A Posteriori Error Analysis for a Discontinuous Galerkin Method Applied to Hyperbolic Problems on Tetrahedral Meshes

Idir Mechai

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Slimane Adjerid, Chair
George A. Hagedorn
Tao Lin
Yuriko Renardy

March 19, 2012
Blacksburg, Virginia

Keywords: Discontinuous Galerkin method, hyperbolic problems, superconvergence, *a posteriori* error estimation, tetrahedral meshes.

A Posteriori Error Analysis for a Discontinuous Galerkin Method Applied to Hyperbolic Problems on Tetrahedral Meshes

Idir Mechai

ABSTRACT

In this thesis, we present a simple and efficient *a posteriori* error estimation procedure for a discontinuous finite element method applied to scalar first-order hyperbolic problems on structured and unstructured tetrahedral meshes. We present a local error analysis to derive a discontinuous Galerkin orthogonality condition for the leading term of the discretization error and find basis functions spanning the error for several finite element spaces. We describe an implicit error estimation procedure for the leading term of the discretization error by solving a local problem on each tetrahedron. Numerical computations show that the implicit *a posteriori* error estimation procedure yields accurate estimates for linear and nonlinear problems with smooth solutions. Furthermore, we show the performance of our error estimates on problems with discontinuous solutions.

We investigate pointwise superconvergence properties of the discontinuous Galerkin (DG) method using enriched polynomial spaces. We study the effect of finite element spaces on the superconvergence properties of DG solutions on each class and type of tetrahedral elements. We show that, using enriched polynomial spaces, the discretization error on tetrahedral elements having one *inflow* face, is $O(h^{p+2})$ superconvergent on the three edges of the *inflow* face, while on elements with one *inflow* and one *outflow* faces the DG solution is $O(h^{p+2})$ superconvergent on the *outflow* face in addition to the three edges of the *inflow* face. Furthermore, we show that, on tetrahedral elements with two *inflow* faces, the DG solution is $O(h^{p+2})$ superconvergent on the edge shared by two of the *inflow* faces. On elements with two *inflow* and one *outflow* faces and on elements with three *inflow* faces, the DG solution is $O(h^{p+2})$ superconvergent on two edges of the *inflow* faces. We also show that using enriched polynomial spaces lead to a simpler *a posteriori* error estimation procedure.

Finally, we extend our error analysis for the discontinuous Galerkin method applied to linear three-dimensional hyperbolic systems of conservation laws with smooth solutions. We perform a local error analysis by expanding the local error as a series and showing that its leading term is $O\left(h^{p+1}\right)$. We further simplify the leading term and express it in terms of an optimal set of polynomials which can be used to estimate the error.

In the name of Allah, the most gracious, the most merciful.

...وَقُل رَّبِّ زِدْنِي عِلْمًا ﴿١١٤﴾

من سورة: طه

... and say: My Lord! Increase me in knowledge.
Chapter Ta-Ha, verse (114)

*This work is dedicated to **my beloved parents**, **my sisters**,and **my brothers** for their prayers and continuous support. They have always been present with their affection, encouragement and providing me the best conditions for success.*

Acknowledgments

First of all, all praise and thanks are due to Allah, for his compassion and mercy and for giving me the ability to successfully complete this dissertation.

In addition, I would like to give a special acknowledgement and thanks to my advisor, Professor Slimane Adjerid, for his continued excellent guidance, patience, endless support, and for his invaluable spiritual and academic advice during my research study. His trust and scientific excitement inspired me throughout the completion of this thesis. I am grateful to him for helping me become a better researcher, teacher, writer, and presenter. I appreciate him greatly and I am blessed and honored to be his student.

I would also like to thank Professor Tao Lin, Professor George Hagedorn and Professor Yuriko Renardy for agreeing to serve on my dissertation committee and for their efforts in reviewing my work. It is an honor to have worked and learned from them. I want also to thank Professor Nick Loehr for the time he spent in answering my abstract algebra questions and helped us achieve part of this work.

I have to give a very special thanks and appreciation to professor Mohamed Medjden, for his help and encouragement for my academic pursuit as a graduate student at Virginia Tech.

This section would be incomplete without acknowledging all the professors of our department for teaching me and advising me throughout the years, as well as all the department staff. Special thanks go to Robert Wheeler, John Burkardt, Lizette Zietsman, Eileen Shugart, Nicole Sutphin, Bill Reilly for teaching, helping, and supporting me on various aspects of academic development.

I would like to thank all students and friends I have met at Virginia Tech. Special thanks go to Mohamed Ben Romdhane, Steven Boyce, Kapil Ahuja, Hans-Werner Van Wyk, Xu Zhang, Alan Lattimer, and Nabil Chaabane all deserve a special mention. I also would like to thank all my friends in Blacksburg for the good time that we shared together in the past five years, namely, Amara Loulizi, Helmi Temimi, Mahboub Baccouch, Toufik Laadj, Naim Dakhli, Idris Adjerid , Khaled Adjerid for their help and advice, especially during my first year in the United States. I also extend my thanks to Saher Lahouar, Faycel Beji, Michael Tomaszewicz, Omar Al-Abbasi, Amr Hilal, Ahmad Athamneh, Ahmed Kamal Ali, Khoirul Muslim, and Ali Abdullah Albassami. Being surrounded with such nice people helped me a lot to overcome the difficulty of being away from home.

Special thanks, beyond what any words can express, go to all my family without whose support and patience I would have never reached this point. Special thanks go to my parents for their prayers and continuous support. My thanks also go to my brothers Amar and Samir, my friends Abdennour Alimohad, Abdallah Benostmane, they have always been present with their affection and encouragement.

I also address my sincere acknowledgements and gratitude to Virginia Tech, as well as to the US National Science Foundation (NSF) for supporting this work by grants DMS 0809262.

Contents

1 Introduction **1**

1.1 Review of Past Work and Historical Perspective 1

1.2 Problem Statement 4

1.3 Research Goals 5

1.4 Outline 6

2 The Discontinuous Galerkin Method for Hyperbolic Problems **8**

2.1 Notations 8

2.2 The Discontinuous Galerkin Method 9

2.3 Orthogonal Polynomial Basis on Tetrahedra 10

2.4 Numerical Integration Rules 12

3 Discontinuous Galerkin Error Estimation for the Space $\mathcal{P}_p$ **15**

3.1 Discontinuous Galerkin Formulation 15

3.2 Local DG Error Analysis 21

3.3 *A Posteriori* Error Estimation 30

3.3.1 Error Basis Functions 31

3.3.2 Error Estimation Procedure 32

3.4 Computational Examples 35

3.5 Conclusion 54

4 Superconvergence Error Analysis **55**

4.1 Introduction 55

4.2 Preliminary Results 59

4.3 DG Superconvergence Error Analysis 62

4.4 Factoring the Leading Term of the DG Error 83

4.4.1 Basic Algebraic Geometry and Notations 84

4.4.2 Groebner Bases 85

4.4.3 Factoring Polynomials in $\mathbb{C}[\xi, \eta, \zeta]$ 86

4.5 Basis Functions for DG Error Estimation 91

4.5.1 Basis Functions for Element of *Class I, II* and *III* 93

4.5.2 Error Estimation Procedure 99

4.6 Computational Examples 100

5 The Discontinuous Galerkin Method for Nonlinear Problems 122

5.1 Nonlinear Hyperbolic Problems 122

5.2 Error Estimation Procedure 124

5.3 Computational Examples 125

5.4 Conclusion 128

6 *A Posteriori* Error Estimation for Linear Hyperbolic systems 131

6.1 DG Formulation and Local Error Analysis 131

6.2 *A Posteriori* Error Estimation 137

6.3 Solution and Error Estimation Procedure 139

6.4 Conclusion 140

7 Conclusions and Future Work 141

7.1 Contributions 141

7.2 Future Work 142

Bibliography 144

List of Tables

2.1 The space $\mathcal{P}_4$. 11

2.2 Orthogonal basis functions for the space $\mathcal{P}_3$ on the reference tetrahedron. . . 12

2.3 A 45-point quadrature on a tetrahedron 14

3.1 Reference element for each class and type. 26

3.2 Error basis functions on a reference tetrahedron of *Class I*. 32

3.3 Error basis functions on a reference tetrahedron of *Class II*, $\boldsymbol{a}_0 = (\alpha, \beta, \gamma)^T$, $\lambda = \alpha/\beta$, $\mu = \gamma/\beta$ 33

3.4 Error basis functions on a reference tetrahedron of *Class III*, $\boldsymbol{a}_0 = (\alpha, \beta, \gamma)^T$, $\lambda = \alpha/\beta$, $\mu = \gamma/\beta$. 33

3.5 L^2 errors, orders and effectivity indices for the standard DG method applied to problem (3.4.1) on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$. 38

3.6 L^2 errors and effectivity indices for problem (3.4.1) using the modified DG method on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$. 39

3.7 L^2 errors and effectivity indices for problem (3.4.1) using the modified DG method on unstructured meshes having N elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$. 40

3.8 L^2 errors and effectivity indices for problem (3.4.1) using the modified DG method and the approximated weak formulation for the error (3.3.10) on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$. 41

3.9 L^2 errors and effectivity indices for problem (3.4.2) using the modified DG method on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$. 42

3.10 L^2 errors and effectivity indices for problem (3.4.2) using the modified DG method on unstructured meshes having N elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$. 43

3.11 L^2 errors and effectivity indices for problem (3.4.2) using the modified DG method and the approximated weak formulation for the error (3.3.10) on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$. 44

3.12 L^2 errors and effectivity indices for problem (3.4.3) using the modified DG method on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$. 46

3.13 L^2 errors and effectivity indices for problem (3.4.4) using the modified DG method on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$. 47

3.14 L^2 errors and effectivity indices for problem (3.4.5) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$. 48

4.1 Monomials in black for $\mathcal{P}_4$ (left) and $\mathcal{L}_3$ (right). 56

4.2 Monomials in black for $\mathcal{P}_4$ (left) and $\mathcal{U}_3$ (right). 57

4.3 Monomials in black for $\mathcal{P}_4$ (left) and $\mathcal{M}_3$ (right). 57

4.4 Superconvergence points using space $\mathcal{U}_p$. 82

4.5 Error basis functions for elements of *Class I* and *Type 1* using the space $\mathcal{U}_p$. 94

4.6 Error basis functions for elements of *Class I* and *Type 2* using the space $\mathcal{U}_p$. 95

4.7 Error basis functions for elements of *Class I* and *Type 3* using the space $\mathcal{U}_p$, with $\lambda = -\frac{1}{2}$. 96

4.8 Error basis functions for elements of *Class II* and *Type 1* using the space $\mathcal{U}_2$. 96

4.9 Error basis functions for elements of *Class II* and *Type 2* using the space $\mathcal{U}_p$, with $\lambda = \frac{1}{2}$ 96

4.10 Error basis functions for elements of *Class III* and *Type 1* using the space $\mathcal{U}_p$, with $\lambda = \frac{1}{2}$. 97

4.11 L^2 errors and effectivity indices for problem (4.6.1) using the standard and modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{L}_p$, $p = 1, 2, 3$. 102

4.12 L^2 errors and effectivity indices for problem (4.6.1) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$. 103

4.13 Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.1) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$. 104

4.14 L^2 errors and effectivity indices for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$. 106

4.15 Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, on elements of *Type 1*. 107

4.16 Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Type 2*. 108

4.17 Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Type 3*. 109

4.18 Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Class I*. 110

4.19 Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Class II*. 111

4.20 Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Class III*. 112

4.21 L^2 errors and effectivity indices for problem (4.6.2) using the modified DG method on unstructured meshes having N elements for the spaces $\mathcal{U}_p$, $p = 2, 3$. 112

4.22 L^2 errors and effectivity indices for problem (4.6.2) using the modified DG method and the approximated weak formulation for the error (4.5.7) on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3$. 113

4.23 L^2 errors and effectivity indices for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$. 114

4.24 Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Type 1*. 115

4.25 Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Type 2*. 116

4.26 Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Type 3*. 117

4.27 Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Class I*. 118

4.28 Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Class II*. 119

4.29 Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Class III*. 120

5.1 L^2 errors, their orders and effectivity indices for problem (5.3.1) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2$. 126

5.2 L^2 errors, their orders and effectivity indices for problem (5.3.2) using the modified DG method on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1$. 127

List of Figures

3.1 Mapping of physical elements of *Class I* (left) to corresponding reference elements (right). 18

3.2 Mapping of physical elements of *Class II* and *III* (left) to corresponding reference elements (right). 19

3.3 Unstructured tetrahedral meshes (obtained by *COMSOL*) having 943(left) and 8713(right) elements on $[0,1]^3$. 36

3.4 Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.6, with $a = 1$, $b = 0$, $\mathcal{P}_0$ and using uniform meshes having $N = 2058$ elements (left) and $N = 13182$ (right). 49

3.5 Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.6, with $a = 1$, $b = 0$, $\mathcal{P}_1$ and using uniform meshes having $N = 2058$ elements (left) and $N = 13182$ (right). 50

3.6 Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.6, with $a = 1$, $b = 1/3$ using $\mathcal{P}_0$ (left) and $\mathcal{P}_1$ (right) on a uniform mesh. 50

3.7 Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.6, with $a = 1/2$, $b = 0$ using $\mathcal{P}_0$ (left) and $\mathcal{P}_1$ (right) on a unstructured mesh. 51

3.8 Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.7, with $\mathcal{P}_0$ and using uniform meshes having $N = 2058$ elements (left) and $N = 16464$ (right). 52

3.9 Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.7, with $\mathcal{P}_1$ and using uniform meshes having $N = 2058$ elements (left) and $N = 16464$ (right). 52

3.10 Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.7, with $\mathcal{P}_2$ and using uniform meshes having $N = 2058$ elements (left) and $N = 16464$ (right). 53

3.11 Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.7, with $\mathcal{P}_3$ and using uniform meshes having $N = 2058$ elements (left) and $N = 16464$ (right). 53

4.1 Superconvergence points (red) using the space $\mathcal{L}_p$ on elements of *Class I* and *Type 1*. 64

4.2 Superconvergence points at the zero level curves (red) of φ_0^1 (left) and φ_1^0 (right). 66

4.3 Superconvergence points at the zero level curves (red) of φ_0^2 (left), φ_1^1 (center) and φ_2^0 (right). 66

4.4 Superconvergence points at the zero level curves (red) of φ_0^3 (top-left), φ_1^2 (top-right), φ_2^1 (bottom-left) and φ_3^0 (bottom-right). 67

4.5 Superconvergence points at the intersection of the zero level curves (marked by x) of φ_0^1 (blue) and φ_1^0 (red). 68

4.6 Superconvergence points (marked by x) for DG solutions as in Corollary 4.3.1 assertion *(ii)*. 68

4.7 Superconvergence points (marked by x and the dashed line) by combining φ_i^{3-i} and φ_j^{3-j}, for $0 \le i, j \le 3,\ i \ne j$. 69

4.8 Superconvergence points (red) using the space $\mathcal{U}_p$ for each type of element. . 81

4.9 Interior superconvergence points (red) using the space $\mathcal{U}_2$ for elements of *Class I* and *Type 1*. 97

4.10 Interior superconvergence points (red) using the space $\mathcal{U}_2$. 98

5.1 True solution (left), L^2 true local errors (center) and local effectivity indices (right) for (5.3.2) on a uniform mesh having N=13720 tetrahedra for the space $\mathcal{P}_0$ at $t = 0.5$ (top) and $t = 1$ (bottom). 128

5.2 True solution (left), L^2 true local errors (center) and local effectivity indices (right) for (5.3.2) using a uniform mesh having N=13720 tetrahedra for the space $\mathcal{P}_1$ at $t = 0.5$ (top) and $t = 1$ (bottom). 129

5.3 True solution (left), L^2 true local errors (center) and local effectivity indices (right) for (5.3.2) using a uniform mesh having N=5000 tetrahedra for the space $\mathcal{P}_1$ at $t = 0.5$ (top) and $t = 1$ (bottom). 130

Chapter 1

Introduction

Partial differential equations arise in a number of applications, such as fluid flow, heat transfer, solid mechanics, biological processes, and astrophysics. These equations often fall into one of the three types: hyperbolic, parabolic and elliptic equations. Our emphasis in this thesis is on hyperbolic conservation laws, which describe a number of interesting problems in diverse areas in science and engineering.

There are quite a number of different methods for solving differential equation computationally. Among these are the widely used finite difference, finite volume, and finite element methods, which are techniques used to derive discrete representations of the partial derivative operators and give an approximation for the exact solution. If one also needs to advance the equations in time, there is likewise a wide variety of methods for the integration of systems of ordinary differential equations.

Discontinuous Galerkin methods (DG methods) are a family of locally conservative, stable and high-order accurate methods that are easily coupled with other well-known methods and are well-suited to adaptive strategies. For these reasons, they have attracted the attention of many researchers working in computational mechanics, computational mathematics and computer science. They provide an appealing approach to address problems having discontinuities, such as those arising in hyperbolic conservation laws. The DG method does not require the approximate solutions to be continuous across element boundaries; it instead involves a flux term to account for the discontinuities.

1.1 Review of Past Work and Historical Perspective

Over the past several years, significant advances have been made in developing the discontinuous Galerkin finite element method for solving fluid flow and heat transfer problems. Certain unique features of the method have made it attractive as an alternative to other popular methods such as finite volume and finite elements.

DG methods form a class of numerical methods for solving partial differential equations. They combine features of the finite element and the finite volume framework and have been successfully applied to hyperbolic, elliptic and parabolic problems arising from a wide range of applications. DG methods have, in particular, received considerable interest for problems with a dominant first-order part, e.g., in electrodynamics, fluid mechanics and plasma physics. The DG method was developed first to solve the neutron transport equation [53]

$$\sigma u + \nabla \cdot (au) = f$$

where σ is a real constant, $a\left(x\right)$ is a piecewise constant, and u is unknown. The method was then studied for initial-value problems for ordinary differential equations [6, 19, 49, 41, 45], hyperbolic [26, 27, 29, 30, 32, 18, 37] and diffusion and convection-diffusion [13, 16, 17, 20, 24, 31, 35, 36, 38, 39] partial differential equations. The method may be regarded as a cross between a finite volume and finite element method and it has many of the good properties of both, for example, it has the following properties: *(i)* locally conservative, *(ii)* well suited to solve problems on locally refined meshes with hanging nodes, *(iii)* exhibits strong superconvergence that can be used to estimate the discretization error, *(iv)* has a simple communication pattern between elements with a common face that makes it useful for parallel computation, *(v)* can handle problems with complex geometries to high order, and *(vi)* does not require continuity across element boundaries, it instead involves a flux term to account for the discontinuities, which simplify both h-refinement (mesh refinement and coarsening) and p-refinement (method order variation).

The first analysis of this method was presented by LeSaint and Raviart [49], showing a rate of convergence of $O\left(h^p\right)$ in the $\mathcal{L}^2$ norm on a general triangulated grid, where h is the mesh size and optimal convergence rate $O\left(h^{p+1}\right)$, on a Cartesian grid with a local polynomial approximation of order p. Later, this result was improved by Johnson and Pitkaranta [52] to $O\left(h^{p+1/2}\right)$-convergence on general grids. Hesthaven and Gottlieb [51] confirmed the optimality of this convergence rate by a special example. These results assume the exact solution to be smooth, whereas for linear problems with nonsmooth solutions the method were analyzed in [25, 50]. Techniques for postprocessing on Cartesian grids to improve the accuracy to $O\left(h^{p+1}\right)$ for linear problems have been developed in [28, 44, 55].

For nonlinear systems of conservation laws, Chavent and Salzano [23] constructed an explicit DG Method, by discretizing in space using the DG method with piecewise linear elements to obtain an explicit semi-discrete scheme. Then to solve in time, they use a simple Euler forward method. Later, Chavent and Cockburn [22] modified the scheme by introducing a slope limiter, introduced by van Leer [56], to improve the stability of the scheme. The Runge-Kutta Discontinuous Galerkin (RKDG) methods were introduced by Cockburn and Shu [30]. They used a piecewise linear DG method for the space discretization, a special explicit TVD second-order Runge-Kutta time discretization, and a modified slope limiter to maintain formal accuracy of the scheme at extrema. A generalized approach for high-order accurate RKDG methods for scalar conservation laws, was developed by Cockburn and Shu [29]. Cockburn, Lin, and Shu [27] extended the RKDG method to one-dimensional systems, and then Cockburn, Hou and Shu [26] to multi-dimensional scalar equations.

Regardless of which DG method is used, *a posteriori* error estimations are needed to guide adaptivity and stop the refinement process in an adaptive framework and to provide a measure of the numerical solution with respect to the exact solution. According to Adjerid *et al.* [5], a good *a posteriori* error estimate should:

(i) be asymptotically correct in the sense that they converge to the true error under h and p-refinement,

(ii) be computationally inexpensive to compute relative to the solution cost,

(iii) be robust by giving accurate estimates for a wide range of meshes and method orders,

(iv) supply relatively tight upper and lower bounds of the true error in a particular norm, and

(v) provide local error indicators that can be used to compute global error estimates in commonly used norms.

The efficiency of the *a posteriori* error estimates is measured by the effectivity index which is the ratio of the estimated error to the exact error. Ideally, the effectivity indices should approach unity under mesh refinement.

Asymptotically exact *a posteriori* DG error estimates for hyperbolic problems were first constructed for one-dimensional hyperbolic problems by Adjerid *et al.* [6]. Later, Adjerid *et al.* [8, 9, 10, 11, 12] presented asymptotically exact implicit error estimates for multi-dimensional problems on rectangular meshes. Krivodonova and Flaherty [48] showed that the leading term of the local discretization error on triangles having one *outflow* edge is spanned by a suboptimal set of orthogonal polynomials of degree p and $p+1$ and computed DG error estimates by solving local problems. Adjerid *et al.* [1, 7] constructed asymptotically exact *a posteriori* error estimates for a local discontinuous Galerkin method applied to convection and convection-diffusion problems. Adjerid and Baccouch [2, 3, 4] investigated the superconvergence properties of the discontinuous Galerkin method applied to scalar first-order hyperbolic partial differential equations on structured and unstructured triangular meshes. They showed that the discontinuous finite element solution is $O\left(h^{p+2}\right)$ superconvergent at Legendre points on the *outflow* edge for triangles having one *outflow* edge. For triangles having two *outflow* edges the finite element error is $O\left(h^{p+2}\right)$ superconvergent at the end points of the *inflow* edge. With these results they constructed asymptotically correct global *a posteriori* error estimates by solving a local hyperbolic problem on each triangle.

1.2 Problem Statement

Let d denote the space dimension, $\mathbf{x} = (x_1, ..., x_d)^T$ the space variable defined on a domain $\Omega \subset \mathbb{R}^d$, and t the time variable defined on $[0, T]$. Let us consider a system of hyperbolic conservation laws [14, 57]

$$\begin{aligned} \mathbf{u}_t + \nabla \cdot \mathbf{F}(\mathbf{u}) &= \mathbf{u}_t + \sum_{i=1}^{d} \frac{\partial \mathbf{F}(\mathbf{u})}{\partial x_i} = \mathbf{r}(\mathbf{u}), \mathbf{x} \in \mathbb{R}^d, t \in [0, T], && \text{(1.2.1a)} \\ \mathbf{u} &= \mathbf{g}, \text{ at } t = 0, && \text{(1.2.1b)} \end{aligned}$$

with well posed boundary data prescribed on $\partial\Omega$. The symbol ∇ refers the spacial gradient operator, the components of the solution $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), ..., u_n(\mathbf{x}, t))^T$ are the densities of various conserved quantities, and $\nabla \cdot \mathbf{F}(\mathbf{u})$ is the divergence of the flux function $\mathbf{F}(\mathbf{u}) = (\mathbf{F}_1(\mathbf{u}), ..., \mathbf{F}_n(\mathbf{u}))^T$, where $\mathbf{F}_i(\mathbf{u}) : \mathbb{R}^n \longrightarrow \mathbb{R}^d$ is the i^{th} component of the flux $\mathbf{F}$. The function $\mathbf{g}$ describes the initial condition of $\mathbf{u}$.
For instance, the three-dimensional Euler equations have the following form:

$$\mathbf{u}_t + \mathbf{F}(\mathbf{u})_x + \mathbf{G}(\mathbf{u})_y + \mathbf{H}(\mathbf{u})_z = 0, \text{ in } \Omega \subset \mathbb{R}^3, t \in [0, T], \quad \text{(1.2.2)}$$

where

$$\begin{aligned} \mathbf{u} &= (\rho, \rho u, \rho v, \rho w, E)^T, && \text{(1.2.3a)} \\ \mathbf{F}(\mathbf{u}) &= \left(\rho u, \rho u^2 + p, \rho u v, \rho u w, u(E + p)\right)^T, && \text{(1.2.3b)} \\ \mathbf{G}(\mathbf{u}) &= \left(\rho v, \rho u v, \rho v^2 + p, \rho v w, v(E + p)\right)^T, && \text{(1.2.3c)} \\ \mathbf{H}(\mathbf{u}) &= \left(\rho w, \rho u w, \rho v w, \rho w^2 + p, w(E + p)\right)^T, && \text{(1.2.3d)} \end{aligned}$$

and ρ, u, v, w, p, respectively, denote density, x, y, z components of velocity, and pressure. E is the total energy per unit volume, defined as $E = \rho e + \frac{1}{2}\rho(u^2 + v^2 + w^2)$, where e is specific internal energy given by a caloric equation of state $e = \frac{p}{\rho(\gamma-1)}$, with $\gamma = \dfrac{c_p}{c_v}$ denoting the ratio of specific heats.
Although the DG formulation applies to arbitrary conservation laws, our discussion is restricted to the following hyperbolic problems:

(i) A convection reaction problem

$$\boldsymbol{a}.\nabla u + \phi(u) = f(x, y, z), \ (x, y, z) \in \Omega, \quad \text{(1.2.4a)}$$

subject to the boundary condition

$$u|_{\partial\Omega^-} = g(x, y, z), \quad \text{(1.2.5a)}$$

with $\phi(u)$ is a smooth function and $\partial\Omega^- = \{(x, y, z) \in \partial\Omega | \ \boldsymbol{a} \cdot \mathbf{n} < 0\}$, where $\mathbf{n}$ denotes the outward unit normal vector.

(ii) A nonlinear scalar hyperbolic problems of the form

$$\nabla \cdot \mathbf{F}(u) = f(x,y,z), \ (x,y,z) \in \Omega, \tag{1.2.6a}$$

subject to the boundary condition

$$u(x,y,z) = h_0(x,y,z), \ (x,y,z) \in \partial\Omega^-, \tag{1.2.7a}$$

where $\mathbf{F} : \mathbb{R} \longrightarrow \mathbb{R}^3$, $u : \mathbb{R}^3 \longrightarrow \mathbb{R}$, f and h_0 are analytic functions. We further assume that $\mathbf{F}(u)$ is such that the boundary $\partial\Omega$ can be split into inflow $\partial\Omega^-$, outflow $\partial\Omega^+$ and characteristic $\partial\Omega^0$ boundaries using $\boldsymbol{a}(u) = \mathbf{F}'(u)$.

