improved upon herein by accounting for the coupling between the bending and stretching of the plate's middle surface. Two terms in the expansion for the dependent variable representing the bending of the plate are obtained; the second of which gives a correction due to the stretching of the middle surface of the plate. As long as the maximum deflection of the plate does not exceed approximately one-half of its thickness, the correction for the stretching of the middle surface is negligible for this problem. The expansion for the dependent variable which represents the stretching of the plate's middle surface is found to one term. Both expansions are uniformly valid over the extent of the plate.

For significant increases in deformation above the range just discussed the stretching and bending effects are equally important although the latter are confined to narrow boundary-layer zones at the edges of the plate. In this case the analysis of the von Karman equations by perturbation methods leads to a singular perturbation problem in that the interior region of the plate and its edge zones must be investigated individually. Perturbation expansions are obtained which apply separately in these regions of the plate.

Solutions for the first two and, in one case, the first three terms in the expansions of the dependent variables are obtained in closed form. It is found that the expansions which apply in the interior region of the plate, which we refer to as membrane expansions, take one of three possible forms depending on whether Poisson's ratio ν is less than, greater than, or equal to $1/3$. In the case when ν is equal to $1/3$ the algebraic complexities are greatly reduced; in this case we obtain three terms of the expansions. The theory of "Matched Asymptotic Expansions", as popularized by Van Dyke (4) in fluid

mechanics problems, is utilized here in order to determine the necessary conditions, in addition to those provided at the boundaries of the plate, for evaluating the constants of integration in the solution.

Numerical results for the transverse displacement and maximum radial stress are compared to data obtained from the numerical integration of Reissner's equations as presented by other authors (5); and the agreement is very good.

The work most closely related to the above phase of the investigation is that recently conducted by Hart and Evans (5) who investigated the same physical problem as we are concerned with in the case of large deformations. They also attacked the problem by a perturbation analysis of the von Karman equations and obtained a power series solution for the first terms in the perturbation expansions of the dependent variables. However, as evidenced in their report, as well as herein, one term expansions are not adequate over a large range of deformations. A major part of their work is concerned with the numerical integration of the Reissner equations and we draw upon their numerical data to verify our results, as mentioned above.

The only other known work dealing with the same geometry and loading as in this investigation is due to Hamada and Seguchi (6) who solve the von Karman equations by a numerical iteration procedure.

It is appropriate to note that Schwerin (7), in his analysis of a centrally loaded circular membrane, encountered the same basic differential equation as that which governs the first term of one of the membrane expansions herein. His closed form solution, which we draw upon, failed to exist for values of Poisson's ratio ν greater than $1/3$. Later Jahsman, Field and Holmes (8) observed that a certain variable in Schwerin's solution need not be real to yield real solutions. This led to the solution for ν greater than $1/3$. However, it has not been reported that both forms of these solutions become indeterminate when ν is set equal to $1/3$ and, more importantly, it has not been shown that a third form of solution exists, which is extremely

less complicated than the other two, in the case of ν equal to $1/3$. This third form of solution which we present is important due to its simplicity along with the fact that for many engineering materials ν is approximately equal to $1/3$.

The concluding part of this investigation deals with the problem of exceedingly large deformations such that the von Karman equations are no longer valid. As indicated by Reissner (2) this occurs when sine and cosine terms in the variable $\bar{\beta}$, which represents the rotation of the middle surface of the plate, can no longer be approximated by neglecting terms of third order and higher in their series expansions.

Our aim is to formulate the first approximations to Reissner's equations in the case of large rotations and then derive the differential equations on the first terms of perturbation expansions of the dependent variables. A singular perturbation problem arises again as expected. The differential equations of the boundary-layer zones and the interior region are given, as well as the conditions, above those provided at the boundaries, which are required for evaluating the constants of integration. These are obtained by employing the theory of "Matched Asymptotic Expansions". The solution of this problem requires numerical integration since an analytical solution can not be obtained. For this reason results are not presented in this case.

CHAPTER II

FORMULATION OF THE PROBLEM

Starting with the "full" Reissner equations an order of magnitude analysis is performed in this chapter to determine the first approximations to these equations for the separate cases of small finite and large finite deformations. The resulting equations contain two arbitrary parameters which are introduced at the beginning of the order of magnitude analysis. One of these parameters remains unspecified in magnitude and is used as a perturbation parameter to obtain solutions to the equations of the first approximations in terms of perturbation expansions. Thus we obtain solutions for the limiting cases of deformation that exist within the frameworks of the theories for small finite and large finite deformations. In this chapter we limit ourselves to the derivation of the basic differential equations for the first approximations and the formulation of the boundary conditions.

A. Governing Equations

E. Reissner's equations (1) for finite deformations of axially symmetric thin plates are,

$$\frac{d^2\bar{\beta}}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{d\bar{\beta}}{d\bar{r}} - \frac{\cos\bar{\beta}\sin\bar{\beta}}{\bar{r}^2} - \frac{\bar{\psi}\sin\bar{\beta}}{D\bar{r}} = - \frac{V\cos\bar{\beta}}{D} \quad (2.1a)$$

and

$$\frac{d^2\bar{\psi}}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{d\bar{\psi}}{d\bar{r}} - \frac{1}{\bar{r}^2} (\cos^2\bar{\beta} - \nu\bar{r} \frac{d\bar{\beta}}{d\bar{r}} \sin\bar{\beta}) \bar{\psi} + \frac{Eh}{\bar{r}} (1-\cos\bar{\beta}) =$$

$$\left(\frac{\sin\bar{\beta}\cos\bar{\beta}}{\bar{r}^2} + \frac{\nu}{\bar{r}} \frac{d\bar{\beta}}{d\bar{r}} \cos\bar{\beta} \right) (V\bar{r}). \quad (2.1b)$$

In these equations $\bar{\beta}$ represents the rotation of the middle surface during the deformation, $\bar{\psi}$ is a stress function defined as the product of the radial coordinate \bar{r} and the horizontal stress resultant H, and the quantities E, h and ν stand for the modulus of Elasticity, the plate thickness and Poisson's ratio respectively (see Figure 1). V is the vertical stress resultant. For this problem, (see Figure 2), it follows from Statics that,

$$V = \frac{P}{2\pi r}, \quad (2.2)$$

where P is the load applied to the inclusion. The quantity D is defined to be,

$$D = \frac{Eh^3}{12(1-\nu^2)}. \quad (2.3)$$

Once $\bar{\beta}$ and $\bar{\psi}$ are known the stresses and displacements are determinable by direct calculation using the expressions in section A.2 of the appendix. Figure (1) showing the element in the undeformed and deformed positions, contains certain of the above quantities, and others for later reference. Figure (2) shows the plate and rigid inclusion. The letters a and b stand for the outer and inner radii of the plate.

Equations (2.1) are valid for all ranges of deformation as long as the strains are small compared to unity. We consider these equations for the cases of small finite and large finite deformations under the assumption of small strains. The classical case of infinitesimal deformations is treated as a limiting case of the small finite deformation theory.

First non-dimensionalize the equations by setting,

$$\bar{\psi} = B\psi \quad (2.4)$$

PLATE ELEMENT IN THE UNDEFORMED AND DEFORMED POSITION

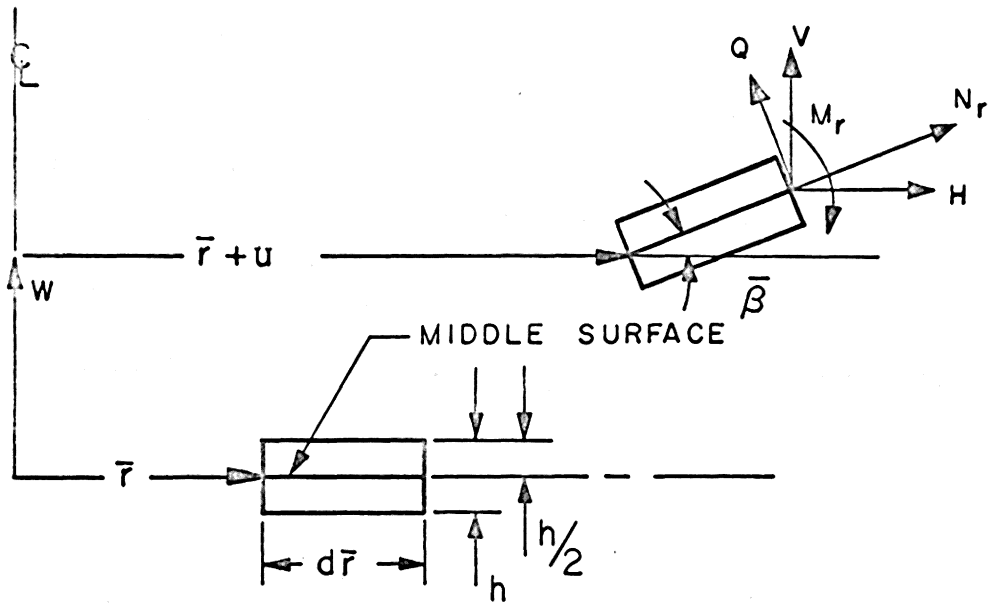


Figure (1).

CIRCULAR PLATE WITH CENTRAL RIGID INCLUSION

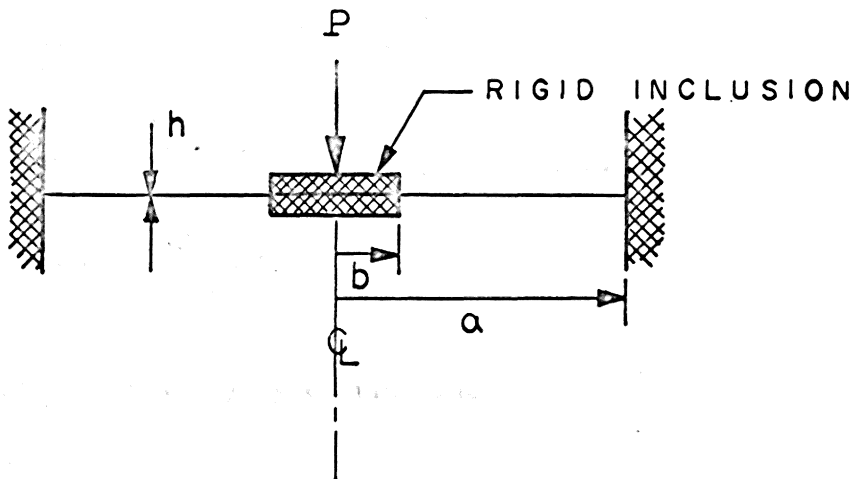


Figure (2)

$$\bar{\beta} = \delta\beta \quad (2.5)$$

and

$$\bar{r} = ar \quad (2.6)$$

where ψ , β and r are new dimensionless variables. B and δ are parametric quantities to be determined in the course of the analysis and a is the outer radius of the plate. Substituting (2.2), (2.4), (2.5) and (2.6) into (2.1) gives us,

$$\frac{d^2\beta}{dr^2} + \frac{1}{r} \frac{d\beta}{dr} - \frac{\cos(\delta\beta)\sin(\delta\beta)}{\delta r^2} - \frac{\lambda^2 \psi \sin(\delta\beta)}{\delta r} = - \frac{Pa \cos(\delta\beta)}{2\pi D \delta r} \quad (2.7a)$$

and

$$\begin{aligned} \frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{1}{r^2} (\cos^2(\delta\beta) - \nu\delta r \frac{d\beta}{dr} \sin(\delta\beta)) \psi + \frac{L^2}{r} (1 - \cos\delta\beta) = \\ \left(\frac{\sin(\delta\beta)\cos(\delta\beta)}{r^2} + \frac{\nu\delta}{r} \frac{d\beta}{dr} \cos(\delta\beta) \right) \frac{Pa}{2\pi D \lambda^2} \end{aligned} \quad (2.7b)$$

in which we have let,

$$\lambda^2 = \frac{Ba}{D} \quad \text{and} \quad L^2 = \frac{aEh}{B} \quad (2.8a, b)$$

λ^2 and L^2 are related through the parameter (a/h) by,

$$\lambda^2 = (a/h)^2 \frac{12(1-\nu^2)}{L^2}, \quad (2.9)$$

which follows from (2.8) by eliminating B .

1. Small finite deformations

Let us assume the parameters B and δ are chosen such that ψ and β are of order unity. Using the symbols $f \ll 1$ and $f \gg 1$, or equivalently the terms "small" and "large", to imply that the function f is approaching zero or infinity respectively let us examine first the case where δ is small i.e. $\bar{\beta} \ll 1$. Replacing the trigonometric terms in (2.7b) by their series expansions and neglecting terms of order δ and higher, the first approximation to (2.7b) becomes,

$$\frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{\psi}{r^2} + \frac{L^2}{r} \left(\frac{\delta^2 \beta^2}{2} \right) = 0 \quad (2.10)$$

as long as we assume that either

$$\frac{Pa}{2\pi D \lambda^2} \ll 1 \quad (2.11a)$$

or

$$\frac{Pa}{2\pi D \lambda^2} = 0(1).^* \quad (2.11b)$$

The last term in (2.10) is retained for further examination since the order of magnitude of L^2 has not been specified. Following the classical procedure we look for the order of magnitude for L^2 which leads to the least degenerate first approximation to (2.7b). This occurs when the quantity $\delta^2 L^2$ is of order unity since β^2 will then remain in the first approximation. Therefore, let us set $\delta^2 L^2$ equal to some

*A function $f(\delta)$ is said to be of order $g(\delta)$ as δ tends to zero if $0 < \lim |f/g| < \infty$ as δ tends to zero. Symbolically we write $f = 0(g)$. In this definition we assume that the limit of $|f/g|$ exists.

constant, preferably unity for convenience. Thus we assume,

$$\delta^2 L^2 = 1. \quad (2.12)$$

or,

$$L = 1/\delta. \quad (2.13)$$

Equation (2.13) shows that L^2 is large when δ^2 is small. Therefore, in this part of the analysis we assume

$$L^2 \gg 1. \quad (2.14)$$

In view of (2.12) the first approximation to (2.7b) becomes,

$$\frac{d^2 \psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{\psi}{r^2} + \frac{\beta^2}{2r} = 0. \quad (2.15)$$

It remains to investigate (2.7a). By replacing the trigonometric functions with their series expansions and neglecting higher order terms the first approximation to (2.7a) for small δ becomes,

$$\frac{d^2 \beta}{dr^2} + \frac{1}{r} \frac{d\beta}{dr} - \frac{\beta}{r^2} - \frac{\lambda^2 \psi \beta}{r} = - \frac{Pa}{2\pi D \delta r}. \quad (2.16)$$

The last two terms in this equation are retained for further examination.

Equations (2.15) and (2.16) represent the first approximations to Reissner's equations for the case of small finite deformations. The two arbitrary parameters B and δ , or equivalently λ^2 and δ , remain to be determined. One of these, δ , has been restricted to be small, but the other one, λ^2 , is completely arbitrary in magnitude. Thus the term in (2.16) containing λ^2 may or may not be significant. Clearly when λ^2

is of order unity the term under discussion is significant. The determination of the parameters δ and λ^2 is postponed to chapters III and IV where (2.15) and (2.16) are solved in terms of perturbation expansions. Let us note that these equations correspond to the von Karman equations (3) as can easily be shown by converting them to the variables $\bar{\psi}$, $\bar{\beta}$, and \bar{r} and using (2.8).

2. Large finite deformations

For this case we assume $\bar{\beta}$ is of order unity. Therefore in (2.5) let us set

$$\delta = 1. \quad (2.17)$$

The basic differential equations (2.7) now become,

$$\frac{d^2\beta}{dr^2} + \frac{1}{r} \frac{d\beta}{dr} - \frac{\cos\beta\sin\beta}{r^2} - \frac{\lambda^2\psi\sin\beta}{r} = \frac{Pa \cos\beta}{2\pi D r}, \quad (2.18a)$$

and,

$$\begin{aligned} \frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{1}{r^2} (\cos^2\beta - \nu r \frac{d\beta}{dr} \sin\beta) \psi + \frac{L^2}{r} (1-\cos\beta) = \\ \left(\frac{\sin\beta\cos\beta}{r^2} + \frac{\nu}{r} \frac{d\beta}{dr} \cos\beta \right) \frac{Pa}{2\pi D \lambda^2}. \end{aligned} \quad (2.18b)$$

Assuming B is chosen such that ψ is of order unity, a plausible first approximation to (2.18a) occurs only if we assume that λ^2 and the quantity $Pa/2\pi D$ simultaneously grow indefinitely large such that their ratio is of order unity. In order to guarantee that this assumption holds we set

$$\lambda^2 = \frac{Pa}{2\pi D}. \quad (2.19)$$

Substituting (2.19) into (2.18) it follows that,

$$\frac{d^2\beta}{dr^2} + \frac{1}{r} \frac{d\beta}{dr} - \frac{\cos\beta\sin\beta}{r^2} - \frac{\lambda^2\psi\sin\beta}{r} = - \frac{\lambda^2\cos\beta}{r} \quad (2.20a)$$

and

$$\begin{aligned} \frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{1}{r^2} (\cos^2\beta - \nu r \frac{d\beta}{dr} \sin\beta)\psi + \frac{L^2}{r}(1-\cos\beta) = \\ \frac{\sin\beta\cos\beta}{r^2} + \frac{\nu}{r} \frac{d\beta}{dr} \cos\beta \end{aligned} \quad (2.20b)$$

Equations (2.20) represent the first approximations to (2.7) for the case when λ^2 is large. Further examination of these equations is delayed to chapter V.

B. Boundary Conditions

At the inner and outer edges of the plate the slope, $\bar{\beta}$, and radial displacement, denoted by U, must vanish (see Figure 2). In view of (2.5) and (2.6) the boundary condition on β becomes

$$\beta = 0 \quad \text{at } r = b/a \quad \text{and } r = 1. \quad (2.21)$$

From (A.45) of section A.2 of the appendix, the radial displacement in terms of $\bar{\beta}$, $\bar{\psi}$ and \bar{r} is,

$$U = \frac{\bar{r}}{Eh} \left(\frac{d\bar{\psi}}{d\bar{r}} - \nu \left(\frac{P}{2\pi\bar{r}} \sin\bar{\beta} + \frac{\bar{\psi}}{\bar{r}} \cos\bar{\beta} \right) \right). \quad (2.22)$$

Using (2.4), (2.5) and (2.6) gives us,

$$U = \frac{rB}{Eh} \left(\frac{d\psi}{dr} - \frac{\nu}{r} \left(\frac{P \sin \delta \beta}{2\pi B} + \psi \cos \delta \beta \right) \right). \quad (2.23)$$

The right hand side of (2.23) must vanish at $r = b/a$ and $r = 1$.

In view of (2.21) the boundary condition on U becomes,

$$\frac{d\psi}{dr} - \frac{\nu\psi}{r} = 0, \quad (2.24)$$

at $r = b/a$ and $r = 1$. By virtue of (2.21) it is noted that the values of the trigonometric terms are independent of δ at the boundaries of the plate, and therefore (2.24) applies for the cases of small finite and large finite deformations.

CHAPTER III
INFINITESIMAL DEFORMATIONS

The limiting case of small finite deformations, where the deformations are considered to be infinitesimal, encompasses the classical linearized bending theory of plates. The classical theory is based on the assumption that the deformations are so small that the stretching of the middle surface of the plate is negligible and that only bending effects are important. In terms of the governing equations of the previous chapter this is equivalent to discarding (2.15) and assuming that λ^2 in (2.16) is small such that the term containing λ^2 may be neglected.

In this chapter we investigate the case of infinitesimal deformations by finding perturbation type solutions to (2.15) and (2.16) for small λ^2 . By considering both equations, i.e. (2.15) as well as (2.16), the stretching of the middle surface of the plate, although small, is accounted for. Two terms in the expansion for β are obtained, the second of which accounts for the coupling of the stretching and bending effects. The expansion for ψ is found to one term. Higher order terms in the expansions follow directly but the algebra becomes exceedingly lengthy.

A. Determination of the Parameters

Thus far the restrictions on the parameters for the small finite deformation theory are that $\delta \ll 1$, $L^2 \gg 1$ and that either (2.11a) or (2.11b) hold true. Also L and δ are related according to (2.13). In view of (2.9) we observe that one can have either,

$$\lambda^2 \ll 1 \qquad (a/h)^2 = 0(1) \qquad (3.1a)$$

or

$$\lambda^2 = 0(1) \qquad (a/h)^2 = 0(L^2) \qquad (3.1b)$$

or

$$\lambda^2 \gg 1 \qquad a/h = 0(L^2) \qquad (3.1c)$$

for $L^2 \gg 1$. As just mentioned the case for infinitesimal deformations corresponds to $\lambda^2 \ll 1$.

In order for the load P to appear in the first approximation to (2.16) for small λ^2 we must have the quantity $(Pa/2\pi D\delta)$ of order unity. Therefore let us set this quantity equal to some constant which we choose to be 4 for later convenience. Thus we set,

$$\frac{Pa}{2\pi D\delta} = 4. \quad (3.2)$$

From (3.2) it follows that,

$$\delta = \frac{Pa}{8\pi D}, \quad (3.3)$$

and upon substituting (2.3) into (3.3) we get,

$$\delta = \frac{P(a/h)^2 12(1-\nu^2)}{8\pi aEh}. \quad (3.4)$$

From (3.4), (2.13), (2.9) and (2.8) we determine that,

$$L^2 = \left(\frac{8\pi aEh}{P(a/h)^2 12(1-\nu^2)} \right)^2, \quad (3.5)$$

and,

$$\lambda^2 = \frac{P^2(a/h)^6 (12(1-\nu^2))^3}{(8\pi aEh)^2}, \quad (3.6)$$

$$B = \left(\frac{P(a/h)^2 12(1-\nu^2)}{8\pi} \right)^2 \frac{1}{aEh}. \quad (3.7)$$

Hence the parameters for the infinitesimal deformation theory are known.

B. Perturbation Expansions

Substituting (3.2) into (2.16) and rewriting (2.15) gives us the following basic differential equations for the infinitesimal deformation theory:

$$\frac{d^2\beta}{dr^2} + \frac{1}{r} \frac{d\beta}{dr} - \frac{\beta}{r^2} - \frac{\lambda^2\psi\beta}{r} = -\frac{4}{r}, \quad (3.8a)$$

and,

$$\frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{\psi}{r^2} + \frac{\beta^2}{2r} = 0, \quad (3.8b)$$

where the parameter λ^2 in (3.8a) is considered to be small. These equations can be simplified by the following transformations which were used by Schwerin (7) in a membrane analysis of a similar problem. Letting ρ , F and α be new variables we set,

$$r^2 = \rho, \quad (3.9)$$

$$\psi = \frac{F(r^2)}{r}, \quad (3.10),$$

and,

$$\beta = \frac{\alpha(r^2)}{r}. \quad (3.11)$$

where F and α are considered to be functions of r^2 . Substituting (3.9), (3.10) and (3.11) into (3.8) gives us the following equations where primes indicate differentiation with respect to ρ :

$$\alpha'' - \frac{\lambda^2 F \alpha}{4\rho^2} = \frac{-1}{\rho}, \quad (3.12a)$$

and,

$$F'' + \frac{\alpha^2}{8\rho^2} = 0. \quad (3.12b)$$

We now consider (3.12) as the basic differential equations. The relations between the original dependent variables $\bar{\beta}$ and $\bar{\psi}$ and the new dependent variables F and α are,

$$\bar{\beta} = \frac{a\delta\alpha}{\bar{r}} \quad (3.13)$$

and

$$\bar{\psi} = \frac{BaF}{\bar{r}}, \quad (3.14)$$

which follow by combining (2.4) through (2.6) with (3.10) and (3.11).

Following the usual procedure we assume that the variables α and F can be expanded in the form,

$$\alpha = \alpha_0 + \alpha_1 \epsilon + \alpha_2 \epsilon^2 + \dots, \quad (3.15a)$$

and,

$$F = F_0 + F_1 \epsilon + F_2 \epsilon^2 + \dots, \quad (3.15b)$$

where ϵ is a small parameter and the coefficients of ϵ and its powers are of order unity and independent of ϵ . We assume that the expansions become asymptotic to the exact solution as ϵ tends to zero. Let us assume,

$$\epsilon = \lambda^2. \quad (3.16)$$

Substituting (3.15) and (3.16) into (3.12) and equating the coefficients of successive powers of ϵ to zero results in the following sets of equations:

Zeroth order equations,

$$\alpha_0'' = -\frac{1}{\rho}, \quad (3.17a)$$

and

$$F_0'' + \frac{\alpha_0^2}{8\rho^2} = 0. \quad (3.17b)$$

First order equations,

$$\alpha_1'' - \frac{F_0 \alpha_0}{4\rho^2} = 0, \quad (3.18a)$$

and

$$F_1'' + \frac{\alpha_0 \alpha_1}{4\rho^2} = 0. \quad (3.18b)$$

Before solving these equations we discuss the boundary conditions as it is most expedient to apply them during the integration process.

1. Boundary conditions

From (2.21), (3.11) and (3.15a) the condition of zero edge rotation is met by setting,

$$\alpha_0 = \alpha_1 = 0 , \quad (3.19a,b)$$

at $\rho = (b/a)^2$ and $\rho = 1$. The condition for zero radial displacement at the edges is given by (2.24). Substituting (3.9) and (3.10) into (2.24) gives us,

$$F' - \left(\frac{1+\nu}{2}\right) \frac{F}{\rho} = 0 . \quad (3.20)$$

Inserting (3.15b) into (3.20) gives us the following conditions which must hold at $\rho = (b/a)^2$ and $\rho = 1$:

$$F'_0 - \left(\frac{1+\nu}{2}\right) \frac{F_0}{\rho} = 0 , \quad (3.21a)$$

and,

$$F'_1 - \left(\frac{1+\nu}{2}\right) \frac{F_1}{\rho} = 0 . \quad (3.21b)$$

2. Solutions

Here we solve (3.17) and (3.18a) to give us the first two terms in (3.15a) and the first term of (3.15b). Higher order terms in both (3.15a) and (3.15b) could be obtained but algebraic difficulties make it unattractive to do so. Our primary interest is to determine the amount of non-linearity in a plot of the transverse displacement W versus the applied load P , and since this non-linear effect is primarily accounted for by the second term of (3.15a) we do not need to proceed any further in the analysis.

Using the method of quadratures the solution to (3.17a) is easily found to be,

$$\alpha_0 = -\rho \ln \rho + \rho + C_1 \rho + C_2 , \quad (3.22)$$

where C_1 and C_2 are constants of integration. Applying (3.19a) at

$\rho = (b/a)^2$ and $\rho = 1$ we determine that,

$$C_2 = - (C_1 + 1) , \quad (3.23a)$$

and,

$$C_2 = (b/a)^2 \ln(b/a)^2 - (b/a)^2 (C_1 + 1). \quad (3.23b)$$

Solving the above two equations simultaneously gives us,

$$C_1 = \frac{1 + (b/a)^2 (\ln(b/a)^2 - 1)}{(b/a)^2 - 1} , \quad (3.24a)$$

and,

$$C_2 = \frac{(b/a)^2 \ln(b/a)^2}{1 - (b/a)^2} . \quad (3.24b)$$

To solve (3.17b) let us first make use of (3.23a) and express α_0 in the following form:

$$\alpha_0 = C_2(1 - \rho) - \rho \ln \rho . \quad (3.25)$$

Substituting (3.25) into (3.17b) and integrating by the method of quadratures we find that,

$$\begin{aligned} F_0 = & C_4 + C_3 \rho + k_1 \rho + k_2 \rho^2 + k_3 \ln \rho + k_4 \rho \ln \rho + k_5 \rho^2 \ln \rho + \\ & k_6 \rho \ln^2 \rho + k_7 \rho^2 \ln^2 \rho , \end{aligned} \quad (3.26)$$

where C_3 and C_4 are constants of integration and,

$$k_1 = (C_2 - C_2^2)/4, \quad (3.27a)$$

$$k_2 = (6C_2 - 2C_2^2 - 7)/32, \quad (3.27b)$$

$$k_3 = C_2^2/8, \quad (3.27c)$$

$$k_4 = (C_2^2 - C_2)/4, \quad (3.27d)$$

$$k_5 = (-2C_2 + 3)/16, \quad (3.27e)$$

$$k_6 = C_2/8, \quad (3.27f)$$

and,

$$k_7 = -1/16. \quad (3.27g)$$

Applying (3.21a) at $\rho = (b/a)^2$ and $\rho = 1$ we determine that,

$$C_3 = \left(\frac{2}{1-\nu}\right) \frac{k_{11} - (a/b)^2 k_8}{1 - (a/b)^2}, \quad (3.28a)$$

and,

$$C_4 = \left(\frac{2}{1+\nu}\right) \frac{k_{11} - k_8}{1 - (a/b)^2}, \quad (3.28b)$$

where,

$$k_8 = (1 - C_2)/4 + (1 + \nu) (-10C_2^2 + 14C_2 - 7)/64, \quad (3.29a)$$

and,

$$k_{11} = \left(\frac{1 + \nu}{2}\right) (a/b)^2 k_{10} - k_9, \quad (3.29b)$$

in which,

$$k_9 = (b/a)^2 (-C_2^2 + 2C_2 - 2)/8 + (a/b)^2 C_2^2/8 + (C_2^2/4) \ln(b/a)^2 +$$

$$(b/a)^2 \ln(b/a)^2 (1 - C_2)/4 + (C_2/8) \ln^2(b/a)^2 -$$

$$((b/a)^2 \ln^2(b/a)^2)/8, \quad (3.29c)$$

and,

$$k_{10} = (b/a)^2 (C_2 - C_2^2)/4 + (b/a)^4 (-2C_2^2 + 6C_2 - 7)/32 +$$

$$(C_2^2/8) \ln(b/a)^2 + ((b/a)^2 (C_2^2 - C_2)/4) \ln(b/a)^2 +$$

$$((b/a)^4 (-2C_2 + 3)/16) \ln(b/a)^2 + (b/a)^2 (C_2/8) \ln^2(b/a)^2 -$$

$$((b/a)^4/16) \ln^2(b/a)^2. \quad (3.29d)$$

This completes the solution for the zeroth order equations (3.17).

