AN ANALYSIS PROCEDURE FOR

CYLINDRICAL AND SPHERICAL SHELLS SUBJECTED TO

MULTI-HARMONIC LOADS

by

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TABLE OF CONTENTS

LIST OF TABLES AND FIGURES

LIST OF SYMBOLS

A_1, A_2	=	Lame' Parameters
$A_{i,n}$	=	Complex Arbitrary Constants
$A_n, A_{n,m}, B_{n,m}$	=	Spherical Harmonic Coefficients
$A(\xi), A_n(\xi), B_n(\xi)$	=	Fourier Coefficients
$AK_{i,n}$	=	Roots of Auxiliary Equation
$AR_{i,n}, AI_{i,n}$	=	Real Arbitrary Contents
$B_{n,m}, D_{n,m}$	=	Coefficients for Particular Solution Series of the Sphere
b	=	$[3(1-\mu^2)]^{1/4}(r/\delta)^{1/2}$
$C_{n,j}$	=	Coefficients for the Particular Solution Series of the Cylinder
$CN_{n,j}$	=	Load Coefficients for Cylinder
C_p	=	Pressure Coefficient
D_i	=	Polynomial Coefficients
E	=	Young's Modulus
$F(\cos\theta, \sin\theta)$	=	$P_m^n(\cos\theta)$
H	=	$M_{12} = M_{21}$
i	=	$\sqrt{-1}$

M_1, M_2, M_{2}, M_2 = Moment Resultants

n Harmonic Number

$P_m^n(\cos\theta)$ = Associated Legendre Polynomials

q_1, q_2, q_3 = Surface Loads

$q_n(\xi)$ = Harmonic Load

R_1, R_2 = Radii of Curvature

r = Radius

$S \approx T_{12} \approx T_{21}$

$T = T_1 + T_2$

$\widetilde{T} = \widetilde{T}_1 + \widetilde{T}_2$

$T_1, T_2, T_{12}, T_{21}, N_1, N_2$ = Force Resultants

$\widetilde{T}_1, \widetilde{T}_2, \widetilde{S}$ = Complex Force Resultants

T_1^*, T_2^*, S^* = Membrane Force Resultants

u, v, w = Displacements

$\widetilde{u}, \widetilde{v}, \widetilde{w}$ = Complex Displacements

α = Angle from Edge of Sphere

α_1, α_2 = Arbitrary Orthogonal Coordinates

$\varepsilon_1, \varepsilon_2, \omega$ = Strains

$\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2, \widetilde{\omega}$ = Complex Strains

δ = Thickness

κ_1, κ_2, τ	=	Changes in Curvature
$\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\tau}$	=	Complex Changes in Curvature
μ	=	Poisson's Ratio
θ, ϕ	=	Spherical Coordinates
ξ, ϕ	=	Cylindrical Coordinates

I. INTRODUCTION

The study of wind forces on structures has been the subject of many investigations. In a paper discussing the history of wind load investigations (2), J. M. Biggs lists twenty-five major references, mostly concerned with results of wind tunnel tests of models. A study of such tests would make it reasonably clear that no simple, general rules for wind load magnitude and distribution are available for the structural designer or analyst.

In an attempt to develop information on the wind load distributions of some basic structures, a test program was undertaken at Virginia Polytechnic Institute in 1963. The program gathered such information for spherical domes of varying rise to span ratios (including hemispheres) within the ordinary practical range. The domes were mounted both on the floor of the wind tunnel and on cylinders of varying heights. The results of the tests have been published and are available in references (17) and (18). In addition, the limiting case of the flat top on cylinders was studied and is presently scheduled for publication (20). This latter paper also contains a mathematical representation of wind distribution on both the cylinder and flat top. Since this information will be drawn on for this paper, a review of the original tests should be in order.

The domes were made of commercially produced spun aluminum with internal plywood stiffeners added at the base. The cylinders were made of commerically produced aluminum sheets with internal plywood stiffeners added at the top and bottom. Pressures along the domes and cylinders were measured along the line lying in a vertical plane which includes the axis of symmetry of the model. Since the models were rotationally symmetric, pressures were measured over the entire surface by rotating the model through 180 degrees at 15 degree intervals. After a correction for the change in static pressure caused by the operation of the wind tunnel, the local pressure values were divided by the dynamic pressure of the undisturbed wind stream to produce a non-dimensional pressure coefficient Cp. The tests were conducted on the floor of the wind tunnel in the presence of a boundary layer about four inches high. The results of a test run on a model composed of a 12 inch diameter hemisphere mounted on a 12 inch high circular cylinder are presented in Figures [1] and [2].[1]

In order to use the data from the model tests, it is necessary to "scale up" to full size. This process is always open to question for two basic reasons. First, the model Reynold's Number is necessarily smaller than that of the prototype, giving rise to the question of

1. Figures 1 and 2 have been previously published in reference (18).

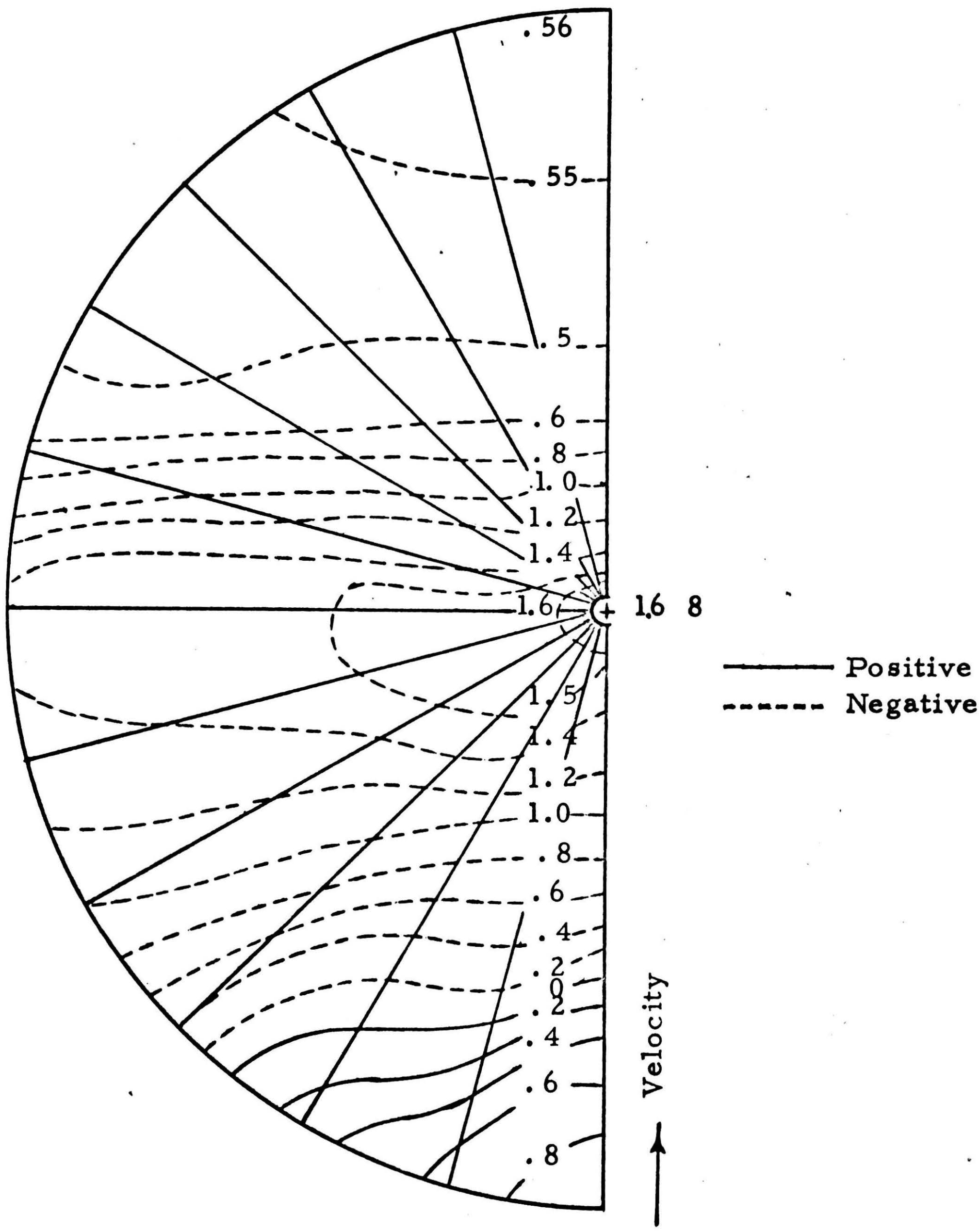

Figure 1. C_p on the Hemisphere

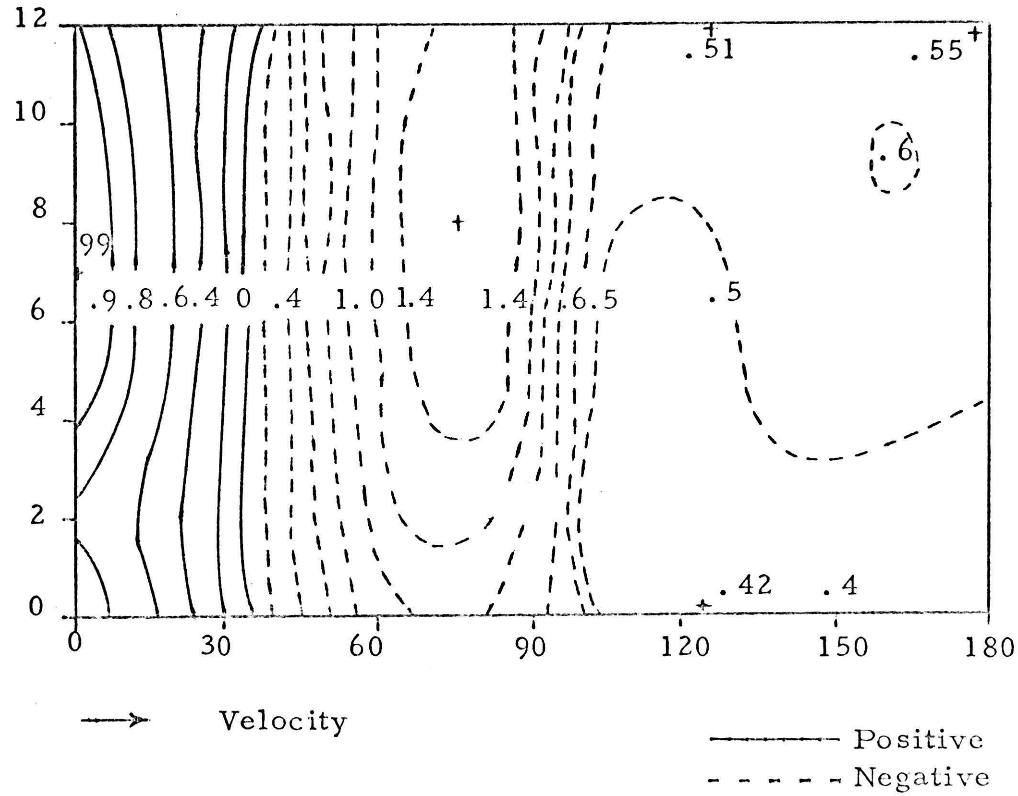

Figure 2. C_p on the Cylinder

whether or not the flow regimes are similar. The second reason concerns itself with the similarity of boundary layers for a ground based structure. In addition, actual structural details can be expected to cause, at least, local disturbances in the pressure patterns obtained with relatively "clean" models.

In spite of these difficulties, it was felt that the results would present designers with basic information useful for at least the preliminary design of similar structures. With the intention of extending the usefulness of the data, methods of representing the pressure loadings mathematically were undertaken with the ultimate goal of using the results for stress and deformation analysis of constant thickness shells so loaded.

The object of this research, then, was to develop a procedure for the stress and displacement analysis of dome and dome-cylinder structures under the action of an actual wind load. The force and moment resultants and the displacements for a dome-cylinder problem were determined by means of a series solution technique utilizing the IBM 7040 computer. Although several other methods of approach were available, namely, the use of finite-elements, finite-differences, and numerical integration, the series solution technique was chosen. This method, although discussed very little in the literature for complex problems, provided several important advantages over alternate approaches. It was possible to determine the accuracy of the solutions obtained, the accuracy could be easily increased, and the computer time was shortened because of the lack of large matrices which require inversion.

The results obtained from the use of finite-element or finite-difference techniques are often of questionable accuracy when load or material parameters are rapidly varying quantities. The wind loads measured were certainly of a rapidly varying nature over a large portion of the surface. Errors of this type can be reduced in the case of finite-difference techniques by reducing the step size (increasing the number of cells). Examination of the problem quickly reveals that the number of cells required would easily push the problem beyond the limits of available computers.

Numerical integration techniques, although perhaps superior to finite-difference approaches, still give results of questionable accuracy, particularly in the case of long shells.

Thin shell theory as developed by Novozhilov (19) is used as the basic method of approach. Novozhilov's shell theory utilizes a complex substitution which reduces the order of the governing differential equations from eighth order to fourth order. This makes the use of a series solution practical, whereas without the use of a complex substitution a series solution would be extremely difficult to obtain because general solutions of the governing equations would be very difficult to generate.

The dome-cylinder problem has four basic parts: the expansion of the wind loads in series form, the solution of the cylinder problem, the solution of the dome problem, and the matching of the two solutions at the junction between the two shells.

II. REVIEW OF LITERATURE

1. Wind Loads

The problem of wind loads on structures has been studied for many years. Most of the studies resulted in the publication of isolated model tests or a series of model tests. There has been, however, two recent attempts to present information covering the entire field of wind loaded structures. In July, 1958, four papers (2), (25), (26), and (30) were published in the Proceedings of the American Society of Civil Engineers Journal of the Structures Division. These four papers provide a review of existing wind load information followed by discussions on the nature of wind and static and dynamic effects of wind. In June, 1963, an international conference on "Wind Effects on Buildings and Structures" was held at the National Physical Laboratory, Teddington, England. Apparently the proceedings of this conference were not published, but a summary of the conference was published in reference (6).

These two presentations define the current state of affairs in wind loaded structures. There was general agreement about many of the problems encountered and general disagreement about one particular area of interest.

There was an agreement of opinions about the shape of a natural boundary layer. The shape of the boundary layer was generally taken in the following form:

$$\left[\frac{V}{V_m}\right] = \left[\frac{H}{H_n}\right]^a$$

where
$a \approx \frac{1}{7}$ for flat open country

$a \approx \frac{1}{4}$ for rough hilly land

$a \approx \frac{1}{3}$ for centers of large cities

H = height above ground

V = velocity at height H

V_m = maximum velocity

H_n = height at which maximum velocity is obtained

Noting that the most rapid changes in velocity occur near the ground, it becomes clear that any model testing should be done in the presence of a boundary layer and that models tested on a ground board which suppresses the boundary layer would probably lead to invalid conclusions. Other areas of agreement include the facts that small changes in architectural details and variations in the Reynold's Number have very little effect on the overall pressure distribution.

There was, however, a large amount of disagreement about the height of the boundary layer. In reference (7), Davenport recommends a boundary layer of 900 feet for flat open country and of 1700 feet for the centers of large cities. Mr. H. S. Saffir, however, states without reference in (22) that the consensus of opinion seems to be that the velocity of a wind storm increases from the surface up to about 3,000 feet.

II. REVIEW OF LITERATURE

2. Stress Analysis

The theory of thin shells has not reached the point where there is only one set of governing equations generally accepted and used by everyone. The set of equations obtained depends upon the basic assumptions made and there are several assumptions which can be varied without changing the solutions of the equations beyond the point where they would no longer be useful in engineering design. Some of the more popular forms of the shell theory equations have existed for many years and are available in the following books: "Stresses in Shells" (8) by W. Flügge, "Theory of Plates and Shells" (22) by Timoshenko and Woinowsky-Krieger, "Theory of Elastic Thin Shells" (10) by Goldenveizer, and "Thin Shell Theory" (19) by Novoshilov. There are other books available on shell theory, but these four seem to be the most significant.

Although the equations have been available for many years, methods by which complicated shell of revolution problems (complicated because of shape, loading, or boundary conditions) could be solved have come into existence only in the past five years. The methods have, in general, used approximate techniques such as numerical or matrix-difference methods to obtain solutions.

In 1962, two papers appeared presenting techniques for solving thin shells of revolution loaded axisymmetrically. Radkowski, Davis, and Bolduc, in reference (21), used a numerical procedure to integrate the governing differential equations. Sepetaski, Pearson, Dingwell, and Adkins, in reference (24), develop a general computer program which uses an elimination method to solve the difference equations obtained from the basic differential equations.