(iii) A first-order symmetric linear hyperbolic system in multiple space dimensions of the form

$$\mathbf{u}_t + \sum_{i=1}^{d} \mathbf{A}_i \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{g}(\mathbf{x}, \mathbf{t}), \mathbf{x} \in \Omega, t \in [0,T], \tag{1.2.8a}$$

with source term $\mathbf{g} : (0,T) \times \Omega \longrightarrow \mathbb{R}^m$, subject to the initial and boundary conditions

$$\begin{aligned} \mathbf{u}(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}), \mathbf{x} \in \Omega, & (1.2.9a)\\ \left(\sum_{i=1}^{d} n_i A_i\right)^- \mathbf{u}(\mathbf{x},t) &= \left(\sum_{i=1}^{d} n_i \mathbf{A}_i\right)^- \mathbf{u}_B(\mathbf{x},t), \mathbf{x} \in \partial\Omega, t \in (0,T), & (1.2.9b) \end{aligned}$$

where $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_d$ are constant matrices in $\mathbb{R}^{m\times m}$, m is the size of the system, $\mathbf{n}$ denotes the unit outward normal on $\partial\Omega$, and

$$\begin{aligned} \text{if } \mathbf{M} &= \mathbf{P} diag(\lambda_1, \lambda_2, \ldots, \lambda_m) \mathbf{P}^T \Longrightarrow \\ \mathbf{M}^- &= \mathbf{P} diag(\min(\lambda_1,0), \min(\lambda_2,0), \ldots, \min(\lambda_m,0)) \mathbf{P}^T, \end{aligned}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$.

1.3 Research Goals

In this thesis we extend the error analysis of Adjerid and Baccouch [2, 3, 4] and Adjerid and Weinhart [54], to three-dimensional hyperbolic problems using general tetrahedral meshes. First, each tetrahedron is of *Class I, II* and *III*, respectively, if it has one, two and three *outflow* faces. Moreover, elements are of *Type 1, 2* and *3*, respectively, if they have one, two and three *inflow* faces. We perform a local error analysis on an arbitrary tetrahedron by constructing a family of similar tetrahedra with size h and having the same center. This family of tetrahedra is such that as $h \to 0$ the limit is the common center. Assuming we compute a *p-degree* DG approximation of a smooth solution, we expand the local error as a power series with respect to h and prove that the leading term of the DG error is a $O(h^{p+1})$ polynomial of degree *p+1*. We further simplify the leading term and express it in terms of an optimal set of polynomials which will be used to estimate the leading term of the error.

We investigate the pointwise superconvergence properties of the DG method, using enriched polynomial spaces. We study the effect of finite element spaces on the superconvergence properties of DG solutions on each class and type of tetrahedral elements. We show that the discretization error on tetrahedral elements having one *inflow* and one *outflow* faces is $O(h^{p+2})$ superconvergent on the *outflow* face and on the three edges of the *outflow* face. On tetrahedral elements with one *inflow* face, we prove that the discretization error is $O(h^{p+2})$ superconvergent on the three edges of the *inflow* face. Furthermore, we show that, on tetrahedral elements with two *inflow* faces, the DG solution is $O(h^{p+2})$ superconvergent on the edge shared by two of the *inflow* faces. On elements with two *inflow* and one *outflow* faces and on elements with three *inflow* faces, we establish $O(h^{p+2})$ superconvergence on two edges of the *inflow* faces.

We further present an error estimation procedure to compute accurate DG error estimates on structured and unstructured tetrahedral meshes and solve several linear and nonlinear problems to validate our theory for smooth and discontinuous solutions.

Finally, we present an *a posteriori* error analysis for the discontinuous Galerkin discretization error of first-order linear symmetric hyperbolic systems of partial differential equations with smooth solutions. We perform a local error analysis by expanding the local error as a series and showing that its leading term is $O\left(h^{p+1}\right)$.

1.4 Outline

In this dissertation, we will discuss the numerical approximation of hyperbolic equations using the DG method. In Chapter 2, we introduce our notations and present the discontinuous Galerkin formulation for three-dimensional conservation laws. This chapter also includes an overview of orthogonal basis functions, and numerical quadratures on triangles and tetrahedra.

In Chapter 3 we present the problem and state its discontinuous Galerkin formulation for the space of polynomials of degree not exceeding p. A local error analysis for each class of elements is presented. Then we construct optimal error basis functions and spaces and present an *a posteriori* error estimation procedure. We present several computational examples to validate our theory.

Chapter 4 is devoted to proving new pointwise superconvergence results for all classes and types of elements using enriched polynomial spaces. After a summary of our superconvergence results, we show how to construct efficient and asymptotically correct *a posteriori* finite element error estimates for all classes and types of elements. Finally, we provide several examples showing the DG errors and their rates of convergence under mesh refinement.

In Chapter 5, we extend our error analysis to three-dimensional nonlinear hyperbolic problems and present an *a posteriori* error estimation procedure. Then we test our *a posteriori* error estimates on problems having smooth and discontinuous solutions to show their efficiency and accuracy under mesh refinement.

In Chapter 6 we present an *a posteriori* error analysis for the discontinuous Galerkin discretization error applied to first-order linear symmetric hyperbolic systems with smooth solutions.

Finally, we conclude and discuss our results in Chapter 7. A short summary and an outline of future work are given.

Chapter 2

The Discontinuous Galerkin Method for Hyperbolic Problems

In this chapter we introduce our notations and present the discontinuous Galerkin formulation for three-dimensional conservation laws. We also include an overview of orthogonal basis functions, and numerical quadratures on triangles and tetrahedra.

2.1 Notations

Throughout this dissertation, we use the $\mathcal{L}^2$ norm of a function $f(x, y, z)$ over a domain Ω defined by

$$\|f\|_{2,\Omega} = \left(\iiint_{\Omega} |f(x, y, z)|^2 \, dxdydz \right)^{\frac{1}{2}}.$$

The local error is denoted by $\epsilon = u - U$, where u and U, respectively, denote the exact and numerical solutions.
The divergence and Jacobian of a differentiable vector function
$\mathbf{a}(x, y, z) = (a_1(x, y, z), a_2(x, y, z), a_3(x, y, z))^T$ are defined by

$$\nabla \cdot \mathbf{a}(x, y, z) = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z},$$

and

$$\mathbf{J}(\mathbf{a}(x, y, z)) = \begin{pmatrix} \frac{\partial a_1}{\partial x} & \frac{\partial a_1}{\partial y} & \frac{\partial a_1}{\partial z} \\ \frac{\partial a_2}{\partial x} & \frac{\partial a_2}{\partial y} & \frac{\partial a_2}{\partial z} \\ \frac{\partial a_3}{\partial x} & \frac{\partial a_3}{\partial y} & \frac{\partial a_3}{\partial z} \end{pmatrix}.$$

The gradient (or gradient vector field) of a scalar function $u(x,y,z)$ is denoted by ∇u and is defined by

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)^T.$$

2.2 The Discontinuous Galerkin Method

To illustrate the basic ideas of the discontinuous finite element method, we consider the hyperbolic conservation laws (1.2.1), and partition the domain Ω into a collection of N elements such that

$$\overline{\Omega} = \bigcup_{j=1}^{N} \overline{\Delta}_j,$$

then multiplying (1.2.1a) by a test function $\mathbf{v}$, integrating over the element Δ_j, and applying Stokes' theorem to write

$$\int_{\Delta_j} \mathbf{v}^T \mathbf{u}_t d\mathbf{x} - \int_{\Delta_j} \mathbf{v}^T \mathbf{F}(\mathbf{u})\, d\mathbf{x} + \int_{\partial\Delta_j} \mathbf{v}^T \mathbf{F}(\mathbf{u}) \cdot \mathbf{n} d\sigma = \int_{\Delta_j} \mathbf{v}^T \mathbf{r}(\mathbf{u})\, d\mathbf{x}, \tag{2.2.1}$$

where $\mathbf{n}$ is the normal vector to $\partial\Delta_j$.
On Δ_j, $\mathbf{u}$ is approximated by $\mathbf{U}_j \in \mathcal{P}_p(\Delta_j)$, where $\mathcal{P}_p$ consists of polynomials of degree not exceeding p on Δ_j, and $\mathbf{v}$ by $\mathbf{V}$ taken from the same function space as $\mathbf{U}_j$, with $j = 1, 2, \ldots, N$. Substituting $\mathbf{u}$ and $\mathbf{v}$, respectively, by $\mathbf{U}$ and $\mathbf{V}$ in equation (2.2.1), we obtain the discontinuous Glerkin finite element formulation

$$\int_{\Delta_j} \mathbf{V}^T \mathbf{U}_{j,t} d\mathbf{x} - \int_{\Delta_j} \mathbf{V}^T \mathbf{F}(\mathbf{U}_j)\, d\mathbf{x} + \int_{\partial\Delta_j} \mathbf{V}^T \hat{\mathbf{F}}(\mathbf{U}_j) \cdot \mathbf{n} d\sigma = \int_{\Delta_j} \mathbf{V}^T \mathbf{r}(\mathbf{U}_j)\, d\mathbf{x}. \tag{2.2.2}$$

In the traditional finite element, the field variable $\mathbf{U}_j$ is forced to be continuous across the boundary. However, the discontinuous Glerkin method allow to $\mathbf{U}_j$ to be discontinuous across the boundary. Therefore, across the element, the following two different values are defined at the two sides of the boundary:

$$\mathbf{U}_j^-(\mathbf{x}) = \lim_{s \longrightarrow 0^+} \mathbf{U}_j(\mathbf{x} + s\mathbf{n}), \text{ and } \mathbf{U}_j^+(\mathbf{x}) = \lim_{s \longrightarrow 0^-} \mathbf{U}_j(\mathbf{x} + s\mathbf{n}).$$

Moreover, we note that $\mathbf{U}_j$ is piecewise smooth function and is discontinuous only at the element boundaries. The solution $\mathbf{u}$ and $\mathbf{F}(\mathbf{u})$ are smooth within (but excluding) the boundary. There is no direct coupling with other elements, except between elements sharing the same boundaries (faces in case of tetrahedral meshes). The field values $\mathbf{F}(\mathbf{U}_j)$ at the interface between two elements, are not unique. Cockburn and Shu [29] present several possible numerical flux functions. An example of numerical flux that can apply to vector systems and employs upwind information is the Lax-Friedrichs function [29]

$$\hat{\mathbf{F}}(\mathbf{U}_j(\mathbf{x})) = \frac{1}{2}\left[\left(\mathbf{F}\left(\mathbf{U}_j^-(\mathbf{x})\right) + \mathbf{F}\left(\mathbf{U}_j^+(\mathbf{x})\right)\right) - \lambda_{\max}\left(\mathbf{F}\left(\mathbf{U}_j^-(\mathbf{x})\right) - \mathbf{F}\left(\mathbf{U}_j^+(\mathbf{x})\right)\right)\right],$$

with

$$\lambda_{\max} = \max_{1 \le i \le m} |\lambda_i|,$$

where λ_i for $1 \le i \le m$ are eigenvalue of the Jacobian matrix $[\mathbf{F} \cdot \mathbf{n}]_{\mathbf{u}} = \dfrac{\partial}{\partial \mathbf{u}} (\mathbf{F}(\mathbf{u}) \cdot \mathbf{n})$.

The discontinuous formulation expressed in (2.2.2), may be viewed from different perspectives, which all involve the cross-element treatments either by weakly imposing the continuity at the element interface, or by using numerical fluxes, or by boundary constraint minimization.

Finally, selecting a time integration strategy, typically, this is performed by classical Runge-Kutta integration scheme [29] with a time step chosen according to the Courant-Friedrichs-Levy (CFL) condition.

2.3 Orthogonal Polynomial Basis on Tetrahedra

In our error analysis we use the complete space of polynomials of degree not exceeding p defined as

$$\mathcal{P}_p = \left\{ q | \, q = \sum_{m=0}^{p} \sum_{i=0}^{m} \sum_{j=0}^{i} c_{i,j}^{m} x^{m-i} y^{i-j} z^{j} \right\}, \tag{2.3.1}$$

and the enriched polynomial spaces $\mathcal{L}_p$, $\mathcal{U}_p$, and $\mathcal{M}_p$ defined in (4.1.1). The spaces $\mathcal{L}_p$, $\mathcal{U}_p$, and $\mathcal{M}_p$ are suboptimal but they lead to a simpler *a posteriori* error estimation which we present in chapter 4.
The finite element space $\mathcal{P}_4$ is shown in Table 2.1 and has dimension

$$\dim(\mathcal{P}_p) \;=\; \frac{(p+1)(p+2)(p+3)}{6}. \tag{2.3.2}$$

Let $P_n^{\delta,\rho}(x)$; $-1 \le x \le 1$, denote the n-degree Jacobi polynomial defined by the Rodrigues' formula [46]:

$$P_n^{\delta,\rho}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\delta} (1+x)^{-\rho} \frac{d^n}{dx^n} \left[(1-x)^{\delta+n} (1+x)^{\rho+n} \right], \; \delta, \, \rho > -1,$$

which can also be computed using the following recursion formula [46]:

$$\begin{aligned}
P_0^{\delta,\rho}(x) &= 1, \\
P_1^{\delta,\rho}(x) &= \frac{1}{2} (\delta - \rho + (\delta + \rho + 2) x), \\
a_n^1 P_{n+1}^{\delta,\rho}(x) &= \left(a_n^2 + a_n^3 x\right) P_n^{\delta,\rho}(x) - a_n^4 P_{n-1}^{\delta,\rho}(x),
\end{aligned}$$

1
x
$y \quad z$
x^2
$xy \quad xz$
$y^2 \quad yz \quad z^2$
x^3
$x^2y \quad x^2z$
$xy^2 \quad xyz \quad x^2z$
$y^3 \quad y^2z \quad yz^2 \quad z^3$
x^4
$x^3y \quad x^3z$
$x^2y^2 \quad x^2yz \quad x^2z^2$
$xy^3 \quad xy^2z \quad xyz^2 \quad xz^3$
$y^4 \quad y^3z \quad y^2z^2 \quad yz^3 \quad z^4$

Table 2.1: The space $\mathcal{P}_4$.

where

$$\begin{aligned}
a_n^1 &= 2\left(n+1\right)\left(n+\delta+\rho+1\right)\left(2n+\delta+\rho\right),\\
a_n^2 &= \left(2n+\delta+\rho+1\right)\left(\delta^2-\rho^2\right),\\
a_n^3 &= \left(2n+\delta+\rho\right)\left(2n+\delta+\rho+1\right)\left(2n+\delta+\rho+2\right),\\
a_n^4 &= 2\left(n+\delta\right)\left(n+\rho\right)\left(2n+\delta+\rho+2\right).
\end{aligned}$$

An orthogonal basis for $\mathcal{P}_p$ on the reference tetrahedron defined by the vertices $\mathbf{v}_1 = (0,0,0)$, $\mathbf{v}_2 = (1,0,0)$, $\mathbf{v}_3 = (0,1,0)$ and $\mathbf{v}_4 = (0,0,1)$ is given in terms of Jacobi polynomials [46] as

$$\varphi_{q,r}^{m}\left(\xi,\eta,\zeta\right) = \bar{P}_m^{0,0}\left(\xi,\eta,\zeta\right)\bar{P}_q^{2m+1,0}\left(\eta,\zeta\right)\bar{P}_r^{2m+2q+2,0}\left(\zeta\right), m,q,r \geq 0,$$

of degree $m+q+r$, where

$$\begin{aligned}
\bar{P}_m^{0,0}\left(\xi,\eta,\zeta\right) &= \left(1-\zeta-\eta\right)^m P_m^{0,0}\left(\frac{2\xi}{1-\eta-\zeta}-1\right),\\
\bar{P}_q^{2m+1,0}\left(\eta,\zeta\right) &= \left(1-\zeta\right)^q P_q^{2m+1,0}\left(\frac{2\eta}{\left(1-\zeta\right)}-1\right),\\
\bar{P}_r^{2m+2q+2,0}\left(\zeta\right) &= P_r^{2m+2q+2,0}\left(2\zeta-1\right),
\end{aligned}$$

and satisfy

$$\int_0^1\int_0^{1-\eta}\int_0^{1-\eta-\eta}\varphi_{i,j}^{m}\varphi_{k,l}^{n}d\zeta d\eta d\xi = c_{ij,kl}^{mn}\delta_{ik}\delta_{jl}\delta_{mn}.$$

$\varphi_{0,0}^{0} =$	1
$\varphi_{0,0}^{1} =$	$\zeta + \eta + 2\xi - 1$
$\varphi_{1,0}^{0} =$	$\zeta + 3\eta - 1$
$\varphi_{0,1}^{0} =$	$4\zeta - 1$
$\varphi_{0,0}^{2} =$	$\zeta^2 + 2\zeta\eta + 6\zeta\xi - 2\zeta + \eta^2 + 6\eta\xi - 2\eta + 6\xi^2 - 6\xi + 1$
$\varphi_{1,0}^{1} =$	$\zeta^2 + 6\zeta\eta + 2\zeta\xi - 2\zeta + 5\eta^2 + 10\eta\xi - 6\eta - 2\xi + 1$
$\varphi_{0,1}^{1} =$	$6\zeta^2 + 6\zeta\eta + 12\zeta\xi - 7\zeta - \eta - 2\xi + 1$
$\varphi_{2,0}^{0} =$	$\zeta^2 + 8\zeta\eta - 2\zeta + 10\eta^2 - 8\eta + 1$
$\varphi_{1,1}^{0} =$	$6\zeta^2 + 18\zeta\eta - 7\zeta - 3\eta + 1$
$\varphi_{0,2}^{0} =$	$15\zeta^2 - 10\zeta + 1$
$\varphi_{0,0}^{3} =$	$\zeta^3 + 3\zeta^2\eta + 12\zeta^2\xi - 3\zeta^2 + 3\zeta\eta^2 + 24\zeta\eta\xi - 6\zeta\eta + 30\zeta\xi^2 - 24\zeta\xi$ $+3\zeta + \eta^3 + 12\eta^2\xi - 3\eta^2 + 30\eta\xi^2 - 24\eta\xi + 3\eta + 20\xi^3 - 30\xi^2 + 12\xi - 1$
$\varphi_{1,0}^{2} =$	$\zeta^3 + 9\zeta^2\eta + 6\zeta^2\xi - 3\zeta^2 + 15\zeta\eta^2 + 48\zeta\eta\xi - 18\zeta\eta + 6\zeta\xi^2 - 12\zeta\xi + 3\zeta$ $+7\eta^3 + 42\eta^2\xi - 15\eta^2 + 42\eta\xi^2 - 48\eta\xi + 9\eta - 6\xi^2 + 6\xi - 1$
$\varphi_{0,1}^{2} =$	$8\zeta^3 + 16\zeta^2\eta + 48\zeta^2\xi - 17\zeta^2 + 8\zeta\eta^2 + 48\zeta\eta\xi - 18\zeta\eta + 48\zeta\xi^2 - 54\zeta\xi$ $+10\zeta - \eta^2 - 6\eta\xi + 2\eta - 6\xi^2 + 6\xi - 1$
$\varphi_{2,0}^{1} =$	$\zeta^3 + 13\zeta^2\eta + 2\zeta^2\xi - 3\zeta^2 + 33\zeta\eta^2 + 24\zeta\eta\xi - 26\zeta\eta - 4\zeta\xi + 3\zeta + 21\eta^3$ $+42\eta^2\xi - 33\eta^2 - 24\eta\xi + 13\eta + 2\xi - 1$
$\varphi_{1,1}^{1} =$	$8\zeta^3 + 48\zeta^2\eta + 16\zeta^2\xi - 17\zeta^2 + 40\zeta\eta^2 + 80\zeta\eta\xi - 54\zeta\eta - 18\zeta\xi$ $+10\zeta - 5\eta^2 - 10\eta\xi + 6\eta + 2\xi - 1$
$\varphi_{0,2}^{1} =$	$28\zeta^3 + 28\zeta^2\eta + 56\zeta^2\xi - 42\zeta^2 - 14\zeta\eta - 28\zeta\xi + 15\zeta + \eta + 2\xi - 1$
$\varphi_{3,0}^{0} =$	$\zeta^3 + 15\zeta^2\eta - 3\zeta^2 + 45\zeta\eta^2 - 30\zeta\eta + 3\zeta + 35\eta^3 - 45\eta^2 + 15\eta - 1$
$\varphi_{2,1}^{0} =$	$8\zeta^3 + 64\zeta^2\eta - 17\zeta^2 + 80\zeta\eta^2 - 72\zeta\eta + 10\zeta - 10\eta^2 + 8\eta - 1$
$\varphi_{1,2}^{0} =$	$28\zeta^3 + 84\zeta^2\eta - 42\zeta^2 - 42\zeta\eta + 15\zeta + 3\eta - 1$
$\varphi_{0,3}^{0} =$	$56\zeta^3 - 63\zeta^2 + 18\zeta - 1$

Table 2.2: Orthogonal basis functions for the space $\mathcal{P}_3$ on the reference tetrahedron.

The orthogonal functions $\varphi_{q,r}^{m}$ of degree ≤ 3 are presented in Table 2.2.

In our analysis we will also use the two-dimensional $(k+l)$-degree orthogonal polynomials [46] given by

$$\varphi_k^l(\xi,\eta) = 2^k P_k^{0,0}\left(\frac{2\xi}{1-\eta} - 1\right)(1-\eta)^k P_l^{2k+1,0}(2\eta - 1), \; k, l \geq 0, \tag{2.3.3}$$

where L_p and $P_p^{\delta,\rho}$, respectively, denote Legendre and Jacobi polynomials shifted to $[0,1]$.

The set of polynomials $\left\{\varphi_k^l, k, l \geq 0\right\}$ satisfy the $\mathcal{L}^2$ orthogonality on the reference triangle defined by the vertices $(0,0)$, $(1,0)$ and $(0,1)$

$$\int_0^1 \int_0^{1-\eta} \varphi_k^l \varphi_p^q d\xi d\eta = c_{kp}^{lq} \delta_{kp} \delta_{lq}. \tag{2.3.4}$$

2.4 Numerical Integration Rules

Numerical integration constitutes an important part of any finite element computation. The most popular methods for approximating integrals are Gauss quadrature, where weights and nodes are selected to obtain the highest degree of precision possible.

In our computations we use Gauss quadrature rules on the canonical triangle $\{(\xi, \eta), -1 \leq \xi \leq 1, -1 \leq \eta \leq -\xi\}$ of the form

$$\int_{-1}^{1} \int_{-1}^{-\xi} f(\xi, \eta)\, d\xi d\eta \approx \sum_{i=1}^{n} w_i f(\xi_i, \eta_i),$$

where n denotes the number of integration points. Gaussian integration points and weights on the canonical triangle are given in [42].

Quadrature on the reference tetrahedron $\{(\xi, \eta, \zeta), 0 \leq \xi \leq 1, 0 \leq \eta \leq 1-\xi, 0 \leq \zeta \leq 1-\xi-\eta\}$ has the form

$$\int_{0}^{1} \int_{0}^{1-\xi} \int_{0}^{1-\xi-\zeta} f(\xi, \eta, \zeta)\, d\zeta d\eta d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i, \eta_i, \zeta_i),$$

where n denotes the number of integration points. We use Keast quadrature formulas with degree of precision 8 for numerical integration over tetrahedra [47]. For the sake of completeness we present the weights and points in Table 2.3; for higher degrees of precision we use Grundmann-Moeller quadrature [43]. Matlab codes for these two quadratures can be found in [21].

ξ_i	η_i	ζ_i	w_i
0.250000000000000	0.250000000000000	0.250000000000000	-0.039327006641293
0.617587190300083	0.127470936566639	0.127470936566639	0.004081316059343
0.127470936566639	0.127470936566639	0.127470936566639	0.004081316059343
0.127470936566639	0.127470936566639	0.617587190300083	0.004081316059343
0.127470936566639	0.617587190300083	0.127470936566639	0.004081316059343
0.903763508822103	0.032078830392632	0.032078830392632	0.000658086773304
0.032078830392632	0.032078830392632	0.032078830392632	0.000658086773304
0.032078830392632	0.032078830392632	0.903763508822103	0.000658086773304
0.032078830392632	0.903763508822103	0.032078830392632	0.000658086773304
0.450222904356719	0.049777095643281	0.049777095643281	0.004384258825123
0.049777095643281	0.450222904356719	0.049777095643281	0.004384258825123
0.049777095643281	0.049777095643281	0.450222904356719	0.004384258825123
0.049777095643281	0.450222904356719	0.450222904356719	0.004384258825123
0.450222904356719	0.049777095643281	0.450222904356719	0.004384258825123
0.450222904356719	0.450222904356719	0.049777095643281	0.004384258825123
0.316269552601450	0.183730447398550	0.183730447398550	0.013830063842510
0.183730447398550	0.316269552601450	0.183730447398550	0.013830063842510
0.183730447398550	0.183730447398550	0.316269552601450	0.013830063842510
0.183730447398550	0.316269552601450	0.316269552601450	0.013830063842510
0.316269552601450	0.183730447398550	0.316269552601450	0.013830063842510
0.316269552601450	0.316269552601450	0.183730447398550	0.013830063842510
0.022917787844817	0.231901089397151	0.231901089397151	0.004240437424684
0.231901089397151	0.022917787844817	0.231901089397151	0.004240437424684
0.231901089397151	0.231901089397151	0.022917787844817	0.004240437424684
0.513280033360881	0.231901089397151	0.231901089397151	0.004240437424684
0.231901089397151	0.513280033360881	0.231901089397151	0.004240437424684
0.231901089397151	0.231901089397151	0.513280033360881	0.004240437424684
0.231901089397151	0.022917787844817	0.513280033360881	0.004240437424684
0.022917787844817	0.513280033360881	0.231901089397151	0.004240437424684
0.513280033360881	0.231901089397151	0.022917787844817	0.004240437424684
0.231901089397151	0.513280033360881	0.022917787844817	0.004240437424684
0.022917787844817	0.231901089397151	0.513280033360881	0.004240437424684
0.513280033360881	0.022917787844817	0.231901089397151	0.004240437424684
0.730313427807538	0.037970048471829	0.037970048471829	0.002238739739614
0.037970048471829	0.730313427807538	0.037970048471829	0.002238739739614
0.037970048471829	0.037970048471829	0.730313427807538	0.002238739739614
0.193746475248804	0.037970048471829	0.037970048471829	0.002238739739614
0.037970048471829	0.193746475248804	0.037970048471829	0.002238739739614
0.037970048471829	0.037970048471829	0.193746475248804	0.002238739739614
0.037970048471829	0.730313427807538	0.193746475248804	0.002238739739614
0.730313427807538	0.193746475248804	0.037970048471829	0.002238739739614
0.193746475248804	0.037970048471829	0.730313427807538	0.002238739739614
0.037970048471829	0.193746475248804	0.730313427807538	0.002238739739614
0.730313427807538	0.037970048471829	0.193746475248804	0.002238739739614
0.193746475248804	0.730313427807538	0.037970048471829	0.002238739739614

Table 2.3: A 45-point quadrature on a tetrahedron

Chapter 3

Discontinuous Galerkin Error Estimation for the Space $\mathcal{P}_p$

In this chapter, we extend the error analysis of Adjerid and Baccouch [2, 3, 4] to three-dimensional hyperbolic problems on general tetrahedral meshes. We perform a local error analysis on an arbitrary tetrahedron by constructing a family of similar tetrahedra with size h and having the same center. This family of tetrahedra is such that as $h \to 0$ the limit is the common center. Assuming we compute a *p-degree* DG approximation of a smooth solution, we expand the local error as a power series with respect to h and prove that the $O(h^{p+1})$ leading term of the DG error is a polynomial of degree *p+1*. We further observe that the leading term of the error satisfies a DG orthogonality condition which simplifies the form of leading term of the error. For instance, on a tetrahedron of *Class I*, the leading term may be written in terms of orthogonal polynomials of degrees p and *p+1* only. We further simplify the leading term and express it in terms of an optimal set of polynomials which will be used to estimate the error. Similarly, optimal error basis functions are derived on elements of *Class II* and *III.* Moreover, on the *outflow* face of an arbitrary element of *Class I* the local error is $\mathcal{O}(h^{2p+2})$ on average. Finally, we present an error estimation procedure to compute accurate DG error estimates on structured and unstructured tetrahedral meshes and solve several linear problems with both smooth and discontinuous solutions. All numerical examples show that the proposed error estimates are very accurate for smooth solutions.