The first order term α_1 is determined in a similar fashion. Substituting (3.25) and (3.26) into (3.18a) and integrating by the method of quadratures again gives us,

$$\begin{aligned} \alpha_1 = & C_6 + C_5\rho + k_{35}\rho + k_{36}\rho^2 + k_{37}\rho^3 + k_{25}\ln\rho + k_{38}\rho\ln\rho + k_{39}\rho^2\ln\rho + \\ & k_{40}\rho^3\ln\rho + k_{41}\rho\ln^2\rho + k_{42}\rho^2\ln^2\rho + k_{43}\rho^3\ln^2\rho + k_{44}\rho^3\ln^3\rho + \\ & k_{45}\rho^2\ln^3\rho + k_{29}\rho\ln^3\rho + k_{46}\ln^2\rho, \end{aligned} \quad (3.30)$$

where C_5 and C_6 are constants of integration and,

$$k_{46} = -k_{16}/2, \quad (3.31a)$$

$$k_{45} = k_{23}/2, \quad (3.31b)$$

$$k_{44} = k_{34}/3, \quad (3.31c)$$

$$k_{43} = (k_{33} - k_{34})/3, \quad (3.31d)$$

$$k_{42} = (2k_{32} - 3k_{23})/4, \quad (3.31e)$$

$$k_{41} = k_{28} - 3k_{29}, \quad (3.31f)$$

$$k_{40} = (3k_{31} - 2k_{33} + 2k_{34})/9, \quad (3.31g)$$

$$k_{39} = (2k_{30} - 2k_{32} + 3k_{23})/4, \quad (3.31h)$$

$$k_{38} = k_{13} - 2k_{28} + 6k_{29}, \quad (3.31i)$$

$$k_{37} = (9k_{27} - 3k_{31} + 2k_{33} - 2k_{34})/27, \quad (3.31j)$$

$$k_{36} = (4k_{26} - 2k_{30} + 2k_{32} - 3k_{23})/8, \quad (3.31k)$$

$$k_{35} = -6k_{29} + 2k_{28} - k_{13}, \quad (3.31l)$$

in which,

$$k_{34} = k_{24}/2, \quad (3.32a)$$

$$k_{33} = (2k_{22} - 3k_{24})/4, \quad (3.32b)$$

$$k_{32} = k_{21} - 3k_{23}, \quad (3.32c)$$

$$k_{31} = (2k_{19} - 2k_{22} + 3k_{24})/4, \quad (3.32d)$$

$$k_{30} = k_{18} - 2k_{21} + 6k_{23}, \quad (3.32e)$$

$$k_{29} = k_{20}/3, \quad (3.32f)$$

$$k_{28} = k_{17}/2, \quad (3.32g)$$

$$k_{27} = (4k_{15} - 2k_{19} + 2k_{22} - 3k_{24})/8, \quad (3.32h)$$

$$k_{26} = k_{14} - k_{18} + 2k_{21} - 6k_{23}, \quad (3.32i)$$

$$k_{25} = -k_{16} - k_{12}, \quad (3.32j)$$

in which,

$$k_{24} = -k_7/4, \quad (3.33a)$$

$$k_{23} = -k_6/4, \quad (3.33b)$$

$$k_{22} = - (C_2 k_7 + k_5)/4, \quad (3.33c)$$

$$k_{21} = (C_2 k_7 - C_2 k_6 - k_4)/4, \quad (3.33d)$$

$$k_{20} = (- k_3 + C_2 k_6)/4, \quad (3.33e)$$

$$k_{19} = - (C_2 k_5 + k_2)/4, \quad (3.33f)$$

$$k_{18} = (C_2 k_5 - k_1 - C_3 - C_2 k_4)/4, \quad (3.33g)$$

$$k_{17} = (C_2 k_4 - C_4 - C_2 k_3)/4, \quad (3.33h)$$

$$k_{16} = C_2 k_3/4, \quad (3.33i)$$

$$k_{15} = - C_2 k_2/4, \quad (3.33j)$$

$$k_{14} = (C_2 k_2 - C_2 k_1 - C_2 C_3)/4, \quad (3.33k)$$

$$k_{13} = (C_2 C_3 + C_2 k_1 - C_2 C_4)/4, \quad (3.33l)$$

and

$$k_{12} = C_2 C_4/4. \quad (3.33m)$$

The constants C_5 and C_6 are determined by setting (3.30) equal to zero for $\rho = 1$ and $\rho = (b/a)^2$, according to (3.19b), and solving the resulting equations simultaneously for C_5 and C_6 . Upon doing so we get,

$$\begin{aligned}
 C_6 = & (k_{36}(b/a)^2((b/a)^2-1) + k_{37}(b/a)^2((b/a)^4-1) + k_{25}\ln(b/a)^2 + \\
 & k_{38}(b/a)^2\ln(b/a)^2 + k_{39}(b/a)^4\ln(b/a)^2 + k_{40}(b/a)^6\ln(b/a)^2 + \\
 & k_{41}(b/a)^2\ln^2(b/a)^2 + k_{42}(b/a)^4\ln^2(b/a)^2 + k_{43}(b/a)^6\ln^2(b/a)^2 + \\
 & k_{44}(b/a)^6\ln^3(b/a)^2 + k_{45}(b/a)^4\ln^3(b/a)^2 + k_{29}(b/a)^2\ln^3(b/a)^2 + \\
 & k_{46}\ln^2(b/a)^2) / ((b/a)^2-1) , \tag{3.34a}
 \end{aligned}$$

and,

$$C_5 = - (C_6 + k_{35} + k_{36} + k_{37}) . \tag{3.34b}$$

This concludes the solution for α_1 .

C. Stresses, displacements and other physical quantities.

The stresses, displacements, stress and moment resultants follow by direct calculation by substituting the expansions (3.15) into the formulas given in section (A.2) of the appendix, and using (3.25), (3.26), (3.30) and the numerous expressions above for the constants. In order to determine the expansion for the transverse displacement W we must evaluate an indefinite integral involving the quantity α and we consider this next. Substituting (3.15a) into (A.49) and keeping only the first two terms of the expansion for α we get,

$$\frac{2WL}{a} = \int^{\rho} \frac{(\alpha_0 + \alpha_1 \epsilon) d\rho}{\rho} , \tag{3.35}$$

where L is given by (3.5). Letting W_0 and W_1 be defined by,

$$W_0 = \int^{\rho} \frac{\alpha_0 d\rho}{\rho} , \tag{3.36a}$$

and,

$$W_1 = \int^{\rho} \frac{\alpha_1 d\rho}{\rho} , \tag{3.36b}$$

we get,

$$\frac{2WL}{a} = W_0 + W_1 \epsilon. \quad (3.37)$$

Substituting (3.25) and (3.30) into (3.36a) and (3.36b) respectively and performing the indicated integration gives us,

$$W_0 = C_2(\ln \rho - \rho) + \rho - \rho \ln \rho + C_7, \quad (3.38)$$

and,

$$\begin{aligned} W_1 = & k_{47}\rho + k_{48}\rho^2 + k_{49}\rho^3 + C_6 \ln \rho + k_{50} \ln^2 \rho + k_{51} \rho \ln \rho + k_{52} \rho^2 \ln \rho + \\ & k_{53} \rho^3 \ln \rho + k_{54} \rho \ln^2 \rho + k_{55} \rho^2 \ln^2 \rho + k_{56} \rho^3 \ln^2 \rho + k_{29} \rho \ln^3 \rho + \\ & k_{57} \rho^2 \ln^3 \rho + k_{58} \rho^3 \ln^3 \rho + k_{59} \ln^3 \rho + C_8, \end{aligned} \quad (3.39)$$

where C_7 and C_8 are constants of integration and,

$$k_{47} = C_5 + k_{35} - k_{38} + 2k_{41} - 6k_{29}, \quad (3.40a)$$

$$k_{48} = (4k_{36} - 2k_{39} + 2k_{42} - 3k_{45})/8, \quad (3.40b)$$

$$k_{49} = (9k_{37} - 3k_{40} + 2k_{43} - 2k_{44})/27, \quad (3.40c)$$

$$k_{50} = k_{25}/2, \quad (3.40d)$$

$$k_{51} = k_{38} - 2k_{41} + 6k_{29}, \quad (3.40e)$$

$$k_{52} = (2k_{39} - 2k_{42} + 3k_{45})/4, \quad (3.40f)$$

$$k_{53} = (3k_{40} - 2k_{43} + 2k_{44})/9, \quad (3.40g)$$

$$k_{54} = k_{41} - 3k_{29}, \quad (3.40h)$$

$$k_{55} = (2k_{42} - 3k_{45})/4, \quad (3.40i)$$

$$k_{56} = (k_{43} - k_{44})/3, \quad (3.40j)$$

$$k_{57} = k_{45}/2, \quad (3.40k)$$

$$k_{58} = k_{44}/3, \quad (3.40l)$$

and,

$$k_{59} = k_{46}/3. \quad (3.40m)$$

To evaluate C_7 and C_8 we imagine that the outer edge of the plate is fixed in space, as illustrated in Figure (2), and that the rigid inclusion moves vertically relative to the outer edge. Therefore we set W equal to zero at $\rho = 1$. From (3.37) we see that W_0 and W_1 must accordingly vanish at $\rho = 1$ which provides the necessary conditions for finding C_7 and C_8 . Applying these conditions, we get,

$$C_7 = C_2 - 1, \quad (3.41a)$$

and,

$$C_8 = -k_{47} - k_{48} - k_{49}, \quad (3.42b)$$

which concludes the solution for W . A plot of W , as given by (3.37), versus the applied load P is given in Figures (5) through (8) in Chapter VI.

CHAPTER IV

SMALL FINITE DEFORMATIONS

In this chapter we consider the equations of the small finite deformation theory in the case when the deformations are becoming increasingly large. Here we are interested in finding perturbation type solutions to (2.15) and (2.16), the von Karman equations, for large values of λ^2 .

First we solve (2.15) and (2.16) by means of a straightforward perturbation scheme. It is found that the derivatives vanish from one of the equations of the first approximations to the von Karman equations and thus two boundary conditions must be abandoned in the first approximations. This of course implies that the straightforward or membrane expansions, as we shall call them, must be restricted to the interior region of the plate away from its edges.

The edge zones of the plate, where boundary-layers exist, are investigated by transforming the von Karman equations to a system of boundary-layer coordinates. Separate perturbation expansions are obtained in the vicinity of each edge of the plate. We shall refer to these as boundary-layer expansions.

After exhausting the boundary conditions on the problem one finds that a number of constants of integration remain undetermined in the boundary-layer expansions. These constants, as well as the ones in the membrane expansions, are determined by matching the boundary-layer and membrane expansions according to the theory of Matched Asymptotic Expansions (4).

It is found that the membrane expansions are in terms of either trigonometric, hyperbolic or polynomial functions depending on whether Poisson's ratio ν is respectively less than, greater than or equal to $1/3$. The first two aforementioned cases are significantly more complicated than the last in that they contain an additional

variable which is related transcendentally to the independent variable. For ν equal to $1/3$ only powers of the independent variable appear. Membrane and boundary-layer expansions for the dependent variables are found for all three ranges of ν but in calculating the stresses and displacements we concern ourselves only with the less complicated case for ν equal to $1/3$. Three-term expansions are found for the case of ν equal to $1/3$; two-term expansions are found for the other two cases.

A. Determination of the Parameters

The restrictions on the parameters δ and L^2 for the small finite deformation theory are that $\delta \ll 1$ and $L^2 \gg 1$. Furthermore either (2.11a) or (2.11b) must hold. In view of (3.1) we see that the magnitude of λ^2 is unrestricted. Dividing (2.16) by λ^2 gives us,

$$\frac{1}{\lambda^2} \left(\frac{d^2\beta}{dr^2} + \frac{1}{r} \frac{d\beta}{dr} - \frac{\beta}{r^2} \right) - \frac{\psi\beta}{r} = - \left(\frac{Pa}{2\pi D\lambda^2} \right) \frac{1}{\delta r} \quad (4.1)$$

It is clear from physical considerations that the stretching of the middle surface of the plate is significant for large deformations and consequently the above term containing ψ , which is directly related to the stretching, must remain in the first approximation to (2.16). Therefore in view of (4.1) we determine that the limiting case for increasingly large deformations corresponds to the case for $\lambda^2 \gg 1$. Furthermore we assume that the quantity $(Pa/2\pi D\lambda^2)$ on the right hand side of (4.1) satisfies (2.11a) and that this quantity and δ approach zero simultaneously such that their ratio is of order unity. To guarantee this set,

$$\delta = \left(\frac{Pa}{2\pi D\lambda^2} \right) (1/4) \quad (4.2)$$

where the factor of 1/4 is added for convenience. Using (2.13) it follows from (4.2) that

$$\lambda^2 = \frac{PaL}{8\pi D}. \quad (4.3)$$

Substituting in (4.3) for λ^2 and L from (2.8) and making use of (2.3) we determine that,

$$B = \left(\frac{P}{8\pi}\right)^{2/3} (aEh)^{1/3}. \quad (4.4)$$

Substituting (4.4) back into (2.8) and using (2.3) again, L^2 and λ^2 are determined to be,

$$L^2 = \left(\frac{8\pi aEh}{P}\right)^{2/3}, \quad (4.5)$$

and,

$$\lambda^2 = \left(\frac{a}{h}\right)^2 \left(\frac{P}{8\pi aEh}\right)^{2/3} 12(1-\nu^2). \quad (4.6)$$

From (2.13) and (4.5) we get,

$$\delta = \left(\frac{P}{8\pi aEh}\right)^{1/3}. \quad (4.7)$$

B. Membrane Expansions

Using (4.2), the basic differential equations (2.15) and (2.16) become,

$$\frac{d^2\beta}{dr^2} + \frac{1}{r} \frac{d\beta}{dr} - \frac{\beta}{r^2} - \frac{\lambda^2 \psi\beta}{r} = -\frac{4\lambda^2}{r}, \quad (4.8a)$$

and,

$$\frac{d^2\psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} - \frac{\psi}{r^2} + \frac{\beta^2}{2r} = 0. \quad (4.8b)$$

These equations are simplified following Schwerin (7) by letting

$$\psi = \frac{F(r^2)}{r}, \quad (4.9)$$

$$\beta = \frac{\alpha(r^2)}{r}, \quad (4.10)$$

and,

$$r^2 = \rho, \quad (4.11)$$

where F , α and ρ are new variables. The relations between the original and new variables are,

$$\bar{\beta} = \frac{a\alpha}{r}, \quad (4.12)$$

$$\bar{\psi} = \frac{BaF}{r}, \quad (4.13)$$

and,

$$\bar{r}^2 = a^2 \rho, \quad (4.14)$$

which follow by combining (2.4) through (2.6) with (4.9) through (4.11). Substituting (4.9), (4.10) and (4.11) into (4.8) we get,

$$\alpha'' - \frac{\lambda^2 F \alpha}{4\rho^2} = - \frac{\lambda^2}{\rho}, \quad (4.15a)$$

and,

$$F'' + \frac{\alpha^2}{8\rho^2} = 0, \quad (4.15b)$$

where primes indicate differentiation with respect to ρ . Equations (4.15) are considered to be the basic differential equations for the analysis of chapter IV.

Following the usual procedure we assume that the variables α and F can be expanded in the form,

$$\alpha = \alpha_0 + \alpha_1 \epsilon + \alpha_2 \epsilon^2 + \dots, \quad (4.16a)$$

and,

$$F = F_0 + F_1 \epsilon + F_2 \epsilon^2 + \dots, \quad (4.16b)$$

where ϵ is a small parameter and the coefficients of ϵ and its powers are of order unity and independent of ϵ . The appropriate choice for ϵ is not obvious at this point. However, in the course of the boundary-layer analysis of section C of this chapter, it is obvious that the boundary-layer expansions should be in powers of $(1/\lambda)$. Let us assume that the same perturbation parameter also is appropriate for (4.16). Later we verify that this is the correct choice for ϵ by virtue of the fact that the boundary-layer and membrane expansions match. Therefore let us set

$$\epsilon = 1/\lambda = \left(\frac{8\pi a E h}{P} \right)^{1/3} \frac{(h/a)}{(12(1-\nu^2))^{1/2}} \quad (4.17)$$

in which we have used (4.6).

Substituting (4.15) into (4.16) we determine the following sets of governing equations for the first three terms in the membrane expansions.

Zeroth order equations

$$\frac{F_0 \alpha_0}{4\rho} = 1 \quad (4.18a)$$

$$F_0'' + \frac{\alpha_0^2}{8\rho^2} = 0 \quad (4.18b)$$

First order equations

$$F_1 \alpha_0 + F_0 \alpha_1 = 0 \quad (4.19a)$$

$$F_1'' + \frac{\alpha_0 \alpha_1}{4\rho^2} = 0 \quad (4.19b)$$

Second order equations

$$4\rho^2 \alpha_0'' - \alpha_1 F_1 - F_0 \alpha_2 - F_2 \alpha_0 = 0 \quad (4.20a)$$

$$F_2'' + \frac{2\alpha_2 \alpha_0 + \alpha_1^2}{8\rho^2} = 0 \quad (4.20b)$$

These equations can be simplified by making use of the algebraic relationships between the variables as given by (4.18a), (4.19a) and (4.20a). We get the following three differential equations:

Zerth order

$$F_0'' + \frac{2}{F_0^2} = 0 \quad (4.21)$$

First order

$$F_1'' - \frac{4F_1}{F_0^3} = 0 \quad (4.22)$$

Second order

$$F_2'' - \frac{4F_2}{F_0^3} = -\frac{3\alpha_1^2}{8\rho^2} - \frac{\alpha_0 \alpha_0''}{F_0} \quad (4.23)$$

The zeroth order equation (4.21) is non-linear, the other two equations are linear. It remains to solve these three equations to obtain three-term membrane expansions. For two-term expansions we must solve only the first two equations.

Herein we show that (4.21) possesses three different forms of solution, which are obtained by the method of quadratures, depending on whether the constant of integration in its first integral is assumed to be positive, negative or zero. It is shown later that these forms of solution correspond respectively to the cases where Poisson's ratio ν is greater than, less than or equal to $1/3$. Two forms of the solution have been reported previously in the literature. In an analysis of a circular membrane problem Schwerin (7) determined the solution to (4.21) for the case where ν is less than $1/3$. Later W. E. Jahsman, F. A. Field and A. M. C. Holmes (8) recognized that a certain variable in Schwerin's solution need not be real to yield real solutions. This led to the solution for ν greater than $1/3$. However, it has not been reported that a third form of solution exists, namely the one corresponding to ν equal to $1/3$. This solution is very important due to its simplicity coupled with the fact that for many engineering materials ν is approximately equal to $1/3$.

The first integral to (4.21) is found by reproducing the steps of Schwerin. First write (4.21) in the form,

$$dF'_0 + \frac{2d\rho}{F_0^2} = 0. \quad (4.24)$$

Multiplying this by $2F'_0$ and expressing each term of the resulting expression in differential form gives us,

$$d(F'_0)^2 + d\left(\frac{-4}{F_0}\right) = 0. \quad (4.25)$$

Integrating this we determine the first integral to (4.21) to be,

$$(F'_0)^2 - \frac{4}{F_0} = \pm 4A_0^2, \quad (4.26)$$

where $4A_0^2$ is a convenient form for the constant of integration. We now consider separately the cases when the minus and plus sign in (4.26) holds and also the case when A_0 is zero. Later, in section D of this chapter, it is shown that these cases apply when ν is less than, greater than and equal to $1/3$ respectively.

1. Membrane expansions for ν less than $1/3$

In this section we solve (4.21) and (4.22) in order to determine two-term membrane expansions for ν less than $1/3$. Assuming that the minus sign in (4.26) holds we write

$$(F'_0)^2 - \frac{4}{F_0} = -4A_0^2. \quad (4.27)'$$

Solving for F'_0 we find that,

$$F'_0 = 2A_0 \sqrt{\frac{1}{A_0^2 F_0} - 1}. \quad (4.28)$$

Multiplying each side of (4.28) by A_0^2 gives us,

$$A_0^2 F'_0 = 2A_0^3 \sqrt{\frac{1}{A_0^2 F_0} - 1}. \quad (4.29)$$

Let us introduce a new variable γ defined by the following equation:

$$A_0^2 F_0 = \sin^2(\gamma/2). \quad (4.30)$$

Substituting (4.30) into (4.29) it is easy to show that,

$$(\sin^2(\gamma/2))' = 2A_0^3 \cot(\gamma/2), \quad (4.31)$$

which can be written as,

$$\frac{d(\sin^2 \gamma/2)}{\cot(\gamma/2)} = 2A_o^3 d\rho . \quad (4.32)$$

Expanding the above differential gives us,

$$(\sin^2 \gamma/2) \frac{d\gamma}{2} = A_o^3 d\rho . \quad (4.33)$$

This is easily integrated giving us,

$$\frac{\gamma}{4} - \frac{\sin \gamma}{4} = A_o^3 \rho - \frac{B_o}{4} , \quad (4.34)$$

where $(-B_o/4)$ is a convenient form for the constant of integration.

Rearranging (4.34) gives the following relation between γ and ρ :

$$4A_o^3 \rho = \gamma - \sin \gamma + B_o . \quad (4.35)$$

Thus the function F_o is therefore expressed through the parameter γ by (4.30) and (4.35), with A_o and B_o to be determined later from the matching conditions. From (4.18a) and (4.30) we get,

$$\alpha_o = \frac{4A_o^2 \rho}{\sin^2(\gamma/2)} . \quad (4.36)$$

We now turn to the solution of (4.22). First F_1'' is replaced by the expression,

$$F_1'' = \frac{(F_o' F_1')' - F_o'' F_1'}{F_o'} \quad (4.37)$$

which follows from the identity,

$$(F'_0 F'_1)' = F'_0 F''_1 + F''_0 F'_1 . \quad (4.38)$$

Therefore (4.22) becomes,

$$(F'_0 F'_1)' - F''_0 F'_1 - \frac{4F_1 F'_0}{F_0^3} = 0 . \quad (4.39)$$

Next F''_0 is replaced by the quantity $-2/F_0^2$ which follows from (4.21).

This gives us,

$$(F'_0 F'_1)' + 2\left(\frac{F'_1}{F_0^2} - \frac{2F_1 F'_0}{F_0^3}\right) = 0 . \quad (4.40)$$

The term in the last set of parenthesis is identically equal to $(F_1/F_0^2)'$.

Therefore the first integral becomes,

$$F'_0 F'_1 + 2 \frac{F_1}{F_0^2} = A_1 , \quad (4.41)$$

where A_1 is a constant of integration. Replacing $(1/F_0^2)$ in the above equation by $(\frac{-F''_0}{2})$ from (4.21) and dividing by $(F'_0)^2$ results in,

$$\frac{F'_1}{F'_0} - \frac{F_1 F''_0}{(F'_0)^2} = \frac{A_1}{(F'_0)^2} . \quad (4.42)$$

The left hand side of the above equation is identically equal to

$(F_1/F'_0)'$. Hence we see that,

$$\left(\frac{F_1}{F'_0}\right)' = \frac{A_1}{(F'_0)^2} . \quad (4.43)$$

Integrating this gives F_1 in terms of the known quantity F'_0 as,

$$F_1 = (F'_0) \left(\int \frac{A_1}{(F'_0)^2} d\rho + B_1 \right), \quad (4.44)$$

where B_1 is a constant of integration. For future reference we note that the above solution for F_1 does not depend on the explicit solution for F_0 or its derivative. It only requires that F_0 be governed by (4.21).

To evaluate the indefinite integral let us express F'_0 in the form,

$$F'_0 = 2A_0 \cot(\gamma/2), \quad (4.45)$$

which follows from substituting (4.30) into (4.31). From (4.35) we find that,

$$d\rho = \frac{(1-\cos\gamma)d\gamma}{4A_0^3}. \quad (4.46)$$

Substituting (4.45) and (4.46) into (4.44) and performing the integration F_1 is found to be,

$$F_1 = \frac{A_1}{2A_0^4} \left(1 + \frac{\sin^2\gamma}{8\sin^2\gamma/2} - \frac{3\gamma\cot\gamma/2}{4} \right) + 2A_0B_1\cot\gamma/2. \quad (4.47)$$

With F_1 known α_1 is easily determined from (4.19a), i.e.

$$\alpha_1 = -\left(\frac{\alpha_0}{F_0}\right)F_1. \quad (4.48)$$

Thus the second terms of the membrane expansions are known. The constants A_1 and B_1 are determined later from matching (see Section D).

2. Membrane expansions for ν greater than $1/3$

The first two terms in the membrane expansions for ν greater than $1/3$ are found in this section. Returning to (4.26) we assume that the plus sign applies and (4.26) becomes,

$$(F'_0)^2 - \frac{4}{F_0} = 4A_0^2 \quad (4.49)$$

Solving this for F'_0 we get,

$$F'_0 = 2A_0 \sqrt{\frac{1}{A_0^2 F_0} + 1} \quad (4.50)$$

Multiplying each side of (4.50) by A_0^2 gives us

$$A_0^2 F'_0 = 2A_0^3 \sqrt{\frac{1}{A_0^2 F_0} + 1} \quad (4.51)$$

We introduce a new variable $\bar{\delta}$ by setting

$$A_0^2 F_0 = \sinh^2(\bar{\delta}/2) \quad (4.52)$$

Substituting (4.52) into (4.51) it follows that,

$$(\sinh^2(\bar{\delta}/2)) = 2A_0^3 \coth(\bar{\delta}/2), \quad (4.53)$$

which can be written as,

$$\frac{d(\sinh^2(\bar{\delta}/2))}{\coth(\bar{\delta}/2)} = 2A_o^3 d\rho. \quad (4.54)$$

Carrying out the differential we get,

$$\sinh^2(\bar{\delta}/2) \frac{d\bar{\delta}}{2} = A_o^3 d\rho. \quad (4.55)$$

From a table of integrals we find that,

$$\frac{\sinh \bar{\delta}}{4} - \frac{\bar{\delta}}{4} = A_o^3 \rho + B_o/4, \quad (4.56)$$

where $(B_o/4)$ is a convenient form for the constant of integration. Rearranging (4.56) we have the following relation between ρ and $\bar{\delta}$:

$$4A_o^3 \rho = \sinh \bar{\delta} - \bar{\delta} - B_o. \quad (4.57)$$

Thus F_o is known from (4.52) and (4.57) through the parameter $\bar{\delta}$. From (4.18a) it follows that α_o is known.

To determine F_1 we refer to (4.44) and the comment following it. Combining (4.52) and (4.53) we determine that,

$$F_o' = 2A_o \coth(\bar{\delta}/2), \quad (4.58)$$

and the following differential comes from (4.57):

$$d\rho = \left(\frac{\cosh \bar{\delta} - 1}{4A_o^3} \right) d\bar{\delta}. \quad (4.59)$$

Substituting (4.58) and (4.59) into (4.44) we find that,

$$F_1 = \frac{A_1}{2A_o^4} \left(1 + \frac{\sinh^2 \bar{\delta}}{8 \sinh^2 \bar{\delta}/2} - \frac{3\bar{\delta} \coth(\bar{\delta}/2)}{4} \right) + 2A_o B_1 \coth \bar{\delta}/2. \quad (4.60)$$

From (4.19a) α_1 is known. Thus the two-term membrane expansions for ν greater than $1/3$ are known in terms of the constants of integration. These constants are determined by matching.

3. Membrane expansions for ν equal to $1/3$.

In this section we determine the first three terms in the membrane expansions for ν equal to $1/3$. The constant A_0^2 in (4.26) is assumed to be zero and therefore we have,

$$(F'_0)^2 - \frac{4}{F_0} = 0. \quad (4.61)$$

It follows from (4.61) that,

$$F_0^{1/2} F'_0 = 2, \quad (4.62)$$

or equivalently,

$$\left(\frac{2}{3} F_0^{3/2} \right)' = 2. \quad (4.63)$$

Integrating this gives us,

$$\frac{2}{3} F_0^{3/2} = 2\rho, \quad (4.64)$$

where we have assumed that the constant of integration is zero.

Therefore F_0 is determined to take the following simple form,

$$F_0 = (3\rho)^{2/3}. \quad (4.65)$$

From (4.18a) we get,

$$\alpha_0 = \frac{4\rho^{1/3}}{(3)^{2/3}}. \quad (4.66)$$

This completes the solution for the zeroth order terms in (4.16). Later when we apply the matching principle we verify that the constants which we have assumed to be zero in (4.61) and (4.64) are such if, and only if, ν is equal to $1/3$.

Referring to (4.44) and the discussion following it, and making use of (4.65), F_1 is easily determined to be,

$$F_1 = \frac{(3)^{4/3} A_1}{10} \rho^{4/3} + \frac{2B_1}{(3)^{1/3}} \rho^{-1/3} \quad (4.67)$$

Letting,

$$k_1 = \frac{(3)^{4/3} A_1}{10} \quad \text{and} \quad k_2 = \frac{2B_1}{(3)^{1/3}} \quad (4.68)$$

equation (4.67) can be written as,

$$F_1 = k_1 \rho^{4/3} + k_2 \rho^{-1/3} \quad (4.69)$$

where k_1 and k_2 are considered to be new constants of integration. The function α_1 is known by virtue of (4.19a), (4.69), (4.66) and (4.65). Thus the first order terms of the membrane expansions are known in terms of k_1 and k_2 to be evaluated later by matching.

The second order terms of the expansions are found by first substituting the known quantities F_0 , α_0 , α_0'' and α_1 into the right hand side of (4.23) to give us,

$$F_2'' - \frac{4F_2}{F_0^3} = - \frac{2}{(3)^{5/3}} (k_1^2 + k_2^2 \rho^{-10/3} + 2k_1 k_2 \rho^{-5/3}) + \frac{32}{81} \rho^{-2} \quad (4.70)$$

Comparison of (4.22) and (4.23) shows that both equations have the same homogeneous solution. Therefore the complete solution for F_2 becomes,

$$F_2 = k_3 \rho^{4/3} + k_4 \rho^{-1/3} - \frac{(3)^{1/3} k_1^2 \rho^2}{7} + \frac{2k_1 k_2 \rho^{1/3}}{(3)^{2/3}} - \frac{k_2^2 \rho^{-4/3}}{4(3)^{2/3}} - 8/9 \quad (4.71)$$

where k_3 and k_4 are constants of integration to be evaluated by matching, and the last four terms above represent the particular solution to (4.70). From (4.20a) α_2 is given in terms of known quantities and thus the solution for the second order terms in (4.16) is complete.