In reference (27), Steele presents a technique which could be used to solve steep shells of revolution, provided that the loads vary slowly in the circumferential direction. This procedure is the well-known method of applying a linear combination of the membrane, inextensional, and edge effect solutions. This technique, although approximate for most shells, provides acceptable solutions for the cases cited above. Steele concentrated primarily on the problems encountered in solving for the arbitrary constants.

Budiansky and Radkowski published reference (4) in 1963. Here, a general numerical procedure is developed for the elastic stress and deflection analysis of shells of revolution subject to arbitrary loads. Solutions are obtained by matrix analysis of the finite-difference form of the governing equations. The procedure involves the expansion of all pertinent load, stress, and deformation variables into Fourier series. An outstanding feature of the technique presented is the use of the equations of shell theory as developed by Sanders (23). The equations developed by Sanders are reported (5) to be the best available.

The multi-segment method of numerical integration was applied to arbitrarily loaded shells of revolution in 1964 by A. Kalnins (16). This method eliminates the loss of accuracy usually encountered in the use of a computer in numerical integration. Prior to Kalnins' use of the multi-segment method, numerical integration could be applied only to shells whose length was small compared to its principal radius.

A conference on "Matrix Methods in Structural Mechanics" was held in October, 1965, at Wright-Patterson Air Force Base, Ohio. Although several of the papers presented (13), (15), and (16) dealt directly with shell analysis, none offered a procedure for the solution of shell of revolution problems which improved on the procedures developed by Budiansky and Radkowski or Kalnins.

Several examples of a stress analysis of wind loaded structures have appeared in the literature. Timoshenko and Woinowsky-Krieger assumed a wind load of the form

$$Z = P \cos \theta \sin \phi$$

acting on a hemisphere. Novozhilov (19) and Flügge (8) presented very similar problems. In reference (11), Gondikas and Salvadori solved a hemispherical dome elastically built-in into a cylinder. The wind loads were taken as

$$Z = P \cos\theta \sin\phi \text{ on the hemisphere and}$$

$$Z = P \cos\theta \text{ on the cylinder.}$$

It was an excellent presentation except that the authors tended to make statements implying that the solutions obtained could be used by the designer for all cases of practical significance. The paper was criticized for this reason in a discussion by Talbes (28). Talbes showed that although the wind load expressions might give reasonably good results in calculation of the resultant drag force, they would be quite unrealistic in stress and deformation analysis. For instance, at $\theta = 90^{o}$ the value of the pressure is zero in the given expressions, whereas the test results give the maximum negative pressures at about 90^{o}.

A Ph. D. thesis by Garcia (9) should also be mentioned here. Garcia obtained solutions for the membrane stresses in a wind loaded hemisphere. The wind load used was an actual pressure distribution obtained from wind tunnel tests at Virginia Polytechnic Institute.

This literature review is presented to show the attempts at development of methods of solution for shells of revolution under wind loads. It should be mentioned, however, that the primary reference used was Novozhilov's book "Thin Shell Theory" (19).

III. SERIES REPRESENTATION OF FUNCTIONS FOR USE IN SHELL ANALYSIS

The data obtained from the wind tunnel tests, which existed only as discrete values at certain points, was put in functional form so that it could be used in the differential equations governing the behavior of the shell. The nature of the loads was such that a closed form representation of the data would not provide the accuracy which was required. It was necessary to use a series representation of the wind loads not only to obtain the accuracy required but also to make solution of the governing equations possible. Two types of series were found which could be used effectively in the equations of shell analysis; the Fourier series with variable coefficients, and the spherical harmonic series.

III. A. FOURIER SERIES

Using the coordinate ξ and θ, where ξ has the units of length and θ is measured in radians, a function $F(\xi, \theta)$ may be expressed in the form of a Fourier series provided that

$$F(\xi, \theta) = F(\xi, \theta + 2\pi).$$

The Fourier series would take the form

$$F(\xi, \theta) = A(\xi) + \sum_{n=1}^{\infty} A_n(\xi) \cos n\theta + \sum_{n=1}^{\infty} B_n(\xi) \sin n\theta \quad (3.01)$$

The Fourier coefficients used in (3.01) are given as follows:

$$A(\xi) = \frac{1}{2\pi}\int_o^{2\pi} F(\xi,\theta)\, d\theta$$

$$A_n(\xi) = \frac{1}{\pi}\int_o^{2\pi} F(\xi,\theta)\cos n\theta\, d\theta \qquad (3.02)$$

$$B_n(\xi) = \frac{1}{\pi}\int_o^{2\pi} F(\xi,\theta)\sin n\theta\, d\theta$$

In the analysis of wind loaded shells of revolution, all functions encountered are of one of two types.

Either $F(\xi, \theta) = F(\xi, -\theta)$

or $F(\xi, \theta) = -F(\xi, -\theta)$.

For the case $F(\xi, \theta) = F(\xi, -\theta)$ the coefficients $B_n(\xi)$ vanish and

$$F(\xi, \theta) = A(\xi) + \sum_{n=1}^{\infty} A_n(\xi)\cos n\theta \qquad (3.03)$$

For the case $F(\xi, \theta) = -F(\xi, -\theta)$ the coefficients $A(\xi)$ and $A_n(\xi)$ vanish and

$$F(\xi, \theta) = \sum_{n=1}^{\infty} B_n(\xi)\sin n\theta$$

The coefficients of the Fourier cosine and the Fourier sine series are now given as:

$$A(\xi) = \frac{1}{\pi}\int_o^{\pi} F(\xi,\theta)\, d\theta$$

$$A_n(\xi) = \frac{2}{\pi}\int_o^{\pi} F(\xi,\theta)\cos n\theta\, d\theta$$

$$B_n(\xi) = \frac{2}{\pi}\int_o^{\pi} F(\xi,\theta)\sin n\theta\, d\theta$$

In order to use equations (3.03) and (3.04), it was first necessary to develop a procedure for expressing $F(\xi, \theta)$. Since the wind tunnel data exists as discrete values known at certain points on the surface, the function $F(\xi, \theta)$ was first expanded into polynomials in ξ for specific values of θ from 0° to 180° in fifteen degree intervals.

$$F(\xi, 15^\circ) = D_o + D_1 \xi + D_2 \xi^2 + D_3 \xi^3 + D_4 \xi^4 + D_5 \xi^5 \qquad (3.04)$$

It was then assumed that for the value of θ used in an equation of the form (3.04) $F(\xi, \theta)$ is only a function of ξ over an arc 7.5° less than to 7.5° greater than the value of θ. This assumption led to the evaluation of $A(\xi)$ as follows:

$$A(\xi) = \frac{1}{\pi} \left[\int_{o}^{7.5^\circ} F(\xi, 0^\circ)\, d\theta + \int_{7.5^\circ}^{22.5^\circ} F(\xi, 15^\circ)\, d\theta + \ldots \right.$$

$$\left. + \int_{157.5^\circ}^{172.5^\circ} F(\xi, 165^\circ)\, d\theta + \int_{172.5^\circ}^{180^\circ} F(\xi, 180^\circ)\, d\theta \right] \qquad (3.05)$$

In a like fashion the value for $A_n(\xi)$ and $B_n(\xi)$ can be found.

III. SPHERICAL HARMONICS

From Hobson (12), a function $F(\theta, \phi)$ may be expanded in a series of the form

$$F(\theta, \phi) = \sum_{m=0}^{\infty} A_m P_m(\cos\theta) + \sum_{m=0}^{\infty} \sum_{n=0}^{n=m} [A_{m,n} \cos n\phi$$

$$+ B_{m,n} \sin n\phi] P_m^n (\cos\theta) \tag{3.06}$$

where θ and ϕ are the usual spherical coordinates and $P_m(\cos\theta)$ and $P_m^n(\cos\theta)$ are the Legendre polynomials and the associated Legendre polynomials, respectively. The coefficients are given as

$$A_n = \frac{2n+1}{4\pi} \int_{-1}^{+1} \int_{o}^{2\pi} P_n(\cos\theta)\, F(\theta, \phi)\, d(\cos\theta)\, d\phi \tag{3.07}$$

$$A_{n,m} = \frac{2n+1}{2\pi} \frac{(n-m)!}{(m+m)!} \int_{-1}^{+1} \int_{o}^{2\pi} P_n^m(\cos\theta) \cos m\phi \;\; F(\theta, \phi)\, d(\cos\theta) d\phi$$

$$B_{n,m} = \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \int_{-1}^{+1} \int_{o}^{2\pi} P_n^m(\cos\theta) \sin m\phi \;\; F(\theta, \phi)\, d(\cos\theta)\, d\phi$$

For functions of the type $F(\theta, \phi) = F(\theta, -\phi)$, $B_{n,m} = 0$, also since $P_n^m(\cos\theta) = P_n^m(-\cos\theta)$ for $(m + n)$ even and because the function $F(\theta, \phi)$ is arbitrary for $\theta > 90^\circ$ the series takes on the following form:

$$F(\theta, \phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{n=m} A_{n,m} \cos n\phi \; P_m^n(\cos\theta) \qquad \begin{array}{l} m-n = 2j \\ j = 0, 1, 2, 3.... \end{array} \tag{3.08}$$

$$A_{n,m} = C \frac{2m+1}{2\pi} \frac{(m-n)!}{(m+n)!} \int_o^{\pi} \int_o^{\pi/2} F(\theta, \phi) P_m^n(\cos\theta) \sin\theta \cos n\phi \; d\theta\, d\phi$$

where $C = 2$ for $n = 0$

and $C = 4$ for $n \neq 0$.

$A_{n,m}$ can now be rewritten as

$$A_{n,m} = C \frac{2m+1}{2\pi} \frac{(m-n)!}{(m+n)!} \int_o^{\pi} Q_{n,m} \cos n\phi \, d\phi \tag{3.09}$$

where

$$Q_{n,m} = \int_o^{\pi/2} F(\theta,\phi) \, P_m^n(\cos\theta) \sin\theta \, d\theta \tag{3.10}$$

Similarly for functions of the type $F(\theta,\phi) = F(\theta,-\phi)$

$$F(\theta,\phi) = \sum_{m=1}^{\infty} \sum_{n=1}^{n=m} B_{n,m} \sin n\phi \, P_m^n(\cos\theta) \qquad \begin{matrix} m-n = 2j \\ j = 0, 1, 2, 3 \ldots \end{matrix} \tag{3.11}$$

where

$$B_{n,m} = \frac{2(2m+1)}{\pi} \frac{(m-n)!}{(m+n)!} \int_o^{\pi} Q_{n,m} \sin n\phi \, d\phi . \tag{3.12}$$

To find the coefficients of the Legendre cosine (3.08) or the Legendre sine (3.11) series, the quantity $Q_{n,m}$ must first be found. The $Q_{n,m}$'s were solved for at constant values of ϕ using a polynomial representation of $F(\theta,\phi)$ in that region.

$$F(\theta, 15^o) = D_o + D_1\theta + D_2\theta^2 + D_3\theta^3 + D_4\theta^4 + D_5\theta^5 \tag{3.13}$$

Combining (3.13) with the values of $P_m^n(\cos\theta)$ given in Table I, equation (3.10) can now be integrated and $Q_{n,m}$ found. With $Q_{n,m}$ known at certain values of ϕ, equation (3.09) or (3.12) can be integrated in the same manner that equation (3.04) was integrated.

Computer programs were written in Fortran IV which would take data existing as discrete values at certain points on a surface and, utilizing the procedure discussed, return either a Fourier cosine, a Fourier sine, a Legendre cosine, or a Legendre sine series. The accuracy of the fit given by series representation is demonstrated in Figure 3. In Figure 3 the solid lines are the original data, the circles are the fit given by a Fourier cosine series and the triangles are the points given by a Legendre cosine series. The data used was a measured wind load over the surface of a hemisphere.

TABLE 1

Certain Associated Legendre Polynomials

n	m	$F(\cos\theta, \sin\theta) = P_m^n \cos\theta$
0	0	1
0	2	$(1/2)(\cos^2\theta - 1)$
0	4	$(1/8)(35\cos^4\theta - 30\cos^2\theta + 3)$
0	6	$(1/16)(231\cos^6\theta - 315\cos^4\theta + 105\cos^2\theta - 5)$
0	8	$(1/128)(6435\cos^8\theta - 12012\cos^6\theta + 6930\cos^4\theta - 1260\cos^2\theta + 35)$
1	1	$\sin\theta$
1	3	$(3/2)(5\cos^2\theta - 1)\sin\theta$
1	5	$(15/8)(21\cos^4\theta - 14\cos^2\theta + 1)\sin\theta$
1	7	$(7/16)(429\cos^6\theta - 495\cos^4\theta + 135\cos^2\theta - 5)\sin\theta$
2	2	$3\sin^2\theta$
2	4	$(15/2)(7\cos^2\theta - 1)\sin^2\theta$
2	6	$(105/8)(33\cos^4\theta - 18\cos^2\theta + 1)\sin^2\theta$
2	8	$(315/16)(143\cos^6\theta - 143\cos^4\theta + 33\cos^2\theta - 1)\sin^2\theta$
3	3	$15\sin^3\theta$
3	5	$(105/2)(9\cos^2\theta - 1)\sin^3\theta$
3	7	$(315/8)(143\cos^4\theta - 66\cos^2\theta + 3)\sin^3\theta$
4	4	$105\sin^4\theta$

TABLE 1, Continued

4	6	$(945/2)(11 \cos^2 \theta - 1) \sin^4 \theta$
4	8	$(10395/8)(65 \cos^4 \theta - 26 \cos^2 \theta + 1) \sin^4 \theta$
5	5	$945 \sin^5 \theta$
5	7	$(10395/2)(13 \cos^2 \theta - 1) \sin^6 \theta$
6	6	$10395 \sin^6 \theta$
6	8	$(135135/2)(15 \cos^2 \theta - 1) \sin^6 \theta$
7	7	$135135 \sin^7 \theta$
7	8	$2027025 \sin^8 \theta$