3.1 Discontinuous Galerkin Formulation

In this section we consider linear first-order hyperbolic scalar problems on a bounded domain $\Omega \subseteq \mathbb{R}^3$. Let $\boldsymbol{a} = (a_1(x,y,z), a_2(x,y,z), a_3(x,y,z))^T$ and $\mathbf{n}$, respectively, denote a non zero velocity vector and the outward unit normal vector. The boundary of Ω can be witten as $\partial\Omega = \partial\Omega^- \cup \partial\Omega^+ \cup \partial\Omega^0$, where $\partial\Omega^- = \{(x,y,z) \in \partial\Omega |\, \boldsymbol{a} \cdot \mathbf{n} < 0\}$, $\partial\Omega^+ =$

$\{(x, y, z) \in \partial\Omega | \, \boldsymbol{a} \cdot \mathbf{n} > 0\}$ and $\partial\Omega^0 = \{(x, y, z) \in \partial\Omega | \, \boldsymbol{a} {\cdot} \mathbf{n} = 0\}$. The boundaries $\partial\Omega^-$, $\partial\Omega^+$, $\partial\Omega^0$, respectively, are called *inflow*, *outflow* and *characteristic*.

Let $u(x, y, z)$ be the solution of the hyperbolic problem

$$\boldsymbol{a} \cdot \nabla u + cu = f(x, y, z), \; (x, y, z) \in \Omega, \tag{3.1.1a}$$

$$\nabla \cdot \boldsymbol{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} = 0, \tag{3.1.1b}$$

$$u|_{\partial\Omega^-} = g(x, y, z), \tag{3.1.1c}$$

and assume that $\boldsymbol{a}(x, y, z)$, $c(x, y, z)$, $f(x, y, z)$ and $g(x, y, z)$ are selected such that the exact solution $u(x, y, z) \in C^\infty(\Omega)$.

In order to obtain the weak discontinuous Galerkin formulation, we partition the domain Ω into a regular mesh having N tetrahedral elements Δ_j, $j = 1, 2, \ldots, N$ and assume, for simplicity, that this can be done without error. Let us further assume that each face in the mesh is either, *inflow, outflow* or *characteristic.*

In the remainder of this chapter we omit the element index and refer to an arbitrary element by Δ whenever confusion in unlikely, with its boundary $\Gamma = \Gamma^+ \cup \Gamma^- \cup \Gamma^0$, where Γ^+, Γ^- and Γ^0, respectively, denote the *outflow*, *inflow* and *characteristic* boundaries.

Multiplying (3.1.1a) by a test function v, integrating over an arbitrary element Δ, and applying Stokes' theorem we write

$$\iint_{\Gamma^-} \boldsymbol{a} \cdot \mathbf{n} uv d\sigma + \iint_{\Gamma^+} \boldsymbol{a} \cdot \mathbf{n} uv d\sigma - \iiint_\Delta (\boldsymbol{a} \cdot \nabla v - cv)\, u dx dy dz = \iiint_\Delta f v dx dy dz. \tag{3.1.2}$$

Now, we approximate $u(x, y, z)$ by a piecewise polynomial function $U(x, y, z)$ whose restriction to Δ is in $\mathcal{P}_p$ consisting of complete polynomials of degree not exceeding p given by (2.3.1).

Next, we define the space $S^{N,p}$ consisting of piecewise polynomial functions

$$S^{N,p} = \{U, U|_\Delta \in \mathcal{P}_p\}, \tag{3.1.3}$$

and consider the discrete DG formulation which consists of determining $U \in S^{N,p}$ such that

$$\begin{aligned} & \iint_{\Gamma^-} \boldsymbol{a} \cdot \mathbf{n} \tilde{U} V d\sigma + \iint_{\Gamma^+} \boldsymbol{a} \cdot \mathbf{n} U V d\sigma - \iiint_\Delta (\boldsymbol{a} \cdot \nabla V - cV)\, U dx dy dz \\ = & \iiint_\Delta f V dx dy dz, \; \forall \; V \in \mathcal{P}_p. \end{aligned} \tag{3.1.4}$$

In order to complete the definition of our DG method we need to select the upwind numerical flux $\tilde{U}$ on Γ^- as

$$\tilde{U} = \begin{cases} u, & \text{if } \Gamma^- \subset \partial\Omega^- \\ U^-, & \textit{otherwise} \end{cases}, \tag{3.1.5a}$$

for the standard DG method and the corrected flux

$$\tilde{U} \;=\; \begin{cases} u, & \text{if } \Gamma^{-} \subset \partial\Omega^{-} \\ U^{-} + E^{-}, & \textit{otherwise} \end{cases}, \tag{3.1.5b}$$

for a modified DG method, where U^{-} is the limit from the *inflow* element sharing Γ^{-}, *i.e.*, if $(x, y, z) \in \Gamma^{-}$, then

$$U^{-}(x, y, z) = \lim_{s\to 0^{+}} U\left((x, y, z) + s\mathbf{n}\right),$$

and E is an *a posteriori* error estimate that will be defined in section 3.3.

Subtracting (3.1.4) from (3.1.2) with $v = V$ to obtain the DG orthogonality condition for the local error $\epsilon = u - U$ for all $V \in \mathcal{P}_p$,

$$\iint_{\Gamma^{-}} \boldsymbol{a}\cdot\mathbf{n}\epsilon^{-} V d\sigma + \iint_{\Gamma^{+}} \boldsymbol{a}\cdot\mathbf{n}\epsilon V d\sigma - \iiint_{\Delta} \left(\boldsymbol{a}\cdot\nabla V - cV\right)\epsilon dxdydz = 0. \tag{3.1.6}$$

We map a physical tetrahedron Δ having vertices $\mathbf{v}_i = (x_i, y_i, z_i)$, $1 \le i \le 4$, into the reference tetrahedron $\hat{\Delta}$ with vertices $\hat{\mathbf{v}}_1 = (0,0,0)$, $\hat{\mathbf{v}}_2 = (1,0,0)$, $\hat{\mathbf{v}}_3 = (0,1,0)$, $\hat{\mathbf{v}}_4 = (0,0,1)$, by the standard affine mapping illustrated in Figures 3.1 and 3.2. The inverse of this mapping $F^{-1} : \hat{\Delta} \longrightarrow \Delta$ is given by

$$\begin{aligned}\left(x(\xi,\eta,\zeta), y(\xi,\eta,\zeta), z(\xi,\eta,\zeta)\right)^{T} &= \frac{\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4}{4} + (\mathbf{v}_2 - \mathbf{v}_1)\left(\xi - \frac{1}{4}\right) \\ &+ (\mathbf{v}_4 - \mathbf{v}_1)\left(\eta - \frac{1}{4}\right) + (\mathbf{v}_3 - \mathbf{v}_1)\left(\zeta - \frac{1}{4}\right),\end{aligned} \tag{3.1.7}$$

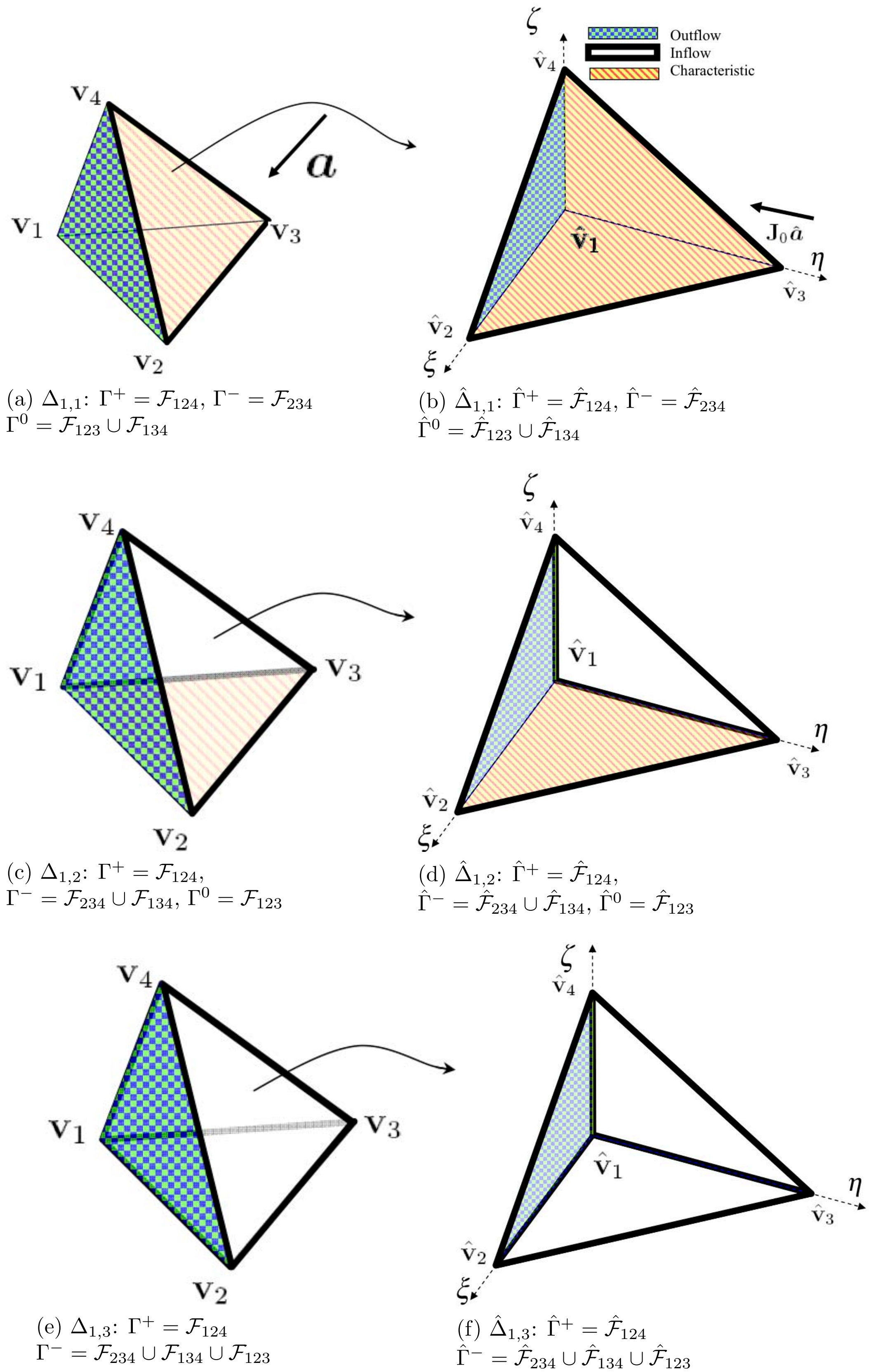
such that

$$F(\mathbf{v}_i) = \hat{\mathbf{v}}_i \text{ and } F(\mathcal{F}_{ijk}) = \hat{\mathcal{F}}_{ijk}, 1 \leqslant i, j, k \leqslant 4,$$

where $\mathcal{F}_{ijk} = \mathbf{v}_i\mathbf{v}_j\mathbf{v}_k$ and $\hat{\mathcal{F}}_{ijk} = \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j\hat{\mathbf{v}}_k$, respectively, denote the faces of Δ and $\hat{\Delta}$.

An element of *Class I* and *Type 1* is shown in Figure 3.1 (upper-left) with $\Gamma^{-} = \mathcal{F}_{234}$, $\Gamma^{+} = \mathcal{F}_{124}$ and $\Gamma^{0} = \mathcal{F}_{123} \cup \mathcal{F}_{134}$ while element of *Class I* and *Type 2* shown in Figure 3.1 (center-left) has $\Gamma^{-} = \mathcal{F}_{234} \cup \mathcal{F}_{134}$, $\Gamma^{+} = \mathcal{F}_{124}$ and $\Gamma^{0} = \mathcal{F}_{123}$. An element of *Class I* and *Type 3* is shown in Figure 3.1 (bottom-left) with $\Gamma^{-} = \mathcal{F}_{234} \cup \mathcal{F}_{123} \cup \mathcal{F}_{134}$ and $\Gamma^{+} = \mathcal{F}_{124}$. In Figure 3.2 (upper-left), we show an element of *Class II* and *Type 1* with $\Gamma^{-} = \mathcal{F}_{234}$, $\Gamma^{+} = \mathcal{F}_{124} \cup \mathcal{F}_{134}$ and $\Gamma^{0} = \mathcal{F}_{123}$ while the element of *Class II* and *Type 2* shown in Figure 3.2 (center-left) has $\Gamma^{-} = \mathcal{F}_{234} \cup \mathcal{F}_{123}$ and $\Gamma^{+} = \mathcal{F}_{124} \cup \mathcal{F}_{134}$. An element of *Class III* and *Type 1* is shown in Figure 3.2 (bottom-left) with $\Gamma^{-} = \mathcal{F}_{234}$ and $\Gamma^{+} = \mathcal{F}_{124} \cup \mathcal{F}_{123} \cup \mathcal{F}_{134}$.

In order to show the explicit dependence of the mapping on $h = diam\,(\Delta)$, we consider a family of similar tetrahedra parameterized by the diameter h such that as $h \to 0$ these



(a) $\Delta_{1,1}$: $\Gamma^+ = \mathcal{F}_{124}$, $\Gamma^- = \mathcal{F}_{234}$
$\Gamma^0 = \mathcal{F}_{123} \cup \mathcal{F}_{134}$

(b) $\hat{\Delta}_{1,1}$: $\hat{\Gamma}^+ = \hat{\mathcal{F}}_{124}$, $\hat{\Gamma}^- = \hat{\mathcal{F}}_{234}$
$\hat{\Gamma}^0 = \hat{\mathcal{F}}_{123} \cup \hat{\mathcal{F}}_{134}$

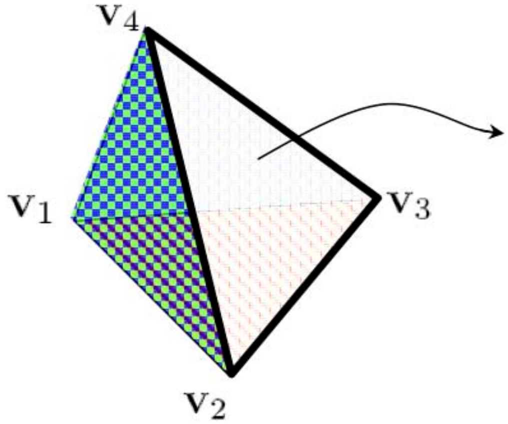
(c) $\Delta_{1,2}$: $\Gamma^+ = \mathcal{F}_{124}$,
$\Gamma^- = \mathcal{F}_{234} \cup \mathcal{F}_{134}$, $\Gamma^0 = \mathcal{F}_{123}$

(d) $\hat{\Delta}_{1,2}$: $\hat{\Gamma}^+ = \hat{\mathcal{F}}_{124}$,
$\hat{\Gamma}^- = \hat{\mathcal{F}}_{234} \cup \hat{\mathcal{F}}_{134}$, $\hat{\Gamma}^0 = \hat{\mathcal{F}}_{123}$

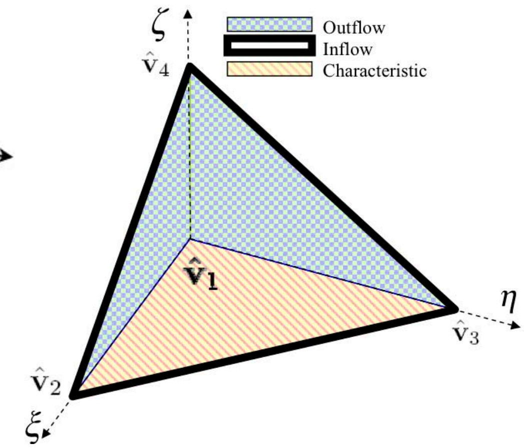
(e) $\Delta_{1,3}$: $\Gamma^+ = \mathcal{F}_{124}$
$\Gamma^- = \mathcal{F}_{234} \cup \mathcal{F}_{134} \cup \mathcal{F}_{123}$

(f) $\hat{\Delta}_{1,3}$: $\hat{\Gamma}^+ = \hat{\mathcal{F}}_{124}$
$\hat{\Gamma}^- = \hat{\mathcal{F}}_{234} \cup \hat{\mathcal{F}}_{134} \cup \hat{\mathcal{F}}_{123}$

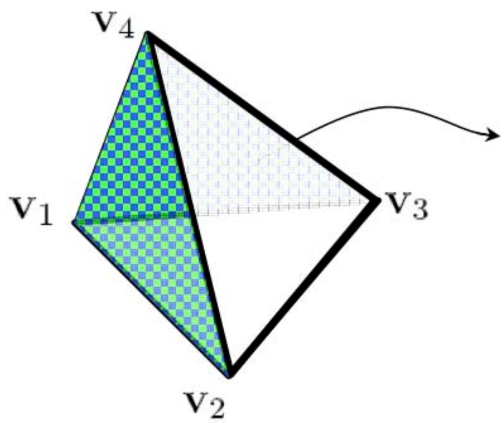
Figure 3.1: Mapping of physical elements of *Class I* (left) to corresponding reference elements (right).



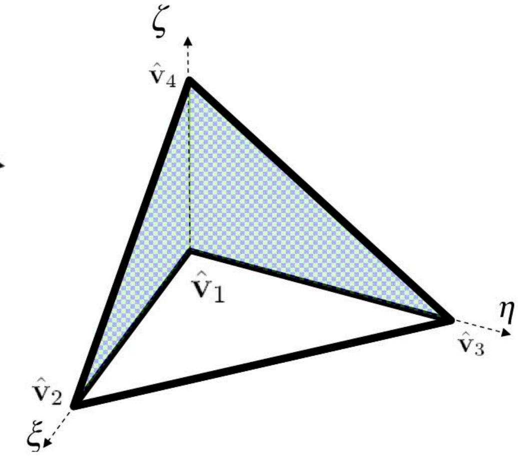
(a) $\Delta_{2,1}$: $\Gamma^+ = \mathcal{F}_{124} \cup \mathcal{F}_{134}$, $\Gamma^- = \mathcal{F}_{234}$, $\Gamma^0 = \mathcal{F}_{123}$



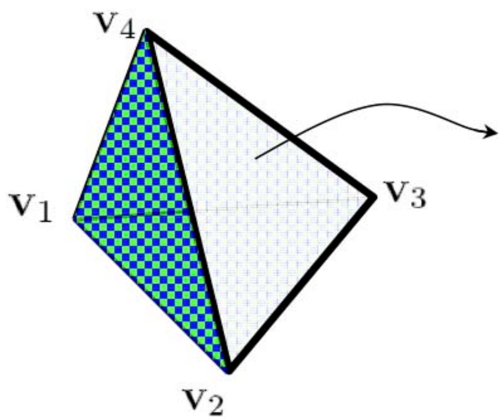
(b) $\hat{\Delta}_{2,1}$: $\hat{\Gamma}^+ = \hat{\mathcal{F}}_{124} \cup \hat{\mathcal{F}}_{134}$, $\hat{\Gamma}^- = \hat{\mathcal{F}}_{234}$, $\hat{\Gamma}^0 = \hat{\mathcal{F}}_{123}$



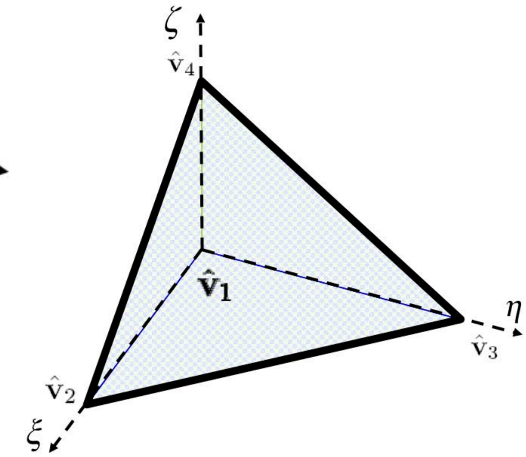
(c) $\Delta_{2,2}$: $\Gamma^+ = \mathcal{F}_{124} \cup \mathcal{F}_{134}$, $\Gamma^- = \mathcal{F}_{234} \cup \mathcal{F}_{123}$



(d) $\hat{\Delta}_{2,2}$: $\hat{\Gamma}^+ = \hat{\mathcal{F}}_{124} \cup \hat{\mathcal{F}}_{134}$, $\hat{\Gamma}^- = \hat{\mathcal{F}}_{234} \cup \hat{\mathcal{F}}_{124}$



(e) $\Delta_{3,1}$: $\Gamma^+ = \mathcal{F}_{124} \cup \mathcal{F}_{134} \cup \mathcal{F}_{123}$, $\Gamma^- = \mathcal{F}_{234}$



(f) $\hat{\Delta}_{3,1}$: $\hat{\Gamma}^+ = \hat{\mathcal{F}}_{124} \cup \hat{\mathcal{F}}_{134} \cup \hat{\mathcal{F}}_{123}$, $\hat{\Gamma}^- = \hat{\mathcal{F}}_{234}$

Figure 3.2: Mapping of physical elements of *Class II* and *III* (left) to corresponding reference elements (right).

tetrahedra converge to the center $\mathbf{v}_{1234} = \frac{\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_4+\mathbf{v}_3}{4}$. By the law of sines and the fact that the angles in these tetrahedra are the same as $h \to 0$ we have

$$\mathbf{v}_2 - \mathbf{v}_1 = h\mathbf{u}_1,\ \mathbf{v}_4 - \mathbf{v}_1 = h\mathbf{u}_2,\ \mathbf{v}_4 - \mathbf{v}_1 = h\mathbf{u}_3,$$

where the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are independent of h.

Letting $\mathbf{z} = \mathbf{u}_1\left(\xi - \frac{1}{4}\right) + \mathbf{u}_2\left(\eta - \frac{1}{4}\right) + \mathbf{u}_3\left(\zeta - \frac{1}{4}\right)$, the affine mapping (3.1.7) is written as

$$\left(x\left(\xi,\eta,\zeta,h\right), y\left(\xi,\eta,\zeta,h\right), z\left(\xi,\eta,\zeta,h\right)\right)^T \quad = \quad \mathbf{v}_{1234} + \mathbf{z}h,$$

and its inverse Jacobian matrix $\mathbf{J} = \frac{1}{h}\mathbf{J}_0$, where $\mathbf{J}_0$ is a 3×3 matrix independent of h.

By the affine mapping the DG orthogonality (3.1.6) becomes

$$\iint_{\hat{\Gamma}^-} (\mathbf{J}\hat{\boldsymbol{a}}) \cdot \hat{\mathbf{n}}\hat{\epsilon}^-\hat{V}d\sigma + \iint_{\hat{\Gamma}^+} (\mathbf{J}\hat{\boldsymbol{a}}) \cdot \hat{\mathbf{n}}\hat{\epsilon}\hat{V}d\sigma - \iiint_{\hat{\Delta}} \left((\mathbf{J}\hat{\boldsymbol{a}}) \cdot \nabla\hat{V} - \hat{c}\hat{V}\right)\hat{\epsilon}d\xi d\eta d\zeta = 0, \quad (3.1.8)$$

for all $\hat{V} \in \mathcal{P}_p$, where $\hat{\Gamma}^-$ and $\hat{\Gamma}^+$, respectively, denote the *inflow* ($\mathbf{J}\hat{\boldsymbol{a}} \cdot \hat{\mathbf{n}} < \mathbf{0}$) and *outflow* ($\mathbf{J}\hat{\boldsymbol{a}} \cdot \hat{\mathbf{n}} > \mathbf{0}$) boundaries of $\hat{\Delta}$ with respect to the vector $\mathbf{J}\hat{\boldsymbol{a}}$ and unit normal $\hat{\mathbf{n}}$ on $\hat{\Delta}$. Here we also have

$$\begin{aligned}
\hat{\boldsymbol{a}}(\xi,\eta,\zeta,h) &= \boldsymbol{a}(x(\xi,\eta,\zeta,h), y(\xi,\eta,\zeta,h), z(\xi,\eta,\zeta,h)),\\
\hat{c}(\xi,\eta,\zeta,h) &= c(x(\xi,\eta,\zeta,h), y(\xi,\eta,\zeta,h), z(\xi,\eta,\zeta,h)),\\
\hat{\epsilon}(\xi,\eta,\zeta,h) &= \epsilon(x(\xi,\eta,\zeta,h), y(\xi,\eta,\zeta,h), z(\xi,\eta,\zeta,h), h).
\end{aligned}$$

Note that $\hat{\epsilon}(\xi,\eta,\zeta,h)$ depends explicitly on h since U depends on x, y, z and h with all derivatives of U with respect to h exist at $h = 0$.

Therefore, the DG orthogonality condition (3.1.8) becomes

$$\iint_{\hat{\Gamma}^-} \tilde{\boldsymbol{a}} \cdot \hat{\mathbf{n}}\hat{\epsilon}^-\hat{V}d\sigma + \iint_{\hat{\Gamma}^+} \tilde{\boldsymbol{a}} \cdot \hat{\mathbf{n}}\hat{\epsilon}\hat{V}d\sigma - \iiint_{\hat{\Delta}} \left(\tilde{\boldsymbol{a}} \cdot \nabla\hat{V} - h\hat{c}\hat{V}\right)\hat{\epsilon}d\xi d\eta d\zeta \quad = \quad 0, \quad (3.1.9)$$

for all $\hat{V} \in \mathcal{P}_p$, where $\tilde{\boldsymbol{a}} = \mathbf{J}_0\hat{\boldsymbol{a}}$.

In the remainder of this chapter we will omit the ˆ unless needed for clarity. In our error analysis we need Taylor series of the analytic function $\mathbf{J}_0\boldsymbol{a}(x,y,z,h)$ about the center of the element Δ. Applying Taylor's theorem we expand $\mathbf{J}_0\boldsymbol{a}$ as

$$\tilde{\boldsymbol{a}}(\xi,\eta,\zeta,h) = \boldsymbol{a}_0 + \sum_{k=1}^{\infty} h^k\boldsymbol{a}_k(\xi,\eta,\zeta), \quad (3.1.10a)$$

where $\boldsymbol{a}_k \in [\mathcal{P}_k]^3$ are obtained by the chain rule as

$$\boldsymbol{a}_k(\xi,\eta,\zeta) \;=\; \frac{1}{k!}\mathbf{J}_0\left.\frac{d^k\boldsymbol{a}\left(x\left(\xi,\eta,\zeta,h\right),y\left(\xi,\eta,\zeta,h\right),z\left(\xi,\eta,\zeta,h\right)\right)}{dh^k}\right|_{h=0},$$

and

$$\boldsymbol{a}_0 = (\alpha,\beta,\gamma)^T = \mathbf{J}_0\boldsymbol{a}\left(1/4,1/4,1/4\right). \tag{3.1.10b}$$

Similarly, Maclaurin series for c and ϵ can be written as

$$c(\xi,\eta,\zeta,h) = \sum_{k=0}^{\infty} h^k C_k(\xi,\eta,\zeta),\; C_k(\xi,\eta,\zeta) \in \mathcal{P}_k, \tag{3.1.11}$$

and

$$\epsilon(\xi,\eta,\zeta,h) = \sum_{k=0}^{\infty} Q_k(\xi,\eta,\zeta)h^k, \quad Q_k(\xi,\eta,\zeta) \in \mathcal{P}_k. \tag{3.1.12}$$

In the next section we will investigate the local DG error on elements of *Class I, II* and *III*.