C. Boundary-Layer Expansions

As discussed at the beginning of the chapter, the membrane expansions must be restricted to the interior of the plate since the equations governing the terms of these expansions, namely (4.18), (4.19) and (4.20) do not yield a sufficient number of constants of integration. In this section we determine two systems of boundary-layer equations which apply separately to the inner and outer edge zones of the plate. These equations are solved in terms of perturbation expansions which we shall call boundary-layer expansions. After imposing the boundary conditions on the problem certain constants in the boundary-layer expansions remain undetermined. These constants, as well as those in the membrane expansions, are determined later from the matching principle (see section D).

1. Boundary-layer expansions for the vicinity of the outer edge of the plate.

a. Boundary-layer coordinate system and equations

Assuming that the boundary-layers are confined to narrow zones at the edges of the plate let us introduce a new outer edge boundary-layer coordinate η which remains of order unity as the widths of the boundary-layers shrink to zero with increasing λ^2 . Set,

$$\eta = \frac{1-\rho}{(1/\lambda^2)^n}, \quad (4.72)$$

where the exponent n is to be determined. Rearranging (4.72) we have,

$$\rho = 1 - (1/\lambda^2)^n \eta. \quad (4.73)$$

For convenience let,

$$\tau = (1/\lambda^2), \quad (4.74)$$

and note that

$$\tau \ll 1. \quad (4.75)$$

Substituting (4.74) into (4.73) we get,

$$\rho = 1 - \tau^n \eta. \quad (4.76)$$

Next the basic differential equations (4.15) are transformed to the coordinate η through (4.76) to give us,

$$\frac{d^2\alpha}{d\eta^2} - \frac{F\alpha\tau^{2n-1}}{4(1-\tau^n\eta)^2} + \frac{\tau^{2n-1}}{(1-\tau^n\eta)} = 0, \quad (4.77a)$$

and

$$\frac{d^2F}{d\eta^2} + \frac{\tau^{2n}\alpha^2}{8(1-\tau^n\eta)^2} = 0. \quad (4.77b)$$

To determine n let us follow the classical procedure and look for the value which leads to the least degenerate form for the first approximations to (4.77) for small τ . Inspection of (4.77a) indicates that the term containing the product of F and α will remain in the first approximations if n is chosen to be $1/2$. Therefore let us set,

$$n = 1/2, \quad (4.78)$$

and the basic differential equations (4.77) for the outer edge boundary-layer become,

$$\frac{d^2\alpha}{d\eta^2} - \frac{F\alpha}{4(1-\epsilon\eta)^2} + \frac{1}{(1-\epsilon\eta)} = 0, \quad (4.79a)$$

and

$$\frac{d^2F}{d\eta^2} + \frac{\epsilon^2\alpha^2}{8(1-\epsilon\eta)^2} = 0, \quad (4.79b)$$

where we have introduced a new quantity ϵ which is defined as,

$$\epsilon = \tau^{1/2}. \quad (4.80)$$

Combining (4.80) and (4.74) we determine that,

$$\epsilon = 1/\lambda. \quad (4.81)$$

From (4.6) it follows that,

$$\epsilon = \left(\frac{8\pi a E h}{P}\right)^{1/3} \frac{(h/a)}{(12(1-\nu^2))^{1/2}}, \quad (4.82)$$

and in view of (4.75) and (4.80) we see that

$$\epsilon \ll 1. \quad (4.83)$$

From the above relations the following expression for ρ is obtained.

$$\rho = 1 - \epsilon \eta. \quad (4.84)$$

Replacing the quantities $(1 - \epsilon \eta)^{-2}$ and $(1 - \epsilon \eta)^{-1}$ in (4.79) with their series expansions and neglecting higher order terms gives us,

$$\frac{d^2 \alpha}{d\eta^2} - \frac{F\alpha}{4} (1 + 2\epsilon\eta + 3\epsilon^2 \eta^2) + (1 + \epsilon\eta + \epsilon^2 \eta^2) + O(\epsilon^3) = 0, \quad (4.85a)$$

and

$$\frac{d^2 F}{d\eta^2} + \frac{\epsilon^2 \alpha^2}{8} + O(\epsilon^3) = 0. \quad (4.85b)$$

Equations (4.85) represent the basic differential equations for the outer edge boundary-layer. These are to be solved in terms of perturbation expansions for small ϵ . Therefore let us assume the functions F and α can be represented by the following expansions which become asymptotic to the exact solution as ϵ tends to zero:

$$\alpha = \alpha_0 + \alpha_1 \epsilon + \alpha_2 \epsilon^2 + \dots; \quad (4.86a)$$

and

$$F = F_0 + F_1 \epsilon + F_2 \epsilon^2 + \dots \quad (4.86b)$$

The parameter ϵ is defined by (4.82). Substituting (4.86) into (4.85) and equating the coefficients of successive powers of ϵ to zero we get

the following sets of equations for three-term boundary-layer expansions:

Zeroth order,

$$\frac{d^2 \alpha_0}{d\eta^2} - \frac{F_0 \alpha_0}{4} + 1 = 0, \quad (4.87a)$$

and

$$\frac{d^2 F_0}{d\eta^2} = 0. \quad (4.87b)$$

First order,

$$\frac{d^2 \alpha_1}{d\eta^2} - \frac{F_0 \alpha_1 + F_1 \alpha_0}{4} - \frac{\eta F_0 \alpha_0}{2} + \eta = 0, \quad (4.88a)$$

and

$$\frac{d^2 F_1}{d\eta^2} = 0, \quad (4.88b)$$

Second order,

$$\frac{d^2 \alpha_2}{d\eta^2} - \frac{1}{4} (F_0 \alpha_2 + F_1 \alpha_1 + F_2 \alpha_0) - \frac{\eta}{2} (F_0 \alpha_1 + F_1 \alpha_0) - \frac{3\eta^2}{4} F_0 \alpha_0 + \eta^2 = 0, \quad (4.89a)$$

and

$$\frac{d^2 F_2}{d\eta^2} + \frac{\alpha_0^2}{8} = 0. \quad (4.89b)$$

The solution to these equations is straightforward. However since it is most expedient to apply the boundary conditions during the integration process we postpone the solution until after a discussion of the boundary conditions.

b. Boundary Conditions

From (2.21) and (4.10) it follows that the condition of zero slope is met by setting

$$\alpha_1 = \alpha_2 = \alpha_3 = 0 \quad (4.90a, b, c)$$

at the outer edge of the plate where $\rho=1$, or equivalently $\eta=0$, as seen from (4.14) and (4.84). The condition for zero radial displacement at the outer edge is expressed by (2.24). Substituting (4.9) and (4.11) into (2.24) we determine that the following condition must hold at the outer edge of the plate:

$$F' - \frac{(1+\nu)F}{2\rho} = 0. \quad (4.91)$$

By transforming (4.91) to the boundary-layer coordinate η by means of (4.84) and setting η equal to zero we get,

$$\frac{dF}{d\eta} + \epsilon \left(\frac{1+\nu}{2} \right) F = 0. \quad (4.92)$$

Substituting (4.86b) into (4.92) one finds that the following conditions must be imposed at $\eta = 0$:

$$\frac{dF}{d\eta} = 0, \quad (4.93a)$$

$$\frac{dF_1}{d\eta} + \left(\frac{1+\nu}{2}\right)F_0 = 0, \quad (4.93b)$$

and

$$\frac{dF_2}{d\eta} + \left(\frac{1+\nu}{2}\right)F_1 = 0. \quad (4.93c)$$

Thus (4.90) and (4.93) represent the boundary conditions at the outer edge of the plate.

c. Solutions to the boundary-layer equations

The solution to the system of equations (4.87), (4.88) and (4.89) is begun by first solving (4.87b) subject to (4.93a), and then substituting the result into (4.87a). The resulting expression for (4.87a) subsequently is solved by elementary methods. After applying the boundary condition (4.90a) on α_0 it is clear that a total of two constants of integration remain undetermined in the solutions to (4.87). One of these constants is set equal to zero by the requirement of finiteness since it multiplies an exponential term to a positive power in the coordinate η . Thus the solutions to the zeroth order equations contain one undetermined constant, which we call C_1 , after exhausting the boundary and finiteness conditions.

Substituting the known results for α_0 and F_0 into (4.88), the same process as described above is used to solve for F_1 and α_1 . Here again we are left with another undetermined constant which we denote as C_2 after exhausting the boundary conditions and applying a finiteness condition. With α_0 , F_0 , α_1 and F_1 known the solutions for F_2 and α_2 are found from (4.89) in a straightforward manner. As in the previous cases one constant of integration, C_3 , remains undetermined.

The solutions for F_0 , α_0 , F_1 , etc. in terms of the undetermined constants C_1 , C_2 , and C_3 are found to be:

Zeroth order,

$$F_0 = C_1^2, \quad (4.94a)$$

and

$$\alpha_0 = (4/C_1^2)(1 - e^{-(C_1/2)\eta}). \quad (4.94b)$$

First order,

$$F_1 = C_2 - (1+\nu)C_1^2\eta/2, \quad (4.95a)$$

and

$$\begin{aligned} \alpha_1 = & -4C_2/C_1^4 - 2(1-\nu)\eta/C_1^2 + e^{-(C_1/2)\eta} (4C_2/C_1^4 + C_2\eta/C_1^3 \\ & + (3-\nu)\eta/(2C_1^2) + (3-\nu)\eta^2/(4C_1)). \end{aligned} \quad (4.95b)$$

Second order,

$$\begin{aligned} F_2 = & C_3 + (6/C_1^5 - (1+\nu)C_2/2)\eta - \eta^2/C_1^4 + (16/C_1^6)e^{-(C_1/2)\eta} \\ & - (2/C_1^6)e^{-C_1\eta}, \end{aligned} \quad (4.96a)$$

and

$$\begin{aligned} \alpha_2 = & K_1 + K_2\eta + K_3\eta^2 + e^{-(C_1/2)\eta} (K_4 + K_5\eta + K_6\eta^2 + K_7\eta^3 + K_8\eta^4) \\ & - (24/C_1^{10})e^{-C_1\eta} + (1/C_1^{10})e^{-(3C_1/2)\eta}, \end{aligned} \quad (4.96b)$$

where,

$$K_1 = -4C_3/C_1^4 + 4C_2^2/C_1^6 - 8(1-v^2)/C_1^4 + 32/C_1^{10}, \quad (4.96c)$$

$$K_2 = 2(1-v)C_2/C_1^4 - 24/C_1^9, \quad (4.96d)$$

$$K_3 = 4/C_1^8 - (1-v^2)/C_1^2, \quad (4.96e)$$

$$K_4 = 4C_3/C_1^4 + 8(1-v^2)/C_1^4 - 4C_2^2/C_1^6 - 9/C_1^{10}, \quad (4.96f)$$

$$K_5 = -5C_2^2/(4C_1^5) - 12/C_1^9 + C_3/C_1^3 - (3-v)C_2/(2C_1^4) \\ + (19-2v-5v^2)/(16C_1^3), \quad (4.96g)$$

$$K_6 = -C_2^2/(8C_1^4) - (3-v)C_2/(4C_1^3) + 2/C_1^8 + (19-2v-5v^2)/(32C_1^2), \quad (4.96h)$$

$$K_7 = -(3-v)C_2/(16C_1^2) + (19-2v-5v^2)/(96C_1) - 1/(3C_1^7), \quad (4.96i)$$

and

$$K_8 = (-9 + 6v - v^2)/128. \quad (4.96j)$$

This completes the solution for the three-term boundary-layer expansions for the vicinity of the outer edge of the plate. The missing conditions for evaluating the undetermined constants are supplied later by the matching principle (see section D).

2. Boundary-layer expansions for the vicinity of the inner edge of the plate.

a. Boundary-layer coordinate system and equations

The expansions for the vicinity of the inner edge of the plate are obtained in identical fashion to the above section. Consequently we present only the important steps for the sake of brevity.

First a new inner edge boundary-layer coordinate ξ is introduced by the following:

$$\rho = b^2/a^2 + \epsilon \xi, \quad (4.97)$$

which is analogous to (4.84). The parameter ϵ is given by (4.82). Substituting (4.97) into (4.15), the basic differential equations for the boundary-layer zone at the inner edge of the plate become,

$$\begin{aligned} \frac{d^2 \alpha}{d\xi^2} - \frac{R\alpha}{4} \left(\left(\frac{a}{b}\right)^4 - 2\left(\frac{a}{b}\right)^6 \epsilon \xi + 3\left(\frac{a}{b}\right)^8 (\epsilon \xi)^2 \right) + \left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^4 \epsilon \xi + \left(\frac{a}{b}\right)^6 (\epsilon \xi)^2 \\ + 0(\epsilon^3) = 0, \end{aligned} \quad (4.98a)$$

and

$$\frac{d^2 F}{d\xi^2} + \frac{\epsilon \alpha^2}{8} \left(\frac{a}{b}\right)^4 + 0(\epsilon^3) = 0. \quad (4.98b)$$

These equations are solved in terms of perturbation expansions by assuming that the dependent variables α and F can be expanded in the following way:

$$\alpha = \alpha_0 + \alpha_1 \epsilon + \alpha_2 \epsilon^2 + \dots, \quad (4.99a)$$

and

$$F = F_0 + F_1 \epsilon + F_2 \epsilon^2 + \dots \quad (4.99b)$$

Substituting (4.99) into (4.98) the following equations are obtained:

Zeroth order,

$$\frac{d^2 \alpha_0}{d \zeta^2} - \frac{F_0 \alpha_0}{4} \left(\frac{a}{b}\right)^4 + \left(\frac{a}{b}\right)^2 = 0, \quad (4.100a)$$

and

$$\frac{d^2 F_0}{d \zeta^2} = 0. \quad (4.100b)$$

First order,

$$\frac{d^2 \alpha_1}{d \zeta^2} - \left(\frac{a}{b}\right)^4 \frac{F_0 \alpha_1 + F_1 \alpha_0}{4} + \frac{\zeta}{2} \left(\frac{a}{b}\right)^6 F_0 \alpha_0 - \left(\frac{a}{b}\right)^4 \zeta = 0, \quad (4.101a)$$

and

$$\frac{d^2 F_1}{d \zeta^2} = 0. \quad (4.101b)$$

Second order,

$$\begin{aligned} \frac{d^2 \alpha_2}{d \zeta^2} - \left(\frac{a}{b}\right)^4 \frac{F_0 \alpha_2 + F_1 \alpha_1 + F_2 \alpha_0}{4} + \left(\frac{a}{b}\right)^6 \frac{\zeta}{2} \frac{F_0 \alpha_1 + F_1 \alpha_0}{2} - \frac{3}{4} \left(\frac{a}{b}\right)^8 \zeta^2 F_0 \alpha_0 \\ + \left(\frac{a}{b}\right)^6 \zeta^2 = 0, \end{aligned} \quad (4.102a)$$

and

$$\frac{d^2 F_2}{d \zeta^2} + \left(\frac{a}{b}\right)^4 \frac{\alpha_0^2}{8} = 0. \quad (4.102b)$$

b. Boundary conditions

Following the same steps as before we find that the following conditions must be satisfied at $\rho = b^2/a^2$, or equivalently at $\xi = 0$:

$$\alpha_0 = \alpha_1 = \alpha_2 = 0, \quad (4.103a,b,c)$$

and

$$\frac{dF_0}{d\xi} = 0, \quad (4.104a)$$

$$\frac{dF_1}{d\xi} - \left(\frac{1+\nu}{2}\right) \left(\frac{a}{b}\right)^2 F_0 = 0, \quad (4.104b)$$

and

$$\frac{dF_2}{d\xi} - \left(\frac{1+\nu}{2}\right) \left(\frac{a}{b}\right)^2 F_1 = 0. \quad (4.104c)$$

c. Solutions to the boundary-layer equations

The solutions to (4.100), (4.101) and (4.102), subject to the boundary conditions (4.103) and (4.104), and finiteness conditions, are found to be:

Zeroth order,

$$F_0 = C_4^2, \quad (4.105a)$$

and

$$\alpha_0 = -\left(\frac{4}{C_4^2}\right) \left(\frac{b}{a}\right)^2 \left(1 - e^{-\left(C_4/2\right) \left(\frac{a}{b}\right)^2 \xi}\right). \quad (4.105b)$$

First order,

$$F_1 = C_5 + (1+\nu)C_4^2 (a/b)^2 \xi / 2, \quad (4.106a)$$

and

$$\begin{aligned} \alpha_1 = & -4C_5(b/a)^2/C_4^2 + 2(1-\nu) \xi / C_4^2 + e^{-(C_4/2)(a/b)^2 \xi} (4C_5(b/a)^2/C_4^4 \\ & + C_5 \xi / C_4^3 - (3-\nu) \xi / (2C_4^2) - (3-\nu)(a/b)^2 \xi^2 / (4C_4)) . \end{aligned} \quad (4.106b)$$

Second order,

$$\begin{aligned} F_2 = & C_6 + (6(b/a)^2/C_4^5 + (1+\nu)(a/b)^2 C_5/2) \xi - \xi^2 / C_4^4 \\ & + (16(b/a)^4/C_4^6) e^{-(C_4/2)(a/b)^2 \xi} - (2(b/a)^4/C_4^6) e^{-C_4(a/b)^2 \xi} \end{aligned} \quad (4.107a)$$

and

$$\begin{aligned} \alpha_2 = & K_9 + K_{10} \xi + K_{11} \xi^2 + e^{-(C_4/2)(a/b)^2 \xi} (K_{12} + K_{13} \xi + K_{14} \xi^2 \\ & + K_{15} \xi^3 + K_{16} \xi^4) - (24(b/a)^6/C_4^{10}) e^{-C_4(a/b)^2 \xi} \\ & + ((b/a)^6/C_4^{10}) e^{-(3C_4/2)(a/b)^2 \xi}, \end{aligned} \quad (4.107b)$$

where C_4 , C_5 and C_6 are undetermined constants of integration analogous to C_1 , C_2 and C_3 , and

$$K_9 = -4(b/a)^2 C_6 / C_4^4 + 4(b/a)^2 C_5^2 / C_4^6 - 8(1-\nu^2)(b/a)^2 / C_4^4 + 32(b/a)^6 / C_4^{10}, \quad (4.107c)$$

$$K_{10} = -2(1-\nu)C_5/C_4^4 - 24(b/a)^4 C_4^9, \quad (4.107d)$$

$$K_{11} = 4(b/a)^2/C_4^8 - (1-v^2)(a/b)^2/C_4^2, \quad (4.107e)$$

$$K_{12} = 4(b/a)^2 C_6/C_4^4 + 8(b/a)^2(1-v^2)/C_4^4 - 4(b/a)^2 C_5^2/C_4^6 - 9(b/a)^6/C_4^{10}, \quad (4.107f)$$

$$K_{13} = -5C_5^2/(4C_4^5) - 12(b/a)^4/C_4^9 + C_6/C_4^3 + (3-v)C_5/(2C_4^4) + (19-2v-5v^2)/(16C_4^3), \quad (4.107g)$$

$$K_{14} = -C_5^2(a/b)^2/(8C_4^4) + (3-v)(a/b)^2 C_5/(4C_4^3) + 2(b/a)^2/C_4^8 + (19-2v-5v^2)(a/b)^2/(32C_4^2), \quad (4.107h)$$

$$K_{15} = (3-v)C_5(a/b)^4/(16C_4^2) + (19-2v-5v^2)(a/b)^4/(96C_4) - 1/(3C_4^7) \quad (4.107i)$$

and

$$K_{16} = (-9 + 6v-v^2)(a/b)^6/128. \quad (4.107j)$$

The conditions for evaluating C_4 , C_5 and C_6 are determined from the matching principle (see Section D).

D. Matching

The undetermined constants in the membrane and boundary-layer expansions are evaluated in this section. The necessary equations for determining these constants are obtained by utilizing the theory of "Matched Asymptotic Expansions" which has been popularized by M. Van Dyke (4) in fluid mechanics problems. Basically the theory assumes the existence of an overlap domain where both representations (in this case, the membrane and boundary-layer expansions) of a given variable are valid. From this assumption M. Van Dyke formulates the following Asymptotic Matching Principle:

The m -term inner expansion of (the n -term outer expansion)
= the n -term outer expansion of (the m -term inner
expansion)

where, by definition,

Outer expansion: The asymptotic expansion for ϵ tending
to zero with outer variables fixed.

Outer variables: Dimensionless independent and dependent
variables based upon the primary reference quantities
in the problem.

Inner expansion: The asymptotic expansion for ϵ tending to
zero with inner variables fixed.

Inner variables: Dimensionless independent and dependent
variables stretched (transformed) by appropriate
functions of ϵ so as to be of order unity in the region
of nonuniformity of the outer expansion.

Here m and n are any two integers. According to these definitions we see that the membrane and boundary-layer expansions correspond to outer and inner expansions respectively and, in addition, that ρ is an outer variable while η and ζ are inner variables. We use the terms "inner" and "outer" and the corresponding terms "boundary-layer" and "membrane" interchangeably in this chapter.

Since there are three outer expansions, one for each of the three ranges of ν which we consider, the matching process is accordingly divided into three parts. First we consider the simpler case where ν is equal to $1/3$.

1. Poisson's ratio equal to 1/3.

Here we determine the constants k_1 through k_4 appearing in (4.69) and (4.71) and the constants C_1 through C_6 of the boundary-layer expansions for the case when ν is equal to 1/3. Furthermore we prove that the constants arising in the integration of (4.21) are zero, as we have assumed, if, and only if, Poisson's ratio is 1/3.

a. Matching conditions for the variable F

First the outer edge boundary-layer and membrane expansions for the variable F are matched. Following Van Dyke let us write,

Three-term outer expansion for F:

$$\begin{aligned}
 & (3\rho)^{2/3} + \epsilon(k_1\rho^{4/3} + k_2\rho^{-1/3}) + \\
 & \epsilon^2(k_3\rho^{4/3} + k_4\rho^{-1/3} - (3)^{1/3}k_1^2\rho^2/7 + 2k_1k_2\rho^{1/3}/(3)^{2/3} \\
 & - k_2^2\rho^{-4/3}/(4(3)^{2/3}) - 8/9) \tag{4.108}
 \end{aligned}$$

See (4.65), (4.69) and (4.71).

Three-term outer expansion written in inner variables:

$$\begin{aligned}
 & (3)^{2/3}(1-\epsilon\eta)^{2/3} + \epsilon(k_1(1-\epsilon\eta)^{4/3} + k_2(1-\epsilon\eta)^{-1/3}) + \\
 & \epsilon^2(k_3(1-\epsilon\eta)^{4/3} + k_4(1-\epsilon\eta)^{-1/3} - (3)^{1/3}k_1^2(1-\epsilon\eta)^2/7 + \\
 & 2k_1k_2(1-\epsilon\eta)^{1/3}/(3)^{2/3} - k_2^2(1-\epsilon\eta)^{-4/3}/(4(3)^{2/3}) - 8/9) \tag{4.109}
 \end{aligned}$$

See (4.84).

Expanded for small ϵ :

$$(3)^{2/3} + \epsilon (-2\eta/(3)^{1/3} + k_1 + k_2) + \epsilon^2 (- (3)^{2/3} \eta^2/9 - 4k_1\eta/3 + k_2\eta/3 + k_3 + k_4 - (3)^{1/3} k_1^2/7 + 2k_1k_2/(3)^{2/3} - k_2^2/(4(3)^{2/3}) - 8/9) \quad (4.110)$$

Three-term inner expansion of the three-term outer expansion:

$$(3)^{2/3} + \epsilon (-2\eta/(3)^{1/3} + k_1 + k_2) + \epsilon^2 (-\eta^2/(3)^{4/3} - 4k_1\eta/3 + k_2\eta/3 + k_3 + k_4 - (3)^{1/3} k_1^2/7 + 2k_1k_2/(3)^{2/3} - k_2^2/(4(3)^{2/3}) - 8/9) \quad (4.111)$$

Next we write:

Three-term inner expansion for F:

$$C_1^2 + \epsilon (C_2 - (1+\nu)C_1^2\eta/2) + \epsilon^2 (C_3 + (6/C_1^5 - (1+\nu)C_2/2)\eta - \eta^2/C_1^4 + (16/C_1^6)e^{-(C_1/2)\eta} - (2/C_1^6)e^{-C_1\eta}) \quad (4.112)$$

See (4.94a), (4.95a) and (4.96a).

Three-term inner expansion written in outer variables:

$$C_1^2 + \epsilon (C_2 - (1+\nu)C_1^2(1-\rho)/(2\epsilon)) + \epsilon^2 (C_3 + (6/C_1^5 - (1+\nu)C_2/2)(1-\rho)/\epsilon - (1-\rho)^2/(\epsilon^2 C_1^4) + (16/C_1^6)e^{-(C_1/2)(1-\rho)/\epsilon} - (2/C_1^6)e^{-C_1(1-\rho)/\epsilon}) \quad (4.113)$$

See (4.84).

Expanded for small ϵ :

$$C_1^2 + \epsilon(C_2 - (1+\nu)C_1^2(1-\rho)/(2\epsilon)) + \epsilon^2(C_3 + (6/C_1^5 - (1+\nu)C_2/2)(1-\rho)/\epsilon - (1-\rho)^2/(\epsilon^2 C_1^4)) + \text{exponentially small terms}^* \quad (4.114)$$

Three-term outer expansion of the three-term inner expansion:

$$C_1^2 + \epsilon(C_2 - (1+\nu)C_1^2(1-\rho)/(2\epsilon)) + \epsilon^2(C_3 + (6/C_1^5 - (1+\nu)C_2/2)(1-\rho)/\epsilon - (1-\rho)^2/(\epsilon^2 C_1^4)). \quad (4.115)$$

According to the matching principle (4.111) and (4.115), in terms of a common variable, should agree to appropriate orders of ϵ . Rewriting (4.115) in terms of η by means of (4.84) and equating the resulting expression to (4.111) we get,

$$C_1^2 + \epsilon(C_2 - (1+\nu)C_1^2 \eta/2) + \epsilon^2(C_3 + (6/C_1^5 - (1+\nu)C_2/2)\eta - \eta^2/C_1^4) = (3)^{2/3} + \epsilon(-2\eta/(3)^{1/3} + k_1 + k_2) + \epsilon^2(-\eta^2/(3)^{4/3} - 4k_1\eta/3 + k_2\eta/3 + k_3 + k_4 - (3)^{1/3}k_1^2/7 + 2k_1k_2/(3)^{2/3} - k_2^2/(4(3)^{2/3}) - 8/9) \quad (4.116)$$

Equating the coefficients of like powers of ϵ on each side of (4.116) we determine from the zero order powers that,

$$C_1 = (3)^{1/3}. \quad (4.117)$$

* A function $f(\epsilon)$ is said to be exponentially small if f/ϵ^N tends to zero as ϵ tends to zero for any N greater than zero.

Equating the coefficients of the first order terms in ϵ from (4.116) one finds that the following two equations must hold:

$$(1+\nu)C_1^2/2 = 2/(3)^{1/3}, \quad (4.118)$$

and

$$C_2 = k_1 + k_2 \quad (4.119)$$

Substituting (4.117) into (4.118) and simplifying, we observe that (4.118) is satisfied if, and only if, ν is equal to $1/3$. We conclude therefore that the analysis of section B, part 3, where certain constants of integration were assumed to be equal to zero, is valid if, and only if, ν is equal to $1/3$.

Equating the coefficients of ϵ^2 in (4.116) one determines that the following relations must hold:

$$C_3 = k_3 + k_4 - (3)^{1/3} k_1^2/7 + 2k_1 k_2 / (3)^{2/3} - k_2^2 / (4(3)^{2/3}) - 8/9, \quad (4.120)$$

$$6/C_1^5 - (1+\nu)C_2/2 = -4k_1/3 + k_2/3, \quad (4.121)$$

and

$$-1/C_1^4 = -1/(3)^{4/3}. \quad (4.122)$$

Equation (4.122) represents an identity obtainable from (4.117) and consequently must be discarded. Thus (4.117), (4.119), (4.120) and (4.121) give us four conditions for evaluating the constants. By matching the inner edge boundary-layer and membrane expansions for F an analogous set of equations is determined. They are:

$$C_4 = (3)^{1/3} (b/a)^{2/3}, \quad (4.123)$$

$$C_5 = k_1 (b/a)^{8/3} + k_2 (b/a)^{-2/3}, \quad (4.124)$$

$$C_6 = k_3 (b/a)^{8/3} + k_4 (b/a)^{-2/3} - (3)^{1/3} k_1^2 (b/a)^{4/3} \\ + 2k_1 k_2 (b/a)^{2/3} / (3)^{2/3} - k_2^2 (b/a)^{-8/3} / (4(3)^{2/3}) - 8/9, \quad (4.125)$$

and

$$6(b/a)^2 / C_4^5 + (1+\nu) (b/a)^{-2} C_5 / 2 = 4k_1 (b/a)^{2/3} / 3 - k_2 (b/a)^{-8/3} / 3. \quad (4.126)$$

In view of (4.117) and (4.123) the constants C_1 and C_4 are known. This leaves a total of eight constants still to be determined and we have at this point six equations, (4.119) through (4.121) and (4.124) through (4.126), for determining them. Normally we would expect to obtain the required additional two conditions from matching the variable α . However, upon doing so, we find that this leads to the same equations which we have already established. Therefore we must match another variable, such as the radial displacement for instance, and see if we obtain equations which are independent of those already established.

b. Matching conditions for the radial displacement U.