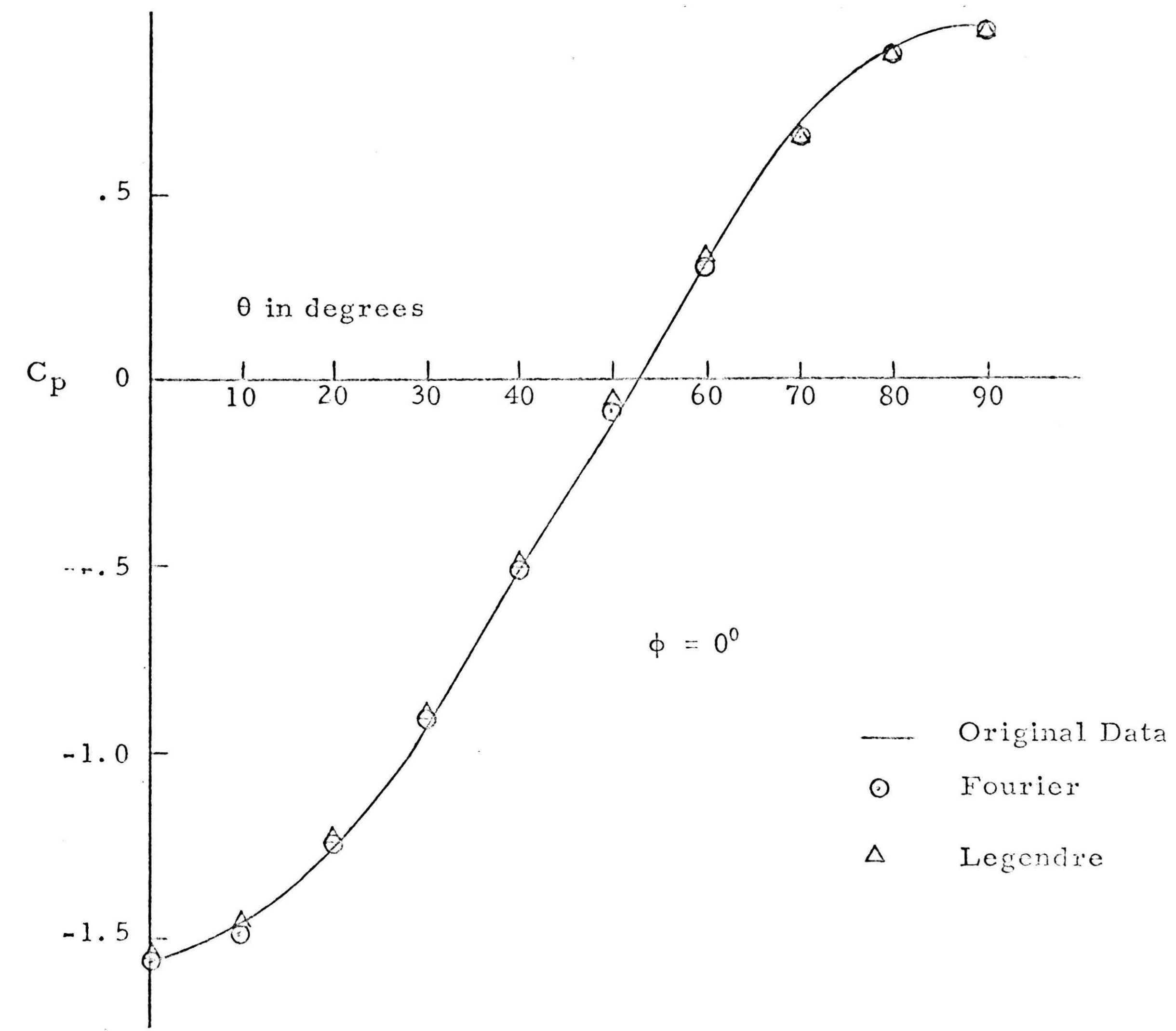

Figure 3. Demonstration of Surface Fit

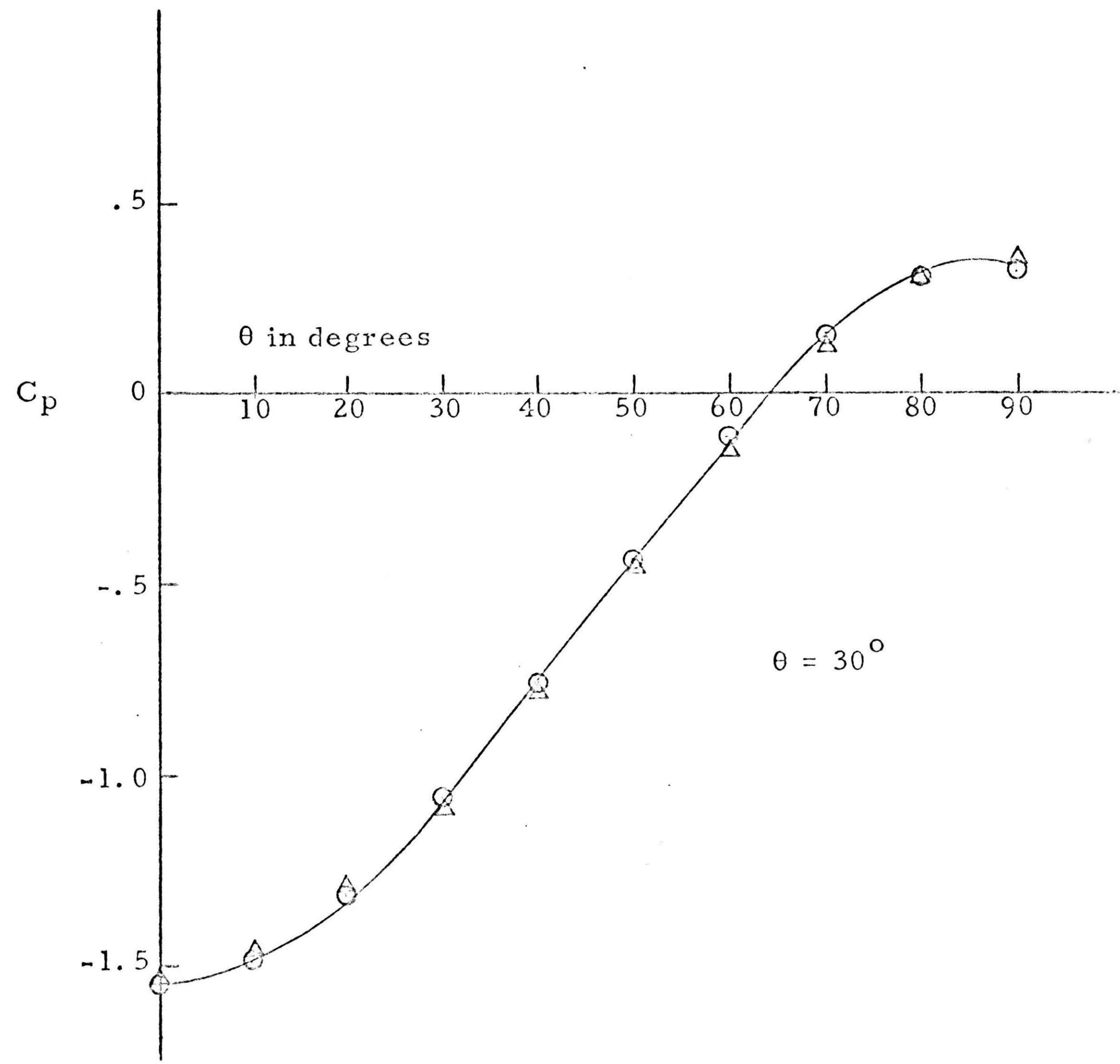

Figure 3. Demonstration of Surface Fit, continued

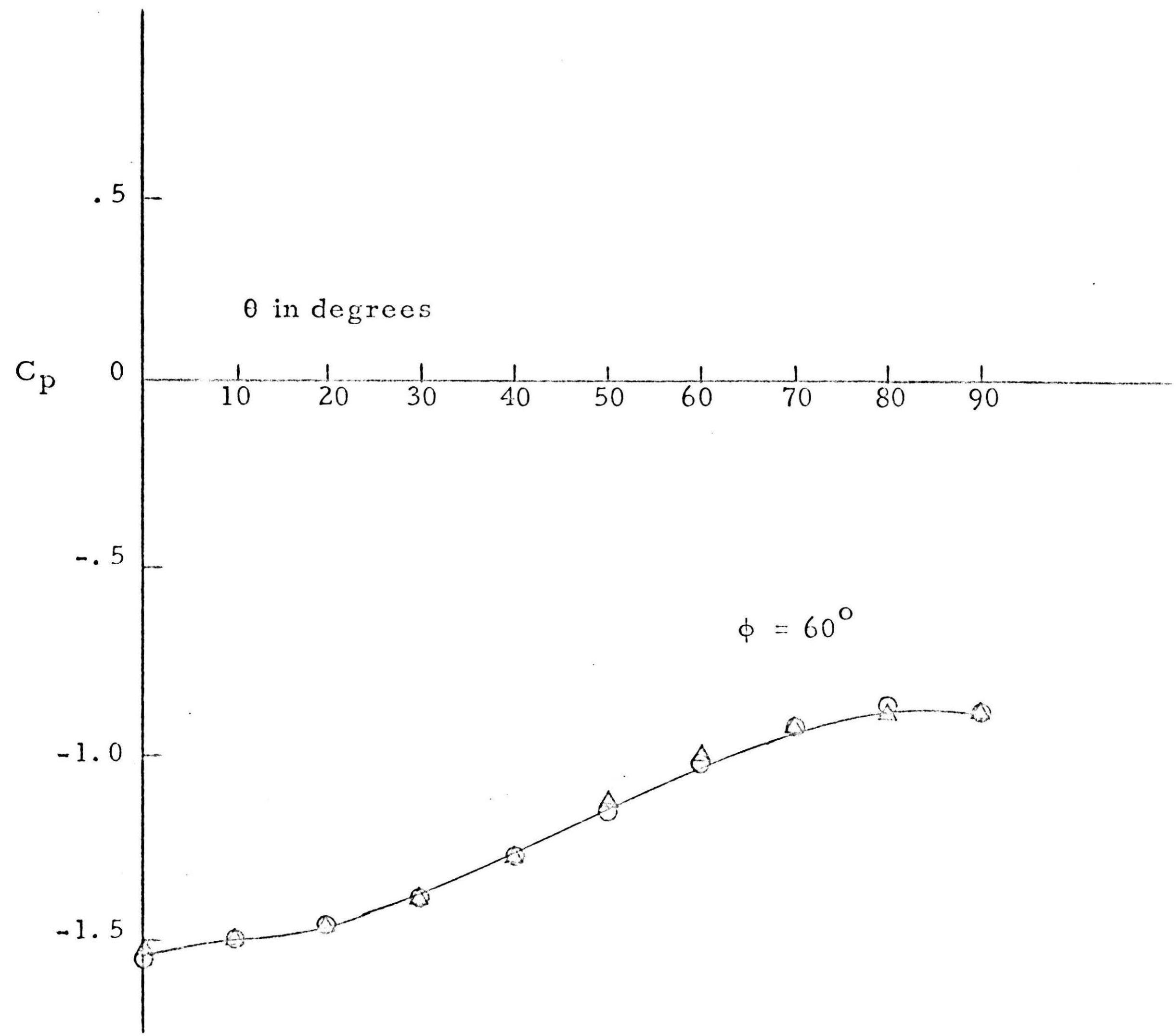

Figure 3. Demonstration of Surface Fit, continued

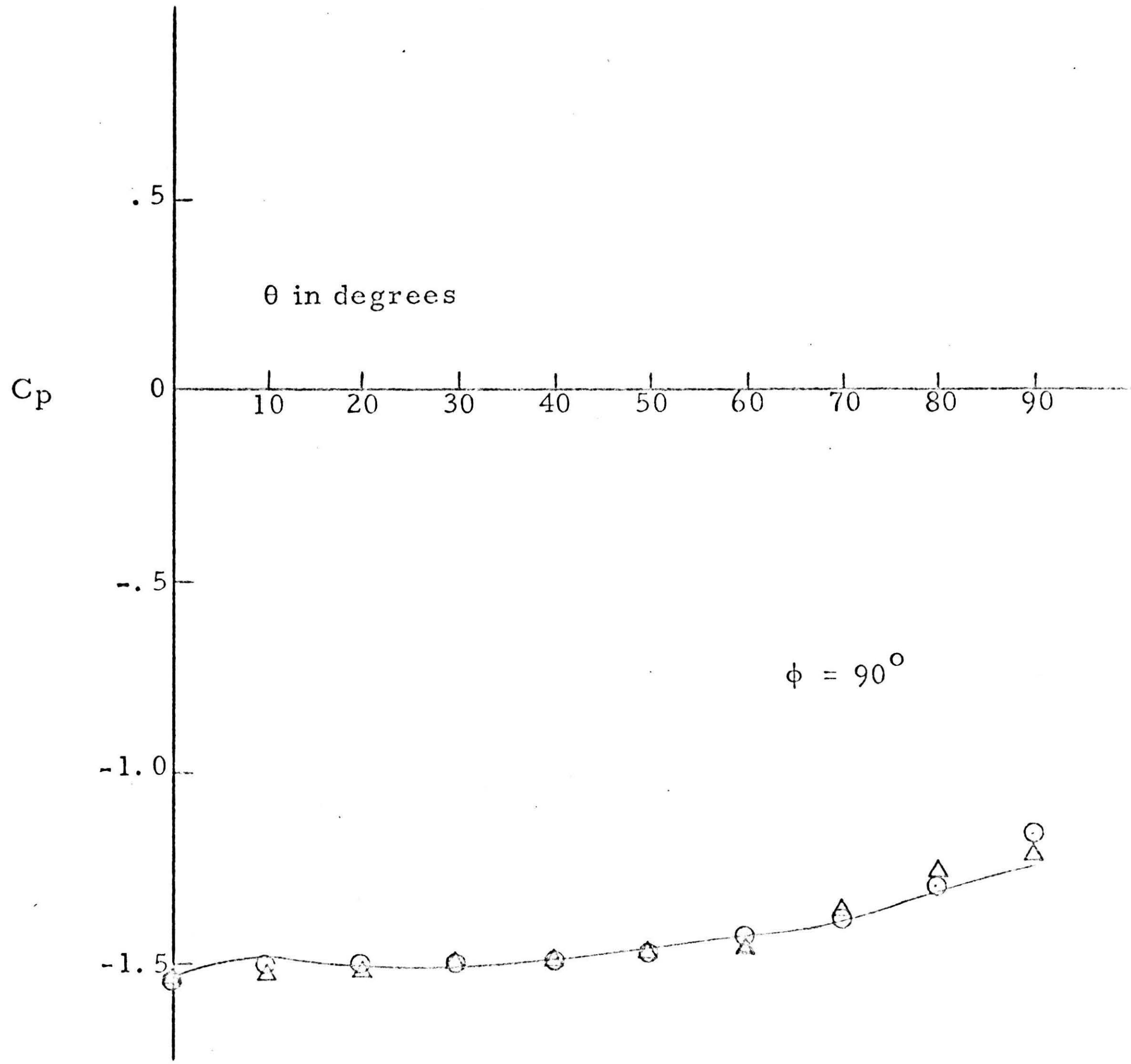

Figure 3. Demonstration of Surface Fit , continued

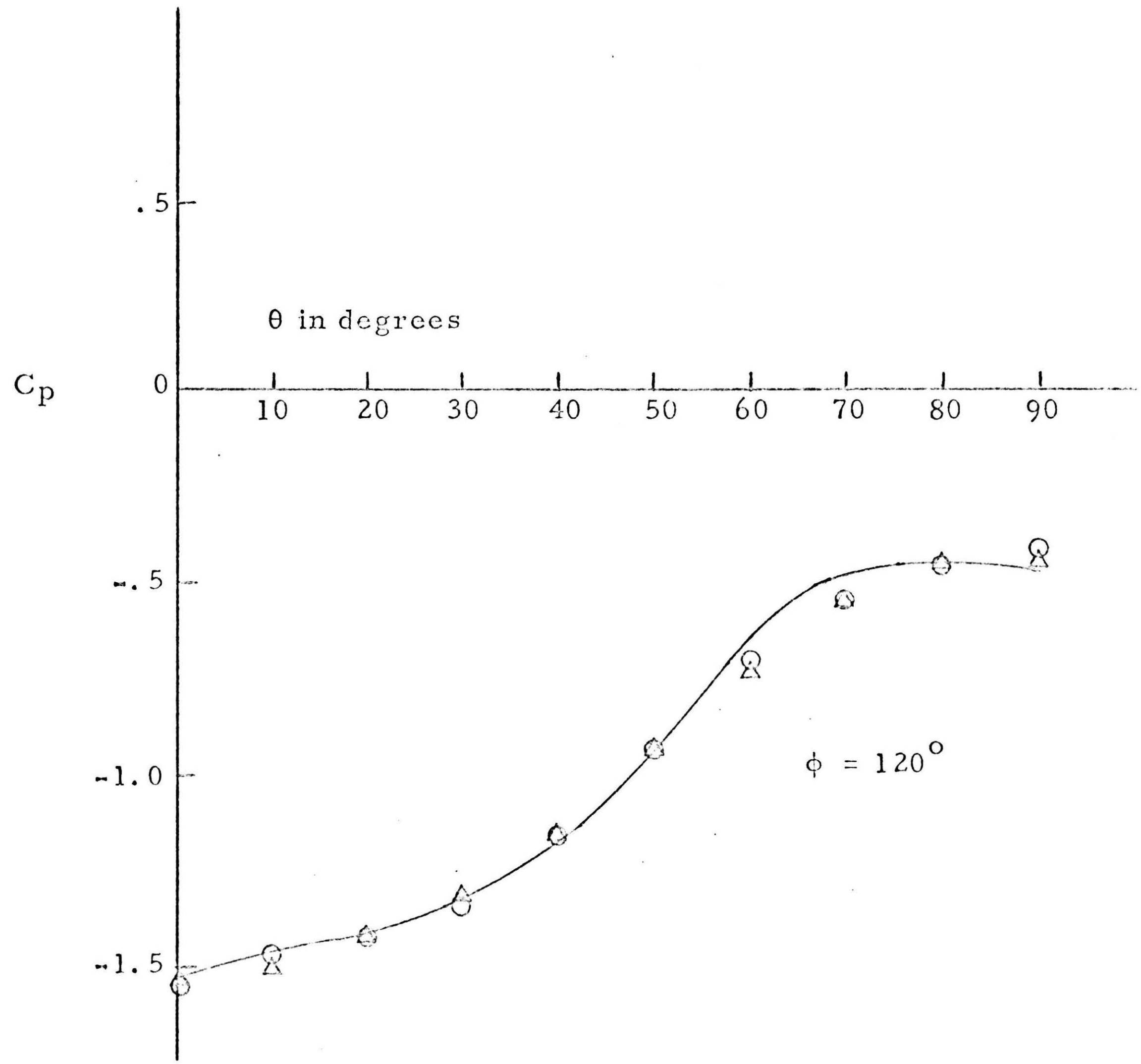

Figure 3. Demonstration of Surface Fit, continued

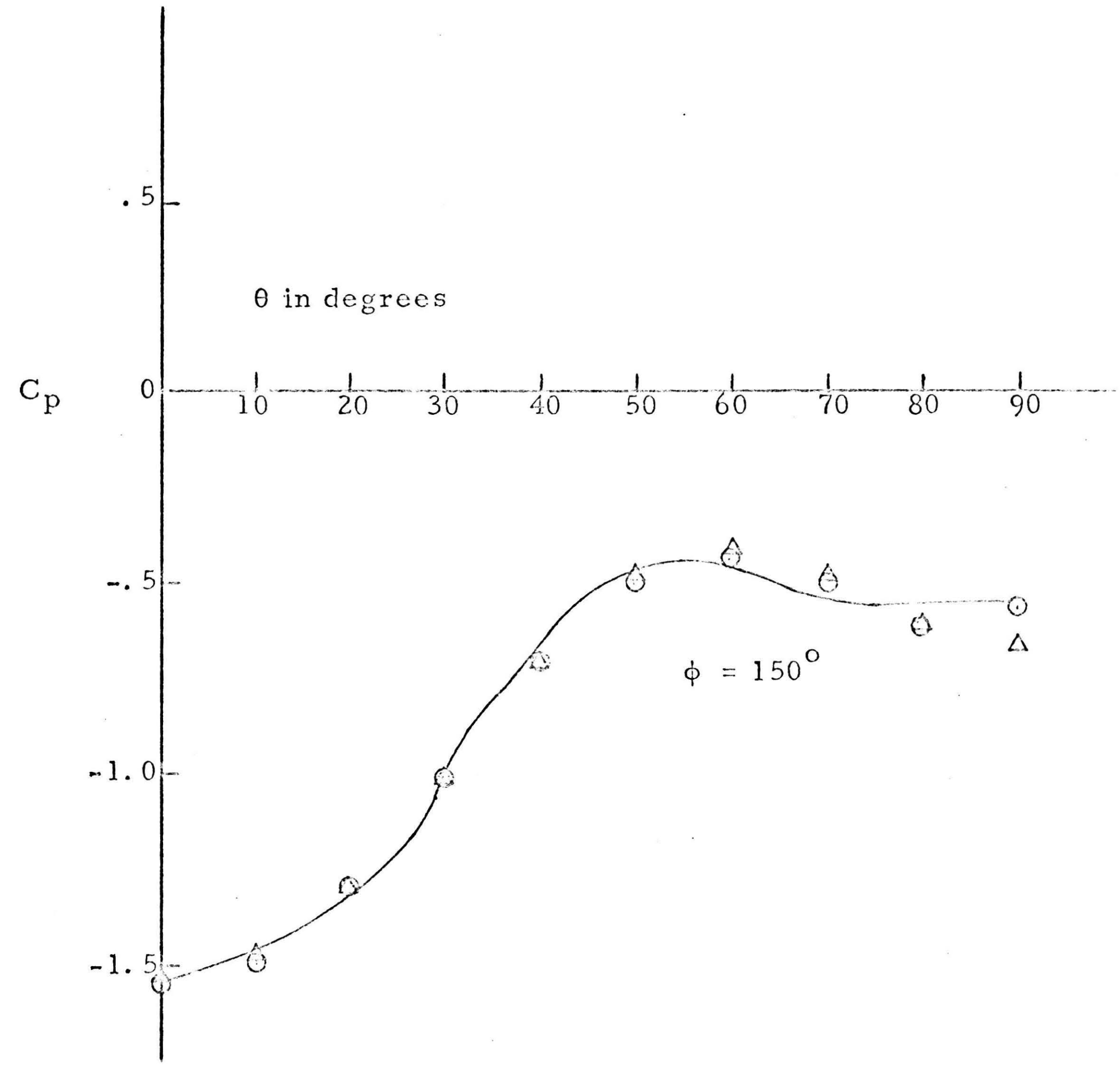

Figure 3. Demonstration of Surface Fit, continued

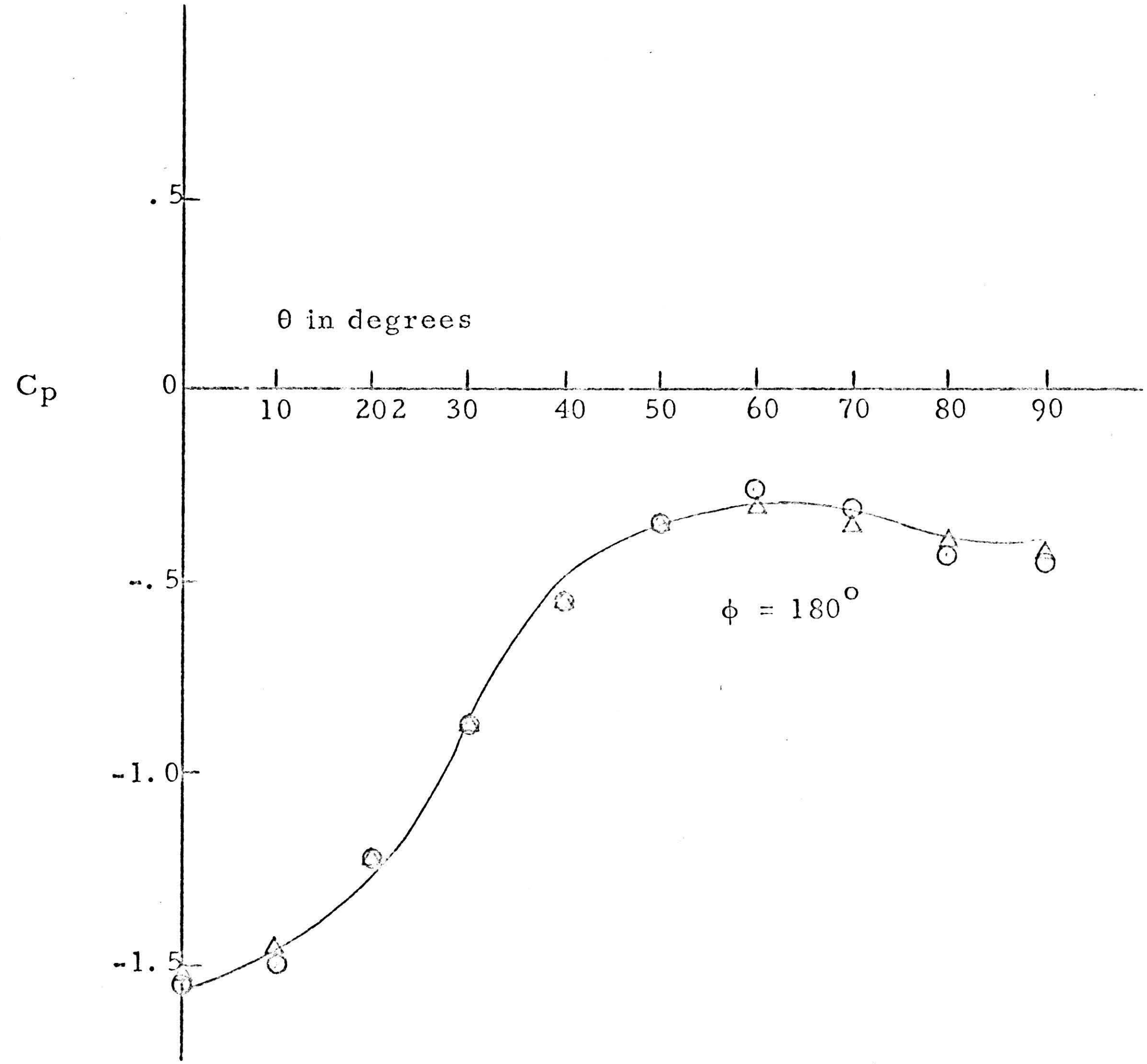

Figure 3. Demonstration of Surface Fit, continued

IV. FORMULATION OF A COMPLEX SHELL THEORY

This section is presented to familiarize the reader with shell theory as developed by V. V. Novozhilov in his book "Thin Shell Theory" (19). Novozhilov's approach to shell theory differs from most approaches, in that he introduces a complex substitution which reduces the order of the governing equations from eighth to fourth order. A brief outline of this shell theory is presented here; for greater detail the reader is referred to Novozhilov's book.

A standard form of the equilibrium equations appear as follows:

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 T_1}{\partial \alpha_1} + \frac{\partial A_1 T_{21}}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2} T_{12} - \frac{\partial A_2}{\partial \alpha_1} T_2\right] + \frac{N_1}{R_1} + q_1 = 0$$

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 T_{12}}{\partial \alpha_1} + \frac{\partial A_1 T_2}{\partial \alpha_2} + \frac{\partial A_2}{\partial \alpha_2} + T_{21} - \frac{\partial A_1}{\partial \alpha_2} T_1\right] + \frac{N_2}{R_2} + q_2 = 0$$

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 N_1}{\partial \alpha_1} + \frac{\partial A_1 N_2}{\partial \alpha_2}\right] - \frac{T_1}{R_1} - \frac{T_2}{R_2} + q_n = 0 \qquad (4.01)$$

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 M_1}{\partial \alpha_1} + \frac{\partial A_1 M_{21}}{\partial \alpha_2} + \frac{\partial A_1}{\partial A_2} M_{12} - \frac{\partial A_2}{\partial \alpha_1} M_2\right] - N_1 = 0$$

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 M_{12}}{\partial \alpha_1} + \frac{\partial A_1 M_2}{\partial \alpha_2} + \frac{\partial A_2}{\partial \alpha_1} M_{21} - \frac{\partial A_1}{\partial \alpha_2} M_1\right] - N_2 = 0$$

$$T_{12} - T_{21} + \frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} = 0$$

The positive directions of the force and moment resultants are defined in Figure 4.

A simplified form (certain small terms are neglected) of the equations of compatibility in terms of forces and moments

$$(1+\mu) N_1 \approx \frac{1}{A_1} \frac{\partial M}{\partial a_1} - \frac{\delta^2}{12} \frac{1}{R_1 A_1} \frac{\partial T}{\partial a_1}$$

$$(1+\mu) N_2 \approx \frac{1}{A_2} \frac{\partial M}{\partial a_2} - \frac{\delta^2}{12} \frac{1}{R_2 A_1} \frac{\partial T}{\partial a_2} \qquad (4.02)$$

$$\frac{M_1 - \mu M_2}{R_2} + \frac{M_2 - \mu M_1}{R_1} + \frac{\delta^2}{12} \Delta(T) = \frac{\delta^2}{12} \frac{(1+\mu)}{A_1 A_2} \left[\frac{\partial A_2 q_1}{\partial a_1} + \frac{\partial A_1 q_1}{\partial a_2} \right]$$

was next used to rewrite the equilibrium equations.

$$\frac{1}{A_1 A_2} \left[\frac{\partial A_2 T_1}{\partial a_1} + \frac{\partial A_1 S}{\partial a_2} + \frac{\partial A_1 S}{\partial a_2} - \frac{\partial A_2}{\partial A_1} T_2 \right] + \frac{1}{1+\mu} \frac{1}{R_1 A_1} \frac{\partial M}{\partial a_1} + q_1 = 0$$

$$\frac{1}{A_1 A_2} \left[- \frac{\partial A_2 (M_2 - \mu M_1)}{\partial a_1} + (1+\mu) \left(\frac{\partial A_1 H}{\partial a_3} + \frac{\partial A_1}{\partial a_2} H \right) + \frac{\partial A_2}{\partial a_1} (M_1 - \mu M_2) \right] + \frac{\delta^2}{12 R_1 A_1} \frac{\partial T}{\partial a_1} = 0 \qquad (4.03)$$

$$\frac{1}{A_1 A_2} \left[\frac{\partial A_2 S}{\partial a_1} + \frac{\partial A_1 T_2}{\partial a_2} + \frac{\partial A_2}{\partial a_1} S - \frac{\partial A_1}{\partial a_2} T_1 \right] + \frac{1}{1+\mu} \frac{1}{R_2 A_2} \frac{\partial M}{\partial a_2} + q_2 = 0$$

$$\frac{1}{A_1 A_2} \left[- \frac{\partial A_1 (M_1 - \mu M_2)}{\partial a_2} + (1+\mu) \left(\frac{\partial A_2 H}{\partial a_1} + \frac{\partial A_2}{\partial a_1} H \right) + \frac{\partial A_1}{\partial a_2} (M_2 - \mu M_1) \right] + \frac{\delta^2}{12 R_2 A_2} \frac{\partial T}{\partial a_2} = 0$$

$$\frac{T_1}{R_1} + \frac{T_2}{R_2} - \frac{1}{1+\mu} \Delta(M) - q_3 = 0$$

$$\frac{M_2 - \mu M_1}{R_1} + \frac{M - \mu M_2}{R_2} + \frac{\delta^2}{12} \Delta T = - \frac{\delta^2}{12} \frac{1+\mu}{A_1 A_2} \left[\frac{\partial A_2 q_1}{\partial a_1} + \frac{\partial A_1 q_2}{\partial a_1} \right]$$

