STATISTICAL MODELS WITH DICHOTOMOUS RESPONSE

by

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(ABSTRACT)

Statistical models with dichotomous response are investigated. Many techniques for modeling such situations are discussed which involve various types of distributions. The two distributions which shall be discussed further are those related to probit and logit analysis. Several estimation methods are developed for determining the parameters in the two distributions mentioned.

Examples are given for each of these estimation methods. And finally, the estimation methods are compared.

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Chapter I

INTRODUCTION

A regression model can generally be defined as a statistical model that embodies assumptions about a dependent relationship between one variable (the response or dependent variable, y) and one or more other variables (the explanatory or independent variables, x's).

Much of the research in regression involves consideration of continuous responses. For example, given three variables such as height, diameter, and age of a tree, a typical response is merchantable volume, which is measured on a continuum. But there are many statistical models whose response is a dichotomous variable. These types of models are found in many fields such as forestry, medicine, biology, and engineering. For example, after applying a certain insecticide to an individual beetle, the beetle either dies or it does not die. Or, in a medical application, a new treatment is administered to a patient and results in the patient being cured or not cured.

The analysis discussed in this paper, however, is designed to estimate the probability that any given observation will assume one of two distinct classes. This analysis is different from discriminant analysis, which is used to

assign individual observations to one of a finite number of classes.

The theory of discrete dependent variable regression has its genesis in bioassay (Finney (1952)). It should be noted that discrete models are synonymous with quantal, categorical, qualitative response, or qualitative choice models (Capps (1983)).

Several conventional methods are available to estimate the parameters in a regression model when the response variable is continuous. If large samples are available and the dependent variable is discrete, a transformation of the dependent variable followed by conventional (or weighted) least squares estimation is a reasonable solution to the regression problem (Press (1972)). It will be shown that when samples are small or moderately large and when the dependent variable can assume one of two possible values, conventional estimation techniques are inappropriate, but alternatives are available (Press (1972)). The purpose of this paper is to discuss these alternatives for the dichotomous case.

Up to this point in the discussion, each illustration involves responses based on individuals run without grouping. The difficulties with estimating probability models with a dichotomous response variable are lessened with grouped data, where the dependent variable is the proportion of the

individuals making a given choice or falling into a particular category, rather than the dichotomous variable observed at the individual level. The problems with individual data, however, are not absent even with grouped data even though grouped data may be easier to solve. Note that individual level data contains more information than grouped data, and thus individual data is to be preferred when available.

The dichotomous model where the discrete response variable can take on only one of two values is also known as a binary choice model. For example, suppose n people are asked whether or not they favor a particular issue. Their responses are classifiable as dichotomous, or binary (excluding such categories as "no response" or "no opinion"). The procedures described in this paper will allow the statistician to relate responses to some other explanatory variables. In the above illustration, the explanatory variables may be socioeconomic and demographic characteristics.

In this work we attempt to:

1). present a theoretical review of dichotomous response models

2). discuss estimation methods

3). compare estimation methods by use of sample problems

Chapter II

THEORETICAL REVIEW

In the dichotomous model only one of two responses are possible. We shall denote the ith response of n possible observations as y_i (i=1,...,n). For convenience, we shall denote the two possible response values as 0 or 1 (though any other pair of real numbers could be used (Capps (1983))). Hence,

$$y_i = \begin{cases} 1 & \text{if a particular event occurs on the ith observation} \\ 0 & \text{if the event does not occur on the ith observation} \end{cases} \quad (1)$$

Also available are k explanatory variables for each of the i response measurements, y_i, which we shall denote as x_{ij} (i=1,...,n;j=1,...,k). The response variable, y, is a dependent variable while the explanatory variables, x's, are independent variables.

The problem that one faces is to find a way to approximate the relationship between the response and the explanatory variables.

2.1 LINEAR PROBABILITY MODEL

It is natural to attempt to relate the variables by a regression equation. Let us first suppose that the y_i's are regressed on the x_{ij}'s in the conventional linear regression framework. When the dependent variable, y_i, takes on only the values 1 or 0, as in the dichotomous model, the conventional linear regression model is termed the linear probability model (Goldberger (1964)). What follows is a brief review of linear regression and then presented are the short comings of the linear probability model for approximating the dichotomous model.

The linear regression model is

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_k x_{ik} + \varepsilon_i \qquad (2)$$

where y_i = response for ith observation

β_j = unknown coefficients (j=0,1,...,k)

x_{ij} = jth explanatory variable for the ith observation

ε_i = random error component for the ith observation

The model can be written in vector notation as

$$y_i = \underline{x}'_i \underline{\beta} + \varepsilon_i \qquad (i=1,\ldots,n) \qquad (3)$$

where $\underline{x}'_i = [1, x_{i1}, x_{i2}, \ldots, x_{ik}]$ (px1 vector)

$\underline{\beta}' = [\beta_0, \beta_1, \ldots, \beta_k]$ (px1 vector)

$p = k + 1$

or in matrix notation as

$$\underline{y} = X\underline{\beta} + \underline{\varepsilon} \qquad (4)$$

where $\underline{y}' = [y_1, y_2, \ldots, y_n]$ (nx1 vector)

$$X = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & x_{2k} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix} \qquad \text{(nxp matrix)}$$

$\underline{\varepsilon}' = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n]$ (nx1 vector)

The parameter β_j represents the expected change in the response y_i per unit change in the x_{ij} when all remaining regressor variables are held constant.

It is convenient to view the explanatory variables, the x_{ij}'s, as established by the experimenter and measured with negligible error (constants) (Assumption #1).

The regression model describes a stochastic relationship between the y_i's and the x_{ij}'s and this implies that for every $\underline{x}_i$, there is a probability distribution of possible values of y_i.

As a result the value of y_i can never be forecast exactly. This results from the fact that the stochastic disturbance ε_i is random and thus imparts randomness to y_i.

In the linear regression model therefore, the probability distribution of y_i is determined by the value of $\underline{x}_i$ and by the distribution of ε_i. Because of this relationship a full

specification of a regression model must state not only the form of the regression equation but also a set of assumptions concerning the probability distribution of ε_i which are as follow:

- The error mean must be zero [$E(\varepsilon_i)=0$]
  (Assumption #2)

- The error variance must be constant (also known as the homoscedasticity assumption) [$Var(\varepsilon_i)=\sigma^2$]
  (Assumption #3)

- The error terms must be uncorrelated
  [$E(\varepsilon_i\varepsilon_j)=0$ for every i,j ($i\neq j$)]
  (Assumption #4)

Therefore, the probability distribution for y_i at each possible value for x_{ij} has mean

$$E(y_i|\underline{x}_i) = \underline{x}'_i\underline{\beta} \tag{5}$$

and variance

$$V(y_i|\underline{x}_i) = V(\varepsilon_i) = \sigma^2 \tag{6}$$

Thus, the mean of y_i is a linear function of $\underline{x}_i$ while the variance of y_i does not depend on the value of $\underline{x}_i$. Furthermore, because the errors are uncorrelated (Assumption #4), the responses are also uncorrelated.

Suppose we consider the estimation of the unknown β_j's using the classical ordinary least squares (OLS) method. The idea behind this method is to find the vector of least squares estimators, $\underline{\hat{\beta}}$, which minimizes

$$\Sigma\varepsilon_i^2 = \underline{\varepsilon}'\underline{\varepsilon} = (\underline{y}-X\underline{\beta})'(\underline{y}-X\underline{\beta}) \tag{7}$$

Note that all summations, Σ's, unless otherwise specified are summed from i = 1 to n. Also note that (7) can be written as

$$\begin{aligned} &\underline{y}'\underline{y} - \underline{\beta}'X'\underline{y} - \underline{y}'X\underline{\beta} + \underline{\beta}'X'X\underline{\beta} \\ &\qquad = \underline{y}'\underline{y} - 2\underline{\beta}'X'\underline{y} + \underline{\beta}'X'X\underline{\beta} \\ &\qquad = S(\underline{y},X,\underline{\beta}) \end{aligned} \tag{8}$$

since $\underline{\beta}'X'\underline{y}$ is a scalar and hence its transpose, $(\underline{\beta}'X'\underline{y})'=\underline{y}'X\underline{\beta}$, is the same scalar.

The least squares estimators must satisfy

$$\frac{\delta}{\delta\underline{\beta}} S(\underline{y},X,\underline{\beta})\Big|_{\underline{\hat{\beta}}} = -2X'\underline{y} + 2X'X\underline{\hat{\beta}} = \underline{0} \tag{9}$$

which simplifies to

$$X'X\underline{\hat{\beta}} = X'\underline{y} \tag{10}$$

This equation is known as the normal equations. To solve the normal equations multiply both sides of (10) by the inverse of X'X. Thus the least squares estimator of $\underline{\beta}$ is

$$\underline{\hat{\beta}} = (X'X)^{-1}X'\underline{y} \tag{11}$$

provided that $(X'X)^{-1}$ exists.

The fitted regression model corresponding to the explanatory variables $\underline{x}'_i = [1, x_{i1}, x_{i2}, \ldots, x_{ik}]$ is

$$\hat{y}_i = \underline{x}'_i \hat{\underline{\beta}} \qquad (12)$$

The vector of fitted values $\hat{y}_i$ corresponding to the observed values y_i is

$$\hat{\underline{y}} = X\hat{\underline{\beta}} \qquad (13)$$

The difference between the observed value y_i and the corresponding fitted value $\hat{y}_i$ is the residual

$$e_i = y_i - \hat{y}_i \qquad (14)$$

The n residuals may be conveniently written in vector notation as

$$\underline{e} = \underline{y} - \hat{\underline{y}} \qquad (15)$$

In view of the dichotomous nature of the response variable, the conditional expectation of y_i given $\underline{x}_i$ may be interpreted as the conditional probability that the event occurs in the ith sample given $\underline{x}_i$.

$$E(y_i|\underline{x}_i) = 1*P(y_i=1|\underline{x}_i) + 0*P(y_i=0|\underline{x}_i) \qquad (16)$$
$$= P(y_i=1|\underline{x}_i) = P_i$$

By referring to (5) we have

$$P_i = E(y_i | \underline{x}_i) = \underline{x}'_i \underline{\beta} \tag{17}$$

Then the predicted value of y_i,

$$\hat{y}_i = \underline{x}'_i \hat{\underline{\beta}} = \hat{P}_i \tag{18}$$

is interpreted as the estimated probability that the event will occur given $\underline{x}_i$.

To illustrate the linear probability model, we draw on a study of new 1952 auto purchases (Goldberger (1964)). With x = disposable income in \$100 and y=1 for purchasers and 0 for nonpurchasers, the function estimated by classical least squares is

$$\hat{y} = -0.008 + 0.0022x \tag{19}$$

Thus the estimate of the probability that an individual with \$10,000 disposable income will buy a new car in a year is

$$-0.008 + 0.0022(100) = 0.212 \tag{20}$$

The linear probability model, however, suffers from a number of deficiencies.

The first problem we encounter is that the constant error variance assumption (#3) is violated. This follows from the fact that the error term

$$\varepsilon_i = y_i - P_i \tag{21}$$

can in this case have one of two possible values:

$$\varepsilon_i = \begin{cases} 1 - P_i & \text{if } y_i=1 \\ 0 - P_i & \text{if } y_i=0 \end{cases} \tag{22}$$

Then since ε_i is assumed to have expectation zero ($E(\varepsilon_i)=0$ from Assumption #2):

$$E(\varepsilon_i) = P_i(1-P_i) + (1-P_i)(-P_i) = 0 \tag{23}$$

The respective probabilities of these events are as follows

ε_i	$f(\varepsilon_i)$
$1 - P_i$	$P(y_i=1\|\underline{x}_i) = P_i$
$- P_i$	$P(y_i=0\|\underline{x}_i) = 1 - P_i$

(24)

with variance

$$\begin{aligned} V(\varepsilon_i) &= E(\varepsilon_i^2) - [E(\varepsilon_i)]^2 \\ &= E(\varepsilon_i^2) \\ &= (1-P_i)^2(P_i) + (-P_i)^2(1-P_i) \\ &= P_i(1-P_i) \\ &= E(y_i)[1 - E(y_i)] \end{aligned} \tag{25}$$

Clearly the error variance is not a constant, but varies systematically with $\underline{x}_i$. This violates assumption #3.

The presence of heteroscedasticity results in the loss of efficiency of the parameter estimates, the $\hat{\beta}_j$'s. Addition-

ally, the standard deviation of the OLS parameter estimates are biased because of the heteroscedasticity problem (Goldberger (1964); Press (1972)). Consequently, if OLS estimation of the unknown parameters is used, then the OLS parameter estimates, the $\hat{\beta}_j$'s, although unbiased, are not the minimum variance of all linear unbiased estimators of $\underline{\beta}$ (not B.L.U.E.) (Graybill (1976); Wrigley(1976)).

The second problem we will encounter concerns the predicted values, $\hat{y}_i$'s, which are generated using the linear probability model. That is, OLS estimators of $\underline{x}'_i\underline{\beta}$ are often unreasonable since probability is defined to lie between 0 and 1 and the above formulation allows the $\hat{y}_i$'s to be unbounded taking values from $-\infty$ to $+\infty$. Consequently, the predictions may lie outside the meaningful range of probability and thus be inconsistent with the probability interpretation.

The linear probability model has frequently been used in many types of applications because of computational simplicity. The OLS method yields consistent and unbiased estimates of $\underline{\beta}$ (Hanushek and Jackson (1977)).

Linear regression models with nonconstant error variance can be fitted by the method of weighted or generalized least squares (WLS or GLS). Although this method circumvents the heteroscedasticity problem, it is still possible to generate

estimated values of y_i that lie outside the unit interval. In this method of estimation, the deviation between observed and expected values of y_i is multiplied by a weight w_i chosen inversely proportional to the variance of y_i. The weighted least squares function is

$$\Sigma w_i \varepsilon_i^2 = \underline{\varepsilon}' W \underline{\varepsilon} = (\underline{y} - X\underline{\beta})' W (\underline{y} - X\underline{\beta}) \tag{26}$$

where $W = \begin{bmatrix} w_1 & & 0 \\ & w_2 & \\ & & \ddots \\ 0 & & w_n \end{bmatrix}$ (nxn matrix)

$$w_i = \frac{1}{P_i (1 - P_i)}$$

and the weighted least squares normal equations are given by the solution to:

$$(X'WX)\hat{\underline{\beta}}^* = X'W\underline{y} \tag{27}$$

and the solution to these equations is

$$\hat{\underline{\beta}}^* = (X'WX)^{-1} X'W\underline{y} \tag{28}$$

A possible alternative is the use of resticted least squares estimation in which $\underline{\beta}$ is estimated by ordinary least squares, subject to the constraint that $0 \leq \underline{x}'_i \underline{\beta} \leq 1$. However, when the heteroscadasticity difficulty is accounted for (by generalized least squares) the resulting programming problem becomes especially complex (Press (1972)).

We shall now consider an alternative model which overcomes the inequality constraint $0 \leq \hat{P}_i \leq 1$ of the linear probability model. In the case of $\hat{P}_i = \underline{x}'_i\hat{\underline{\beta}}$ (as given before), the simplest way to satisfy this condition is to impose the following arbitrary definition of $\hat{P}_i$

$$\hat{P}_i = \begin{cases} 0 & \text{if } \underline{x}'_i\hat{\underline{\beta}} < 0 \\ \hat{Y}_i & \text{if } 0 \leq \underline{x}'_i\hat{\underline{\beta}} \leq 1 \\ 1 & \text{if } 1 < \underline{x}'_i\hat{\underline{\beta}} \end{cases} \tag{29}$$

and to use straight forward ordinary or weighted least squares estimation of the regression parameters, $\hat{\underline{\beta}}$. Unfortunately although it appears to be a simple solution to the predicted probabilities problem, the model has a number of serious limitations (Domencich and McFadden (1975)). An important problem occurs for example if a number of the samples have values of the x'_is where the probabilities take the extreme values. In this case, this model will produce parameter estimates which are substantially biased below their true values.

2.2 INTRODUCTION TO PROBIT AND LOGIT MODELS

Since the linear probability model has many shortcomings, alternative approaches to the dichotomous regression problem need to be addressed.

In the bioassay type formulation of the problem, the inequality constraint ($0 \leq \underline{x}'_i\underline{\beta} \leq 1$) is obviated, but at the cost of assuming a probabilistic functional form.

It assumed that the probability P_i is some function of the independent variables $\underline{x}_i$ and of the unknown parameters $\underline{\beta}$,

$$P_i = g(x_{i1},\ldots,x_{ik};\beta_1,\ldots,\beta_k) \qquad (i=1,\ldots,n) \qquad (30)$$

for some properly chosen function, g. Then P will be near zero over a certain part of the domain of g, near one over another part, and will increase from near zero to near one over an intermediate part of the domain. A reasonable assumption is that P_i follows a symmetric sigmoid curve as is shown in Figure 1 for the k=1 case.

Assume a model with the argument as the linear function $t_i = \underline{x}'_i\underline{\beta}$ of the independent variables, x's, and of the p unknown parameters, $\underline{\beta}$. We then seek an estimate of $\underline{\beta}$ which will provide a satisfactory fit of the observations, $\hat{P}_i$'s, to such a symmetric sigmoid curve.

The symmetric sigmoid curve is often represented by distribution functions. The mechanics of some of these transformations are as follows (Goldberger (1964); Tobin (1955)).

Let t_i, any arbitrary index, be a linear function of the regressors,

$$t_i = \underline{x}'_i\underline{\beta} = \beta_0+\beta_1x_{1i}+\ldots+\beta_kx_{ki} \qquad (31)$$

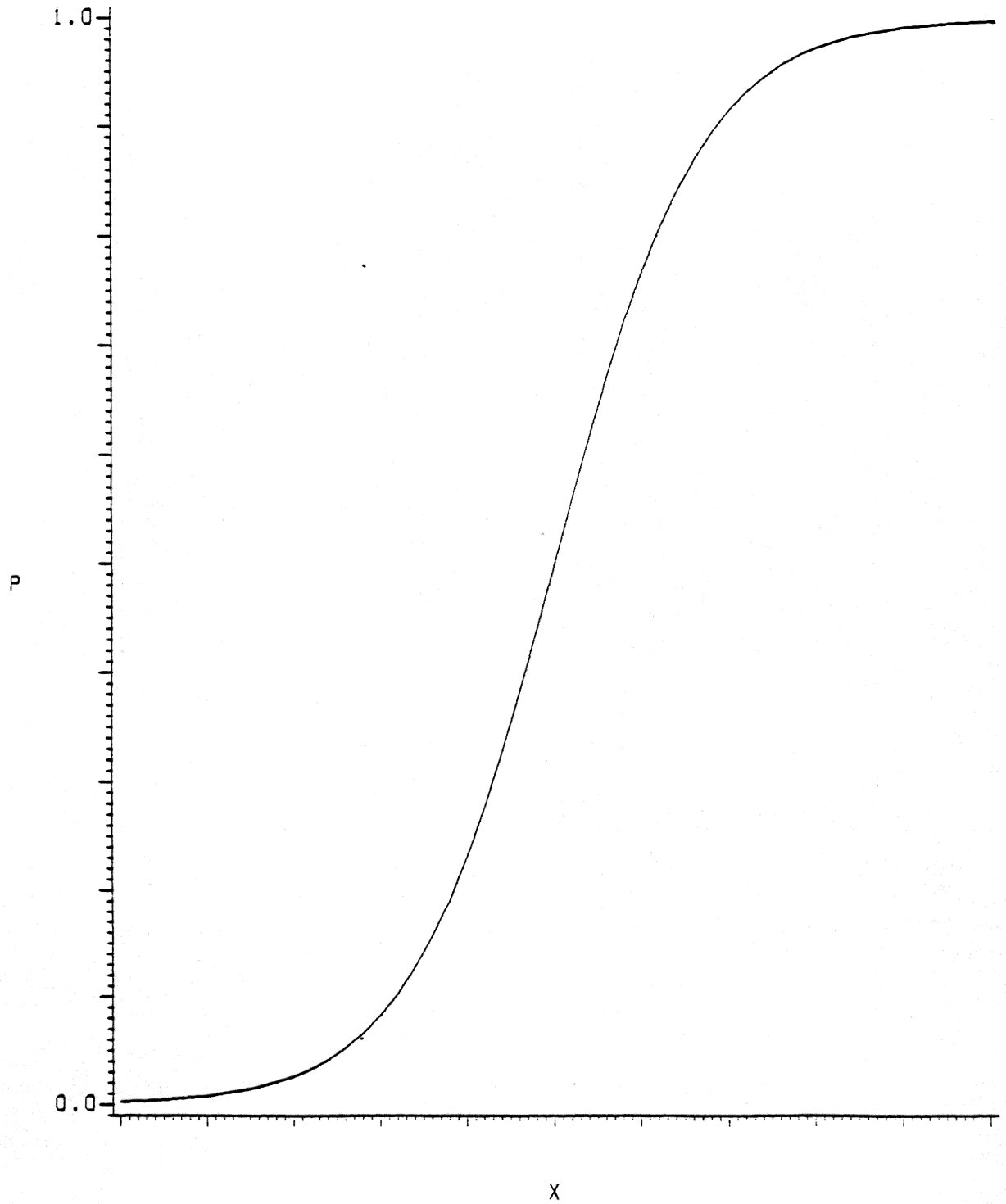


Figure 1. Symmetric Sigmoid Curve

and let t_i^* be a random variable with probability density function (pdf) f. In addition, let the value of y_i be such that

$$y_i = \begin{cases} 1 & \text{if } t_i^* \leq t_i \\ 0 & \text{if } t_i^* > t_i \end{cases} \qquad (32)$$

This relationship is shown in Figure 2.

Each y_i is thus a function of the x's, by way of t_i, and of t_i^*. The t_i^*'s may be interpreted as critical or threshold values of the index t_i. For example, if y = car ownership and x = income, the ith individual with a high t_i^* would own a car only if his income x_i is so high that $t_i \geq t_i^*$ ($t_i = \beta_0 + \beta_1 x_i$) (Goldberger (1964). Or for another example, let y = death of an animal, x1 = dosage, and x2 = age of animal, then the ith animal with a high t_i^* would die only if his dosage ($x1_i$) and age ($x2_i$) are so large that $t_i \geq t_i^*$ ($t_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i}$).

Note that ideally the threshold value, t_i, is what is really of interest but unfortunately this threshold value is not observable. All that is observed is if the response is above the threshold value (y_i=0) or below the threshold value (y_i=1).

Let $F(t_i)$ be the value of the cumulative distribution function (cdf) at t_i given $\underline{x}_i$. That is

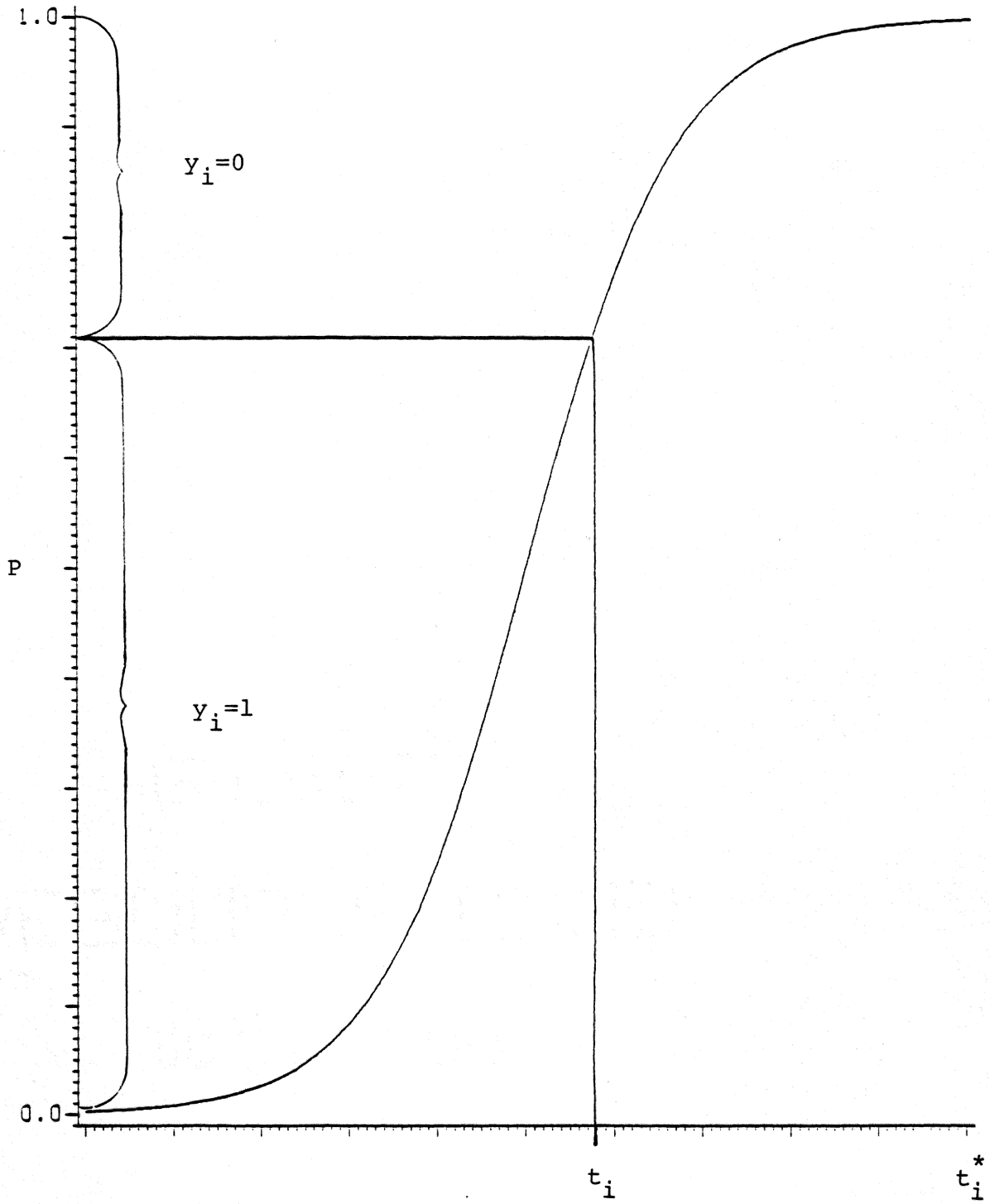


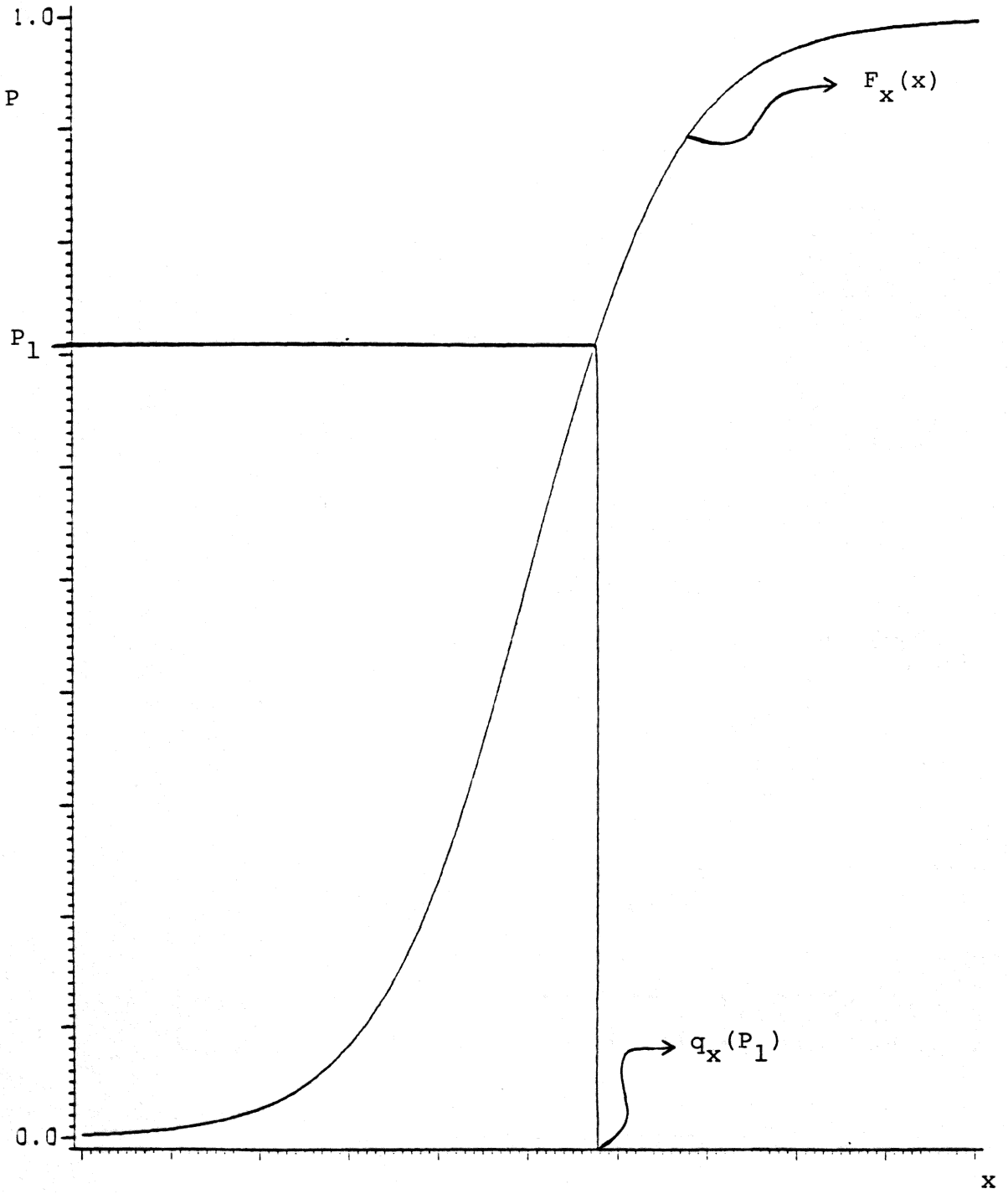
Figure 2. Relationship between y, t, and t^*

$$P(y_i=1|t_i) = P(t_i^* \leq t_i|\underline{x}_i) = F(t_i)$$
$$P(y_i=0|t_i) = P(t_i^* > t_i|\underline{x}_i) = 1 - F(t_i) \quad (33)$$

In short, the index t_i, which has a range on the real line is translated to a unit-interval range by the use of the cdf, $F(t_i)$, which is monotone nondecreasing, so $0 \leq F(t_i) \leq 1$, and thus the inequality constraint is automatically satisfied. These cdf's can also be thought of in the context of probability plotting. Figure 3 illustrates this concept.