3.2 Local DG Error Analysis

For the sake of the local error analysis we solve a problem on one element of size h with $\tilde{U}|_{\Gamma^-} = u$, *i.e.*, $\epsilon^- = 0$ while in practice we only need $\epsilon^- = \mathcal{O}\left(h^{p+2}\right)$. We start by stating and proving the following preliminary result.

Lemma 3.2.1. *Let $\boldsymbol{a}_0 \neq 0$ in $\mathbb{R}^3$, Δ the reference tetrahedron with Γ^- ($\boldsymbol{a}_0\cdot\mathbf{n} < 0$) and Γ^+ ($\boldsymbol{a}_0\cdot\mathbf{n} > 0$), respectively, are the* inflow *and* outflow *boundaries and $\mathbf{n}$ be its unit normal vector. If $Q_k \in \mathcal{P}_k$, $k = 0,1,\ldots,p$ satisfies*

$$\iint_{\Gamma^+} \boldsymbol{a}_0\cdot\mathbf{n}Q_kV\,d\sigma + \iiint_{\Delta}\left(-\boldsymbol{a}_0\cdot\nabla V\right)Q_k d\xi d\eta d\zeta = 0,\;\forall\; V \in \mathcal{P}_p, \tag{3.2.1}$$

then

$$Q_k = 0,\; 0 \le k \le p. \tag{3.2.2}$$

Proof. Using Stokes' theorem we write (3.2.1) as

$$-\iint_{\Gamma^-} \boldsymbol{a}_0\cdot\mathbf{n}Q_kV\,d\sigma + \iiint_{\Delta} \boldsymbol{a}_0\cdot\nabla Q_kV\,d\xi d\eta d\zeta = 0,\;\forall\; V \in \mathcal{P}_p. \tag{3.2.3}$$

Adding (3.2.1) to (3.2.3) with $V = Q_k$ we obtain

$$-\iint_{\Gamma^-} \boldsymbol{a}_0\cdot\mathbf{n}Q_k^2 d\sigma + \iint_{\Gamma^+} \boldsymbol{a}_0\cdot\mathbf{n}Q_k^2 d\sigma = \iint_{\Gamma}\left|\boldsymbol{a}_0\cdot\mathbf{n}\right|Q_k^2 d\sigma = 0,\; 0 \le k \le p.$$

This leads to $Q_k = 0$ on $\Gamma^-\cup\Gamma^+$ which, when combined with (3.2.3) for $V = \boldsymbol{a}_0\cdot\nabla Q_k$, yields $\boldsymbol{a}_0\cdot\nabla Q_k = 0$ on Δ and completes the proof. $\square$

In the next theorem we state and prove several orthogonality conditions for the leading DG error term.

Theorem 3.2.1. *Let $u \in C^{\infty}(\Omega)$ and $U \in \mathcal{P}_p$, respectively, be the solutions of (3.1.1a), (3.1.1c) and (3.1.4) with $\tilde{U}|_{\Gamma^-} = u$. Then the local finite element error can be written as*

$$\epsilon(\xi,\eta,\zeta,h) = \sum_{k=p+1}^{\infty} h^k Q_k(\xi,\eta,\zeta), \tag{3.2.4}$$

where

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+1} d\sigma = 0, \ (if \ c \neq 0), \tag{3.2.5}$$

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_k d\sigma = 0, \ k \geq p+1, \ (if \ c = 0). \tag{3.2.6}$$

Furthermore, on the outflow *boundary the local error satisfies*

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} \epsilon d\sigma = O\left(h^{p+2}\right), \ (if \ c \neq 0), \tag{3.2.7}$$

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} \epsilon d\sigma = 0, \ (if \ c = 0). \tag{3.2.8}$$

Proof. Substituting (3.1.10), (3.1.11) and (3.1.12) in (3.1.9) and collecting terms having the same power of h we obtain the following series

$$\sum_{k=0}^{\infty} Z_k h^k = 0, \ \forall \ V \in \mathcal{P}_p, \tag{3.2.9}$$

where

$$Z_0 = \iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_0 V d\sigma - \iiint_{\triangle} \boldsymbol{a}_0 \cdot \nabla V Q_0 d\xi \, d\eta d\zeta, \qquad V \in \mathcal{P}_p,$$

and

$$Z_k = \iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_k V d\sigma - \iiint_{\triangle} [\boldsymbol{a}_0 \cdot \nabla V Q_k - C_0 Q_{k-1} V] d\xi \, d\eta d\zeta, \qquad V \in \mathcal{P}_p, \ k > 0.$$

Thus $Z_k = 0, \ \forall \ V \in \mathcal{P}_p$. By Lemma 3.2.1, $Q_k = 0, \ k = 0, 1, \ldots, p$ which proves (3.2.4). The leading term satisfies

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+1} V d\sigma - \iiint_{\Delta} (\boldsymbol{a}_0 \cdot \nabla V) Q_{p+1} d\xi d\eta d\zeta = 0, \forall \ V \in \mathcal{P}_p, \tag{3.2.10}$$

while for $k \geq p+2$

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_k V d\sigma - \iiint_{\Delta} ((\boldsymbol{a}_0 \cdot \nabla V) Q_k - C_0 V Q_{k-1}) d\xi d\eta d\zeta = 0, \forall \ V \in \mathcal{P}_p. \tag{3.2.11}$$

Testing against $V = 1$ in (3.2.10) yields (3.2.5), similarly, we prove (3.2.6) by testing against $V = 1$ in (3.2.11) with $c = 0$.

Multiplying (3.2.4) by $\boldsymbol{a}_0 \cdot \mathbf{n}$ and integrating over Γ^+ we obtain

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n}\epsilon d\sigma = \sum_{k=p+1}^{\infty} h^k \iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_k d\sigma,$$

which by (3.2.5) establishes (3.2.7). The orthogonality condition (3.2.8) follows from (3.2.6). □

In particular for elements of *Class I*, we state and prove the following theorem.

Theorem 3.2.2. *Let $u \in C^\infty(\Omega)$ and $U \in \mathcal{P}_p$, respectively, be the solutions of (3.1.1a), (3.1.1c) and (3.1.4), with $\tilde{U}\Big|_{\Gamma^-} = u$. Let Δ be a tetrahedron of Class I, then the leading term Q_{p+1} of the local finite element error satisfies*

$$\iiint_\Delta Q_{p+1} V d\xi d\eta d\zeta = 0, \; \forall \; V \in \mathcal{P}_{p-1}, \tag{3.2.12}$$

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+1} V d\sigma = 0, \; \forall \; V \in \mathcal{P}_p, \tag{3.2.13}$$

and for $p + 2 \le k \le 2p + 1$

$$\iiint_\Delta Q_k V d\xi d\eta d\zeta = 0, \; \forall \; V \in \mathcal{P}_{2p-k}, \tag{3.2.14}$$

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_k V d\sigma = 0, \; \forall \; V \in \mathcal{P}_{2p-k+1}. \tag{3.2.15}$$

Furthermore, on the outflow boundary the local error satisfies

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n}\epsilon d\sigma = O\left(h^{2p+2}\right). \tag{3.2.16}$$

Proof. On an element of *Class I* with one *outflow* face at $\eta = 0$, (3.2.10) becomes for all $V \in \mathcal{P}_p$

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+1}(\xi, 0, \zeta) V(\xi, 0, \zeta) d\xi d\zeta + \iiint_\Delta (-\boldsymbol{a}_0 \cdot \nabla V) Q_{p+1} d\xi d\eta d\zeta = 0. \tag{3.2.17}$$

We now consider the set Π_q of all monomials of degree q and establish (3.2.12) and (3.2.13) by proving

$$\iiint_\Delta Q_{p+1} V d\xi d\eta d\zeta = 0, \; \forall \; V \in \Pi_{q-1}, \; 1 \le q \le p, \tag{3.2.18}$$

and

$$\iint_{\Gamma^+} Q_{p+1}(\xi,0,\zeta)\, V(\xi,0,\zeta)\, d\xi d\zeta = 0,\ \forall\ V \in \Pi_q,\ 0 \le q \le p. \tag{3.2.19}$$

Setting $V = 1$ in (3.2.17) yields

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+1}(\xi,0,\zeta)\, d\xi d\zeta = 0, \tag{3.2.20}$$

thus, satisfying (3.2.19) for $q = 0$.

Testing against $V = \xi^{q-i}\eta^{i-j}\zeta^j$, $1 \le i \le q$, $q > 0$ and $0 \le j \le i-1$ and substituting the resulting equation into (3.2.17) to obtain

$$\iiint_{\Delta} Q_{p+1}(\xi,\eta,\zeta)\, \boldsymbol{a}_0 \cdot \nabla\left(\xi^{q-i}\eta^{i-j}\zeta^j\right) d\xi d\eta d\zeta = 0,\ 1 \le i \le q \text{ and } 0 \le j \le i-1, \tag{3.2.21}$$

where the face integral in (3.2.17) has vanished ($V(\xi,0,\zeta) = 0$). The condition (3.2.21) for $V = \xi^{q-i}\eta^{i-j}\zeta^j$, $1 \le i \le q$ and $0 \le j \le i-1$, is equivalent to (3.2.18) for all $V \in \Pi_{q-1}$. Thus, we have established (3.2.18).

Testing against $V = \xi^{q-i}\zeta^j$, is redundant in the sense that $\boldsymbol{a}_0 \cdot \nabla\left(\xi^{q-i}\eta^{i-j}\zeta^j\right)$, $1 \le i \le q$ and $0 \le j \le i-1$, form a basis for Π_{q-1}, and $\boldsymbol{a}_0 \cdot \nabla\left(\xi^{q-i}\zeta^j\right)$ can be expressed in terms of this basis. Thus, (3.2.21) is also satisfied when $j = i$. Using (3.2.18) in (3.2.17) yields (3.2.19). Combining results for $q = 0, 1, \ldots, p$ proves (3.2.12) and (3.2.13).

For $k = p+2$, (3.2.11) yields

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+2} V d\sigma - \iiint_{\Delta} \left(\left(\boldsymbol{a}_0 \cdot \nabla V\right) Q_{p+2} - C_0 V Q_{p+1}\right) d\xi d\eta d\zeta = 0,\ \forall\ V \in \mathcal{P}_p. \tag{3.2.22}$$

Using the orthogonality conditions (3.2.12) we obtain

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+2} V d\sigma - \iiint_{\Delta} \left(\boldsymbol{a}_0 \cdot \nabla V\right) Q_{p+2} d\xi d\eta d\zeta = 0,\ \forall\ V \in \mathcal{P}_{p-1}. \tag{3.2.23}$$

By induction we can show that for $p+2 \le k \le 2p+1$

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_k V d\sigma - \iiint_{\Delta} \left(\boldsymbol{a}_0 \cdot \nabla V\right) Q_k d\xi d\eta d\zeta = 0,\ \forall\ V \in \mathcal{P}_{2p-k+1}. \tag{3.2.24}$$

Using (3.2.24) and the same argument as in (3.2.12) and (3.2.13) we establish (3.2.14) and (3.2.15).

Finally, testing against $V = 1$ in (3.2.15) we obtain

$$\sum_{k=p+1}^{\infty} h^k \iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_k(\xi,\eta,\zeta)\, d\sigma = O\left(h^{2p+2}\right),$$

which completes the proof. $\square$

A direct consequence of Theorem 3.2.2 is the following corollary.

Corollary 3.2.1. *Under the assumptions of Theorem 3.2.2 the leading term Q_{p+1} of the local finite element error can be written as*

$$Q_{p+1}(\xi,\eta,\zeta) = \sum_{i=0}^{p}\sum_{j=0}^{i} c_{i,j}^{p}\varphi_{i-j,j}^{p-i} + \sum_{i=0}^{p+1}\sum_{j=0}^{i} c_{i,j}^{p+1}\varphi_{i-j,j}^{p+1-i}, \tag{3.2.25a}$$

$$Q_{p+1}(\xi,0,\zeta) = \sum_{i=0}^{p+1} C_i^{p+1}\varphi_{p+1-i}^{i}(\xi,\zeta), \tag{3.2.25b}$$

$$Q_{p+1}(\xi,\eta,\zeta) = \sum_{i=0}^{p+1} C_i^{p+1}\varphi_{p+1-i}^{i}(\xi,\zeta) + \eta\sum_{l=0}^{p}\sum_{i=0}^{l}\sum_{j=0}^{i} C_{i,j}^{l}\varphi_{i-j,j}^{l-i}(\xi,\eta,\zeta). \tag{3.2.25c}$$

Furthermore, for $p+2 \le k \le 2p+1$

$$Q_k(\xi,\eta,\zeta) = \sum_{m=2p-k+1}^{k} \psi^m(\xi,\eta,\zeta), \tag{3.2.26a}$$

$$Q_k(\xi,0,\zeta) = \sum_{m=2p-k+2}^{k}\sum_{i=0}^{m} C_i^{m}\varphi_{m-i}^{i}(\xi,\zeta), \tag{3.2.26b}$$

$$Q_k(\xi,\eta,\zeta) = Q_k(\xi,0,\zeta) + \eta\sum_{m=2p-k+1}^{k-1}\Psi^m(\xi,\eta,\zeta), \tag{3.2.26c}$$

where ψ^m and Ψ^m are in the span of $\{\varphi_{i-j,j}^{m-i},\ i=0,1,\ldots,m,\ j=0,1,\ldots,i\}$.

Proof. Since $\left\{\varphi_{ij}^{m}\right\}$ are orthogonal, (3.2.12) yields (3.2.25a). Splitting the leading term Q_{p+1} as

$$Q_{p+1}(\xi,\eta,\zeta) = Q_{p+1}(\xi,0,\zeta) + \eta q_p(\xi,\eta,\zeta),\ q_p \in \mathcal{P}_p, \tag{3.2.27}$$

and applying (3.2.13) leads to (3.2.25b). Equation (3.2.25c) follows directly from (3.2.27) and (3.2.25b).

Following the same line of reasoning, we establish (3.2.26a), (3.2.26b) and (3.2.26c) from the orthogonality conditions (3.2.14) and (3.2.15). □

In the remainder of this section we state and prove several orthogonality conditions on elements of *Class II* and *III*. Let $\Gamma_1,\ \Gamma_2,\ \Gamma_3$ and Γ_4, respectively, denote the faces $\eta = 0,\ \zeta = 0,\ \xi = 0$ and $1-\xi-\eta-\zeta = 0$ of the reference element Δ shown in Table 3.1.

In the following theorem we state several orthogonality results on Δ of *Class II.*

	Type 1			*Type 2*		*Type 3*
	Class I	*Class II*	*Class III*	*Class I*	*Class II*	*Class I*
	$\Delta_{1,1}$	$\Delta_{2,1}$	$\Delta_{3,1}$	$\Delta_{1,2}$	$\Delta_{2,2}$	$\Delta_{1,3}$
Inflow	Γ_4	Γ_4	Γ_4	Γ_4, Γ_2	Γ_4, Γ_2	$\Gamma_4, \Gamma_2, \Gamma_3$
Outflow	Γ_1	Γ_1, Γ_3	$\Gamma_1, \Gamma_2, \Gamma_3$	Γ_1	Γ_1, Γ_3	Γ_1
Characteristic	Γ_2, Γ_3	Γ_2	–	Γ_3	–	–

Table 3.1: Reference element for each class and type.

Theorem 3.2.3. *Let $u \in C^\infty(\Omega)$ and $U \in \mathcal{P}_p$, respectively, be the solutions of (3.1.1a), (3.1.1c) and (3.1.4), with $\tilde{U}\Big|_{\Gamma^-} = u$. Let $\boldsymbol{a}_0 = (\alpha, \beta, \gamma)^T$ be such that Δ is a tetrahedron of Class II with $\Gamma^+ = \Gamma_1 \cup \Gamma_3$. Then the leading term Q_{p+1} of the local finite element error satisfies the following orthogonality conditions for all $V \in \{\xi^i \eta^j\} \otimes \mathcal{P}_{p-(i+j)}$ with $i, j = 0, 1$,*

$$(1-j)\iint_{\Gamma_1} \beta Q_{p+1} V \, d\sigma + (1-i)\iint_{\Gamma_3} \alpha Q_{p+1} V \, d\sigma + \iiint_{\Delta} Q_{p+1}\left(\boldsymbol{a}_0 \cdot \nabla V\right) d\xi d\eta d\zeta = 0, \tag{3.2.28}$$

$$\iiint_{\Delta} \left(\alpha\eta + \beta\xi\right) Q_{p+1} d\xi d\eta d\zeta = 0, \tag{3.2.29}$$

and

$$\iint_{\Gamma_3} \alpha\eta Q_{p+1} d\sigma + \beta \iiint_{\Delta} Q_{p+1} d\xi d\eta d\zeta = 0, \tag{3.2.30}$$

$$\iint_{\Gamma_1} \beta\xi Q_{p+1} d\sigma + \alpha \iiint_{\Delta} Q_{p+1} d\xi d\eta d\zeta = 0. \tag{3.2.31}$$

Furthermore, for $1 \le k \le p$ we have

$$\iint_{\Gamma_3} \alpha^{k+1}\eta^k Q_{p+1} V \, d\sigma + (-1)^k \iint_{\Gamma_1} \beta^{k+1}\xi^k Q_{p+1} V \, d\sigma + \iiint_{\Delta} \left(\alpha\eta - \beta\xi\right)^k Q_{p+1}\left(\boldsymbol{a}_0 \cdot \nabla V\right) d\xi d\eta d\zeta = 0, \ \forall\ V \in \mathcal{P}_{p-k}\ , \tag{3.2.32}$$

and

$$\iint_{\Gamma_3} \alpha^{k+1}\eta^k Q_{p+1} d\sigma + (-1)^k \iint_{\Gamma_1} \beta^{k+1}\xi^k Q_{p+1} d\sigma = 0. \tag{3.2.33}$$

Finally, for elements of Class II *and* Type 1, *(*i.e. $\gamma = 0$*) we have*

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+1} \zeta^k d\sigma = 0, \ k = 0, 1, \ldots, p. \tag{3.2.34}$$

Proof. Testing against $\xi^i\eta^j V$ in (3.2.10) and noting that $\xi^i\eta^j V|_{\Gamma_1} = 0$ for $j = 1$ and $\xi^i\eta^j V|_{\Gamma_3} = 0$ for $i = 1$ yields (3.2.28), which, in turn, since $\nabla(\xi\eta V) = V\nabla(\xi\eta) + \xi\eta\nabla V$, yields

$$\iiint_\Delta (\alpha\eta + \beta\xi) Q_{p+1} V d\xi d\eta d\zeta + \iiint_\Delta \xi\eta Q_{p+1} (\boldsymbol{a}_0 \cdot \nabla V) d\xi d\eta d\zeta = 0, \forall V \in \mathcal{P}_{p-2} , \tag{3.2.35}$$

which for $V = 1$ proves (3.2.29).
Testing against ξV (with $(i,j) = (0,1)$) and ηV (with $(i,j) = (1,0)$), (3.2.28) gives

$$\begin{aligned} &\iint_{\Gamma_3} \alpha\eta Q_{p+1}(0,\eta,\zeta) V d\sigma + \beta \iiint_\Delta Q_{p+1} V d\xi d\eta d\zeta \\ &+ \iiint_\Delta \eta Q_{p+1} (\boldsymbol{a}_0 \cdot \nabla V) d\xi d\eta d\zeta = 0, \forall V \in \mathcal{P}_{p-1} , \end{aligned} \tag{3.2.36}$$

and

$$\begin{aligned} &\iint_{\Gamma_1} \beta\xi Q_{p+1}(\xi,0,\zeta) V d\sigma + \alpha \iiint_\Delta Q_{p+1} V d\xi d\eta d\zeta \\ &+ \iiint_\Delta \xi Q_{p+1} (\boldsymbol{a}_0 \cdot \nabla V) d\xi d\eta d\zeta = 0, \forall V \in \mathcal{P}_{p-1} . \end{aligned} \tag{3.2.37}$$

Testing against $V = 1$ in (3.2.36) and (3.2.37), respectively, yields (3.2.30) and (3.2.31).

Next, multiplying (3.2.36) by α and (3.2.37) by β and subtracting the resulting equations, we obtain (3.2.32) for $k = 1$. Now, using (3.2.32) (with $k = 1$) and taking as test functions ξV, ηV for $V \in \mathcal{P}_{p-2}$ to find

$$\begin{aligned} &-\iint_{\Gamma_1} \beta^2\xi^2 Q_{p+1}(\xi,0,\zeta) V d\xi d\zeta + \iiint_\Delta \alpha(\alpha\eta - \beta\xi) Q_{p+1} V d\xi d\eta d\zeta \\ &+ \iiint_\Delta (\alpha\eta - \beta\xi)\xi Q_{p+1} (\boldsymbol{a}_0 \cdot \nabla V) d\xi d\eta d\zeta = 0, \forall V \in \mathcal{P}_{p-2} , \end{aligned} \tag{3.2.38}$$

and

$$\begin{aligned} &\iint_{\Gamma_3} \alpha^2\eta^2 Q_{p+1}(0,\eta,\zeta) V d\eta d\zeta + \iiint_\Delta \beta(\alpha\eta - \beta\xi) Q_{p+1} V d\xi d\eta d\zeta \\ &+ \iiint_\Delta (\alpha\eta - \beta\xi)\eta Q_{p+1} (\boldsymbol{a}_0 \cdot \nabla V) d\xi d\eta d\zeta = 0, \forall V \in \mathcal{P}_{p-1}. \end{aligned} \tag{3.2.39}$$

Multiplying (3.2.38) by β and (3.2.39) by α and subtracting the resulting equations, we obtain

$$\begin{aligned} &\iint_{\Gamma_1} \beta^3\xi^2 Q_{p+1}(\xi,0,\zeta) V d\xi d\zeta + \iint_{\Gamma_3} \alpha^3\eta^2 Q_{p+1}(0,\eta,\zeta) V d\eta d\zeta \\ &+ \iiint_\Delta (\alpha\eta - \beta\xi)^2 Q_{p+1} (\boldsymbol{a}_0 \cdot \nabla V) d\xi d\eta d\zeta = 0, \forall V \in \mathcal{P}_{p-2} . \end{aligned}$$

Repeating the same argument k times for $2 \leq k \leq p$, we establish (3.2.32). Testing against $V = 1$ in (3.2.32) yields (3.2.33). Finally, testing against ζ^k in (3.2.10), for $k = 0, 1, \ldots, p$ and noting that $\boldsymbol{a}_0 \cdot \nabla(\zeta^k) = 0$, we obtain (3.2.34) which completes the proof. $\square$

In the next theorem we state and prove new orthogonality conditions for elements of *Class III*.

Theorem 3.2.4. *Let $u \in C^\infty(\Omega)$ and $U \in \mathcal{P}_p$, respectively, be the solutions of (3.1.1a), (3.1.1c) and (3.1.4), with $U^-|_{\Gamma^-} = u$. Let $\boldsymbol{a}_0 = (\alpha, \beta, \gamma)^T$ be such that Δ is a tetrahedron of Class III, with $\Gamma^+ = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Then the leading term Q_{p+1} of the local finite element error satisfies the following orthogonality conditions for all $V \in \left\{\xi^i \eta^j \zeta^l\right\} \otimes \mathcal{P}_{p-(i+j+l)}$ with $i, j, l = 0, 1$*

$$(1-j) \iint_{\Gamma_1} \beta Q_{p+1} V d\sigma + (1-i) \iint_{\Gamma_3} \alpha Q_{p+1} V d\sigma + (1-l) \iint_{\Gamma_2} \gamma Q_{p+1} V d\sigma$$
$$+ \iiint_\Delta Q_{p+1} \left(\boldsymbol{a}_0 \cdot \nabla V\right) d\xi d\eta d\zeta = 0, \tag{3.2.40}$$
$$\iiint_\Delta \left(\alpha\eta\zeta + \beta\xi\zeta + \gamma\xi\eta\right) Q_{p+1} d\xi d\eta d\zeta = 0 \ , \tag{3.2.41}$$

and

$$(1-j) \iint_{\Gamma_1} \beta \left(\xi^i \zeta^l\right)^k Q_{p+1} V d\sigma + (1-i) \iint_{\Gamma_3} \alpha \left(\eta^j \zeta^l\right)^k Q_{p+1} V d\sigma$$
$$+ (1-l) \iint_{\Gamma_2} \gamma \left(\xi^i \eta^j\right)^k Q_{p+1} V d\sigma + \iiint_\Delta \left(\boldsymbol{a}_0 \cdot \nabla \left(\zeta^l \eta^j \xi^i\right)^k\right) Q_{p+1} V d\xi d\eta d\zeta$$
$$+ \iiint_\Delta \left(\xi^i \eta^j \zeta^l\right)^k Q_{p+1} \left(\boldsymbol{a}_0 \cdot \nabla V\right) d\xi d\eta d\zeta = 0, \tag{3.2.42}$$

for all $V \in \mathcal{P}_{p-nk}$ with $1 \leq n \leq nk \leq p$, $n = (i + j + l)$ and $i, j, l = 0, 1$.
Furthermore, for $1 \leq k \leq p$ and $V \in \mathcal{P}_{p-k}$ we have

$$\iint_{\Gamma_1} \beta^2 \xi \left(\beta\xi - \alpha\eta\right)^{k-1} Q_{p+1} V d\sigma - \iint_{\Gamma_3} \alpha^2 \eta \left(\beta\xi - \alpha\eta\right)^{k-1} Q_{p+1} V d\sigma \tag{3.2.43}$$
$$+ \iint_{\Gamma_2} \gamma \left(\alpha\eta - \beta\xi\right)^k Q_{p+1} V d\sigma + \iiint_\Delta \left(\alpha\eta - \beta\xi\right)^k Q_{p+1} \left(\boldsymbol{a}_0 \cdot \nabla V\right) d\xi d\eta d\zeta = 0.$$

Finally, on the outflow *boundary we have*

$$\iint_{\Gamma_1} \beta^{k+1} \xi^k Q_{p+1} d\sigma + (-1)^k \iint_{\Gamma_3} \alpha^{k+1} \eta^k Q_{p+1} d\sigma$$
$$+ \iint_{\Gamma_2} \gamma \left(\alpha\eta - \beta\xi\right)^k Q_{p+1} V d\sigma = 0 \text{ for } 1 \leq k \leq p. \tag{3.2.44}$$