where $T = T_1 + T_2$

$M = M_1 + M_2$

μ = Poisson's Ratio

$S = T_{12} \approx T_{21}$

$H = M_{12} = M_{21}$

$$\Delta = \frac{\partial^2}{\partial a_1{}^2} (\ ) + \frac{\partial^2}{\partial a_2{}^2} (\ ).$$

Introducting the auxiliary functions

$$\begin{aligned} \tilde{T}_1 &= T_1 - \frac{i}{c} \frac{M_2 - \mu M_1}{1 - \mu^2} \\ \tilde{T}_2 &= T_2 - \frac{i}{c} \frac{M_1 - \mu M_2}{1 - \mu^2} \\ \tilde{S} &= S + \frac{i}{c} \frac{H}{1 - \mu} \\ c &= \frac{\delta}{\sqrt{12(1-\mu^2)}} \end{aligned} \qquad (4.04)$$

into equations (4.03) and simplifying, the equilibrium equations are found in the form:

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 \widetilde{T}_1}{\partial a_1} + \frac{\partial A_1 \widetilde{S}}{\partial a_2} + \frac{\partial A_1 \widetilde{S}}{\partial a_2} - \frac{\partial A_2}{2a_1} T_2\right] + \frac{i c}{R_1 A_1}\frac{\partial \widetilde{T}}{\partial a_1} + q_1 = 0$$

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 \widetilde{S}}{\partial a_1} + \frac{\partial A_1 \widetilde{T}_2}{\partial a_2} + \frac{\partial A_2 \widetilde{S}}{\partial a_1} - \frac{\partial A_1}{\partial a_2} \widetilde{T}_1\right] + \frac{i c}{R_2 A_2}\frac{\partial \widetilde{T}}{\partial a_2} + q_2 = 0 \qquad (4.05)$$

$$\frac{\widetilde{T}_1}{R_1} + \frac{\widetilde{T}_2}{R_2} - i c \Delta \widetilde{T} = \widetilde{q}_3$$

where $$\widetilde{q}_3 = q_3 + i \frac{(1+\mu)}{A_1 A_2} c \left[\frac{\partial A_2 q_1}{\partial a_2} + \frac{\partial A_1 q_2}{\partial a_2}\right]$$

Although equations (4.05) are a complete set of equations, two other systems of equations will be needed for a complete analysis. These two other systems of equations are the strain-displacement equations

$$\varepsilon_1 = \frac{1}{A_1}\frac{\partial u}{\partial a_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial a_2} v + \frac{w}{R_1}$$

$$\varepsilon_2 = \frac{1}{A_1}\frac{\partial v}{\partial a_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial a_1} u + \frac{w}{R_2}$$

$$\omega = \frac{A_2}{A_1}\frac{\partial}{\partial a_1}\left[\frac{v}{A_2}\right] + \frac{A_1}{A_2}\frac{\partial}{\partial a_3}\left[\frac{u}{A_1}\right] \qquad (4.06)$$

$$\kappa_1 = -\frac{1}{A_1}\frac{\partial}{\partial a_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial a_1} - \frac{u}{R_1}\right) - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial a_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial a_2} - \frac{v}{R_2}\right)$$

$$\kappa_2 = -\frac{1}{A_2}\frac{\partial}{\partial a_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial a_2} - \frac{v}{R_2}\right) - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial a_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial a_1} - \frac{u}{R_1}\right)$$

$$\tau = -\frac{1}{A_1 A_2}\left(\frac{\partial^2 w}{\partial a_1 \partial_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial a_2}\frac{\partial w}{\partial a_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial a_1}\frac{\partial w}{\partial a_2}\right) + \frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial u}{\partial a_2} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial a_2} u\right)$$

$$+ \frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial v}{\partial a_1} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial a_1} v\right)$$

and the force-strain relationships.

$$\tilde{\varepsilon}_1 = \frac{1}{E\delta}(\tilde{T}_1 - \mu\tilde{T}_2)$$

$$\tilde{\varepsilon}_2 = \frac{1}{E\delta}(\tilde{T}_2 - \mu\tilde{T}_1)$$

$$\tilde{\omega} = \frac{(1+\mu)}{E\delta}\tilde{S} \tag{4.07}$$

$$\tilde{\kappa}_1 = \frac{i}{c}\frac{1}{E\delta}(\tilde{T}_2 - T_2{}^*)$$

$$\tilde{\kappa}_2 = \frac{i}{c}\frac{1}{E\delta}(\tilde{T}_1 - T_1{}^*)$$

$$\tilde{\tau} = -\frac{i}{c}\frac{1}{E\delta}(\tilde{S} - S^*)$$

The symbols $T_1{}^*$, $T_2{}^*$, and S^* are solutions to the membrane equations which can be written as

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 T_1{}^*}{\partial a_1} + \frac{\partial A_1 S^*}{\partial a_2} + \frac{\partial A_1}{\partial a_2} S^* - \frac{\partial A_2}{\partial a_1} T_2{}^*\right] + q_1 = 0$$

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 S^*}{\partial a_1} + \frac{\partial A_1 T_2{}^*}{\partial a_2} + \frac{\partial A_2}{\partial a_1} S^* - \frac{\partial A_1}{\partial a_2} T_1{}^*\right] + q_2 = 0 \tag{4.08}$$

$$\frac{T_1{}^*}{R_1} + \frac{T_2{}^*}{R_2} = q_n$$

and $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$, $\tilde{\omega}$, $\tilde{\kappa}_1$, $\tilde{\kappa}_2$, and $\tilde{\tau}$ are the complex strains, the real part of which are the strains ε_1, ε_2, ω, κ_1, κ_2 and $\tilde{\tau}$.