To estimate $\underline{\beta}$ it will be necessary to specify F. Although infinitely many cumulative distribution functions provide suitable transformations, two are more popular than others: the probit model and the logit model. When F is taken to be the standard normal cdf, then $F=\Phi$ results in probit analysis. And when F is taken as the standard logistic cdf, then $F=\Psi$ results in logit analysis.

The probit and logit models have received considerable attention in the literature, for example Berkson (1953), Berkson (1955b), Berkson (1957a), Cox (1970), Finney (1964), Finney (1971).



where $F_x(x)$ = cdf of x

$q_x(P_1) = F^{-1}(P_1)$ = quantile

Figure 3. Probability Plotting

2.2.1 Probit Model

The probit (or normit) model, with a long history in biometrics (Finney (1952)), is associated with the standard normal cumulative distribution function where for the one regressor situation

$$P_i = F(x_i) \tag{34}$$
$$= \Phi[(x_i-\mu)/\sigma]$$
$$= \Phi[-\mu/\sigma + (1/\sigma)x_i]$$
$$= \Phi[\beta_0 + \beta_1 x_i]$$
$$= \Phi[v_i]$$

where F is the cdf of $N(\mu,\sigma)$, Φ is the cdf of the $N(0,1)$, and $v_i \sim N(0,1)$

It follows that

$$v_i = \Phi^{-1}(P_i) \tag{35}$$
$$= -\mu/\sigma + (1/\sigma)x_i$$
$$= \beta_0 + \beta_1 x_i$$
$$= \text{'normit'}$$
$$\beta_0 = -\mu/\sigma$$
$$\beta_1 = 1/\sigma$$
$$x_i = \mu + \sigma t_i$$
$$t_i = (x_i-\mu)/\sigma$$

Note that $v_i = t_i$. Also note that by construction, the inequality constraint is automatically satisfied since P_i will lie in the unit interval. In Figure 3 the quantile for F is called the normit when $F=\Phi$.

If the normit (also referred to as the normal equivalent deviate (NED)) of P_i, v_i, is plotted against x_i, the points will fall on a straight line with β_0 as the intercept and β_1 as the slope.

Bliss (1934) suggested something slightly different. He defined the probit of P_i ('probit' = probability unit) as u_i where

$$P_i = \Phi(u_i - 5) \qquad (36)$$
$$= \Phi(v_i)$$

So for any P_i, the probit is simply the normit increased by 5:

$$\begin{aligned} \text{Probit}(P_i) &= u_i \qquad (37) \\ &= v_i + 5 \\ &= \frac{x_i - \mu}{\sigma} + 5 \\ &= ((-\mu/\sigma) + 5) + (1/\sigma)x_i \\ &= \beta_0^* + \beta_1 x_i \end{aligned}$$

where β_0^* and β_1 become new parameters in place of μ and σ. Note that $\beta_0^* = \beta_0 + 5$.

Experimental results can lead to an estimate of (37), and the parameters of the distribution can then be estimated.

The normit is negative if P_i is less than 50%, whereas the probit is always positive unless P_i is exceedingly small (causing $v_i < -5$). So the main reason probits were intro-

duced in the 1930's when most people lacked even simple calculating machines, was the probit's avoidance of negative quantities (Finney (1971)). Today, vastly improved computing facilities and increased familiarity with statistical methods would make the advantage rest with the more simply defined normit, except for the fact that 'probit analysis' is so well established as a name for a group of methods. All the tables required for normits occupy about half as much space as for the probits because of symmetry, and the 5 in the probit is untidy in theoretical discussions. To abandon probits now might cause confusion (Finney (1971)).

Another way to express the definition of a probit is to say that the probit of P_i, u_i, is the abscissa which corresponds to a probability P_i in a normal distribution with mean 5 and variance 1 (mean 0 and variance 1 in the normit situation). This relationship is shown in Figure 4.

Although the definition is rather complex, in practice probits for specified values of P_i are available from tables (See Appendix A).

Using the definition of the probit, a probit regression model for the ith observation can be written as

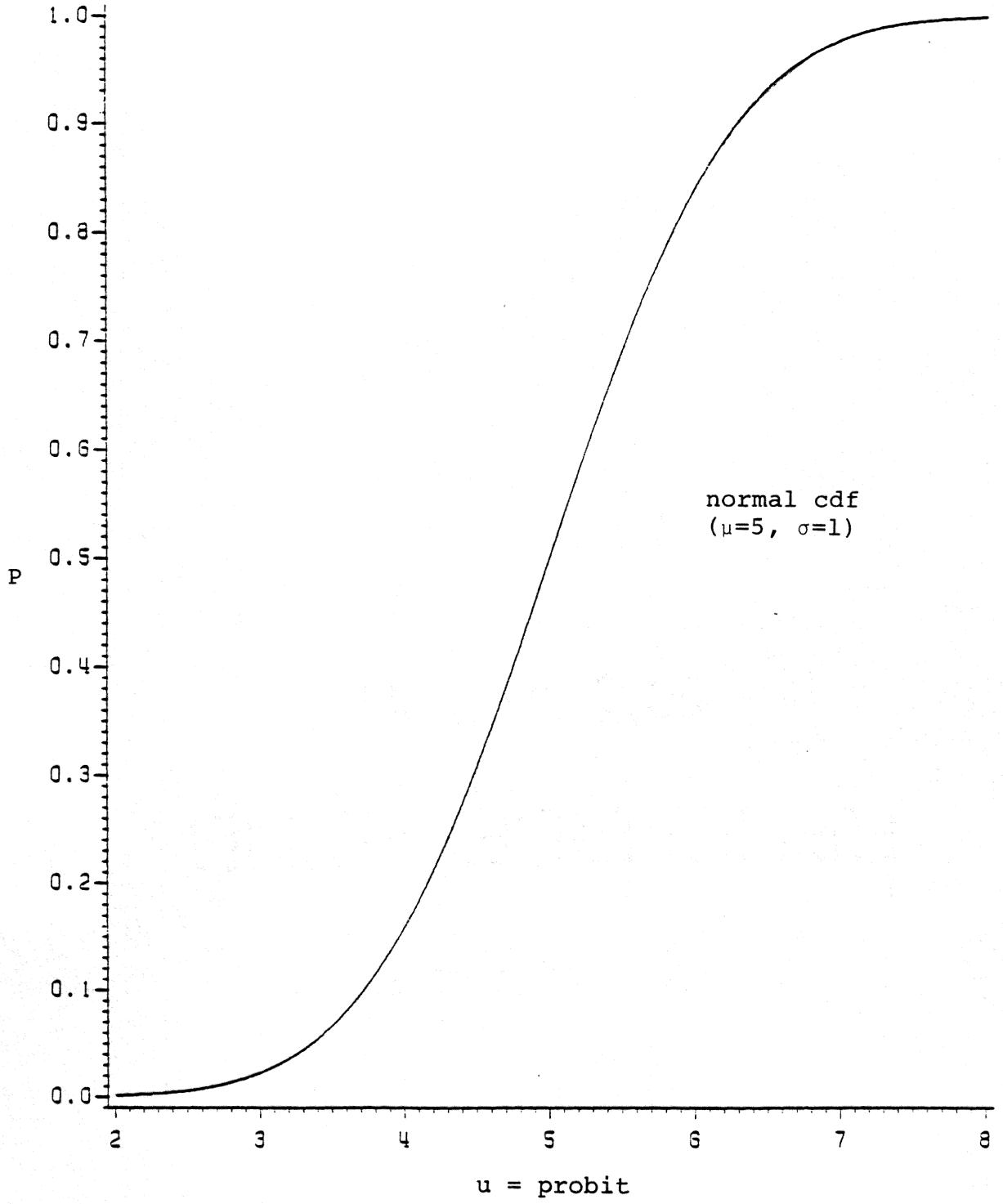


Figure 4. Relationship between p and u

$$u_i = v_i + 5 \qquad (38)$$
$$= \underline{x}'_i \underline{\beta}^*$$

where $\underline{\beta}^{*\prime} = [\beta_0^*, \beta_1, \ldots, \beta_k]$

This is called the probit transformation.

A more prudent alternative is to select a cdf which has a simple algebraic form. For this purpose the logistic distribution suggested by Berkson (1944) serves admirably well.

2.2.2 Logit Model

The logit model is associated with the standard logistic cumulative distribution function

$$F(t_i) = \Psi(t_i) = \frac{e^{t_i}}{1 + e^{t_i}} = \frac{1}{1 + e^{-t_i}} \qquad (-\infty < t_i < \infty) \qquad (39)$$

recalling that $t_i = \underline{x}'_i \underline{\beta}$. Or in other words, for the logit model P_i, the probability that the event occurs on the ith observation, is given by

$$P_i = \Psi(t_i) = \frac{e^{t_i}}{1 + e^{t_i}} = \frac{1}{1 + e^{-t_i}} \qquad (-\infty < t_i < \infty) \qquad (40)$$

Similar to the probit model, P_i lies in the unit interval by construction.

Although (39) is a nonlinear model, it is a simple matter to rewrite it as

$$\frac{P_i}{1-P_i} = \exp(-\underline{x}'_i\underline{\beta}) \tag{41}$$

The ratio $P_i/(1-P_i)$ represents the probability that an event will occur (P_i) to the probability that it will not occur ($1-P_i$).

Through the use of algebraic manipulation,

$$L_i = \ln\left[\frac{P_i}{1-P_i}\right] = \underline{x}'_i\underline{\beta} = t_i \tag{42}$$

This natural logarithm of the odds, L_i, is called the logit. From (43), we have achieved a linear model in which the left hand side is a transformation of P_i. This transformation has the property of increasing from $-\infty$ to ∞ as P_i increases from 0 to 1. This is called the logit transformation. By noting that

$$L_i = \underline{x}'_i\underline{\beta} \tag{43}$$

we have what is termed the linear logit model for the ith observation.

The appeal of the logit model follows from the fact that this formulation L_i translates the problem of predicting probabilities within the unit interval to the problem of predicting the natural logarithm of the odds of the event occuring to the event not occuring. Tables giving logits for specified values of P_i are available (Berkson (1953)).

The predictions of the logit model

$$\hat{L}_i = \underline{x}'_i \hat{\underline{\beta}} \tag{44}$$

can take any value in the range $-\infty$ to ∞ but the predicted probabilities which can be found by substituting $\underline{\beta}$ of (43) into (39) are confined to the range $0 \leq P_i \leq 1$ thus satisfying the inequality constraint. The linear logit model or its nonlinear alternative (39) which we call the nonlinear logit model thus overcomes the second of the problems we identified in an earlier section and represent feasible alternative regression models.

In terms of probability plotting (for the one regressor situation), the logit model is formulated as follows where $t_i = \beta_0 + \beta_1 x_i$.

$$\begin{aligned} P_i &= F(x_i) \\ &= \Psi[(x_i - \mu)/\sigma] \\ &= \Psi[-\mu/\sigma + (1/\sigma)x_i] \\ &= \Psi[\beta_0 + \beta_1 x_i] \\ &= \Psi[L_i] \end{aligned} \tag{45}$$

where F is the logistic cdf and Ψ is the standardized logistic cdf

$$\begin{aligned} L_i &= \Psi^{-1}(P_i) \\ &= -\mu/\sigma + (1/\sigma)x_i \\ &= \beta_0 + \beta_1 x_i \\ &= \text{'logit'} \end{aligned} \tag{46}$$

2.2.3 LD50

Another parameter that is also of interest to estimate in both the probit and logit models is the median effective dose (ED50) which is also known as the lethal dose (LD50). In either case, this quantity, γ, is defined in terms of bioassay as the 'dose' which is effective or lethal to just 50 percent of the population 'exposed'. In other words, the value of x which results in P equal to 0.5.

So in the normit model, we want

$$P_i = F(t_i) = \Phi(v_i) = 0.5 \tag{47}$$

which is equivalent to saying we want

$$v_i = 0$$

$$\beta_0 + \beta_1 \gamma = 0$$

$$\gamma = -\beta_0/\beta_1$$

In the probit model, we want

$$P_i = F(t_i) = \Phi(u_i - 5) = 0.5 \tag{48}$$

which is equivalent to saying we want

$$u_i - 5 = 0$$

$$u_i = 5$$

$$\beta_0^* + \beta_1 \gamma = 5$$

$$\gamma = (-\beta_0^* + 5)/\beta_1 = -\beta_0/\beta_1$$

In the logit model, we want

$$P_i = F(t_i) = \Psi(t_i) = 0.5 \tag{49}$$

$$\ln\left[\frac{P_i}{Q_i}\right] = \beta_0 + \beta_1\gamma = \ln\left[\frac{.5}{.5}\right] = \ln(1) = 0$$

$$\gamma = -\beta_0/\beta_1$$

Hence, in all three models the LD50 is

$$\gamma = -\beta_0/\beta_1 \tag{50}$$

which can be estimated by

$$\hat{\gamma} = x_{(50)} = -\hat{\beta}_0/\hat{\beta}_1 \tag{51}$$

Note that the LD50 is only defined in the one regressor situation (k=1).

2.2.4 Comparison of Probit and Logit Models

Since the logistic density function closely resembles the t distribution with seven degrees of freedom (Hanushek and Jackson (1977)), the probit and logit formulations are quite similar.

Both the standard normal and the logistic distributions are symmetric about 0, the mean of the respective distributions. The variance of the standard normal distribution is 1, while the variance of the standard logistic distribution is $\pi^2/3$ ($\simeq 3.29$) Maddala (1983). Furthermore, the distribution functions are very close in mid-range. The only major difference is that the logistic distribution has slightly heavier tails than the standard normal distribution (Capps

(1983)). See Cox (1970:28) for a tabular comparison. Thus, unless interest must focus on the extreme tails, there is no reason to prefer the normal over the logistic functional form. "The logistic distribution is scarcely distinguishable from the normal between response rates of 0.01 and 0.99, ... the choice [may be] made primarily on the score of convenience" (Finney (1964:460)). This relationship is shown in Figure 5.

Clearly while the index t_i is a linear function of the x's ($t_i = \underline{x}'_i\underline{\beta}$), the probabilities derived from the probit model and the logit model are not. In addition, for the probit model and the logit model, the assumption that $E(\varepsilon_i)=0$ (as in the case for the linear probability model) is no longer tenable (Goldfeld and Quandt (1972)). (The exception being when $t \approx .78$ in the probit model and $t \approx .66$ in the logit model (Capps (1983)).

The logistic function is more tractable mathematically and better suited to computer operation because of its cdf being in closed form while the cdf for the normal distribution involves integration.

Although there are other models which automatically satisfy the inequality constraint and from which other transformations with similar properties to the probit and logit can be derived, the probit and logit are by far the most

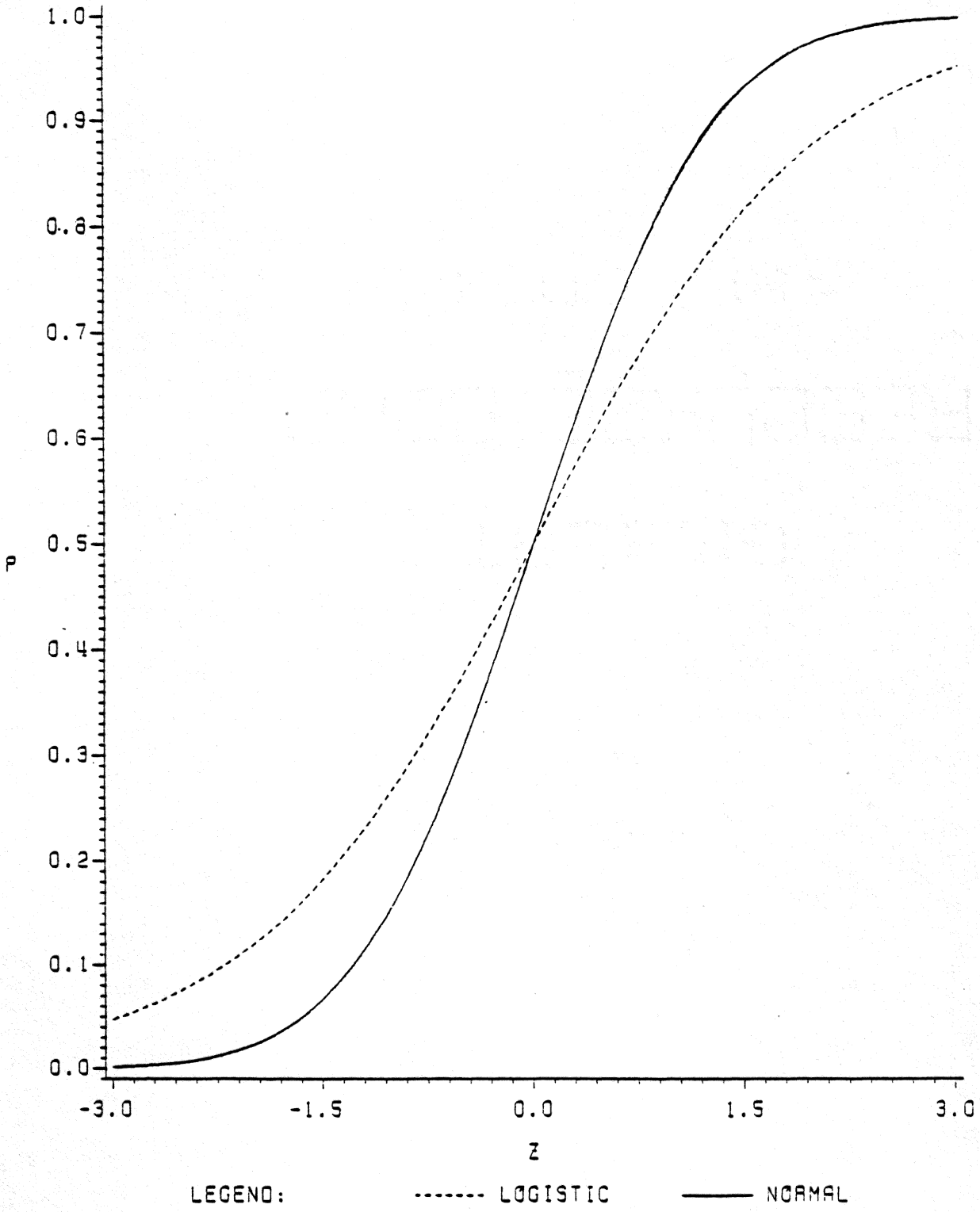


Figure 5. Normal and Logistic CDF's

widely used (Wrigley (1976:11)). Unless the probability P_i is very close to 1 or 0, both transformations are virtually equivalent. As a result, the choice between probit and logit models is essentially a matter of computational convenience. "Logit models ... are without doubt computationally more convenient than the equivalent probit models" (Wrigley (1976)).

So unless there is an underlying distribution assumed in the data, i.e. the normal distribution, the logit analysis is preferable over probit (normit) analysis.

2.3 OTHER MODELS

Other transformations which have been used mainly for their arithmetical simplicity are mentioned below.

1). Urban (1910) made some use of

$$P_i = \tfrac{1}{2} + (1/\pi)\tan^{-1} Y_i \qquad (52)$$

This is known as the Cauchy distribution where

$$f(0) = \frac{1}{\pi(1+0^2)} \qquad (53)$$

2). Wilson and Worcester (1943b) suggested

$$P_i = \tfrac{1}{2}\left[1+ \frac{Y_i}{\sqrt{1+Y_i^2}}\right] \qquad \text{where} \qquad (54)$$

$$f(0) = \tfrac{1}{2}\,(1+0^2)^{-3/2} \qquad (55)$$

3). Knudsen and Curtis (1947) and Anscombe (1956)

proposed the following Angular Transformation

$$P_i = \begin{cases} 0 & \text{for } Y_i \leq 0 \quad \text{where} \\ \sin Y_i & \text{for } 0 < Y_i < \pi/2 \\ 1 & \text{for } Y_i \geq \pi/2 \end{cases} \tag{56}$$

$$f(0) = \begin{cases} \sin 2\theta & 0 \leq \theta \leq \pi/2 \\ 0 & \theta < 0,\ \theta > \pi/2 \end{cases} \tag{57}$$

This transformation is used because of its property of making Var(Y) independent of E(Y). It also has a certain amount of numerical simplicity in the calculations because the weighting coefficient is independent of P.

4). Martin (1951) proposed

$$P_i = \begin{cases} 0 & \text{for } Y_i \leq -2.5 \\ 0.5+0.4Y_i+0.08Y_i^2 & \text{for } -2.5 < Y_i \leq 0 \\ 0.5+0.4Y_i-0.08Y_i^2 & \text{for } 0 < Y_i < 2.5 \\ 1 & \text{for } Y_i \geq 2.5 \end{cases} \tag{58}$$

This transformation corresponds to approximating the 'tolerance' distribution by an equilateral triangle.

where $Y_i = \beta_0 + \beta_1 x_i$ in all these transformations ($t_i = Y_i$).

"Almost certainly, an attempt to choose the transformation that best agrees with the results of a particular experiment is doomed to failure because they are practically

indistinguishable over a wide range. If arithmetical simplicity is not a criterion, only the probit and logit seem worth serious consideration as no theoretical basis has been suggested for others" (Finney (1971:49).

2.4 THE NOTION OF GROUPED DATA

In the definition of the probit or logit, the P_i represents the true probability that the given event will occur at $x=x_i$. If there is only one observation at each $x=x_i$, and the only possible responses are that the event occurs ($y_i=1$) or that the event does not occur ($y_i=0$), then the estimate of the proportion of observations at x_i will be limited to the values of 1 or 0. To circumvent this poor estimation problem, n_i observations will be taken at each x_i.

In this statistical model it is assumed that p_i can be considered a random variable binomially distributed:

$$p_i \sim B(P_i, \sigma^2_{p_i}) \tag{59}$$

where P_i = true probability that the given event will occur

$Q_i = 1 - P_i$ = true probability that the given event will not occur

n_i = number of observations at each x_i

r_i = number of observations out of n_i that are "affected" at x_i

$s_i = n_i - r_i$ = number of observations out of n_i that are not "affected" at x_i

$p_i = r_i/n_i$ = the observed proportion of observations that are "affected" out of n_i at x_i

$q_i = 1 - p_i$ = the observed proportion of observations that are not "affected" out of n_i at x_i

$$\sigma^2_{p_i} = \frac{P_i Q_i}{n_i}$$

Note that $r_i = n_i p_i$. Also note that $\Sigma n_i = N$

where N = total number of individuals sampled

n_i = number of individuals sampled at x_i

n = total number of different x values.

Chapter III

ESTIMATION METHODS AND PROCEDURES

Now that the theoretical groundwork has been laid, estimation methods for both probit and logit analysis will be presented. Examples of these methods can be found in Appendix C. After all the estimation methods are presented, they will subsequently be compared based on a number of different criteria.

3.1 PROBIT MODEL

3.1.1 Maximum Likelihood for Grouped Data

An important method of estimating the unknown parameters is that of maximum likelihood (MLE). The likelihood that a set of parameters should have any particular values is defined to be a quantity whose magnitude is proportional to the probability that the totality of observations should be the data recorded. The factor of proportionality is usually fixed by the condition that the maximum value of the likelihood is unity. The method of maximum likelihood allows for the estimated values of the parameters to be chosen that the likelihood takes its maximum value.

Maximum likelihood estimates have a number of important properties (Finney (1964)):

i). The MLE is consistent; that is to say, if the

number of observations is increased without limit, the estimate is almost certain to be within any specified narrow interval that includes the true value.

ii). The frequency distribution of the MLE of a parameter is asymptotically normal. In other words, if the number of observations is increased without limit, this distribution approaches the normal form.

iii). The MLE of a parameter is asymptotically efficient; that is to say, if the number of observations is increased without limit, the ratio of the sampling variance of the estimate to the sampling variance of any alternative estimate tends to a limit not greater than unity.

iv). The estimate will be efficient in finite samples, if an efficient estimate for the particular problem exists.

v). If a sufficient estimate of the parameter exists, the MLE will either be equal to it or be a function of it.

Since it is assumed that p_i is distributed binomially in the grouped data situation, the probability that r_i 'respond' is the binomial function

$$P(r_i|n_i) = \binom{n_i}{r_i} P_i^{r_i} Q_i^{n_i-r_i} \tag{60}$$

The joint probability of these results, the likelihood function, is

$$L = \Pi \binom{n_i}{r_i} P_i^{r_i} Q_i^{n_i-r_i} \tag{61}$$

$$\ln L = \Sigma r_i \ln P_i + \Sigma (n_i - r_i) \ln Q_i + \Sigma \ln \binom{n_i}{r_i} \tag{62}$$

The likelihood is a maximum when L (and hence ln L) is a maximum. Hence, if θ is any parameter of the distribution, the maximum likelihood estimate of θ must satisfy

$$\frac{\delta \ln L}{\delta \theta} = \Sigma \frac{r_i}{P_i} \frac{\delta P_i}{\delta \theta} + \Sigma \frac{n_i - r_i}{Q_i} \frac{\delta Q_i}{\delta \theta} \tag{63}$$

$$= \Sigma \frac{n_i(p_i - P_i)}{P_i Q_i} \frac{\delta P_i}{\delta \theta} = 0$$

If the distribution involves more than one unknown parameter, a set of equations of this form must be satisfied simultaneously. Explicit solution of the unknown parameters of such equations is seldom possible, but iterative methods can give successive approximations converging to the solutions. What follows is an illustration of this method for two parameters β_0 and β_1.