Proof. Testing against $\xi^i\eta^j\zeta^l V$ in (3.2.10), for $V \in \mathcal{P}_{p-(i+j+k)}$, yields (3.2.40).
From (3.2.40) with $i = j = l = 1$ and $V \in \mathcal{P}_{p-3}$ we have

$$\iiint_\Delta (\alpha\eta\zeta + \beta\xi\zeta + \gamma\xi\eta)\, Q_{p+1} V d\xi d\eta d\zeta + \iiint_\Delta \xi\eta\zeta Q_{p+1} \left(\boldsymbol{a}_0 \cdot \nabla V\right) d\xi d\eta d\zeta = 0, \quad (3.2.45)$$

which, for $V = 1$, yields (3.2.41).
Again, testing against $\left(\xi^i\eta^j\zeta^l\right)^k V$, $V \in \mathcal{P}_{p-nk}$ with $k \geq 1$ and $n \leq nk \leq p$, (3.2.40) yields (3.2.42).
Next, testing in (3.2.40) against ξV ($(i,j,l) = (1,0,0)$) and ηV ($(i,j,l) = (0,1,0)$) we obtain

$$\begin{aligned}&\iint_{\Gamma_1} \beta\xi Q_{p+1} V d\sigma + \iint_{\Gamma_2} \gamma\xi Q_{p+1} V d\sigma + \alpha \iiint_\Delta Q_{p+1} V d\xi d\eta d\zeta \\ &+ \iiint_\Delta \xi Q_{p+1} \left(\boldsymbol{a}_0 \cdot \nabla V\right) d\xi d\eta d\zeta \, = 0, \, \forall \, V \in \mathcal{P}_{p-1}, \end{aligned} \quad (3.2.46)$$

$$\begin{aligned}&\iint_{\Gamma_3} \alpha\eta Q_{p+1} V d\sigma + \iint_{\Gamma_2} \gamma\eta Q_{p+1} V d\sigma + \beta \iiint_\Delta Q_{p+1} V d\xi d\eta d\zeta \\ &+ \iiint_\Delta \eta Q_{p+1} \left(\boldsymbol{a}_0 \cdot \nabla V\right) d\xi d\eta d\zeta \, = 0, \, \forall \, V \mathcal{P}_{p-1}. \end{aligned} \quad (3.2.47)$$

Multiplying (3.2.46) by β and (3.2.47) by α, subtracting the resulting equations to obtain (3.2.43) for $k = 1$. Now, use (3.2.43) (with $k = 1$) and testing against ξV, ηV, $V \in \mathcal{P}_{p-2}$ we find

$$\begin{aligned}&\iint_{\Gamma_1} \beta^2\xi^2 Q_{p+1} V d\sigma - \iint_{\Gamma_3} \alpha^2\eta\xi Q_{p+1} V d\sigma + \iint_{\Gamma_2} \gamma\xi \left(\beta\xi - \alpha\eta\right) Q_{p+1} V d\sigma \\ &+ \iiint_\Delta \alpha \left(\beta\xi - \alpha\eta\right) Q_{p+1} V d\xi d\eta d\zeta + \iiint_\Delta \left(\beta\xi - \alpha\eta\right) \xi Q_{p+1} \left(\boldsymbol{a}_0 \cdot \nabla V\right) d\xi d\eta d\zeta \, = 0, \end{aligned} \quad (3.2.48)$$

$$\begin{aligned}&\iint_{\Gamma_1} \beta^2\xi\eta Q_{p+1} V d\sigma - \iint_{\Gamma_3} \alpha^2\eta^2 Q_{p+1} V d\sigma + \iint_{\Gamma_2} \gamma\eta \left(\beta\xi - \alpha\eta\right) Q_{p+1} V d\sigma \\ &+ \iiint_\Delta \beta \left(\beta\xi - \alpha\eta\right) Q_{p+1} V d\xi d\eta d\zeta + \iiint_\Delta \eta \left(\beta\xi - \alpha\eta\right) Q_{p+1} \left(\boldsymbol{a}_0 \cdot \nabla V\right) d\xi d\eta d\zeta \, = 0. \end{aligned} \quad (3.2.49)$$

Multiplying (3.2.48) by β and (3.2.49) by α and subtracting the resulting equations, we obtain

$$\begin{aligned}&\iint_{\Gamma_1} \beta^3\xi^2 Q_{p+1}\left(\xi, 0, \zeta\right) V d\xi d\zeta + \iint_{\Gamma_3} \alpha^3\eta^2 Q_{p+1}\left(0, \eta, \zeta\right) V d\eta d\zeta \\ &\iiint_\Delta \left(\alpha\eta - \beta\xi\right)^2 Q_{p+1} \left(\boldsymbol{a}_0 \cdot \nabla V\right) d\xi d\eta d\zeta \, = 0, \, \forall \, V \in \mathcal{P}_{p-2} \ . \end{aligned}$$

Repeating the same argument k times for $3 \leq k \leq p$, we establish (3.2.43). Finally, testing against $V = 1$ in (3.2.43) yields (3.2.32) and completes the proof. $\square$

3.3 *A Posteriori* Error Estimation

In this section we present an error estimation procedure by first constructing bases functions for the leading term of the DG error and stating a weak problem on each element to compute error estimates. In order to construct efficient and asymptotically exact *a posteriori* error estimates for the leading term Q_{p+1} on each element, we assume that the global error has the same behavior as the local error holds on all elements, *i.e.*, we write

$$(u-U)(x,y,z,h) \approx E(\xi,\eta,\zeta,h) = Q_{p+1}(\xi,\eta,\zeta)h^{p+1} = \sum_{i=0}^{p+1}\sum_{j=0}^{i}\sum_{k=0}^{j} c_{j,k}^{i}\varphi_{j-k,k}^{i-j}(\xi,\eta,\zeta),$$

where $Q_{p+1} \in \mathcal{P}_{p+1}$ and E satisfies the following orthogonality conditions on the reference element

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} E V d\sigma - \iiint_{\Delta} (\boldsymbol{a}_0 \cdot \nabla V)\, E d\xi d\eta d\zeta = 0,\ \forall\ V \in \mathcal{P}_p. \tag{3.3.1}$$

Testing against $V = \varphi_{r-s,s}^{q-r}$, $0 \le s \le r \le q \le p$, yields

$$\sum_{i,j,k} c_{j,k}^{i} \left[\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} \varphi_{j-k,k}^{i-j}\varphi_{r-s,s}^{q-r} d\sigma - \iiint_{\Delta} \left(\boldsymbol{a}_0 \cdot \nabla\varphi_{r-s,s}^{q-r}\right) \varphi_{j-k,k}^{i-j} d\xi d\eta d\zeta\right] = 0. \tag{3.3.2}$$

Let $m = \dim \mathcal{P}_p = (p+1)(p+2)(p+3)/6$, $n = \dim \mathcal{P}_{p+1} = (p+2)(p+3)(p+4)/6$ such that $n - m = (p+2)(p+3)/2$. Thus, if

$$\boldsymbol{C} = (\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{p+1})^T \in \mathbb{R}^n,$$

and

$$\boldsymbol{\Phi} = (\Phi_1, \Phi_2, \ldots, \Phi_n)^T = \left(\boldsymbol{\phi}_0, \boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{p+1}\right)^T,$$

where

$$\mathbf{c}_l = (c_{0,0}^l, c_{1,0}^l, c_{1,1}^l, \ldots, c_{l,0}^l, c_{l,1}^l, \ldots, c_{l,l}^l),$$

and

$$\boldsymbol{\phi}_l = (\varphi_{0,0}^l, \varphi_{1,0}^{l-1}, \varphi_{0,1}^{l-1}, \ldots, \varphi_{l,0}^0, \ldots, \varphi_{l-1,1}^0, \ldots, \varphi_{l-j,j}^0, \ldots, \varphi_{0,l}^0),$$

the orthogonality conditions (3.3.2) may be written in a matrix form $\boldsymbol{AC} = 0$ where

$$a_{ij} = \iint_{\Gamma}^{+} \boldsymbol{a}_0 \cdot \mathbf{n}\Phi_j\Phi_i d\sigma - \iiint_{\Delta} (\boldsymbol{a}_0 \cdot \nabla\Phi_j)\, \Phi_i d\xi d\eta d\zeta,\ j = 1, 2, \ldots, n,\ i = 1, 2, \ldots, m. \tag{3.3.3}$$

A direct computation reveals that the dimension of $\mathcal{N}(\boldsymbol{A})$, the null space of $\boldsymbol{A}$, is $(p+2)(p+3)/2$, $p = 0, 1, 2, 3$ on all elements.
Without loss of generality we assume $\boldsymbol{a}_0 = (\alpha, \beta, \gamma)^T$ such that $\beta \neq 0$ and let $\lambda = \frac{\alpha}{\beta}$, $\mu = \frac{\gamma}{\beta}$. We now define the finite element space

$$\mathcal{E} = \left\{\boldsymbol{C}^T\boldsymbol{\Phi},\ \boldsymbol{C} \in \mathcal{N}(\boldsymbol{A})\right\},$$

for the error and state the following lemma.

Lemma 3.3.1. *The polynomial space $\mathcal{E}$ is isomorphic to the null space $\mathcal{N}(\boldsymbol{A})$, and the leading term Q_{p+1} may be written as*

$$E = Q_{p+1}h^{p+1} = \sum_{i=1}^{n-m} d_i \chi_i,$$

where $\chi_i = \boldsymbol{C}_i^T \boldsymbol{\Phi},\ i = 1, 2, \ldots, n-m$, with $\{\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_{n-m}\}$ being a basis of $\mathcal{N}(\boldsymbol{A})$. Furthermore, on elements of Class I, there exists a basis $\{\chi_1, \chi_2, \ldots, \chi_{n-m}\}$ independent of λ and μ.

Proof. The application $F : \mathcal{N}(\boldsymbol{A}) \longrightarrow \mathcal{E}$ such that $F(\boldsymbol{C}) = \boldsymbol{C}^T\boldsymbol{\Phi}$ is clearly an isomorphism. Thus, every basis $\{\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_{n-m}\}$ for $\mathcal{N}(\boldsymbol{A})$ is mapped into a basis $\{\chi_1 = \boldsymbol{C}_1^T\boldsymbol{\Phi}, \chi_2 = \boldsymbol{C}_2^T\boldsymbol{\Phi}, \ldots, \chi_{n-m} = \boldsymbol{C}_{n-m}^T\boldsymbol{\Phi}\}$ of $\mathcal{E}$.

Applying (3.2.25a) on an element of *Class I* and noting that for $V \in \mathcal{P}_p,\ \boldsymbol{a}_0 \cdot \nabla V \in \mathcal{P}_{p-1}$ is orthogonal to Q_{p+1} on Δ, (3.3.3) with $\beta \neq 0$ may be written as

$$a_{ij} \ = \ \iint_{\hat{\Gamma}^+} \beta \Phi_j \Phi_i d\sigma,\ j = 1, 2, \ldots, n,\ i = 1, 2, \ldots, m.$$

Thus, the null space of $\boldsymbol{A}$ is independent of λ and μ which completes the proof of the lemma. $\square$

Remark 3.3.1. *Lemma 3.3.1 gives an optimal basis for the leading error term Q_{p+1} and reduces the number of degrees of freedom from $n = \dfrac{(p+2)(p+3)(p+4)}{6}$ to $m = \dfrac{(p+2)(p+3)}{2}$, which leads to a more efficient error estimation procedure.*

In the following section we construct the error basis functions χ_i.

3.3.1 Error Basis Functions

We follow Lemma 3.3.1 to find a basis of $\mathcal{N}(\boldsymbol{A})$ and construct basis functions for $\mathcal{E}$. For instance, a basis of $\mathcal{N}(\boldsymbol{A})$ for $p = 0$ on a reference element of *Class I* is given in terms of the canonical vectors $e_i,\ i = 1, 2, 3, 4$ in $\mathbb{R}^4$ as

$$\boldsymbol{C}_1 = e_2,\ \boldsymbol{C}_2 = \tfrac{2}{3}e_1 + e_3,\ \boldsymbol{C}_3 = -\tfrac{1}{3}e_1 + e_4,$$

$$\boldsymbol{\Phi} = (\varphi_{0,0}^0, \varphi_{0,0}^1, \varphi_{1,0}^0, \varphi_{0,1}^0)^T,$$

and $\chi_i = \boldsymbol{C}_i^T\boldsymbol{\Phi},\ i = 1, 2, 3.$

$p=0$	$\chi_1 = \varphi^1_{0,0}$ $\chi_2 = \frac{2}{3}\varphi^0_{0,0} + \varphi^0_{1,0}$ $\chi_3 = -\frac{1}{3}\varphi^0_{0,0} + \varphi^0_{0,1}$
$p=1$	$\chi_1 = \varphi^2_{0,0}$ $\chi_2 = \frac{4}{5}\varphi^1_{0,0} + \varphi^1_{1,0}$ $\chi_3 = -\frac{1}{5}\varphi^1_{0,0} + \varphi^1_{0,1}$ $\chi_4 = \frac{1}{10}\varphi^0_{0,1} + \frac{4}{5}\varphi^0_{1,0} + \varphi^0_{2,0}$ $\chi_5 = \frac{3}{5}\varphi^0_{0,1} - \frac{1}{5}\varphi^0_{1,0} + \varphi^0_{1,1}$ $\chi_6 = -\frac{1}{2}\varphi^0_{0,1} + \varphi^0_{0,2}$
$p=2$	$\chi_1 = \varphi^3_{0,0}$ $\chi_2 = \frac{6}{7}\varphi^2_{0,0} + \varphi^2_{1,0}$ $\chi_3 = -\frac{1}{7}\varphi^2_{0,0} + \varphi^2_{0,1}$ $\chi_4 = \frac{2}{21}\varphi^1_{0,1} + \frac{6}{7}\varphi^1_{1,0} + \varphi^1_{2,0}$ $\chi_5 = \frac{16}{21}\varphi^1_{0,1} - \frac{1}{7}\varphi^1_{1,0} + \varphi^1_{1,1}$ $\chi_6 = \frac{-1}{3}\varphi^1_{0,1} + \varphi^1_{0,2}$ $\chi_7 = \frac{2}{105}\varphi^0_{0,2} + \frac{2}{21}\varphi^0_{1,1} + \frac{6}{7}\varphi^0_{2,0} + \varphi^0_{3,0}$ $\chi_8 = \frac{16}{105}\varphi^0_{0,2} + \frac{16}{21}\varphi^0_{1,1} - \frac{1}{7}\varphi^0_{2,0} + \varphi^0_{2,1}$ $\chi_9 = \frac{8}{15}\varphi^0_{0,2} - \frac{1}{3}\varphi^0_{1,1} + \varphi^0_{1,2}$ $\chi_{10} = -\frac{3}{5}\varphi^0_{0,2} + \varphi^0_{0,3}$

Table 3.2: Error basis functions on a reference tetrahedron of *Class I*.

If $e_i,\ i = 1, 2, \ldots, 10$ are the canonical vectors in $\mathbb{R}^{10}$, a set of basis vectors in $\mathcal{N}(\boldsymbol{A})$ for $p = 1$ are

$$\boldsymbol{C}_1 = e_5,\ \boldsymbol{C}_2 = \tfrac{4}{5}e_2 + e_6,\ \boldsymbol{C}_3 = -\tfrac{1}{5}e_2 + e_7,$$

$$\boldsymbol{C}_4 = \tfrac{4}{5}e_3 + \tfrac{1}{10}e_4 + e_8,\ \boldsymbol{C}_5 = -\tfrac{1}{5}e_3 + \tfrac{3}{5}e_4 + e_9,\ \boldsymbol{C}_6 = -\tfrac{1}{2}e_4 + e_{10}.$$

With

$$\boldsymbol{\Phi} = (\varphi^0_{0,0}, \varphi^1_{0,0}, \varphi^0_{1,0}, \varphi^0_{0,1}, \varphi^2_{0,0}, \varphi^1_{1,0}, \varphi^1_{0,1}, \varphi^0_{2,0}, \varphi^0_{1,1}, \varphi^0_{0,2})^T,$$

the basis functions for $\mathcal{E}$ are given by $\chi_i = \boldsymbol{C}_i^T \boldsymbol{\Phi},\ i = 1, 2, \ldots, 6$.

Error basis functions for $p = 2$ are computed using *Mathematica* and are shown in Table 3.2.

Error basis functions on a reference tetrahedron of *Class II* are obtained following the previous lemma, *i.e.*, we compute a basis for $\mathcal{N}(\boldsymbol{A})$ which depends on λ and μ and construct the basis functions for $\mathcal{E}$ shown in Table 3.3.

Similarly, error basis functions for $p = 0, 1$ on a reference tetrahedron of *Class III* are shown in Table 3.4. Error basis functions for higher-degree polynomials are computed using *Mathematica* and are not presented here.

3.3.2 Error Estimation Procedure

Integrating (3.1.4) by parts shows that the DG solution U on a physical element Δ satisfies

$$\iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}\left(\tilde{U} - U\right) V\,d\sigma + \iiint_{\Delta} (\boldsymbol{a}\cdot\nabla U + cU)\, V\,dxdydz = \iiint_{\Delta} fV\,dxdydz, \qquad (3.3.5)$$

$p=0$	$\chi_1 = \frac{\lambda}{3(\lambda+1)}\varphi^0_{0,0} + \varphi^1_{0,0}, \quad \chi_2 = -\frac{(\lambda-2)}{3(\lambda+1)}\varphi^0_{0,0} + \varphi^0_{1,0},$ $\chi_3 = -\frac{1}{3}\varphi^0_{0,0} + \varphi^0_{0,1},$
$p=1$	$\chi_1 = -\frac{\lambda}{10(\lambda+1)^2}\varphi^0_{0,0} + \frac{\lambda\left(7\lambda+\mu+3\lambda^2+4\right)}{5(\lambda+1)^3}\varphi^1_{0,0} + \frac{\lambda\left(3\lambda+\mu+\lambda^2+2\right)}{15(\lambda+1)^3}\varphi^0_{1,0}$ $\quad + \frac{\lambda\left(3\lambda+4\mu+\lambda^2+2\right)}{30(\lambda+1)^3}\varphi^0_{0,1} + \varphi^2_{0,0},$ $\chi_2 = +\frac{\lambda}{2(\lambda+1)^2}\varphi^0_{0,0} - \frac{\left(3\lambda^2+\lambda^3+\lambda(5\mu-2)-4\right)}{5(\lambda+1)^3}\varphi^1_{0,0} + \frac{\lambda\left(\lambda-\mu+\lambda^2\right)}{3(\lambda+1)^3}\varphi^0_{1,0}$ $\quad - \frac{\lambda\left(7\lambda+20\mu+\lambda^2+6\right)}{30(\lambda+1)^3}\varphi^0_{0,1} + \varphi^1_{1,0},$ $\chi_3 = -\frac{1}{5}\varphi^1_{0,0} + \frac{3\lambda}{10(\lambda+1)}\varphi^0_{0,1} + \varphi^1_{0,1},$ $\chi_4 = -\frac{\lambda}{(\lambda+1)^2}\varphi^0_{0,0} + \frac{2\lambda(\lambda+\mu+1)}{(\lambda+1)^3}\varphi^1_{0,0} + \frac{\left(6\lambda^2-8\lambda^3+2\lambda(5\mu+13)+12\right)}{15(\lambda+1)^3}\varphi^0_{1,0}$ $\quad + \frac{\left(15\lambda^2+\lambda^3+\lambda(40\mu+17)+3\right)}{30(\lambda+1)^3}\varphi^0_{0,1} + \varphi^0_{2,0},$ $\chi_5 = -\frac{1}{5}\varphi^0_{1,0} - \frac{3(\lambda-2)}{10(\lambda+1)}\varphi^0_{0,1} + \varphi^0_{1,1},$ $\chi_6 = -\frac{1}{2}\varphi^0_{0,1} + \varphi^0_{0,2},$

Table 3.3: Error basis functions on a reference tetrahedron of *Class II*, $\boldsymbol{a}_0 = (\alpha,\beta,\gamma)^T$, $\lambda = \alpha/\beta$, $\mu = \gamma/\beta$
.

$p=0$	$\chi_1 = \frac{\lambda}{3(\lambda+\mu+1)}\varphi^0_{0,0} + \varphi^1_{0,0} \quad \chi_2 = -\frac{(\lambda-2)}{3(\lambda+\mu+1)}\varphi^0_{0,0} + \varphi^0_{1,0}$ $\chi_3 = -\frac{\lambda-3\mu+1}{3(\lambda+\mu+1)}\varphi^0_{0,0} + \varphi^0_{0,1}$
$p=1$	$\chi_1 = -\frac{\lambda(\mu+1)}{10(\lambda+\mu+1)^2}\varphi^0_{0,0} + \frac{\lambda(3\lambda+4\mu+4)}{5(\lambda+\mu+1)^2}\varphi^1_{0,0} + \frac{\lambda(\lambda+2\mu+2)}{15(\lambda+\mu+1)^2}\varphi^0_{1,0}$ $\quad + \frac{\lambda(\lambda+2\mu+2)}{30(\lambda+\mu+1)^2}\varphi^0_{0,1} + \varphi^2_{0,0}$ $\chi_2 = \frac{\lambda(\mu+5)}{10(\lambda+\mu+1)^2}\varphi^0_{0,0} - \frac{\left(-4\mu+\lambda^2+2\lambda(\mu+1)-4\right)}{5(\lambda+\mu+1)^2}\varphi^1_{0,0} + \frac{\lambda(5\lambda+4\mu)}{15(\lambda+\mu+1)^2}\varphi^0_{1,0}$ $\quad - \frac{\lambda(\lambda+2\mu+6)}{30(\lambda+\mu+1)^2}\varphi^0_{0,1} + \varphi^1_{1,0}$ $\chi_3 = \frac{3\lambda\mu}{5(\lambda+\mu+1)^2}\varphi^0_{0,0} - \frac{\left(-4\mu+\lambda^2-5\mu^2+2\lambda(\mu+1)+1\right)}{5(\lambda+\mu+1)^2}\varphi^1_{0,0} - \frac{2\lambda\mu}{5(\lambda+\mu+1)^2}\varphi^0_{1,0}$ $\quad + \frac{\lambda(3\lambda+\mu+3)}{10(\lambda+\mu+1)^2}\varphi^0_{0,1} + \varphi^1_{0,1}$ $\chi_4 = -\frac{(3\mu+\lambda(\mu+10))}{10(\lambda+\mu+1)^2}\varphi^0_{0,0} + \frac{(3\mu+\lambda(\mu+10))}{5(\lambda+\mu+1)^2}\varphi^1_{0,0} + \frac{\left(15\mu-8\lambda^2-7\lambda(\mu-2)+12\right)}{15(\lambda+\mu+1)^2}\varphi^0_{1,0}$ $\quad + \frac{\left(6\mu+\lambda^2+2\lambda(\mu+7)+3\right)}{30(\lambda+\mu+1)^2}\varphi^0_{0,1} + \varphi^0_{2,0}$ $\chi_5 = -\frac{3\mu(\lambda-2)}{5(\lambda+\mu+1)^2}\varphi^0_{0,0} + \frac{6\mu(\lambda-2)}{5(\lambda+\mu+1)^2}\varphi^1_{0,0} - \frac{\left(\lambda^2-5\mu^2+\lambda(-6\mu+2)+1\right)}{5(\lambda+\mu+1)^2}\varphi^0_{1,0}$ $\quad - \frac{(\lambda-2)(3\lambda+\mu+3)}{10(\lambda+\mu+1)^2}\varphi^0_{0,1} + \varphi^0_{1,1},$ $\chi_6 = -\frac{3\mu(\lambda+1)}{2(\lambda+\mu+1)^2}\varphi^0_{0,0} + \frac{3\mu(\lambda+1)}{(\lambda+\mu+1)^2}\varphi^1_{0,0} + \frac{\mu(\lambda+1)}{(\lambda+\mu+1)^2}\varphi^0_{1,0} +$ $\quad \frac{\left(2\mu-\lambda^2+2\mu^2+2\lambda(\mu-1)-1\right)}{2(\lambda+\mu+1)^2}\varphi^0_{0,1} + \varphi^0_{0,2}$

Table 3.4: Error basis functions on a reference tetrahedron of *Class III*, $\boldsymbol{a}_0 = (\alpha,\beta,\gamma)^T$, $\lambda = \alpha/\beta$, $\mu = \gamma/\beta$.

where the numerical flux $\tilde{U}$ is given by (3.1.5a) or (3.1.5b).
Next, we consider the following weak formulation for the exact solution u

$$\iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}u^- V d\sigma + \iint_{\Gamma^+} \boldsymbol{a}\cdot\mathbf{n}uVd\sigma - \iiint_\Delta (\boldsymbol{a}\cdot\nabla V + cV)\, udxdydz = \iiint_\Delta fVdxdydz, \tag{3.3.6}$$

with $u = U + e$ and $u^- = U^- - e^-$, respectively, the finite element error on Δ and Γ^-.
The weak formulation (3.3.6) can be written as

$$\begin{aligned} &\iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}\left(U^- + e^-\right) V d\sigma + \iint_{\Gamma^+} \boldsymbol{a}\cdot\mathbf{n}\,(U+e)\, V d\sigma \\ &- \iiint_\Delta (\boldsymbol{a}\cdot\nabla V + cV)\,(U+e)\, dxdydz \;=\; \iiint_\Delta fVdxdydz. \end{aligned} \tag{3.3.7}$$

Applying Stokes' theorem to write

$$\begin{aligned} &\iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}\left((e^- - e)\right) V d\sigma + \iiint_\Delta (\boldsymbol{a}\cdot\nabla\,(e))\, V dxdydz \\ &= \iiint_\Delta rVdxdydz - \iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}\left(U^- - U\right) V d\sigma, \end{aligned} \tag{3.3.8}$$

where $r = (f - \boldsymbol{a}\cdot\nabla U - cU)$ is the interior residual.
In order to estimate the finite element error $e = u - U$ on Δ we assume that the leading term of the DG error exhibits the same asymptotic behavior as the local error on Δ. Thus, the DG error e on Δ is approximated by

$$E\,(x,y,z) = \sum_{i=1}^{K} d_i \chi_i(\xi(x,y,z), \eta(x,y,z), \zeta(x,y,z)). \tag{3.3.9a}$$

and determined by solving the weak finite element problem

$$\begin{aligned} &\iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}\left(E^- - E\right) V d\sigma + \iiint_\Delta (\boldsymbol{a}\cdot\nabla E)\, V dxdydz \\ = \;&\iiint_\Delta rVdxdydz - \iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}\left(U^- - U\right) V d\sigma, \ \forall\ V \in \mathcal{E}, \end{aligned} \tag{3.3.9b}$$

This local weak formulation can be approximated by

$$\iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}\left(E^- - E\right) V d\sigma = -\iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}\left(U^- - U\right) V d\sigma, \ \forall\ V \in \mathcal{E}. \tag{3.3.10}$$

The accuracy of *a posteriori* error estimates is measured by the ratio of the error estimate over the true error. In this manuscript we use the element effectivity indices in the L^2 norm as

$$\theta_j = \frac{\|E\|_{2,\Delta_j}}{\|e\|_{2,\Delta_j}}, \tag{3.3.11}$$

and the global effectivity index

$$\theta = \frac{\|E\|_{2,\Omega}}{\|e\|_{2,\Omega}}. \tag{3.3.12}$$

Ideally, the effectivity indices should approach unity under mesh refinement, thus, the error estimates are asymptotically exact.

The following are the main steps of our modified DG method with error estimation.

1. Create a mesh for the domain Ω
2. Find the set $\mathcal{Z}^0$ of elements whose *inflow* faces are on the domain *inflow* boundary $\partial\Omega^-$.
3. For $k = 1, 2, \ldots,$ find the set $\mathcal{Z}^k$ of all elements not in $\mathcal{Z}^{k-1}$ whose *inflow* faces are either on $\partial\Omega^-$ or are shared by an element from $\mathcal{Z}^{k-1}$.
4. For $k = 0, 1, \ldots,$ find the DG solution and error estimate:
   (a) Compute the DG solution U on each element in $\mathcal{Z}^k$ by solving the DG finite element problem (3.1.4) with boundary conditions (3.1.5a) or (3.1.5b), respectively, for standard and the modified DG method.
   (b) Compute the error estimate E on each element of $\mathcal{Z}^k$ by solving (3.3.9).

3.4 Computational Examples

We solve several linear hyperbolic problems on uniform and general unstructured tetrahedral meshes and test our *a posteriori* error estimation for the standard and modified DG methods. In order to test the robustness of our procedures we use the following three families of meshes.