The system (4.05) is a general system of equations governing the behavior of thin shells under static loading. After a general solutio to a particular problem has been obtained from the system (4.05), the strain-displacement equations (4.06), the force-strain relationships (4.07), and the membrane equations (4.08) are used to obtain the complete solution.

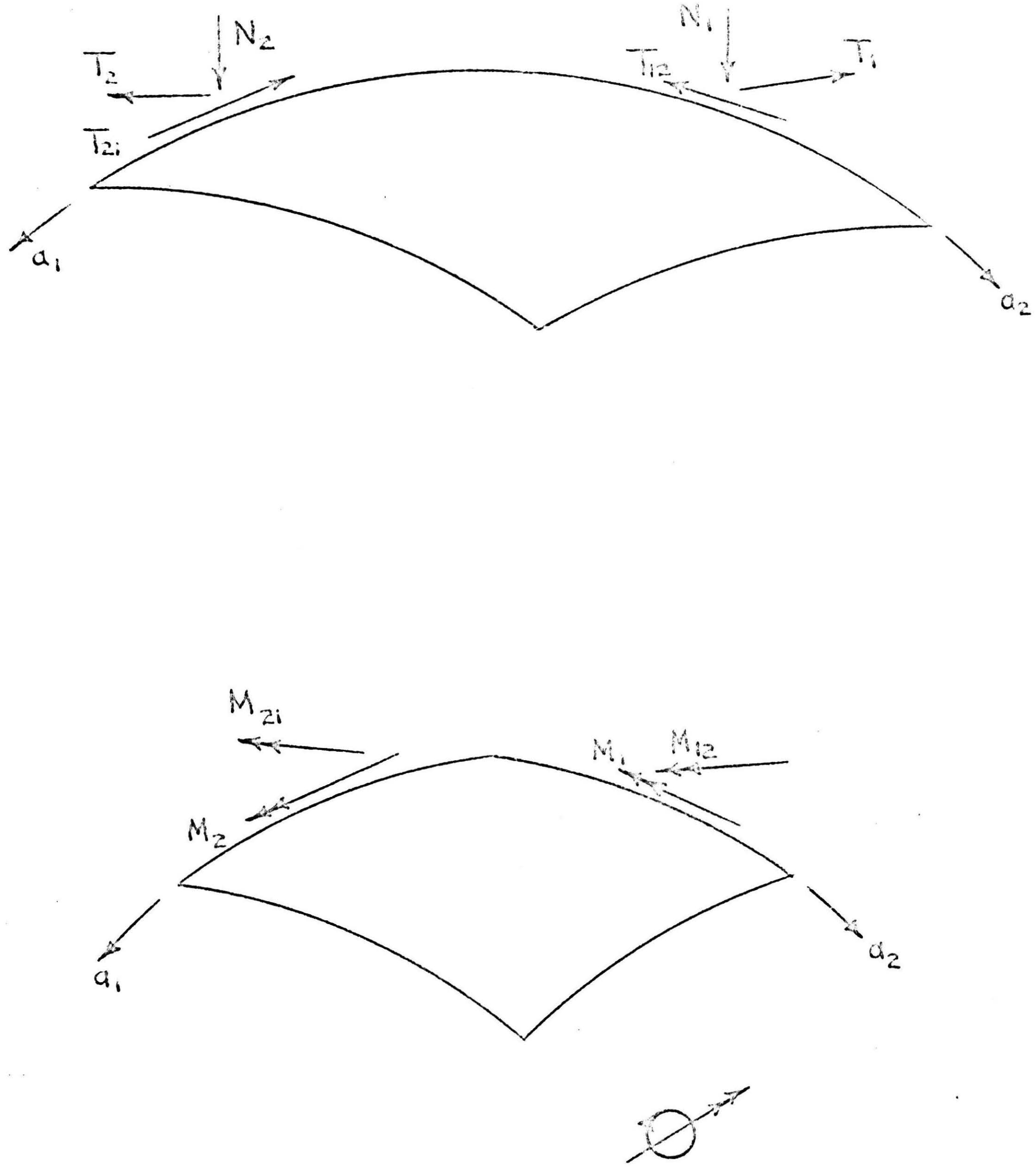

Figure 4. Positive Direction of the Force and Moment

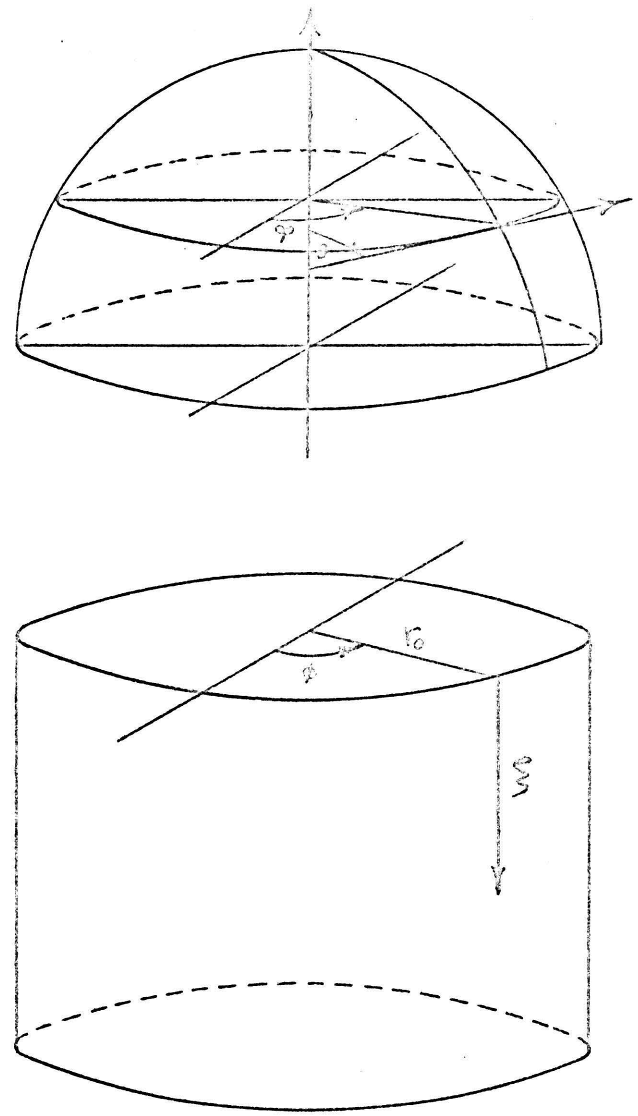

Figure 5. Coordinates on the Cylinder and Sphere

V. ANALYSIS OF THE CYLINDRICAL SHELL

For the special case of the cylindrical shell,

$$R_1 = \infty$$

$$R_2 = \rho r_o$$

$$A_1 = r_o$$

$$A_2 = r_o$$

$$\alpha_1 = \xi = x/r_o$$

$$\alpha_2 = \eta = s/r_o$$

$$\rho = r/r_0,$$

where r_o is a conveniently chosen constant and x, r, and s are given in Figure ( ). Equations (4.05), (4.06), and (4.07) take on the following form:

$$\begin{aligned} &\frac{\partial T_1}{\partial \xi} + \frac{\partial \tilde{S}}{\partial \eta} = -q_1 r_o \\ &\frac{\partial \tilde{S}}{\partial \xi} + \frac{\partial \tilde{T}_2}{\partial \eta} + \frac{i}{\partial b^2} \frac{1}{\rho} \frac{\partial \tilde{T}}{\partial \eta} = -q_2 r_o \\ &\tilde{T}_2 - \frac{i}{2b^2} \rho \Delta \tilde{T} = \rho \tilde{q}_3 r_o \end{aligned} \tag{5.01}$$

and

$$r_o \tilde{\varepsilon}_1 = \frac{\partial \tilde{u}}{\partial \xi} = \frac{r_o}{E\delta} [\tilde{T}_1 - \mu \tilde{T}_2]$$

$$r_o \tilde{\varepsilon}_2 = \frac{\partial \tilde{v}}{\partial \eta} + \frac{\tilde{w}}{\rho} = \frac{r_o}{E\delta} [\tilde{T}_2 - \mu \tilde{T}_1]$$

$$r_o \tilde{\omega} = \frac{\partial \tilde{u}}{\partial \eta} + \frac{\partial \tilde{v}}{\partial \xi} = \frac{2(1+\mu)}{E\delta} r_o \tilde{S} \qquad (5.02)$$

$$r_o^2 \tilde{\kappa}_1 = - \frac{\partial^2 \tilde{w}}{\partial \xi^2} = i \frac{2b^2 r_o}{E\delta} [\tilde{T}_2 - T_2 *]$$

$$r_o^2 \tilde{\kappa}_2 = - \frac{\partial}{\partial \eta} [\frac{\partial \tilde{w}}{\partial \eta} - \frac{\tilde{v}}{\rho}] = \frac{i\, 2 b^2 r_o}{E\delta} [\tilde{T}_1 - T_1 *]$$

$$r_o^2 \tilde{\tau} = - \frac{\partial^2 \tilde{w}}{\partial \xi \partial \eta} + \frac{1}{\rho} \frac{\partial \tilde{v}}{\partial \xi} = - i \frac{\partial b^2 r_o}{E\delta} [\tilde{S} - S*]$$

where

$$4b^4 = 12(1-\mu)(\frac{r_o}{\delta})^2$$

$$\Delta(\ ) = \frac{\partial^2}{\partial \xi^2} (\ ) + \frac{\partial^2}{\partial \eta^2} (\ )$$

$$\tilde{T} = \tilde{T}_1 + \tilde{T}_2$$

and T_1*, T_2*, and $S*$ are particular solutions of the membrane equations

$$\frac{\partial T_1^*}{\partial \xi} + \frac{\partial S*}{\partial \eta} = -q_1 r_o$$

$$\frac{\partial S*}{\partial \xi} + \frac{\partial T_2 *}{\partial \eta} = -q_1 r_o \qquad (5.03)$$

$$T_2 * = q_3 \rho r_o$$

The solution of the system (5.01) must be accomplished in two parts. First consider the case of a circular cylindrical shell under the action of an axisymmetrical load of the form

$$q_1 = q_2 = 0$$

$$q_3 = q_3(\xi)$$

Equations (5.01) would take on the following form:

$$\frac{\partial \tilde{T}_1}{\partial \xi} = 0$$

$$\frac{\partial \tilde{S}}{\partial \xi} = 0 \qquad (5.04)$$

$$\tilde{T}_2 - \frac{i}{2b^2}\frac{\partial^2 \tilde{T}_2}{\partial \xi^2} = q_3 r_o$$

With the application of the boundary conditions, this sytem of equations (5.04) can be used to solve circular cylindrical shell problems loaded symmetrically with only normal loads.

For cylindrical shells with non-symmetric loads, equations (5.01) can be combined to obtain

$$\Delta[\rho \Delta \tilde{T}] + \frac{\partial}{\partial \eta}\left[\frac{1}{\rho}\frac{\partial \tilde{T}}{\partial \eta}\right] + i2b^2 \frac{\partial^2 \tilde{T}}{\partial \xi^2}$$

$$= i2b^2 r_o \left[\Delta(\rho \tilde{q}_3) + \frac{\partial q_2}{\partial \eta} - \frac{\partial q_1}{\partial \xi}\right] \qquad (5.05)$$

Reducing equation (5.05) to govern circular cylindrical shells under the action of normal surface loads, the following equation is obtained:

$$\Delta^4 \tilde{T} + \frac{\partial^2}{\partial \eta^2} \tilde{T} + i\,2b^2 \frac{\partial^2 \tilde{T}}{\partial \xi^2} = i\,2b^2 r_o \Delta q_3 \tag{5.06}$$

For the load q_3 given in the form

$$q_3 = q_o(\xi) + \sum_{n=1}^{\infty} q_n(\xi) \cos n\eta \tag{5.07}$$

solutions are obtained from equations (5.04) and equations (5.06) in the form

$$\tilde{T} = T_o(\xi) + \sum_{n=1}^{\infty} \tilde{T}_n(\xi) \cos n\eta \tag{5.08}$$

Equations (5.06) can now be reduced to a system of n equations governing the coefficients of the solution series (5.08).

$$\frac{d^4 \tilde{T}_n}{d\xi^4} + [\,i\,2b^2 - 2n^2\,] \frac{d^2 \tilde{T}_n}{d\xi^2} + [\,n^4 - n^2\,] \tilde{T}_n = i\,2b^2 r_o \left[\frac{d^2 q_n}{d\xi^2} - n^2 q_n \right] \tag{5.09}$$

These linear fourth order equations with constant coefficients can be solved for each $\tilde{T}_n$ if $q_n(\xi)$ is a function such that it can be generated from a particular solution of equation (5.09).

Because real physical problems must be dealt with, it is necessary to truncate the series expansions for q_3 and $\tilde{T}$. Whenever a series is truncated the question of accuracy must always be considered. The determination of the accuracy of the loading series presents no fundamental problems. It is only necessary to expand the load in a series containing a preselected number of terms and to compare values of the loading from the series with the actual load to determine if the required accuracy has been obtained. The accuracy of the solution series, however, must be considered in greater detail.

Stresses in a shell problem are induced by surface loads and by boundary conditions. In order to use a series type of solution, both the boundary conditions and the surface loadings must also be written in series form. Since both the boundary conditions and the surface loads are known quantities, the accuracy of the series representing them can be determined. The next step (solving for the coefficients of the solution series) would cause no loss of accuracy, for exact solutions are obtained for each harmonic of the loading and boundary condition series. It is there seen that the question of series accuracy arises in the loading and boundary conditions of the shell, and that no errors are induced in going from these series to the solution series.

For convenience of notation, n was altered to start at one rather than zero. The loading series then took on the form

$$q_3 = \sum_{n=1}^{9} q_n(\xi) \cos\left[(n-1)\eta\right] \tag{5.10}$$

It was convenient to express the loading series coefficients $q_n(\xi)$ in the form of polynomials as discussed in Section III. Substitution in equation (5.10) gave

$$q_3 = \sum_{n=1}^{9} \sum_{i=1}^{6} CN_{n,i}\ \xi^{(i-1)} \cos[(n-1)\eta] \qquad (5.11)$$

where the $CN_{n,i}$ are constants.

5a. Solutions for n=1

For symmetrical loads (n=1), solutions can be obtained from equations (5.04).

$$\tilde{T}_2 = \sum_{i=1}^{2} A_{i,1}\ e^{AK_{i,1}\xi} + \sum_{j=1}^{8} C_{1,j}\ \xi^{(j-1)}$$

$$\tilde{T}_1 = A_{3,1}$$

$$\tilde{S} = 0$$

where $A_{i,1}$ are the arbitrary constants, $AK_{i,1}$ are the roots of the auxiliary equation

$$m^2 + 2b^2 i = 0$$

and the term

$$\sum_{j=1}^{8} C_{1,j}\ \xi^{(j-1)}$$

is a particular solution. The equations governing the $C_{1,j}$ are given in Table 2.