First we match the outer edge boundary-layer and membrane expansions of the radial displacement U in nondimensional form. From section A.2 of the appendix we have the following nondimensional expression for the radial displacement:

$$\frac{EhU}{2B} = \rho^{1/2} \left(F' - \left(\frac{1+\nu}{2} \right) \frac{F}{\rho} \right). \quad (4.127)$$

In order to obtain a three-term outer expansion for the quantity $EhU/2B$ let us write (4.127) as,

$$\frac{EhU}{2B} = \rho^{1/2} (F'_0 + F'_1 \epsilon + F'_2 \epsilon^2 - \frac{(1+\nu)}{2\rho} (F_0 + F_1 \epsilon + F_2 \epsilon^2)) , \quad (4.128)$$

where the functions F_0 , F_1 and F_2 in (4.128) are given by (4.65), (4.69) and (4.71) respectively. Making these substitutions and setting ν equal to $1/3$ the following expansion is obtained:

Three-term outer expansion for $EhU/2B$:

$$\begin{aligned} & \epsilon(2k_1 \rho^{5/6}/3 - k_2 \rho^{-5/6}) + \epsilon^2(2k_3 \rho^{5/6}/3 - k_4 \rho^{-5/6} - 4k_1^2 \rho^{3/2}/(7(3)^{2/3}) \\ & - 2k_1 k_2 \rho^{-1/6}/(3)^{5/3} + k_2^2 \rho^{-11/6}/(2(3)^{2/3}) + 16\rho^{-1/2}/27). \end{aligned} \quad (4.129)$$

We observe that the zeroth order term in ϵ has vanished from the above expansion. Expressing (4.129) in terms of the inner variable η by means of (4.84) and expanding the resulting expression for small ϵ we obtain:

Three-term inner expansion of the three-term outer expansion:

$$\begin{aligned} & \epsilon(2k_1/3 - k_2) + \epsilon^2(-5k_1 \eta/9 - 5k_2 \eta/6 + 2k_3/3 - k_4 - 4k_1^2/(7(3)^{2/3}) \\ & - 2k_1 k_2/(3)^{5/3} + k_2^2/(2(3)^{2/3}) + 16/27) \end{aligned} \quad (4.130)$$

Next we find the three-term inner expansion of the quantity $EhU/2B$. This is accomplished by first converting (4.127) to inner variables by means of (4.84). We get,

$$\frac{EhU}{2B} = -(1-\epsilon\eta)^{1/2} \left(\frac{1}{\epsilon} \frac{dF}{d\eta} + \frac{2}{3} \frac{F}{(1-\epsilon\eta)} \right), \quad (4.131)$$

where we have set ν equal to $1/3$ and it is understood that the inner representation of the function F applies. Substituting the formal expansion for F as given by (4.86b) into (4.131) we get,

$$\frac{EhU}{2B} = -(1-\epsilon\eta)^{1/2} \left(\frac{1}{\epsilon} \frac{dF_0}{d\eta} + \frac{dF_1}{d\eta} + \epsilon \frac{dF_2}{d\eta} + \frac{2}{3} \frac{F_0 + \epsilon F_1 + \epsilon^2 F_2}{(1-\epsilon\eta)} \right). \quad (4.132)$$

By virtue of (4.94a) it is seen that the above derivative of F_0 is zero and consequently the term in (4.132) containing $1/\epsilon$ vanishes. Replacing $(1-\epsilon\eta)^{1/2}$ and $(1-\epsilon\eta)^{-1}$ by their series expansions for small ϵ we arrive at the following three-term inner expansion for $EhU/2B$ in terms of F_0 , F_1 , F_2 and their derivatives, and, in addition, the quantity $dF_3/d\eta$.

$$\begin{aligned} \frac{EhU}{2B} = & -\frac{2}{3} F_0 - \frac{dF_1}{d\eta} + \epsilon \left(-\frac{\eta F_0}{3} - \frac{2F_1}{3} + \frac{\eta}{2} \frac{dF_1}{d\eta} - \frac{dF_2}{d\eta} \right) + \epsilon^2 \left(-\frac{\eta^2 F_0}{4} \right. \\ & \left. - \frac{\eta F_1}{3} + \frac{\eta^2}{8} \frac{dF_1}{d\eta} - \frac{2F_2}{3} + \frac{\eta}{2} \frac{dF_2}{d\eta} - \frac{dF_3}{d\eta} \right) \end{aligned} \quad (4.133)$$

Before proceeding further we must digress momentarily and calculate $dF_3/d\eta$ which represents the derivative of the third order term of (4.86b). From (4.86b) and (4.85b) the governing equation on F_3 is determined to be,

$$\frac{d^2 F_3}{d\eta^2} + \frac{\alpha_0 \alpha_1}{4} + \frac{\eta}{4} \alpha_0^2 = 0 \quad (4.134)$$

Substituting in (4.134) for α_0 and α_1 from (4.94b) and (4.95b), and setting ν equal to $1/3$ wherever it appears, gives us,

$$\begin{aligned} \frac{d^2 F_3}{d\eta^2} &= 4C_2/C_1^6 - 8\eta/(3C_1^4) - (8C_2/C_1^6) - (16/3 - C_2/C_1)(\eta/C_1^4) \\ &+ (2\eta^2/(3C_1^3)) e^{-(C_1/2)\eta} + (4C_2/C_1^6 + (C_2/C_1 - 8/3)(\eta/C_1^4)) \\ &+ 2\eta^2/(3C_1^3) e^{-C_1\eta}. \end{aligned} \quad (4.135)$$

The first integral to (4.135) is easily found to be,

$$\begin{aligned} \frac{dF_3}{d\eta} &= 4C_2 \eta/C_1^6 - 4\eta^2/(3C_1^4) - (4C_2/C_1^7) e^{-C_1\eta} + (16C_2/C_1^7) e^{-(C_1/2)\eta} \\ &- (16/3 - C_2/C_1)(4/C_1^6 + 2\eta/C_1^5) e^{-(C_1/2)\eta} \\ &- (C_2/C_1 - 8/3)(1/C_1^6 + \eta/C_1^5) e^{-C_1\eta} \\ &- (2/(3C_1^3))(2/C_1^3 + 2\eta/C_1^2 - \eta^2/C_1) e^{-C_1\eta} \\ &+ (2/(3C_1^3))(16/C_1^3 + 8\eta/C_1^2 + 2\eta^2/C_1) e^{-(C_1/2)\eta} + C_7, \end{aligned} \quad (4.136)$$

where C_7 is a constant of integration which we determine immediately.

As determined from (4.92) and (4.86b), $dF_3/d\eta$ must satisfy the following condition at $\eta = 0$ for ν equal to $1/3$:

$$\frac{dF_3}{d\eta} + \frac{2}{3} F_2 = 0. \quad (4.137)$$

Substituting (4.96a) and (4.136) into (4.137) and setting η equal to zero we find that,

$$C_7 = -15C_2/C_1^7 - 2C_3/3. \quad (4.138)$$

Thus $df_3/d\eta$ is known in terms of the original constants C_1 , C_2 , and C_3 .

Having determined $dF_3/d\eta$ the outer representation of (4.133) can now be obtained. The steps are as follows:

- (a.) Substitute (4.94a), (4.95a), (4.96a) and (4.136) into (4.133).
- (b.) Rewrite the resulting expression in terms of the outer variable ρ by means of (4.84), and then
- (c.) Expand the result for small ϵ and neglect terms of exponential order.

The result of all of this, rewritten in terms of η , gives us the following expansion for $EhU/2B$ in which we have used (4.117).

Three-term outer expansion of the three-term inner expansion:

$$-2/(3)^{2/3} \epsilon + (5C_2/(3)^{4/3} - 2C_2\eta/3 - \eta/(3)^{5/3}) \epsilon^2 \quad (4.139)$$

According to the matching principle (4.130) and (4.139) must agree to appropriate orders of ϵ . Thus for the zeroth order terms in ϵ to agree we must have

$$-2/(3)^{2/3} = 2k_1/3 - k_2 \quad (4.140)$$

By combining (4.19) and (4.21) it is easy to show that (4.140) represents an identity obtainable from known results and therefore it must be discarded. By equating the second order terms in (4.130) and (4.139) the following two equations are obtained:

$$-5k_1/9 - 5k_2/6 = -2C_2/3 - 1/(3)^{5/3}, \quad (4.141)$$

and,

$$2k_3/3 - k_4 - 4k_1^2/(7(3)^{2/3}) - 2k_1k_2/(3)^{5/3} + k_2^2/(2(3)^{2/3}) + 16/27 = 5C_2/(3)^{4/3}. \quad (4.142)$$

Equation (4.141) can be obtained by combining (4.119) and (4.121) and hence it must be discarded. We find that (4.142) is independent of any previous relations and hence it represents our seventh condition for determining the constants.

The eighth and final equation is obtained similarly by matching the inner edge boundary-layer and membrane expansions for the quantity $EhU/2B$. Upon doing so our final independent equation becomes,

$$2k_3(b/a)^{5/3}/3 - k_4(b/a)^{-5/3} - 4k_1^2(b/a)^3/(7(3)^{2/3}) - 2k_1k_2(b/a)^{-1/3}/(3)^{5/3} + k_2^2(b/a)^{-11/3}/(2(3)^{2/3}) + 16(b/a)^{-1}/27 = -5C_5(b/a)^{-5/3}/(3)^{4/3}. \quad (4.143)$$

Summarizing, we have shown that the solutions for F_0 , α_0 , F_1, \dots, α_2 of Section B, part 3, are valid if, and only if, ν is equal to $1/3$; and have established the necessary conditions for evaluating the constants of the corresponding three-term boundary-layer and membrane expansions.

c. Solutions for the constants of integration

From (4.117) and (4.123) we have,

$$C_1 = (3)^{1/3}, \quad (4.144)$$

and,

$$C_4 = (3)^{1/3} (b/a)^{2/3}. \quad (4.145)$$

Solving (4.119), (4.121), (4.124) and (4.126) simultaneously we get,

$$C_2 = -(3)^{1/3} (1 + 5(b/a)^{4/3}/3 + 2(b/a)^{10/3}/3) / (1 - (b/a)^{10/3}), \quad (4.146)$$

$$C_5 = -(3)^{1/3} (b/a)^4 (1 + 5(b/a)^{-4/3}/3 + 2(b/a)^{-10/3}/3) / (1 - (b/a)^{10/3}), \quad (4.147)$$

$$k_1 = -(3)^{1/3} (1 + (b/a)^{4/3}) / (1 - (b/a)^{10/3}), \quad (4.148)$$

and,

$$k_2 = -(2)(3)^{-2/3} (b/a)^{4/3} (1 + (b/a)^2) / (1 - (b/a)^{10/3}). \quad (4.149)$$

Solving (4.142) and (4.143) simultaneously gives us,

$$\begin{aligned} k_3 = & (4k_1^2 ((b/a)^{-5/3} - (b/a)^3) \gamma(7(3)^{2/3}) + 2k_1 k_2 ((b/a)^{-5/3} - (b/a)^{-1/3}) / \\ & (3)^{5/3} - k_2^2 ((b/a)^{-5/3} - (b/a)^{-11/3}) \gamma(2(3)^{2/3}) - 16((b/a)^{-5/3} - (b/a)^{-1}) / 27 \\ & + 5(b/a)^{-5/3} (C_2 + C_5) / (3)^{4/3} / ((b/a)^{-5/3} - (b/a)^{5/3}) (2/3)), \quad (4.150) \end{aligned}$$

and

$$\begin{aligned} k_4 = & (4k_1^2 ((b/a)^3 - (b/a)^{5/3}) + 2k_1 k_2 ((b/a)^{-1/3} - (b/a)^{5/3}) / (3)^{5/3} \\ & + k_2^2 ((b/a)^{5/3} - (b/a)^{-11/3}) / (2(3)^{2/3}) + 16((b/a)^{5/3} - (b/a)^{-1}) / 27 \\ & - 5(C_2 (b/a)^{5/3} + C_5 (b/a)^{-5/3}) / (3)^{4/3} / ((b/a)^{5/3} - (b/a)^{-5/3}). \quad (4.151) \end{aligned}$$

For evaluating C_3 and C_6 we use (4.120) and (4.125).

2. Poisson's ratio less than 1/3.

Herein we develop the necessary conditions for evaluating the constants A_0 , B_0 , A_1 and B_1 of section B, part 1, and C_1 , C_2 , C_4 and C_5 of the boundary-layer expansions for the case when ν is less than 1/3. The resulting system of equations is found to contain certain transcendental relationships which preclude a closed form solution. However it is not difficult to "solve" the equations by numerical methods and this is demonstrated in section A.1 of the appendix. At the conclusion of the section herein we verify that the analysis of section B, part 1 is valid if, and only if, Poisson's ratio is 1/3.

a. Matching conditions for the variable F

The outer edge boundary-layer and membrane expansions are matched first. From (4.30) and (4.47) let us write the

Two-term outer expansion for F:

$$\frac{\sin^2(\gamma/2)}{A_0^2} + \epsilon \left(\frac{A_1}{2A_0^4} \left(1 + \frac{\sin^2 \gamma}{8 \sin^2 \gamma/2} - \frac{3\gamma \cot \gamma/2}{4} \right) + 2A_0 B_1 \cot \gamma/2 \right) \quad (4.152)$$

Clearly the application of the matching principle becomes more complicated than before due to the presence of the parameter γ in the above expansion.

In order to follow the step by step procedure used previously the outer variable γ must be put in terms of the inner variable η and therein lies the problem. From (4.35) and (4.84) we see that γ and η are related by the following equation:

$$4A_0^3(1-\epsilon\eta) = \gamma - \sin\gamma + B_0. \quad (4.153)$$

This cannot be inverted in closed form. However since we are looking for the inner representation of (4.152) in powers of ϵ as ϵ tends to zero, it suffices to determine γ in the following form:

$$\gamma = \gamma_0 + \gamma_1\epsilon + \gamma_2\epsilon^2 + \dots \quad (4.154)$$

where $\gamma_0, \gamma_1, \gamma_2$, etc. are functions of η and independent of ϵ . To determine the functions $\gamma_0, \gamma_1, \gamma_2$, etc. first substitute (4.154) into (4.153) and group the argument of the sine function as indicated below.

$$4A_0^3(1-\epsilon\eta) = \gamma_0 + \gamma_1\epsilon + \gamma_2\epsilon^2 + \dots - \sin(\gamma_0 + \epsilon(\gamma_1 + \gamma_2\epsilon + \dots)) + B_0. \quad (4.155)$$

Using the trigonometric identity for the sine of a sum of two arguments (4.155) can be written as,

$$4A_0^3 - 4A_0^3\epsilon\eta = \gamma_0 + \gamma_1\epsilon + \gamma_2\epsilon^2 + \dots - \sin\gamma_0 \cos(\epsilon(\gamma_1 + \gamma_2\epsilon + \dots)) - \cos\gamma_0 \sin(\epsilon(\gamma_1 + \gamma_2\epsilon + \dots)) + B_0. \quad (4.156)$$

Next we replace the trigonometric functions involving ϵ with their series representations and neglect higher order terms to get,

$$4A_0^3 - 4A_0^3\epsilon\eta = \gamma_0 + \gamma_1\epsilon + \gamma_2\epsilon^2 - \sin(\gamma_0)(1 - \gamma_1^2\epsilon^2/2) - \cos(\gamma_0)(\gamma_1\epsilon + \gamma_2\epsilon^2) + B_0 + O(\epsilon^3). \quad (4.157)$$

Equating coefficients of like powers of ϵ from each side of (4.157) we get the following:

$$4A_o^3 - B_o = \gamma_o - \sin\gamma_o, \quad (4.158)$$

$$-4A_o^3 \eta = \gamma_1 (1 - \cos\gamma_o), \quad (4.159)$$

$$\gamma_2 (1 - \cos\gamma_o) + \frac{\gamma_1^2}{2} \sin\gamma_o = 0, \quad (4.160)$$

The first equation shows γ_o is a constant which we later identify as being the value of γ at the outer edge of the plate. From the last two equations we get,

$$\gamma_1 = \frac{-4A_o^3 \eta}{1 - \cos\gamma_o}, \quad (4.161)$$

and

$$\gamma_2 = -\frac{8A_o^6 \sin\gamma_o \eta^2}{(1 - \cos\gamma_o)^3}. \quad (4.162)$$

Therefore the expansion for γ takes the form,

$$\gamma = \gamma_o - \frac{4A_o^3 \eta \epsilon}{(1 - \cos\gamma_o)} - \frac{8A_o^6 \sin\gamma_o \eta^2 \epsilon^2}{(1 - \cos\gamma_o)^3} + 0(\epsilon^3). \quad (4.163)$$

By setting η equal to zero in (4.163) we identify γ_o as being the value of γ at the outer edge of the plate. Denoting this value of γ by γ_a , (4.158) and (4.163) can be written as,

$$4A_o^3 - B_o = \gamma_a - \sin\gamma_a, \quad (4.164)$$

and

$$\gamma = \gamma_a - \frac{4A_o^3 \eta \epsilon}{(1 - \cos\gamma_a)} - \frac{8A_o^6 \sin\gamma_a \eta^2 \epsilon^2}{(1 - \cos\gamma_a)^3} + 0(\epsilon^3). \quad (4.165)$$

By substituting (4.165) into (4.152) and expanding the trigonometric functions in the same fashion as shown in going from (4.155) to (4.157) we determine the

Two-term inner expansion of the two-term outer expansion:

$$\frac{\sin^2(\gamma_a/2)}{A_o^2} + \epsilon \left(-\frac{2A_o \sin \gamma_a \eta}{(1-\cos \gamma_a)} + \frac{A_1}{2A_o^4} \left(1 + \frac{\sin^2 \gamma_a}{8\sin^2(\gamma_a/2)} - \frac{3\gamma_a \cot(\gamma_a/2)}{4} \right) + 2A_o B_1 \cot(\gamma_a/2) \right) \quad (4.166)$$

From (4.94a) and (4.95a) we next determine the

Two-term inner expansion for F:

$$C_1^2 + \epsilon (C_2 - (1+\nu)C_1^2 \eta / 2) \quad (4.167)$$

Writing (4.167) in terms of the outer variable ρ from (4.84), expanding the result for small ϵ , and rewriting the resulting expression in terms of η we get the

Two-term outer expansion of the two-term inner expansion:

$$C_1^2 + \epsilon (C_2 - (1+\nu)C_1^2 \eta / 2) \quad (4.168)$$

According to the matching principle (4.166) and (4.168) must agree to appropriate orders of ϵ . Thus we must have,

$$C_1^2 = \sin^2(\gamma_a/2)/A_o^2 \quad (4.169)$$

which follows by equating the zeroth order terms in ϵ . From the first order terms in ϵ we get the following two equations:

$$C_2 = \frac{A_1}{2A_o^4} \left(1 + \frac{\sin^2 \gamma_a}{8 \sin^2(\gamma_a/2)} - \frac{3\gamma_a \cot(\gamma_a/2)}{4} \right) + 2A_o B_1 \cot(\gamma_a/2), \quad (4.170)$$

and

$$(1+\nu)C_1^2/2 = \frac{2A_o \sin \gamma_a}{(1-\cos \gamma_a)}. \quad (4.171)$$

In view of (4.47) it follows that (4.170) can be expressed as,

$$C_2 = F_1(\gamma_a), \quad (4.172)$$

where $F_1(\gamma_a)$ represents the value of the function F_1 at the outer edge of the plate. Using the identity,

$$\sin^2(\gamma/2) = (1-\cos \gamma)/2, \quad (4.173)$$

and (4.169), equation (4.171) can be written in the form,

$$A_o^3 \sin \gamma_a - (1+\nu)(1-\cos \gamma_a)^2/8 = 0. \quad (4.174)$$

Summarizing the results obtained thus far, we have,

$$C_1^2 = \sin^2(\gamma_a/2)/A_o^2, \quad (4.175)$$

$$4A_o^3 - B_o = \gamma_a - \sin \gamma_a, \quad (4.176)$$

$$A_o^3 \sin \gamma_a - (1+\nu)(1-\cos \gamma_a)^2/8 = 0, \quad (4.177)$$

and ,

$$C_2 = F_1(\gamma_a) , \quad (4.178)$$

which follow from (4.169), (4.164), (4.174) and (4.172).

A similar process at the inner edge of the plate yields analogous equations. They are,

$$C_4^2 = \sin^2(\gamma_b/2)/A_0^2 , \quad (4.179)$$

$$4A_0^3 (b/a)^2 - B_0 = \gamma_b - \sin\gamma_b , \quad (4.180)$$

$$A_0^3 \sin\gamma_b - (1+\nu)(b/a)^{-2}(1-\cos\gamma_b)^2/8 = 0 \quad (4.181)$$

and

$$C_5 = F_1(\gamma_b) , \quad (4.182)$$

where γ_b and $F_1(\gamma_b)$ represent the values of γ and F_1 at the inner edge of the plate respectively.

The above eight equations contain ten unknowns, two of which, A_1 and B_1 , do not appear explicitly as they are contained in the expression for F_1 . Thus two additional equations are required. These are obtained in the same fashion as before in section 1, part b, by matching the expansions for the radial displacement.

b. Matching conditions for the radial displacement U.

First we match the membrane and outer edge boundary-layer expansions for the nondimensional form of the radial displacement, $EhU/2B$. From (4.127), the two-term outer expansion for this quantity can be expressed formally as,

$$\frac{EhU}{2B} = \rho^{1/2} (F'_0 + F'_1 \epsilon - \frac{(1+\nu)}{2\rho} (F_0 + F_1 \epsilon)) \quad (4.183)$$

where it is understood that F_0 and F_1 are given by (4.30) and (4.47). For determining the derivatives in (4.183), let us write,

$$F'_0 = \frac{dF}{d\gamma} \frac{d\gamma}{d\rho} = \frac{\sin\gamma}{2A_0^2} \frac{d\gamma}{d\rho}, \quad (4.184)$$

where we have used (4.30). The derivative $d\gamma/d\rho$ is found from (4.35) to be,

$$\frac{d\gamma}{d\rho} = \frac{4A_0^3}{1-\cos\gamma}. \quad (4.185)$$

Therefore (4.184) becomes,

$$F'_0 = \frac{2A_0 \sin\gamma}{1-\cos\gamma}. \quad (4.186)$$

The derivative of F_1 is most easily evaluated by using (4.41) in the following form:

$$F'_1 = \frac{A_1}{F'_0} - \frac{2F_1}{F_0^2 F'_0}. \quad (4.187)$$

From (4.30) and (4.186) we can write (4.187) as,

$$F'_1 = \frac{A_1(1-\cos\gamma)}{2A_0 \sin\gamma} - \frac{A_0^3(1-\cos\gamma)F_1}{(\sin\gamma)\sin^4(\gamma/2)}. \quad (4.188)$$

Using the identity,

$$\sin^4(\gamma/2) = (1-\cos\gamma)^2/4, \quad (4.189)$$

(4.188) becomes,

$$F_1' = \frac{A_1(1-\cos\gamma)}{2A_o \sin\gamma} - \frac{4A_o^3 F_1}{\sin\gamma(1-\cos\gamma)}. \quad (4.190)$$

Substituting (4.186) and (4.190) into (4.183) gives us the

Two-term outer expansion for the quantity EhU/2B:

$$\rho^{1/2} \left(\frac{2A_o \sin\gamma}{1-\cos\gamma} - \left(\frac{1+\nu}{2}\right) \left(\frac{\sin^2 \gamma/2}{A_o^2 \rho}\right) + \epsilon \left(\frac{A_1(1-\cos\gamma)}{2A_o \sin\gamma} - \frac{4A_o^3 F_1}{\sin\gamma(1-\cos\gamma)} - \frac{(1+\nu)F_1}{2\rho} \right) \right) \quad (4.191)$$

In order to find the inner representation of (4.191) let us first expand the trigonometric functions therein, as demonstrated below for $\sin \gamma$. From (4.165) we determine that,

$$\sin\gamma = \sin\gamma_a - \frac{4A_o^3 \cos(\gamma_a)\eta \epsilon}{1-\cos\gamma_a} + O(\epsilon^2). \quad (4.192)$$

The above expansion is substituted in (4.191) when $\sin\gamma$ occurs in the numerator of a term. For the case when $\sin\gamma$ appears in the denominator we factor (4.192) as,

$$\sin\gamma = \sin\gamma_a \left(1 - \frac{4A_o^3 \cos(\gamma_a)\eta \epsilon}{\sin\gamma_a(1-\cos\gamma_a)} + O(\epsilon^2) \right) \quad (4.193)$$

and then use the series expansion for one over the quantity in brackets. Thus we obtain,

$$(\sin\gamma)^{-1} = (\sin\gamma_a)^{-1} \left(1 + \frac{4A_o^3 \cos(\gamma_a)\eta \epsilon}{\sin\gamma_a(1-\cos\gamma_a)} + O(\epsilon^2) \right). \quad (4.194)$$

Wherever powers of ρ appear we use (4.84) and expand accordingly.

In this way we arrive at the

Two-term inner expansion of the two-term outer expansion:

$$\epsilon \left(\frac{8A_0^4 \eta}{(1-\cos\gamma_a)^2} - \frac{(1-\nu)A_0 \sin\gamma_a \eta}{(1-\cos\gamma_a)} + \frac{A_1(1-\cos\gamma_a)}{2A_0 \sin\gamma_a} - \frac{4A_0^3 F_1(\gamma_a)}{\sin\gamma_a (1-\cos\gamma_a)} - \frac{(1+\nu)F_1(\gamma_a)}{2} \right), \quad (4.195)$$

in which the zeroth order terms in ϵ vanish.

The two-term inner expansion for $EhU/2B$ is obtained exactly as before in section 1, part b except that we do not set ν equal to $1/3$. The following result, in terms of F_0 , F_1 and F_2 , and their derivatives, is obtained:

$$\frac{EhU}{2B} = - \left(\frac{dF_1}{d\eta} + \left(\frac{1+\nu}{2}\right)F_0 \right) + \left(- \frac{(1+\nu)}{4} \eta F_0 - \left(\frac{1+\nu}{2}\right) F_1 + \frac{\eta}{2} \frac{dF_1}{d\eta} - \frac{dF_2}{d\eta} \right) \epsilon, \quad (4.196)$$

where it is understood that F_0 , F_1 and F_2 are given by (4.94a), (4.95a) and (4.96a). Making these substitutions in (4.196) we get the

Two-term inner expansion for the quantity $EhU/2B$:

$$\epsilon \left(- \frac{6}{C_1^5} + \frac{2\eta}{C_1^4} + \frac{(1-\nu^2)C_1^2 \eta}{4} + \frac{8e}{C_1^5} - \frac{(C_1/2)\eta}{C_1^5} - \frac{C_1 \eta}{C_1^5} \right), \quad (4.197)$$

where the zeroth order term in (4.197) has vanished. The outer representation of (4.197) is obtained by expressing it in terms of the outer variable ρ by means of (4.84) and expanding the result for small ϵ . Upon doing so, and rewriting the result in terms of η , we get the,

Two-term outer expansion of the two-term inner expansion:

$$\epsilon \left(-6/C_1^5 + 2\eta /C_1^4 - (1-\nu^2)C_1^2\eta /4 \right), \quad (4.198)$$

Note that the outer representations of the last two terms in (4.197) are of exponential order and consequently these terms do not enter into (4.198). According to the matching principle (4.195) and (4.198) must agree. Thus the following two equations must hold:

$$8A_o^4/(1-\cos\gamma_a)^2 - (1-\nu)A_o \sin\gamma_a / (1-\cos\gamma_a) = 2/C_1^4 - (1-\nu^2)C_1^2/4, \quad (4.199)$$

and

$$A_1(1-\cos\gamma_a)/(2A_o \sin\gamma_a) - 4A_o^3 F_1(\gamma_a)/(\sin\gamma_a (1 - \cos\gamma_a)) - (1+\nu)F_1(\gamma_a)/2 = -6/C_1^5 \quad (4.200)$$

From (4.169), (4.171) and (4.173) it can be shown that (4.199) represents an identity based on previous results and therefore it is discarded. Thus (4.200) represents the ninth condition for determining the constants.

The tenth and final condition is obtained similarly by matching the two-term inner edge boundary-layer and membrane expansions for EhU/2B. We find that the tenth condition is given by

$$A_1(1-\cos\gamma_b)/(2A_o \sin\gamma_b) - 4A_o^3 F_1(\gamma_b)/(\sin\gamma_b (1-\cos\gamma_b)) - (1+\nu)(b/a)^{-2} F_1(\gamma_b)/2 = 6(b/a)^2/C_4^5. \quad (4.201)$$

Summarizing, the ten conditions for determining the constants are given by (4.175) through (4.182), (4.200), and (4.201) where the quantity F_1 contained therein is given by (4.47). As mentioned previously the procedure for solving these equations is given in Section A.1 of the appendix.

From (4.176), (4.177), (4.180) and (4.181) it is easy to show that the maximum value of Poisson's ratio admitted by the solutions in section B, part 1, is ν equal to 1/3. From the second and last of the above mentioned equations we get,

$$(b/a)^2 = \frac{\sin\gamma_a (1 - \cos\gamma_b)^2}{\sin\gamma_b (1 - \cos\gamma_a)^2} \quad (4.202)$$

Eliminating B_0 between (4.176) and (4.180), and using (4.181) gives us,

$$(b/a)^2 = 1 - \left(\frac{2}{1 + \nu}\right) \frac{\sin\gamma_a}{(1 - \cos\gamma_a)^2} (\gamma_a - \gamma_b - (\sin\gamma_a - \sin\gamma_b)). \quad (4.203)$$

A limiting process on (4.202) shows that the ratio (b/a) approaches zero as γ_b approaches zero. Therefore, in (4.203) let us set (b/a) and γ_b equal to zero and solve the resulting expression for $(1 + \nu)/2$ to get,

$$\frac{1 + \nu}{2} = \frac{\sin\gamma_a (\gamma_a - \sin\gamma_a)}{(1 - \cos\gamma_a)^2} \quad (4.204)$$

Equation (4.204) may be viewed as an expression defining Poisson's ratio ν as a function of γ_a . For values of γ_a which lead to meaningful values for ν , i.e. for $0 \leq \gamma_a \leq \pi$, the right hand side of (4.204) assumes a maximum value of 2/3 which represents its limit as γ_a approaches zero. Thus we conclude that the maximum value of ν allowed by the solution of section B, part 1, is 1/3. However it can be shown that the solution under discussion becomes indeterminate at ν equal to 1/3 and therefore must be restricted to values of ν less than 1/3.

3. Poisson's ratio greater than 1/3

The conditions for determining the constants A_0 , B_0 , A_1 and B_1 of section B, part 2, and C_1 , C_2 , C_4 and C_5 of the boundary-layer expansions for the case when ν is greater than 1/3, are presented in this section. Since the matching process follows the above section step by step we shall present only the results for the sake of brevity.

As in the previous case we find that the system of equations herein is transcendental in nature. This of course precludes a closed form solution and the equations must be "solved" by numerical methods. Such methods would closely parallel those given in section A.1 of the appendix and we therefore do not attempt to solve the equations given below.

a. Matching conditions for the variable F

By matching the two-term outer edge boundary-layer and membrane expansions for F we find that the following conditions must hold:

$$C_1^2 = \sinh^2(\bar{\delta}_a/2)/A_0^2, \quad (4.205)$$

$$A_0^3 \sinh \bar{\delta}_a - (1 + \nu)(1 - \cosh \bar{\delta}_a)^2/8 = 0, \quad (4.206)$$

and

$$C_2 = F_1(\bar{\delta}_a), \quad (4.207)$$

where $\bar{\delta}_a$ represents the value of $\bar{\delta}$ at the outer edge of the plate.