Since

$$\frac{\delta P_i}{\delta \beta_0} = P_i Q_i \quad \text{and} \quad \frac{\delta P_i}{\delta \beta_1} = P_i Q_i x_i \tag{64}$$

it is readily found that

$$\frac{\delta \ln L}{\delta \beta_0} = \Sigma\, n_i (p_i - P_i) \tag{65}$$

$$\frac{\delta \ln L}{\delta \beta_1} = \Sigma\, n_i x_i (p_i - P_i) \tag{66}$$

The estimating equations for the MLE can be written (Berkson (1957)):

$$\Sigma n_i (p_i - \hat{p}_i) = 0 \tag{67}$$

N.E.

$$\Sigma n_i x_i (p_i - \hat{p}_i) = 0 \tag{68}$$

where $\hat{p}_i = F(\hat{\beta}_0 + \hat{\beta}_1 x_i)$

= the estimate of P_i, obtained with the MLE's $\hat{\beta}_0$, $\hat{\beta}_1$.

No explicit solution of (67) and (68) exist, and iterative methods are employed.

Suppose that $\hat{\beta}_{00}$ and $\hat{\beta}_{10}$ are any initial approximations to the solutions of the equations (63), obtained perhaps by rough graphical or arithmetical estimation. By the Taylor

series expansion (Finney (1971)) to the first order of small quantities, second approximations will be

$$\begin{aligned} \hat{\beta}_{01} &= \hat{\beta}_{00} + \Delta\beta_0 \\ \hat{\beta}_{11} &= \hat{\beta}_{10} + \Delta\beta_1 \end{aligned} \tag{69}$$

where $\Delta\beta_0$ and $\Delta\beta_1$ are obtained from

$$\begin{aligned} \frac{\delta \ln L}{\delta\hat{\beta}_{00}} + \Delta\beta_0 \frac{\delta^2 \ln L}{\delta\hat{\beta}_{00}^2} + \Delta\beta_1 \frac{\delta^2 \ln L}{\delta\hat{\beta}_{00}\,\delta\hat{\beta}_{10}} &= 0 \\ \frac{\delta \ln L}{\delta\hat{\beta}_{10}} + \Delta\beta_0 \frac{\delta^2 \ln L}{\delta\hat{\beta}_{00}\,\delta\hat{\beta}_{10}} + \Delta\beta_1 \frac{\delta^2 \ln L}{\delta\hat{\beta}_{10}^2} &= 0 \end{aligned} \tag{70}$$

the addition of the suffix to $\hat{\beta}_0$ and $\hat{\beta}_1$ indicates the first approximations are to be substituted after differentiation.

The second-order differential coefficients

$$\frac{\delta^2 \ln L}{\delta\hat{\beta}_{00}^2} \qquad \frac{\delta^2 \ln L}{\delta\hat{\beta}_{00}\,\delta\hat{\beta}_{10}} \qquad \frac{\delta^2 \ln L}{\delta\hat{\beta}_{10}^2}$$

may be simplified by putting p=P after differentiation (Finney (1971)). Equations (58.7) thus become

$$\begin{aligned} \Sigma \frac{n_i(p_i - P_i)}{P_{i0}Q_{i0}} \frac{\delta P_{i0}}{\delta\hat{\beta}_{00}} + \Delta\beta_0 \Sigma \frac{n_i}{P_{i0}Q_{i0}} \left(\frac{\delta P_{i0}}{\delta\hat{\beta}_{00}}\right)^2 + \Delta\beta_1 \Sigma \frac{n_i}{P_{i0}Q_{i0}} \frac{\delta P_{i0}\delta P_{i0}}{\delta\hat{\beta}_{00}\delta\hat{\beta}_{10}} &= 0 \\ \Sigma \frac{n_i(p_i - P_{i0})}{P_{i0}Q_{i0}} \frac{\delta P_{i0}}{\delta\hat{\beta}_{10}} + \Delta\beta_0 \Sigma \frac{n_i}{P_{i0}Q_{i0}} \frac{\delta P_{i0}\delta P_{i0}}{\delta\hat{\beta}_{00}\delta\hat{\beta}_{10}} + \Delta\beta_1 \Sigma \frac{n_i}{P_{i0}Q_{i0}} \left(\frac{\delta P_{i0}}{\delta\hat{\beta}_{10}}\right)^2 &= 0 \end{aligned} \tag{71}$$

two linear equations that can be solved for $\Delta\beta_0$ and $\Delta\beta_1$.

The process may now repeat with $\hat{\beta}_{01}$ and $\hat{\beta}_{11}$ in place of $\hat{\beta}_{00}$ and $\hat{\beta}_{10}$, and further cycles computed until the latest set of adjustments is negligible.

The asymptotic variance and covariances of the estimates are the elements of the inverse matrix of second differential coefficients (Finney (1971)):

$$V = \begin{bmatrix} \dfrac{-\delta^2 L}{\delta\hat{\beta}_0^2} & \dfrac{-\delta^2 L}{\delta\hat{\beta}_0\delta\hat{\beta}_1} \\ \dfrac{-\delta^2 L}{\delta\hat{\beta}_0\delta\hat{\beta}_1} & \dfrac{-\delta^2 L}{\delta\hat{\beta}_1^2} \end{bmatrix} \tag{72}$$

The sufficient statistics for the estimates of the parameters in the grouped data situation. These are (Silverstone (1957)):

$$\Sigma n_i p_i = \Sigma r_i \qquad \text{for } \beta_0 \tag{73}$$

sufficient statistics

$$\Sigma n_i p_i x_i = \Sigma r_i x_i \qquad \text{for } \beta_1 \tag{74}$$

Sufficient estimators are those that utilize all the information on the value of the parameter that is contained in a particular set of observations (Finney (1964), p. 82).

It is known that where sufficient statistics exist, the maximum likelihood estimates are functions of the sufficient

statistics (Savage (1954)). However, the MLE is not necessarily, and is not in the present instance, a one-to-one function of the sufficient statistic and therefore the MLE is itself not sufficient (Rao (1952)).

The equations above apply whatever the form of P_i.

In the probit situation, P_i is given by (37) as

$$P_i = \Phi(\beta_0+\beta_1 x_i) \tag{75}$$

Hence,

$$\frac{\delta P_i}{\delta \beta_0} = \frac{1}{\sqrt{2\pi}} e^{-Y_i/2} = z_i \tag{76}$$

$$\frac{\delta P_i}{\delta \beta_1} = \frac{1}{\sqrt{2\pi}} e^{-Y_i/2} = z_i x_i \tag{77}$$

where z_i is the ordinate to the normal curve at the point with abscissa Y_i.

If

$$Y_{i0} = \hat{\beta}_{00} + \hat{\beta}_{10} x_i \tag{78}$$

is a first approximation to the maximum likelihood estimate of (58.11), equations (58.8) become

$$\Sigma n_i w_i \left(\frac{p_i - P_i}{z_i}\right) + \Delta\beta_0 \Sigma n_i w_i + \Delta\beta_1 \Sigma n_i w_i x_i = 0$$

$$\Sigma n_i w_i x_i \left(\frac{p_i - P_i}{z_i}\right) + \Delta\beta_0 \Sigma n_i w_i x_i + \Delta\beta_1 \Sigma n_i w_i x_i^2 = 0 \qquad (79)$$

where $w_i = z_i^2 / p_i q_i$

The equations for the adjustments $\Delta\beta_0$ and $\Delta\beta_1$ are in form the same as for calculation of a linear regression of $(p_i - P_i)/z_i$ upon x_i with weight $n_i w_i$ for each value of $(p_i - P_i)/z_i$.

Instead of using the above method as it is, the working probit is introduced at each x_i (Finney (1971))

$$y_i = Y_{i0} + \frac{p_i - P_i}{z_i} \qquad (80)$$

If $\Sigma n_i w_i Y_{i0}$ is added to each side of the first equation, and $\Sigma n_i w_i x_i Y_{i0}$ to each side of the second equation in (79), they become

$$\Sigma n_i w_i y_i + (\hat{\beta}_{00} + \Delta\beta_0) \Sigma n_i w_i + (\hat{\beta}_{10} + \Delta\beta_1) \Sigma n_i w_i x_i = 0$$

$$\Sigma n_i w_i y_i + (\hat{\beta}_{00} + \Delta\beta_0) \Sigma n_i w_i x_i + (\hat{\beta}_{10} + \Delta\beta_1) \Sigma n_i w_i x_i^2 = 0 \qquad (81)$$

Writing

$$\hat{\beta}_{01} = \hat{\beta}_{00} + \Delta\beta_0$$
$$\hat{\beta}_{11} = \hat{\beta}_{10} + \Delta\beta_1 \qquad (82)$$

and defining

$$\bar{x} = \frac{\Sigma n_i w_i x_i}{\Sigma n_i w_i} \qquad (83)$$

$$\bar{y} = \frac{\Sigma n_i w_i y_i}{\Sigma n_i w_i} \qquad (84)$$

Standard algebra of weighted regression gives

$$\hat{\beta}_{11} = \frac{\Sigma n_i w_i (x_i - \bar{x})(y_i - \bar{y})}{\Sigma n_i w_i (x_i - \bar{x})^2} \qquad (85)$$

$$\hat{\beta}_{01} = \bar{y} - \hat{\beta}_1 \bar{x} \qquad (86)$$

In other words, the revised relation

$$Y_{i1} = \hat{\beta}_{01} + \hat{\beta}_{11} x_i \qquad (87)$$

is calculated as the weighted linear regression of y_i on x_i, the weights being $n_i w_i$.

The iterative routine is now clear, for the whole cycle of arithmetic can be repeated with Y_{i1} in place of Y_{i0}. Usually successive iterations converge rapidly, even if Y_0 is a poor approximation (for example, $\hat{\beta}_{00}=0$ and $\hat{\beta}_{10}=0$), but

the process is much speeded by starting from moderately good graphical estimates. The limit must be the maximum likelihood estimates.

If symbols now refer to the limiting values,

$$V(\bar{y}) = \frac{1}{\Sigma n_i w_i} \tag{88}$$

$$V(\hat{\beta}_1) = \frac{1}{\Sigma n_i w_i (x_i - \bar{x})^2} \tag{89}$$

with zero covariance. Strictly, these variances are applicable only for large values of each n_i.

3.1.2 Maximum Likelihood for Individual Data

The concept of maximum likelihood, as was introduced earlier, will now be presented for the probit model with individual data. Let us denote by $\phi(.)$ and $\Phi(.)$ the density function and the distribution function, respectfully, of the standard normal recalling that

$$P_i = \Phi(t_i) \tag{90}$$

Then for the probit model the likelihood function is (Maddala (1983)):

$$L = \Pi [\Phi(\underline{x}'_i \underline{\beta})]^{y_i} [1 - \Phi(\underline{x}'_i \underline{\beta})]^{1-y_i} \tag{91}$$

and the log-likelihood is

$$\ln(L) = \Sigma\, y_i \ln[\Phi(\underline{x}'_i\underline{\beta})] + \Sigma\, (1-y_i) \ln[1-\Phi(\underline{x}'_i\underline{\beta})] \qquad (92)$$

Differentiating ln L with respect to β yields

$$S(\underline{\beta}) = \delta \ln L \,/\, \delta\underline{\beta} \qquad (93)$$

$$= \Sigma \frac{[y_i - \Phi(\underline{x}'_i\underline{\beta})]}{\Phi(\underline{x}'_i\underline{\beta})[1 - \Phi(\underline{x}'_i\underline{\beta})]}\phi(\underline{x}'_i\underline{\beta})x_i$$

The MLE for β can be obtained as a solution of the equations $S(\underline{\beta}) = \underline{0}$.

These equations are nonlinear in $\underline{\beta}$, and thus have to be solved by an iterative procedure (usually following the Newton-Raphson method).

This method, as do all MLE methods, allows the estimators to have asymptotic properties. The asymptotic covariance matrix of the MLE's is the inverse of the I matrix, $I^{-1}(\underline{\beta})$ and is consistently estimated by $I^{-1}(\hat{\underline{\beta}})$ where (Maddala (1983)):

$$I(\underline{\beta}) = E\left[-\frac{\delta^2 \ln L}{\delta\underline{\beta}\delta\underline{\beta}'}\right] \qquad (94)$$

$$= \Sigma \frac{[\phi(\underline{x}'_i\underline{\beta})]^2}{\Phi(\underline{x}'_i\underline{\beta})[1 - \Phi(\underline{x}'_i\underline{\beta})]}\underline{x}_i\underline{x}'_i$$

3.1.3 Minimum Normit χ^2 for Grouped Data

The distribution function used in this method is given by

$$P_i = \Phi(\beta_0+\beta_1 x_i) \tag{95}$$

The minimum normit χ^2 estimate of the normit model is defined by minimization of the following quantity, called the "normit χ^2" (Berkson (1955b)):

$$\chi^2 \text{ (normit)} = \Sigma n_i \frac{z_i^2}{p_i q_i} (v_i-\hat{v}_i)^2 \tag{96}$$

where v_i = the observed normit at x_i
$= \Phi^{-1}(P_i)$

$\hat{v}_i$ = the estimate of the normit at x_i
$= \Phi^{-1}(\hat{P}_i) = \hat{\beta}_0+\hat{\beta}_1 x_i$

$w_i = z_i^2/p_i q_i$

z_i = the ordinate of the unit normal curve at the point where its area is divided into p_i and q_i, and is given by

$$z_i = \frac{1}{\sqrt{2\pi}} e^{-v_i^2/2}$$

The normal equations for obtaining the estimates of β_0 and β_1 for the normit χ^2 are (Berkson (1955b)):

$$\Sigma n_i (z_i^2/p_i q_i)(v_i - \hat{v}_i) = 0 \qquad (97)$$

N.E.

$$\Sigma n_i (z_i^2/p_i q_i) x_i (v_i - \hat{v}_i) = 0 \qquad (98)$$

The evaluation of (97) and (98) leads to a procedure that amounts simply to obtaining a least squares solution of the straight line

$$\hat{v}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i \qquad (99)$$

with $n_i (z_i^2/p_i q_i)$ as weight of the observation v_i.

The values of $\hat{\beta}_0$ and $\hat{\beta}_1$, which are the estimates respectively of β_0 and β_1 are given by the following non-iterative equations

$$\hat{\beta}_1 = \frac{\Sigma n_i w_i (v_i - \bar{v})(x_i - \bar{x})}{\Sigma n_i w_i (x_i - \bar{x})^2} \qquad (100)$$

$$= \frac{\Sigma n_i w_i v_i x_i - \dfrac{\Sigma n_i w_i v_i \; \Sigma n_i w_i x_i}{\Sigma n_i w_i}}{\Sigma n_i w_i x_i^2 - \dfrac{(\Sigma n_i w_i x_i)^2}{\Sigma n_i w_i}}$$

$$\hat{\beta}_0 = \bar{v} - \hat{\beta}_1 \bar{x} = \frac{\Sigma n_i w_i v_i - \hat{\beta}_1 \Sigma n_i w_i v_i}{\Sigma n_i w_i} \qquad (101)$$

where $w_i = z_i^2/p_i q_i$

$$\bar{v} = \frac{\Sigma n_i w_i v_i}{\Sigma n_i w_i}$$

$$\bar{x} = \frac{\Sigma n_i w_i x_i}{\Sigma n_i w_i}$$

We may write the estimated normit linear equation as

$$\hat{v}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i = \hat{\beta}'_0 + \hat{\beta}_1 (x_i - \bar{x}) \tag{102}$$

where $\hat{\beta}'_0 = \bar{v} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$

Formulas for variances of the estimates of the parameters may be written as follows (Berkson (1955b)):

$$S^2_{\hat{\beta}'_0} = \frac{1}{\Sigma n_i w_i} \tag{103}$$

$$S^2_{\hat{\beta}_0} = S^2_{\hat{\beta}'_0} + \bar{x}^2 S^2_{\hat{\beta}_1} \tag{104}$$

$$S^2_{\hat{\beta}_1} = \frac{1}{\Sigma n_i w_i (x_i - \bar{x})^2} \tag{105}$$

$$S^2_{\hat{\gamma}} = \frac{1}{\hat{\beta}_1^2} \left[S^2_{\hat{\beta}'_0} + S^2_{\hat{\beta}_1} (x_{50} - \bar{x})^2 \right] \tag{106}$$

Appendix A contains tables giving the normit, v_i, for

argument p_i (Table A.1), the normit weights $w_i=z_i^2/p_iq_i$ and $w_iv_i=(z_i^2/p_iq_i)*v_i$ for argument p_i (Table A.2), and the antinormit, p_i, for argument v_i (Table A.3).

When the observed p_i is either zero or 100 percent, v_i is negatively or positively infinite and w_iv_i is zero. Berkson (1955b) substitutes in a working value which is equal to $1/2n_i$ or $1 - 1/2n_i$, respectively.

We can also test the validity of this model with the value of the normit χ^2 since it is asymptotically distributed as χ^2 (Berkson (1955b)) with degrees of freedom equal to the number of 'doses (n) minus the number of parameters to be estimated (p) (Wrigley (1976)). Hence

$$\chi^2(\text{normit}) \overset{\text{asym}}{\sim} \chi^2(n-p) \qquad (107)$$

If $\chi^2(\text{normit}) > \chi^2(1-\alpha,n-p)$, there is an indication of significant model inadequacy at the $100\alpha\%$ level of significance, i.e. the tolerance distribution is non-normal. If $\chi^2(\text{normit}) \leq \chi^2(1-\alpha,n-p)$, there is no indication of model inadequacy.

The minimum normit χ^2 estimate has the same asymptotic properties as the maximum likelihood estimate (Berkson (1955b)). The minimum normit χ^2 estimate, as well as the maximum likelihood estimate, is therefore asymptotically efficient.

3.2 LOGIT MODEL

3.2.1 Maximum Likelihood for Grouped Data

Once again, the idea of estimating by means of the maximum likelihood estimates for grouped data as defined for the probit model earlier shall be used here except the function P_i here is the logistic function:

$$P_i = \Psi(\underline{x}'_i\underline{\beta}) = \frac{1}{1 + \exp(-(\underline{x}'_i\underline{\beta}))} \tag{108}$$

$$L_i = \ln\left[\frac{P_i}{Q_i}\right] = \underline{x}'_i\underline{\beta} \tag{109}$$

where L_i = the logit of P_i

Following the general framework described in section (3.1.1)

$$\frac{\delta P_i}{\delta \beta_0} = P_i Q_i \quad \text{and} \quad \frac{\delta P_i}{\delta \beta_1} = P_i Q_i x_i \tag{110}$$

it is readily found that

$$\frac{\delta \ln L}{\delta \beta_0} = \Sigma\, n_i(p_i - P_i) \tag{111}$$

$$\frac{\delta \ln L}{\delta \beta_1} = \Sigma\, n_i x_i(p_i - P_i) \tag{112}$$

The estimating equations for the MLE can be written (Berkson (1957)):

$$\Sigma n_i(p_i-\hat{p}_i) = 0 \tag{113}$$

N.E.

$$\Sigma n_i x_i(p_i-\hat{p}_i) = 0 \tag{114}$$

where $\hat{p}_i = \Psi(\hat{\beta}_0+\hat{\beta}_1)$

= the estimate of P_i, obtained with the MLE's $\hat{\beta}_0$, $\hat{\beta}_1$.

No explicit solution of (113) and (114) exist, and iterative methods are employed. Let

$\hat{\beta}_{00}$, $\hat{\beta}_{10}$ be starting values for the estimates $\hat{\beta}_0$, $\hat{\beta}_1$

$\hat{l}_{01} = \hat{\beta}_{00} + \hat{\beta}_{10}x_i$, the corresponding value for l_i, the logit estimate

$\hat{p}_{01} = 1 - \hat{q}_{01}$, the corresponding starting value for $\hat{p}_i$.

Then if for $(\hat{p}_0-\hat{p}_i)$ we use the approximation (Berkson (1957)):

$$(\hat{p}_0-\hat{p}_i) \simeq (\hat{l}_0-\hat{l}_i)\hat{p}_0\hat{q}_0 \tag{115}$$

given by the first term of a Taylor series expansion, the estimating equations for $\Delta\hat{\beta}_0$, $\Delta\hat{\beta}_1$, the corrections to the provisional estimates $\hat{\beta}_{00}$, $\hat{\beta}_{10}$ are the solutions of

$$\Sigma n_i p_i - \Sigma n_i \hat{p}_i - \Delta\hat{\beta}_0 \Sigma n_i \hat{p}_{01}\hat{q}_{01} - \Delta\hat{\beta}_1 \Sigma n_i \hat{p}_{01}\hat{q}_{01}x_i = 0 \qquad (116)$$

$$\Sigma n_i p_i x_i - \Sigma n_i \hat{p}_i x_i - \Delta\hat{\beta}_0 \Sigma n_i \hat{p}_{01}\hat{q}_{01}x_i - \Delta\hat{\beta}_1 \Sigma n_i \hat{p}_{01}\hat{q}_{01}x_i^2 = 0 \qquad (117)$$

These yield for the corrections

$$\Delta\hat{\beta}_1 = \Sigma n_i x_i (p_i - \hat{p}_{i0}) - \frac{(\Sigma n_i (p_i - \hat{p}_{i0}))(\Sigma n_i x_i w_i^*)}{\Sigma n_i w_i^*} \qquad (118)$$

$$\Delta\hat{\beta}_0 = \frac{\Sigma n_i (p_i - \hat{p}_{i0}) - \Delta\hat{\beta}_1 (\Sigma n_i x_i w_i^*)}{\Sigma n_i w_i^*} \qquad (119)$$

where $w_i^* = \hat{p}_{0i}\hat{q}_{0i}$, corresponding to x_i.

The estimates obtained with these corrections are used as new starting values in an iterative procedure which, when it converges to finite values, yields in the limit the maximum likelihood estimates.

The values $\hat{p}_{0i}$ and $w_i^* = \hat{p}_{0i}\hat{q}_{0i}$ will be obtained from $\hat{l}_{i0}$; it is therefore convenient to have tables which, for argument l_i gives p_i and $w_i^* = p_i q_i$. These tables are given in Appendix B.

The maximum likelihood estimation procedure assures the large-sample properties of consistency and asymptotic normality of the parameter vector $\hat{\underline{\beta}}$.

3.2.2 Maximum Likelihood for Individual Data

The concept of maximum likelihood, as was introduced earlier, will now be presented as an estimation procedure for the logit model where

$$P_i = \frac{1}{1+\exp(-\underline{x}'_i\underline{\beta})} = \frac{\exp(\underline{x}'_i\underline{\beta})}{1+\exp(\underline{x}'_i\underline{\beta})} \quad \text{if } y_i = 1 \tag{120}$$

$$1 - P_i = \frac{1}{1+\exp(\underline{x}'_i\underline{\beta})} \quad \text{if } y_i = 0 \tag{121}$$

Hence, the likelihood is

$$L = \prod \left[\frac{\exp(\underline{x}'_i\underline{\beta})}{1+\exp(\underline{x}'_i\underline{\beta})}\right]^{y_i} \left[\frac{1}{1+\exp(\underline{x}'_i\underline{\beta})}\right]^{(1-y_i)} \tag{122}$$

$$= \frac{\prod [\exp(\underline{x}'_i\underline{\beta})]^{y_i}}{\prod [1+\exp(\underline{x}'_i\underline{\beta})]}$$

$$= \frac{\prod [\exp(\underline{x}'_i\underline{\beta}\, y_i]}{\prod [1+\exp(\underline{x}'_i\underline{\beta})]}$$

$$= \frac{\exp(\Sigma\, \underline{x}'_i\underline{\beta}\, y_i)}{\prod [1+\exp(\underline{x}'_i\underline{\beta})]}$$

$$= \frac{\exp[\sum_j \beta_j t_j]}{\prod [1+\exp(\underline{x}'_i \underline{\beta})]}$$

where $t_j = \sum x_{ij} y_i \qquad (j=0,\ldots,k)$

The log likelihood function therefore, is (Maddala (1983)):

$$\ln(L) = \sum_j \beta_j t_j - \sum \ln(1+\exp(\underline{x}'_i \underline{\beta})) \qquad (123)$$

Hence,

$$\frac{\delta[\ln L]}{\delta \beta_j} = t_j - \sum \frac{x_{ij}\exp(\underline{x}'_i \underline{\beta})}{1+\exp(\underline{x}'_i \underline{\beta})} \qquad (j=0,\ldots,k) \qquad (124)$$

The maximum likelihood estimate (MLE) $\hat{\underline{\beta}}$ satisfies the equations

$$\left[\frac{\delta[\ln L]}{\delta \beta_j}\right]_{\underline{\beta}=\hat{\underline{\beta}}} = 0 \qquad (125)$$

The MLE of $\underline{\beta}$, $\hat{\underline{\beta}}$, are obtained by iterative use of the above two equations (usually following the Newton-Raphson method).

This method, as do all MLE methods, allows the estimators to have asymptotic properties. The asymptotic covariance matrix of the MLE's is the inverse of the I matrix, $I^{-1}(\underline{\beta})$ and is consistently estimated by $I^{-1}(\hat{\underline{\beta}})$ where (Maddala (1983)):

$$I_{j_1 j_2}(\underline{\beta}) = E\left[\frac{\delta^2 \ln L}{\delta\beta_{j_1}\delta\beta_{j_2}}\right] = \Sigma\left[\frac{x_{ij_1}x_{ij_2}.\exp(\underline{x}'_i\underline{\beta})}{[1+\exp(\underline{x}'_i\underline{\beta})]^2}\right] \qquad (126)$$

Usually, this method is reasonably effective. However, as most other methods of functional iteration to systems of equations, it is subject to convergence limitations. In general, a good initial estimate of β is very helpful. In some cases it can be obtained graphically (if the number of explanatory variables is small) or by ordinary or weighted least squares.

3.2.3 Weighted Least Squares for Individual Data

For this method

$$P_i = P[Y_i=1|\underline{X}_i=\underline{x}_i] = f(\underline{x}_i,\underline{\beta}) = \frac{1}{1+e^{-\underline{x}'_i\underline{\beta}}} \qquad (127)$$

$$y_i = f(\underline{x}_i,\underline{\beta}) + \varepsilon_i \qquad (i=1,\ldots,n) \qquad (128)$$

where $E(\varepsilon_i) = 0$

$Var(\varepsilon_i) = P_iQ_i$

Expanding the non-linear f in a Taylor series around some initial value of $\underline{\beta}$, say $\underline{\beta}_0$, and writing $f'(\underline{x}_i,\underline{\beta}_0)$ for $\delta f(\underline{x}'_i\underline{\beta})/\delta\underline{\beta}$ at $\underline{\beta} = \underline{\beta}_0$ and obtain

$$y_i \simeq f(\underline{x}_i,\underline{\beta}_0) + f'(\underline{x}_i,\underline{\beta}_0)(\underline{\beta} - \underline{\beta}_0) + \varepsilon_i \qquad (129)$$

$$\text{where } f'(\underline{x}_i, \underline{\beta}_0) = P_{0i} Q_{0i} \underline{x}'_i \qquad (130)$$

$$P_{0i} = \frac{1}{1 + e^{-\underline{x}'_i \underline{\beta}_0}} = f(\underline{x}_i, \underline{\beta}_0)$$

$$Q_{0i} = 1 - P_{0i}$$

In the non-linear situation, both f and f' are functions of $\underline{\beta}$ and a closed form solution generally does not exist. Thus an iterative process by the Gauss-Newton procedure is used: a starting value for $\underline{\beta}$ is chosen and continually improved until the error sum of squares is minimized. So, due to the non-constant error variance, weighted iterative non-linear least squares procedures, using a diagonal weight matrix W determined as the inverse of the variance matrix of the vector $\underline{\varepsilon}$, hence

$$w_{ii} = \frac{1}{P_{0i} Q_{0i}} \qquad (i=1,\ldots,n) \qquad (131)$$

3.2.4 Minimum χ^2 for Grouped Data

The logistic function used in this method is given by

$$P_i = \frac{1}{1+\exp(-(\beta_0 + \beta_1 x_i))} \qquad (132)$$

The minimum χ^2 estimate of the logit model is defined by minimization of the following quantity (Berkson(1953)):

$$\chi^2 = \Sigma \left[\frac{(r_i - \hat{p}_i n_i)^2}{\hat{p}_i n_i} + \frac{(s_i - \hat{q}_i n_i)^2}{\hat{q}_i n_i} \right] \tag{133}$$

where $\hat{p}_i$ = "expected" value of P_i at x_i, obtained by inserting the estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ for β_0 and β_1 in the logistic function

$\hat{q}_i = 1 - \hat{p}_i$ = "expected" number affected at x_i

$\hat{p}_i n_i$ = "expected" number affected at x_i

$\hat{q}_i n_i$ = "expected" number not affected at x_i

The χ^2 can be evaluated directly from (133). However, it is easily shown that (133) is identically equal to

$$\chi^2 = \Sigma \frac{n_i}{p_i q_i} (p_i - \hat{p}_i)^2 \tag{134}$$

Since (134) is easier to compute than (133), a direct calculation of χ^2 can appropriately employ (134).