1. A family of uniform meshes obtained by partitioning the domain $[0, 1]^3$ into n^3 cubes, $n = 7, 8, \ldots, 16$ and subdividing each cube into five tetrahedra [40]. The resulting meshes have $N = 5n^3 = 1715, 2560, \ldots, 20480$ tetrahedral elements with diameter $h_{max} = \frac{\sqrt{2}}{n}$.
2. A family of uniform meshes obtained by partitioning the domain $[0, 1]^3$ into n^3 cubes, $n = 7, 8, \ldots, 16$ and subdividing each cube into six tetrahedra [40]. These meshes have $N = 6n^3 = 2058, 3072, \ldots, 24576$ tetrahedral elements with diameter $h_{max} = \frac{\sqrt{2}}{n}$.
3. A family of unstructured meshes generated by COMSOL software [33] with maximum mesh size $h_{\max} = 1/n,\ n = 1, 2, \ldots, 10$ which yields ten unstructured meshes having $N = 24, 192, 476,\ 943, 2121, 3731, 5846, 8713,\ 12525, 17120$ tetrahedral elements. See Figure 3.3 for two typical meshes having 943 and 8713 elements generated by *COMSOL*.

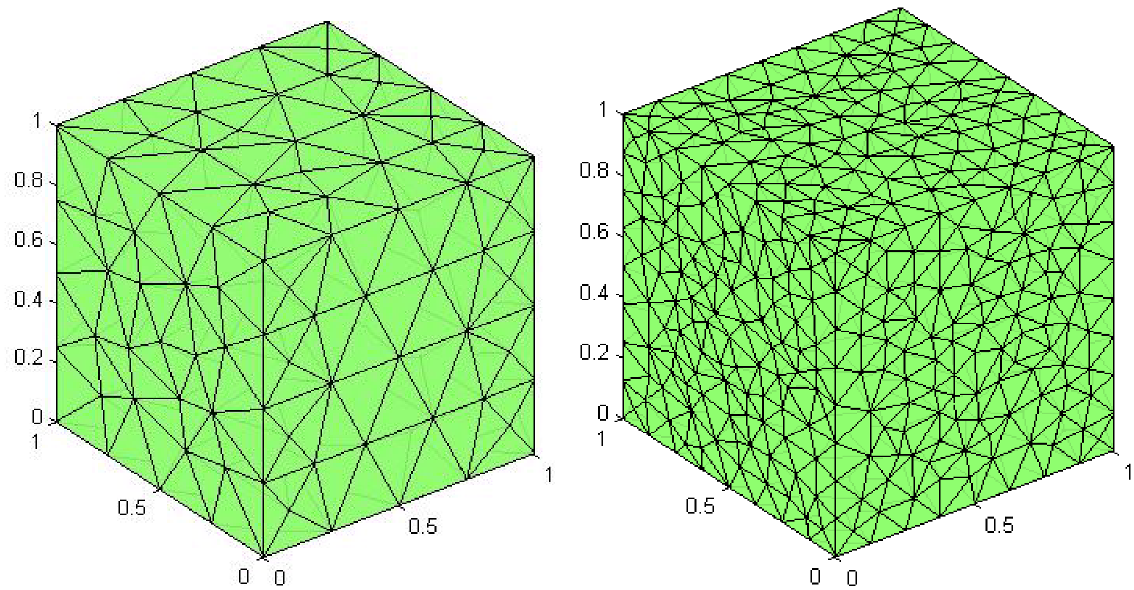


Figure 3.3: Unstructured tetrahedral meshes (obtained by *COMSOL*) having 943(left) and 8713(right) elements on $[0,1]^3$.

Example 3.4.1.

Let us consider the following linear hyperbolic problem

$$-3u_x - 7u_y + 13u_z = 3e^{x+y+z},\ (x,y,z) \in \Omega = [0,1]^3\,, \tag{3.4.1a}$$

subject to the boundary conditions on the *inflow* boundary Γ^- such that the exact solution is given by

$$u\,(x,y,z) = e^{x+y+z}. \tag{3.4.1b}$$

We solve (3.4.1) with the exact *inflow* boundary condition, $U^- = u$, using the standard and the modified DG method on uniform tetrahedral meshes having $N = 1715, 2560, \ldots, 20480$ elements and $p = 0,1,2,3$. We present the L^2 errors $||e||$ and $||e_c|| = ||u - U - E||$, their orders of convergence, the maximum and minimum element effectivity indices, and the global effectivity indices in Tables 3.5 and 3.6.

Next, we solve (3.4.1) using the modified DG method on unstructured tetrahedral meshes having $N = 24, 192, 476, 943, 2121$,$3731, 5846, 8713,\ 12525, 17120$ elements and present the L^2 errors $||e||$ and $||e_c|| = ||u - U - E||$, their orders of convergence, the maximum and minimum element effectivity indices, and the global effectivity indices in Table 3.7.

We observe that the effectivity indices for the standard DG method deviate significantly from unity for all meshes and do not converge to unity under mesh refinement. This confirms that the DG error on an element does not behave like the local DG error unless the flux is an $O(h^{p+2})$ approximation of the true flux. This motivated us to construct the new modified

DG method with a numerical flux corrected by the error estimate and which yields much more accurate results. We note that the modified DG errors are smaller than the standard errors for the same meshes and polynomial degrees. The corrected modified DG solution $U+E$ is $O(h^{p+2})$ convergent to the true solution while the modified DG solution is only $O(h^{p+1})$ for constant coefficient problems. Moreover, the effectivity indices are close to unity for all meshes and polynomial degrees and they converge to unity under mesh refinement.

We also solve problem (3.4.1) using the modified DG method and the approximated weak formulation for the error (3.3.10) on uniform meshes having $N = 5n^3$ elements with $p = 0, 1, 2, 3$. The L^2 errors, their orders and effectivity indices presented in Table 3.8 show that modified DG method exhibit optimal $O(h^{p+1})$ convergence rates and the effectivity indices approach unity under mesh refinement.

Example 3.4.2.

Let us consider the following linear hyperbolic problem

$$-u_x - 2u_y + 3u_z = -\frac{-x-2y+3z}{(1+x^2+y^2+z^2)^{\frac{3}{2}}}, \ (x,y,z) \in \Omega = [0,1]^3, \tag{3.4.2a}$$

and select the boundary conditions such that the exact solution is

$$u(x,y,z) = \frac{1}{\sqrt{1+x^2+y^2+z^2}}. \tag{3.4.2b}$$

We solve problem (3.4.2) using the modified DG method on uniform meshes having $N = 5n^3$ elements with $p = 0, 1, 2, 3$. The L^2 errors, their orders and effectivity indices presented in Table 3.9 show that modified DG method exhibit optimal $O(h^{p+1})$ convergence rates. Furthermore, the effectivity indices for the modified DG are close to unity and converge under mesh refinement.

Next, we solve (3.4.2) using the modified DG method on the unstructured tetrahedral meshes having $N = 24, 192, 476, 943, 2121$,$3731, 5846, 8713$, $12525, 17120$ elements and present the L^2 errors $||e||$ and $||e_c|| = ||u - U - E||$, their orders of convergence, the maximum and minimum element effectivity indices, and the global effectivity indices in Table 3.10. Again the method exhibits optimal convergence rates and the effectivity indices converge to unity under mesh refinement.

We also solve problem (3.4.2) using the modified DG method and the approximated weak formulation for the error (3.3.10) on uniform meshes having $N = 5n^3$ elements with $p = 0, 1, 2, 3$. The L^2 errors, their orders and effectivity indices presented in Table 3.11. We observe that modified DG method exhibits optimal $O(h^{p+1})$ convergence rates and the effectivity indices approach unity and mesh refinement. We note that $U + E$ is $O(h^{p+1})$ convergent to the true solution, while it is $O(h^{p+2})$ convergent if E is computed using the weak formulation (3.3.9b) as shown in Table 3.9.

N	$\|e\|_{2,\Omega}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
p=0							
1715	3.4639e-01	-	3.0300e-01	-	0.0668	1.7226	0.3984
2560	3.0492e-01	0.9549	2.6829e-01	0.9111	0.0667	1.7406	0.3901
3645	2.7232e-01	0.9600	2.4040e-01	0.9319	0.0665	1.7536	0.3831
5000	2.4605e-01	0.9627	2.1797e-01	0.9296	0.0664	1.7640	0.3777
6655	2.2441e-01	0.9663	1.9923e-01	0.9429	0.0663	1.7724	0.3730
8640	2.0627e-01	0.9682	1.8356e-01	0.9416	0.0662	1.7792	0.3691
10985	1.9085e-01	0.9709	1.7011e-01	0.9508	0.0662	1.7849	0.3657
13720	1.7758e-01	0.9722	1.5855e-01	0.9500	0.0661	1.7897	0.3629
16875	1.6604e-01	0.9743	1.4842e-01	0.9568	0.0661	1.7939	0.3603
20480	1.5591e-01	0.9754	1.3954e-01	0.9562	0.0660	1.7974	0.3581
p=1							
1715	1.6053e-02	-	1.4907e-02	-	0.0424	1.0042	0.2810
2560	1.2524e-02	1.8594	1.1664e-02	1.8371	0.0420	1.0046	0.2794
3645	1.0047e-02	1.8706	9.3771e-03	1.8531	0.0419	1.0049	0.2783
5000	8.2401e-03	1.8820	7.7041e-03	1.8652	0.0420	1.0052	0.2777
6655	6.8805e-03	1.8919	6.4413e-03	1.8782	0.0421	1.0054	0.2773
8640	5.8321e-03	1.8999	5.4661e-03	1.8868	0.0422	1.0056	0.2771
10985	5.0066e-03	1.9068	4.6966e-03	1.8955	0.0423	1.0057	0.2769
13720	4.3450e-03	1.9125	4.0792e-03	1.9016	0.0423	1.0058	0.2768
16875	3.8065e-03	1.9178	3.5760e-03	1.9083	0.0424	1.0060	0.2767
20480	3.3624e-03	1.9224	3.1607e-03	1.9131	0.0424	1.0060	0.2767
p=2							
1715	3.5980e-04	-	3.4323e-04	-	0.0238	1.0498	0.3315
2560	2.4287e-04	2.9433	2.3252e-04	2.9163	0.0236	1.0507	0.3360
3645	1.7105e-04	2.9765	1.6406e-04	2.9610	0.0234	1.0507	0.3403
5000	1.2502e-04	2.9754	1.2010e-04	2.9603	0.0233	1.0501	0.3428
6655	9.4271e-05	2.9617	9.0664e-05	2.9501	0.0232	1.0494	0.3442
8640	7.2938e-05	2.9485	7.0234e-05	2.9343	0.0231	1.0485	0.3450
10985	5.7596e-05	2.9504	5.5518e-05	2.9375	0.0230	1.0476	0.3460
13720	4.6238e-05	2.9639	4.4615e-05	2.9503	0.0229	1.0468	0.3474
16875	3.7635e-05	2.9841	3.6340e-05	2.9734	0.0229	1.0459	0.3492
20480	3.1013e-05	2.9985	2.9965e-05	2.9890	0.0228	1.0455	0.3510
p=3							
1715	6.3532e-06	-	6.0112e-06	-	0.0157	1.0111	0.3502
2560	3.7927e-06	3.8634	3.6049e-06	3.8293	0.0154	1.0109	0.3505
3645	2.3955e-06	3.9012	2.2831e-06	3.8780	0.0153	1.0109	0.3523
5000	1.5843e-06	3.9242	1.5134e-06	3.9028	0.0152	1.0108	0.3539
6655	1.0887e-06	3.9357	1.0415e-06	3.9209	0.0151	1.0107	0.3554
8640	7.7314e-07	3.9340	7.4055e-07	3.9189	0.0150	1.0107	0.3561
10985	5.6441e-07	3.9314	5.4117e-07	3.9186	0.0149	1.0106	0.3566
13720	4.2178e-07	3.9306	4.0485e-07	3.9163	0.0149	1.0106	0.3569
16875	3.2144e-07	3.9380	3.0881e-07	3.9250	0.0148	1.0106	0.3575
20480	2.4910e-07	3.9502	2.3952e-07	3.9370	0.0148	1.0105	0.3582

Table 3.5: L^2 errors, orders and effectivity indices for the standard DG method applied to problem (3.4.1) on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$.

p=0							
N	$\|e\|_{2,\Omega}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
1715	2.9960e-01	-	1.6053e-02	-	0.9476	1.1220	1.0198
2560	2.6212e-01	1.0009	1.2524e-02	1.8594	0.9538	1.1070	1.0181
3645	2.3306e-01	0.9974	1.0047e-02	1.8706	0.9586	1.0952	1.0168
5000	2.0975e-01	1.0002	8.2401e-03	1.8820	0.9626	1.0858	1.0156
6655	1.9071e-01	0.9983	6.8805e-03	1.8919	0.9658	1.0780	1.0146
8640	1.7482e-01	0.9999	5.8321e-03	1.8999	0.9686	1.0716	1.0137
10985	1.6139e-01	0.9988	5.0066e-03	1.9068	0.9709	1.0661	1.0129
13720	1.4986e-01	0.9998	4.3450e-03	1.9125	0.9729	1.0614	1.0122
16875	1.3988e-01	0.9990	3.8065e-03	1.9178	0.9746	1.0573	1.0116
20480	1.3114e-01	0.9998	3.3624e-03	1.9224	0.9762	1.0538	1.0110
p=1							
1715	9.7472e-03	-	3.5980e-04	-	0.9083	1.0519	1.0101
2560	7.4596e-03	2.0031	2.4287e-04	2.9433	0.9192	1.0458	1.0093
3645	5.8961e-03	1.9970	1.7105e-04	2.9765	0.9278	1.0409	1.0086
5000	4.7751e-03	2.0014	1.2502e-04	2.9754	0.9348	1.0370	1.0079
6655	3.9471e-03	1.9982	9.4271e-05	2.9617	0.9405	1.0337	1.0074
8640	3.3164e-03	2.0009	7.2938e-05	2.9485	0.9453	1.0310	1.0070
10985	2.8261e-03	1.9989	5.7596e-05	2.9504	0.9494	1.0287	1.0066
13720	2.4367e-03	2.0005	4.6238e-05	2.9639	0.9529	1.0267	1.0062
16875	2.1227e-03	1.9991	3.7635e-05	2.9841	0.9560	1.0249	1.0059
20480	1.8657e-03	2.0002	3.1013e-05	2.9985	0.9587	1.0234	1.0056
p=2							
1715	2.2252e-04	-	6.3532e-06	-	0.8752	1.0305	1.0042
2560	1.4897e-04	3.0048	3.7927e-06	3.8634	0.8890	1.0277	1.0038
3645	1.0466e-04	2.9976	2.3955e-06	3.9012	0.9004	1.0252	1.0035
5000	7.6278e-05	3.0023	1.5843e-06	3.9242	0.9096	1.0232	1.0032
6655	5.7317e-05	2.9985	1.0887e-06	3.9357	0.9173	1.0214	1.0029
8640	4.4143e-05	3.0015	7.7314e-07	3.9340	0.9238	1.0199	1.0027
10985	3.4721e-05	2.9993	5.6441e-07	3.9314	0.9293	1.0186	1.0025
13720	2.7797e-05	3.0013	4.2178e-07	3.9306	0.9341	1.0174	1.0024
16875	2.2601e-05	2.9997	3.2144e-07	3.9380	0.9383	1.0164	1.0023
20480	1.8621e-05	3.0009	2.4910e-07	3.9502	0.9420	1.0155	1.0022
p=3							
1715	3.8956e-06	-	8.4228e-08	-	0.7860	1.0156	1.0030
2560	2.2825e-06	4.0035	4.3753e-08	4.9049	0.8149	1.0136	1.0026
3645	1.4254e-06	3.9973	2.4512e-08	4.9193	0.8371	1.0121	1.0023
5000	9.3502e-07	4.0018	1.4587e-08	4.9264	0.8547	1.0109	1.0021
6655	6.3872e-07	3.9986	9.1105e-09	4.9384	0.8690	1.0099	1.0019
8640	4.5094e-07	4.0011	5.9244e-09	4.9459	0.8807	1.0091	1.0017
10985	3.2741e-07	3.9992	3.9871e-09	4.9477	0.8906	1.0084	1.0016
13720	2.4341e-07	4.0008	2.7628e-09	4.9497	0.8990	1.0078	1.0015
16875	1.8471e-07	3.9995	1.9629e-09	4.9546	0.9062	1.0073	1.0014
20480	1.4268e-07	4.0006	1.4252e-09	4.9598	0.9124	1.0068	1.0013

Table 3.6: L^2 errors and effectivity indices for problem (3.4.1) using the modified DG method on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$.

N	$\|e\|_{2,\Omega}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
p=0							
24	1.2423	-	1.9127e-01	-	0.8581	1.2251	1.0625
192	6.1422e-01	1.0162	5.5121e-02	1.7950	0.9238	1.1620	1.0425
476	4.7248e-01	0.6471	3.2789e-02	1.2811	0.8782	1.1442	1.0118
943	3.6819e-01	0.8669	1.9908e-02	1.7345	0.8809	1.1883	1.0046
2121	2.8091e-01	1.2125	1.2014e-02	2.2634	0.9037	1.1082	1.0050
3731	2.3522e-01	0.9736	8.6014e-03	1.8327	0.9151	1.0995	1.0043
5846	2.0161e-01	1.0002	6.4742e-03	1.8429	0.9092	1.0918	1.0022
8713	1.7554e-01	1.0370	4.8318e-03	2.1913	0.9328	1.0682	1.0030
12525	1.5818e-01	0.8840	3.9963e-03	1.6120	0.9451	1.0718	1.0019
17120	1.4036e-01	1.1347	3.1523e-03	2.2517	0.9427	1.0720	1.0020
p=1							
24	1.5617e-01	-	1.5700e-02	-	0.8968	1.1535	1.0213
192	3.8243e-02	2.0299	2.3789e-03	2.7223	0.9456	1.1220	1.0200
476	2.5148e-02	1.0338	1.3144e-03	1.4632	0.8883	1.1427	1.0084
943	1.4583e-02	1.8943	6.0792e-04	2.6803	0.8649	1.2019	1.0032
2121	8.6147e-03	2.3588	2.7321e-04	3.5842	0.8933	1.1333	1.0028
3731	5.9949e-03	1.9886	1.6530e-04	2.7562	0.9177	1.1039	1.0028
5846	4.4744e-03	1.8976	1.1182e-04	2.5354	0.9019	1.1424	1.0013
8713	3.3727e-03	2.1168	7.3732e-05	3.1189	0.9045	1.0882	1.0018
12525	2.7415e-03	1.7594	5.4546e-05	2.5589	0.9245	1.0831	1.0009
17120	2.1551e-03	2.2842	3.8053e-05	3.4173	0.9333	1.0890	1.0014
p=2							
24	1.3130e-02	-	1.0244e-03	-	0.9173	1.1337	1.0264
192	1.6180e-03	3.0206	7.5291e-05	3.7662	0.9569	1.0816	1.0161
476	9.9069e-04	1.2098	4.3144e-05	1.3732	0.8670	1.1226	1.0079
943	4.1263e-04	3.0445	1.4898e-05	3.6963	0.8224	1.1343	1.0024
2121	1.8990e-04	3.4777	5.3293e-06	4.6068	0.7354	1.1254	1.0021
3731	1.0915e-04	3.0376	2.6278e-06	3.8781	0.8753	1.1288	1.0021
5846	7.1829e-05	2.7143	1.5817e-06	3.2931	0.8976	1.1021	1.0013
8713	4.6621e-05	3.2370	8.7999e-07	4.3913	0.7475	1.0834	1.0014
12525	3.4164e-05	2.6394	5.9287e-07	3.3530	0.8337	1.0926	1.0007
17120	2.3753e-05	3.4498	3.6549e-07	4.5912	0.8221	1.0860	1.0012
p=3							
24	8.3118e-04	-	5.2796e-05	-	0.9308	1.1141	1.0230
192	5.1438e-05	4.0142	1.8696e-06	4.8196	0.9642	1.0625	1.0132
476	3.1925e-05	1.1764	1.1723e-06	1.1512	0.7970	1.1083	1.0070
943	9.2356e-06	4.3114	2.9574e-07	4.7873	0.7303	1.1497	1.0020
2121	3.3315e-06	4.5695	8.7329e-08	5.4664	0.5758	1.0968	1.0020
3731	1.5809e-06	4.0884	3.6117e-08	4.8426	0.7082	1.0887	1.0014
5846	9.2656e-07	3.4660	1.9267e-08	4.0764	0.6201	1.2761	1.0011
8713	5.1258e-07	4.4336	8.8672e-09	5.8115	0.5423	1.0764	1.0012
12525	3.3928e-07	3.5033	5.5606e-09	3.9620	0.6686	1.1702	1.0007
17120	2.0813e-07	4.6380	3.0086e-09	5.8299	0.4434	1.0761	1.0009

Table 3.7: L^2 errors and effectivity indices for problem (3.4.1) using the modified DG method on unstructured meshes having N elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$.

p=0							
N	$\|e\|_{2,\Omega}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
1715	2.9974e-01	-	1.4991e-02	-	0.9213	1.1082	1.0081
2560	2.6223e-01	1.0011	1.1621e-02	1.9069	0.9306	1.0952	1.0078
3645	2.3316e-01	0.9976	9.2721e-03	1.9174	0.9379	1.0850	1.0075
5000	2.0984e-01	1.0004	7.5701e-03	1.9248	0.9438	1.0768	1.0071
6655	1.9079e-01	0.9985	6.2966e-03	1.9327	0.9487	1.0700	1.0068
8640	1.7489e-01	1.0002	5.3197e-03	1.9375	0.9528	1.0643	1.0065
10985	1.6145e-01	0.9990	4.5537e-03	1.9426	0.9563	1.0595	1.0062
13720	1.4991e-01	1.0000	3.9421e-03	1.9460	0.9593	1.0554	1.0060
16875	1.3993e-01	0.9993	3.4459e-03	1.9499	0.9619	1.0517	1.0057
20480	1.3118e-01	1.0000	3.0379e-03	1.9525	0.9642	1.0486	1.0055
24565	1.2347e-01	0.9994	2.6983e-03	1.9556	0.9663	1.0458	1.0052
29160	1.1661e-01	0.9999	2.4126e-03	1.9577	0.9681	1.0433	1.0050
34295	1.1048e-01	0.9995	2.1700e-03	1.9601	0.9697	1.0411	1.0048
40000	1.0495e-01	0.9999	1.9623e-03	1.9618	0.9712	1.0390	1.0047
p=1							
1715	9.7643e-03	-	4.0733e-04	-	0.8607	1.0675	1.0004
2560	7.4708e-03	2.0050	2.7346e-04	2.9840	0.8769	1.0594	1.0011
3645	5.9039e-03	1.9985	1.9192e-04	3.0063	0.8898	1.0530	1.0015
5000	4.7808e-03	2.0027	1.4003e-04	2.9915	0.9002	1.0479	1.0016
6655	3.9513e-03	1.9993	1.0545e-04	2.9757	0.9089	1.0437	1.0018
8640	3.3196e-03	2.0019	8.1463e-05	2.9665	0.9162	1.0401	1.0019
10985	2.8286e-03	1.9999	6.4189e-05	2.9774	0.9224	1.0371	1.0020
13720	2.4387e-03	2.0015	5.1412e-05	2.9950	0.9277	1.0345	1.0021
16875	2.1243e-03	2.0000	4.1757e-05	3.0151	0.9324	1.0323	1.0021
20480	1.8670e-03	2.0010	3.4354e-05	3.0235	0.9365	1.0303	1.0021
24565	1.6538e-03	2.0000	2.8602e-05	3.0229	0.9401	1.0286	1.0020
29160	1.4751e-03	2.0008	2.4076e-05	3.0137	0.9434	1.0270	1.0020
34295	1.3239e-03	2.0000	2.0464e-05	3.0066	0.9463	1.0256	1.0019
40000	1.1948e-03	2.0007	1.7542e-05	3.0033	0.9489	1.0243	1.0019
p=2							
1715	2.2287e-04	-	7.7680e-06	-	0.7942	1.0229	0.9950
2560	1.4916e-04	3.0071	4.6294e-06	3.8762	0.8154	1.0213	0.9962
3645	1.0477e-04	2.9996	2.9133e-06	3.9321	0.8335	1.0191	0.9970
5000	7.6345e-05	3.0037	1.9208e-06	3.9535	0.8482	1.0175	0.9974
6655	5.7362e-05	2.9995	1.3169e-06	3.9604	0.8607	1.0169	0.9978
8640	4.4174e-05	3.0022	9.3402e-07	3.9482	0.8713	1.0169	0.9980
10985	3.4744e-05	3.0001	6.8122e-07	3.9429	0.8804	1.0167	0.9982
13720	2.7814e-05	3.0021	5.0859e-07	3.9435	0.8883	1.0164	0.9985
16875	2.2613e-05	3.0006	3.8700e-07	3.9602	0.8953	1.0160	0.9987
20480	1.8630e-05	3.0018	2.9929e-07	3.9821	0.9014	1.0156	0.9989
24565	1.5532e-05	3.0005	2.3476e-07	4.0060	0.9069	1.0151	0.9990
29160	1.3083e-05	3.0011	1.8660e-07	4.0169	0.9118	1.0147	0.9991
34295	1.1124e-05	3.0001	1.5018e-07	4.0154	0.9162	1.0143	0.9991
40000	9.5373e-06	3.0008	1.2231e-07	4.0019	0.9202	1.0139	0.9992
p=3							
1715	3.8988e-06	-	1.0151e-07	-	0.7656	1.0149	0.9966
2560	2.2840e-06	4.0047	5.2896e-08	4.8817	0.7921	1.0134	0.9972
3645	1.4261e-06	3.9984	2.9670e-08	4.9090	0.8131	1.0126	0.9977
5000	9.3545e-07	4.0025	1.7668e-08	4.9202	0.8306	1.0115	0.9980
6655	6.3898e-07	3.9991	1.1037e-08	4.9362	0.8454	1.0106	0.9982
8640	4.5110e-07	4.0015	7.1819e-09	4.9386	0.8578	1.0098	0.9984
10985	3.2752e-07	3.9996	4.8359e-09	4.9410	0.8685	1.0091	0.9986
13720	2.4348e-07	4.0012	3.3526e-09	4.9432	0.8778	1.0085	0.9987
16875	1.8476e-07	4.0000	2.3824e-09	4.9517	0.8859	1.0082	0.9988
20480	1.4271e-07	4.0010	1.7298e-09	4.9595	0.8930	1.0081	0.9990
24565	1.1198e-07	4.0001	1.2799e-09	4.9684	0.8993	1.0079	0.9990
29160	8.9090e-08	4.0007	9.6320e-10	4.9741	0.9048	1.0077	0.9991
34295	7.1764e-08	4.0000	7.3593e-10	4.9776	0.9098	1.0075	0.9992
40000	5.8451e-08	4.0005	5.7012e-10	4.9769	0.9144	1.0073	0.9992

Table 3.8: L^2 errors and effectivity indices for problem (3.4.1) using the modified DG method and the approximated weak formulation for the error (3.3.10) on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$.