5b. Solutions for n>1

For n> 1, solutions are obtained for $\tilde{T}$ from equations (5.09)

$$\tilde{T} = \sum_{n=2}^{10} \sum_{i=1}^{2} A_{i,n}\, e^{AK_{i,n}\xi} \cos[(n-1)\eta] + \sum_{i=3}^{4} A_{i,2}\, \xi^{(i=3)} \cos[(n-1)\eta]$$

$$+ \sum_{n=3}^{10} \sum_{i=3}^{4} A_{i,n}\, e^{AK_{i,n}\xi} \cos[(n-1)\eta] + \sum_{n=2}^{10} \sum_{j=1}^{8} C_{n,j}\, \xi^{j-1} \cos[(n-1)\eta] \qquad (5.13)$$

where $A_{i,n}$ are the arbitrary constants, $AK_{i,n}$ are the roots of the auxiliary equation

$$m^4 + [i\, 2b^2 - 2n^2]\, m^2 + [n^4 - n^2] = 0$$

and the term

$$\sum_{n=2}^{10} \sum_{j=1}^{8} C_{n,j}\, \xi^{(j-1)} \cos[(n-1)\eta]$$

is a particular solution which generates the terms on the right-hand side of equation (5.09).

The rather unusual form of equation (5.13) results from the fact that double roots (=0) were obtained for the case n = 2.

It is noted that the coefficients $C_{n,j}$ of equations (5.12) and (5.13) are directly related to the coefficients $CN_{n,i}$ of equation (5.11). Three systems of six equations with six unknowns ($C_{n,j}$) are presented in Table 2.

TABLE 2

Particular Solution Equations

$n = 1$

$$2b^2 i\, C_{1,1} + 2\, C_{1,3} = i2b^2\, r_0\, CN_{1,1}$$

$$2b^2 i\, C_{1,2} + 6\, C_{1,4} = i2b^2\, r_0\, CN_{1,2}$$

$$2b^2 i\, C_{1,3} + 12\, C_{1,5} = i2b^2\, r_0\, CN_{1,3}$$

$$2b^2 i\, C_{1,4} + 20\, C_{1,6} = i2b^2\, r_0\, CN_{1,4}$$

$$2b^2 i\, C_{1,5} = i2b^2\, r_0\, CN_{1,5}$$

$$2b^2 i\, C_{1,6} = i2b^2\, r_0\, CN_{1,6}$$

$n = 2$

$$2(2b^2 i-2)\, C_{2,3} + 24\, C_{2,5} = i2b^2\, r_0\, (-CN_{2,1} + 2\, CN_{2,3})$$

$$6(2b^2 i-2)\, C_{2,4} + 120\, C_{2,6} = i2b^2\, r_0 (-CN_{2,2} + 6\, CN_{2,4})$$

$$12(2b^2 i-2)\, C_{2,5} + 360\, C_{2,7} = i2b^2\, r_0\, (-CN_{2,3} + 12\, CN_{2,5})$$

$$20(2b^2 i-2)\, C_{2,6} + 840\, C_{2,8} = i2b^2\, r_0 (-CN_{2,4} + 20\, CN_{2,6})$$

$$30(2b^2 i-2) C_{2,7} = i2b^2\, r_0 (-CN_{2,5})$$

$$42(2b^2 i-2) C_{2,8} = i2b^2\, r_0 (-CN_{2,6})$$

TABLE 2, continued

$n > 2$

$$(n^4 - n^2)C_{n,1} + [i2b^2 - 2n^2]2C_{n,3} + 24\,C_{n,5} = i2b^2 r_0 [2CN_{n,3} - n^2 CN_{n,1}]$$

$$(n^4 - n^2)C_{n,2} + [i2b^2 - 2n^2]\,6C_{n,4} + 120\,C_{n,6} = i2b^2 r_0 [6CN_{n,4} - n^2 CN_{n,2}]$$

$$(n^4 - n^2)C_{n,3} + [i2b^2 - 2n^2]\,12C_{n,5} = i2b^2 r_0 [12CN_{n,5} - n^2 CN_{n,3}]$$

$$(n^4 - n^2)C_{n,4} + [i2b^2 - 2n^2]\,20C_{n,6} = i2b^2 r_0 [20CN_{n,6} - n^2 CN_{n,4}]$$

$$(n^4 - n^2)\,C_{n,5} = i2b^2 r_0 [-n^2 CN_{n,5}]$$

$$(n^4 - n^2)C_{n,6} = i2b^2 r_0 [-n^2 CN_{n,6}]$$

The complex forces (for $n > 1$) can be obtained from the following euqations which were obtained in the development of equation (5.05).

$$T_2 = \frac{i}{2b^2} \Delta \widetilde{T} + q_3 r_o$$

$$\widetilde{T}_1 = \widetilde{T} - \widetilde{T}_2$$

$$\frac{\partial \widetilde{S}}{\partial \eta} = - \frac{\partial \widetilde{T}_1}{\partial \xi} .$$

Using the solutions (5.13) the complex forces for $n > 1$ take on the following form:

$$\widetilde{T}_1 = \sum_{n=2}^{10} \sum_{i=1}^{2} ATO_{i,n} A_{i,n} e^{AK_{i,n} \xi} \cos [(n-1)\eta]$$

$$+ \sum_{n=3}^{10} \sum_{i=3}^{4} ATO_{i,n} A_{i,n} e^{AK_{i,n} \xi} \cos [(n-1)\eta]$$

$$+ \sum_{i=3}^{4} DTO_{i,2} A_{i,2} \xi^{(i-3)} \cos [\eta]$$

$$+ \sum_{n=2}^{10} \sum_{j=1}^{8} BTO_{n,j} \xi^{(j-1)} \cos [(n-1)\eta]$$

$$+ \sum_{n=2}^{10} \sum_{j=3}^{8} CTO_{n,j} \xi^{(j-3)} \cos [(n-1)\eta],$$

$$\tilde{T}_2 = \sum_{n=2}^{10} \sum_{i=1}^{2} ATT_{i,n} A_{i,n} e^{AK_{i,n}\xi} \cos[(n-1)\eta]$$

$$+ \sum_{n=3}^{10} \sum_{i=3}^{4} ATT_{i,n} A_{i,n} e^{AK_{i,n}\xi} \cos[(n-1)\eta]$$

$$+ \sum_{i=3}^{4} DTT_{i,2} A_{i,2} \xi^{(i-3)} \cos[\eta]$$

$$+ \sum_{n=2}^{10} \sum_{j=1}^{8} BTT_{n,j} \xi^{(j-1)} \cos[(n-1)\eta]$$

$$+ \sum_{n=2}^{10} \sum_{j=3}^{8} CTT_{n,j} \xi^{(j-3)} \cos[(n-1)\eta],$$

and

$$\tilde{S} = \sum_{n=2}^{10} \sum_{i=1}^{2} AS_{i,n} A_{i,n} e^{AK_{i,n}\xi} \sin[(n-1)\eta]$$

$$+ \sum_{n=3}^{10} \sum_{i=3}^{4} AS_{i,n} A_{i,n} e^{AK_{i,n}\xi} \sin[(n-1)\eta]$$

$$+ \sum_{i=3}^{4} DS_{i,2} A_{i,2} \xi^{(i-3)} \sin[\eta]$$

$$+ \sum_{n=2}^{10} \sum_{j=1}^{8} BS_{n,j} \xi^{(j-1)} \sin[(n-1)\eta]$$

$$+ \sum_{n=2}^{10} \sum_{j=3}^{8} CS_{n,j} \xi^{(j-3)} \sin[(n-1)\eta].$$

The complex displacements for $n > 1$ are now obtained in two parts from equations (5.02).

For $n = 2$

$$\tilde{w} = \sum_{i=1}^{2} AW_{i,2} A_{i,2} e^{AK_{i,2}\xi} \cos[\eta]$$

$$+ \sum_{i=3}^{4} DW_{i,2} A_{i,2} \xi^{(i-1)} \cos[\eta]$$

$$+ \sum_{j=3}^{8} cW_{2,j} \xi^{j-3} \cos[\eta],$$

$$\tilde{v} = \sum_{i=1}^{2} AV_{i,2} A_{i,2} e^{AK_{i,2}\xi} \sin[\eta]$$

$$+ \sum_{i=3}^{4} DVO_{i,2} A_{i,2} \xi^{(i-1)} \sin[\eta]$$

$$+ \sum_{i=3}^{4} DVT_{i,2} A_{i,2} \xi^{(i-3)} \sin[\eta]$$

$$+ \sum_{j=1}^{8} BV_{2,j} \xi^{(j-1)} \sin[\eta]$$

$$+ \sum_{j=3}^{8} CV_{2,j} \xi^{(j-3)} \sin[\eta],$$

and $$\tilde{u} = \sum_{i=1}^{2} AU_{i,2}\, A_{i,2}\, e^{AK_{i,2}\xi} \cos[\eta]$$

$$+ \sum_{i=3}^{4} DU_{i,2}\, A_{i,2}\, \xi^{(i-2)} \cos[\eta]$$

$$+ \sum_{j=1}^{8} BU_{2,j}\, \xi^{(j-1)} \cos[\eta] \;.$$

For $n>2$

$$\tilde{w} = \sum_{n=3}^{9} \sum_{i=1}^{4} AW_{i,n}\, A_{i,n}\, e^{AK_{i,n}\xi} \cos[(n-1)\eta]$$

$$+ \sum_{n=3}^{9} \sum_{j=1}^{8} BW_{n,j}\, \xi^{(j-1)} \cos[(n-1)\eta]$$

$$+ \sum_{n=3}^{9} \sum_{j=3}^{8} CW_{n,j}\, \xi^{(j-3)} \cos[(n-1)\eta] \,,$$

$$\tilde{v} = \sum_{n=3}^{9} \sum_{i=1}^{4} AV_{i,n}\, A_{i,n}\, e^{AK_{i,n}\xi} \sin[(n-1)\eta]$$

$$+ \sum_{n=3}^{9} \sum_{j=1}^{8} BV_{n,j}\, \xi^{(j-1)} \sin[(n-1)\eta]$$

$$+ \sum_{n=3}^{9} \sum_{j=3}^{8} CV_{n,j}\, \xi^{(j-3)} \sin[(n-1)\eta] \,,$$

$$\tilde{u} = \sum_{n=3}^{9} \sum_{i=1}^{4} AU_{i,n}\, A_{i,n}\, e^{AK_{i,n}\xi} \cos[(n-1)\eta]$$

$$+ \sum_{n=3}^{9} \sum_{j=1}^{8} BU_{n,j}\, \xi^{(j-1)} \cos[(n-1)\eta]$$

$$+ \sum_{n=3}^{9} \sum_{j=3}^{8} CU_{n,j}\, \xi^{(i-3)} \cos[(n-1)\eta] .$$

The coefficients of these series are defined in Table 3.

The total solution is obtained by combining the solutions for $n = 1$ with the solutions for $n > 1$. The real forces and displacements are obtained from the complex forces and displacements using the following equations:

$$\tilde{T}_1 = T_1 - \frac{i\,2b^2}{r_o} \frac{M_2 - \mu M_1}{1 - \mu^2}$$

$$\tilde{T}_2 = T_2 - \frac{i\,2b^2}{r_o} \frac{M_1 - \mu M_2}{1 - \mu^2}$$

$$\tilde{S} = S + \frac{i\,2b^2}{r_o} \frac{H}{1 - \mu}$$

$$u = \text{REAL}\ (\tilde{u})$$

$$v = \text{REAL}\ (\tilde{v})$$

$$w = \text{REAL}\ (\tilde{w})$$

The solutions for the complex forces and displacements contain 40 arbitrary constants $(A_{i,n})$. The constants must be determined from the boundary conditions of the shell. The boundary conditions are normally prescribed in terms of real quantities. They might take the form of the displacements, moments or forces at the edges of the shell.

The complex constants must also be separated into real and imaginary parts.

TABLE 3

Coefficients of Cylinder Solutions

$$ATT_{i,n} = \frac{i}{2b^2}\left[AK^2_{i,n} - (n-1)^2\right]$$

$$DTT_{i,n} = \frac{-i}{2b^2}(n-1)^2$$

$$BTT_{n,j} = rCN_{n,j} - \frac{i}{2b^2}(n-1)^2 C_{n,j}$$

$$CTT_{n,j} = \frac{i}{2b^2} C_{n,j}(j-1)(j-2)$$

$$ATO_{i,n} = 1 - ATT_{i,n}$$

$$DTO_{i,n} = 1 - DTT_{i,n}$$

$$BTO_{n,j} = C_{n,j} - BTT_{n,j}$$

$$CTO_{n,j} = -CTT_{n,j}$$

$$AS_{i,n} = -ATO_{i,n}\ AK_{i,n}$$

$$DS_{i,n} = -DTO_{i,n}$$

$$BS_{n,j} = -BTO_{n,j-1}(j)$$

$$CS_{n,j} = -CTO_{n,j-1}(j)$$

For $n = 2$

$$AW_{i,2} = -i\,2b^2 \frac{R}{E\delta} \frac{ATT_{i,2}}{AK^2_{i,2}}$$

$$DW_{i,2} = -i\,2b^2 \frac{R}{E\delta} DTT_{i,2}$$

$$CW_{2,j} = -i\,2b^2 \frac{R}{E\delta} [BTT_{2,j} - CN_{2,j}\, r] \frac{1}{j(j+1)}$$

$$AV_{i,2} = -AW_{i,2} + \frac{R}{E\delta} [ATT_{i,2} - \mu ATO_{i,2}]$$

$$DVO_{i,2} = -DW_{i,2}$$

$$DVT_{i,2} = \frac{R}{E\delta} [DTT_{i,2} - \mu DTO_{i,2}]$$

$$BV_{2,j} = \frac{R}{E\delta} [BTT_{2,j} - \mu BTO_{2,j}]$$

$$CV_{2,j} = -CW_{2,j}$$

$$AU_{i,2} = \frac{R}{E\delta} [ATO_{i,2} - \mu ATT_{i,2}] \frac{1}{AK^2_{i,2}}$$

$$DU_{i,2} = \frac{R}{E\delta} [DTO_{i,2} - \mu DTT_{i,2}] \frac{1}{i-2}$$

$$BU_{2,j+1} = \frac{R}{E\delta} [BTO_{2,j} - \mu BTT_{2,j}] \frac{1}{j}$$

For $n > 2$

$$AW_{i,n} = \frac{R}{E\delta}\frac{1}{[-(n-1)^2+1]}[(-\mu - i\,2b^2)\,ATO_{i,n} + ATT_{i,n}]$$

$$BW_{m,j} = \frac{R}{E\delta}\frac{1}{[-(n-1)^2+1]}[-\mu - i\,2b^2)BTO_{n,j} + BTT_{n,j}]$$

$$CW_{n,j} = \frac{-R^2}{E\delta}(n-1)^2 \frac{CN_{n,j}}{(j-1)(j-2)}\frac{C\,2b^2}{[-(n-1)^2+1]}$$

$$AV_{i,n} = [-AW_{i,n} + \frac{R}{E\delta}(ATT_{i,n} - \mu ATO_{i,n})]\frac{1}{n-1}$$

$$BV_{n,j} = [-BW_{n,j} + \frac{R}{E\delta}(BTT_{n,j} - \mu\ BTT_{n,j} - \mu BTO_{n,j})]\ \frac{1}{n-1}$$

$$CV_{n,j} = -CW_{n,j}\frac{1}{n-1}$$

$$AU_{i,n} = [AV_{i,n}\,AK_{i,n} - 2\,(1+\mu)\frac{R}{E\delta}AS_{i,n}]\frac{1}{n-1}$$

$$BU_{n,j} = [BV_{n,j+1}\,j - 2\,(1+\mu)\frac{R}{E\delta}\,BS_{n,j}]\ \frac{1}{n-1}$$

$$CU_{n,j} = CV_{n,j+1}\,j\,\frac{1}{(n-1)}$$

$$A_{i,n} = AR_{i,n} + iAI_{i,n}$$

This results in 80 real arbitrary constants or 8 constants for each harmonic. In order to determine these constants, up to four boundary conditions must be given at each of the two ends of the cylinder. This results in ten systems, eight linear algebraic equations with eight unknowns each, which permit the determination of all constants of integration in the solution. The boundary conditions will be discussed further in Section VII.