For the minimum χ^2 estimate, the equations to be solved are (Berkson (1955a)):

$$\Sigma\, n_i \frac{(\hat{p}_i q_i + \hat{q}_i p_i)}{\hat{p}_i \hat{q}_i} (p_i - \hat{p}_i) = 0 \qquad (135)$$

N.E.

$$\Sigma\, n_i \frac{(\hat{p}_i q_i + \hat{q}_i p_i)}{\hat{p}_i \hat{q}_i} x_i (p_i - \hat{p}_i) = 0 \qquad (136)$$

Since the coefficients of these equations are functions of the p_i's, which are to be estimated, as well as because the p_i's are not linear in the parameters, the minimum χ^2 estimates require iterative methods for solution.

General statistical theory shows that estimates of parameters based upon the minimization of χ^2 tend to equality with maximum likelihood estimates as the number of observations is increased, and that they also have the properties of consistency, asymptotic normality of distribution, and asymptotic efficiency (Berkson (1955a), Finney (1964)). Minimum χ^2 estimates will not usually have the properties listed as (iv) and (v) in section 3.1.1, but the existence of efficient (in finite samples) or sufficient estimates for the dichotomous response problem has not been 'stipulated' in the model so that the maximum likelihood procedure has no advantage for this situation.

Instructions and tables to assist the use of minimum χ^2 seem unnecessary in the absence of evidence of its superiority to maximum likelihood (Finney (1964)) and hence a procedure for determining the estimates of the parameters will not be developed here.

3.2.5 Minimum Logit χ^2 for Grouped Data

The logistic function used in this method is given by

$$P_i = \Psi(\beta_0+\beta_1 x_i) \tag{137}$$

The minimum logit χ^2 estimate is based on minimization of the following quantity, called the "logit χ^2" (Berkson (1953)):

$$\chi^2(\text{logit}) = \Sigma n_i p_i q_i (l_i-\hat{l}_i)^2 \tag{138}$$

where $l_i = \ln(p_i/q_i)$ = the logit of p_i

$\hat{l}_i = \ln(\hat{p}_i/\hat{q}_i) = \hat{\beta}_0+\hat{\beta}_1 x_i$ = logit of $\hat{p}_i$

$\hat{p}_i$ = the estimate of P_i at x_i, given by (137) with estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ replacing the parameters β_0 and β_1.

The normal equations for obtaining the minimum logit χ^2 estimate of β_0 and β_1 are (Berkson (1953)):

$$\Sigma n_i p_i q_i (l_i-\hat{l}_i) = 0 \tag{139}$$

N.E.

$$\Sigma n_i p_i q_i x_i (l_i-\hat{l}_i) = 0 \tag{140}$$

The evaluation of (124) and (125) leads to a procedure that amounts simply to obtaining a weighted least squares solution of the straight line

$$\hat{l}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i \tag{141}$$

using $n_i p_i q_i$ as weights of the observation l_i since

$$\text{Var}\,(l_i) = \frac{1}{n_i p_i q_i} \tag{142}$$

Since the coefficients are in terms of the known observations, and l_i is linear in the parameters, (139) and (140) can be solved directly and simply for $\hat{\beta}_0$ and $\hat{\beta}_1$.

The estimates then are

$$\hat{\beta}_1 = \frac{\Sigma n_i w_i (l_i - \bar{l})(x_i - \bar{x})}{\Sigma n_i w_i (x_i - \bar{x})^2} \tag{143}$$

$$= \frac{\Sigma n_i w_i l_i x_i - \dfrac{\Sigma n_i w_i l_i \; \Sigma n_i w_i x_i}{\Sigma n_i w_i}}{\Sigma n_i w_i x_i^2 - \dfrac{(\Sigma n_i w_i x_i)^2}{\Sigma n_i w_i}}$$

$$\hat{\beta}_0 = \bar{l} - \hat{\beta}_1 \bar{x} = \frac{\Sigma n_i w_i l_i - \hat{\beta}_1 \Sigma n_i w_i l_i}{\Sigma n_i w_i} \tag{144}$$

where $w_i = p_i q_i$

$$\bar{l} = \frac{\Sigma n_i w_i l_i}{\Sigma n_i w_i}$$

$$\bar{x} = \frac{\Sigma n_i w_i x_i}{\Sigma n_i w_i}$$

We may write the estimated logit linear equation as

$$\hat{l}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i = \hat{\beta}'_0 + \hat{\beta}_1 (x_i - \bar{x}) \tag{145}$$

where $\hat{\beta}'_0 = \bar{l} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$

Formulas for variances of the estimates of the parameters may be written as follow (Berkson (1953)):

$$S^2_{\hat{\beta}'_0} = \frac{1}{\Sigma n_i w_i} \tag{146}$$

$$S^2_{\hat{\beta}_0} = S^2_{\hat{\beta}'_0} + \bar{x}^2 S^2_{\hat{\beta}_1} \tag{147}$$

$$S^2_{\hat{\beta}_1} = \frac{1}{\Sigma n_i w_i (x_i - \bar{x})^2} \tag{148}$$

$$S^2_{\hat{\gamma}} = \frac{1}{\hat{\beta}_1^2}\left[S^2_{\hat{\beta}'_0} + S^2_{\hat{\beta}_1}(x_{50} - \bar{x})^2\right] \tag{149}$$

These formulas provide closely accurate estimates of the variances, under ideal conditions in which:

a). the true P_i's are given exactly by (137)

b). the samples are random at each x_i, the x_i's themselves being fixed quantities, and

c). the number of observations at each x_i is large

Berkson (1953) is opposed to using observations of zero or 100 percent response. However, such observations may occur and Berkson has made provisions for utilizing them. For an observation $p_i=0$ or $p_i=1$, the corresponding logit is infinite, the weight p_iq_i is zero, while the weighted logit $p_iq_il_i$ approaches the limit zero as $p\to0$, or $p\to1$. If with these observations, we use for $p_iq_il_i$ the limiting value zero, the observations are effectively eliminated and hence actually are not used. The definitive procedure for dealing with zero's and 100 percent observations in the minimum logit χ^2 method is to use the rule of substituting $1/2(n_i)$ for zero and $1 - 1/2n_i$ for 100 percent observations (Berkson (1953)).

The minimum logit χ^2 estimate has the following properties (Berkson (1953)):

1). The logit χ^2 (3) is distributed asymptotically as χ^2, as is the classic Pearson χ^2, and for small numbers is very close to the Pearson χ^2. Hence, the validity of the model can be tested in the same manner as was done for the normit χ^2.

2). The estimate is asymptotically efficient, and therefore as the number of experimental runs at each dose $n_i\to\infty$, the value of the variance of the asymptotic distribution is minimum and given by $1/I(0)$, where

$$I(\theta) = E\left[\frac{\delta \ln \phi}{\delta \theta}\right]^2$$

ϕ being the probability of the total sample and θ the parameter β_0 or β_1 which is to be estimated.

3). It is sufficient and therefore extracts the total amount of information available in the sample which is relevant to the estimated parameters.

4). For finite samples

a). it has smaller sampling variance than 1/I, and

b). has smaller sampling error (MSE) and smaller variance about the mean than the maximum likelihood estimate.

Appendix B contains tables giving the logit, l_i, for argument p_i (Table B.1), the logistic weights $w_i = p_i q_i$ and $w_i l_i = p_i q_i l_i$ for argument p_i (Table B.2), and the antilogit, p_i, for argument l_i (Table B.3).

3.3 COMPARISON OF METHODS

Now that all estimation methods have been presented, three data sets have been chosen to illustrate these methods. The data set descriptions as well as the worked out examples are in Appendix C.

Table 1 gives the parameter estimates, the $\hat{\beta}_j$'s, for each of the three data sets used in the examples. The $\hat{\beta}_j$'s should only be compared within the probit analysis and within the logit analysis, not between the two analyses.

In the probit analysis, the standard deviations for all parameter estimates is smaller for the minimum normit χ^2 method than in the maximum likelihood method.

In the logit analysis, the parameter estimates for the individual and grouped data maximum likelihood methods are the same. The standard deviations for the parameter estimates in the logit weighted least squares method is smaller than for either of the logit maximum likelihood methods. And the standard deviations for all the parameter estimates are smaller for the minimum logit χ^2 method than for any other logit method.

A point needs to be made that the Wrigley (1976) data when grouped is done in such a way as to lose information due to averaging, whereas the Berkson (1953) and Montgomery and Peck (1982) data sets are grouped due to replication.

Table 1. Comparison of Methods: Parameter Estimates ($\hat{\beta}_j$'s)

Data Set	Estimator	ESTIMATION METHOD					
		Probit Analysis		Logit Analysis			
		Maximum Likelihood	Minimum Normit X^2	Individual Max. Likelihood	Grouped Max. Likelihood	Weighted Least Squares	Minimum Logit X^2
Berkson (1953)	$\hat{\beta}_0$	-1.1375	-1.1359	-1.9184	-1.9184	-1.9501	-1.9046
	$S_{\hat{\beta}_0}$	0.1670	0.1215	0.2982	0.2982	0.2076	0.2034
	$\hat{\beta}_1$	4.2155	4.1957	7.1495	7.1495	7.3464	7.0605
	$S_{\hat{\beta}_1}$	0.4785	0.3582	0.8930	0.8930	0.6343	0.5998
Montgomery & Peck (1982)	$\hat{\beta}_0$	-3.2712	-3.3142	-5.3397	-5.3397	-5.3471	-5.3361
	$S_{\hat{\beta}_0}$	0.3209	0.1155	0.5457	0.5457	0.1186	0.1170
	$\hat{\beta}_1$	0.0949	0.0959	0.1548	0.1548	0.1551	0.1547
	$S_{\hat{\beta}_1}$	0.0093	0.0033	0.0157	0.0157	0.0034	0.0034
Wrigley (1976)	$\hat{\beta}_0$		-3.7943	-10.0862		-5.7925	-7.0558
	$S_{\hat{\beta}_0}$		2.3189	2.9504		2.1502	4.0788
	$\hat{\beta}_1$		0.1155	0.2117		0.1442	0.1999
	$S_{\hat{\beta}_1}$		0.0253	0.0381		0.0280	0.0446
	$\hat{\beta}_2$		0.0439	0.1318		0.0702	0.0842
	$S_{\hat{\beta}_2}$		0.0381	0.0489		0.0363	0.0671

This information loss of the Wrigley (1976) data can be seen in the difference of parameter estimates for the individual data method (logit: individual maximum likelihood) and the grouped data methods (logit: weighted least squares and minimum logit χ^2).

The parameter estimates, the $\hat{\beta}_j$'s, are not as important in comparing these methods as are the probability estimates that the given event will occur, the $\hat{p}_i$'s. Table 2 gives the tabular comparison of the $\hat{p}_i$'s for all the estimation methods for the Berkson (1953) and Montgomery and Peck (1982) data sets. Figures 6 and 7 illustrate this information graphically. Figure 6 contains the x_i's plotted against the $\hat{p}_i$'s for all six estimation methods using the Berkson (1953) data. Figure 7 contains the same information but for the Montgomery and Peck (1982) data. All estimation methods appear to equally approximate the actual p_i's. Hence, the choice of estimation method should be based on more than the actual estimates since they are so similar.

In the bioassay situation, the LD50 would be of interest as another means of estimation method comparison. These LD50 estimates are given in Table 3. The estimated LD50 values are indistinguishably close for each of the methods.

χ^2 goodness-of-fit tests were applied with the χ^2 values for the Berkson (1953) and the Montgomery and Peck (1982)

Table 2. Comparison of Methods: Probability Estimates ($\hat{p}_i$'s)

Data Set	X_i	Actual P_i	ESTIMATION METHOD					
			Probit Analysis		Logit Analysis			
			Maximum Likelihood	Minimum$_2$ Normit X^2	Individual Maximum Likelihood	Grouped Maximum Likelihood	Weighted Least Squares	Minimum$_2$ Logit X^2
Berkson (1953)	0.000	0.1200	0.1277	0.1280	0.1280	0.1280	0.1245	0.1295
	0.164	0.3333	0.3276	0.3272	0.3217	0.3217	0.3218	0.3215
	0.292	0.5217	0.5372	0.5358	0.5422	0.5422	0.5486	0.5392
	0.471	0.8571	0.8018	0.7996	0.8098	0.8098	0.8191	0.8055
	0.593	0.8800	0.9134	0.9118	0.9106	0.9106	0.9173	0.9074
Montgomery & Peck (1982)	25	0.2000	0.1844	0.1796	0.1870	0.1870	0.1870	0.1872
	27	0.2429	0.2392	0.2343	0.2386	0.2386	0.2388	0.2388
	29	0.3000	0.3018	0.2970	0.2993	0.2993	0.2996	0.2995
	31	0.3500	0.3710	0.3664	0.3680	0.3680	0.3684	0.3681
	33	0.4500	0.4445	0.4406	0.4424	0.4424	0.4430	0.4425
	35	0.5059	0.5201	0.5169	0.5196	0.5196	0.5203	0.5196
	37	0.6000	0.5949	0.5925	0.5958	0.5958	0.5967	0.5958
	39	0.6600	0.6664	0.6649	0.6676	0.6676	0.6686	0.6676
	41	0.7500	0.7323	0.7316	0.7325	0.7325	0.7334	0.7324
	43	0.7846	0.7909	0.7909	0.7886	0.7886	0.7895	0.7885

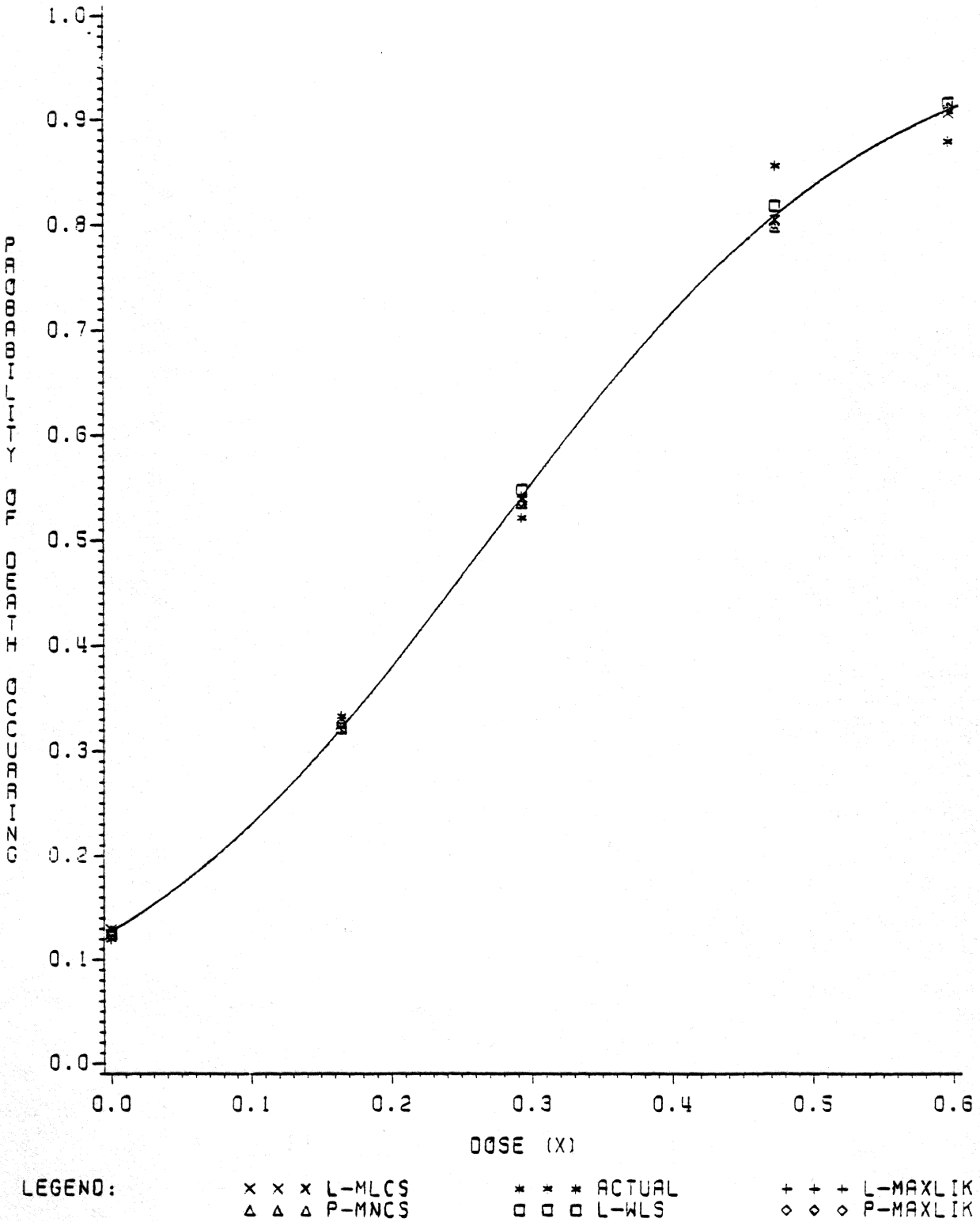


Figure 6. Comparison of Methods Probability Estimates ($\hat{p}_i$'s)

(Data Set = Berkson (1953))

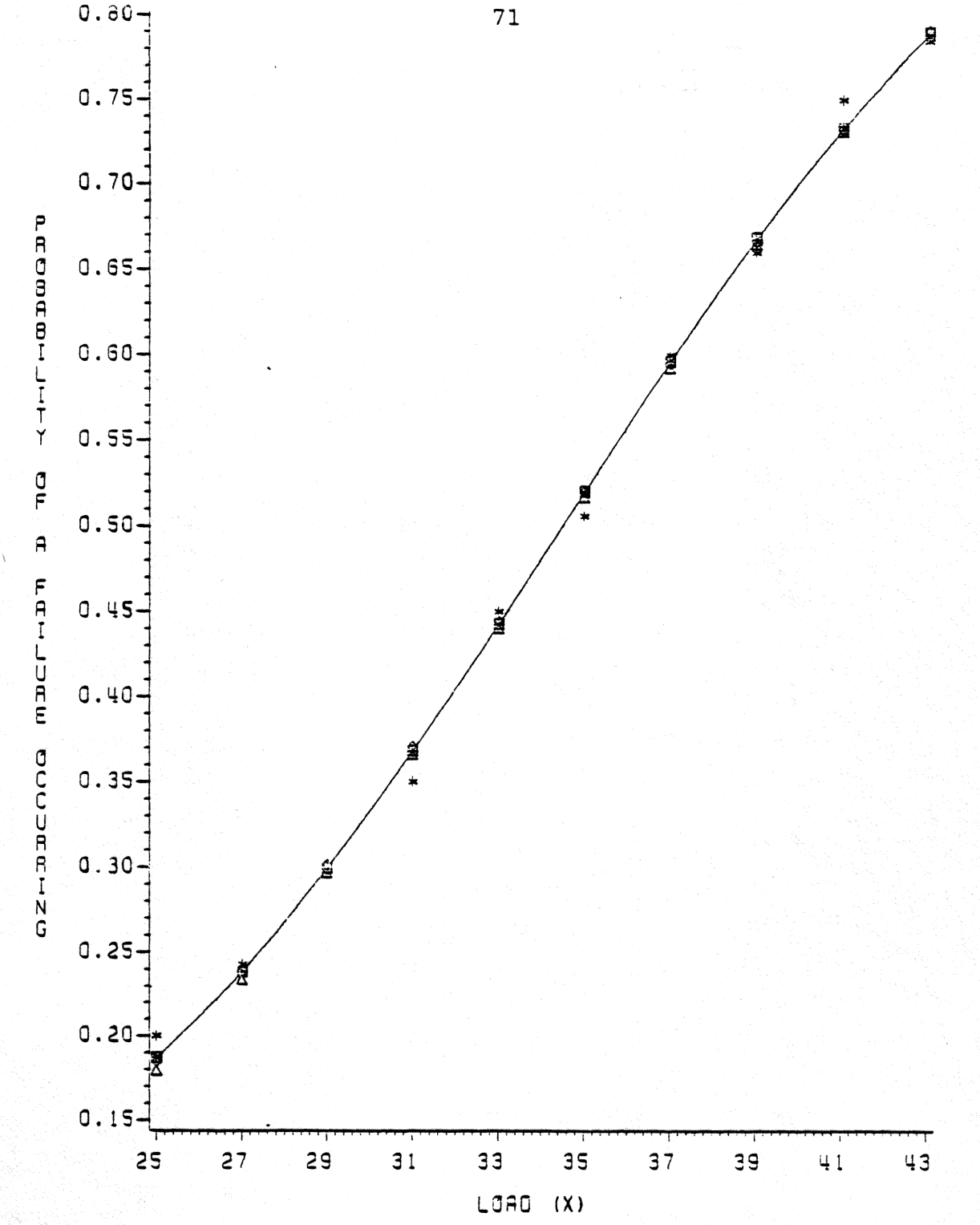


Figure 7. Comparison of Methods Probability Estimates ($\hat{p}_i$'s)

(Data Set = Montgomery & Peck (1982))

Table 3. Comparison of Methods: LD50 and χ^2 Estimates for Grouped Data (k=1)

Data Set	Estimator	ESTIMATION METHOD					
		Probit Analysis		Logit Analysis			
		Maximum Likelihood	Minimum Normit X^2	Individual Maximum Likelihood	Grouped Maximum Likelihood	Weighted Least Squares	Minimum Logit X^2
Berkson (1953)	$\hat{\gamma}$	0.2698	0.2707	0.2683	0.2683	0.2654	0.2654
	X^{2}*	1.74	1.71	1.41	1.41	1.57	1.40
Montgomery & Peck (1982)	$\hat{\gamma}$	34.4700	34.5589	34.4942	34.4942	34.4752	34.4932
	X^{2}**	0.47	0.48	0.37	0.37	0.37	0.37

$$X^2 = \Sigma \frac{n_i}{\hat{p}_i \hat{q}_i} (p_i - \hat{p}_i)^2$$

*$X_3^{2(.05)} = 7.81$

**$X_8^{2(.05)} = 15.51$

data sets given in Table 3. This goodness-of-fit test shows no significant lack-of-fit for all estimation methods at the 0.05 significance level. The logit analysis methods give consistently lower χ^2 values than do the probit analysis methods.

All estimation methods have the asymptotic properties for their estimators. So when the sample size is 'large enough' all methods should be equivalent as far as asymptotic properties are concerned. The sample sizes, the n_i's in the Berkson (1953) and the Montgomery and Peck (1982) data sets must be 'large enough' since the estimated parameters and their associated variances are equivalent for all methods. So as long as the data set to be used has 'large enough' n_i's, there is no clear cut choice of method based on the asymptotic properties of the estimators. If data sets contain small n_i's, then all methods should be employed before a choice is made.

Even though some methods are iterative and others are non-iterative, the accuracy and relative ease of modern day computer packages favor none of the methods over the others on a computational standpoint.

Recall that the minimum normit χ^2 and the minimum logit χ^2 methods must be used with caution due to their inappropriateness at extreme response (p_i near zero or 100%).

Perhaps the most serious objection to any type of minimum χ^2 approach is that when the n_i per x_i is small, the expected frequencies (responses and non-responses) will be small and the value of χ^2 unstable. The difficulty of using any minimum χ^2 when the n_i are small may constitute a serious practical difficulty in data of biological assay (Finney (1964)).

The maximum likelihood method is more generally applicable than any minimum χ^2 method for probit or logit analysis both because it does not run into difficulties when the n_i's are small, and because it can usually be used for individual data as well as for grouped data.

3.4 COMPUTER PROGRAMS

A number of computer software programs for qualitative response models are available from several sources:

1). PROBIT: The PROBIT program (entirely in FORTRAN IV) handles dichotomous probit models with one or more independent variables. (Source: Capps (1983)).

2). MAXLIK: The multi-purpose MAXLIK program carries out the estimation of statistical models in general by the method of maximum likelihood. (Source: Capps (1983))

3). QUAIL: The QUAIL program (the acronym stands for Qualitative, Intermittant, and Limited Dependent Variable Statistical Program) handles the analysis of non-continuous dependent variables. (Source: Capps (1983))

4). SHAZAM: The econometric package SHAZAM (the acronym corresponds to six subroutines with reference to Greek mythological characters - Solomon, Hercules, Atlas, Zeus, Achilles, and Mercury), handles the analysis of dichotomous logit and probit models with multiple independent variables as well as the analysis of non-continuous dependent variables (Source: Capps (1983))

5). SAS: The statistical package SAS (the acronym stands for Statistical Analysis System), deals not only with dichotomous probit models with only single regressors but also dichotomous logit models with multiple regressors. (See PROC PROBIT, PROC LOGIST, and PROC NLIN)

6). LLR: The LLR program (the acronym stands for linear logistic regression) performs regression analysis utilizing Cox's linear logistic model (Source: Lee (1974)). The program has two versions and they are:

a). LOGIST: which fits all the k explanatory variables at the same time, and

b). STLOG: follows a forward selection procedure so that the first variable determined is the most important single characteristic in predicting probability of response, the second variable is second most important after the first one has been included in the equation, and so on.

The program gives maximum likelihood estimates and interval estimates for the regression coefficients, significance level for hypothesis tested, and predicted probability of success and residual for each case.

7). SCREEN: The computer program SCREEN identifies predictors of dichotomous dependent variables. (Source: Hamilton and Wendt (1975))

This enumeration by no means constitutes the exhaustive list of computer programs to handle qualitative response models.

Chapter IV

SUMMARY AND CONCLUSIONS

Many techniques have been presented on modeling the situation where a dichotomous response variable is obtained. Conventional methods such as generalized or ordinary least squares for estimating the relationship (the linear probability model) are inadequate for many reasons.

The notion of the symmetric sigmoid curve was then introduced as well as the concept of probability plotting. Both of these concepts introduce a probabilistic formulation which is estimated by distribution functions. Any cumulative distribution function (cdf) would suffice for these formulations, but the normal and logistic cdf's are by far the two most widely used.

The standardized normal and logistic distributions are very similar, especially in midrange. They are both symmetric about their mean 0. The variance of the normal distribution is 1 while that for the logistic distribution is approximately 3.29. The logistic distribution has slightly heavier tails than does the normal distribution. Computationally, the logistic distribution is much easier to work with since it is in closed form while the normal distribution involves integration.

When the normal cdf is utilized, the resulting estimation method is called probit (normit) analysis while the logistic cdf leads to logit analysis. The remaining portion of the paper discusses specific methods for estimating the parameters in these distributions.