p=0							
N	$\|e\|_{2,\Omega}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
1715	1.0803e-02	-	5.3947e-04	-	0.9169	1.2099	0.9944
2560	9.4551e-03	0.9981	4.1486e-04	1.9669	0.9145	1.2244	0.9952
3645	8.4053e-03	0.9992	3.2889e-04	1.9714	0.9127	1.2351	0.9958
5000	7.5657e-03	0.9988	2.6714e-04	1.9738	0.9114	1.2431	0.9962
6655	6.8782e-03	0.9995	2.2127e-04	1.9767	0.9105	1.2492	0.9966
8640	6.3055e-03	0.9992	1.8628e-04	1.9783	0.9098	1.2540	0.9969
10985	5.8206e-03	0.9997	1.5897e-04	1.9803	0.9092	1.2578	0.9972
13720	5.4051e-03	0.9994	1.3726e-04	1.9814	0.9087	1.2608	0.9974
16875	5.0448e-03	0.9998	1.1971e-04	1.9829	0.9083	1.2633	0.9976
20480	4.7296e-03	0.9996	1.0533e-04	1.9838	0.9080	1.2654	0.9977
p=1							
1715	4.3912e-04	-	1.7130e-05	-	0.9667	1.0237	0.9998
2560	3.3617e-04	2.0007	1.1555e-05	2.9487	0.9713	1.0212	0.9998
3645	2.6559e-04	2.0008	8.1585e-06	2.9549	0.9743	1.0193	0.9999
5000	2.1511e-04	2.0008	5.9721e-06	2.9609	0.9769	1.0180	0.9999
6655	1.7776e-04	2.0009	4.5022e-06	2.9643	0.9788	1.0164	0.9999
8640	1.4936e-04	2.0008	3.4774e-06	2.9683	0.9801	1.0154	0.9999
10985	1.2726e-04	2.0008	2.7415e-06	2.9708	0.9816	1.0145	0.9999
13720	1.0972e-04	2.0008	2.1993e-06	2.9733	0.9829	1.0136	0.9999
16875	9.5574e-05	2.0008	1.7912e-06	2.9752	0.9838	1.0130	1.0000
20480	8.3996e-05	2.0007	1.4781e-06	2.9772	0.9848	1.0123	1.0000
p=2							
1715	1.2847e-05	-	5.5804e-07	-	0.9438	1.0720	0.9985
2560	8.6110e-06	2.9963	3.2908e-07	3.9551	0.9448	1.0775	0.9987
3645	6.0499e-06	2.9971	2.0640e-07	3.9604	0.9455	1.0814	0.9988
5000	4.4113e-06	2.9979	1.3592e-07	3.9651	0.9460	1.0842	0.9989
6655	3.3148e-06	2.9983	9.3112e-08	3.9687	0.9463	1.0863	0.9990
8640	2.5535e-06	2.9987	6.5903e-08	3.9721	0.9465	1.0878	0.9991
10985	2.0086e-06	2.9989	4.7946e-08	3.9743	0.9467	1.0890	0.9992
13720	1.6083e-06	2.9992	3.5708e-08	3.9767	0.9468	1.0900	0.9993
16875	1.3077e-06	2.9993	2.7136e-08	3.9785	0.9470	1.0907	0.9993
20480	1.0775e-06	2.9995	2.0989e-08	3.9801	0.9470	1.0914	0.9994
p=3							
1715	4.3921e-07	-	2.0224e-08	-	0.9650	1.0266	0.9987
2560	2.5753e-07	3.9980	1.0474e-08	4.9277	0.9697	1.0249	0.9990
3645	1.6080e-07	3.9985	5.8543e-09	4.9388	0.9724	1.0212	0.9991
5000	1.0551e-07	3.9991	3.4765e-09	4.9462	0.9756	1.0196	0.9992
6655	7.2070e-08	3.9994	2.1685e-09	4.9522	0.9786	1.0181	0.9993
8640	5.0887e-08	3.9997	1.4088e-09	4.9568	0.9808	1.0167	0.9994
10985	3.6946e-08	3.9998	9.4712e-10	4.9607	0.9822	1.0157	0.9995
13720	2.7468e-08	4.0001	6.5561e-10	4.9639	0.9828	1.0148	0.9995
16875	2.0843e-08	4.0001	4.6540e-10	4.9667	0.9838	1.0139	0.9996
20480	1.6101e-08	4.0002	3.3772e-10	4.9690	0.9849	1.0133	0.9996

Table 3.9: L^2 errors and effectivity indices for problem (3.4.2) using the modified DG method on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$.

p=0							
N	$\|e\|_{2,\Omega}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
24	4.4631e-02	-	7.2492e-03	-	0.9072	1.0660	0.9883
192	2.2423e-02	0.9931	2.0201e-03	1.8434	0.9266	1.0743	0.9946
476	1.6478e-02	0.7597	1.3667e-03	0.9638	0.8488	1.1024	0.9959
943	1.3271e-02	0.7525	9.2814e-04	1.3450	0.8124	1.1570	0.9993
2121	1.0015e-02	1.2614	4.8471e-04	2.9113	0.8705	1.1205	1.0000
3731	8.3494e-03	0.9976	3.4583e-04	1.8517	0.8938	1.2834	1.0000
5846	7.1101e-03	1.0423	2.5820e-04	1.8956	0.7843	1.1655	0.9995
8713	6.2194e-03	1.0023	1.9864e-04	1.9640	0.8445	1.1282	0.9997
12525	5.5558e-03	0.9579	1.5306e-04	2.2129	0.8020	1.2234	1.0000
17120	4.9715e-03	1.0547	1.2556e-04	1.8801	0.8397	1.1901	0.9999
p=1							
24	6.4496e-03	-	1.0624e-03	-	0.9036	1.0364	0.9788
192	1.6743e-03	1.9456	1.4584e-04	2.8649	0.9423	1.0573	0.9962
476	1.0889e-03	1.0611	6.9086e-05	1.8427	0.9402	1.1042	1.0021
943	7.4144e-04	1.3360	3.7764e-05	2.0995	0.9010	1.0934	0.9987
2121	3.8325e-04	2.9573	1.5933e-05	3.8673	0.9434	1.0518	0.9987
3731	2.6385e-04	2.0477	9.3452e-06	2.9264	0.9387	1.0536	0.9994
5846	1.9917e-04	1.8241	5.8635e-06	3.0238	0.9409	1.0583	1.0001
8713	1.5413e-04	1.9199	4.0410e-06	2.7878	0.9563	1.0359	1.0001
12525	1.1827e-04	2.2484	2.8100e-06	3.0844	0.9634	1.0350	0.9997
17120	9.6815e-05	1.9000	2.0712e-06	2.8953	0.9633	1.0334	1.0000
p=2							
24	9.6475e-04	-	9.9804e-05	-	0.9338	1.0929	0.9937
192	1.1810e-04	3.0301	8.1798e-06	3.6090	0.9325	1.0579	0.9977
476	5.3176e-05	1.9680	3.9522e-06	1.7940	0.9220	1.0526	0.9933
943	2.8492e-05	2.1690	2.1122e-06	2.1779	0.8200	1.1014	1.0018
2121	1.1337e-05	4.1299	5.1692e-07	6.3080	0.9467	1.0927	0.9995
3731	6.5722e-06	2.9903	2.6299e-07	3.7064	0.9347	1.0784	0.9999
5846	4.1194e-06	3.0305	1.4453e-07	3.8836	0.8717	1.0947	0.9997
8713	2.8160e-06	2.8487	9.0756e-08	3.4846	0.8904	1.0511	0.9994
12525	1.9291e-06	3.2115	5.3179e-08	4.5381	0.8749	1.1863	0.9999
17120	1.4051e-06	3.0083	3.7066e-08	3.4259	0.9237	1.1098	0.9998
p=3							
24	8.5074e-05	-	1.9427e-05	-	0.8686	1.0589	0.9617
192	6.6187e-06	3.6841	8.3147e-07	4.5463	0.9354	1.0482	0.9928
476	3.0334e-06	1.9243	2.8148e-07	2.6713	0.9294	1.1184	1.0057
943	1.6934e-06	2.0262	1.1230e-07	3.1941	0.8150	1.0814	0.9905
2121	3.7579e-07	6.7468	2.0285e-08	7.6689	0.9318	1.0800	0.9977
3731	1.8445e-07	3.9034	9.3848e-09	4.2277	0.9079	1.1340	0.9984
5846	1.0102e-07	3.9057	4.0051e-09	5.5239	0.9269	1.0812	0.9995
8713	6.4030e-08	3.4145	2.4732e-09	3.6102	0.8566	1.1009	0.9997
12525	3.6863e-08	4.6877	1.1388e-09	6.5841	0.9377	1.0544	0.9995
17120	2.4935e-08	3.7107	7.1453e-10	4.4243	0.9245	1.0453	0.9999

Table 3.10: L^2 errors and effectivity indices for problem (3.4.2) using the modified DG method on unstructured meshes having N elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$.

p=0							
N	$\|e\|_{2,\Omega}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
1080	1.2694e-02	-	1.2321e-03	-	0.6826	1.0952	0.9610
1715	1.0882e-02	0.9994	9.9228e-04	1.4045	0.6693	1.0973	0.9612
2560	9.5233e-03	0.9986	8.2935e-04	1.3432	0.6863	1.0987	0.9614
3645	8.4654e-03	0.9998	7.1245e-04	1.2900	0.6950	1.0997	0.9616
5000	7.6194e-03	0.9992	6.2461e-04	1.2489	0.7028	1.1004	0.9617
6655	6.9268e-03	0.9999	5.5637e-04	1.2138	0.7030	1.1010	0.9618
8640	6.3498e-03	0.9996	5.0182e-04	1.1861	0.7086	1.1014	0.9619
10985	5.8613e-03	1.0000	4.5723e-04	1.1625	0.7110	1.1017	0.9619
13720	5.4428e-03	0.9997	4.2008e-04	1.1434	0.7132	1.1020	0.9620
16875	5.0799e-03	1.0001	3.8866e-04	1.1270	0.7166	1.1022	0.9620
20480	4.7625e-03	0.9998	3.6171e-04	1.1135	0.7193	1.1023	0.9621
24565	4.4823e-03	1.0001	3.3834e-04	1.1017	0.7224	1.1025	0.9621
29160	4.2333e-03	0.9999	3.1787e-04	1.0918	0.7221	1.1026	0.962
p=1							
1080	5.9807e-04	-	2.9725e-05	-	0.9498	1.0638	0.9972
1715	4.3934e-04	2.0008	1.9163e-05	2.8478	0.9539	1.0537	0.9974
2560	3.3634e-04	2.0007	1.3165e-05	2.8116	0.9557	1.0451	0.9975
3645	2.6572e-04	2.0008	9.5031e-06	2.7672	0.9563	1.0433	0.9975
5000	2.1522e-04	2.0007	7.1318e-06	2.7245	0.9568	1.0418	0.9976
6655	1.7785e-04	2.0007	5.5209e-06	2.6862	0.9584	1.0391	0.9976
8640	1.4944e-04	2.0007	4.3868e-06	2.6426	0.9591	1.0391	0.9976
10985	1.2732e-04	2.0007	3.5612e-06	2.6049	0.9596	1.0381	0.9976
13720	1.0978e-04	2.0006	2.9440e-06	2.5683	0.9599	1.0373	0.9976
16875	9.5626e-05	2.0006	2.4721e-06	2.5324	0.9607	1.0372	0.9976
20480	8.4043e-05	2.0006	2.1038e-06	2.4993	0.9609	1.0363	0.9977
24565	7.4444e-05	2.0006	1.8113e-06	2.4692	0.9612	1.0360	0.9977
29160	6.6400e-05	2.0005	1.5756e-06	2.4392	0.9616	1.0357	0.9977
p=2							
1080	2.0415e-05	-	1.5050e-06	-	0.9134	1.1048	0.9881
1715	1.2865e-05	2.9954	8.5917e-07	3.6364	0.9239	1.0877	0.9882
2560	8.6220e-06	2.9970	5.3299e-07	3.5757	0.9260	1.0831	0.9883
3645	6.0572e-06	2.9977	3.5282e-07	3.5026	0.9275	1.0752	0.9884
5000	4.4165e-06	2.9984	2.4541e-07	3.4456	0.9285	1.0669	0.9884
6655	3.3186e-06	2.9986	1.7746e-07	3.4012	0.9292	1.0627	0.9884
8640	2.5564e-06	2.9991	1.3255e-07	3.3537	0.9297	1.0733	0.9885
10985	2.0108e-06	2.9992	1.0163e-07	3.3184	0.9302	1.0695	0.9885
13720	1.6100e-06	2.9994	7.9653e-08	3.2877	0.9305	1.0704	0.9885
16875	1.3090e-06	2.9995	6.3619e-08	3.2579	0.9308	1.0705	0.9885
20480	1.0786e-06	2.9997	5.1634e-08	3.2342	0.9310	1.0667	0.9886
24565	8.9928e-07	2.9997	4.2493e-08	3.2140	0.9312	1.0710	0.9886
29160	7.5758e-07	2.9998	3.5403e-08	3.1938	0.9313	1.0676	0.9886
p=3							
1080	8.1482e-07	-	5.8877e-08	-	0.9254	1.0664	0.9910
1715	4.4003e-07	3.9970	2.8875e-08	4.6219	0.9263	1.0600	0.9912
2560	2.5800e-07	3.9981	1.5712e-08	4.5576	0.9282	1.0551	0.9913
3645	1.6110e-07	3.9986	9.2597e-09	4.4891	0.9299	1.0576	0.9915
5000	1.0570e-07	3.9992	5.8047e-09	4.4324	0.9297	1.0501	0.9915
6655	7.2202e-08	3.9994	3.8211e-09	4.3871	0.9304	1.0559	0.9916
8640	5.0981e-08	3.9997	2.6186e-09	4.3431	0.9299	1.0574	0.9917
10985	3.7014e-08	3.9998	1.8552e-09	4.3061	0.9293	1.0522	0.9917
13720	2.7519e-08	4.0000	1.3512e-09	4.2774	0.9305	1.0575	0.9917
16875	2.0882e-08	4.0001	1.0079e-09	4.2480	0.9302	1.0572	0.9917
20480	1.6131e-08	4.0002	7.6738e-10	4.2249	0.9308	1.0541	0.9918
24565	1.2657e-08	4.0002	5.9470e-10	4.2051	0.9304	1.0577	0.9918

Table 3.11: L^2 errors and effectivity indices for problem (3.4.2) using the modified DG method and the approximated weak formulation for the error (3.3.10) on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$.

Example 3.4.3.

Next, we consider the following linear hyperbolic problem with variable coefficients

$$(x+3)u_x + (y+3)u_y + (z+3)u_z = f(x,y,z)\,,\ (x,y,z) \in \Omega = [0,1]^3\,, \tag{3.4.3}$$

where we select f and the boundary conditions such that the exact solution is given by (3.4.1b).

We solve problem (3.4.3) using the modified DG method on uniform meshes having $N = 5n^3$ tetrahedral elements for $p = 0, 1, 2, 3$. We present the errors, orders and effectivity indices in Table 3.12. We again observe that the error estimates are accurate and asymptotically exact under mesh refinement.

Example 3.4.4.

We consider the following divergence-free linear hyperbolic problem with variable coefficients

$$(y+3)u_x + (z+3)u_y + (x+3)u_z = f(x,y,z)\,,\ (x,y,z) \in \Omega = [0,1]^3\,, \tag{3.4.4}$$

where we select f and the boundary conditions such that the exact solution is given by (3.4.2b).

We solve problem (3.4.4) using the modified DG method on uniform meshes having $N = 5n^3$ tetrahedral elements for $p = 0, 1, 2, 3$. We present the errors, orders and effectivity indices in Table 3.13. We again observe that the error estimates are accurate and asymptotically exact under mesh refinement.

Example 3.4.5.

We consider the following divergence-free linear convection-reaction problem

$$2u_x + u_y + u_z + \pi u = f(x,y,z)\,,\ (x,y,z) \in \Omega = [0,1]^3\,, \tag{3.4.5}$$

and select f and the boundary conditions such that the exact solution is given by (3.4.2b). We solve problem (3.4.5) using the modified DG method on uniform meshes having $N = 6n^3$, $n = 7, 8, \ldots, 16$ tetrahedral elements for $p = 0, 1, 2, 3$ and present the errors, orders and effectivity indices in Table 3.14. We again observe that the error estimates are accurate and asymptotically exact under mesh refinement for reaction diffusion problems. We also note that we solved the weak error problem (3.3.9) where we dropped the reaction term and expect to obtain more accurate estimates by including the reaction term in the error estimation problem.

Example 3.4.6.

N	$\|e\|_{2,\Omega}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
p=0							
1715	4.0253e-01	-	4.5996e-02	-	0.8555	1.0331	0.9640
2560	3.5193e-01	1.0062	3.9695e-02	1.1033	0.8554	1.0339	0.9606
3645	3.1236e-01	1.0125	3.4938e-02	1.0837	0.8556	1.0316	0.9579
5000	2.8095e-01	1.0060	3.1231e-02	1.0646	0.8559	1.0315	0.9557
6655	2.5516e-01	1.0101	2.8246e-02	1.0541	0.8555	1.0363	0.9539
8640	2.3378e-01	1.0056	2.5794e-02	1.0436	0.8552	1.0339	0.9524
10985	2.1566e-01	1.0085	2.3738e-02	1.0375	0.8545	1.0398	0.9512
13720	2.0017e-01	1.0052	2.1992e-02	1.0312	0.8541	1.0375	0.9501
16875	1.8674e-01	1.0073	2.0487e-02	1.0274	0.8538	1.0425	0.9492
20480	1.7501e-01	1.0048	1.9177e-02	1.0235	0.8534	1.0404	0.9484
p=1							
1715	1.1524e-02	-	5.2082e-04	-	0.9739	1.0796	1.0278
2560	8.8166e-03	2.0052	3.6182e-04	2.7280	0.9676	1.0792	1.0245
3645	6.9598e-03	2.0079	2.6390e-04	2.6794	0.9661	1.0778	1.0218
5000	5.6347e-03	2.0046	2.0038e-04	2.6133	0.9629	1.0779	1.0197
6655	4.6539e-03	2.0066	1.5692e-04	2.5651	0.9610	1.0766	1.0179
8640	3.9092e-03	2.0041	1.2615e-04	2.5090	0.9587	1.0776	1.0165
10985	3.3294e-03	2.0056	1.0355e-04	2.4664	0.9573	1.0758	1.0152
13720	2.8700e-03	2.0037	8.6543e-05	2.4206	0.9556	1.0774	1.0141
16875	2.4992e-03	2.0049	7.3413e-05	2.3849	0.9546	1.0754	1.0132
20480	2.1961e-03	2.0033	6.3089e-05	2.3481	0.9532	1.0773	1.0124
p=2							
1715	2.4991e-04	-	1.0260e-05	-	0.9581	1.1026	1.0195
2560	1.6727e-04	3.0064	6.3174e-06	3.6320	0.9515	1.1031	1.0170
3645	1.1739e-04	3.0069	4.1396e-06	3.5889	0.9516	1.1022	1.0150
5000	8.5527e-05	3.0053	2.8567e-06	3.5204	0.9468	1.1026	1.0135
6655	6.4221e-05	3.0061	2.0507e-06	3.4780	0.9475	1.1019	1.0122
8640	4.9446e-05	3.0046	1.5226e-06	3.4223	0.9436	1.1024	1.0112
10985	3.8874e-05	3.0053	1.1612e-06	3.3853	0.9446	1.1017	1.0103
13720	3.1115e-05	3.0041	9.0652e-07	3.3409	0.9414	1.1022	1.0095
16875	2.5290e-05	3.0047	7.2144e-07	3.3101	0.9426	1.1016	1.0088
20480	2.0833e-05	3.0037	5.8399e-07	3.2749	0.9397	1.1020	1.0083
p=3							
1715	4.1702e-06	-	1.5088e-07	-	0.9480	1.1119	1.0189
2560	2.4425e-06	4.0060	8.2495e-08	4.5214	0.9428	1.1135	1.0166
3645	1.5240e-06	4.0049	4.8737e-08	4.4685	0.9414	1.1120	1.0147
5000	9.9940e-07	4.0046	3.0645e-08	4.4035	0.9381	1.1135	1.0133
6655	6.8231e-07	4.0045	2.0221e-08	4.3622	0.9372	1.1121	1.0121
8640	4.8159e-07	4.0039	1.3892e-08	4.3140	0.9349	1.1135	1.0111
10985	3.4954e-07	4.0040	9.8609e-09	4.2824	0.9342	1.1122	1.0103
13720	2.5980e-07	4.0034	7.1988e-09	4.2460	0.9326	1.1136	1.0096
16875	1.9710e-07	4.0036	5.3799e-09	4.2214	0.9321	1.1123	1.0089
20480	1.5222e-07	4.0030	4.1043e-09	4.1934	0.9310	1.1136	1.0084

Table 3.12: L^2 errors and effectivity indices for problem (3.4.3) using the modified DG method on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$.

p=0							
N	$\|e\|_{2,\Omega}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
1715	1.2873e-02	-	1.3952e-03	-	0.7377	1.1044	0.9837
2560	1.1260e-02	1.0029	1.2623e-03	0.7499	0.7190	1.1178	0.9815
3645	1.0007e-02	1.0015	1.1734e-03	0.6198	0.7036	1.1551	0.9795
5000	9.0080e-03	0.9980	1.1114e-03	0.5155	0.6967	1.1886	0.9776
6655	8.1922e-03	0.9961	1.0666e-03	0.4313	0.6608	1.2315	0.9757
8640	7.5142e-03	0.9928	1.0333e-03	0.3644	0.6505	1.2599	0.9739
10985	6.9414e-03	0.9905	1.0080e-03	0.3103	0.6216	1.2973	0.9720
13720	6.4517e-03	0.9872	9.8822e-04	0.2668	0.6062	1.3274	0.9702
16875	6.0281e-03	0.9846	9.7258e-04	0.2312	0.5857	1.3572	0.9683
20480	5.6582e-03	0.9811	9.5998e-04	0.2021	0.5675	1.3911	0.9664
p=1							
1715	4.1383e-04	-	4.1523e-05	-	0.8482	1.2147	1.0032
2560	3.1727e-04	1.9898	3.0976e-05	2.1944	0.8440	1.2171	1.0036
3645	2.5095e-04	1.9908	2.4083e-05	2.1372	0.8385	1.2029	1.0039
5000	2.0346e-04	1.9915	1.9196e-05	2.1524	0.8343	1.2244	1.0040
6655	1.6827e-04	1.9919	1.5695e-05	2.1130	0.8310	1.2023	1.0040
8640	1.4149e-04	1.9922	1.3059e-05	2.1133	0.8283	1.2282	1.0040
10985	1.2064e-04	1.9923	1.1054e-05	2.0815	0.8261	1.2099	1.0040
13720	1.0408e-04	1.9923	9.4830e-06	2.0690	0.8242	1.2305	1.0039
16875	9.0712e-05	1.9922	8.2384e-06	2.0393	0.8198	1.2152	1.0038
20480	7.9768e-05	1.9921	7.2330e-06	2.0167	0.8162	1.2320	1.0036
p=2							
1715	1.3201e-05	-	1.3271e-06	-	0.8277	1.1806	1.0078
2560	8.8522e-06	2.9928	8.6794e-07	3.1799	0.8142	1.1951	1.0066
3645	6.2208e-06	2.9950	5.9748e-07	3.1702	0.8199	1.2029	1.0059
5000	4.5374e-06	2.9950	4.2879e-07	3.1489	0.7949	1.2072	1.0052
6655	3.4102e-06	2.9964	3.1795e-07	3.1380	0.8028	1.2094	1.0047
8640	2.6275e-06	2.9963	2.4221e-07	3.1270	0.7849	1.2105	1.0043
10985	2.0671e-06	2.9971	1.8875e-07	3.1157	0.7879	1.2108	1.0039
13720	1.6554e-06	2.9970	1.4989e-07	3.1101	0.7792	1.2128	1.0035
16875	1.3461e-06	2.9976	1.2105e-07	3.0980	0.7786	1.2157	1.0032
20480	1.1094e-06	2.9976	9.9125e-08	3.0958	0.7757	1.2175	1.0030
p=3							
1715	4.3286e-07	-	5.9356e-08	-	0.7749	1.2296	0.9980
2560	2.5414e-07	3.9880	3.5130e-08	3.9278	0.7485	1.2753	0.9983
3645	1.5884e-07	3.9903	2.2162e-08	3.9113	0.7185	1.2658	0.9987
5000	1.0431e-07	3.9913	1.4696e-08	3.8991	0.7568	1.2635	0.9990
6655	7.1295e-08	3.9928	1.0125e-08	3.9084	0.7160	1.2632	0.9992
8640	5.0368e-08	3.9934	7.2040e-09	3.9124	0.7542	1.2606	0.9995
10985	3.6584e-08	3.9945	5.2600e-09	3.9292	0.7473	1.2520	0.9997
13720	2.7209e-08	3.9949	3.9285e-09	3.9385	0.7548	1.2457	0.9999
16875	2.0654e-08	3.9957	2.9908e-09	3.9530	0.7722	1.2408	1.0000
20480	1.5958e-08	3.9960	2.3161e-09	3.9614	0.7680	1.2384	1.0002

Table 3.13: L^2 errors and effectivity indices for problem (3.4.4) using the modified DG method on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$.

N	$\|e\|_{2,\Omega}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
p=0							
2058	1.4554e-02	-	3.5720e-03	-	0.9678	1.2684	1.0635
3072	1.2736e-02	0.9994	3.1487e-03	0.9447	0.9715	1.2789	1.0674
4374	1.1321e-02	0.9994	2.8157e-03	0.9490	0.9745	1.2880	1.0704
6000	1.0190e-02	0.9994	2.5468e-03	0.9528	0.9769	1.2944	1.0728
7986	9.2639e-03	0.9995	2.3249e-03	0.9561	0.9789	1.3005	1.0748
10368	8.4923e-03	0.9995	2.1388e-03	0.9590	0.9806	1.3052	1.0765
13182	7.8394e-03	0.9995	1.9804e-03	0.9616	0.9820	1.3096	1.0779
16464	7.2796e-03	0.9995	1.8438e-03	0.9639	0.9832	1.3130	1.0791
20250	6.7945e-03	0.9996	1.7249e-03	0.9660	0.9843	1.3162	1.0802
24576	6.3701e-03	0.9996	1.6205e-03	0.9679	0.9853	1.3189	1.0811
p=1							
2058	3.4699e-04	-	4.8510e-05	-	0.8394	1.1896	0.9636
3072	2.6691e-04	1.9650	3.5227e-05	2.3961	0.8383	1.1693	0.9687
4374	2.1163e-04	1.9702	2.6719e-05	2.3470	0.8542	1.1622	0.9727
6000	1.7188e-04	1.9747	2.0952e-05	2.3078	0.8581	1.1809	0.9761
7986	1.4235e-04	1.9776	1.6873e-05	2.2714	0.8541	1.1701	0.9789
10368	1.1982e-04	1.9803	1.3883e-05	2.2420	0.8637	1.1670	0.9812
13182	1.0224e-04	1.9822	1.1627e-05	2.2154	0.8656	1.1763	0.9833
16464	8.8261e-05	1.9840	9.8826e-06	2.1936	0.8619	1.1711	0.9850
20250	7.6964e-05	1.9853	8.5061e-06	2.1740	0.8686	1.1699	0.9866
24576	6.7703e-05	1.9865	7.4004e-06	2.1576	0.8697	1.1738	0.9879
p=2							
2058	1.8959e-05	-	1.5185e-06	-	0.8940	1.0458	0.9690
3072	1.2725e-05	2.9860	9.5542e-07	3.4698	0.8914	1.0471	0.9731
4374	8.9500e-06	2.9876	6.3911e-07	3.4136	0.9002	1.0507	0.9763
6000	6.5322e-06	2.9888	4.4836e-07	3.3644	0.8962	1.0534	0.9789
7986	4.9124e-06	2.9899	3.2671e-07	3.3210	0.9026	1.0556	0.9810
10368	3.7869e-06	2.9907	2.4551e-07	3.2837	0.8998	1.0574	0.9828
13182	2.9805e-06	2.9915	1.8926e-07	3.2512	0.9055	1.0588	0.9843
16464	2.3878e-06	2.9921	1.4904e-07	3.2234	0.9008	1.0601	0.9856
20250	1.9423e-06	2.9927	1.1952e-07	3.1991	0.9055	1.0611	0.9868
24576	1.6011e-06	2.9931	9.7358e-08	3.1783	0.9019	1.0620	0.9878
p=3							
2058	5.5606e-07	-	3.9205e-08	-	0.8819	1.0465	0.9724
3072	3.2719e-07	3.9717	2.0612e-08	4.8148	0.8557	1.0759	0.9758
4374	2.0482e-07	3.9768	1.1710e-08	4.8005	0.8968	1.0535	0.9784
6000	1.3466e-07	3.9805	7.0730e-09	4.7853	0.8803	1.0493	0.9805
7986	9.2120e-08	3.9833	4.4899e-09	4.7680	0.8989	1.0459	0.9823
10368	6.5125e-08	3.9854	2.9697e-09	4.7507	0.9062	1.0489	0.9837
13182	4.7332e-08	3.9871	2.0334e-09	4.7321	0.9089	1.0417	0.9850
16464	3.5220e-08	3.9884	1.4338e-09	4.7139	0.9145	1.0474	0.9861
20250	2.6745e-08	3.9895	1.0371e-09	4.6948	0.9165	1.0424	0.9870
24576	2.0673e-08	3.9905	7.6695e-10	4.6762	0.9223	1.0415	0.9878

Table 3.14: L^2 errors and effectivity indices for problem (3.4.5) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2, 3$.