The function F_1 in (4.207) is given by (4.60) and the symbol $F_1(\bar{\delta}_a)$ of course stands for the value of F_1 at the outer edge of the plate.

In arriving at the equations above we obtain one other equation which we use in finding the constants. It is,

$$4A_0^3 + B_0 = \sinh \bar{\delta}_a - \bar{\delta}_a, \quad (4.208)$$

which is also obtainable from (4.57).

From matching the two-term inner edge boundary-layer and membrane expansions for F we obtain four more conditions which are similar to the above. They are,

$$C_4^2 = \sinh^2(\bar{\delta}_b/2)/A_0^2, \quad (4.209)$$

$$A_0^3 \sinh \bar{\delta}_b - (1 + \nu)(a/b)^2(1 - \cosh \bar{\delta}_b)^2/8 = 0, \quad (4.210)$$

$$C_5 = F_1(\bar{\delta}_b), \quad (4.211)$$

and

$$4A_0^3(b/a)^2 + B_0 = \sinh \bar{\delta}_b - \bar{\delta}_b, \quad (4.212)$$

in which $\bar{\delta}_b$ represents the value of $\bar{\delta}$ at the inner edge of the plate.

b. Matching conditions for the radial displacement U

The final two equations for finding the constants are obtained by matching the expansions for the radial displacement U . From matching the outer edge boundary-layer and membrane expansions we find that,

$$-6/C_1^5 = 4A_0^3 F_1(\bar{\delta}_a) / ((1 - \cosh \bar{\delta}_a) \sinh \bar{\delta}_a) - ((1 + \nu)/2) F_1(\bar{\delta}_a) -$$

$$A_1(1 - \cosh \bar{\delta}_a) / (2A_0 \sinh \bar{\delta}_a), \quad (4.213)$$

and by matching the inner edge boundary-layer and membrane expansions we get,

$$6(b/a)^2/C_4^5 = 4A_0^3 F_1(\bar{\delta}_b) / ((1 - \cosh\bar{\delta}_b) \sinh\bar{\delta}_b) -$$

$$((1 + \nu)/2)(a/b)^2 F_1(\bar{\delta}_b) - A_1(1 - \cosh\bar{\delta}_b) / (2A_0 \sinh\bar{\delta}_b). \quad (4.214)$$

Thus the ten equations above provide the necessary conditions for finding the eight constants of integration and the values of $\bar{\delta}_a$ and $\bar{\delta}_b$. By going through an analysis similar to that at the conclusion of the previous section it can be shown that the analysis of section B, part 2, is applicable only for ν greater than 1/3.

E. Summary

We have found solutions to (2.15) and (2.16) in the form of perturbation expansions which apply separately in the boundary-layer and interior regions of the plate when the deformations are large. The solutions for the interior region of the plate take one of three possible forms depending on whether Poisson's ratio ν is less than, greater than or equal to 1/3 whereas the solutions for the boundary-layer zones do not change in form.

The conditions for finding the constants of integration in the expansions are derived, and in one case, where ν equals 1/3, closed form expressions are obtained for the constants. For the case when ν is less than 1/3 the necessary (approximate) numerical solution for the constants is given in the appendix (see section A.1), whereas for the remaining case of ν greater than 1/3 the solution is not shown.

Three terms are found for the boundary-layer expansions. The membrane expansions are found to two terms except for the case where ν is equal to 1/3 where we obtain three terms of the expansions. These results are next summarized in more detail.

1. Poisson's ratio less than 1/3

The expansions for the dependent variables α and F , which are given in terms of the physical variables $\bar{\beta}$ and $\bar{\psi}$ by (4.12) and (4.13) respectively, are as follows:

a. Membrane expansions

$$\begin{aligned} \alpha = & 4A_0^4 \rho / \sin^2(\gamma/2) + \epsilon (3A_1 \gamma \rho \cos(\gamma/2) / (2 \sin^5(\gamma/2)) - \\ & 2A_1 \rho / (\sin^4(\gamma/2)) - A_1 \rho \sin^2 \gamma / (4 \sin^6(\gamma/2)) - \\ & 8A_0^5 B_1 \rho \cos(\gamma/2) / \sin^5(\gamma/2)) + O(\epsilon^2), \end{aligned} \quad (4.215)$$

and

$$\begin{aligned} F = & \sin^2(\gamma/2) / A_0^2 + \epsilon (A_1 / (2A_0^4) + A_1 \sin^2 \gamma / (16A_0^4 \sin^2(\gamma/2)) - \\ & 3A_1 \gamma \cot(\gamma/2) / (8A_0^4) + 2A_0 B_1 \cot(\gamma/2)) + O(\epsilon^2). \end{aligned} \quad (4.216)$$

The above are obtained from (4.36), (4.48), (4.30) and (4.47). The independent variable ρ is related to the parameter γ by (4.35) and ρ is found in terms of the radial coordinate \bar{r} through (4.11) and (2.6).

b. Boundary-layer expansions

For the outer edge boundary-layer zones we have the following expansions for α and F which follow from (4.94) and (4.95),

$$\alpha = (4/C_1^2)(1 - e^{-(C_1/2)\eta}) + \epsilon (-4C_2/C_1^4 - 2(1 - \nu)\eta/C_1^2 + (4C_2/C_1^4 + C_2\eta/C_1^3 + (3 - \nu)\eta/(2C_1^2) + (3 - \nu)\eta^2/(4C_1)) e^{-(C_1/2)\eta} + 0(\epsilon^2), \quad (4.217)$$

and

$$F = C_1^2 + \epsilon (C_2 - (1 + \nu)C_1^2\eta/2) + 0(\epsilon^2). \quad (4.218)$$

In the above equations ρ and η are related by (4.84). From (4.105) and (4.106) the following expansions are obtained which apply in the boundary-layer zone at the inner edge of the plate.

$$\alpha = (4/C_4^2)(b/a)^2(1 - e^{-(C_4/2)(a/b)^2\zeta}) + \epsilon (-4C_5(b/a)^2/C_4^2 + 2(1 - \nu)\zeta/C_4^2 + (4C_5(b/a)^2/C_4^4 + C_5\zeta/C_4^3 - (3 - \nu)\zeta/(2C_4^2) - (3 - \nu)(a/b)^2\zeta^2/(4C_4)) e^{-(C_4/2)(a/b)^2\zeta} + 0(\epsilon^2), \quad (4.219)$$

and,

$$F = C_4^2 + \epsilon (C_5 + (1 + \nu)C_4^2(a/b)^2\zeta/2) + 0(\epsilon^2), \quad (4.220)$$

where ρ and ζ are related by (4.97). The constants A_0 , C_1 , C_2 , C_4 , C_5 appearing above, as well as B_0 , are found according to the method described in section A.1 of the appendix. The stresses, displacements, etc. are found by using the formulas of section A.2 of the appendix and by following the same procedure which is discussed in part 3 below.

2. Poisson's ratio greater than 1/3

a. Membrane expansions

First we list the membrane expansions which follow from (4.52), (4.18a), (4.19a) and (4.60). They are,

$$\begin{aligned} \alpha = & 4A_o^2 \rho / \sinh^2(\bar{\delta}/2) + \epsilon (3A_1 \bar{\delta} \rho \cosh(\bar{\delta}/2) / (2 \sinh^5(\bar{\delta}/2)) - \\ & 2A_1 \rho / \sinh^4(\bar{\delta}/2) - A_1 \rho \sinh^2 \bar{\delta} / (4 \sinh^6(\bar{\delta}/2)) - \\ & 8A_o^5 B_1 \rho \cosh(\bar{\delta}/2) / \sinh^5(\bar{\delta}/2)) + O(\epsilon^2), \end{aligned} \quad (4.221)$$

and,

$$\begin{aligned} F = & \sinh^2(\bar{\delta}/2) / A_o^2 + \epsilon (A_1 / (2A_o^4) + A_1 \sinh^2 \bar{\delta} / (16A_o^4 \sinh^2(\bar{\delta}/2)) - \\ & 3A_1 \bar{\delta} \coth(\bar{\delta}/2) / (8A_o^4) + 2A_o B_1 \coth(\bar{\delta}/2)) + O(\epsilon^2). \end{aligned} \quad (4.222)$$

The independent variable ρ and the parameter $\bar{\delta}$ are related through (4.57).

b. Boundary-layer expansions

The boundary-layer expansions for the case under discussion coincide with those given by (4.217) through (4.220). However, the constants appearing in those equations, as well as A_o , B_o , A_1 , B_1 , $\bar{\delta}_a$ and $\bar{\delta}_b$ are to be determined by solving (4.205) through (4.214). Stresses, displacements, etc. follow according to the discussion at the conclusion of the following section.

3. Poisson's ratio equal to 1/3

As we recall the expansions are found to three terms for ν

equal to $1/3$. Since the third terms in the expansions are very lengthy we do not write these terms explicitly, but simply refer to the appropriate expressions for them which appear in previous sections.

a. Membrane expansions

First we find from (4.19a), (4.65), (4.66) and (4.67) that α_1 takes the form,

$$\alpha_1 = -4(k_1\rho + k_2\rho^{-2/3})/(3)^{4/3}, \quad (4.223)$$

and from (4.20a) we get,

$$\alpha_2 = (4\rho^2\alpha_0'' - \alpha_1 F_1 - \alpha_0 F_2)/F_0. \quad (4.224)$$

Therefore we can write,

$$\alpha = 4\rho^{1/3}/(3)^{2/3} + \epsilon(-4(k_1\rho + k_2\rho^{-2/3})/(3)^{4/3}) + \epsilon^2\alpha_2 + O(\epsilon^3) \quad (4.225)$$

in which α_2 is calculated from (4.224) with the aid of (4.65), (4.66), (4.223), (4.69) and (4.71). For the variable F we have,

$$F = (3\rho)^{2/3} + \epsilon(k_1\rho^{4/3} + k_2\rho^{-1/3}) + \epsilon^2 F_2 + O(\epsilon^3), \quad (4.226)$$

in which F_2 is given by (4.71).

b. Boundary-layer expansions

The boundary-layer expansions for the vicinity of the outer edge of the plate are identical to (4.217) and (4.218) except that we attach the third term to the expansions. Thus let us simply indicate that,

$$\alpha = \alpha_0 + \alpha_1 \epsilon + \alpha_2 \epsilon^2 + O(\epsilon^3) , \quad (4.227)$$

and ,

$$F = F_0 + F_1 \epsilon + F_2 \epsilon^2 + O(\epsilon^3) , \quad (4.228)$$

where α_0 , α_1 , α_2 , F_0 , F_1 and F_2 are given by (4.94b), (4.95b), (4.96b), (4.94a), (4.95a) and (4.96a) respectively with ν equal to 1/3.

For the vicinity of the inner edge of the plate we of course have the same type of expansions as shown in (4.227) and (4.228) except that the terms α_0 , α_1 , etc. are given by (4.105b), (4.106b), (4.107b), (4.105a), (4.106a) and (4.107a) respectively.

The constants which appear in the expressions in this section are known in closed form and these are given by (4.144) through (4.151) and by (4.120) and (4.125).

c. Composite Expansions

In making numerical computations for the stresses, etc. it is convenient to work with expansions which are uniformly valid over the total extent of the plate (see Van Dyke (4) p. 83). This eliminates the problem of having to average the results of the boundary-layer and membrane expansions in the regions of the plate slightly away from its edges where it is not clear which expansion should be used. Thus we construct from the boundary-layer and membrane expansions what is referred to as a composite expansion which is uniformly valid. Letting the subscript "c" stand for "composite expansion" we determine from sections (a) and (b) above that the three-term composite expansion for F becomes,

$$\begin{aligned}
 F_c = & (3\rho)^{2/3} + \epsilon (k_1\rho^{4/3} + k_2\rho^{-1/3}) + \epsilon^2 (k_3\rho^{4/3} + k_4\rho^{-1/3} - \\
 & (3)^{1/3} k_1^2 \rho^2 / 7 + 2k_1 k_2 \rho^{1/3} / (3)^{2/3} - k_2^2 \rho^{-4/3} / (4(3)^{2/3}) - \\
 & 8/9 - (2/C_1^6) e^{-C_1(1-\rho)/\epsilon} + (16/C_1^6) e^{-(C_1/2)(1-\rho)/\epsilon} - \\
 & (2/C_4^6) (b/a)^4 e^{-C_4(a/b)^2(\rho-(b/a)^2)/\epsilon} + \\
 & (16/C_4^6) (b/a)^4 e^{-(C_4/2)(a/b)^2(\rho-(b/a)^2)/\epsilon}) + O(\epsilon^3), \quad (4.229)
 \end{aligned}$$

which is written in terms of the variable ρ by means of (4.84) and (4.97). The three-term composite expansion for α becomes,

$$\begin{aligned}
 \alpha_c = & (4\rho^{1/3}) / (3)^{2/3} - (4/C_1^2) e^{-(C_1/2)(1-\rho)/\epsilon} - \\
 & (4/C_4^2) (b/a)^2 e^{-(C_4/2)(a/b)^2(\rho-(b/a)^2)/\epsilon} + \\
 & \epsilon (-4(k_1\rho + k_2\rho^{-2/3}) / (3)^{4/3} + \phi_1 e^{-(C_1/2)(1-\rho)/\epsilon} + \\
 & \phi_2 e^{-(C_4/2)(a/b)^2(\rho-(b/a)^2)/\epsilon}) + \epsilon^2 (4F_1^2 / (9\rho) - 4F_2 / ((3)^{4/3} \rho^{1/3}) - \\
 & 32 / ((3)^{10/3} \rho^{1/3}) - (24/C_1^{10}) e^{-C_1(1-\rho)/\epsilon} + (1/C_1^{10}) e^{-(3C_1/2)(1-\rho)/\epsilon} - \\
 & 24(b/a)^6 / (C_4^{10}) e^{-C_4(a/b)^2(\rho-(b/a)^2)/\epsilon} + \\
 & (b/a)^6 / C_4^{10} e^{-(3C_4/2)(a/b)^2(\rho-(b/a)^2)/\epsilon} + \\
 & \phi_3 e^{-(C_1/2)(1-\rho)/\epsilon} + \phi_4 e^{-(C_4/2)(a/b)^2(\rho-(b/a)^2)/\epsilon}) + O(\epsilon^3),
 \end{aligned}$$

(4.230)

in which we have expressed all quantities in terms of ρ . The quantities F_1 and F_2 above are given by (4.69) and (4.71) and ϕ_1 , ϕ_2 , ϕ_3 and ϕ_4 are given by,

$$\phi_1 = 4C_2/C_1^4 + C_2(1-\rho)/(C_1^3\epsilon) + 2(1-\rho)^2/(3C_1\epsilon^2) + 4(1-\rho)/(3C_1^2\epsilon), \quad (4.231a)$$

$$\begin{aligned} \phi_2 = & 4C_5(b/a)^2/C_4^4 + C_5(\rho - (b/a)^2)/(C_4^3\epsilon) - 2(a/b)^2(\rho - (b/a)^2)^2/(3C_4\epsilon^2) \\ & - 4(\rho - (b/a)^2)(3C_4^2\epsilon), \end{aligned} \quad (4.231b)$$

$$\begin{aligned} \phi_3 = & 4C_3/C_1^4 + 64/(9C_1^4) - 4C_2^2/C_1^6 - 9/C_1^{10} + \\ & \frac{(1-\rho)}{\epsilon} \left(-5C_2^2/(4C_1^5) - 12/C_1^9 + C_3/C_1^3 - 4C_2/(3C_1^4) + \right. \\ & \left. 10/(9C_1^3) \right) + \frac{(1-\rho)^2}{\epsilon^2} \left(-C_2^2/(8C_1^4) - 2C_2/(3C_1^3) + \right. \\ & \left. 2/C_1^8 + 5/(9C_1^2) \right) + \frac{(1-\rho)^3}{\epsilon^3} \left(-C_2/(6C_1^2) + 5/(27C_1) - \right. \\ & \left. 1/(3C_1^7) \right) - (1-\rho)^4/(18\epsilon^4), \end{aligned} \quad (4.231c)$$

and ,

$$\begin{aligned}
 \phi_4 = & 4(b/a)^2 C_6 / C_4^4 + 64(b/a)^2 / (9C_4^4) - 4(b/a)^2 C_5^2 / C_4^6 - \\
 & 9(b/a)^6 / C_4^{10} + \frac{(\rho - (b/a)^2)}{\epsilon} (- 5C_5^2 / (4C_4^5) - 12(b/a)^4 / C_4^9 + \\
 & C_6 / C_4^3 + 4C_5 / (3C_4^4) + 10 / (9C_4^3)) + \frac{(\rho - (b/a)^2)^2}{\epsilon^2} (- C_5^2 / (8(b/a)^2 C_4^4) + \\
 & 2C_5 / (3C_4^3) + 2(b/a)^2 / C_4^8 + 5 / ((b/a)^2 9C_4^2)) + \\
 & \frac{(\rho - (b/a)^2)^3}{\epsilon^3} (C_5 / ((b/a)^4 6C_4^2) + 5 / ((b/a)^4 27C_4) - \\
 & 1 / (3C_4^7)) + (\rho - (b/a)^2)^4 / (\epsilon^4 18(b/a)^6). \tag{4.231d}
 \end{aligned}$$

d. Stresses and displacements

In section A.2 of the appendix we see that the expressions for the radial and circumferential bending and direct stresses are given in terms of F , α and their first derivatives and the same is true for the radial displacement U . Thus, for these quantities their calculation is straight-forward. However, in order to calculate the transverse displacement W it is necessary to evaluate an integral function of α as seen from (A.49). The most efficient manner for determining W is to integrate separately the boundary-layer and membrane expansions for α . This results in separate expansions for W in the interior and both boundary-layer zones of the plate. Upon doing this we find in the boundary-layer zone near the outer edge of the plate that,

$$\begin{aligned}
 (WL/a) = & \epsilon \left(-2(1 - \rho)/(C_1^2 \epsilon) - (4/C_1^3) e^{-\frac{(C_1/2)(1 - \rho)}{\epsilon}} \right) + \\
 & \epsilon^2 \left(2C_2(1 - \rho)/(C_1^4 \epsilon) - 2(1 - \rho)^2/(3C_1^2 \epsilon^2) \right) + \\
 & \left(6C_2/C_1^5 + C_2(1 - \rho)/(C_1^4 \epsilon) + 2(1 - \rho)^2/(3C_1^2 \epsilon^2) \right) e^{-\frac{(C_1/2)(1 - \rho)}{\epsilon}} + \\
 & X + O(\epsilon^3)
 \end{aligned} \tag{4.232}$$

which follows by substituting the first two terms of (4.227) into (A49), integrating in terms of η , and then expressing the results in terms of ρ . The quantity X is a constant of integration. In the same fashion we obtain an analogous expression for the displacement in the inner edge boundary-layer zone. It is,

$$\begin{aligned}
 (WL/a) = & \epsilon \left(2(\rho - (b/a)^2)/(C_4^2 \epsilon) + (4(b/a)^2/C_4^3) e^{-\frac{(C_4/2)(a/b)^2(\rho - (b/a)^2)}{\epsilon}} \right) \\
 & + \epsilon^2 \left(-2C_5(\rho - (b/a)^2)/(C_4^4 \epsilon) - 2(\rho - (b/a)^2)^2/(3C_4^2 (b/a)^2 \epsilon^2) \right) \\
 & + \left(-6C_5(b/a)^2/C_4^5 - C_5(\rho - (b/a)^2)/(C_4^4 \epsilon) + \right. \\
 & \left. 2(\rho - (b/a)^2)^2/(3C_4^2 \epsilon^2) \right) e^{-\frac{(C_4/2)(a/b)^2(\rho - (b/a)^2)}{\epsilon}} + \\
 & Y + O(\epsilon^3) ,
 \end{aligned} \tag{4.233}$$

in which Y is a constant of integration.

In the interior region of the plate W is obtained by substituting (4.226) into (A49) and integrating in terms of ρ . We get,

$$\begin{aligned}
 (WL/a) = & (2)(3\rho)^{1/3} + \epsilon (k_2/(3\rho^2)^{1/3} - 2k_1\rho/(3)^{4/3}) + \\
 & \epsilon^2 (4k_1^2\rho^{5/3}/21 - k_2^2/(6\rho^{5/3}) + k_4/(3\rho^2)^{1/3} - \\
 & 2k_3\rho/(3)^{4/3}) + Z + O(\epsilon^3), \tag{4.234}
 \end{aligned}$$

in which Z stands for the constant of integration.

We determine X in (4.232) by setting W as given by (4.232) equal to zero at $\rho = 1$ since the plate is considered to be fixed at its outer edge. Upon doing so it follows that,

$$X = 4\epsilon/C_1^3 - 6C_2\epsilon^2/C_1^5. \tag{4.235}$$

The constants Y and Z are found by first matching (4.232) and (4.234), which gives us Z, after which we then match (4.233) and (4.234) to obtain Y. The results of this are,

$$\begin{aligned}
 Y = & 2(3)^{1/3}((b/a)^{2/3} - 1) + \epsilon(2/3 - 2k_1(b/a)^2/(3)^{4/3} + \\
 & k_2/(3)^{1/3}(b/a)^{4/3}) + \epsilon^2(-6C_2/C_1^5 + 4k_1^2((b/a)^{10/3} - 1)/21 + \\
 & k_2^2(1 - (b/a)^{-10/3})/6 + k_4((b/a)^{-4/3} - 1)/(3)^{1/3} + \\
 & 2k_3(1 - (b/a)^2)/(3)^{4/3}), \tag{4.236}
 \end{aligned}$$

and,

$$\begin{aligned}
 Z = & -2(3)^{1/3} + 2\epsilon/3 + \epsilon^2(-6C_2/C_1^5 - 4k_1^2/21 + k_2^2/6 - \\
 & k_4/(3)^{1/3} + 2k_3/(3)^{4/3}). \tag{4.237}
 \end{aligned}$$

From (4.232), (4.233), and (4.234) we next obtain a composite expansion for WL/a which is uniformly valid. In terms of ρ , we find that,

$$\begin{aligned}
 (WL/a)_c &= (WL/a)_m + \epsilon \left((-4/C_1^3) e^{-(C_1/2)(1-\rho)/\epsilon} + \right. \\
 &\quad \left. (4/C_4^3) (b/a)^2 e^{-(C_4/2)(a/b)^2(\rho-(b/a)^2)/\epsilon} \right) + \\
 &\quad \epsilon^2 \left((6C_2/C_1^5 + C_2(1-\rho)/(C_1^4\epsilon) + 2(1-\rho)^2/(3C_1^2\epsilon^2)) e^{-(C_1/2)(1-\rho)/\epsilon} + \right. \\
 &\quad \left. (-6C_5(b/a)^2/C_4^5 - C_5(\rho-(b/a)^2)/(C_4^4\epsilon) + \right. \\
 &\quad \left. 2(\rho-(b/a)^2)^2/(3C_4^2(b/a)^2\epsilon^2)) e^{-(C_4/2)(a/b)^2(\rho-(b/a)^2)/\epsilon} \right) + O(\epsilon^3),
 \end{aligned}
 \tag{4.238}$$

where the subscript "m" stands for "membrane," and $(WL/a)_m$ stands for the expression on the right hand side of (4.234).

CHAPTER V
LARGE FINITE DEFORMATIONS

In this chapter we consider (2.20) as the basic differential equations for the deformations. These equations are to be used when the magnitude of deformation is so large that (2.15) and (2.16), which represent the von Karman equations, are no longer valid or, as pointed out by Reissner (2), when the approximations

$$\sin\beta = \beta \text{ and } \cos\beta = 1 - \beta^2/2$$

are no longer adequate.

We concern ourselves with finding perturbation type solutions to (2.20) for large λ^2 . As closed form solutions are not obtainable here we limit ourselves to one-term expansions; or equivalently we consider only the first approximations to (2.20).

The analysis is similar to that of the previous chapter in that a singular perturbation problem results. The basic differential equations for the membrane and boundary-layer regions of the plate, and the conditions determined from matching are derived. These equations are discussed in terms of a numerical integration procedure although the actual integration is not performed. However, no difficulties are foreseen in a numerical integration process.

A. Determination of the Parameters

From (2.19) and (2.3) we have,

$$\lambda^2 = \frac{P}{2\pi a E h} \left(\frac{a}{h}\right)^2 12(1 - \nu^2). \quad (5.1)$$

It follows from (2.8) and (5.1) that,

$$B = \frac{P}{2\pi}, \quad (5.2)$$

and,

$$L^2 = \frac{2\pi a E h}{P}, \quad (5.3)$$

and we recall from (2.17) that δ is equal to unity.

B. Membrane Expansions

The first approximations to (2.20) for large λ^2 follow by inspection. However for conciseness we introduce the following perturbation expansions for β and ψ . Let us set,

$$\beta = \beta_0 + \beta_1 \epsilon + \beta_2 \epsilon^2 + \dots, \quad (5.4a)$$

and

$$\psi = \psi_0 + \psi_1 \epsilon + \psi_2 \epsilon^2 + \dots, \quad (5.4b)$$

and assume that,

$$\epsilon = 1/\lambda. \quad (5.5)$$

By expanding the trigonometric functions in (2.20) as follows,

$$\sin \beta = \sin \beta_0 + \epsilon \beta_1 \cos \beta_0 + O(\epsilon^2), \quad (5.6a)$$

and,

$$\cos \beta = \cos \beta_0 - \epsilon \beta_1 \sin \beta_0 + O(\epsilon^2), \quad (5.6b)$$

and substituting these in (2.20a) we determine the first approximation to (2.20a) to be,

$$\psi_0 \sin \beta_0 = \cos \beta_0, \quad (5.7)$$

or,

$$\psi_0 = \cot \beta_0. \quad (5.8)$$

The first approximation to (2.20b) follows similarly. The quantity L^2 in (2.20b) is to be considered to be of order unity or small, (or else

the first approximation to (2.20b) would imply that β is small which is contrary to our assumptions) and therefore the first approximation to (2.20b) becomes,

$$\frac{d^2\psi_o}{dr^2} + \frac{1}{r} \frac{d\psi_o}{dr} - \frac{1}{r^2} (\cos^2 \beta_o - \nu r \frac{d\beta_o}{dr} \sin \beta_o) \psi_o + \frac{L^2}{r} (1 - \cos \beta_o) = \frac{\sin \beta_o \cos \beta_o}{r^2} + \frac{\nu}{r} \frac{d\beta_o}{dr} \cos \beta_o . \quad (5.9)$$

Before proceeding further, let us note that the terms on the right hand side of (5.9), which are neglected in the more recent theory of Reissner (2), become significant in the case of large finite deformations and therefore should be retained in the theory.

From (5.8) it follows that,

$$\sin \beta_o = \frac{1}{\sqrt{1 + \psi_o^2}} \text{ and } \cos \beta_o = \frac{\psi_o}{\sqrt{1 + \psi_o^2}} . \quad (5.10)$$

Substituting (5.10) into (5.9) we get the following simplified equation on ψ_o :

$$\frac{d^2\psi_o}{dr^2} + \frac{1}{r} \frac{d\psi_o}{dr} - \frac{\psi_o}{r^2} + \frac{L^2}{r} \left(1 - \frac{\psi_o}{\sqrt{1 + \psi_o^2}}\right) = 0 \quad (5.11)$$

Thus to determine the first term, ψ_o , in the expansion for ψ , equation (5.11) must be solved subject to certain conditions which we determine later in section D. Upon determining ψ_o , then β_o follows directly from (5.8).

C. Boundary-Layer Expansions

1. Boundary-layer expansions for the vicinity of the outer edge of the plate.

a. Boundary-layer coordinate system and equations

Following the same procedure as in the previous chapter we let,

$$r = 1 - \epsilon\eta, \tag{5.12}$$

for the vicinity of the outer edge of the plate. The parameter ϵ in (5.12) is given by (5.5).

Let us assume that β and ψ can be expanded in the form,

$$\beta = \tilde{\beta}_0 + \tilde{\beta}_1\epsilon + \tilde{\beta}_2\epsilon^2 + \dots, \tag{5.13a}$$

and,

$$\psi = \tilde{\psi}_0 + \tilde{\psi}_1\epsilon + \tilde{\psi}_2\epsilon^2 + \dots, \tag{5.13b}$$

where the tilde is used to indicate that the functions apply to the vicinity of the outer edge of the plate. Substituting (5.12) and (5.13) into (2.20), and expanding the trigonometric functions therein in a form analogous to (5.6), we get the following basic differential equations for the first approximations,

$$\frac{d^2\tilde{\beta}_0}{d\eta^2} - \tilde{\psi}_0 \sin \tilde{\beta}_0 + \cos \tilde{\beta}_0 = 0, \tag{5.14a}$$

and,

$$\frac{d^2\tilde{\psi}_0}{d\eta^2} = 0. \tag{5.14b}$$

b. Boundary conditions

Equations (5.14) are subject to the following boundary conditions which follow from (2.21), (2.24), (5.12) and (5.13):

$$\tilde{\beta}_0 = 0, \tag{5.15a}$$

and,

$$\frac{d\tilde{\psi}_0}{d\eta} = 0, \tag{5.15b}$$

at $r = 1$ or equivalently at $\eta = 0$.

c. Solutions to the boundary-layer equations

The solution to (5.14b), subject to (5.15b) takes the following form:

$$\tilde{\psi}_0 = C_1, \quad (5.16)$$

where C_1 is a constant of integration to be determined later (see section D). Substituting (5.16) into (5.14a) we get,

$$\frac{d^2 \tilde{\beta}_0}{d\eta^2} - C_1 \sin \tilde{\beta}_0 + \cos \tilde{\beta}_0 = 0. \quad (5.17)$$

Let us introduce a quantity $\tilde{\gamma}_0$ defined as,

$$\tilde{\gamma}_0 = \frac{d\tilde{\beta}_0}{d\eta}. \quad (5.18)$$

Therefore, from (5.17) we get,

$$\frac{d\tilde{\gamma}_0}{d\eta} = C_1 \sin \tilde{\beta}_0 - \cos \tilde{\beta}_0, \quad (5.19)$$

and multiplying this by $\tilde{\gamma}_0$ gives us,

$$\tilde{\gamma}_0 \frac{d\tilde{\gamma}_0}{d\eta} = \tilde{\gamma}_0 (C_1 \sin \tilde{\beta}_0 - \cos \tilde{\beta}_0). \quad (5.20)$$

Replacing the quantity $\tilde{\gamma}_0$ on the right hand side of (5.20) by (5.18), and multiplying the resulting equation by $d\eta$ gives us;

$$\tilde{\gamma}_0 d\tilde{\gamma}_0 = (C_1 \sin \tilde{\beta}_0 - \cos \tilde{\beta}_0) d\tilde{\beta}_0, \quad (5.21)$$

which can be written as,

$$d(\tilde{\gamma}_0^2/2) = d(-C_1 \cos \tilde{\beta}_0 - \sin \tilde{\beta}_0). \quad (5.22)$$

Integrating (5.22) and replacing $\tilde{\gamma}_0$ by the right hand side of (5.18) we get,

$$\left(\frac{d\tilde{\beta}_0}{d\eta}\right)^2 = -2C_1 \cos \tilde{\beta}_0 - 2 \sin \tilde{\beta}_0 + 2C_2 \quad (5.23)$$

where C_2 is a constant of integration which we determine in section D. From (5.23) it follows that,

$$d\eta = \frac{d\tilde{\beta}_0}{\sqrt{2(C_2 - C_1 \cos \tilde{\beta}_0 - \sin \tilde{\beta}_0)}} \quad (5.24)$$

and therefore,

$$\eta = \int^{\tilde{\beta}_0} \frac{d\tilde{\beta}_0}{\sqrt{2(C_2 - C_1 \cos \tilde{\beta}_0 - \sin \tilde{\beta}_0)}} + C_3 \quad (5.25)$$

in which C_3 is a constant of integration. From (5.15a) it follows that C_3 is zero. Therefore we get,

$$\eta = \int^{\tilde{\beta}_0} \frac{d\tilde{\beta}_0}{\sqrt{2(C_2 - C_1 \cos \tilde{\beta}_0 - \sin \tilde{\beta}_0)}} \quad (5.26)$$

Thus the quantity $\tilde{\beta}_0$ is known as a function of η through the above integral.