Two distribution parameter estimation procedures are discussed for probit analysis: maximum likelihood and minimum normit χ^2. Four estimation procedures are presented for logit analysis: maximum likelihood, weighted least squares, minimum χ^2, and minimum logit χ^2. Some of these methods are based on individual data, although the majority of them are based on grouped data. In the grouped data situation, and since there are only two possible responses, the binomial distribution is assumed for the proportion 'responding' with the probability of the event occuring determined by the cdf that is chosen. Another parameter of interest to be estimated in the one regressor situation is the lethal dose (LD50), that is, the 'dose' which is lethal to one-half of the population.

Choosing between these methods should be based on two different criteria. The first is to pick the distribution and the second is to choose the estimation method for the chosen distribution. The differences in the distributions are discussed in chapter 2. What follows is the discussion

of the differences in estimation methods. The first quantity compared between the methods is the estimated distribution parameters. Very little difference was found between these estimates. The second quantity compared were the probability estimates. All estimation methods appear to equally approximate the actual probabilities. So goodness-of-fit tests were performed. All estimation methods showed no significant lack-of-fit at the 0.05 level although the logit models displayed consistently lower χ^2 values than for the probit models.

For data sets with one regressor, another means of estimation method comparison is looking at the LD50. The estimated LD50 values are indistinguishably close for each of the methods.

All estimation methods have the same asymptotic properties for their estimators. So as long as the data set to be used has 'large enough' samples, there is no clear cut choice of method based on the asymptotic properties of the estimators.

The maximum likelihood method is more generally applicable than any minimum χ^2 method because it does not run into difficulties when the samples are small, and because it can usually be used for individual data as well as for grouped data.

The purpose of this paper is not to be a comprehensive source for all dichotomous response models. In fact, this work just scratches the surface. This paper is designed to lay the groundwork for further research in this area. Other topics include items such as: forward and backward selection of regressors in the logit situation, joint estimation, multivariate estimation, and probit estimation by graphical means (k=1) just to list a few. Also, the polychotomous model is not discussed but is, in fact, an extension of the dichotomous model situation.

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Appendix A

NORMIT ANALYSIS TABLES

Table A.1
Normits (v)

For p less than .50 on left, normit is negative.
For p greater than .50 on right, normit is positive.

p	Thousandths, for p in left column											
	0	1	2	3	4	5	6	7	8	9	—	
.00	∞	3.09023	2.87816	2.74778	2.65207	2.57583	2.51214	2.45726	2.40892	2.36562	2.32635	.99
.01	2.32635	2.29037	2.25713	2.22621	2.19729	2.17009	2.14441	2.12007	2.09693	2.07485	2.05375	.98
.02	2.05375	2.03352	2.01409	1.99539	1.97737	1.95996	1.94313	1.92684	1.91104	1.89570	1.88079	.97
.03	1.88079	1.86630	1.85218	1.83842	1.82501	1.81191	1.79912	1.78661	1.77438	1.76241	1.75069	.96
.04	1.75069	1.73920	1.72793	1.71689	1.70604	1.69540	1.68494	1.67466	1.66456	1.65463	1.64485	.95
.05	1.64485	1.63523	1.62576	1.61644	1.60725	1.59819	1.58927	1.58047	1.57179	1.56322	1.55477	.94
.06	1.55477	1.54643	1.53820	1.53007	1.52204	1.51410	1.50626	1.49851	1.49085	1.48329	1.47579	.93
.07	1.47579	1.46838	1.46106	1.45381	1.44663	1.43953	1.43250	1.42554	1.41865	1.41183	1.40507	.92
.08	1.40507	1.39838	1.39174	1.38517	1.37866	1.37220	1.36581	1.35946	1.35317	1.34694	1.34076	.91
.09	1.34076	1.33462	1.32854	1.32251	1.31652	1.31058	1.30469	1.29884	1.29303	1.28727	1.28155	.90
.10	1.28155	1.27587	1.27024	1.26464	1.25908	1.25357	1.24808	1.24264	1.23723	1.23186	1.22653	.89
.11	1.22653	1.22123	1.21596	1.21073	1.20553	1.20036	1.19522	1.19012	1.18504	1.18000	1.17499	.88
.12	1.17499	1.17000	1.16505	1.16012	1.15522	1.15035	1.14551	1.14069	1.13590	1.13113	1.12639	.87
.13	1.12639	1.12168	1.11699	1.11232	1.10768	1.10306	1.09847	1.09390	1.08935	1.08482	1.08032	.86
.14	1.08032	1.07584	1.07138	1.06694	1.06252	1.05812	1.05374	1.04939	1.04505	1.04073	1.03643	.85
.15	1.03643	1.03215	1.02789	1.02365	1.01943	1.01522	1.01103	1.00686	1.00271	0.99858	0.99446	.84
.16	0.99446	0.99036	0.98627	0.98220	0.97815	0.97411	0.97009	0.96609	0.96210	0.95812	0.95417	.83
.17	0.95417	0.95022	0.94629	0.94238	0.93848	0.93459	0.93072	0.92686	0.92301	0.91918	0.91537	.82
.18	0.91537	0.91156	0.90777	0.90399	0.90023	0.89647	0.89273	0.88901	0.88529	0.88159	0.87790	.81
.19	0.87790	0.87422	0.87055	0.86689	0.86325	0.85962	0.85600	0.85239	0.84879	0.84520	0.84162	.80
.20	0.84162	0.83805	0.83450	0.83095	0.82742	0.82389	0.82038	0.81687	0.81338	0.80990	0.80642	.79
.21	0.80642	0.80296	0.79950	0.79606	0.79262	0.78919	0.78577	0.78237	0.77897	0.77557	0.77219	.78
.22	0.77219	0.76882	0.76546	0.76210	0.75875	0.75542	0.75208	0.74876	0.74545	0.74214	0.73885	.77
.23	0.73885	0.73556	0.73228	0.72900	0.72574	0.72248	0.71923	0.71599	0.71275	0.70952	0.70630	.76
.24	0.70630	0.70309	0.69988	0.69668	0.69349	0.69031	0.68713	0.68396	0.68080	0.67764	0.67449	.75
.25	0.67449	0.67135	0.66821	0.66508	0.66196	0.65884	0.65573	0.65262	0.64952	0.64643	0.64335	.74
.26	0.64335	0.64027	0.63719	0.63412	0.63106	0.62801	0.62496	0.62191	0.61887	0.61584	0.61281	.73
.27	0.61281	0.60979	0.60678	0.60376	0.60076	0.59776	0.59477	0.59178	0.58879	0.58581	0.58284	.72
.28	0.58284	0.57987	0.57691	0.57395	0.57100	0.56805	0.56511	0.56217	0.55924	0.55631	0.55338	.71
.29	0.55338	0.55047	0.54755	0.54464	0.54174	0.53884	0.53594	0.53305	0.53016	0.52728	0.52440	.70
.30	0.52440	0.52153	0.51866	0.51579	0.51293	0.51007	0.50722	0.50437	0.50153	0.49869	0.49585	.69
.31	0.49585	0.49302	0.49019	0.48736	0.48454	0.48173	0.47891	0.47610	0.47330	0.47050	0.46770	.68
.32	0.46770	0.46490	0.46211	0.45933	0.45654	0.45376	0.45099	0.44821	0.44544	0.44268	0.43991	.67
.33	0.43991	0.43715	0.43440	0.43164	0.42889	0.42615	0.42340	0.42066	0.41793	0.41519	0.41246	.66
.34	0.41246	0.40974	0.40701	0.40429	0.40157	0.39886	0.39614	0.39343	0.39073	0.38802	0.38532	.65
.35	0.38532	0.38262	0.37993	0.37722	0.37454	0.37186	0.36917	0.36649	0.36381	0.36113	0.35846	.64
.36	0.35846	0.35579	0.35312	0.35045	0.34779	0.34513	0.34247	0.33981	0.33716	0.33450	0.33185	.63
.37	0.33185	0.32921	0.32656	0.32392	0.32128	0.31864	0.31600	0.31337	0.31074	0.30811	0.30548	.62
.38	0.30548	0.30286	0.30023	0.29761	0.29499	0.29237	0.28976	0.28715	0.28454	0.28193	0.27932	.61
.39	0.27932	0.27671	0.27411	0.27151	0.26891	0.26631	0.26371	0.26112	0.25853	0.25594	0.25335	.60
.40	0.25335	0.25076	0.24817	0.24559	0.24301	0.24043	0.23785	0.23527	0.23269	0.23012	0.22754	.59
.41	0.22754	0.22497	0.22240	0.21983	0.21727	0.21470	0.21214	0.20957	0.20701	0.20445	0.20189	.58
.42	0.20189	0.19934	0.19678	0.19422	0.19167	0.18912	0.18657	0.18402	0.18147	0.17892	0.17637	.57
.43	0.17637	0.17383	0.17128	0.16874	0.16620	0.16366	0.16112	0.15858	0.15604	0.15351	0.15097	.56
.44	0.15097	0.14843	0.14590	0.14337	0.14084	0.13830	0.13577	0.13324	0.13072	0.12819	0.12566	.55
.45	0.12566	0.12314	0.12061	0.11809	0.11556	0.11304	0.11052	0.10799	0.10547	0.10295	0.10043	.54
.46	0.10043	0.09791	0.09540	0.09288	0.09036	0.08784	0.08533	0.08281	0.08030	0.07778	0.07527	.53
.47	0.07527	0.07276	0.07024	0.06773	0.06522	0.06271	0.06020	0.05768	0.05517	0.05266	0.05015	.52
.48	0.05015	0.04764	0.04513	0.04263	0.04012	0.03761	0.03510	0.03259	0.03008	0.02758	0.02507	.51
.49	0.02507	0.02256	0.02005	0.01755	0.01504	0.01253	0.01003	0.00752	0.00501	0.00251	0.00000	.50
	—	9	8	7	6	5	4	3	2	1	0	p
	Thousandths, for p in right column											

Source: Berkson (1955b:540-541)

Table A.2
Normit Weights

Upper figure in table is $w=z^2/pq$; lower figure is wv.
For p less than .50 on left, wv is negative.
For p greater than .50 on right, wv is positive.

p	Thousandths, for p on left											p
	0	1	2	3	4	5	6	7	8	9		
.00		.0113	.0201	.0280	.0352	.0420	.0485	.0546	.0605	.0662	.0718	.99
		.0351	.0580	.0780	.0954	.1083	.1218	.1342	.1458	.1567	.1669	
.01	.0718	.0771	.0823	.0873	.0923	.0971	.1018	.1064	.1109	.1153	.1196	.98
	.1669	.1768	.1857	.1944	.2027	.2106	.2182	.2255	.2325	.2392	.2456	
.02	.1196	.1239	.1280	.1321	.1362	.1401	.1440	.1479	.1517	.1554	.1591	.97
	.2456	.2519	.2579	.2637	.2693	.2747	.2799	.2850	.2899	.2946	.2992	
.03	.1591	.1627	.1663	.1698	.1733	.1768	.1802	.1835	.1869	.1901	.1934	.96
	.2992	.3037	.3080	.3122	.3163	.3203	.3242	.3279	.3316	.3351	.3386	
.04	.1934	.1966	.1998	.2029	.2060	.2091	.2121	.2151	.2181	.2210	.2239	.95
	.3386	.3419	.3452	.3483	.3514	.3544	.3574	.3602	.3630	.3657	.3683	
.05	.2239	.2268	.2297	.2325	.2353	.2381	.2408	.2436	.2463	.2489	.2516	.94
	.3683	.3709	.3734	.3758	.3782	.3805	.3829	.3850	.3871	.3892	.3912	
.06	.2516	.2542	.2568	.2594	.2620	.2645	.2670	.2695	.2720	.2745	.2769	.93
	.3912	.3932	.3951	.3969	.3988	.4006	.4022	.4039	.4056	.4071	.4087	
.07	.2769	.2793	.2817	.2841	.2865	.2889	.2912	.2935	.2958	.2980	.3003	.92
	.4087	.4102	.4116	.4131	.4144	.4158	.4171	.4184	.4196	.4208	.4219	
.08	.3003	.3025	.3048	.3070	.3092	.3113	.3135	.3156	.3178	.3199	.3220	.91
	.4219	.4231	.4241	.4252	.4262	.4272	.4282	.4291	.4300	.4309	.4317	
.09	.3220	.3241	.3261	.3282	.3302	.3323	.3343	.3363	.3383	.3403	.3422	.90
	.4317	.4325	.4333	.4340	.4348	.4355	.4361	.4368	.4374	.4380	.4386	
.10	.3422	.3442	.3461	.3480	.3499	.3518	.3537	.3556	.3573	.3593	.3612	.89
	.4386	.4391	.4396	.4401	.4406	.4411	.4415	.4419	.4423	.4426	.4430	
.11	.3612	.3630	.3648	.3666	.3684	.3702	.3720	.3737	.3755	.3772	.3789	.88
	.4430	.4433	.4436	.4439	.4441	.4444	.4446	.4448	.4450	.4451	.4453	
.12	.3789	.3807	.3824	.3841	.3858	.3874	.3891	.3908	.3924	.3940	.3957	.87
	.4453	.4454	.4455	.4456	.4456	.4457	.4457	.4457	.4457	.4457	.4457	
.13	.3957	.3973	.3989	.4005	.4021	.4037	.4053	.4068	.4084	.4099	.4115	.86
	.4457	.4456	.4456	.4455	.4454	.4453	.4452	.4450	.4449	.4447	.4445	
.14	.4115	.4130	.4145	.4160	.4173	.4190	.4205	.4220	.4235	.4249	.4264	.85
	.4445	.4443	.4441	.4439	.4436	.4431	.4431	.4425	.4425	.4422	.4419	
.15	.4264	.4278	.4293	.4307	.4321	.4335	.4349	.4363	.4377	.4391	.4405	.84
	.4419	.4416	.4412	.4409	.4405	.4401	.4397	.4393	.4389	.4385	.4380	
.16	.4405	.4418	.4432	.4446	.4460	.4472	.4486	.4499	.4512	.4525	.4538	.83
	.4380	.4376	.4371	.4367	.4362	.4357	.4352	.4346	.4341	.4336	.4330	
.17	.4538	.4551	.4564	.4577	.4590	.4602	.4615	.4628	.4640	.4653	.4665	.82
	.4330	.4325	.4319	.4313	.4307	.4301	.4295	.4289	.4283	.4277	.4270	
.18	.4665	.4677	.4689	.4702	.4714	.4726	.4738	.4750	.4761	.4773	.4785	.81
	.4270	.4264	.4257	.4250	.4243	.4236	.4229	.4222	.4215	.4208	.4201	
.19	.4785	.4797	.4809	.4820	.4831	.4843	.4854	.4865	.4876	.4886	.4899	.80
	.4201	.4193	.4186	.4178	.4170	.4163	.4155	.4147	.4139	.4131	.4123	
.20	.4899	.4910	.4921	.4932	.4943	.4953	.4964	.4973	.4985	.4996	.5007	.79
	.4123	.4115	.4106	.4098	.4090	.4081	.4072	.4064	.4055	.4046	.4037	
.21	.5007	.5017	.5028	.5038	.5048	.5058	.5068	.5078	.5088	.5099	.5109	.78
	.4037	.4029	.4020	.4010	.4001	.3992	.3983	.3974	.3964	.3955	.3946	
.22	.5109	.5119	.5129	.5139	.5149	.5158	.5168	.5178	.5187	.5197	.5206	.77
	.3946	.3936	.3926	.3916	.3906	.3897	.3887	.3877	.3867	.3857	.3847	
.23	.5206	.5216	.5225	.5234	.5244	.5253	.5262	.5271	.5280	.5289	.5298	.76
	.3847	.3836	.3826	.3816	.3806	.3795	.3785	.3774	.3764	.3753	.3742	
.24	.5298	.5307	.5316	.5325	.5334	.5343	.5351	.5360	.5369	.5377	.5386	.75
	.3742	.3732	.3721	.3710	.3699	.3688	.3677	.3666	.3655	.3644	.3633	
		9	8	7	6	5	4	3	2	1	0	p
		Thousandths, for p on right										

Source: Berkson (1955b:542-543)

Table A.2 (continued)
Normit Weights

p	Thousandths, for p on left											p
	0	1	2	3	4	5	6	7	8	9		
.25	.5336	.5394	.5403	.5411	.5419	.5428	.5436	.5444	.5452	.5460	.5468	.74
	.3633	.3621	.3610	.3599	.3587	.3576	.3564	.3553	.3541	.3530	.3518	
.26	.5468	.5476	.5484	.5492	.5500	.5508	.5516	.5524	.5532	.5539	.5547	.73
	.3518	.3506	.3495	.3483	.3471	.3460	.3447	.3435	.3423	.3411	.3399	
.27	.5347	.5554	.5562	.5569	.5577	.5584	.5592	.5599	.5606	.5614	.5621	.72
	.3399	.3387	.3375	.3363	.3350	.3338	.3326	.3313	.3301	.3288	.3276	
.28	.5621	.5628	.5635	.5642	.5649	.5656	.5663	.5670	.5677	.5684	.5691	.71
	.3276	.3261	.3251	.3238	.3225	.3213	.3200	.3188	.3175	.3162	.3149	
.29	.5691	.5697	.5704	.5711	.5718	.5724	.5731	.5737	.5744	.5750	.5757	.70
	.3149	.3136	.3123	.3110	.3097	.3084	.3071	.3058	.3045	.3032	.3019	
.30	.5757	.5763	.5769	.5776	.5782	.5788	.5794	.5801	.5807	.5813	.5819	.69
	.3019	.3006	.2992	.2979	.2966	.2952	.2939	.2925	.2912	.2899	.2885	
.31	.5819	.5825	.5831	.5837	.5843	.5848	.5854	.5860	.5866	.5871	.5877	.68
	.2885	.2872	.2858	.2845	.2831	.2817	.2804	.2790	.2776	.2762	.2749	
.32	.5877	.5883	.5888	.5894	.5899	.5905	.5910	.5916	.5921	.5926	.5932	.67
	.2749	.2735	.2721	.2707	.2693	.2679	.2665	.2652	.2638	.2624	.2609	
.33	.5932	.5937	.5942	.5947	.5953	.5958	.5963	.5968	.5973	.5978	.5983	.66
	.2609	.2595	.2581	.2567	.2553	.2539	.2525	.2511	.2496	.2482	.2468	
.34	.5983	.5988	.5993	.5998	.6002	.6007	.6012	.6017	.6021	.6025	.6031	.65
	.2468	.2453	.2439	.2425	.2410	.2396	.2382	.2367	.2353	.2338	.2324	
.35	.6031	.6035	.6040	.6044	.6049	.6053	.6058	.6062	.6066	.6071	.6075	.64
	.2324	.2309	.2295	.2280	.2265	.2251	.2236	.2222	.2207	.2192	.2178	
.36	.6075	.6079	.6083	.6087	.6092	.6096	.6100	.6104	.6108	.6112	.6116	.63
	.2178	.2163	.2148	.2133	.2119	.2104	.2089	.2074	.2059	.2044	.2030	
.37	.6116	.6120	.6124	.6127	.6131	.6135	.6139	.6142	.6146	.6150	.6153	.62
	.2030	.2015	.2000	.1985	.1970	.1955	.1940	.1925	.1910	.1895	.1880	
.38	.6153	.6157	.6161	.6164	.6168	.6171	.6174	.6178	.6181	.6185	.6188	.61
	.1880	.1865	.1850	.1835	.1819	.1804	.1789	.1774	.1759	.1744	.1729	
.39	.6188	.6191	.6194	.6198	.6201	.6204	.6207	.6210	.6213	.6216	.6219	.60
	.1729	.1713	.1698	.1683	.1667	.1652	.1637	.1622	.1606	.1591	.1576	
.40	.6219	.6222	.6225	.6228	.6231	.6234	.6236	.6239	.6242	.6245	.6247	.59
	.1576	.1560	.1545	.1530	.1514	.1499	.1483	.1468	.1452	.1437	.1422	
.41	.6247	.6250	.6253	.6255	.6258	.6260	.6263	.6265	.6268	.6270	.6272	.58
	.1422	.1406	.1391	.1375	.1360	.1344	.1329	.1313	.1297	.1282	.1266	
.42	.6272	.6275	.6277	.6279	.6282	.6284	.6286	.6288	.6290	.6293	.6295	.57
	.1266	.1251	.1235	.1220	.1204	.1188	.1173	.1157	.1142	.1125	.1110	
.43	.6295	.6297	.6299	.6301	.6303	.6304	.6306	.6308	.6310	.6312	.6314	.56
	.1110	.1095	.1079	.1063	.1047	.1032	.1016	.1000	.0985	.0969	.0953	
.44	.6314	.6315	.6317	.6319	.6320	.6322	.6324	.6325	.6327	.6328	.6330	.55
	.0953	.0937	.0922	.0906	.0890	.0874	.0859	.0843	.0827	.0811	.0795	
.45	.6330	.6331	.6333	.6334	.6335	.6337	.6338	.6339	.6341	.6342	.6343	.54
	.0795	.0780	.0764	.0748	.0732	.0716	.0700	.0685	.0669	.0653	.0637	
.46	.6343	.6344	.6345	.6346	.6347	.6348	.6349	.6350	.6351	.6352	.6353	.53
	.0637	.0621	.0605	.0589	.0574	.0558	.0542	.0526	.0510	.0494	.0478	
.47	.6353	.6354	.6355	.6356	.6356	.6357	.6358	.6359	.6359	.6360	.6360	.52
	.0478	.0462	.0446	.0430	.0415	.0399	.0383	.0367	.0351	.0334	.0319	
.48	.6360	.6361	.6361	.6362	.6362	.6363	.6363	.6364	.6364	.6364	.6365	.51
	.0319	.0303	.0287	.0271	.0255	.0239	.0223	.0207	.0191	.0175	.0160	
.49	.6365	.6365	.6365	.6365	.6366	.6366	.6366	.6366	.6366	.6366	.6366	.50
	.0160	.0144	.0128	.0112	.0096	.0080	.0064	.0048	.0032	.0016	.0000	
		9	8	7	6	5	4	3	2	1	0	
		Thousandths, for p on right										p

Table A.3
Antinormits

The table gives the value of p for a specified value of the normit v.
If v is negative, p is 1 minus the tabled value.

v	Thousandths										v
	0	1	2	3	4	5	6	7	8	9	
.00	.50000	.50040	.50080	.50120	.50160	.50199	.50239	.50279	.50319	.50359	.00
.01	.50399	.50439	.50479	.50519	.50559	.50598	.50638	.50678	.50718	.50758	.01
.02	.50798	.50838	.50878	.50918	.50957	.50997	.51037	.51077	.51117	.51157	.02
.03	.51197	.51237	.51276	.51316	.51356	.51396	.51436	.51476	.51516	.51556	.03
.04	.51595	.51635	.51675	.51715	.51755	.51795	.51835	.51874	.51914	.51954	.04
.05	.51994	.52034	.52074	.52113	.52153	.52193	.52233	.52273	.52313	.52352	.05
.06	.52392	.52432	.52472	.52512	.52552	.52591	.52631	.52671	.52711	.52751	.06
.07	.52790	.52830	.52870	.52910	.52950	.52989	.53029	.53060	.53109	.53148	.07
.08	.53188	.53228	.53268	.53307	.53347	.53387	.53427	.53466	.53506	.53546	.08
.09	.53586	.53625	.53665	.53705	.53745	.53784	.53824	.53864	.53903	.53943	.09
.10	.53983	.54023	.54062	.54102	.54142	.54181	.54221	.54261	.54300	.54340	.10
.11	.54380	.54419	.54459	.54498	.54538	.54579	.54617	.54657	.54696	.54736	.11
.12	.54776	.54815	.54855	.54895	.54934	.54974	.55013	.55053	.55093	.55132	.12
.13	.55172	.55211	.55251	.55290	.55330	.55369	.55409	.55448	.55488	.55529	.13
.14	.55567	.55607	.55646	.55686	.55725	.55764	.55804	.55843	.55883	.55922	.14
.15	.55962	.56001	.56041	.56080	.56120	.56159	.56198	.56238	.56277	.56317	.15
.16	.56356	.56395	.56435	.56474	.56513	.56553	.56592	.56632	.56671	.56710	.16
.17	.56750	.56789	.56828	.56867	.56907	.56946	.56985	.57025	.57064	.57103	.17
.18	.57142	.57182	.57221	.57260	.57299	.57339	.57378	.57417	.57456	.57495	.18
.19	.57535	.57574	.57613	.57652	.57691	.57730	.57770	.57809	.57848	.57887	.19
.20	.57926	.57965	.58004	.58043	.58082	.58121	.58160	.58200	.58239	.58273	.20
.21	.58317	.58356	.58396	.58434	.58473	.58512	.58551	.58590	.58629	.58668	.21
.22	.58706	.58745	.58784	.58823	.58862	.58901	.58940	.58979	.59018	.59057	.22
.23	.59095	.59134	.59173	.59212	.59251	.59290	.59329	.59367	.59406	.59445	.23
.24	.59484	.59522	.59561	.59600	.59638	.59677	.59716	.59755	.59793	.59832	.24
.25	.59871	.59909	.59948	.59987	.60025	.60064	.60102	.60141	.60180	.60218	.25
.26	.60257	.60295	.60334	.60372	.60411	.60450	.60488	.60527	.60565	.60604	.26
.27	.60642	.60680	.60719	.60757	.60796	.60834	.60873	.60911	.60949	.60988	.27
.28	.61026	.61065	.61103	.61141	.61180	.61218	.61256	.61294	.61333	.61371	.28
.29	.61409	.61447	.61486	.61524	.61562	.61600	.61639	.61677	.61715	.61753	.29
.30	.61791	.61829	.61867	.61905	.61944	.61982	.62020	.62058	.62096	.62134	.30
.31	.62172	.62210	.62248	.62286	.62324	.62362	.62400	.62438	.62476	.62514	.31
.32	.62552	.62590	.62627	.62665	.62703	.62741	.62779	.62817	.62854	.62892	.32
.33	.62930	.62968	.63006	.63043	.63081	.63119	.63156	.63194	.63232	.63270	.33
.34	.63307	.63345	.63382	.63420	.63458	.63495	.63533	.63570	.63608	.63646	.34
.35	.63683	.63721	.63758	.63796	.63833	.63871	.63908	.63945	.63983	.64020	.35
.36	.64058	.64095	.64132	.64170	.64207	.64244	.64282	.64319	.64356	.64394	.36
.37	.64431	.64468	.64505	.64543	.64580	.64617	.64654	.64691	.64729	.64766	.37
.38	.64803	.64840	.64877	.64914	.64951	.64988	.65025	.65062	.65099	.65136	.38
.39	.65173	.65210	.65247	.65284	.65321	.65358	.65395	.65432	.65469	.65505	.39
.40	.65542	.65579	.65616	.65653	.65690	.65726	.65763	.65800	.65836	.65873	.40
.41	.65910	.65946	.65983	.66020	.66056	.66093	.66130	.66166	.66203	.66239	.41
.42	.66276	.66312	.66349	.66385	.66122	.66458	.66495	.66531	.66567	.66604	.42
.43	.66640	.66677	.66713	.66749	.66786	.66822	.66858	.66894	.66931	.66967	.43
.44	.67003	.67039	.67076	.67112	.67148	.67184	.67220	.67256	.67292	.67328	.44
.45	.67385	.67401	.67437	.67473	.67509	.67545	.67581	.67616	.67652	.67688	.45
.46	.67724	.67760	.67796	.67832	.67868	.67903	.67939	.67975	.68011	.68047	.46
.47	.68082	.68118	.68154	.68189	.68225	.68261	.68296	.68332	.68368	.68403	.47
.48	.68439	.68474	.68510	.68545	.68581	.68616	.68652	.68687	.68723	.68758	.48
.49	.68793	.68829	.68864	.68900	.68935	.68970	.69006	.69041	.69076	.69111	.49
	0	1	2	3	4	5	6	7	8	9	

Source: Berkson (1955b:544-548)