VI. ANALYSIS OF THE SPHERICAL SHELL

Adapting equations (4.05) to the particular case of the spherical shell, the following system of equation in terms of the complex is obtained. Where

$$R = R_2 = R_1$$

$$A_1 = R$$

$$A_2 = R \sin \theta$$

$$a_1 = \theta$$

$$a_2 = \phi .$$

$$\frac{1}{r}\frac{\partial \tilde{T}_1}{\partial \theta} + \frac{\cotan \theta}{r}(\tilde{T}_1 - \tilde{T}_2) + \frac{1}{r \sin \theta}\frac{\partial S}{\partial \phi} + \frac{c}{r^2}\frac{\partial \tilde{T}}{\partial \theta} = -q_1$$

$$\frac{1}{r}\frac{\partial \tilde{S}}{\partial \theta} + \frac{2 \cotan \theta}{r}\tilde{S} + \frac{1}{r \sin \theta}\frac{\partial \tilde{T}_2}{\partial \phi} + \frac{i c}{r^2 \sin \theta}\frac{\partial \tilde{T}}{\partial \phi} = -q_2 \qquad (6.01)$$

$$\frac{\tilde{T}_1 + \tilde{T}_2}{r} - i c \, \Delta \tilde{T} = q_3$$

Where

$$\Delta(\ldots) = \frac{1}{r^2}\frac{\partial^2(\ldots)}{\partial \theta^2} + \frac{\cotan \theta}{r^2}\frac{\partial(\ldots)}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2(\ldots)}{\partial \phi^2}$$

$$c = \frac{\delta}{\sqrt{12(1-\mu^2)}}$$

and θ and ϕ are the usual spherical coordinates.

Equations (4.06) and (4.07) combine to take on the form:

$$\frac{\partial \tilde{u}}{\partial \theta} + \tilde{w} = \frac{r}{E\delta}(\tilde{T}_1 - \mu \tilde{T}_2)$$

$$\frac{1}{\sin\theta}\frac{\partial \tilde{v}}{\partial \theta} + \tilde{u}\,\mathrm{cotan}\,\theta + \tilde{w} = \frac{r}{E\delta}(\tilde{T}_2 - \mu \tilde{T}_1)$$

$$\frac{\partial \tilde{v}}{\partial \theta} - \tilde{v}\,\mathrm{cotan}\,\theta + \frac{1}{\sin\theta}\frac{\partial \tilde{u}}{\partial \phi} = \frac{2(1+\mu)\,r}{E\delta}\tilde{S} \tag{6.02}$$

$$-\frac{1}{r^2}\frac{\partial}{\partial \theta}\left[\frac{\partial \tilde{w}}{\partial \theta} - \tilde{u}\right] = \frac{i}{cE\delta}[\tilde{T}_2 - T_2 *]$$

$$-\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \phi}\left[\frac{1}{\sin\theta}\frac{\partial \tilde{w}}{\partial \phi} - \tilde{v}\right] - \frac{\mathrm{cotan}\,\theta}{r^2}\left[\frac{\partial \tilde{w}}{\partial \theta} - \tilde{u}\right] = \frac{i}{cE\delta}[T_1 - T_1 *]$$

$$-\frac{1}{r^2}\frac{\partial}{\partial \theta}\left[\frac{1}{\sin\theta}\frac{\partial \tilde{w}}{\partial \phi} - \tilde{v}\right] + \frac{1}{r^2 \sin\theta}\left[\frac{\partial \tilde{u}}{\partial \phi} - \tilde{v}\cos\theta\right] = \frac{-i}{cE\delta}[\tilde{S} - \delta *]$$

Introducing two new auxiliary variables

$$\tilde{U} = T_1 r \sin^2\theta - i c \sin\theta \cos\theta \frac{\partial \tilde{T}}{\partial \theta}$$

$$\tilde{V} = \tilde{S} r^2 \sin^2\theta$$

into equations (6.01) and assuming solutions of form

$$\tilde{U} = \sum_{n=o}^{\infty} U_n \cos n\phi$$

$$\tilde{T} = \sum_{n=o}^{\infty} \tilde{T}_n \cos n\phi$$

$$\tilde{V} = \sum_{n=1}^{\infty} \tilde{V}_n \sin n\phi$$

the system (6.01) can be reduced to two equations and two unknowns ($\tilde{T}_k$ and $\tilde{U}_k$).

$$\frac{d^2 \widetilde{T}_k}{d\theta^2} + \cot an\,\theta \,\frac{d}{d\theta}\, T - \frac{k^2}{\sin^2\theta}\, T_k + i\, 2a^2\, \widetilde{T} = i\, 2a^2\, r\, q_k \qquad (6.03)$$

$$\frac{d^2}{d\theta^2}\, \widetilde{U} + \cot an\,\theta\, \frac{dU_k}{d\theta} - \frac{k^2}{\sin^2\theta}\, \widetilde{U} + r\, k^2\, \widetilde{T} = r\, f_k(\theta)$$

Where

$$f_k(\theta) = \frac{r}{\sin\theta}\, \frac{d}{d\theta}\, [q_k(\theta) \cos\theta \sin^2\theta]\,.$$

The homogenous solutions of (6.03) have been developed by Novozhilov (19). The coefficients of the following solutions were developed by Novozhilov (19) and are presented in Table (4):

$$u = \sum_{n=o}^{\infty} u_k \cos n\phi$$

$$v = \sum_{n=1}^{\infty} v_k \sin n\phi$$

$$w = \sum_{n=o}^{\infty} w_k \cos n\phi$$

$$T_1 = \sum_{n=o}^{\infty} T_{1,k} \cos n\phi$$

$$T_2 = \sum_{n=o}^{\infty} T_{2,k} \cos n\phi$$

$$S = \sum_{n=1}^{\infty} S_k \sin n\phi$$

$$M_1 = \sum_{n=o}^{\infty} M_{1,k} \cos n\phi$$

$$M_2 = \sum_{n=o}^{\infty} M_{2,k} \cos n\phi$$

$$H = \sum_{n=1}^{\infty} H_k \sin \phi$$

TABLE 4

Coefficients of the Homogeneous Solution Series

$$u_k = -[A_1 \sin\theta + A_2 \frac{k(k+\cos\theta)}{\sin\theta}]\tan^k(\frac{\theta}{2})$$

$$+\frac{R}{E\delta}\frac{(1+\mu)}{2a}[-(A_3 b_2 + A_4 b_1)\cos b_2 a\alpha$$

$$+ (A_3 b_1 - A_4 b_2)\sin b_2 a\alpha]e^{-b_1 a}$$

$$v_k = -[A_1 \sin\theta - A_2 (\frac{k(k+\cos\theta)}{\sin\theta} - \sin\theta)]\tan^k(\frac{\theta}{2})$$

$$-\frac{R}{E\delta}\frac{1+H}{2a^2}\frac{k}{\sin\theta}[-A_4 \cos b_2 a\alpha + A_3 \sin b_2 a\alpha]e^{-b_1 a\alpha}$$

$$w_k = A_1 (k+\cos\theta)\tan^k[\frac{\theta}{2}]$$

$$+\frac{R}{E\delta}[A_3 \cos b_2 a\alpha + A_4 \sin b_2 a\alpha]e^{-b_1 a\alpha}$$

$$T_{1,k} = -\frac{E\delta}{R(1+\mu)} A_2 \frac{k(k^2-1)}{\sin^2\theta}\tan^k[\frac{\theta}{2}]$$

$$+\frac{1}{2a}[(A_3 b_2 + A_4 b_1)\cos b_2 a\alpha - (A_3 b_1 - A_4 b_2)$$

$$\sin b_2 a\alpha]\text{cotan}\,\theta + \frac{k^2}{a\sin^2\theta}(-A_4 \cos b_2 a\alpha$$

$$+ A_3 \sin b_2 a\alpha)\ e^{-b_1 a\alpha}$$

$$T_{2,k} = \frac{E\delta}{R(1+\mu)} A_2 \frac{k(k^2-1)}{\sin^2\theta} \tan^k(\frac{\theta}{2}) + [A_3 \cos b_2 a \alpha$$

$$+ A_4 \sin b_2 a \alpha + \frac{k^2}{2a^2 \sin^2\theta} (A_4 \cos b_2 a \alpha$$

$$- A_3 \sin b_2 a \beta)] e^{-b_1 a \alpha}$$

$$S_k = \frac{E\delta}{R(1+\mu)} A_2 \frac{k(k^2-1)}{\sin^2\theta} \tan^k(\frac{\theta}{2}) + \frac{1}{2a} \frac{k}{\sin\theta} [(A_3 b_2$$

$$+ A_4 b_1) \cos b_2 a \alpha - (A_3 b_1 - A_4 b_2) \sin b_2 a \alpha] e^{-b_1 a \alpha}$$

$$M_{1,k} = \frac{-E\delta^3}{12(1+\mu)R^2} A_1 \frac{k(k^2-1)}{\sin^2\theta} \tan^k(\frac{\theta}{2})$$

$$+ \frac{R}{2a^2} [A_4 \cos b_2 a \alpha - A_3 \sin b_2 a \alpha$$

$$- \frac{1-\mu}{2a^2} \frac{k^2 \alpha}{\sin^2\theta} (A_3 \cos b_2 a \alpha + A_4 \sin b_2 a \alpha)] e^{-b_1 a \alpha}$$

$$M_{2,k} = \frac{E\delta}{12(1+\mu)R^2} A_1 \frac{k(k^2-1)}{\sin^2\theta} \tan^k(\frac{\theta}{2})$$

$$+ \frac{R}{2a^2} [\mu(A_4 \cos b_2 a \omega - A_3 \sin b_2 a \alpha)$$

$$+ \frac{1-\mu}{2a^2} \frac{k^2}{\sin^2\theta} (A_3 \cos b_2 a \alpha + A_4 \sin b_2 a \alpha)] e^{-b_1 a \alpha}$$

$$H_k = \frac{E\delta^3}{12(1+\mu)R^2} \, A_1 \, \frac{k(k^2-1)}{\sin^2\theta} \tan^k(\frac{\theta}{2})$$

$$+ \frac{(1-\mu)}{4a^3} \, R \, \frac{k}{\sin\theta} [(A_3 b_1 - A_4 b_2) \cos b_2 \, a\alpha$$

$$+ (A_3 b_2 + A_4 b_1) \sin b_2 \, a\alpha] \, e^{-b_1 a \alpha}$$

The particular solution for $\tilde{T}_k$ is found directly from the first of equations (6.03) by expanding q_3 into a series of spherical harmonics as discussed in Section III. This is done by noting that

$$\frac{d^2}{d\theta^2}[P_n^k(\cos\theta)] + \cotan\theta \frac{d}{d\theta}[P_n^k(\cos\theta)] - \frac{k^2}{\sin^2\theta}[P_n^k(\cos\theta)]$$

$$= -n(n+1)\, P_n^k(\cos\theta)$$

Therefore if the loading is in the form

$$q_3 = \sum_{n=o}^{8} \sum_{n=o}^{n=m} A_{n,m} \cos n\phi \, P_m^n(\cos\theta) \qquad \begin{array}{l} m-n = 2j \\ j = 0, 1, 2, 3, \ldots \end{array}$$

the particular solution for T is found in the form

$$\tilde{T} = \sum_{n=o}^{8} \sum_{n=o}^{n=m} B_{n,m} \cos n\phi \, P_m^n(\cos\theta) \qquad (6.04)$$

Where the relationship between $B_{n,m}$ and $A_{n,m}$ is found as

$$B_{n,m} = \frac{An,m}{-m(m+1) + i\, 2a^2}$$

The particular solution for $\tilde{U}_k$ which must be found from the second of equations (6.03) presents more difficulty than is encountered in finding the particular solution for $\tilde{T}_k$. A particular solution which would develop the function $rf_k(\theta)$ could not be found. However, it is possible to alter the form of $rf_k(\theta)$ by using the actual values of loads and the values of sin θ and cos θ. The values of the combined function are next placed in the form of a spherical harmonic series using the techniques developed in Section III. With $rf_k(\theta)$ existing in the form

$$rf_k(\theta, \phi) = \sum_{n=o}^{8} r f_n(\theta) \cos n\phi$$

$$= \sum_{m=o}^{8} \sum_{n=o}^{n=m} C_{n,m} \cos n\phi P_m^n (\cos \theta) \qquad \begin{matrix} m-n=2j \\ j, 0, 1, 2, 3 \ldots \end{matrix}$$

and $\widetilde{T}_k$ existing in the form of equation (6. 04), the particular solution for $\widetilde{U}_k$ is found in the form:

$$\widetilde{U} = \sum_{m=o}^{8} \sum_{n=o}^{n=m} D_{n,m} \cos n\phi P_m^n (\cos\theta)$$

Where the relationship between $D_{n,m}$ and $C_{n,m}$ is

$$D_{n,m} = \frac{r^2 n^2 B_{n,m} - C_{n,m}}{m(m+1)}$$

The particular solution for the complex forces can now be found directly using equations which were produced in the development of equations (6. 03).

$$\widetilde{T}_{1,k} = \frac{U_k}{\sin^2\theta} + \frac{i}{2a^2} \operatorname{cotan}\theta \frac{dT_k}{d\theta}$$

$$\widetilde{T}_{2,k} = \widetilde{T}_k - \widetilde{T}_{1,k}$$

$$\widetilde{S}_k = \frac{1}{n r \sin\theta} \frac{d\widetilde{U}_n}{d\theta} + \frac{i C n \cos\theta}{r \sin^2\theta} \widetilde{T}_n + q_n \cos\theta \frac{r}{n}$$

It is convenient to first convert the particular solutions for $\tilde{T}$ and $\tilde{U}$ from spherical harmonics to Fourier series with polynomial coefficients. The particular solutions for the complex forces are then found in the form:

$$\tilde{T}_1 = \sum_{n=1}^{9} \sum_{j=1}^{6} TO_{n,j} \; \theta^{(j-1)} \cos[(n-1)\phi]$$

$$\tilde{T}_2 = \sum_{n=1}^{9} \sum_{j=1}^{6} TT_{n,j} \; \theta^{(j-1)} \cos[(n-1)\phi]$$

$$\tilde{S} = \sum_{n=2}^{9} \sum_{j=1}^{6} S_{n,j} \; \theta^{(j-1)} \sin[(n-1)\phi]$$

where the $TO_{n,j}$, $TT_{n,j}$ and $S_{n,j}$ are constants.

The particular solutions for the complex displacements are now obtained from equations (6.02) using the complex forces.

$$\tilde{u} = \sum_{n=1}^{9} \sum_{j=1}^{6} UU_{n,j} \; \theta^{(j-1)} \cos[(n-1)\phi]$$

$$\tilde{v} = \sum_{n=2}^{9} \sum_{j=1}^{6} VV_{n,j} \; \theta^{(j-1)} \sin[(n-1)\phi]$$

$$\tilde{w} = \sum_{n=1}^{9} \sum_{j=1}^{6} WW_{n,j} \; \theta^{(j-1)} \cos[(n-1)\phi]$$

where the $UU_{n,j}$, $VV_{n,j}$, and $WW_{n,j}$ are constants.

The homogeneous and particular solutions are now combined to form the total solutions which contain four arbitrary constants for each harmonic. The solution for the arbitrary constants can now be obtained from the boundary conditions of the shell.

VII. BOUNDARY AND CONTINUITY CONDITIONS

Figure (4) demonstrates that if α_1 = constant is a boundary of the shell, then the quantities T_{12}, T_1, N_1, M_1 and M_{12} exist on the edge of the shell. It can be shown, however, that M_{12} can be replaced by corresponding distributed, tangential, and transverse forces at the boundary. This results in three forces, T_1, $T_{12} + \dfrac{M_{12}}{R_2}$, $N_2 + \dfrac{1}{A_2}\dfrac{\partial M_{12}}{\partial \alpha_2}$ and one moment M_1 existing at the boundary α_1 = constant. These four quantities must be described directly or indirectly at the boundary in order to completely determine the state of stress at the edge of the shell.

A few of the more obvious possibilities of boundary conditions (with α_1 = constant as the edge of the shell) are:

a) Free edge

$$T_1 = 0, \quad T_{12} + \frac{M_{12}}{A_2} = 0, \quad N_1 + \frac{1}{A_2}\frac{\partial M_{12}}{\partial \alpha_2} = 0, \quad M_1 = 0;$$

b) Hinged edge with fixed support

$$M_1 = 0, \quad u = 0, \quad v = 0, \quad w = 0.$$

c) Clamped edge

$$u = 0, \quad v = 0, \quad w = 0, \quad \theta = -\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} + \frac{u}{R_1} = 0.$$

Using the fourth equilibrium equation (4.01)

$$\frac{1}{A_1 A_2}\left[\frac{\partial A_2 M_1}{\partial a_1} + \frac{\partial A_1 M_{21}}{\partial a_2} + \frac{\partial A_1}{\partial a_2} M_{12} - \frac{\partial A_2}{\partial a_1} M_2\right] - N_1 = 0$$

and the definitions of S and H

$$S = T_{12} - \frac{M_{21}}{R_2} = T_{21} - \frac{M_{12}}{R_1}$$

$$H = \frac{M_{12} + M_{21}}{2}$$

two of the four quantities which exist on the boundary of the shell may be rewritten.

$$T_{12} + \frac{M_{12}}{R_2} = S + \frac{2H}{R_2}$$

$$N_1 + \frac{1}{A_2}\frac{\partial M_{12}}{\partial a_2} = \frac{1}{A_1 A_2}\left[2\frac{\partial A_2 H}{\partial a_1} + \frac{\partial A_1 M_2}{\partial a_1} - \frac{\partial A_1}{\partial a_2} M_1\right]$$

Therefore, it is seen that the forces T_{12}, T_{21}, N_1, and N_2 and the moments M_{12} and M_{21} do not appear separately in the boundary conditions, but only in combinations of S and H.