2. Boundary-layer expansions for the vicinity of the inner edge of the plate.

a. Boundary-layer coordinate system and equations

Here we let,

$$r = (b/a) + \epsilon \xi \quad (5.27)$$

and assume that we can expand β and ψ as,

$$\beta = \bar{\bar{\beta}}_0 + \bar{\bar{\beta}}_1 \epsilon + \bar{\bar{\beta}}_2 \epsilon^2 + \dots , \quad (5.28a)$$

and,

$$\psi = \bar{\bar{\psi}}_0 + \bar{\bar{\psi}}_1 \epsilon + \bar{\bar{\psi}}_2 \epsilon^2 + \dots , \quad (5.28b)$$

where the double bar is used to indicate that the functions apply in the vicinity of the inner edge of the plate. Substituting (5.27) and (5.28) into (2.20), and using expansions for the trigonometric terms therein similar to (5.6), we obtain the following first approximations:

$$\frac{d^2 \bar{\bar{\beta}}_0}{d\zeta^2} - \frac{a}{b} \bar{\bar{\psi}}_0 \sin \bar{\bar{\beta}}_0 + \frac{a}{b} \cos \bar{\bar{\beta}}_0 = 0 , \quad (5.29a)$$

and,

$$\frac{d^2 \bar{\bar{\psi}}_0}{d\zeta^2} = 0. \quad (5.29b)$$

b. Boundary conditions

The boundary conditions for $\bar{\bar{\beta}}_0$ and $\bar{\bar{\psi}}_0$ are found by substituting (5.27) and (5.28) into (2.21) and (2.24) to give us,

$$\bar{\bar{\beta}}_0 = 0, \quad (5.30a)$$

and

$$\frac{d\bar{\bar{\psi}}_0}{d\zeta} = 0 , \quad (5.30b)$$

at $r = b/a$ or $\zeta = 0$.

c. Solutions to the boundary-layer equations

The solution to (5.29b), subject to (5.30b), becomes,

$$\bar{\bar{\psi}}_0 = C_4 , \quad (5.31)$$

in which C_4 represents a constant of integration which we evaluate

in section D. To solve (5.29a) we use the same procedure as in part (1) for finding $\bar{\beta}_0$. Omitting the details we obtain,

$$\zeta = \int_{\bar{\beta}_0}^{\bar{\beta}_0} \frac{d\bar{\beta}_0}{\sqrt{2\left(\frac{a}{b}\right) (-C_4 \cos \bar{\beta}_0 - \sin \bar{\beta}_0) + 2C_5}}, \quad (5.32)$$

where C_5 is a constant of integration which we evaluate in section D. In arriving at (5.32) equations (5.30a) and (5.31) have been used. This concludes the analysis of the boundary-layer equations.

D. Matching

The conditions for determining C_1 , C_2 , C_4 and C_5 , as well as the conditions on ψ_0 , are derived in this section. Clearly a total of six conditions are required. These are obtained by matching the quantities ψ , β , and a nondimensional form of the radial displacement.

1. Matching conditions for the variable ψ

First the outer edge boundary-layer and membrane expansions for ψ are matched. Since the membrane expansion for ψ is not known in closed form we represent the function ψ by its Taylor series expansion. Following the formal matching procedure we first write,

One-term outer expansion for ψ :

$$\psi_0(r) \quad (5.33)$$

One-term outer expansion written in inner variables:

$$\psi_0(1 - \epsilon\eta) \quad (5.34)$$

Expanded for small ϵ :

$$\psi_0(1) - \epsilon\eta \frac{d\psi_0}{dr}(1) + O(\epsilon^2) \quad (5.35)$$

One-term inner expansion of the one-term outer expansion:

$$\psi_0(1). \tag{5.36}$$

Next we write,

One-term inner expansion for ψ_0 :

$$C_1 \tag{5.37}$$

In view of (5.37) it is clear that (5.37) also represents the one-term outer expansion of the one-term inner expansion for ψ . According to the matching principle we therefore set,

$$\psi_0(1) = C_1. \tag{5.38}$$

Hence the constant C_1 is equal to the value of ψ_0 at $r=1$. By a similar process at the inner edge of the plate we find that

$$\psi_0(b/a) = C_4, \tag{5.39}$$

or C_4 is equal to the value of ψ_0 at $r = b/a$. Therefore once (5.11) is integrated (numerically) the constants C_1 and C_4 are known.

2. Matching conditions for the variable β

The outer edge boundary-layer and membrane expansions for β are matched first. The process is basically the same as in the preceding section.

One-term outer expansion for β :

$$\beta_0(r) \tag{5.40}$$

One-term outer expansion written in inner variables:

$$\beta_0(1 - \epsilon\eta) \tag{5.41}$$

Expanded for small ϵ :

$$\beta_0(1) = \epsilon \eta \frac{d\beta_0(1)}{dr} + O(\epsilon^2) \quad (5.42)$$

One-term inner expansion of the one-term outer expansion:

$$\beta_0(1) . \quad (5.43)$$

Following the usual procedure we next write,

One-term inner expansion for β :

$$\tilde{\beta}_0(\eta) \quad (5.44)$$

One-term inner expansion written in outer variables:

$$\tilde{\beta}_0\left(\frac{1-r}{\epsilon}\right) \quad (5.45)$$

By expanding (5.45) for small ϵ , and rewriting the result in terms of η for convenience, we get the,

One-term outer expansion of the one-term inner expansion:

$$\lim_{\eta \rightarrow \infty} \tilde{\beta}_0(\eta) \quad (5.46)$$

Equating (5.43) and (5.46) according to the matching principle we get,

$$\beta_0(1) = \lim_{\eta \rightarrow \infty} \tilde{\beta}_0(\eta) . \quad (5.47)$$

By a similar process for the inner edge of the plate we determine that,

$$\beta_0(b/a) = \lim_{\zeta \rightarrow \infty} \bar{\beta}_0(\zeta) . \quad (5.48)$$

Equations (5.47) and (5.48) can be expressed more conveniently in the following form:

$$\frac{d\tilde{\beta}_0}{d\eta} = 0 \quad \text{at} \quad \eta = \infty, \quad (5.49)$$

and,

$$\frac{d\bar{\beta}_0}{d\zeta} = 0 \quad \text{at} \quad \zeta = \infty. \quad (5.50)$$

3. Matching conditions for the radial displacement

Here we match a non-dimensional form of the radial displacement U . From (2.23) the radial displacement is given by,

$$U = \frac{rB}{Eh} \left(\frac{d\psi}{dr} - \frac{\nu}{r} \left(\frac{P \sin \beta}{2\pi B} + \psi \cos \beta \right) \right), \quad (5.51)$$

where we have set $\delta = 1$. From (2.19) and (2.8), equation (5.51) takes the following form,

$$U = \frac{rB}{Eh} \left(\frac{d\psi}{dr} - \frac{\nu}{r} (\sin \beta + \psi \cos \beta) \right), \quad (5.52)$$

and upon multiplying by (Eh/B) we get,

$$\frac{EhU}{B} = r \left(\frac{d\psi}{dr} - \frac{\nu}{r} (\sin \beta + \psi \cos \beta) \right). \quad (5.53)$$

The membrane and outer edge boundary-layer representations of (5.53) are matched next. In view of the steps in going from (5.33) to (5.36), and (5.40) to (5.42), we can immediately write the

One-term inner expansion of the one-term outer expansion:

$$\frac{d\psi_0(1)}{dr} - \nu(\sin\beta_0(1) + \psi_0(1) \cos\beta_0(1)). \quad (5.54)$$

By using (5.12), and in view of (5.16), we determine the

One-term inner expansion:

$$-\frac{d\tilde{\psi}_1}{d\eta} - \nu(\sin \tilde{\beta}_0 + \tilde{\psi}_0 \cos \tilde{\beta}_0) \quad (5.55)$$

Inspection of (5.55) shows we must next evaluate the term $\frac{d\tilde{\psi}_1}{d\eta}$, the derivative of the second term in (5.13b).

From (2.20b), (5.12), and (5.13) the governing equation on $\tilde{\psi}_1$ is determined to be,

$$\frac{d^2\tilde{\psi}_1}{d\eta^2} - \frac{d\tilde{\psi}_0}{d\eta} - \nu \frac{d\tilde{\beta}_0}{d\eta} \tilde{\psi}_0 \sin \tilde{\beta}_0 = -\nu \frac{d\tilde{\beta}_0}{d\eta} \cos \tilde{\beta}_0. \quad (5.56)$$

Using the fact that $\tilde{\psi}_0$ is a constant, namely C_1 by (5.16), we get,

$$\frac{d^2\tilde{\psi}_1}{d\eta^2} - \nu \frac{d\tilde{\beta}_0}{d\eta} (C_1 \sin \tilde{\beta}_0 - \cos \tilde{\beta}_0) = 0. \quad (5.57)$$

Multiplying (5.57) by $d\eta$ and expressing the result in differential form gives us,

$$d\left(\frac{d\tilde{\psi}_1}{d\eta}\right) - \nu d(-C_1 \cos \tilde{\beta}_0 - \sin \tilde{\beta}_0) = 0. \quad (5.58)$$

Upon integrating (5.58) we get,

$$\frac{d\tilde{\psi}_1}{d\eta} + \nu (C_1 \cos \tilde{\beta}_0 + \sin \tilde{\beta}_0) = k, \quad (5.59)$$

where k is a constant of integration. From (2.24) and (5.12) the boundary condition on $\tilde{\psi}_1$ takes the form,

$$\frac{d\tilde{\psi}_1}{d\eta} + \nu \tilde{\psi}_0 = 0, \quad (5.60)$$

at $r = 1$. In view of (5.15a) and (5.16) we see that k is equal to zero. Therefore from (5.59) we can write,

$$-\frac{d\tilde{\psi}_1}{d\eta} = \nu(C_1 \cos \tilde{\beta}_0 + \sin \tilde{\beta}_0) . \quad (5.61)$$

Substituting (5.61) into (5.55) and using (5.16) we see that the one-term inner expansion for (EhU/B) vanishes. Clearly this implies that (5.54) must also vanish. Therefore we set,

$$\frac{d\psi_0}{dr}(1) - \nu(\sin \beta_0(1) + \psi_0(1) \cos \beta_0(1)) = 0. \quad (5.62)$$

Before proceeding further let us note, by comparison of (5.62) and (5.53), that the boundary condition of zero radial displacement at the outer edge of the plate carries over to the membrane expansion in the first approximation. This particular feature of the problem also occurred in Chapter IV, although much more markedly. There, the membrane expansion gives zero radial displacement in the first approximation everywhere, in addition to at the edges of the plate, as can be seen, for example, by the absence of the zeroth order term from (4.129). From physical considerations concerning the width of the boundary-layers for increasing deformation, we expect that the boundary condition on U should carry over here also; as it did. However, had we not retained the terms on the right hand side of (5.9), the boundary condition on U would not have carried over and hence we have further justification for retaining the right hand side of (5.9). Clearly the matching principle, which leads us to (5.62), is a powerful tool.

Returning to (5.62), and using (5.10), we obtain the following condition on ψ_0 :

$$\frac{d\psi_0}{dr} - \nu \sqrt{1 + \psi_0^2} = 0 , \quad (5.63)$$

at $r = 1$. By matching the inner edge and membrane expansions for

(EhU/B) we obtain a similar result. It is,

$$\frac{d\psi_0}{dr} - \left(\frac{a}{b}\right) \sqrt{1 + \psi_0^2} = 0. \quad (5.64)$$

Equations (5.63) and (5.64) provide the necessary conditions for the numerical integration of (5.11).

Using (5.49) and (5.50) we are now able to evaluate the constants C_2 and C_5 in (5.26) and (5.32). From (5.23) and (5.49) the following equation is obtained,

$$0 = -2C_1 \cos \tilde{\beta}_0(\infty) - 2 \sin \tilde{\beta}_0(\infty) + 2C_2. \quad (5.65)$$

Substituting (5.47) into (5.65) we get, upon rearranging,

$$C_2 = C_1 \cos \beta_0(1) + \sin \beta_0(1). \quad (5.66)$$

Using (5.10) and (5.38) gives us,

$$C_2 = \sqrt{1 + \psi_0^2(1)}. \quad (5.67)$$

Therefore once (5.11) is integrated C_2 is known. By going through a similar process we find that C_5 is given by,

$$C_5 = \frac{a}{b} (C_4 \cos \beta_0(b/a) + \sin \beta_0(b/a)). \quad (5.68)$$

Using (5.10) and (5.39) we get,

$$C_5 = \left(\frac{a}{b}\right) \sqrt{1 + \psi_0^2(b/a)}. \quad (5.69)$$

E. Summary for the Numerical Integration Process

1. Membrane expansions

The solution for ψ_0 in the interior region of the plate is obtained by numerically integrating (5.11) over the interval $(b/a) \leq r \leq 1$ subject to the conditions that (5.63) and (5.64) are satisfied at $r = 1$

and $r = b/a$ respectively. Using (5.8) we then obtain β_0 over the interval $(b/a) \leq r \leq 1$. Thus the basic equations for the interior region of the plate are the following:

$$\frac{d^2\psi_0}{dr^2} + \frac{1}{r} \frac{d\psi_0}{dr} - \frac{\psi_0}{r^2} + \frac{L^2}{r} \left(1 - \frac{\psi_0}{\sqrt{1 + \psi_0^2}} \right) = 0, \quad (5.11)$$

$$\frac{d\psi_0}{dr} - \sqrt{1 + \psi_0^2} = 0 \quad (\text{at } r = 1) , \quad (5.63)$$

$$\frac{d\psi_0}{dr} - \left(\frac{a}{b}\right) \sqrt{1 + \psi_0^2} = 0 \quad (\text{at } r = b/a) , \quad (5.64)$$

and

$$\cot\beta_0 = \psi_0 . \quad (5.8)$$

2. Boundary-layer expansions

a. Boundary-layer expansions for the vicinity of the outer edge of the plate

The solution for $\tilde{\psi}_0$, which we recall stands for the function ψ_0 in the vicinity of the outer edge of the plate, is obtained from (5.16) and (5.38). We get,

$$\tilde{\psi}_0 = C_1 = \psi_0(1) , \quad (5.70)$$

where $\psi_0(1)$ represents the value of ψ_0 at $r = 1$ as obtained in part (1) above. Thus $\tilde{\psi}_0$, a constant, is known.

To determine $\tilde{\beta}_0$ we evaluate the integral in (5.26) which, by using (5.66) and (5.38), we can express as,

$$\eta = \int^{\tilde{\beta}_0} \frac{d\tilde{\beta}_0}{\sqrt{2 (\psi_0(1)(\cos \beta_0(1) - \cos \tilde{\beta}_0) + (\sin \beta_0(1) - \sin \tilde{\beta}_0))}} \quad (5.71)$$

Therefore, since $\psi_0(1)$ and $\beta_0(1)$ are considered to be known from part (1), (5.71) gives the solution for $\tilde{\beta}_0$ as a function of η .

b. Boundary-layer expansions for the vicinity of the inner edge of the plate

The function $\bar{\psi}_0$, which stands for the function ψ_0 in the vicinity of the inner edge of the plate, is obtained from (5.31) and (5.39) as,

$$\bar{\psi}_0 = C_4 = \psi_0(b/a), \quad (5.72)$$

in which $\psi_0(b/a)$ represents the value of ψ_0 at $r = b/a$ as determined from part 1. Hence, $\bar{\psi}_0$, a constant, is known.

In order to obtain $\bar{\beta}_0$ we evaluate the following integral,

$$\zeta = \int^{\bar{\beta}_0} \frac{d\bar{\beta}_0}{\sqrt{2 \left(\frac{a}{b}\right) (\psi_0(b/a)(\cos \beta_0(b/a) - \cos \bar{\beta}_0) + (\sin \beta_0(b/a) - \sin \bar{\beta}_0))}} \quad (5.73)$$

which follows from (5.32), (5.39) and (5.68). Since $\psi_0(b/a)$ and $\beta_0(b/a)$ are considered to be known from part (1), (5.73) gives us $\bar{\beta}_0$ as a function of ζ . Thus, in theory, the problem is solved.

CHAPTER VI

DISCUSSION OF RESULTS AND CONCLUSIONS

In Figures (3) and (4) we demonstrate the accuracy of the present method. Here we compare the results of the large deformation analysis of the von Karman equations as given in chapter IV with data from the numerical integration of the Reissner equations which is presented by Hart and Evans (5). Our results are calculated from two-term expansions of the pertinent quantities and are based on the analysis for the case when Poisson's ratio is equal to $1/3$. Hart and Evans used a value of $.3$ for Poisson's ratio, however, as discussed below, the results are practically insensitive to slight variations of Poisson's ratio and we accordingly disregard any effect due to this difference.

With regard to Figure (3) we show the sum of the direct and bending radial stresses at the inner edge of the plate where the maximum stress always occurs. Here the calculations are based on the boundary-layer expansions which apply in the vicinity of the inner edge of the plate. In all cases the agreement is seen to be very good. The maximum transverse displacement to plate thickness ratios are indicated for the first and last points on each curve. With regard to the two curves for $b/a = .626$ it is observed that the agreement is good even when the maximum transverse displacement is less than one-half of the plate thickness. Thus we conclude that the perturbation analysis of the von Karman equations for large deformations is accurate. Furthermore it is clear that the von Karman theory itself gives an accurate first approximation to the Reissner equations for the range of deformation shown in Figure (3). Unfortunately this latter point could not be investigated more thoroughly since Hart and Evans did not consider larger values of maximum transverse displacement to thickness ratios than those shown in Figure (3).

In Figure (4) we verify the accuracy of our solution as a function of the radial coordinate.

Figures (5) through (8) serve to illustrate how the results of the infinitesimal and large deformation analyses of the von Karman equations, as given in chapters III and IV respectively, tend to merge together. This occurs in a region where the expansions of the infinitesimal deformation analysis are beginning to diverge. In these figures the term "first approximation" is used to indicate that the expansions are found to one term; "second approximation" implies that two terms of the expansions are used, and so forth. The terms "small deformations" and "large deformations" are used to refer to the analyses of chapters III and IV respectively.

It is clear from these figures that the classical linearized bending theory of plates, which is represented by the first approximation of the infinitesimal deformation analysis of chapter III, is no longer valid when the maximum transverse displacement of the plate exceeds approximately four-tenths of the plate thickness.

Figures (9) and (10) are included in order to show the stress levels, as predicted by the large deformation analysis of the von Karman equations, associated with the ranges of loads considered in Figures (5) and (8). The classical linear theory is included also for comparison.

With regard to the analysis of chapter V, which applies when the deformations are so large that the von Karman equations are no longer valid, we have established the basic differential equations for the boundary-layer zones and the interior region of the plate as well as the conditions for determining the constants of integration. The numerical integration of (5.11) is proposed for future work. Once this is done we can determine the region where the von Karman theory is no longer adequate and, in addition, obtain solutions of the related membrane problem for very large deformations.

The comment above concerning the fact that the solution as given in chapter IV is practically insensitive to slight variations of Poisson's ratio is quite evident if we examine the numerical values of one of the dependent variables for different values of ν . For instance, by evaluating F_0 at the inner edge of the plate for ν equal to $1/3$, $.3$ and $.25$, we find that F_0 is equal to $.613$, $.613$ and $.614$ respectively. Here we have arbitrarily taken b/a to be equal to $.4$. The latter two values above are determined from (4.30) and by using Figures (A.1) and (A.2) of the appendix.