Table A.3 (continued)
Antinormits

p	Thousandths										p
	0	1	2	3	4	5	6	7	8	9	
.50	.69146	.69181	.69217	.69252	.69287	.69322	.69357	.69392	.69427	.69462	.50
.51	.69497	.69532	.69567	.69602	.69637	.69672	.69707	.69742	.69777	.69812	.51
.52	.69847	.69882	.69917	.69951	.69986	.70021	.70056	.70090	.70125	.70160	.52
.53	.70194	.70229	.70264	.70298	.70333	.70368	.70402	.70437	.70471	.70506	.53
.54	.70540	.70575	.70609	.70644	.70678	.70712	.70747	.70781	.70815	.70850	.54
.55	.70884	.70918	.70953	.70987	.71021	.71055	.71089	.71124	.71158	.71192	.55
.56	.71225	.71260	.71294	.71328	.71362	.71396	.71430	.71464	.71498	.71532	.56
.57	.71566	.71600	.71634	.71668	.71702	.71735	.71769	.71803	.71837	.71871	.57
.58	.71904	.71938	.71972	.72006	.72039	.72073	.72106	.72140	.72173	.72207	.58
.59	.72240	.72274	.72307	.72341	.72374	.72408	.72441	.72475	.72508	.72541	.59
.60	.72575	.72608	.72641	.72675	.72708	.72741	.72774	.72807	.72841	.72874	.60
.61	.72907	.72940	.72973	.73006	.73039	.73072	.73105	.73138	.73171	.73204	.61
.62	.73237	.73270	.73303	.73336	.73369	.73401	.73434	.73467	.73500	.73533	.62
.63	.73565	.73598	.73631	.73663	.73696	.73729	.73761	.73794	.73825	.73859	.63
.64	.73891	.73924	.73956	.73989	.74021	.74054	.74086	.74118	.74151	.74183	.64
.65	.74215	.74248	.74280	.74312	.74344	.74377	.74409	.74441	.74473	.74505	.65
.66	.74537	.74569	.74601	.74633	.74666	.74698	.74729	.74761	.74793	.74825	.66
.67	.74857	.74889	.74921	.74953	.74984	.75016	.75048	.75080	.75111	.75143	.67
.68	.75175	.75206	.75239	.75270	.75301	.75333	.75364	.75396	.75427	.75459	.68
.69	.75490	.75522	.75553	.75585	.75616	.75647	.75679	.75710	.75741	.75772	.69
.70	.75804	.75835	.75866	.75897	.75928	.75960	.75991	.76022	.76053	.76084	.70
.71	.76115	.76146	.76177	.76208	.76239	.76270	.76300	.76331	.76362	.76393	.71
.72	.76421	.76455	.76485	.76516	.76547	.76577	.76608	.76639	.76669	.76700	.72
.73	.76731	.76761	.76792	.76822	.76853	.76883	.76913	.76944	.76974	.77005	.73
.74	.77035	.77065	.77096	.77125	.77156	.77186	.77217	.77247	.77277	.77307	.74
.75	.77337	.77367	.77397	.77428	.77458	.77488	.77518	.77548	.77577	.77607	.75
.76	.77637	.77667	.77697	.77727	.77757	.77786	.77816	.77846	.77876	.77905	.76
.77	.77935	.77965	.77994	.78024	.78053	.78083	.78113	.78142	.78172	.78201	.77
.78	.78230	.78260	.78289	.78319	.78348	.78377	.78407	.78436	.78465	.78494	.78
.79	.78524	.78553	.78582	.78611	.78640	.78669	.78698	.78727	.78757	.78786	.79
.80	.78814	.78843	.78872	.78901	.78930	.78959	.78988	.79017	.79045	.79074	.80
.81	.79103	.79132	.79160	.79189	.79218	.79246	.79275	.79304	.79332	.79361	.81
.82	.79389	.79418	.79446	.79475	.79503	.79531	.79560	.79588	.79617	.79646	.82
.83	.79673	.79701	.79730	.79758	.79786	.79814	.79842	.79870	.79898	.79927	.83
.84	.79955	.79983	.80011	.80039	.80067	.80094	.80122	.80150	.80178	.80206	.84
.85	.80234	.80262	.80289	.80317	.80345	.80372	.80400	.80428	.80455	.80483	.85
.86	.80511	.80538	.80566	.80593	.80621	.80648	.80676	.80703	.80730	.80758	.86
.87	.80785	.80812	.80840	.80867	.80894	.80921	.80949	.80976	.81003	.81030	.87
.88	.81057	.81084	.81111	.81138	.81165	.81192	.81219	.81218	.81273	.81300	.88
.89	.81327	.81354	.81380	.81407	.81434	.81461	.81487	.81514	.81541	.81567	.89
.90	.81594	.81621	.81647	.81674	.81700	.81727	.81753	.81780	.81806	.81833	.90
.91	.81859	.81885	.81912	.81938	.81964	.81990	.82017	.82043	.82069	.82095	.91
.92	.82121	.82148	.82174	.82200	.82226	.82252	.82278	.82304	.82330	.82356	.92
.93	.82381	.82407	.82433	.82159	.82485	.82511	.82536	.82562	.82588	.82613	.93
.94	.82639	.82665	.82690	.82716	.82742	.82767	.82793	.82819	.82844	.82869	.94
.95	.82894	.82920	.82945	.82971	.82996	.83021	.83046	.83072	.83097	.83122	.95
.96	.83147	.83172	.83198	.83223	.83248	.83273	.83298	.83323	.83348	.83373	.96
.97	.83398	.83423	.83447	.83472	.83497	.83522	.83547	.83572	.83596	.83621	.97
.98	.83646	.83670	.83695	.83720	.83744	.83769	.83793	.83818	.83842	.83867	.98
.99	.83891	.83916	.83940	.83965	.83989	.84013	.84038	.84062	.84086	.84110	.99
	0	1	2	3	4	5	6	7	8	9	

Table A.3 (continued)
Antinormits

p	Thousandths										p
	0	1	2	3	4	5	6	7	8	9	
1.00	.34134	.34159	.34183	.34207	.34231	.34255	.34279	.34303	.34327	.34351	1.00
1.01	.34375	.34399	.34423	.34447	.34471	.34495	.34519	.34542	.34566	.34590	1.01
1.02	.34614	.34637	.34661	.34685	.34708	.34732	.34755	.34779	.34802	.34826	1.02
1.03	.34850	.34873	.34896	.34920	.34943	.34967	.34990	.35013	.35037	.35060	1.03
1.04	.35083	.35106	.35129	.35153	.35176	.35199	.35222	.35245	.35268	.35291	1.04
1.05	.35314	.35337	.35360	.35383	.35406	.35429	.35452	.35474	.35497	.35520	1.05
1.06	.35543	.35566	.35588	.35611	.35634	.35656	.35679	.35701	.35724	.35747	1.06
1.07	.35769	.35792	.35814	.35836	.35859	.35881	.35904	.35926	.35948	.35971	1.07
1.08	.35993	.36015	.36037	.36060	.36082	.36104	.36126	.36148	.36170	.36192	1.08
1.09	.36214	.36236	.36258	.36280	.36302	.36324	.36346	.36368	.36390	.36412	1.09
1.10	.36433	.36455	.36477	.36499	.36520	.36542	.36564	.36585	.36607	.36629	1.10
1.11	.36650	.36672	.36693	.36715	.36736	.36758	.36779	.36800	.36822	.36843	1.11
1.12	.36864	.36886	.36907	.36928	.36949	.36971	.36992	.37013	.37034	.37055	1.12
1.13	.37076	.37097	.37118	.37139	.37160	.37181	.37202	.37223	.37244	.37265	1.13
1.14	.37286	.37307	.37327	.37348	.37369	.37390	.37410	.37431	.37452	.37472	1.14
1.15	.37493	.37513	.37534	.37555	.37575	.37596	.37616	.37636	.37657	.37677	1.15
1.16	.37698	.37718	.37738	.37759	.37779	.37799	.37819	.37840	.37860	.37880	1.16
1.17	.37900	.37920	.37940	.37960	.37980	.38000	.38020	.38040	.38060	.38080	1.17
1.18	.38100	.38120	.38140	.38160	.38179	.38199	.38219	.38239	.38258	.38278	1.18
1.19	.38298	.38317	.38337	.38357	.38376	.38396	.38415	.38435	.38454	.38474	1.19
1.20	.38493	.38512	.38532	.38551	.38571	.38590	.38609	.38628	.38648	.38667	1.20
1.21	.38686	.38705	.38724	.38744	.38763	.38782	.38801	.38820	.38839	.38858	1.21
1.22	.38877	.38896	.38915	.38934	.38952	.38971	.38990	.39009	.39028	.39046	1.22
1.23	.39065	.39084	.39103	.39121	.39140	.39158	.39177	.39196	.39214	.39233	1.23
1.24	.39251	.39270	.39288	.39307	.39325	.39343	.39362	.39380	.39398	.39417	1.24
1.25	.39435	.39453	.39472	.39490	.39508	.39526	.39544	.39562	.39580	.39599	1.25
1.26	.39617	.39635	.39653	.39671	.39689	.39706	.39724	.39742	.39760	.39778	1.26
1.27	.39796	.39814	.39831	.39849	.39867	.39885	.39902	.39920	.39938	.39955	1.27
1.28	.39973	.39990	.40008	.40025	.40043	.40060	.40078	.40095	.40113	.40130	1.28
1.29	.40147	.40165	.40182	.40199	.40217	.40234	.40251	.40268	.40286	.40303	1.29
1.30	.40320	.40337	.40354	.40371	.40388	.40405	.40422	.40439	.40456	.40473	1.30
1.31	.40490	.40507	.40521	.40541	.40558	.40575	.40591	.40608	.40625	.40642	1.31
1.32	.40658	.40675	.40692	.40709	.40725	.40741	.40758	.40775	.40791	.40808	1.32
1.33	.40824	.40841	.40857	.40873	.40890	.40906	.40921	.40939	.40955	.40971	1.33
1.34	.40988	.41004	.41020	.41036	.41052	.41069	.41085	.41101	.41117	.41133	1.34
1.35	.41149	.41165	.41181	.41197	.41213	.41229	.41245	.41261	.41277	.41293	1.35
1.36	.41309	.41324	.41340	.41356	.41372	.41387	.41403	.41419	.41434	.41450	1.36
1.37	.41466	.41481	.41497	.41512	.41528	.41543	.41559	.41574	.41590	.41605	1.37
1.38	.41621	.41636	.41651	.41667	.41682	.41697	.41713	.41728	.41743	.41758	1.38
1.39	.41774	.41789	.41804	.41819	.41834	.41849	.41864	.41879	.41894	.41909	1.39
1.40	.41924	.41939	.41954	.41969	.41984	.41999	.42014	.42029	.42043	.42058	1.40
1.41	.42073	.42088	.42103	.42117	.42132	.42147	.42161	.42176	.42190	.42205	1.41
1.42	.42220	.42234	.42249	.42263	.42278	.42292	.42307	.42321	.42335	.42350	1.42
1.43	.42364	.42379	.42393	.42407	.42421	.42436	.42450	.42464	.42478	.42492	1.43
1.44	.42507	.42521	.42535	.42549	.42563	.42577	.42591	.42605	.42619	.42633	1.44
1.45	.42647	.42661	.42675	.42689	.42703	.42717	.42730	.42744	.42758	.42772	1.45
1.46	.42786	.42799	.42813	.42827	.42840	.42854	.42868	.42881	.42895	.42908	1.46
1.47	.42922	.42935	.42949	.42962	.42975	.42989	.43003	.43016	.43030	.43043	1.47
1.48	.43056	.43070	.43083	.43096	.43110	.43123	.43136	.43149	.43162	.43176	1.48
1.49	.43189	.43202	.43215	.43229	.43241	.43254	.43267	.43280	.43293	.43306	1.49
	0	1	2	3	4	5	6	7	8	9	

Table A.3 (continued)
Antinormits

z	Thousandths										z
	0	1	2	3	4	5	6	7	8	9	
1.50	.93319	.93332	.93345	.93358	.93371	.93384	.93397	.93409	.93422	.93435	1.50
1.51	.93448	.93461	.93473	.93486	.93499	.93511	.93524	.93537	.93549	.93562	1.51
1.52	.93574	.93587	.93600	.93612	.93625	.93637	.93650	.93662	.93674	.93687	1.52
1.53	.93699	.93712	.93724	.93736	.93749	.93761	.93773	.93785	.93798	.93810	1.53
1.54	.93822	.93834	.93846	.93858	.93871	.93883	.93895	.93907	.93919	.93931	1.54
1.55	.93943	.93955	.93967	.93979	.93991	.94003	.94015	.94027	.94038	.94050	1.55
1.56	.94062	.94074	.94086	.94097	.94109	.94121	.94133	.94144	.94156	.94168	1.56
1.57	.94179	.94191	.94202	.94214	.94225	.94237	.94249	.94260	.94272	.94283	1.57
1.58	.94295	.94306	.94318	.94329	.94340	.94352	.94363	.94374	.94386	.94397	1.58
1.59	.94408	.94420	.94431	.94442	.94453	.94464	.94476	.94487	.94498	.94509	1.59
1.60	.94520	.94531	.94542	.94553	.94561	.94573	.94586	.94597	.94608	.94619	1.60
1.61	.94630	.94641	.94652	.94663	.94674	.94684	.94695	.94706	.94717	.94728	1.61
1.62	.94738	.94749	.94760	.94771	.94781	.94792	.94803	.94813	.94824	.94834	1.62
1.63	.94845	.94856	.94866	.94877	.94887	.94898	.94908	.94919	.94929	.94939	1.63
1.64	.94950	.94960	.94971	.94981	.94991	.95002	.95012	.95022	.95032	.95043	1.64
1.65	.95053	.95063	.95073	.95083	.95094	.95104	.95114	.95124	.95134	.95141	1.65
1.66	.95154	.95164	.95174	.95184	.95194	.95204	.95214	.95221	.95231	.95214	1.66
1.67	.95254	.95264	.95274	.95284	.95293	.95303	.95313	.95323	.95333	.95342	1.67
1.68	.95352	.95362	.95372	.95381	.95391	.95401	.95410	.95420	.95429	.95439	1.68
1.69	.95449	.95458	.95468	.95477	.95487	.95496	.95506	.95515	.95525	.95534	1.69
1.70	.95543	.95553	.95562	.95572	.95581	.95590	.95600	.95609	.95618	.95627	1.70
1.71	.95637	.95646	.95655	.95664	.95674	.95683	.95692	.95701	.95710	.95719	1.71
1.72	.95728	.95737	.95747	.95756	.95765	.95774	.95783	.95792	.95801	.95810	1.72
1.73	.95819	.95827	.95836	.95845	.95854	.95863	.95872	.95881	.95889	.95898	1.73
1.74	.95907	.95916	.95925	.95933	.95942	.95951	.95959	.95968	.95977	.95985	1.74
1.75	.95994	.96002	.96011	.96020	.96028	.96037	.96046	.96054	.96063	.96071	1.75
1.76	.96080	.96088	.96097	.96105	.96113	.96122	.96130	.96139	.96147	.96155	1.76
1.77	.96164	.96172	.96180	.96189	.96197	.96205	.96213	.96222	.96230	.96238	1.77
1.78	.96246	.96254	.96263	.96271	.96279	.96287	.96295	.96303	.96311	.96319	1.78
1.79	.96327	.96335	.96343	.96351	.96359	.96367	.96375	.96383	.96391	.96399	1.79
1.80	.96407	.96415	.96423	.96431	.96438	.96446	.96454	.96462	.96470	.96477	1.80
1.81	.96485	.96493	.96501	.96508	.96516	.96524	.96532	.96539	.96547	.96554	1.81
1.82	.96562	.96570	.96577	.96585	.96592	.96600	.96606	.96615	.96623	.96630	1.82
1.83	.96638	.96645	.96652	.96660	.96667	.96675	.96682	.96690	.96697	.96704	1.83
1.84	.96712	.96719	.96725	.96734	.96741	.96748	.96755	.96763	.96770	.96777	1.84
1.85	.96784	.96792	.96799	.96806	.96813	.96820	.96827	.96834	.96842	.96849	1.85
1.86	.96856	.96863	.96870	.96877	.96884	.96891	.96898	.96905	.96912	.96919	1.86
1.87	.96925	.96933	.96940	.96947	.96954	.96960	.96967	.96974	.96981	.96988	1.87
1.88	.96995	.97001	.97008	.97015	.97022	.97029	.97035	.97042	.97049	.97055	1.88
1.89	.97062	.97069	.97075	.97082	.97089	.97095	.97102	.97109	.97115	.97122	1.89
1.90	.97128	.97135	.97141	.97148	.97155	.97161	.97168	.97174	.97180	.97187	1.90
1.91	.97193	.97200	.97206	.97213	.97219	.97225	.97232	.97238	.97244	.97251	1.91
1.92	.97257	.97263	.97270	.97276	.97282	.97289	.97295	.97301	.97307	.97313	1.92
1.93	.97320	.97326	.97332	.97338	.97344	.97351	.97357	.97363	.97369	.97375	1.93
1.94	.97381	.97387	.97393	.97399	.97405	.97411	.97417	.97423	.97429	.97435	1.94
1.95	.97441	.97447	.97453	.97459	.97465	.97471	.97477	.97483	.97489	.97494	1.95
1.96	.97500	.97506	.97512	.97518	.97524	.97529	.97535	.97541	.97547	.97552	1.96
1.97	.97558	.97564	.97570	.97573	.97581	.97587	.97592	.97598	.97604	.97609	1.97
1.98	.97615	.97620	.97626	.97632	.97637	.97643	.97648	.97654	.97659	.97665	1.98
1.99	.97670	.97676	.97681	.97687	.97692	.97698	.97703	.97709	.97714	.97720	1.99
	0	1	2	3	4	5	6	7	8	9	

Table A.3 (continued)
Antinormits

p	Thousandths										p
	0	1	2	3	4	5	6	7	8	9	
2.00	.97725	.97730	.97736	.97741	.97746	.97752	.97757	.97763	.97768	.97773	2.00
2.01	.97778	.97784	.97789	.97794	.97800	.97805	.97810	.97815	.97820	.97826	2.01
2.02	.97831	.97836	.97841	.97846	.97851	.97857	.97862	.97867	.97872	.97877	2.02
2.03	.97882	.97887	.97892	.97897	.97902	.97907	.97912	.97917	.97923	.97927	2.03
2.04	.97932	.97937	.97942	.97947	.97952	.97957	.97962	.97967	.97972	.97977	2.04
2.05	.97982	.97987	.97992	.97996	.98001	.98006	.98011	.98016	.98020	.98025	2.05
2.06	.98031	.98035	.98040	.98044	.98049	.98054	.98059	.98063	.98068	.98073	2.06
2.07	.98077	.98082	.98087	.98091	.98096	.98101	.98106	.98110	.98115	.98119	2.07
2.08	.98124	.98128	.98133	.98137	.98142	.98147	.98151	.98156	.98160	.98165	2.08
2.09	.98169	.98174	.98178	.98183	.98187	.98191	.98196	.98200	.98205	.98209	2.09
2.10	.98214	.98218	.98222	.98227	.98231	.98235	.98240	.98244	.98248	.98253	2.10
2.11	.98257	.98261	.98266	.98270	.98274	.98279	.98283	.98287	.98291	.98295	2.11
2.12	.98300	.98304	.98308	.98312	.98316	.98321	.98325	.98329	.98333	.98337	2.12
2.13	.98341	.98346	.98350	.98354	.98358	.98362	.98366	.98370	.98374	.98378	2.13
2.14	.98382	.98386	.98390	.98394	.98398	.98402	.98406	.98410	.98414	.98418	2.14
2.15	.98422	.98426	.98430	.98434	.98438	.98442	.98446	.98450	.98454	.98457	2.15
2.16	.98461	.98465	.98469	.98473	.98477	.98481	.98484	.98488	.98492	.98496	2.16
2.17	.98500	.98503	.98507	.98511	.98515	.98518	.98522	.98526	.98530	.98533	2.17
2.18	.98537	.98541	.98545	.98548	.98552	.98556	.98560	.98563	.98567	.98570	2.18
2.19	.98574	.98577	.98581	.98585	.98588	.98592	.98595	.98599	.98603	.98606	2.19
2.20	.98610	.98613	.98617	.98620	.98624	.98627	.98631	.98634	.98638	.98641	2.20
2.21	.98645	.98648	.98652	.98655	.98659	.98662	.98665	.98669	.98672	.98676	2.21
2.22	.98679	.98682	.98686	.98689	.98693	.98696	.98699	.98703	.98706	.98709	2.22
2.23	.98713	.98716	.98719	.98723	.98726	.98729	.98732	.98736	.98739	.98742	2.23
2.24	.98745	.98749	.98752	.98755	.98758	.98762	.98765	.98768	.98771	.98774	2.24
2.25	.98778	.98781	.98784	.98787	.98790	.98793	.98796	.98800	.98803	.98806	2.25
2.26	.98809	.98812	.98815	.98818	.98821	.98824	.98827	.98830	.98834	.98837	2.26
2.27	.98840	.98843	.98846	.98849	.98852	.98855	.98858	.98861	.98864	.98867	2.27
2.28	.98870	.98873	.98876	.98878	.98881	.98884	.98887	.98890	.98893	.98896	2.28
2.29	.98899	.98902	.98905	.98908	.98910	.98913	.98916	.98919	.98922	.98925	2.29
2.30	.98928	.98930	.98933	.98936	.98939	.98942	.98944	.98947	.98950	.98953	2.30
2.31	.98956	.98958	.98961	.98964	.98967	.98969	.98972	.98975	.98978	.98980	2.31
2.32	.98983	.98986	.98988	.98991	.98994	.98996	.98999	.99002	.99004	.99007	2.32
2.33	.99010	.99012	.99015	.99018	.99020	.99023	.99025	.99028	.99031	.99033	2.33
2.34	.99036	.99038	.99041	.99044	.99046	.99049	.99051	.99054	.99056	.99059	2.34
2.35	.99061	.99064	.99066	.99069	.99071	.99074	.99076	.99079	.99081	.99084	2.35
2.36	.99086	.99089	.99091	.99094	.99096	.99098	.99101	.99103	.99106	.99108	2.36
2.37	.99110	.99113	.99115	.99118	.99120	.99122	.99125	.99127	.99130	.99132	2.37
2.38	.99134	.99137	.99139	.99141	.99144	.99146	.99148	.99151	.99153	.99155	2.38
2.39	.99158	.99160	.99162	.99164	.99167	.99169	.99171	.99174	.99176	.99178	2.39
2.40	.99180	.99182	.99185	.99187	.99189	.99191	.99194	.99196	.99198	.99200	2.40
2.41	.99202	.99204	.99207	.99209	.99211	.99213	.99215	.99218	.99220	.99222	2.41
2.42	.99224	.99226	.99229	.99230	.99232	.99234	.99237	.99239	.99241	.99243	2.42
2.43	.99245	.99247	.99249	.99251	.99253	.99256	.99257	.99260	.99262	.99264	2.43
2.44	.99266	.99268	.99270	.99272	.99274	.99276	.99278	.99280	.99282	.99284	2.44
2.45	.99286	.99288	.99290	.99292	.99294	.99296	.99298	.99299	.99301	.99303	2.45
2.46	.99305	.99307	.99309	.99311	.99313	.99315	.99317	.99319	.99321	.99322	2.46
2.47	.99324	.99326	.99328	.99330	.99332	.99334	.99336	.99338	.99339	.99341	2.47
2.48	.99343	.99346	.99347	.99349	.99350	.99352	.99354	.99356	.99358	.99359	2.48
2.49	.99361	.99363	.99365	.99367	.99368	.99370	.99372	.99374	.99376	.99377	2.49
	0	1	2	3	4	5	6	7	8	9	

Appendix B

LOGIT ANALYSIS TABLES

Table B.1
Logits (l)

For p less than .50 on left, logit is negative.
For p greater than .50 on right, logit is positive.

p	Thousandths, for p in left column											p
	0	1	2	3	4	5	6	7	8	9		
.00	—	6.90675	6.21261	5.80614	5.51745	5.29330	5.10998	4.95482	4.82028	4.70149	4.59512	.99
.01	4.59512	4.49880	4.41073	4.32972	4.25460	4.18459	4.11904	4.05740	3.99922	3.94413	3.89182	.98
.02	3.89182	3.84201	3.79447	3.74899	3.70541	3.66356	3.62331	3.58453	3.54715	3.51103	3.47610	.97
.03	3.47610	3.44228	3.40950	3.37769	3.34680	3.31673	3.28737	3.25914	3.23143	3.20441	3.17805	.96
.04	3.17805	3.15232	3.12718	3.10260	3.07857	3.05505	3.03202	3.00947	2.98736	2.96569	2.94444	.95
.05	2.94444	2.92358	2.90311	2.88301	2.86326	2.84385	2.82477	2.80601	2.78756	2.76941	2.75154	.94
.06	2.75154	2.73394	2.71662	2.69955	2.68273	2.66616	2.64982	2.63371	2.61783	2.60215	2.58669	.93
.07	2.58669	2.57143	2.55637	2.54149	2.52681	2.51231	2.49798	2.48382	2.46984	2.45601	2.44235	.92
.08	2.44235	2.42884	2.41548	2.40227	2.38920	2.37627	2.36348	2.35083	2.33830	2.32591	2.31363	.91
.09	2.31363	2.30149	2.28946	2.27754	2.26574	2.25406	2.24248	2.23101	2.21965	2.20839	2.19722	.90
.10	2.19722	2.18616	2.17520	2.16433	2.15355	2.14286	2.13227	2.12176	2.11133	2.10100	2.09074	.89
.11	2.09074	2.08057	2.07047	2.06046	2.05052	2.04066	2.03087	2.02115	2.01151	2.00193	1.99243	.88
.12	1.99243	1.98299	1.97363	1.96432	1.95508	1.94591	1.93680	1.92775	1.91876	1.90983	1.90096	.87
.13	1.90096	1.89215	1.88339	1.87469	1.86605	1.85745	1.84892	1.84043	1.83200	1.82362	1.81529	.86
.14	1.81529	1.80701	1.79878	1.79059	1.78246	1.77437	1.76632	1.75833	1.75037	1.74247	1.73460	.85
.15	1.73460	1.72678	1.71900	1.71125	1.70357	1.69591	1.68830	1.68072	1.67318	1.66569	1.65823	.84
.16	1.65823	1.65081	1.64342	1.63607	1.62876	1.62149	1.61425	1.60704	1.59987	1.59273	1.58563	.83
.17	1.58563	1.57856	1.57152	1.56451	1.55754	1.55060	1.54369	1.53681	1.52996	1.52314	1.51635	.82
.18	1.51635	1.50959	1.50286	1.49615	1.48948	1.48283	1.47621	1.46962	1.46306	1.45652	1.45001	.81
.19	1.45001	1.44353	1.43707	1.43063	1.42423	1.41784	1.41148	1.40515	1.39884	1.39256	1.38629	.80
.20	1.38629	1.38006	1.37384	1.36765	1.36148	1.35533	1.34921	1.34310	1.33702	1.33096	1.32493	.79
.21	1.32193	1.31891	1.31291	1.30694	1.30098	1.29505	1.28913	1.28324	1.27736	1.27150	1.26567	.78
.22	1.26567	1.25985	1.25405	1.24827	1.24251	1.23678	1.23104	1.22533	1.21964	1.21397	1.20831	.77
.23	1.20831	1.20267	1.19705	1.19145	1.18586	1.18029	1.17474	1.16920	1.16368	1.15817	1.15268	.76
.24	1.15268	1.14720	1.14175	1.13630	1.13087	1.12546	1.12006	1.11468	1.10931	1.10396	1.09861	.75
.25	1.09861	1.09329	1.08797	1.08268	1.07739	1.07212	1.06686	1.06162	1.05639	1.05117	1.04597	.74
.26	1.04597	1.04078	1.03560	1.03043	1.02528	1.02014	1.01501	1.00990	1.00479	0.99970	0.99462	.73
.27	0.99462	0.98955	0.98450	0.97945	0.97442	0.96940	0.96439	0.95939	0.95440	0.94943	0.94446	.72
.28	0.94446	0.93951	0.93456	0.92963	0.92471	0.91979	0.91489	0.91000	0.90512	0.90025	0.89538	.71
.29	0.89538	0.89053	0.88569	0.88086	0.87604	0.87122	0.86642	0.86162	0.85684	0.85206	0.84730	.70
.30	0.84730	0.84254	0.83779	0.83305	0.82832	0.82360	0.81889	0.81418	0.80949	0.80480	0.80012	.69
.31	0.80012	0.79545	0.79079	0.78613	0.78148	0.77685	0.77222	0.76759	0.76298	0.75837	0.75377	.68
.32	0.75377	0.74918	0.74460	0.74002	0.73545	0.73089	0.72633	0.72179	0.71724	0.71271	0.70819	.67
.33	0.70819	0.70367	0.69915	0.69465	0.69015	0.68566	0.68117	0.67669	0.67222	0.66775	0.66329	.66
.34	0.66329	0.65884	0.65439	0.64995	0.64552	0.64109	0.63667	0.63225	0.62784	0.62344	0.61904	.65
.35	0.61904	0.61465	0.61026	0.60588	0.60150	0.59713	0.59277	0.58841	0.58406	0.57971	0.57536	.64
.36	0.57536	0.57103	0.56669	0.56237	0.55804	0.55373	0.54942	0.54511	0.54081	0.53651	0.53222	.63
.37	0.53222	0.52793	0.52365	0.51937	0.51509	0.51083	0.50656	0.50230	0.49805	0.49379	0.48955	.62
.38	0.48955	0.48531	0.48107	0.47683	0.47260	0.46838	0.46416	0.45994	0.45573	0.45152	0.44731	.61
.39	0.44731	0.44311	0.43891	0.43472	0.43053	0.42634	0.42216	0.41798	0.41381	0.40963	0.40547	.60
.40	0.40547	0.40130	0.39714	0.39298	0.38883	0.38467	0.38053	0.37638	0.37224	0.36810	0.36397	.59
.41	0.36397	0.35983	0.35570	0.35158	0.34745	0.34333	0.33922	0.33510	0.33099	0.32688	0.32277	.58
.42	0.32277	0.31867	0.31457	0.31047	0.30637	0.30228	0.29819	0.29410	0.29002	0.28593	0.28185	.57
.43	0.28185	0.27777	0.27370	0.26962	0.26555	0.26148	0.25741	0.25333	0.24928	0.24522	0.24116	.56
.44	0.24116	0.23710	0.23305	0.22900	0.22494	0.22089	0.21685	0.21280	0.20875	0.20471	0.20067	.55
.45	0.20067	0.19663	0.19259	0.18856	0.18452	0.18049	0.17646	0.17243	0.16840	0.16437	0.16034	.54
.46	0.16034	0.15632	0.15229	0.14827	0.14425	0.14023	0.13621	0.13219	0.12818	0.12416	0.12014	.53
.47	0.12014	0.11613	0.11212	0.10811	0.10409	0.10008	0.09607	0.09206	0.08805	0.08405	0.08004	.52
.48	0.08004	0.07504	0.07203	0.06803	0.06402	0.06002	0.05601	0.05201	0.04801	0.04401	0.04001	.51
.49	0.04001	0.03600	0.03200	0.02800	0.02400	0.02000	0.01600	0.01200	0.00800	0.00400	0.00000	.50
		9	8	7	6	5	4	3	2	1	0	
		Thousandths, for p in right column										p