The problem of a junction between two shells presents an unusual problem. Consider first two shells of the same thickness rigidly welded together and having a common tangent at the junction. In this case, it is necessary not only that the four usual quantities (three forces and one moment) be continuous at the joint, but also the displacements and rotations resulting from these forces and moments must be continuous. One form of the eight necessary quantities would be

$$T_1,\ S+\frac{2H}{R_2},\ M_1,\ \frac{1}{A_1A_2}\left[\frac{\partial A_2M_1}{\partial a_1}+2\frac{\partial A_1H}{\partial a_2}-\frac{\partial A_2}{\partial a_1}M_2\right],\ u,\ v,\ w,\ \text{and } \theta.$$

Consider next, two shells (A and B) which form an angle ψ at the junction ($\psi = 0^{\circ}$ for the case of the common tangent). If $\phi = 90^{\circ} - \psi$ the equations governing continuity at the joint would take the form:

$$(T_1)_A = +(T_1)_B \sin\phi + \left[\frac{1}{A_1A_2}\left(\frac{\partial A_2M_1}{\partial a_1}+2\frac{\partial A_1H}{\partial a_1}M_2\right)\right]\cos\phi$$

$$(M_1)_A = (M_1)_B$$

$$\left(S+\frac{2H}{R_2}\right)_A = \left(S+\frac{2H}{R_2}\right)_B$$

$$\left[\frac{1}{A_1A_2}\left(\frac{\partial A_2M_1}{\partial a_1}+2\frac{\partial A_1H}{\partial a_2}-\frac{\partial A_2}{\partial a_1}M_2\right]_A$$

$$= \left[\frac{1}{A_1A_2}\left(\frac{\partial A_2M_1}{\partial a_1}+2\frac{\partial A_1H}{\partial a_1}-\frac{\partial A_2}{\partial a_1}M_2\right)\right]_B \sin\phi$$

$$-(T_1)_B\cos\phi$$

$$u_A = +u_B\sin\phi - w_B\cos\phi$$

$$v_A = v_B$$

$$w_A = w_B\sin\phi - u_B\cos\phi$$

$$\theta_A = \theta_B$$

VIII. THE COMPUTER PROGRAM

A computer program consisting of 3000 cards in 40 subroutines was written in Fortran IV using the theory outlined in the previous chapters. The program is designed to give forces, moments, and displacements for circular cylinders, hemispheres, spherical segments, and dome-cylinder combination structures under the action of continuous normal surface loads. Either pinned or clamped boundary conditions are allowed at the edges.

The input for the computer program is given in Table (4). The input required is in a form that does not require any knowledge about the program to run a particular problem. The output gives values for the forces T_1, T_2, T_{12} in forces per unit length, the moments M_1, M_2, M_{12} in moment per unit length, and the displacement w, u, v in length at points over the entire surface.

A set of solutions obtained from the program is compared with a set of solutions obtained from a program using the procedure outlined by Kalnin's in reference (16). Table (5) presents the two sets of solutions for a circular cylinder made of steel ($E=30x10^6$, $\mu = .3$) with a 1200 inch radius, 600 inches high, one inch thick, and a uniform load of 30 pounds per square inch.

TABLE 5 INPUT FOR THE COMPUTER PROGRAM

1. Card 1   FORMAT (I10)

   READ INDI

   INDI = 1 for a cylinder problem,

   INDI = 2 for a sphere problem,

   and INDI = 3 for a dome cylinder problem.

2. Card 2   FORMAT (3I10)

   READ NBCB, NBCT, NBCS

   NBCB = Cylinder boundary condition at ξ=XMAX,

   NBCT = Cylinder boundary condition at ξ=0,

   and NBCS = Sphere boundary condition
   1= clamped edge
   2= pinned edge

3. Card 3   FORMAT (5F10. 0)

   READ E, POIS, XNOC, XNOS, WIND

   E = Young's Modulus

   POIS = Poisson's ratio

   XNOC = The number of data input points on the cylinder

   XNOS = The number of data input points on the sphere

   WIND = Load scale factor to be used if the load is read in terms of coefficients. If the actual load is read in, set WIND = 1. 0.

4. Card 4   FORMAT (5F10. 0)

   READ XMAX, RADC, RADS, OMAX, THIC

XMAX = Length of the cylinder,

RADC = Radius of the cylinder,

RADS = Radius of the sphere,

OMAX = Value of θ at the edge of the sphere in radians,

THIC = Thickness of the shell.

5. The cylinder loads (if any)

```
FORMAT (8F10.0)
      DO 1 I=1, NOC
1 READ X(I), (Y(I,N), N=1,13)
```

X(I) is a location along a generator,

Y(I, 1) is the load at that point for $\phi=0^{\circ}$,

Y(I, 2) is the load at that point for $\phi=15^{\circ}$,

.

.

.

.

Y(I, 13) is the load at that point for $\phi=180^{\circ}$.

6. The sphere loads (if any)

```
FORMAT (8F10.0)
      DO 2 I=1, NOS
2 READ X(I), (Y(I,N), N=1.13)
```

X(I) is a location along a line of constant ϕ in radians,

Y(I, 1) is the load at that point for $\phi=0^{\circ}$,

Y(I, 2) is the load at that point for $\phi=15^{\circ}$,

.

.

.

Y(I, 13) is the load at that point for $\phi=180^{\circ}$

TABLE 6

COMPARISON OF PRESENT SOLUTIONS WITH SOLUTIONS FROM KALNIN'S PROGRAM

x	w_K	w
0.0	0.0	0.0
60.0	1.4021	1.5087
120.0	1.3197	1.2996
180.0	1.3143	1.3095
240.0	1.3157	1.3106
300.0	1.3158	1.3104
350.0	1.3157	1.3106
420.0	1.3143	1.3095
480.0	1.3197	1.2996
540.0	1.4021	1.5087
600.0	0.0	0.0

Where x is in inches

w_K is the deflection w in inches from Kalnin's Program

and w is the deflection w in inches from the procedures outlined in this thesis.

IX. RESULTS

In order to demonstrate the capability of the technique and the computer program, the results of a problem are presented in this section. The particular problem chosen was a combination hemisphere-cylinder problem with the base of the cylinder clamped. The surface load used was the measured wind load presented in Figures 1 and 2. The thickness of the structure was taken as one inch, the radius as 600 inches, and the height of the cylinder as 1200 inches. Young's modulus was set equal to $30. \times 10^6$ and Poisson's ratio equal to .3.

The results are given over the entire surface in Tables 6, 7, and 8. The results appear to behave as expected although they seem to have deteriorated slightly as far as 10° from the apex of the sphere.

There are certain errors inherent in the basic equations and, although certain other errors are induced in the expansion of the surface loads into series form, no other errors are induced in the actual solution of the governing equations.

TABLE 7

THE DISPLACEMENTS

The Displacement w on the Sphere

$\phi =$	0°	30°	60°	90°	120°	150°	180°
$\theta = 10^\circ$	+.0123	+.0116	+.0097	+.0074	+.0045	+.0025	+.0015
20°	+.0218	+.0209	+.0182	+.0137	+.0079	+.0034	+.0016
30°	+.0314	+.0312	+.0290	+.0234	+.0151	+.0083	+.0057
40°	+.0344	+.0359	+.0373	+.0331	+.0239	+.0161	+.0133
50°	+.0296	+.0344	+.0423	+.0421	+.0328	+.0254	+.0231
60°	+.0197	+.0290	+.0462	+.0511	+.0407	+.0345	+.0334
70°	+.0078	+.0231	+.0515	+.0616	+.0462	+.0417	+.0421
80°	-.0045	+.0183	+.0593	+.0747	+.0468	+.0453	+.0474
90°	0.0	0.0	0.0	0.0	0.0	0.0	0.0

TABLE 7

THE DISPLACEMENTS (Continued)

The Displacement w on the Cylinder

$\phi =$	0°	30°	60°	90°	120°	150°	180°
x = 0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
120	-.055	-.040	+.017	+.047	+.00	+.023	+.030
240	-.079	-.071	+.002	+.078	+.005	+.035	+.052
360	-.093	-.093	-.016	+.105	+.010	+.041	+.069
480	-.098	-.106	-.028	+.124	+.012	+.044	+.078
600	-.095	-.105	-.030	+.130	+.010	+.040	+.076

TABLE 8

THE FORCE RESULTANTS

The Force T_2 on the Sphere

$\phi =$	0°	30°	60°	90°	120°	150°	180°
$\theta=10^\circ$	+3404.5	+3183.4	+2713.1	+2380.0	+2359.2	+2465.8	+2514.1
20°	+2526.1	+2358.9	+2005.9	+1774.6	+1785.4	+1772.4	+1723.4
30°	+2067.5	+1951.5	+1766.9	+1633.3	+1575.3	+1358.6	+1231.2
40°	+1646.2	+1628.4	+1635.7	+1604.9	+1440.0	+1127.1	+1004.3
50°	+1152.0	+1277.6	+1496.4	+1564.2	+1291.3	+1015.5	+ 961.3
60°	+ 636.0	+ 921.9	+1380.8	+1528.2	+1153.3	+ 981.2	+1001.6
70°	+ 197.2	+ 633.2	+1369.2	+1574.3	+1074.2	+ 986.1	+1035.5
80°	− 134.4	+ 446.3	+1492.7	+1765.0	+1029.8	+ 983.2	+1013.4
90°	+ 2.2	− 40.3	− 55.6	− 188.8	− 131.1	− 194.3	− 233

TABLE 8

The Force Resultants (Continued)

The Force Resultant T_2 on the Cylinder

$\phi =$	0°	30°	60°	90°	120°	150°	180°
x = 0	+6501	+5572	+2893	-243.2	-2904	-5334	-6160
x = 120	-541.5	-116.2	+717.5	+713.2	+ 257.5	+ 311.3	+ 313.3
x = 240	-557.1	-126.9	+716.8	+713.7	+ 261.3	+ 322.8	+ 325.8
x = 360	-559.3	-127.1	+719.6	+712.7	+ 259.6	+ 323.8	+ 326.4
x = 480	-560.9	-127.2	+721.6	+711.9	+ 258.4	+ 324.5	+ 326.7
x = 600	-561.3	-127.2	+722.3	+711.8	+ 258.0	+ 324.7	+ 326.8

TABLE 9

THE MOMENT RESULTANTS

The Moment Resultant M_1 on the Sphere

$\phi =$	0°	30°	60°	90°	120°	150°	180°
$\theta = 10^\circ$	+.57	+.53	+.27	+.13	+.32	+.64	+.71
20°	+.49	+.40	+.22	+.12	+.22	+.17	+.08
30°	+.39	+.19	+.12	+.15	+.05	-.38	-.56
40°	+.14	+.08	+.02	+.21	-.12	-.54	-.62
50°	-.075	+.06	-.06	+.25	-.24	-.31	-.21
60°	-.25	+.03	-.10	+.19	-.26	+.05	+.25
70°	-.34	-.11	-.11	+.02	-.17	+.21	+.34
80°	-.40	-.35	+.11	0.0	+.15	+.17	+.09
90°	+138.3	-104.9	-504.8	-675.0	-301.8	-336.6	-363.4

TABLE 9

THE MOMENT RESULTANTS (continued)

The Moment Resultant M_2 on the Sphere

$\phi =$	0°	30°	60°	90°	120°	150°	180°
$\theta = 10^\circ$	- 2.75	- .71	+ 2.23	+ 2.49	+ .80	- .91	- 1.50
20°	- 4.45	- .21	+ 3.12	+ 2.15	+ .29	- .78	- 1.04
30°	- 7.92	+ 1.18	+ 3.85	+ 1.70	- .27	- .98	- 1.06
40°	- 15.4	+ 5.28	+ 3.91	+ 1.04	- .58	- 1.03	- 1.15
50°	- 32.5	+ 16.4	+ 1.91	- .32	- .05	- .67	- 1.76
60°	- 75.6	+ 46.5	- 5.50	- 4.78	+ 3.87	+ .59	- 5.5
70°	- 193.2	+ 130.8	- 25.9	- 23.33	+ 20.9	+ 5.47	- 22.6
80°	- 542.6	+ 380.4	- 74.5	-104.2	+ 89.8	+27.0	- 94.4
90°	-1641.2	+1148.8	-323.4	-667.5	+282.2	+24.6	-509.7

TABLE 9

THE MOMENT RESULTANTS (Continued)

The Moment Resultants M_1 on the Cylinder

ϕ =	0°	30°	60°	90°	120°	150°	180°
x = 0	+2189.	+1761.	+612.5	-345.8	-919.7	-1710.6	-1963.7
120	+ 7.85	+5.27	+.91	+1.13	+2.38	+6.37	-6.74
240	+ 9.40	+2.33	-7.82	+5.82	+2.73	+5.96	-3.61
360	+14.2	+2.63	-14.6	+9.30	+5.89	+8.39	-3.93
480	+17.5	+2.86	-19.2	+11.6	+8.04	+10.0	-4.13
600	+18.6	+2.94	-20.8	+12.4	+8.80	+10.6	-4.18

TABLE 9

THE MOMENT RESULTANTS (Continued)

The Moment Resultant M_2 on the Cylinder

ϕ =	0°	30°	60°	90°	120°	150°	180°
x = 0	+656.6	+528.3	+183.8	-103.6	-275.9	-513.2	-589.1
120	+ 10.6	+ .94	- 12.8	- 7.64	+ 5.27	- 5.62	- 2.09
240	+ 23.9	+ 1.02	- 33.1	- 17.3	+ 15.7	- 11.7	- 2.45
360	+ 37.7	+ 2.16	- 52.1	+ 26.1	+ 25.4	- 18.5	- 3.69
480	+ 47.1	+ 3.01	- 65.4	+ 32.1	+ 32.1	- 23.2	- 4.47
600	+ 50.5	+ 3.31	- 70.0	+ 34.3	+ 34.6	- 25.0	- 4.74

X. CONCLUSIONS

There does not exist a good technique for the solution of shell problems which have complicated multi-harmonic surface loads such as actual wind loads. In this thesis a practical technique is developed for the solution of such problems. This technique is the application of series solutions to the complex equations of shell theory as developed by Novozhilov (19). This makes possible the development of a computer program which is not difficult to input and which gives solutions without excessive use of computer time.

A computer program was developed to give force and moment resultants and displacement for cylinders and spheres loaded with up to nine harmonics with pinned or clamped boundary conditions. Although the technique could be expanded to include other shapes, in-plane surface loads, and complicated boundary conditions; changes in the material make-up of the shell such as non-isotropic behavior is impossible.

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AN ANALYSIS PROCEDURE FOR

CYLINDRICAL AND SPHERICAL SHELLS SUBJECTED TO

MULTI-HARMONIC LOADS

by

David M. Purdy

ABSTRACT

The analysis of shells of revolution subjected to complicated multi-harmonic surface loads is a very complex problem. Although several techniques exist for the solution of complex shell problems, all of these techniques present certain difficulties which make them impractical for use with complex surface loads. In an attempt to avoid these difficulties, an analysis procedure is developed to analyze cylindrical and spherical shells of this type.

The surface loads of particular interest are the wind loads obtained from model tests in the stability wind tunnel at Virginia Polytechnic Institute. Although designed for use with wind loads the procedure could be applied equally well to any surface load which could be effectively represented by a Fourier series with variable coefficients.

The surface loads are expanded into a Fourier series with polynomial coefficients for the cylindrical shell and into spherical harmonics for the spherical shell. The series are truncated after ten terms and give results which fit the wind loads within about 3% for all but a few points.

Exact solutions are obtained for the series loadings from the complex shell equations developed by Novozhilov. Novozhilov's equations are developed using a complex substitution which reduces the order of the governing equations from eighth to fourth order. This substitution simplifies the equations in such a manner that it becomes possible to obtain solutions for isotropic, uniform thickness shells without resorting to approximate techniques such as finite differences, finite elements, or numerical integration.

The solutions are obtained in series form for single layered, isotropic, uniform thickness shells with arbitrary boundary conditions. The solutions are as accurate as the load representation and the theory allows.

A computer program was written in Fortran IV to develop the loading series and to obtain numerical answers from the solution series. The program although cumbersome in size gives solutions without excessive use of computer time. The input of the program is not difficult to develop and does not require any knowledge of shell theory or of the solution technique.