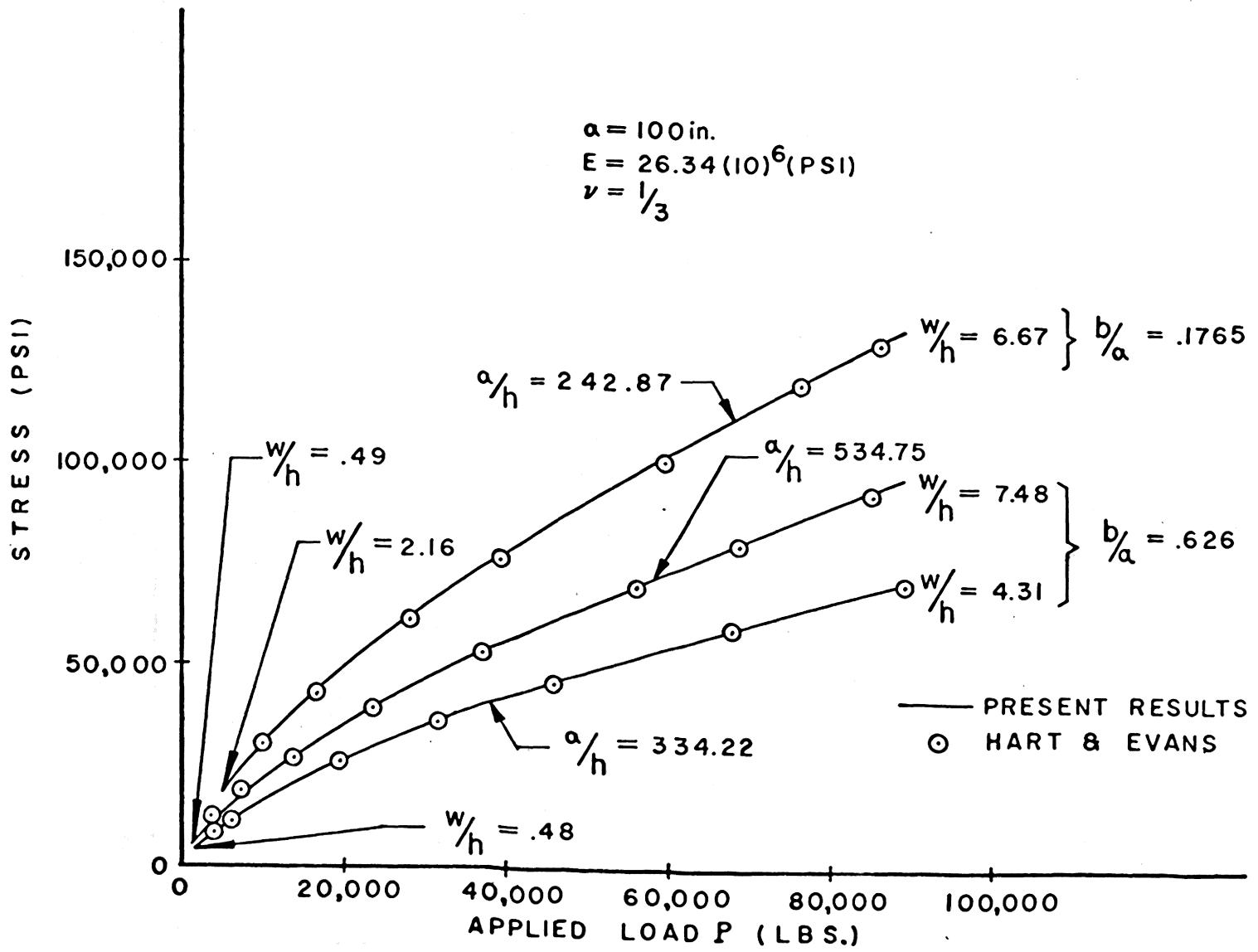


Figure (3) - MAXIMUM STRESS AT THE INNER EDGE OF THE PLATE

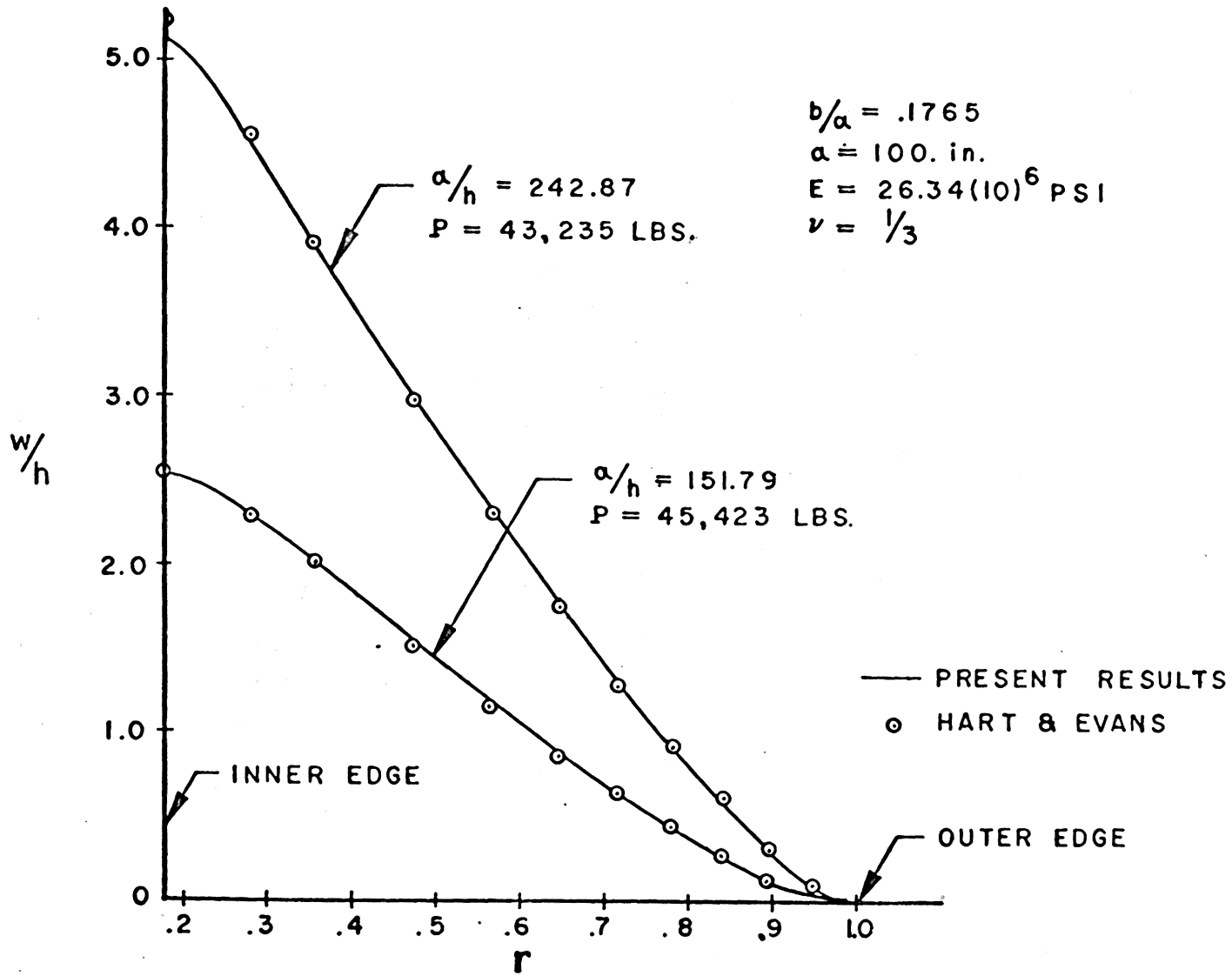


Figure (4) - LATERAL DISPLACEMENT PROFILE

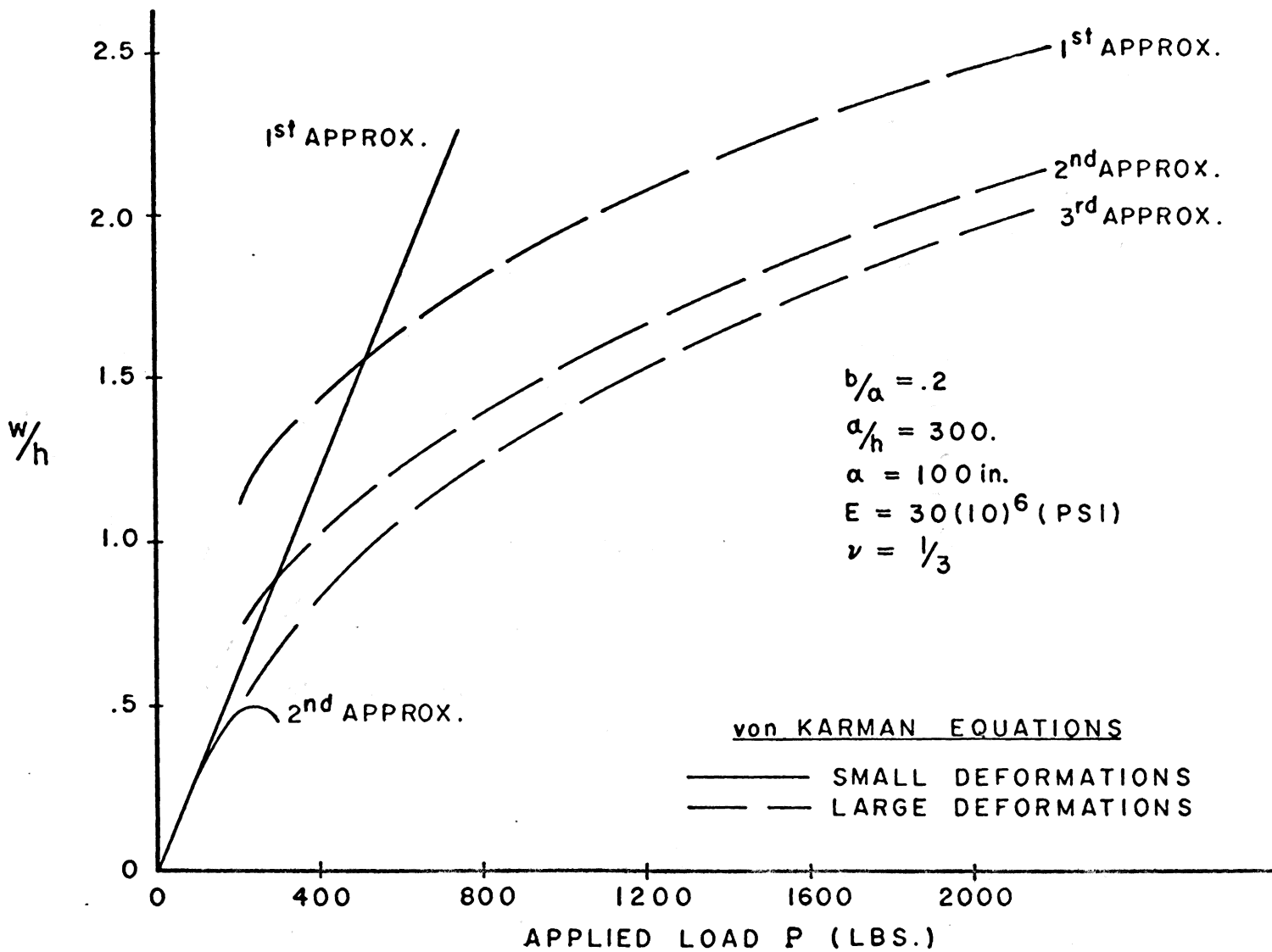


Figure (5) - TRANSVERSE DISPLACEMENT AT THE INNER EDGE OF THE PLATE FOR $b/a = .2$

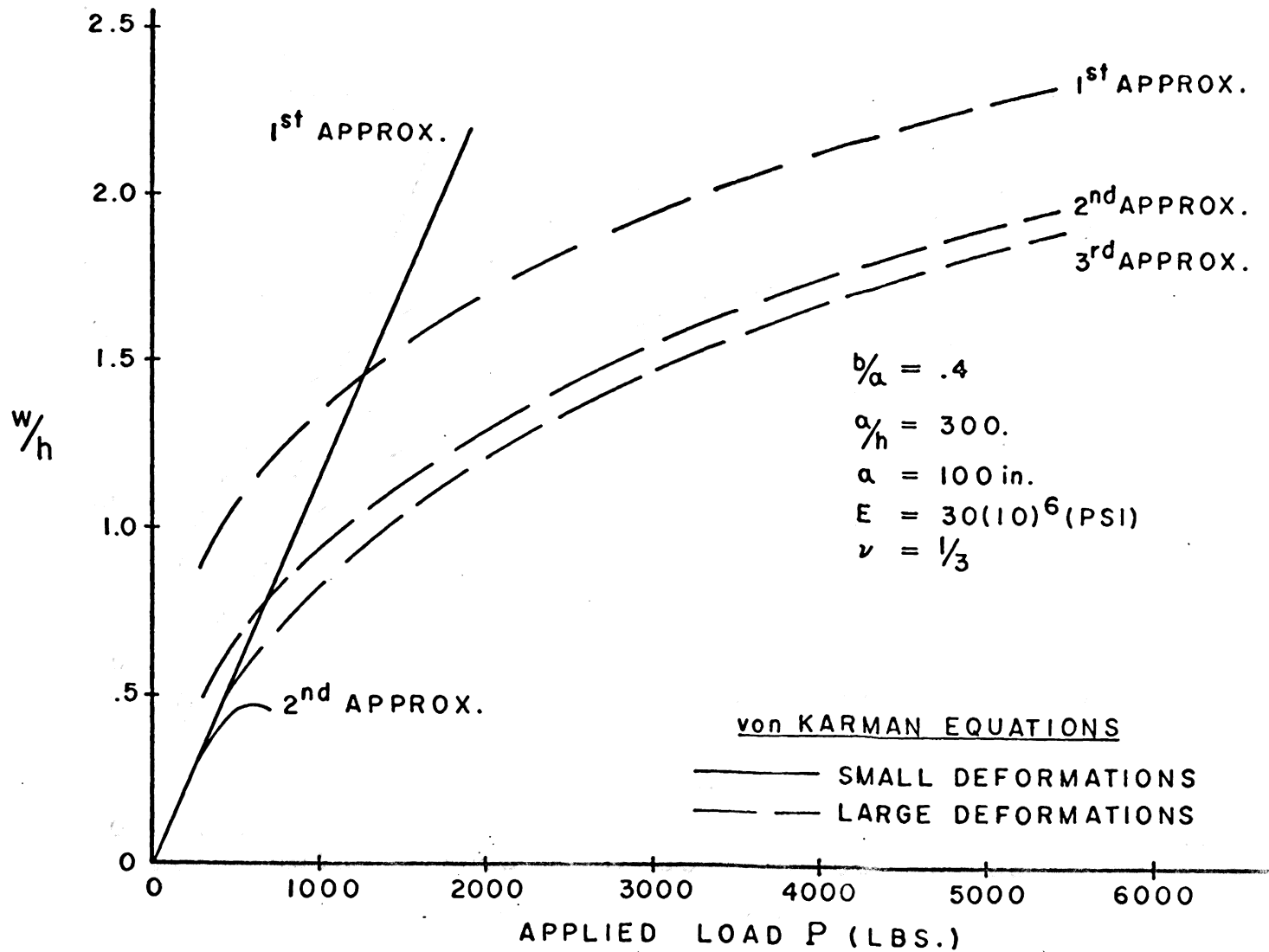


Figure (6) - TRANSVERSE DISPLACEMENT AT THE INNER EDGE OF THE PLATE FOR $b/a = .4$

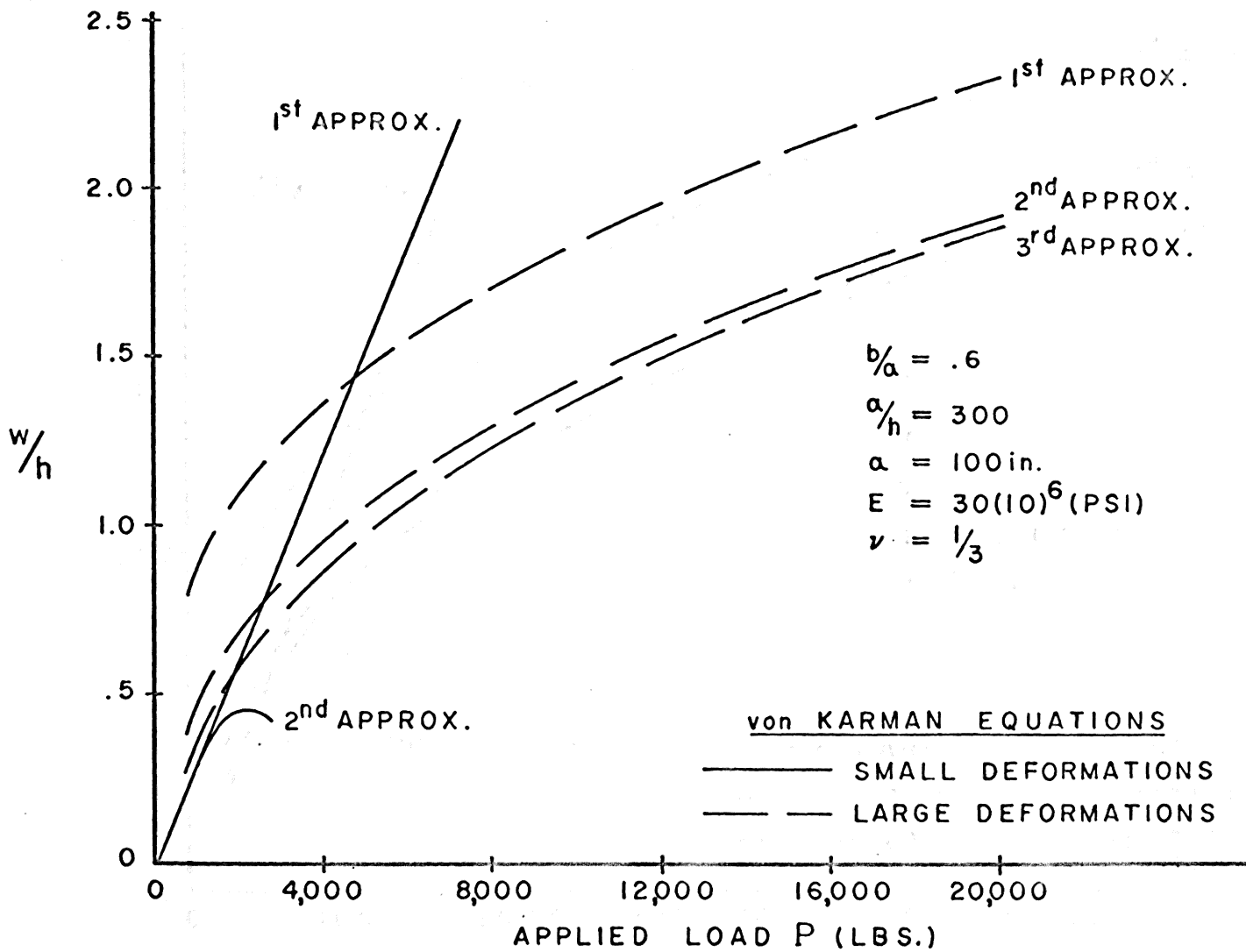


Figure (7) - TRANSVERSE DISPLACEMENT AT THE INNER EDGE OF THE PLATE FOR $b/a = .6$

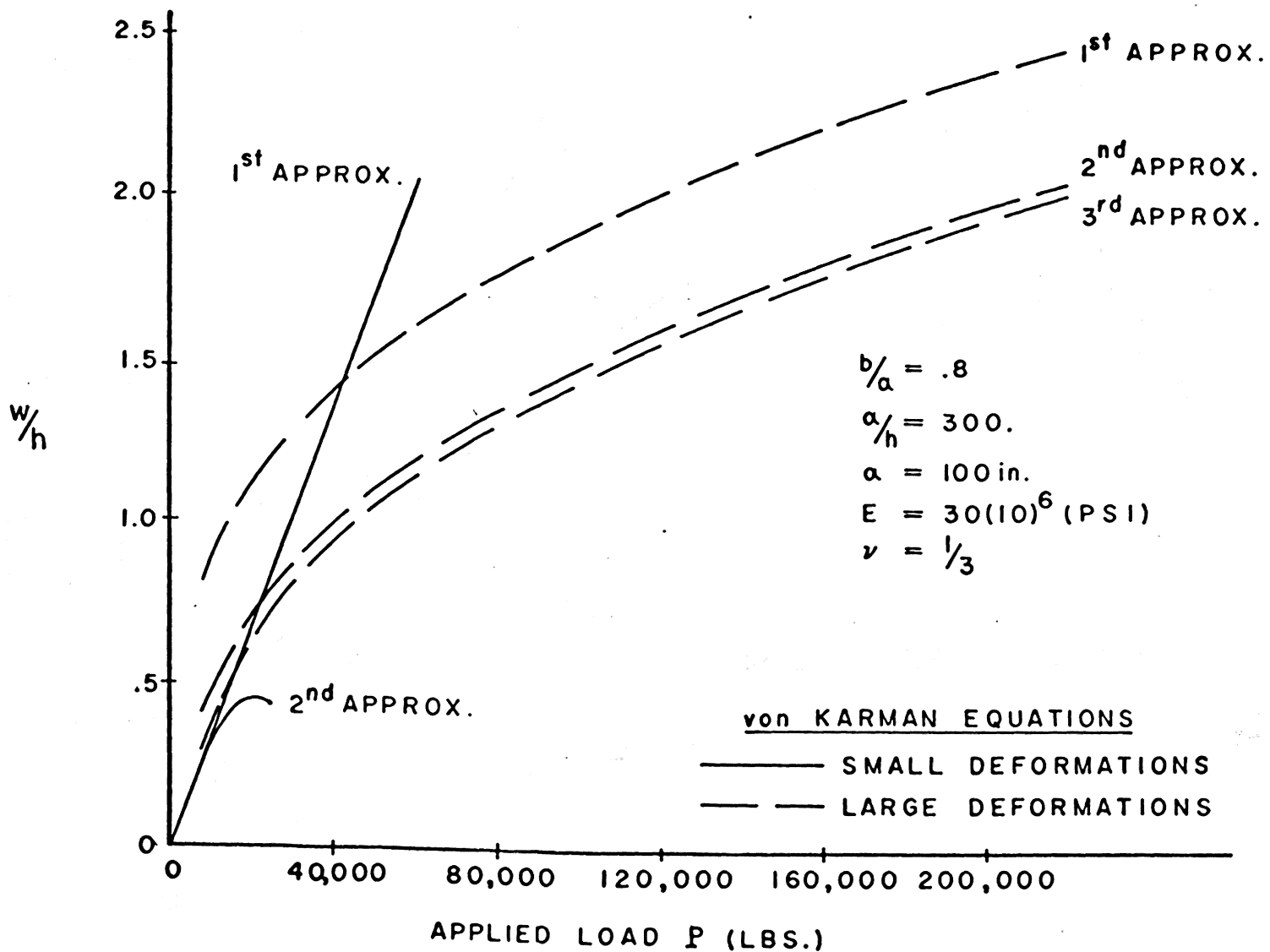


Figure (8) - TRANSVERSE DISPLACEMENT AT THE INNER EDGE OF THE PLATE FOR $b/a = .8$

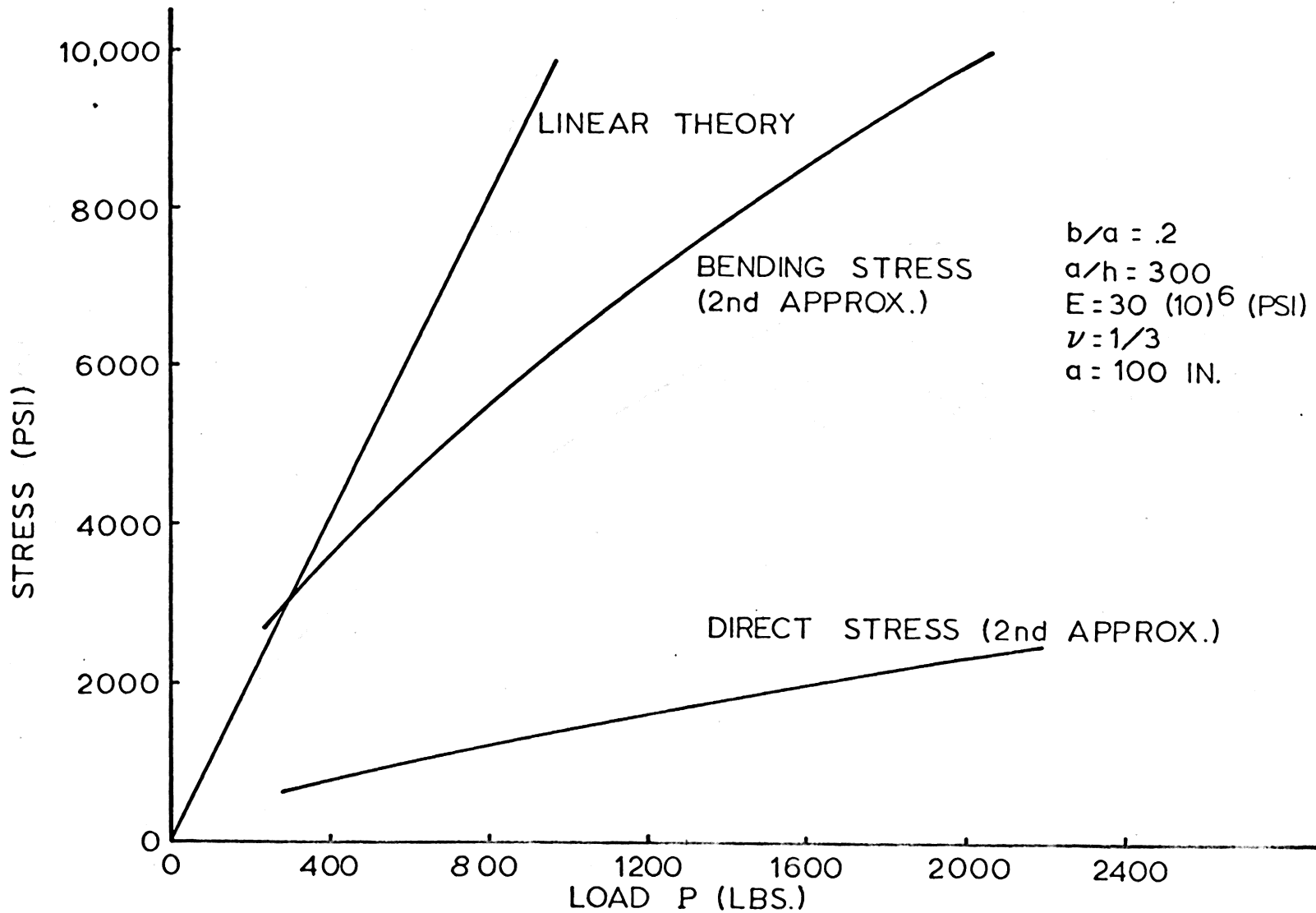


Figure (9) - BENDING AND DIRECT STRESSES AT THE INNER EDGE OF THE PLATE FOR $b/a = .2$

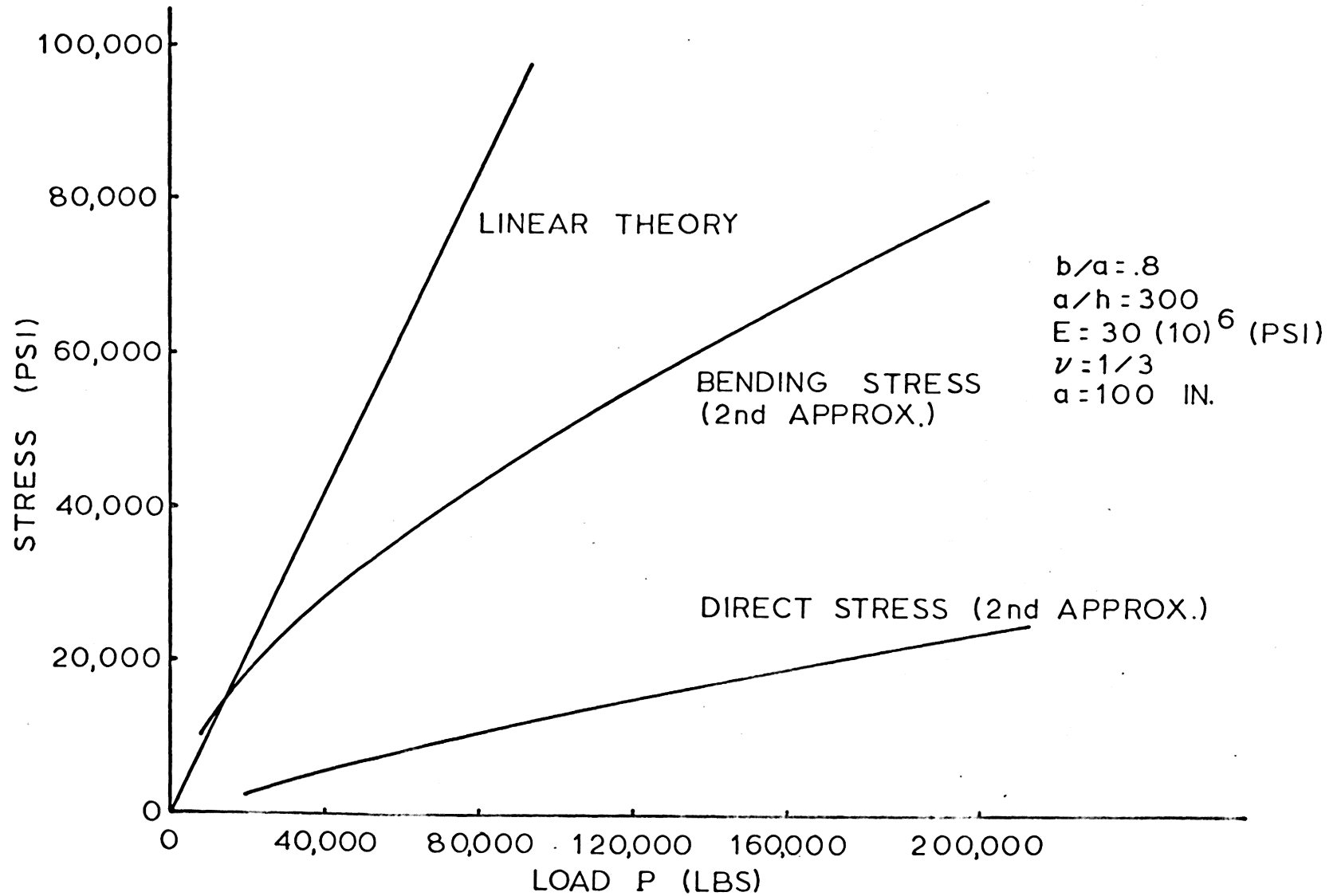


Figure (10) - BENDING AND DIRECT STRESSES AT THE INNER EDGE OF THE PLATE FOR $b/a = .8$

VII

ACKNOWLEDGMENTS

The author wishes to express his sincere thanks to his advisor, Dr. R. T. Davis, for suggesting the topic and for his invaluable guidance, encouragement and enthusiasm throughout this study. He also expresses his gratitude to the members of his graduate committee, Professors C. W. Smith, G. W. Swift, R. P. McNitt and L. McFadden for their criticisms of this work. It has been a privilege to study under and to be associated with the faculty at Virginia Polytechnic Institute.

VIII
BIBLIOGRAPHY

1. Reissner, E. Finite Deflections of Circular Plates. Proc. Symp. in Appl. Math. Vol. 1 (1949). pp. 213-219.
2. Reissner, E. Rotationally Symmetric Problems in the Theory of Thin Elastic Shells. Proc. of Third U. S. National Congress of Applied Mechanics. 1958. pp. 51-69.
3. Karman, Th. von. Encyklopadie der Mathematischen Wissenschaften. Vol. IV. 1910. p. 349.
4. Van Dyke, M. Perturbation Methods in Fluid Mechanics. Academic Press, New York. 1964.
5. Hart, V. G. and Evans, D. J. Non-Linear Bending of an Annular Plate by Transverse Edge Forces. Journal of Mathematics and Physics. Vol. 43. 1964. pp. 275-303.
6. Hamada, M. and Seguchi, Y. Large Deflection Analysis of Circular Ring Plates Under Uniform Transverse Force Along the Inner Edge. Bull. of Japan Society of Mechanical Engineers. Vol. 8. No. 31. 1965. pp. 344-352.
7. Schwerin, E. Uber Spannungen und Formanderung Kreisringformiger Membranen. Z. Tech. Phys. No. 12. 1929. pp. 651-659.
8. Jahsman, W. E., Field, F. A. and Holmes, A. M. C. Finite Deformations in a Prestressed, Centrally Loaded, Circular Elastic Membrane. Proc. of Fourth U. S. National Congress of Applied Mechanics. 1962. pp. 585-594.

**The vita has been removed from
the scanned document**

APPENDIX

A.1 Evaluation of the Constants of Integration for the Case when Poisson's Ratio is less than 1/3

In this section we describe the procedure for determining the constants of integration, and two other related constants, for the case when Poisson's ratio is less than 1/3. From (4.175) through (4.182) and (4.200) and (4.201), we have ten equations to solve for A_0 , B_0 , A_1 , B_1 , C_1 , C_2 , C_4 , C_5 , γ_a and γ_b . The first four constants appear in the membrane expansion, the next four in the boundary-layer expansions, and γ_a and γ_b , as we know, represent the values of γ at the outer and inner edges of the plate respectively.

Once A_0 , B_0 , γ_a and γ_b are known the remaining quantities above are easily obtained as shown next. From (4.175), (4.179), (4.178), (4.182), (4.200) and (4.201) respectively we see that,

$$C_1^2 = \sin^2(\gamma_a/2)/A_0^2, \quad (A.1)$$

$$C_4^2 = \sin^2(\gamma_b/2)/A_0^2, \quad (A.2)$$

$$C_2 = A_1/(2A_0^4) (1 + \sin^2(\gamma_a)/(8 \sin^2(\gamma_a/2)) - 3\gamma_a \cot(\gamma_a/2)/4) + 2B_1A_0 \cot(\gamma_a/2), \quad (A.3)$$

$$C_5 = A_1/(2A_0^4) (1 + \sin^2(\gamma_b)/(8 \sin^2(\gamma_b/2)) - 3\gamma_b \cot(\gamma_b/2)/4) + 2B_1A_0 \cot(\gamma_b/2), \quad (A.4)$$

$$-6/C_1^5 = A_1(1 - \cos\gamma_a)/(2A_0 \sin\gamma_a) - C_2((1 + \nu)/2 +$$

$$4A_0^3/(\sin\gamma_a(1 - \cos\gamma_a))) , \quad (A.5)$$

and,

$$6(b/a)^2/C_4^5 = A_1(1 - \cos\gamma_b)/(2A_0 \sin\gamma_b) - C_5((1 + \nu)/(2(b/a)^2) +$$

$$4A_0^3/(\sin\gamma_b(1 - \cos\gamma_b))) , \quad (A.6)$$

where, in the last two equations, we have made use of the fact that $F_1(\gamma_a)$ and $F_1(\gamma_b)$ are equal to C_2 and C_5 respectively as seen from (4.178) and (4.182). Thus, from (A.1) and (A.2) it is clear that C_1 and C_4 are known once we determine γ_a , γ_b and A_0 , and upon substituting these five quantities into (A.3) through (A.6) we are left with four linear algebraic equations for determining A_1 , B_1 , C_2 and C_5 . Thus the only difficulty lies in solving for the quantities A_0 , B_0 , γ_a and γ_b from the following four equations, which follow from (4.176), (4.180), (4.177) and (4.181):

$$4A_0^3 - B_0 = \gamma_a - \sin\gamma_a , \quad (A.7)$$

$$4A_0^3(b/a)^2 - B_0 = \gamma_b - \sin\gamma_b , \quad (4.8)$$

$$A_0^3 \sin\gamma_a - (1 + \nu)(1 - \cos\gamma_a)^2/8 = 0 , \quad (4.9)$$

and

$$A_0^3 \sin\gamma_b - (1 + \nu)(1 - \cos\gamma_b)^2/(8(b/a)^2) = 0 . \quad (A.10)$$

The transcendental nature of these equations of course prohibits a closed form solution and we accordingly seek an approximate numerical solution to them. By eliminating A_0^3 and B_0 from the above equations we get the following expressions for the quantity $(b/a)^2$:

$$(b/a)^2 = 1 - \frac{2 \sin \gamma_a}{(1 + \nu)(1 - \cos \gamma_a)^2} (\gamma_a - \gamma_b - (\sin \gamma_a - \sin \gamma_b)), \quad (A.11)$$

and

$$(b/a)^2 = \frac{\sin \gamma_a (1 - \cos \gamma_b)^2}{\sin \gamma_b (1 - \cos \gamma_a)^2} \quad (A.12)$$

The above two equations are programmed on a computer and by trial and error we determine, for a given value of ν , corresponding values of γ_a and γ_b which give the same values of $(b/a)^2$, to some prescribed degree of accuracy, from both equations. Thus we obtain a tabulation of corresponding values of (b/a) , γ_a and γ_b for a given value of ν . In Figures (A.1) and (A.2) we show the results of this process for four values of ν . In Figure (A.1) we plot (b/a) versus γ_b while in Figure (A.2) we plot (b/a) versus γ_a . Thus, for a given value of (b/a) and ν , one determines the corresponding values of γ_a and γ_b from these figures. Then by substituting (b/a) , γ_a and γ_b into (A.7) and (A.8), A_0 and B_0 are easily determined.

The trial and error procedure used in obtaining these curves is greatly facilitated by the fact that we are able to establish bounds on the admissible* values of γ_a and γ_b . These bounds are determined by observing the behavior of plots of (b/a) versus γ_b , with γ_a fixed, from both (A.11) and (A.12). In Figure (A.3) we show four sets or pairs of curves, each pair corresponding to a different value of γ_a . The upper curve in each set is obtained from (A.11), the lower curve from (A.12). If the value of γ_a is an admissible value then the curves in a set will intersect each other and the values of γ_a , γ_b and (b/a) at this intersection point of course represent a solution to (A.11) and (A.12).

*For a given value of ν there are fixed ranges for the values of γ_a and γ_b that satisfy (A.11) and (A.12). Values of γ_a and γ_b within these ranges are called admissible values.

If the curves of a set do not intersect one another then we know that the value of γ_a for that set is outside the range of admissible values. The pair of curves for $\gamma_a = 1.0$ in Figure (A.3) are seen to cross one another at $\gamma_b \approx .6$ and $b/a \approx .46$ and therefore these three corresponding values give a solution to (A.11) and (A.12). Clearly the values of 1.5, 2.0, and 2.5 for γ_a are not admissible.