Source: Berkson (1953:568-569)

Table B.2
Logistic Weights

Upper figure is $w^*=pq$; lower figure is $w^*=pql$.
For p less than .50 on left, w^*l is negative.
For p greater than .50 on right, w^*l is positive.

p	Thousandths, for p on left											p
	0	1	2	3	4	5	6	7	8	9		
.00	.0000	.0010	.0020	.0030	.0040	.0050	.0060	.0070	.0079	.0089	.0099	.99
	—	.0069	.0124	.0174	.0220	.0263	.0305	.0344	.0383	.0419	.0455	
.01	.0099	.0109	.0119	.0128	.0138	.0148	.0157	.0167	.0177	.0186	.0196	.98
	.0455	.0489	.0523	.0556	.0587	.0618	.0649	.0678	.0707	.0735	.0763	
.02	.0196	.0206	.0215	.0225	.0234	.0244	.0253	.0263	.0272	.0282	.0291	.97
	.0763	.0790	.0816	.0842	.0868	.0893	.0918	.0942	.0966	.0989	.1012	
.03	.0291	.0300	.0310	.0319	.0329	.0338	.0347	.0356	.0366	.0375	.0384	.96
	.1012	.1034	.1056	.1078	.1099	.1120	.1141	.1161	.1181	.1201	.1220	
.04	.0384	.0393	.0402	.0412	.0421	.0430	.0439	.0448	.0457	.0466	.0475	.95
	.1220	.1239	.1258	.1277	.1295	.1313	.1331	.1348	.1365	.1382	.1399	
.05	.0475	.0484	.0493	.0502	.0511	.0520	.0529	.0538	.0546	.0555	.0564	.94
	.1399	.1415	.1431	.1447	.1463	.1478	.1493	.1508	.1523	.1538	.1552	
.06	.0564	.0573	.0582	.0590	.0599	.0608	.0616	.0625	.0634	.0642	.0651	.93
	.1552	.1566	.1580	.1594	.1607	.1620	.1633	.1646	.1659	.1672	.1684	
.07	.0651	.0660	.0668	.0677	.0685	.0694	.0702	.0711	.0719	.0728	.0736	.92
	.1684	.1696	.1708	.1720	.1731	.1743	.1754	.1765	.1778	.1787	.1798	
.08	.0736	.0744	.0753	.0761	.0769	.0778	.0786	.0794	.0803	.0811	.0819	.91
	.1798	.1808	.1818	.1828	.1838	.1848	.1858	.1867	.1877	.1886	.1896	
.09	.0819	.0827	.0835	.0844	.0852	.0860	.0868	.0876	.0884	.0892	.0900	.90
	.1896	.1904	.1913	.1921	.1930	.1938	.1946	.1954	.1962	.1970	.1977	
.10	.0900	.0908	.0916	.0924	.0932	.0940	.0948	.0956	.0963	.0971	.0979	.89
	.1977	.1985	.1992	.2000	.2007	.2014	.2021	.2027	.2034	.2040	.2047	
.11	.0979	.0987	.0995	.1002	.1010	.1018	.1025	.1033	.1041	.1048	.1056	.88
	.2047	.2053	.2059	.2065	.2071	.2077	.2083	.2088	.2093	.2099	.2104	
.12	.1056	.1064	.1071	.1079	.1086	.1094	.1101	.1109	.1116	.1124	.1131	.87
	.2104	.2109	.2114	.2119	.2124	.2128	.2133	.2137	.2142	.2146	.2150	
.13	.1131	.1138	.1146	.1153	.1160	.1168	.1173	.1182	.1190	.1197	.1204	.86
	.2150	.2154	.2158	.2162	.2165	.2169	.2173	.2176	.2179	.2182	.2186	
.14	.1204	.1211	.1218	.1225	.1233	.1240	.1247	.1254	.1261	.1268	.1275	.85
	.2186	.2189	.2192	.2194	.2197	.2200	.2202	.2205	.2207	.2209	.2212	
.15	.1275	.1282	.1289	.1296	.1303	.1310	.1317	.1324	.1330	.1337	.1344	.84
	.2212	.2214	.2216	.2218	.2219	.2221	.2223	.2224	.2226	.2227	.2229	
.16	.1344	.1351	.1358	.1364	.1371	.1378	.1384	.1391	.1398	.1404	.1411	.83
	.2229	.2230	.2231	.2232	.2233	.2234	.2235	.2236	.2236	.2237	.2237	
.17	.1411	.1418	.1424	.1431	.1437	.1444	.1450	.1457	.1463	.1470	.1476	.82
	.2237	.2238	.2238	.2238	.2239	.2239	.2239	.2239	.2239	.2238	.2238	
.18	.1476	.1482	.1489	.1495	.1501	.1508	.1514	.1520	.1527	.1533	.1539	.81
	.2238	.2238	.2237	.2237	.2236	.2236	.2235	.2234	.2233	.2233	.2232	
.19	.1539	.1545	.1551	.1558	.1564	.1570	.1576	.1582	.1588	.1594	.1600	.80
	.2232	.2231	.2229	.2228	.2227	.2226	.2224	.2223	.2221	.2220	.2218	
.20	.1600	.1606	.1612	.1618	.1624	.1630	.1636	.1642	.1647	.1653	.1659	.79
	.2218	.2216	.2215	.2213	.2211	.2209	.2207	.2205	.2203	.2200	.2198	
.21	.1659	.1665	.1671	.1676	.1682	.1688	.1693	.1699	.1705	.1710	.1716	.78
	.2198	.2196	.2193	.2191	.2188	.2186	.2183	.2180	.2178	.2175	.2172	
.22	.1716	.1722	.1727	.1733	.1738	.1744	.1749	.1754	.1760	.1766	.1771	.77
	.2172	.2169	.2166	.2163	.2160	.2157	.2153	.2150	.2147	.2143	.2140	
.23	.1771	.1776	.1782	.1787	.1792	.1798	.1803	.1808	.1814	.1819	.1824	.76
	.2140	.2136	.2133	.2129	.2126	.2122	.2118	.2114	.2110	.2106	.2102	
.24	.1824	.1829	.1834	.1840	.1845	.1850	.1855	.1860	.1865	.1870	.1875	.75
	.2102	.2098	.2094	.2090	.2086	.2082	.2078	.2073	.2069	.2064	.2060	
		9	8	7	6	5	4	3	2	1	0	
		Thousandths, for p on right										p

Source: Berkson (1953:571-572)

Table B.2 (continued)
Logistic Weights

p	Thousandths, for p on left											p
	0	1	2	3	4	5	6	7	8	9		
.25	.1875	.1880	.1885	.1890	.1895	.1900	.1905	.1910	.1914	.1919	.1924	.74
	.2060	.2055	.2051	.2046	.2041	.2037	.2032	.2027	.2022	.2017	.2012	
.26	.1924	.1929	.1934	.1938	.1943	.1948	.1952	.1957	.1962	.1966	.1971	.73
	.2012	.2007	.2002	.1997	.1992	.1987	.1982	.1978	.1971	.1966	.1960	
.27	.1971	.1976	.1980	.1985	.1989	.1994	.1998	.2003	.2007	.2012	.2016	.72
	.1960	.1955	.1949	.1944	.1938	.1933	.1927	.1921	.1916	.1910	.1904	
.28	.2016	.2020	.2025	.2029	.2033	.2038	.2042	.2046	.2051	.2055	.2059	.71
	.1904	.1898	.1892	.1886	.1880	.1874	.1868	.1862	.1856	.1850	.1844	
.29	.2059	.2063	.2067	.2072	.2076	.2080	.2084	.2088	.2092	.2096	.2100	.70
	.1844	.1837	.1831	.1825	.1818	.1812	.1806	.1799	.1792	.1786	.1779	
.30	.2100	.2104	.2108	.2112	.2116	.2120	.2124	.2128	.2131	.2135	.2139	.69
	.1779	.1773	.1766	.1759	.1753	.1746	.1739	.1732	.1725	.1719	.1711	
.31	.2139	.2143	.2147	.2150	.2154	.2158	.2161	.2165	.2169	.2172	.2176	.68
	.1711	.1704	.1697	.1690	.1683	.1676	.1669	.1662	.1655	.1647	.1640	
.32	.2176	.2180	.2183	.2187	.2190	.2194	.2197	.2201	.2204	.2208	.2211	.67
	.1640	.1633	.1625	.1618	.1611	.1603	.1596	.1588	.1581	.1573	.1566	
.33	.2211	.2214	.2218	.2221	.2224	.2228	.2231	.2234	.2238	.2241	.2244	.66
	.1566	.1558	.1551	.1543	.1535	.1527	.1520	.1512	.1504	.1496	.1488	
.34	.2244	.2247	.2250	.2254	.2257	.2260	.2263	.2266	.2269	.2272	.2275	.65
	.1488	.1481	.1473	.1465	.1457	.1449	.1441	.1433	.1425	.1416	.1408	
.35	.2275	.2278	.2281	.2284	.2287	.2290	.2293	.2296	.2298	.2301	.2304	.64
	.1408	.1400	.1392	.1384	.1376	.1367	.1359	.1351	.1342	.1334	.1326	
.36	.2304	.2307	.2310	.2312	.2315	.2318	.2320	.2323	.2326	.2328	.2331	.63
	.1326	.1317	.1309	.1300	.1292	.1283	.1275	.1266	.1258	.1249	.1241	
.37	.2331	.2334	.2336	.2339	.2341	.2344	.2346	.2349	.2351	.2354	.2356	.62
	.1241	.1232	.1223	.1215	.1206	.1197	.1189	.1180	.1171	.1162	.1153	
.38	.2356	.2358	.2361	.2363	.2365	.2368	.2370	.2372	.2373	.2377	.2379	.61
	.1153	.1145	.1136	.1127	.1118	.1109	.1100	.1091	.1082	.1073	.1064	
.39	.2379	.2381	.2383	.2386	.2388	.2390	.2392	.2394	.2396	.2398	.2400	.60
	.1064	.1055	.1046	.1037	.1028	.1019	.1010	.1001	.0991	.0982	.0973	
.40	.2400	.2402	.2404	.2406	.2408	.2410	.2412	.2414	.2415	.2417	.2419	.59
	.0973	.0964	.0955	.0945	.0936	.0927	.0918	.0908	.0899	.0890	.0880	
.41	.2419	.2421	.2423	.2424	.2426	.2428	.2429	.2431	.2433	.2434	.2436	.58
	.0880	.0871	.0862	.0852	.0843	.0834	.0824	.0815	.0805	.0796	.0786	
.42	.2436	.2438	.2439	.2441	.2442	.2444	.2445	.2447	.2448	.2450	.2451	.57
	.0786	.0777	.0767	.0758	.0748	.0739	.0729	.0720	.0710	.0700	.0691	
.43	.2451	.2452	.2454	.2455	.2456	.2458	.2459	.2460	.2462	.2463	.2464	.56
	.0691	.0681	.0672	.0662	.0652	.0643	.0633	.0623	.0614	.0604	.0594	
.44	.2464	.2465	.2466	.2468	.2469	.2470	.2471	.2472	.2473	.2474	.2475	.55
	.0594	.0584	.0575	.0565	.0555	.0546	.0536	.0526	.0516	.0506	.0497	
.45	.2475	.2476	.2477	.2478	.2479	.2480	.2481	.2482	.2482	.2483	.2484	.54
	.0497	.0487	.0477	.0467	.0457	.0448	.0438	.0428	.0418	.0408	.0398	
.46	.2484	.2485	.2486	.2486	.2487	.2488	.2488	.2489	.2490	.2490	.2491	.53
	.0398	.0388	.0379	.0369	.0359	.0349	.0339	.0329	.0319	.0309	.0299	
.47	.2491	.2492	.2492	.2493	.2493	.2494	.2494	.2495	.2495	.2496	.2496	.52
	.0299	.0289	.0279	.0269	.0260	.0250	.0240	.0230	.0220	.0210	.0200	
.48	.2496	.2496	.2497	.2497	.2497	.2498	.2498	.2498	.2499	.2499	.2499	.51
	.0200	.0190	.0180	.0170	.0160	.0150	.0140	.0130	.0120	.0110	.0100	
.49	.2499	.2499	.2499	.2500	.2500	.2500	.2500	.2500	.2500	.2500	.2500	.50
	.0100	.0090	.0080	.0070	.0060	.0050	.0040	.0030	.0020	.0010	.0000	
	9	8	7	6	5	4	3	2	1	0		p
	Thousandths, for p on right											

Table B.3
Antilogits

Entries give value of p for specified positive value of logit L; if L is negative, p is 1 minus the tabled value.

L	0	1	2	3	4	5	6	7	8	9
0.0	.50000	.50250	.50500	.50750	.51000	.51250	.51500	.51749	.51999	.52248
0.1	.52498	.52747	.52996	.53245	.53194	.53743	.53991	.54240	.54488	.54736
0.2	.54983	.55231	.55478	.55725	.55971	.56218	.56464	.56709	.56955	.57200
0.3	.57444	.57689	.57932	.58176	.58419	.58662	.58904	.59146	.59387	.59628
0.4	.59869	.60109	.60348	.60587	.60826	.61064	.61301	.61538	.61775	.62011
0.5	.62246	.62481	.62715	.62948	.63181	.63414	.63645	.63876	.64107	.64337
0.6	.64566	.64794	.65022	.65249	.65475	.65701	.65925	.66150	.66374	.66597
0.7	.66819	.67040	.67261	.67481	.67700	.67918	.68135	.68352	.68568	.68783
0.8	.68997	.69211	.69424	.69635	.69847	.70057	.70266	.70473	.70682	.70889
0.9	.71095	.71300	.71504	.71708	.71910	.72112	.72312	.72512	.72711	.72909
1.0	.73106	.73302	.73497	.73692	.73885	.74077	.74269	.74460	.74649	.74838
1.1	.75026	.75213	.75399	.75584	.75768	.75951	.76133	.76315	.76495	.76674
1.2	.76852	.77030	.77206	.77382	.77556	.77730	.77903	.78074	.78245	.78415
1.3	.78583	.78751	.78918	.79084	.79249	.79413	.79576	.79738	.79899	.80059
1.4	.80218	.80377	.80534	.80690	.80845	.81000	.81153	.81306	.81457	.81608
1.5	.81757	.81906	.82054	.82201	.82346	.82491	.82635	.82778	.82920	.83062
1.6	.83202	.83341	.83480	.83617	.83753	.83889	.84024	.84158	.84290	.84422
1.7	.84553	.84684	.84813	.84941	.85069	.85195	.85321	.85446	.85570	.85693
1.8	.85815	.85936	.86057	.86176	.86295	.86413	.86530	.86646	.86761	.86876
1.9	.86989	.87102	.87214	.87325	.87435	.87545	.87653	.87761	.87868	.87974
2.0	.88080	.88184	.88288	.88391	.88493	.88595	.88695	.88795	.88894	.88993
2.1	.89090	.89187	.89283	.89379	.89473	.89567	.89660	.89752	.89844	.89935
2.2	.90025	.90114	.90203	.90291	.90378	.90465	.90551	.90636	.90721	.90805
2.3	.90888	.90970	.91052	.91133	.91214	.91293	.91373	.91451	.91529	.91606
2.4	.91683	.91759	.91834	.91909	.91983	.92056	.92129	.92201	.92273	.92344
2.5	.92414	.92484	.92553	.92622	.92690	.92757	.92824	.92891	.92958	.93022
2.6	.93086	.93150	.93214	.93277	.93339	.93401	.93462	.93523	.93584	.93643
2.7	.93703	.93761	.93820	.93877	.93935	.94091	.94048	.94103	.94159	.94213
2.8	.94268	.94321	.94373	.94425	.94480	.94532	.94583	.94634	.94685	.94735
2.9	.94785	.94834	.94883	.94931	.94979	.95026	.95073	.95120	.95166	.95212
3.0	.95257	.95302	.95347	.95391	.95435	.95478	.95521	.95564	.95606	.95648
3.1	.95689	.95730	.95771	.95811	.95851	.95891	.95930	.95969	.96007	.96046
3.2	.96083	.96121	.96158	.96195	.96231	.96267	.96303	.96339	.96374	.96408
3.3	.96443	.96477	.96511	.96544	.96578	.96610	.96643	.96675	.96707	.96739
3.4	.96770	.96802	.96832	.96863	.96893	.96923	.96953	.96982	.97011	.97040
3.5	.97069	.97097	.97125	.97153	.97180	.97208	.97235	.97262	.97289	.97314
3.6	.97340	.97366	.97392	.97417	.97442	.97467	.97491	.97516	.97540	.97564
3.7	.97587	.97611	.97634	.97657	.97680	.97702	.97725	.97747	.97769	.97790
3.8	.97812	.97833	.97854	.97873	.97896	.97916	.97937	.97957	.97977	.97996
3.9	.98016	.98035	.98054	.98073	.98092	.98111	.98129	.98148	.98166	.98184
4.0	.98201	.98219	.98236	.98254	.98271	.98288	.98304	.98321	.98337	.98354
4.1	.98370	.98386	.98402	.98417	.98433	.98448	.98463	.98478	.98493	.98508
4.2	.98523	.98537	.98551	.98566	.98580	.98594	.98607	.98621	.98635	.98648
4.3	.98661	.98674	.98687	.98700	.98713	.98726	.98738	.98751	.98763	.98775
4.4	.98787	.98799	.98811	.98823	.98834	.98846	.98857	.98868	.98879	.98890
4.5	.98901	.98912	.98923	.98933	.98944	.98954	.98965	.98975	.98985	.98995
4.6	.99005	.99015	.99024	.99034	.99043	.99053	.99062	.99071	.99081	.99090
4.7	.99099	.99108	.99116	.99125	.99134	.99142	.99151	.99159	.99167	.99176
4.8	.99184	.99192	.99200	.99208	.99215	.99223	.99231	.99239	.99246	.99253
4.9	.99261	.99268	.99275	.99283	.99290	.99297	.99304	.99310	.99317	.99324
L	0	1	2	3	4	5	6	7	8	9

Source: Berkson (1953:570)

Appendix C

EXAMPLES

Data Set = Berkson (1953)

Different doses of an insecticide, applied under standardized conditions to samples of an insect species. The rotenone was applied in a medium of 0.5% of alcohol. Insects were examined and classified one day after spraying. (A test of the toxicity of rotenone to Macrosiphoniella sanborni.)

x=dose	n	r=dead
0.000	50	6
0.164	48	16
0.292	46	24
0.471	49	42
0.593	50	44

Data Set = Montgomery & Peck (1982)

The compressive strength of an alloy fastener used in aircraft construction is being studied. Ten loads were selected over the range 25 to 43 (measured in 100's of PSI's), and a number of fastener tested at those loads.

x=load	n	r=fail
25	50	10
27	70	17
29	100	30
31	60	21
33	40	18
35	85	43
37	90	54
39	50	33
41	80	60
43	65	51

Data Set = Wrigley (1976)

The data represents the presence (y(i)=1) or absence (y(i)=0) of chronic bronchitis in a sample of 212 adult males. Also given are smoke level values (in milligrams/m**3) for the locality where each respondent lives and the respondent's cigarette consumption. The presence or absence of chronic bronchitis in the sample was determined using a special form of questionnaire devised for use in the investigation of chronic bronchitis by the Medical Research Council. The cigarettes consumption values were determined from the average number of cigarettes smoked per day and the number of years a respondent had smoked, expressed in hundreds of cigarettes. Consumption of pipe tobacco was converted into cigarette equivalents. The survey was restricted to men over fifteen years of age who had lived at their location at the time of the survey for at least ten years.

Data Set = Wrigley (1976) (continued)

OBS	Y=BRON	X1=CIG	X2=SMOKE	OBS	Y=BRON	X1=CIG	X2=SMOKE
1	0	5.15	67.1	54	0	0.00	62.1
2	1	0.00	66.9	55	0	14.55	61.7
3	0	2.50	66.7	56	1	11.00	61.0
4	0	1.75	65.8	57	1	6.75	62.7
5	0	6.75	64.4	58	0	0.00	62.7
6	0	0.00	64.4	59	1	0.00	61.7
7	1	0.00	65.1	60	0	1.75	60.9
8	1	9.50	66.2	61	0	2.40	60.6
9	0	0.00	65.9	62	0	10.05	60.4
10	0	0.75	67.1	63	1	12.75	61.7
11	0	5.25	67.9	64	0	0.00	61.9
12	1	8.00	68.1	65	0	5.00	61.3
13	1	5.14	67.0	66	0	0.60	60.7
14	1	30.00	66.3	67	0	0.00	60.8
15	0	0.00	65.7	68	0	0.85	60.5
16	0	0.00	65.2	69	0	0.90	59.7
17	0	5.25	64.2	70	0	0.00	59.5
18	0	10.05	64.6	71	1	8.75	59.6
19	0	0.00	63.5	72	0	0.80	59.1
20	1	3.40	63.0	73	1	6.60	59.4
21	0	0.00	62.7	74	0	1.00	58.5
22	0	0.55	62.7	75	0	0.00	60.0
23	1	9.50	62.1	76	1	8.15	59.8
24	1	12.50	63.7	77	0	0.00	59.7
25	0	0.00	63.1	78	1	5.00	59.4
26	0	3.40	63.0	79	0	2.55	59.2
27	0	2.20	62.7	80	0	1.20	58.6
28	0	6.70	63.1	81	0	0.00	60.8
29	0	1.10	62.4	82	1	11.25	60.4
30	0	1.80	64.4	83	0	0.00	60.2
31	0	0.00	64.2	84	0	2.00	60.0
32	1	3.60	64.2	85	0	1.90	59.4
33	0	1.60	63.0	86	0	0.45	59.8
34	0	6.20	62.2	87	1	0.00	59.7
35	0	14.75	62.3	88	0	0.00	59.0
36	0	0.35	63.7	89	1	6.90	59.0
37	1	13.75	63.8	90	0	2.35	58.6
38	0	0.00	63.1	91	0	3.95	59.7
39	1	7.50	62.7	92	0	0.60	59.6
40	0	1.00	62.9	93	1	15.00	59.4
41	0	0.00	62.5	94	0	0.00	59.4
42	1	14.80	61.7	95	0	0.95	59.4
43	1	3.50	61.6	96	0	0.00	59.3
44	0	0.00	61.6	97	0	1.40	54.2
45	0	0.00	61.4	98	0	0.50	54.0
46	0	0.25	61.4	99	0	0.60	53.8
47	0	1.55	62.0	100	0	0.00	53.7
48	1	0.00	61.8	101	0	2.45	53.7
49	0	0.00	60.9	102	0	1.75	53.1
50	0	5.90	60.8	103	0	0.00	54.4
51	0	16.45	60.6	104	0	3.10	54.2
52	0	2.65	62.9	105	0	10.05	53.9
53	1	12.50	62.6	106	0	0.55	53.2

Data Set = Wrigley (1976) (continued)

OBS	Y=BRON	X1=CIG	X2=SMOKE	OBS	Y=BRON	X1=CIG	X2=SMOKE
107	0	0.85	53.2	160	0	0.80	52.9
108	0	1.10	54.9	161	0	0.55	52.7
109	0	0.00	54.5	162	0	0.95	52.6
110	0	0.00	54.5	163	0	0.00	52.1
111	0	1.45	54.2	164	0	3.10	54.1
112	0	2.05	54.2	165	0	0.80	53.7
113	1	10.50	54.0	166	0	1.55	53.1
114	0	0.50	55.8	167	0	0.40	53.3
115	1	9.20	55.5	168	0	6.20	53.0
116	0	0.55	55.6	169	0	0.60	53.0
117	0	0.00	55.5	170	0	0.40	53.9
118	0	0.96	54.9	171	1	7.50	53.7
119	0	1.00	54.6	172	0	7.15	53.4
120	0	0.00	56.9	173	0	0.25	53.2
121	0	5.25	56.4	174	0	3.60	53.4
122	1	0.00	55.9	175	0	0.95	53.2
123	0	9.00	55.8	176	0	2.80	54.9
124	0	1.60	55.6	177	1	20.25	54.9
125	1	10.90	57.6	178	0	0.95	54.6
126	0	0.00	57.7	179	0	4.25	54.1
127	0	0.00	57.6	180	0	4.15	54.2
128	0	2.25	57.8	181	0	10.00	57.4
129	0	2.65	57.8	182	0	3.40	57.3
130	0	0.55	58.4	183	0	0.00	57.3
131	0	0.00	58.2	184	0	3.60	56.7
132	1	4.50	58.0	185	0	0.90	56.5
133	0	15.00	58.1	186	0	0.00	56.8
134	0	0.00	57.9	187	0	0.00	56.6
135	0	0.00	57.3	188	1	6.40	56.5
136	0	4.20	58.3	189	0	0.95	56.3
137	0	0.55	58.1	190	0	1.06	56.3
138	1	10.00	57.9	191	0	13.30	56.2
139	0	0.00	57.6	192	0	1.10	56.6
140	0	7.10	57.3	193	0	17.20	55.9
141	0	3.20	57.1	194	0	1.65	56.0
142	1	0.00	58.9	195	1	5.00	55.8
143	1	6.80	58.6	196	0	2.10	55.7
144	0	0.00	58.7	197	0	0.60	57.0
145	0	0.00	57.5	198	1	8.25	56.7
146	0	2.35	57.2	199	0	0.90	56.4
147	0	24.90	58.0	200	0	0.00	56.5
148	0	2.65	57.9	201	1	12.30	55.2
149	1	3.70	57.2	202	0	1.15	56.9
150	0	17.10	57.3	203	0	2.20	56.7
151	0	0.00	57.5	204	0	3.60	56.0
152	0	0.95	57.2	205	1	10.00	55.5
153	0	10.05	53.1	206	0	0.60	55.3
154	0	1.15	53.0	207	0	9.50	56.5
155	1	18.25	53.0	208	0	0.70	56.3
156	0	10.00	52.9	209	1	9.00	56.1
157	0	0.75	52.6	210	0	0.00	55.9
158	0	0.00	53.1	211	0	0.50	55.5
159	0	4.20	53.0	212	0	0.90	55.4

Since the Wrigley data has two regressors, it has been grouped in the following way:

Smoke Level (X2)		Cigarette Consumption (X1)				
		$\leq$ 0.75	0.76-3.75	3.76-6.75	6.76-10.74	$\geq$10.75
$\geq$ 61.0	i	1	2	3	4	5
	n_i	23	13	9	5	9
	r_i	4	3	2	4	7
	X_{i1}	0.0826	2.3114	5.8000	8.9100	15.1778
	X_{i2}	63.4826	63.4308	64.4333	64.7400	62.8222
56.0 -60.9	i	6	7	8	9	10
	n_i	33	30	3	11	8
	r_i	2	1	4	7	3
	X_{i1}	0.1227	1.8970	5.2250	8.5909	15.4875
	X_{i2}	58.5545	57.9467	58.5625	58.1182	58.4500
$\leq$ 55.9	i	11	12	13	14	15
	n_i	22	23	5	9	4
	r_i	1	0	1	4	3
	X_{i1}	0.3068	1.6243	4.7600	9.2722	17.0000
	X_{i2}	54.1591	54.0609	54.0200	54.2000	54.7500

n_i = number sampled in "block" i

r_i = number who developed chronic bronchitis out of n_i

X_{ij} = average value of the explanatory variables for each of the i blocks (i = 1, 15; j = 1,2)

Example Set 1

Probit: Maximum Likelihood for Grouped Data

The examples of this method utilize the SAS (Statistical Analysis System) computer package PROC PROBIT. Note that this method cannot be generalized to the multiple regressor situation (k>1) in SAS's PROC PROBIT.