The important feature to be observed in Figure (A.3) is that the tangents to each curve in a set, at $b/a = 1.0$, continually approach each other as we move closer to an admissible value of γ_a . Therefore it appears that a limiting condition exists when these tangents coincide with each other. From (A.11) and (A.12) we find that the slope, $d(b/a)/d\gamma_b$, of the upper and lower curves in a set are given by,

$$\frac{d(b/a)}{d\gamma_b} = \frac{a \sin \gamma_a (1 - \cos \gamma_b)}{b(1 + \nu)(1 - \cos \gamma_a)^2}, \quad (A.13)$$

and,

$$\frac{d(b/a)}{d\gamma_b} = \frac{a \sin \gamma_a (1 - \cos \gamma_b)}{b(1 - \cos \gamma_a)^2} - \frac{a \sin \gamma_a \cos \gamma_b (1 - \cos \gamma_b)^2}{2b \sin^2 \gamma_b (1 - \cos \gamma_a)^2}, \quad (A.14)$$

respectively. Before setting (A.13) and (A.14) equal to one another we observe from (A.12) that at $(b/a) = 1.0$, $\gamma_a = \gamma_b$. Therefore, by letting γ stand for the value of γ_a and γ_b at $b/a = 1.0$, (A.13) and (A.14) reduce to,

$$\frac{d(b/a)}{d\gamma_b} = \frac{\sin \gamma}{(1 + \nu)(1 - \cos \gamma)}, \quad (A.15)$$

and,

$$\frac{d(b/a)}{d\gamma_b} = \frac{\sin \gamma}{(1 - \cos \gamma)} - \frac{\cos \gamma}{2 \sin \gamma}. \quad (A.16)$$

By equating (A.15) to (A.16) we get the following equation which represents a limiting condition on γ ,

$$\cos \gamma = \frac{2\nu}{1 - \nu} . \quad (\text{A.17})$$

In order to interpret the meaning of (A.17) let us first set ν equal to .25 and solve for γ . We find that γ must be approximately .84. From Figure (A.3), for which ν is equal to .25, we recall that a solution to (A.11) and (A.12) is given by $\gamma_a = 1.0$ and $\gamma_b = .6$ (approximately). Then obviously (A.17) does not give an upper limit to γ_a nor a lower limit to γ_b . Accordingly we assume that (A.17) gives the maximum and minimum admissible values for γ_b and γ_a respectively. Moreover this implies that the ranges for admissible values of γ_a and γ_b do not overlap although they have a common value which occurs when b/a is equal to 1. Also γ_a is always greater than γ_b , except of course when b/a is equal to one. In addition it is apparent that as b/a decreases from unity, corresponding values of γ_a and γ_b steadily become further apart from each other which leads us to suspect that the maximum value of γ_a and the minimum value of γ_b occur simultaneously. As we recall (see (4.202) and the comment following it) γ_b assumes the value of zero when b/a is equal to zero and this represents its minimum. Therefore in (A.11) let us set γ_b and b/a equal to zero and rearrange the equation to the following form:

$$\frac{1 + \nu}{2} = \frac{\sin \gamma_a (\gamma_a - \sin \gamma_a)}{(1 - \cos \gamma_a)^2} . \quad (\text{A.18})$$

Equation (A.18) is viewed as a relation for determining the maximum value of γ_a for a given value of ν . In Figure (A.4) we show a plot of this equation.

Summarizing, the bounds on admissible values of γ_a and γ_b for a given value of ν are found as follows:

- (a) The maximum value of γ_a is obtained from (A.18), or more conveniently from Figure (A.4); its minimum value is found from (A.17).

- (b) The maximum value of γ_b is found from (A.17); its minimum value is zero.

With these bounds the task of finding solutions to (A.11) and (A.12) is simplified considerably.

A.2 Expressions for the Stresses and Displacements

In this section we give the expressions for the stresses and displacements in terms of the dependent variables of chapters III and IV, namely α and F .

(a) Direct stresses

The direct or in-plane circumferential and radial stresses follow from equations (2.4) and (2.7) of (). There we see that N_θ and N_r , the circumferential and radial stress resultants, are given by,

$$N_\theta = \frac{d\bar{\psi}}{d\bar{r}} \quad , \quad (A.19)$$

and,

$$N_r = V \sin\bar{\beta} + H \cos\bar{\beta} \quad , \quad (A.20)$$

respectively, where the quantities $\bar{\psi}$, $\bar{\beta}$, V and H are defined in chapter II.

Substituting in (A.19) for $\bar{\psi}$ and \bar{r} from (2.4) and (2.6), and then using (3.9) and (3.10), or equivalently (4.9) and (4.11), gives us,

$$N_\theta = \frac{2B}{a} \left(F' - \frac{F}{2\rho} \right) \quad , \quad (A.21)$$

where we recall the prime means differentiation with respect to ρ .

Since B is equal to $(\lambda^2 D/a)$, as is seen from (2.8), we get,

$$N_{\theta} = \frac{2\lambda^2 D}{a^2} \left(F' - \frac{F}{2\rho} \right) \quad (\text{A.22})$$

Using the definitions of V and H from chapter II, (A.20) becomes,

$$N_r = \frac{P}{2\pi \bar{r}} \sin \bar{\beta} + \frac{\bar{\psi}}{\bar{r}} \cos \bar{\beta} \quad (\text{A.23})$$

where P is the applied load (see Figure 2). By means of (2.4) through (2.6) the above equation can be written as,

$$N_r = \frac{P}{2\pi ar} \sin(\delta\beta) + \frac{B\psi}{ar} \cos(\delta\beta) , \quad (\text{A.24}) ,$$

and for the case when δ is small, we find that,

$$N_r = \frac{P\delta\beta}{2\pi ar} + \frac{B\psi}{ar} . \quad (\text{A.25})$$

The above approximation for small δ is consistent with the derivation of (2.15) and (2.16), which are the basic differential equations of chapters III and IV. Therefore (A.25) is valid with respect to the aforementioned chapters, although the exact expression given by (A.24) should be used in conjunction with chapter V.

By using (3.9) through (3.11), or equivalently (4.9) through (4.11), (A.25) becomes,

$$N_r = \frac{B}{a\rho} \left(\left(\frac{P\delta}{B} \right) \frac{\alpha}{2\pi} + F \right) . \quad (\text{A.26})$$

The first term in (A.26) is shown to be insignificant compared to the quantity F. From (3.3), (2.8) and (2.9) the quantity $(P\delta/B)$ can be expressed as,

$$\frac{P\delta}{B} = \frac{2\pi}{3(1-\nu^2)} \left(\frac{h}{a} \right)^2 , \quad (\text{A.27a})$$

which holds true for the analysis of chapter III. Therefore, as long as we are considering a very thin plate such that $(h/a)^2$ is small, we can neglect the term in (A.26) containing $(P\delta/B)$ compared to the quantity F . We note that under some conditions though the term containing $(P\delta/B)$ may be significant. For the analysis of chapter IV we find that,

$$\frac{P\delta}{B} = 8\pi\delta^2, \quad (\text{A.27b})$$

which follows from (4.3) and (2.8). Since δ is considered to be small in chapter IV the term in (A.26) containing $(P\delta/B)$ may therefore be neglected compared to F in this case without qualification. Thus, in accordance with the above discussion, we have,

$$N_r = \frac{BF}{a\rho}. \quad (\text{A.28})$$

Substituting in (A.28) for B from (2.8) we get,

$$N_r = \frac{\lambda^2 DF}{a^2 \rho}. \quad (\text{A.29})$$

The circumferential and radial direct stresses, $\sigma_{\theta D}$ and σ_{rD} , are obtained by dividing N_{θ} and N_r by the plate thickness h . Therefore it follows that,

$$\sigma_{\theta D} = \frac{2\lambda^2 D}{a^2 h} \left(F' - \frac{F}{2\rho} \right), \quad (\text{A.30})$$

and,

$$\sigma_{rD} = \frac{\lambda^2 DF}{a^2 h \rho}. \quad (\text{A.31})$$

By substituting for D from (2.3), (A.30) and (A.31) become,

$$\sigma_{\theta D} = \frac{\lambda^2 E}{6(1 - \nu^2)(a/h)^2} \left(F' - \frac{F}{2\rho} \right), \quad (\text{A.32})$$

and,

$$\sigma_{rD} = \frac{\lambda^2_{EF}}{12(1 - \nu^2)(a/h)^2 \rho} \quad (A.33)$$

b. Bending stresses

The circumferential and radial bending stress resultants, M_θ and M_r , are found from (2.9) of (2) to be,

$$M_\theta = -D \left(\frac{\sin \bar{\beta}}{\bar{r}} + \nu \frac{d\bar{\beta}}{d\bar{r}} \right), \quad (A.34)$$

and,

$$M_r = -D \left(\frac{d\bar{\beta}}{d\bar{r}} + \nu \frac{\sin \bar{\beta}}{\bar{r}} \right), \quad (A.35)$$

where D is given by (2.3). From (2.5) and (2.6) the above equations can be written as,

$$M_\theta = -D \left(\frac{\sin(\delta\beta)}{ar} + \frac{\nu\delta}{a} \frac{d\beta}{dr} \right), \quad (A.36)$$

and

$$M_r = -D \left(\frac{\delta}{a} \frac{d\beta}{dr} + \frac{\nu \sin(\delta\beta)}{ar} \right). \quad (A.37)$$

For small δ , as considered in chapters III and IV, we get,

$$M_\theta = -D \left(\frac{\delta\beta}{ar} + \frac{\nu\delta}{a} \frac{d\beta}{dr} \right), \quad (A.38)$$

and,

$$M_r = -D \left(\frac{\delta}{a} \frac{d\beta}{dr} + \frac{\nu\delta\beta}{ar} \right). \quad (A.39)$$

Substituting in the above two equations for β and r from (3.9) and (3.11), or equivalently from (4.10) and (4.11), and using (2.13), (2.9) and (2.3) gives us,

$$M_\theta = - \frac{Eh^2\lambda}{(a/h)^2 (12(1 - \nu^2))^{3/2}} \left((1 - \nu) \frac{\alpha}{\rho} - 2\nu \alpha' \right), \quad (A.40)$$

and,

$$M_r = \frac{Eh^2\lambda}{(a/h)^2(12(1-\nu^2))^{3/2}} \left((1-\nu)\frac{\alpha}{\rho} - 2\alpha' \right) \quad (A.41)$$

The circumferential and radial bending stresses, $\sigma_{\theta B}$ and σ_{rB} , are found from (A.40) and (A.41) by multiplying them by $(6/h^2)$. Therefore,

$$\sigma_{\theta B} = - \frac{6E\lambda}{(a/h)^2(12(1-\nu^2))^{3/2}} \left((1-\nu)\frac{\alpha}{\rho} + 2\nu\alpha' \right), \quad (A.42)$$

and,

$$\sigma_{rB} = \frac{6E\lambda}{(a/h)^2(12(1-\nu^2))^{3/2}} \left((1-\nu)\frac{\alpha}{\rho} - 2\alpha' \right). \quad (A.43)$$

c. Displacements

The radial and transverse displacements, U and W , are obtained from (2.11) and (2.14) of (2). For the radial displacement we have,

$$U = \frac{\bar{r}}{Eh} (N_{\theta} - \nu N_r), \quad (A.44)$$

and upon substituting (A.19) and (A.23) into the above expression we get,

$$U = \frac{\bar{r}}{Eh} \left(\frac{d\bar{\psi}}{d\bar{r}} - \nu \left(\frac{P \sin\bar{\beta}}{2\pi\bar{r}} + \frac{\bar{\psi}}{\bar{r}} \cos\bar{\beta} \right) \right) \quad (A.45)$$

For the case when δ is small, we use (A.21) and (A.28) to obtain,

$$U = \frac{2B_0^{1/2}}{Eh} \left(F' - \left(\frac{1+\nu}{2} \right) \frac{F}{\rho} \right), \quad (A.46)$$

which we have expressed in terms of ρ . Since we have used (A.28) in arriving at the equation above let us note that, in the case of infinitesimal deformations as discussed in chapter III, the above expression is valid only as long as $(h/a)^2$ is small, i.e. the plate is very thin. The given expression above applies without qualification with regard to the range of deformations considered in chapter IV.

Expressing U in terms of λ^2 by means of (2.8) we determine that,

$$U = \frac{a\lambda^2 \rho^{1/2}}{(a/h)^2 6(1 - \nu^2)} \left(F' - \left(\frac{1 + \nu}{2} \right) \frac{F}{\rho} \right) . \quad (\text{A.47})$$

The transverse displacement is given by

$$W = \int_{\bar{r}}^{\bar{r}} \left(1 + \frac{dU}{d\bar{r}} \right) \tan \bar{\beta} d\bar{r} . \quad (\text{A.48})$$

For the case when δ is small we find that,

$$\frac{WL}{a} = \int_{\rho}^{\rho} \frac{\alpha}{2\rho} d\rho \quad (\text{A.49})$$

which is obtained by a process similar to that in arriving at the final expression for N_r . The derivative of U appearing in (A.48) vanishes from (A.49) since it is found to be proportional to δ^2 with regard to chapter IV and for chapter III we make the same assumption concerning the ratio $(h/a)^2$ as is included in the discussion following (A.27a).

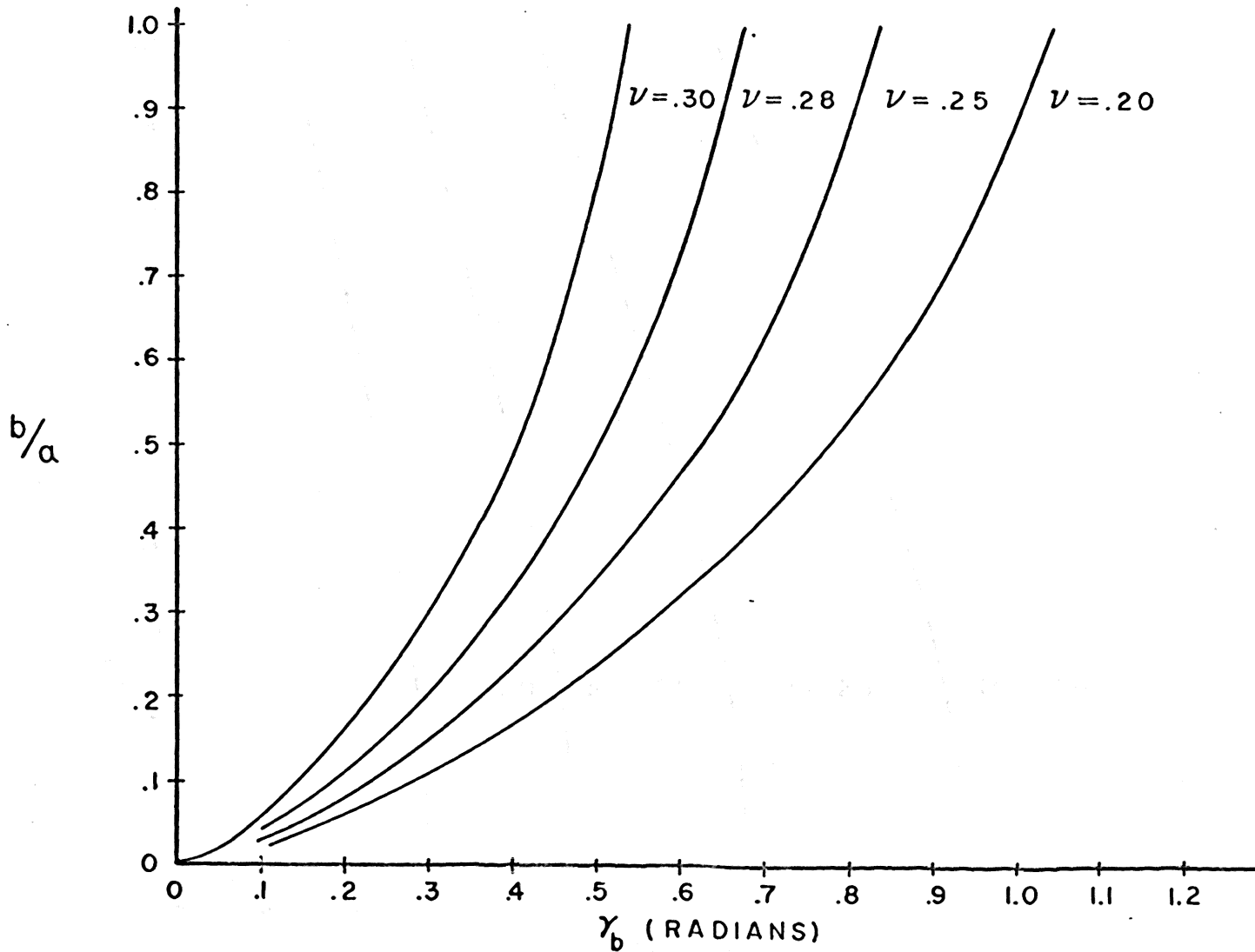


Figure (A.1) - CURVES FOR DETERMINING CORRESPONDING VALUES OF b/a AND γ_b

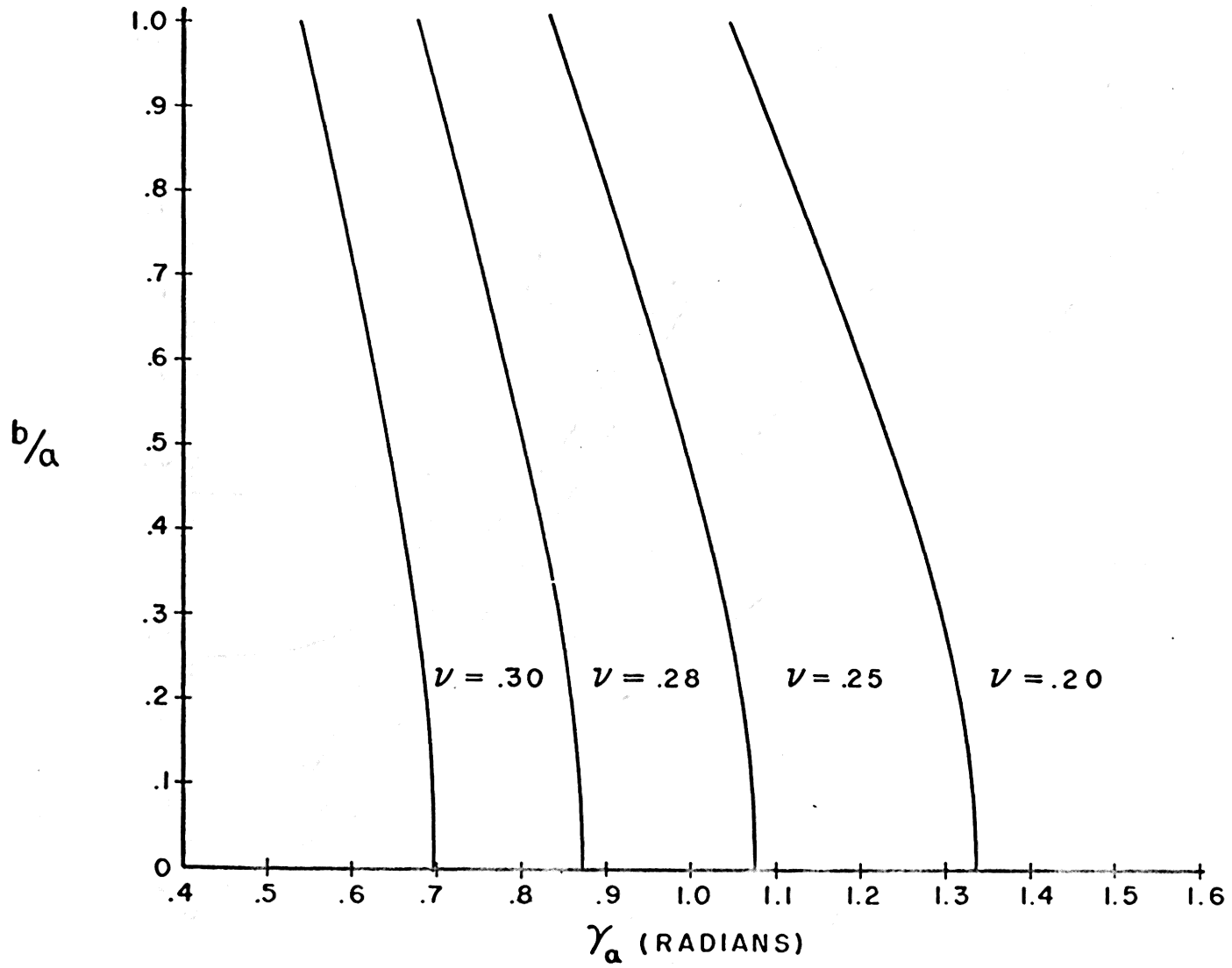


Figure (A.2) - CURVES FOR DETERMINING CORRESPONDING VALUES OF b/a AND γ_a

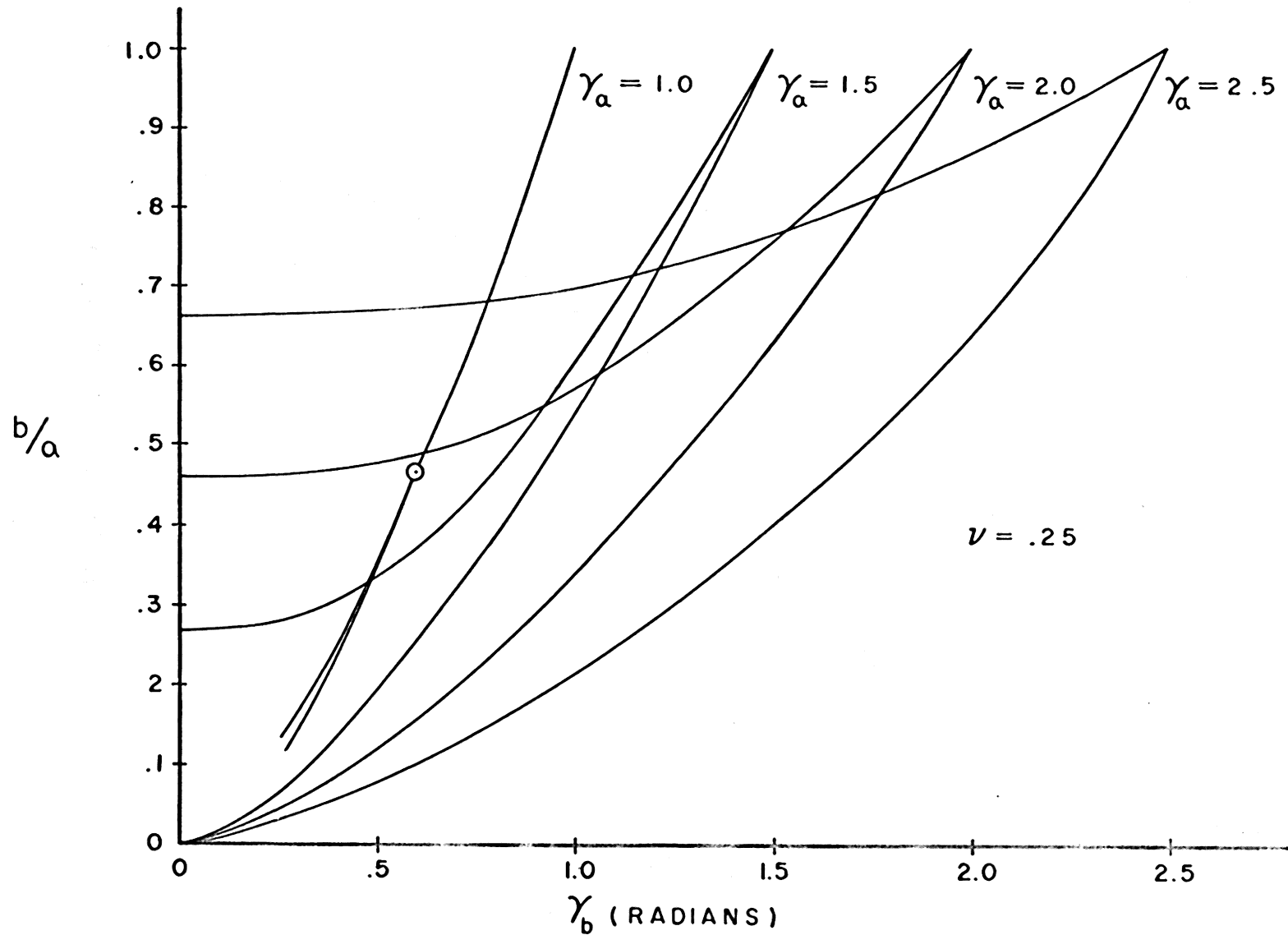


Figure (A.3) - CURVES ILLUSTRATING THE TANGENCY CONDITION AT $b/a = 1.0$

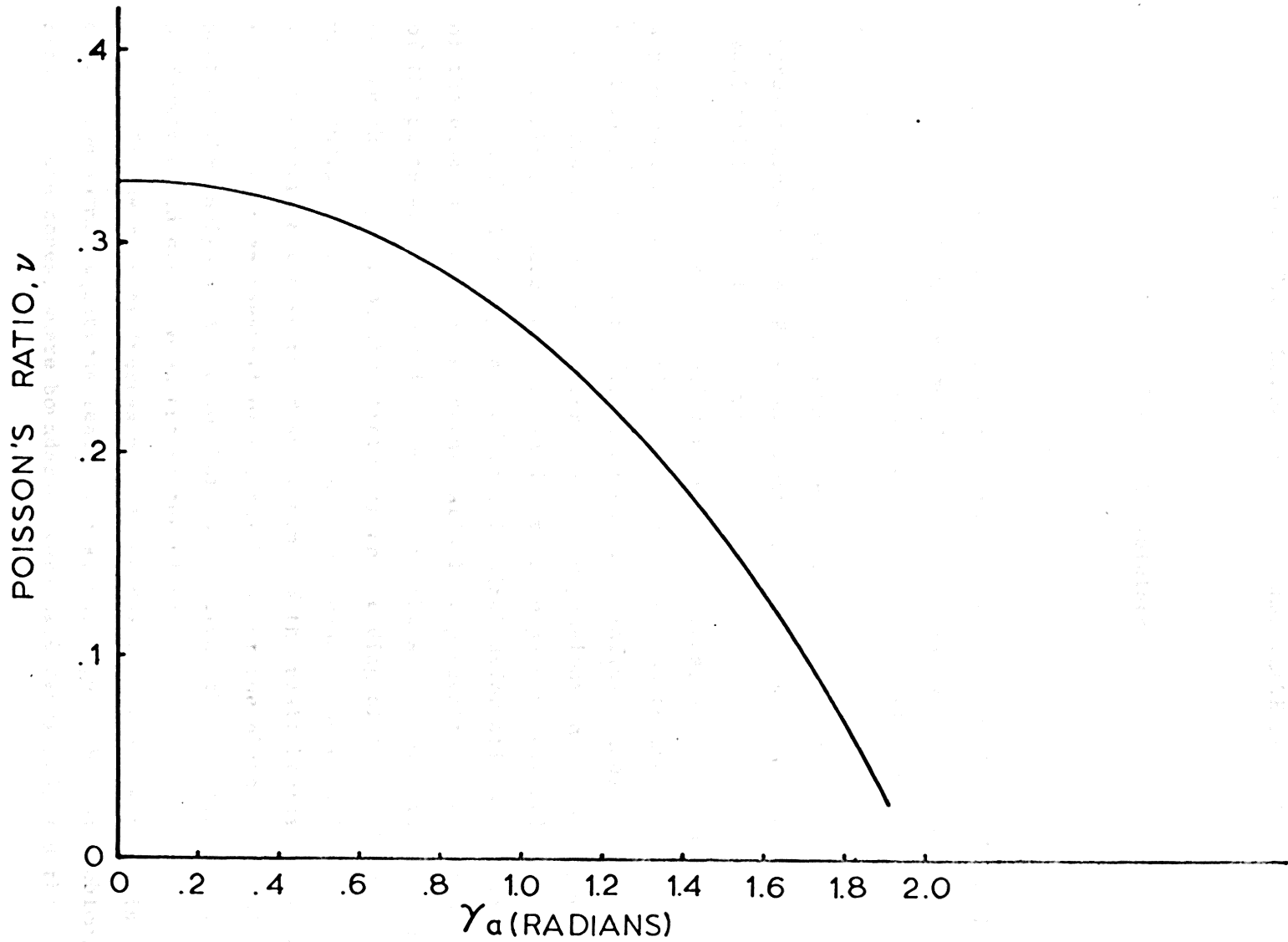


Figure (A.4) - MAXIMUM VALUE OF γ_a AS A FUNCTION OF POISSON'S RATIO

ASYMPTOTIC SOLUTIONS OF A CIRCULAR PLATE PROBLEM

by

Donald C. Tolefson

ABSTRACT

The rotationally symmetric small and large finite deformations of an annular plate, clamped at its outside edge and containing a central rigid inclusion to which a normal load is applied, are considered.

Perturbation techniques are applied to analyze the first approximations to the Reissner equations for the separate cases of small and large finite deformations.

In the case of small finite deformations, where the first approximations to Reissner's equations correspond to the von Karman equations, we examine the limiting cases of infinitesimal and increasingly large deformations. Perturbation expansions in terms of powers of a small parameter are obtained for the dependent variables.

In the limiting case of infinitesimal deformations two terms of the expansion of the dependent variable representing the bending of the plate are obtained in closed form; the second of which arises due to the coupling of the stretching and bending of the plate's middle surface. This second term of the expansion becomes important when the maximum transverse displacement of the plate begins to exceed slightly less than one-half of the thickness of the plate. For this part of the analysis of the von Karman equations the expansions are uniformly valid over the extent of the plate.

In the case of increasingly large deformations the analysis of the von Karman equations leads to a singular perturbation problem. Here the edge zones, where boundary-layers have developed, and the

interior region of the plate are investigated individually. Separate perturbation expansions are obtained in these regions of the plate. The theory of "Matched Asymptotic Expansions" is utilized here in order to evaluate certain constants of integration which remain undetermined after exhausting the boundary conditions on the problem. Two and, in one case, three terms of the expansions are found in closed form. It is found that for the special case where Poisson's ratio is equal to $1/3$ the results are extremely simple and here we obtain three terms in the expansions. Numerical results for the stresses and transverse displacement are compared with data obtained from the numerical integration of the Reissner equations by other authors and the agreement is very good.

Lastly we examine the case of large finite deformations where the deformations have exceeded the range of validity of the von Karman equations.

A perturbation analysis of the first approximations to Reissner's equations for this case leads to a singular perturbation problem as expected. The basic differential equations for the boundary-layer zones and interior region of the plate are derived as well as the required conditions, in addition to those provided at the edges of the plate, for determining the constants of integration which arise in the analysis. Closed form solutions for one of the dependent variables can not be obtained here and numerical integration is required. For this reason numerical results are not given.

In this latter part of the investigation we observe that the first approximations to the original form of Reissner's equations contain certain terms which he neglects in his more recent theory. We, therefore, conclude that these terms which are missing from his more recent theory are important if the deformations are extremely large and should be retained in this case.