Note that $\hat{\beta}_0$ and $\hat{\beta}_1$ (or $\hat{\beta}_0^*$) are found first, then $\hat{\mu}$ and $\hat{\sigma}$ are determined via the transformations.

Example Set 1 (continued)

Probit: Maximum Likelihood for Grouped Data

PROBIT: MLE FOR GROUPED DATA (PROC PROBIT)
DATA SET = BERKSON (1953)

PROBIT ANALYSIS ON X

ITERATION	INTERCEPT	SLOPE	MU	SIGMA
0	3.86731251	4.18078919	0.27092672	0.23918929
1	3.86251662	4.21538301	0.26984105	0.23722637
2	3.86249332	4.21552463	0.26983751	0.23721840
3	3.86249332	4.21552463	0.26983751	0.23721840

COVARIANCE MATRIX

	INTERCEPT	SLOPE
INTERCEPT	0.02788137	-0.06649846
SLOPE	-0.06649846	0.22897667

COVARIANCE MATRIX

	MU	SIGMA
MU	0.00048766	-0.00006290
SIGMA	-0.00006290	0.00072508

INTERCEPT = $\hat{\beta}_0^* = 3.8625 \pm 0.1670$ $\quad \hat{\mu} = 0.2698$

SLOPE = $1/\hat{\sigma} = 4.2155 \pm 0.4785$ $\quad \hat{\sigma} = 0.2372$

$\hat{\beta}_0 = -\hat{\mu}/\hat{\sigma} = \hat{\beta}_0^* - 5 = -1.1375 \pm 0.1670$ $\quad \hat{\gamma} = 0.2698$

Example Set 1 (continued)

Probit: Maximum Likelihood for Grouped Data

PROBIT: MLE FOR GROUPED DATA (PROC PROBIT)
DATA SET = MONTGOMERY & PECK (1982)

PROBIT ANALYSIS ON X

ITERATION	INTERCEPT	SLOPE	MU	SIGMA
0	1.76215304	0.09396249	34.45893164	10.64254488
1	1.72883337	0.09488416	34.47537262	10.53916738
2	1.72879119	0.09488532	34.47539495	10.53903831

COVARIANCE MATRIX

	INTERCEPT	SLOPE
INTERCEPT	0.10300134	-0.00293919
SLOPE	-0.00293919	0.00008597

COVARIANCE MATRIX

	MU	SIGMA
MU	0.28061787	0.02903021
SIGMA	0.02903021	1.06064945

INTERCEPT = $\hat{\beta}_0^* = 1.7288 \pm 032090$ $\quad\quad$ $\hat{\mu} = 34.4754$

SLOPE = $1/\hat{\sigma} = 0.0949 \pm 0.0093$ $\quad\quad$ $\hat{\sigma} = 10.5390$

$\hat{\beta}_0 = -\hat{\mu}/\hat{\sigma} = \hat{\beta}_0^* - 5 = -3.2712 \pm 0.3209$ $\quad\quad$ $\hat{\gamma} = 34.4700$

Example Set 1 (continued)
Probit: Maximum Likelihood for Grouped Data
Computer Program (SAS's PROC PROBIT)

```
TITLE PROBIT: MLE FOR GROUPED DATA (PROC PROBIT);
TITLE2 DATA SET =                ;
OPTIONS LS=80;
DATA ONE;
INPUT X N R;
CARDS;
  .
  .
PROC PRINT;
PROC PROBIT; VAR X N R;
```

Example Set 2

Probit: Minimum Normit χ^2 for Grouped Data

PROBIT: WEIGHTED LINEAR NORMIT LEAST SQUARES FOR GROUPED DATA
DATA SET = BERKSON (1953)

GENERAL LINEAR MODELS PROCEDURE

DEPENDENT VARIABLE: V
WEIGHT: W

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	F VALUE
MODEL	1	22.78024985	22.78024985	137.22
ERROR	3	0.49802877	0.16600959	PR > F
CORRECTED TOTAL	4	23.27827863		0.0013

R-SQUARE	C.V.	STD DEV	V MEAN
0.978605	295.0346	0.40744275	0.13810000

SOURCE	DF	TYPE I SS	F VALUE	PR > F
X	1	22.78024985	137.22	0.0013

SOURCE	DF	TYPE IV SS	F VALUE	PR > F
X	1	22.78024985	137.22	0.0013

PARAMETER	ESTIMATE	T FOR H0: PARAMETER=0	PR > |T|	STD ERROR OF ESTIMATE
INTERCEPT	-1.13590171	-9.35	0.0026	0.12149316
X	4.19570928	11.71	0.0013	0.35817291

$$\hat{\gamma} = 0.2707$$

Example Set 2 (continued)

Probit: Minimum Normit χ^2 for Grouped Data

PROBIT: WEIGHTED LINEAR NORMIT LEAST SQUARES FOR GROUPED DATA
DATA SET = MONTGOMERY & PECK (1982)

GENERAL LINEAR MODELS PROCEDURE

DEPENDENT VARIABLE: V
WEIGHT: W

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	F VALUE
MODEL	1	37.25971804	37.25971804	825.35
ERROR	8	0.36115205	0.04514401	PR > F
CORRECTED TOTAL	9	37.62087010		0.0001

R-SQUARE	C.V.	STD DEV	V MEAN
0.990400	415.0639	0.21247119	-0.05119000

SOURCE	DF	TYPE I SS	F VALUE	PR > F
X	1	37.25971804	825.35	0.0001

SOURCE	DF	TYPE IV SS	F VALUE	PR > F
X	1	37.25971804	825.35	0.0001

PARAMETER	ESTIMATE	T FOR H0: PARAMETER=0	PR > |T|	STD ERROR OF ESTIMATE
INTERCEPT	-3.31419534	-28.70	0.0001	0.11548268
X	0.09589331	28.73	0.0001	0.00333786

$$\hat{\gamma} = 34.5589$$

Example Set 2 (continued)

Probit: Minimum Normit χ^2 for Grouped Data

PROBIT: WEIGHTED LINEAR NORMIT LEAST SQUARES FOR GROUPED DATA
DATA SET = WRIGLEY (1976)

GENERAL LINEAR MODELS PROCEDURE

DEPENDENT VARIABLE: V
WEIGHT: W

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	F VALUE
MODEL	2	8.85133538	4.42566769	10.39
ERROR	12	5.11070648	0.42589221	PR > F
CORRECTED TOTAL	14	13.96204186		0.0024

R-SQUARE	C.V.	STD DEV	V MEAN
0.633957	107.7399	0.65260417	-0.60572200

SOURCE	DF	TYPE I SS	F VALUE	PR > F
X1	1	8.28543922	19.45	0.0008
X2	1	0.56589616	1.33	0.2715

SOURCE	DF	TYPE IV SS	F VALUE	PR > F
X1	1	8.84546018	20.77	0.0007
X2	1	0.56589616	1.33	0.2715

PARAMETER	ESTIMATE	T FOR H0: PARAMETER=0	PR > |T|	STD ERROR OF ESTIMATE
INTERCEPT	-3.79430648	-1.64	0.1277	2.31886770
X1	0.11547292	4.56	0.0007	0.02533785
X2	0.04394357	1.15	0.2715	0.03812210

Example Set 2 (continued)

Probit: Minimum Normit χ^2 for Grouped Data
Computer Program (SAS)

```
TITLE PROBIT: WEIGHTED LINEAR NORMIT LEAST SQUARES
                          FOR GROUPED DATA;
TITLE2 DATA SET =                ;
OPTIONS LS=80;
DATA ONE;
INPUT X N R V;
P=R/N;
W=N*P*(1-P);
CARDS;
  .
  .
PROC PRINT;
PROC GLM; MODEL V=X; WEIGHT W;
```

Example Set 3
Logit: Maximum Likelihood for Individual Data

LOGIT: MLE FOR INDIVIDUAL DATA
DATA SET = BERKSON (1953)
PROC LOGIST

LOGISTIC REGRESSION PROCEDURE

DEPENDENT VARIABLE: Y

243 OBSERVATIONS
111 Y = 0
132 Y = 1
0 OBSERVATIONS DELETED DUE TO MISSING VALUES

VARIABLE	MEAN	MINIMUM	MAXIMUM	RANGE
X	0.304663	0	0.593	0.593

-2 LOG LIKELIHOOD FOR MODEL CONTAINING INTERCEPT ONLY= 335.05

MODEL CHI-SQUARE= 84.80 WITH 1 D.F. (SCORE STAT.) P=0.0 .
CONVERGENCE OBTAINED IN 6 ITERATIONS. R= 0.528.
MAX ABSOLUTE DERIVATIVE=0.0 . -2 LOG L= 239.79.
MODEL CHI-SQUARE= 95.26 WITH 1 D.F. (-2 LOG L.R.) P=0.0 .

VARIABLE	BETA	STD. ERROR	CHI-SQUARE	P	R
INTERCEPT	-1.91839809	0.29823891	41.38	0.0000	
X	7.14950214	0.89297899	64.10	0.0000	0.431

$$\hat{\gamma} = 0.2683$$

Example Set 3 (continued)
Logit: Maximum Likelihood for Individual Data

LOGIT: MLE FOR INDIVIDUAL DATA
DATA SET = MONTGOMERY & PECK (1982)
PROC LOGIST

LOGISTIC REGRESSION PROCEDURE

DEPENDENT VARIABLE: Y

690 OBSERVATIONS
353 Y = 0
337 Y = 1
0 OBSERVATIONS DELETED DUE TO MISSING VALUES

VARIABLE	MEAN	MINIMUM	MAXIMUM	RANGE
X	34.1304	25	43	18

-2 LOG LIKELIHOOD FOR MODEL CONTAINING INTERCEPT ONLY= 956.17

MODEL CHI-SQUARE= 107.07 WITH 1 D.F. (SCORE STAT.) P=0.0 .
CONVERGENCE OBTAINED IN 5 ITERATIONS. R= 0.340.
MAX ABSOLUTE DERIVATIVE=0.1931D-09. -2 LOG L= 843.71.
MODEL CHI-SQUARE= 112.46 WITH 1 D.F. (-2 LOG L.R.) P=0.0 .

VARIABLE	BETA	STD. ERROR	CHI-SQUARE	P	R
INTERCEPT	-5.33971151	0.54569325	95.75	.	
X	0.15484336	0.01575379	96.61	.	0.315

$$\hat{\gamma} = 34.4942$$

Example Set 3 (continued)
Logit: Maximum Likelihood for Individual Data

LOGIT: MLE FOR INDIVIDUAL DATA
DATA SET = WRIGLEY (1976)
PROC LOGIST

LOGISTIC REGRESSION PROCEDURE

DEPENDENT VARIABLE: Y

212 OBSERVATIONS
166 Y = 0
46 Y = 1
0 OBSERVATIONS DELETED DUE TO MISSING VALUES

VARIABLE	MEAN	MINIMUM	MAXIMUM	RANGE
X1	3.80099	0	30	30
X2	58.5434	52.1	68.1	16

-2 LOG LIKELIHOOD FOR MODEL CONTAINING INTERCEPT ONLY= 221.78

MODEL CHI-SQUARE= 50.28 WITH 2 D.F. (SCORE STAT.) P=0.0 .
CONVERGENCE OBTAINED IN 5 ITERATIONS. R= 0.443.
MAX ABSOLUTE DERIVATIVE=0.8188D-04. -2 LOG L= 174.21.
MODEL CHI-SQUARE= 47.57 WITH 2 D.F. (-2 LOG L.R.) P=0.0 .

VARIABLE	BETA	STD. ERROR	CHI-SQUARE	P	R
INTERCEPT	-10.08617787	2.95039640	11.69	0.0006	
X1	0.21166138	0.03812979	30.81	0.0000	0.360
X2	0.13178466	0.04894363	7.25	0.0071	0.154

Example Set 3 (continued)
Logit: Maximum Likelihood for Individual Data
Computer Program (SAS's PROC LOGIST)

```
TITLE1 LOGIT: MLE FOR INDIVIDUAL DATA;
TITLE2 DATA SET =                ;
TITLE3 PROC LOGIST;
OPTIONS LS=80;
DATA ONE;
INPUT X Y;
CARDS;
  .
  .
PROC PRINT;
*COMMENT: PROC LOGIST MUST BE RUN USING BATCH;
PROC LOGIST DATA=ONE;
    MODEL Y=X;
```

Example Set 4
Logit: Maximum Likelihood Method for Grouped Data

Data Set = Berkson (1953)

Iteration	$\hat{\beta}_0$	$\hat{\beta}_1$
0	0	0
1	-1.5043	5.5048
2	-1.8661	6.9303
3	-1.9173	7.1449
4	-1.9184	7.1495
5	-1.9184	7.1495

$\hat{\gamma} = 0.2683$

Data Set = Montgomery & Peck (1982)

Iteration	$\hat{\beta}_0$	$\hat{\beta}_1$
0	0	0
1	-4.8203	0.1399
2	-5.3263	0.1545
3	-5.3397	0.1548
4	-5.3397	0.1548

$\hat{\gamma} = 34.4942$

Example Set 4 (continued)
Logit: Maximum Likelihood Method for Grouped Data
Computer Program (SAS)

```
TITLE LOGIT: MLE METHOD FOR GROUPED DATA;
TITLE2 DATA SET =        ;
TITLE3 ITERATION #       ;
OPTIONS LS=80;
DATA ONE;
INPUT X N R;
A=                  ;          *INSERT STARTING;
B=                  ;          *VALUES HERE    ;
P=R/N;
LHAT=A+B*X;
PHAT=1/(1+EXP(-LHAT));
QHAT=1-PHAT;
W=PHAT*QHAT;
NW=N*W;
NX=N*X;
PMP=P-PHAT;
NXPMP=NX*PMP;
NPMP=N*PMP;
NXW=N*X*W;
NX2W=N*X*X*W;
CARDS;
  .
  .
PROC PRINT; VAR X N R;
PROC PRINT; VAR A B X LHAT PHAT W;
PROC MEANS SUM; VAR NW NXPMP NPMP NXW NX2W;
OUTPUT OUT=TWO SUM=SNW SNXPMP SNPMP SNXW SNX2W;

DATA THREE; SET ONE; SET TWO;
NUM=SNXPMP-(SNPMP*SNXW/SNW);
DEN=SNX2W-(((SNXW)**2)/SNW);
DB=NUM/DEN;
DA=(SNPMP-SNXW*DB)/SNW;
NEWA=A+DA;
NEWB=B+DB;
PROC PRINT; VAR A B DA DB NEWA NEWB;
```

Example Set 5
Logit: Weighted Least Squares for Individual Data

LOGIT: WEIGHTED LEAST SQUARES
PROC NLIN WITH WEIGHTS
DATA SET = BERKSON (1953)

NON-LINEAR LEAST SQUARES ITERATIVE PHASE

DEPENDENT VARIABLE: P METHOD: GAUSS-NEWTON

ITERATION	B0	B1	WEIGHTED RESIDUAL SS
0	0.000000E+00	0.000000E+00	193.87121212
1	-1.48654221	5.44558494	7.40358761
2	-1.87114207	6.99859144	1.56700733
3	-1.94443338	7.32046591	1.41129042
4	-1.94986606	7.34515937	1.41045688
5	-1.95012654	7.34632415	1.41045506
6	-1.95013812	7.34637597	1.41045506

NOTE: CONVERGENCE CRITERION MET.

NON-LINEAR LEAST SQUARES SUMMARY STATISTICS DEPENDENT VARIABLE P

SOURCE	DF	WEIGHTED SS	WEIGHTED MS
REGRESSION	2	740.25621161	370.12810580
RESIDUAL	3	1.41045506	0.47015169
UNCORRECTED TOTAL	5	741.66666667	
(CORRECTED TOTAL)	4	186.83012815	

PARAMETER	ESTIMATE	ASYMPTOTIC STD. ERROR	ASYMPTOTIC 95 % CONFIDENCE INTERVAL LOWER	UPPER
B0	-1.95013812	0.20757001	-2.61072969	-1.28954655
B1	7.34637597	0.63427733	5.32778838	9.36496356

ASYMPTOTIC CORRELATION MATRIX OF THE PARAMETERS

	B0	B1
B0	1.000000	-0.847526
B1	-0.847526	1.000000

$$\hat{\gamma} = 0.2654$$

Example Set 5 (continued)
Logit: Weighted Least Squares for Individual Data

LOGIT: WEIGHTED LEAST SQUARES
PROC NLIN WITH WEIGHTS
DATA SET = MONTGOMERY & PECK (1982)

NON-LINEAR LEAST SQUARES ITERATIVE PHASE

DEPENDENT VARIABLE: P METHOD: GAUSS-NEWTON

ITERATION	B0	B1	WEIGHTED RESIDUAL SS
0	0.000000E+00	0.000000E+00	145.97250217
1	-4.78495561	0.13894856	1.48217091
2	-5.32714622	0.15450860	0.37602150
3	-5.34691197	0.15505626	0.37465731
4	-5.34713015	0.15506190	0.37465709
5	-5.34713297	0.15506198	0.37465709

NOTE: CONVERGENCE CRITERION MET.

NON-LINEAR LEAST SQUARES SUMMARY STATISTICS DEPENDENT VARIABLE P

SOURCE	DF	WEIGHTED SS	WEIGHTED MS
REGRESSION	2	938.33862833	469.16931416
RESIDUAL	8	0.37465709	0.04683214
UNCORRECTED TOTAL	10	938.71328542	
(CORRECTED TOTAL)	9	145.41357783	

PARAMETER	ESTIMATE	ASYMPTOTIC STD. ERROR	ASYMPTOTIC 95 % CONFIDENCE INTERVAL LOWER	UPPER
B0	-5.34713297	0.11861632	-5.62066534	-5.07360060
B1	0.15506198	0.00342034	0.14717459	0.16294936

ASYMPTOTIC CORRELATION MATRIX OF THE PARAMETERS

	B0	B1
B0	1.000000	-0.988509
B1	-0.988509	1.000000

$$\hat{\gamma} = 34.4752$$

Example Set 5 (continued)
Logit: Weighted Least Squares for Individual Data

LOGIT: WEIGHTED LEAST SQUARES
PROC NLIN WITH WEIGHTS
DATA SET = WRIGLEY (1976)

NON-LINEAR LEAST SQUARES ITERATIVE PHASE

DEPENDENT VARIABLE: Y METHOD: GAUSS-NEWTON

ITERATION	B0	B1	B2	WEIGHTED RESIDUAL SS
0	0.000000E+00	0.000000E+00	0.000000E+00	212.00000000
1	-5.79250701	0.14421428	0.07024318	151.52469439

NOTE: USING THE ABOVE PARAMETERS, THE PARAMETER MODIFICATION VECTOR (DELTA) WAS HALVED 10 TIMES. NO REDUCTION IN THE RESIDUAL SS WAS FOUND (CONVERGENCE ASSUMED).

NON-LINEAR LEAST SQUARES SUMMARY STATISTICS DEPENDENT VARIABLE Y

SOURCE	DF	WEIGHTED SS	WEIGHTED MS
REGRESSION	3	90.62853381	30.20951127
RESIDUAL	209	151.52469439	0.72499854
UNCORRECTED TOTAL	212	242.15322820	
(CORRECTED TOTAL)	211	199.28837822	

PARAMETER	ESTIMATE	ASYMPTOTIC STD. ERROR	ASYMPTOTIC 95 % CONFIDENCE INTERVAL LOWER	UPPER
B0	-5.79250701	2.15019124	-10.03140013	-1.55361390
B1	0.14421428	0.02798414	0.08904627	0.19938229
B2	0.07024318	0.03625217	-0.00122445	0.14171081

ASYMPTOTIC CORRELATION MATRIX OF THE PARAMETERS

	B0	B1	B2
B0	1.000000	-0.071463	-0.995971
B1	-0.071463	1.000000	0.011075
B2	-0.995971	0.011075	1.000000

Example Set 5 (continued)
Logit: Weighted Least Squares for Individual Data
Computer Program (SAS's PROC NLIN)

```
TITLE LOGIT: WEIGHTED LEAST SQUARES;
TITLE2 PROC NLIN WITH WEIGHTS;
TITLE3 DATA SET =            ;
OPTIONS LS=80;
DATA ONE;
INPUT X1 X2 y;
CARDS;
  .
  .
PROC NLIN;
   PARAMETERS B0=0 B1=0 B2=0;
   E=EXP(-(B0+B1*X1+B2*X2));
   P0=1/(1+E);
   Q0=1-P0;
   W=1/(P0*Q0);
   MODEL P=1/(1+E);
   DER.B0=E/(1+E)**2;
   DER.B1=X1*E/(1+E)**2;
   DER.B2=X2*E/(1+E)**2;
   _WEIGHT_=W;
```

Example Set 6

Logit: Minimum Logit χ^2 for Grouped Data

LOGIT: WEIGHTED LINEAR LOGIT LEAST SQUARES FOR GROUPED DATA
DATA SET = BERKSON (1953)

GENERAL LINEAR MODELS PROCEDURE

DEPENDENT VARIABLE: L
WEIGHT: W

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	F VALUE
MODEL	1	64.50837615	64.50837615	138.58
ERROR	3	1.39644572	0.46548191	PR > F
CORRECTED TOTAL	4	65.90482187		0.0013

R-SQUARE	C.V.	STD DEV	L MEAN
0.978811	287.7230	0.68226235	0.23712473

SOURCE	DF	TYPE I SS	F VALUE	PR > F
X	1	64.50837615	138.58	0.0013

SOURCE	DF	TYPE IV SS	F VALUE	PR > F
X	1	64.50837615	138.58	0.0013

PARAMETER	ESTIMATE	T FOR HO: PARAMETER=0	PR > |T|	STD ERROR OF ESTIMATE
INTERCEPT	-1.90464734	-9.36	0.0026	0.20344014
X	7.06047958	11.77	0.0013	0.59976007

$$\hat{\gamma} = 0.2654$$

Example Set 6 (continued)

Logit: Minimum Logit χ^2 for Grouped Data

LOGIT: WEIGHTED LINEAR LOGIT LEAST SQUARES FOR GROUPED DATA
DATA SET = MONTGOMERY & PECK (1982)

GENERAL LINEAR MODELS PROCEDURE

DEPENDENT VARIABLE: L
WEIGHT: W

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	F VALUE
MODEL	1	97.01599230	97.01599230	2095.16
ERROR	8	0.37043801	0.04630475	PR > F
CORRECTED TOTAL	9	97.38643031		0.0001

R-SQUARE	C.V.	STD DEV	L MEAN
0.996196	304.4888	0.21518539	-0.07067103

SOURCE	DF	TYPE I SS	F VALUE	PR > F
X	1	97.01599230	2095.16	0.0001

SOURCE	DF	TYPE IV SS	F VALUE	PR > F
X	1	97.01599230	2095.16	0.0001

PARAMETER	ESTIMATE	T FOR H0: PARAMETER=0	PR > |T|	STD ERROR OF ESTIMATE
INTERCEPT	-5.33605978	-45.62	0.0001	0.11695790
X	0.15473559	45.77	0.0001	0.00338050

$$\hat{\gamma} = 34.4932$$

Example Set 6 (continued)

Logit: Minimum Logit χ^2 for Grouped Data

LOGIT: WEIGHTED LINEAR LOGIT LEAST SQUARES FOR GROUPED DATA
DATA SET = WRIGLEY (1976)

GENERAL LINEAR MODELS PROCEDURE

DEPENDENT VARIABLE: L
WEIGHT: W

SOURCE	DF	SUM OF SQUARES	MEAN SQUARE	F VALUE
MODEL	2	26.57663414	13.28831707	10.08
ERROR	12	15.81212322	1.31767694	PR > F
CORRECTED TOTAL	14	42.38875736		0.0027

R-SQUARE	C.V.	STD DEV	L MEAN
0.626974	99.7648	1.14790110	-1.15060748

SOURCE	DF	TYPE I SS	F VALUE	PR > F
X1	1	24.50108033	18.59	0.0010
X2	1	2.07555381	1.58	0.2334

SOURCE	DF	TYPE IV SS	F VALUE	PR > F
X1	1	26.50100374	20.11	0.0007
X2	1	2.07555381	1.58	0.2334

PARAMETER	ESTIMATE	T FOR H0: PARAMETER=0	PR > |T|	STD ERROR OF ESTIMATE
INTERCEPT	-7.05581392	-1.73	0.1093	4.07878296
X1	0.19987160	4.48	0.0007	0.04456813
X2	0.08415771	1.26	0.2334	0.06705504

Example Set 6 (continued)

Logit: Minimum Logit χ^2 for Grouped Data
Computer Program (SAS)

```
TITLE LOGIT: WEIGHTED LINEAR LOGIT LEAST SQUARES
                      FOR GROUPED DATA;
TITLE2 DATA SET =                 ;
OPTIONS LS=80;
DATA ONE;
INPUT X N R;
P=R/N;
L=LOG(P/(1-P));
W=N*P*(1-P);
CARDS;
  .
  .
PROC PRINT;
PROC GLM; MODEL L=X; WEIGHT W;
```

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