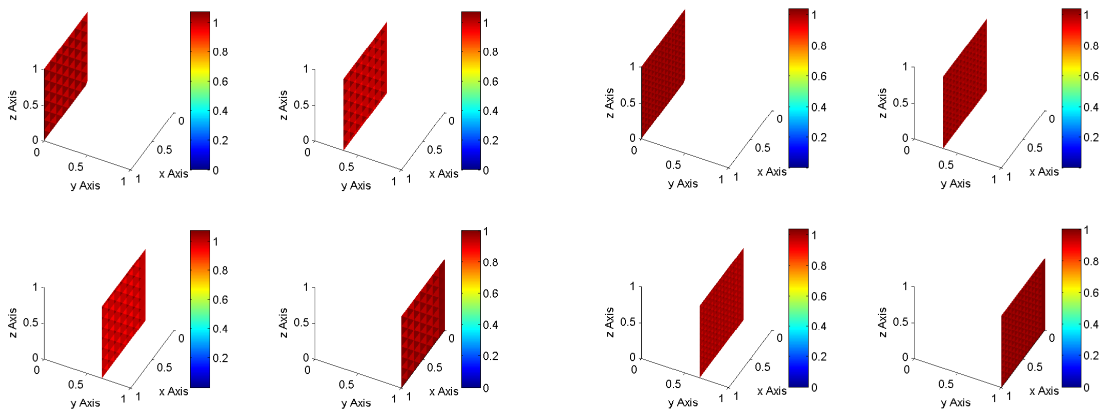


Figure 3.4: Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.6, with $a = 1$, $b = 0$, $\mathcal{P}_0$ and using uniform meshes having $N = 2058$ elements (left) and $N = 13182$ (right).

We consider the following problem with a contact discontinuity

$$u_x + au_z = 0,\ (x, y, z) \in \Omega = [0, 1]^3, \tag{3.4.6a}$$

subject to *inflow* boundary conditions on $\partial\Omega^-$ such that the true solution is given by

$$u(x, y, z) = \begin{cases} e^{x+y-z/a} + 0.25 & \text{if } \ x - z/a < b \\ e^{x+y-z/a} & \text{if } \ x - z/a \geq b \end{cases}. \tag{3.4.6b}$$

The true solution has a contact discontinuity on the plane $x - z/a = b$. If we chose a mesh where all elements are on either side of the discontinuity, the previous theory applies and our error estimates converge to unity under mesh refinement, as shown in Figures 3.4 and 3.5. However, if we allow elements to intersect the discontinuity plane, the error estimator fails to provide an accurate error estimate on elements near the discontinuity which may further pollute neighboring elements with a smooth solution.

First, we consider a special mesh such that every element intersecting the discontinuity plane has one *inflow* face, one *outflow* face and two characteristic faces. Moreover, we assume that the neighbors of each discontinuity element sharing the *inflow* and *outflow* faces are also discontinuity elements. Thus, on this mesh, the elements having a discontinuous solution do not pollute neighboring elements with smooth solutions. We illustrate this by solving (3.4.6) with $a = 1$ and $b = 1/3$ and apply exact *inflow* boundary condition, $U^- = u$, using the modified DG method on uniform meshes having $N = 6n^3 = 16464$ tetrahedral elements and $p = 0,\ 1$. The local effectivity indices of Figure 3.6 show that local and global effectivity indices, as expected, are close to unity on all elements except at the discontinuity.

Next, we solve (3.4.6) with $a = 1/2$ and $b = 0$ on a general unstructured mesh having $N = 17120$ tetrahedral elements with $p = 0,\ 1$ and plot the local effectivity indices in Figure 3.7. These results show that the error estimate, as expected, performs poorly near

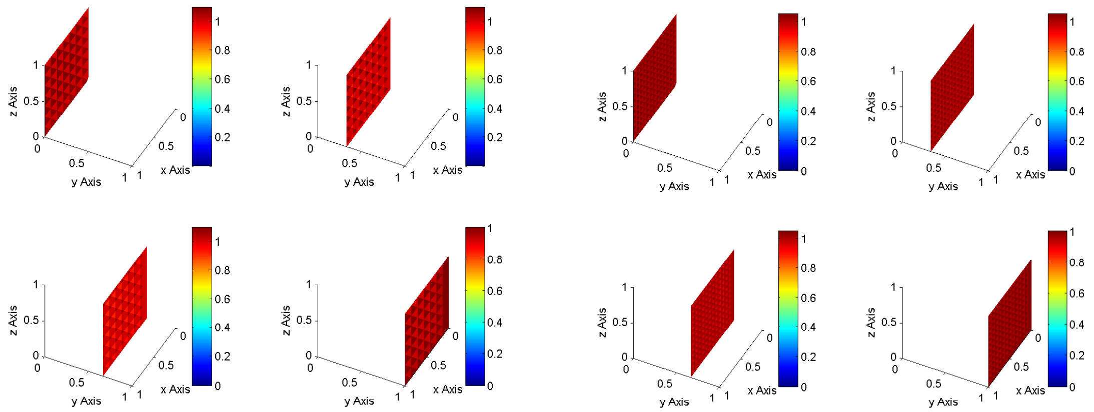


Figure 3.5: Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.6, with $a = 1$, $b = 0$, $\mathcal{P}_1$ and using uniform meshes having $N = 2058$ elements (left) and $N = 13182$ (right).

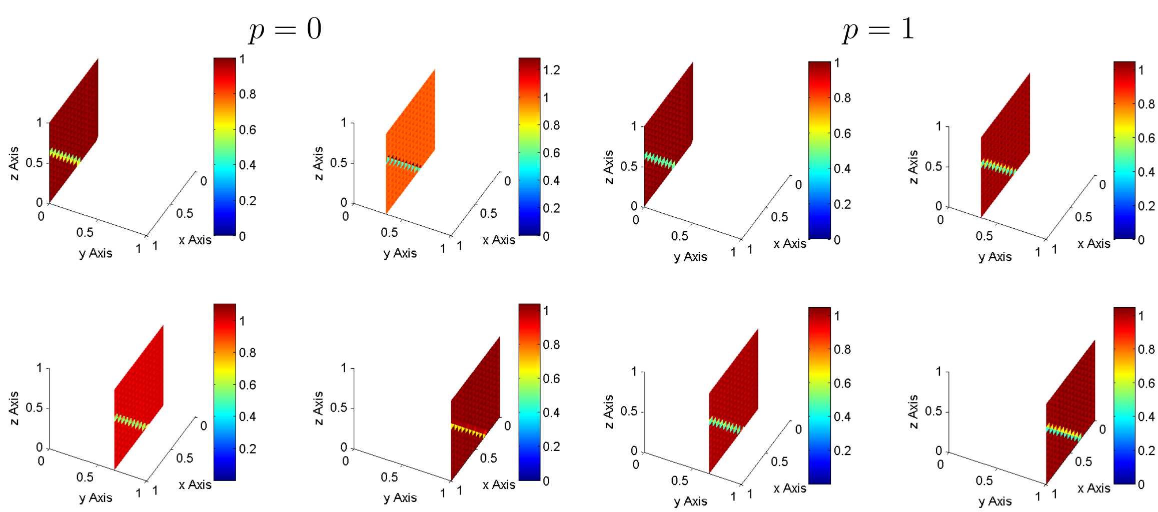


Figure 3.6: Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.6, with $a = 1$, $b = 1/3$ using $\mathcal{P}_0$ (left) and $\mathcal{P}_1$ (right) on a uniform mesh.

the discontinuity and the region polluted by it. These pollution effects can be reduced by local adaptive h-refinement resulting in fine meshes near the discontinuity and/or applying appropriate limiting.

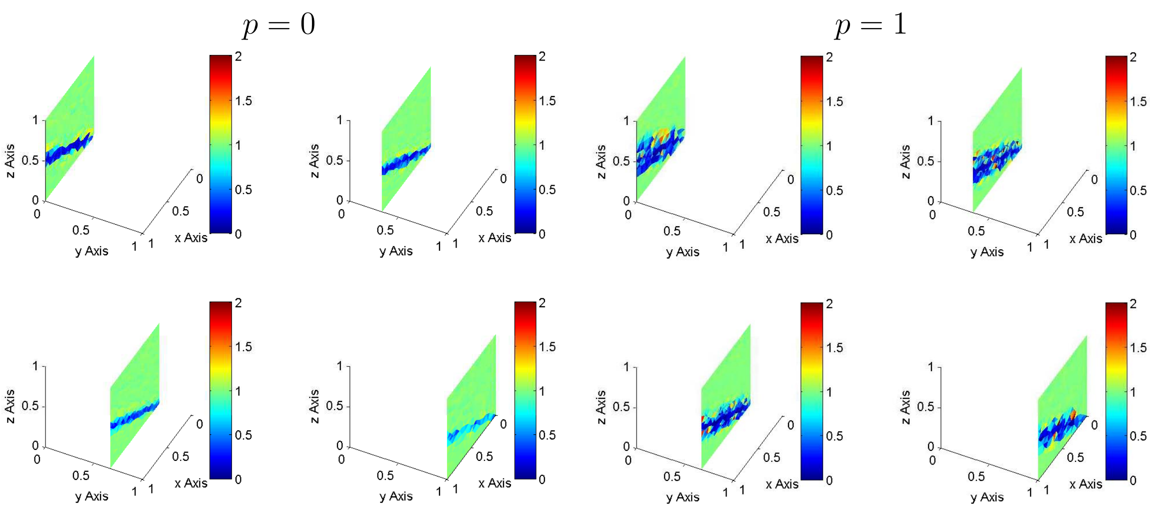


Figure 3.7: Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.6, with $a = 1/2$, $b = 0$ using $\mathcal{P}_0$ (left) and $\mathcal{P}_1$ (right) on a unstructured mesh.

Example 3.4.7.

We consider the following problem with a contact discontinuity

$$u_x = 0,\ (x, y, z) \in \Omega = [0, 1]^3, \tag{3.4.7a}$$

subject to the boundary conditions on the *inflow* boundary Γ^- such that the exact solution is given by

$$u(x, y, z) = \begin{cases} e^{x+y} + 0.25 & \text{if } \ z < \frac{1}{3} \\ e^{x+y} & \text{if } \ z \geq \frac{1}{3} \end{cases}. \tag{3.4.7b}$$

We solve (3.4.7) with the exact *inflow* boundary condition, $U^- = u$, using the modified DG method on uniform tetrahedral meshes for $p = 0, 1, 2, 3$. We present the local effectivity indices in Figures 3.8, 3.9, 3.10 and 3.11.

The exact solution has a contact discontinuity along $z = \frac{1}{3}$. Therefore, the smoothness assumption of Theorem 3.2.2 is violated and as a result we expect our *a posteriori* error estimates to perform poorly near the discontinuity. We observe that the local effectivity indices close to unity in regions where the solution is smooth.

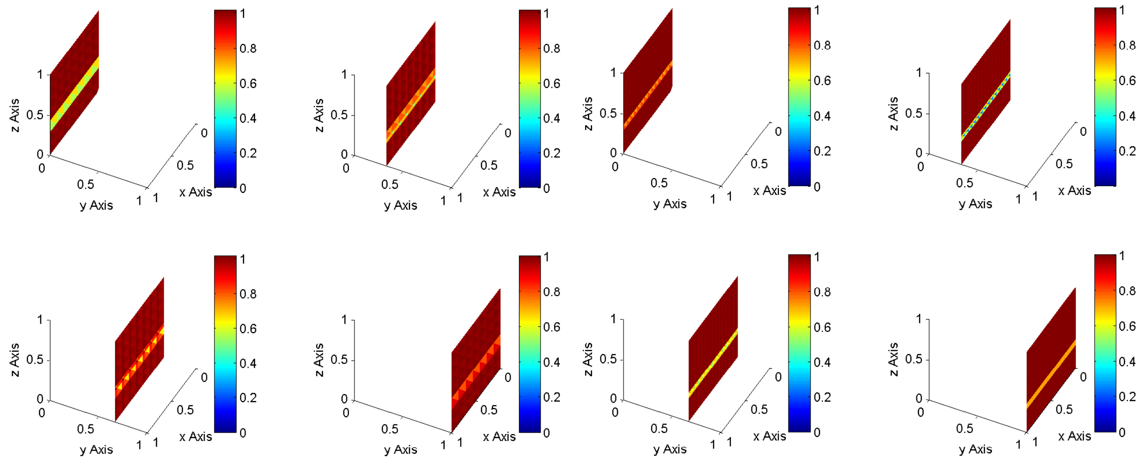


Figure 3.8: Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.7, with $\mathcal{P}_0$ and using uniform meshes having $N = 2058$ elements (left) and $N = 16464$ (right).

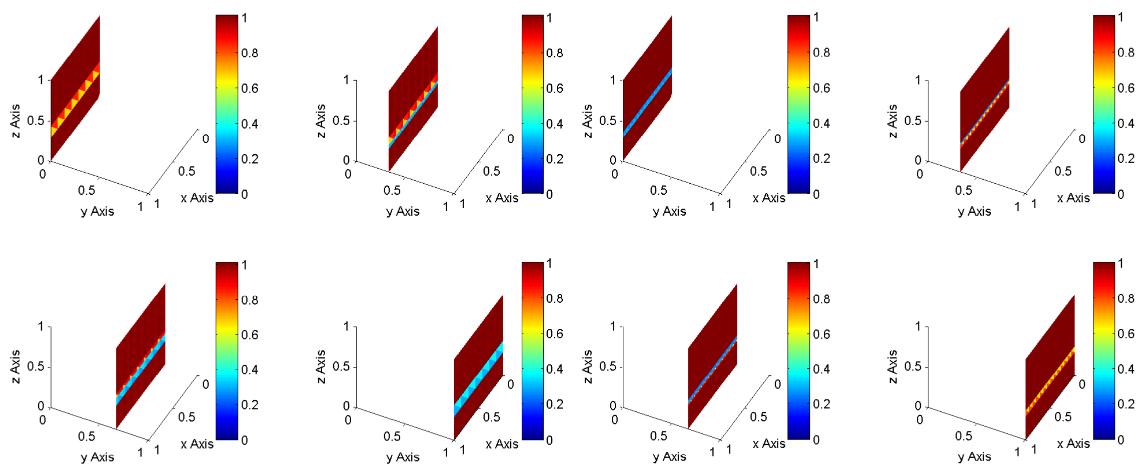


Figure 3.9: Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.7, with $\mathcal{P}_1$ and using uniform meshes having $N = 2058$ elements (left) and $N = 16464$ (right).

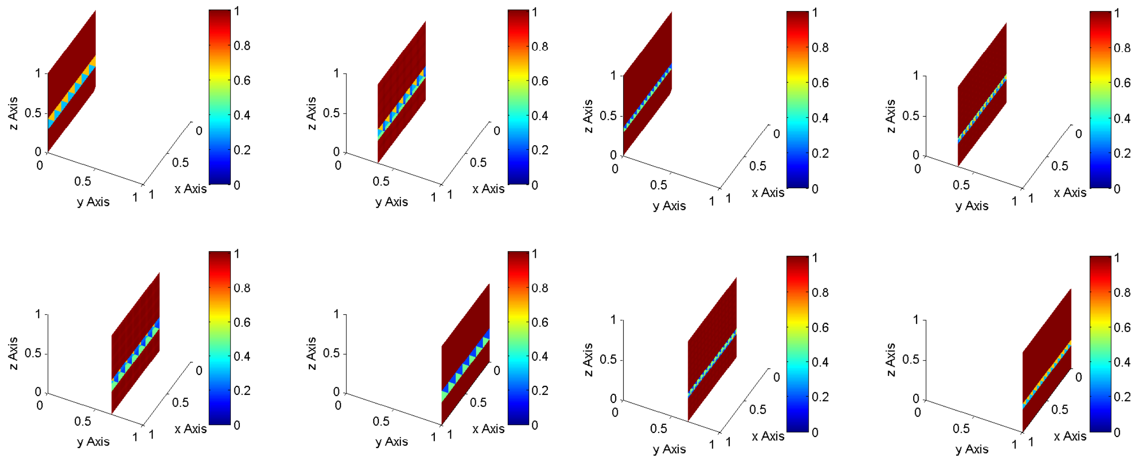


Figure 3.10: Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.7, with $\mathcal{P}_2$ and using uniform meshes having $N = 2058$ elements (left) and $N = 16464$ (right).

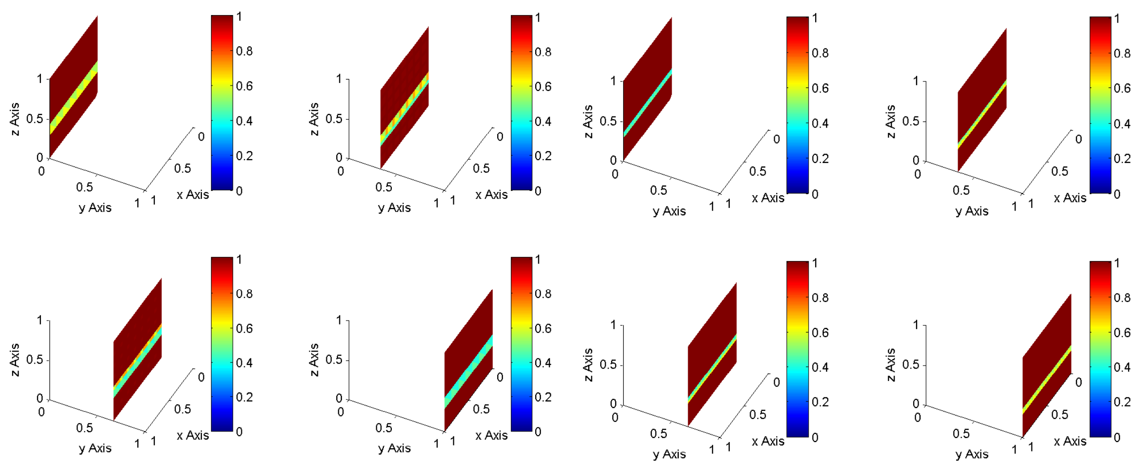


Figure 3.11: Effectivity indices on the planes $y = 0,\ 1/3,\ 2/3,\ 1$ for Example 3.4.7, with $\mathcal{P}_3$ and using uniform meshes having $N = 2058$ elements (left) and $N = 16464$ (right).

3.5 Conclusion

We investigated higher-order discontinuous Galerkin methods for three-dimensional scalar first-order hyperbolic problems on tetrahedral meshes. We constructed simple, efficient and asymptotically correct *a posteriori* error estimates for discontinuous finite element solutions an we explicitly write the basis functions for the error spaces. The leading term of the discretization error on each tetrahedron is estimated by solving a local problem. Our modified DG method and *a posteriori* error estimates are tested on several linear problems to show their efficiency and accuracy under mesh refinement for both smooth and discontinuous solutions.

Chapter 4

Superconvergence Error Analysis

In this chapter, we investigate the pointwise superconvergence properties of the DG method described in section 3.1 using the enriched polynomial spaces $\mathcal{L}_p$ and $\mathcal{U}_p$, defined below. We study the effect of finite element spaces on the superconvergence properties of DG solutions on each class and type of tetrahedral elements. We show that, using the space $\mathcal{L}_p$, the discretization error on tetrahedral elements having one *inflow* and one *outflow* faces, is $O(h^{p+2})$ superconvergent on the *outflow* face. For the space $\mathcal{U}_p$, the discretization error on tetrahedral elements with one *inflow* face, is $O(h^{p+2})$ superconvergent on the three edges of the *inflow* face, while on elements with one *inflow* and one *outflow* faces the DG solution is $O(h^{p+2})$ superconvergent on the *outflow* face in addition to the three edges of the *inflow* face. Furthermore, we show that on tetrahedral elements with two *inflow* faces, the DG solution is $O(h^{p+2})$ superconvergent on the edge shared by two of the *inflow* faces. On elements with two *inflow* and one *outflow* faces and on elements with three *inflow* faces, the DG solution is $O(h^{p+2})$ superconvergent on two edges of the *inflow* faces. Finally, we show that using the enriched space $\mathcal{M}_p$ defined below leads to a simpler *a posteriori* error estimation procedure. In the next section, we define the enriched polynomial spaces $\mathcal{L}_p$, $\mathcal{U}_p$, and $\mathcal{M}_p$, derive the weak DG formulation for the method and present superconvergence analysis.

4.1 Introduction

Before starting our error analysis, let us define the following enriched polynomial spaces on the reference element

$$\mathcal{L}_p = \mathcal{P}_p \cup span\left(\left\{\xi^{p+1-i}\zeta^i, i = 0, 1, \ldots, p+1\right\}\right), \tag{4.1.1a}$$
$$\mathcal{U}_p = \mathcal{P}_p \cup span\left(\left\{\xi^{p+1-i}\eta^i, \xi^{p+1-i}\zeta^i, \eta^{p+1-i}\zeta^i, 0 \le i \le p+1\right\}\right), \tag{4.1.1b}$$
$$\mathcal{M}_p = \mathcal{P}_p \cup span\left(\left\{\eta\left(\xi^{p-i}\eta^{i-j}\zeta^j\right), 0 \le i, j \le p\right\}\right), \tag{4.1.1c}$$

1	1
ξ	ξ
$\eta \quad \zeta$	$\eta \quad \zeta$
ξ^2	ξ^2
$\xi\eta \quad \xi\zeta$	$\xi\eta \quad \xi\zeta$
$\eta^2 \quad \eta\zeta \quad \zeta^2$	$\eta^2 \quad \eta\zeta \quad \zeta^2$
ξ^3	ξ^3
$\xi^2\eta \quad \xi^2\zeta$	$\xi^2\eta \quad \xi^2\zeta$
$\xi\eta^2 \quad \xi\eta\zeta \quad \xi^2\zeta$	$\xi\eta^2 \quad \xi\eta\zeta \quad \xi^2\zeta$
$\eta^3 \quad \eta^2\zeta \quad \eta\zeta^2 \quad \zeta^3$	$\eta^3 \quad \eta^2\zeta \quad \eta\zeta^2 \quad \zeta^3$
ξ^4	ξ^4
$\xi^3\eta \quad \xi^3\zeta$	$\xi^3\eta \quad \xi^3\zeta$
$\xi^2\eta^2 \quad \xi^2\eta\zeta \quad \xi^2\zeta^2$	$\xi^2\eta^2 \quad \xi^2\eta\zeta \quad \xi^2\zeta^2$
$\xi\eta^3 \quad \xi\eta^2\zeta \quad \xi\eta\zeta^2 \quad \xi\zeta^3$	$\xi\eta^3 \quad \xi\eta^2\zeta \quad \xi\eta\zeta^2 \quad \xi\zeta^3$
$\eta^4 \quad \eta^3\zeta \quad \eta^2\zeta^2 \quad \eta\zeta^3 \quad \zeta^4$	$\eta^4 \quad \eta^3\zeta \quad \eta^2\zeta^2 \quad \eta\zeta^3 \quad \zeta^4$

Table 4.1: Monomials in black for $\mathcal{P}_4$ (left) and $\mathcal{L}_3$ (right).

where $\mathcal{P}_p$ is the complete space of polynomials of degree not exceeding p given by (2.3.1). The finite element spaces $\mathcal{L}_p$, $\mathcal{U}_p$ and $\mathcal{M}_p$ are shown in Tables 4.1, 4.2, 4.3, and have the following dimensions

$$\dim(\mathcal{L}_p) = \dim(\mathcal{P}_p) + (p+2), \tag{4.1.2}$$

$$\dim(\mathcal{U}_p) = \dim(\mathcal{P}_p) + 3(p+1), \tag{4.1.3}$$

$$\dim(\mathcal{M}_p) = \dim(\mathcal{P}_p) + \frac{(p+1)(p+2)}{2}. \tag{4.1.4}$$

In order to obtain the weak discontinuous Galerkin formulation for the linear first-order hyperbolic scalar problem (3.1.1), we partition the domain Ω into a regular mesh having N tetrahedral elements Δ_j, $j = 1, 2, \ldots, N$, and assume, for simplicity, that this can be done without error. Assume the mesh is such that each face is either, *inflow*, *outflow* or characteristic, as discussed in chapter 3.

We multiply (3.1.1a) by a test function v, integrate over an arbitrary element Δ, and apply Stokes' theorem to write

$$\iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}uvd\sigma + \iint_{\Gamma^+} \boldsymbol{a}\cdot\mathbf{n}uvd\sigma + \iiint_{\Delta} (-\boldsymbol{a}\cdot\nabla v + cv)\, udxdydz = \iiint_{\Delta} fvdxdydz. \tag{4.1.5}$$

Next we approximate $u(x, y, z)$ by a piecewise polynomial function $U(x, y, z)$ whose restriction to Δ is either in $\mathcal{L}_p$, $\mathcal{U}_p$, or $\mathcal{M}_p$ defined in (4.1.1).

1	1
ξ	ξ
$\eta \quad \zeta$	$\eta \quad \zeta$
ξ^2	ξ^2
$\xi\eta \quad \xi\zeta$	$\xi\eta \quad \xi\zeta$
$\eta^2 \quad \eta\zeta \quad \zeta^2$	$\eta^2 \quad \eta\zeta \quad \zeta^2$
ξ^3	ξ^3
$\xi^2\eta \quad \xi^2\zeta$	$\xi^2\eta \quad \xi^2\zeta$
$\xi\eta^2 \quad \xi\eta\zeta \quad \xi^2\zeta$	$\xi\eta^2 \quad \xi\eta\zeta \quad \xi^2\zeta$
$\eta^3 \quad \eta^2\zeta \quad \eta\zeta^2 \quad \zeta^3$	$\eta^3 \quad \eta^2\zeta \quad \eta\zeta^2 \quad \zeta^3$
ξ^4	ξ^4
$\xi^3\eta \quad \xi^3\zeta$	$\xi^3\eta \quad \xi^3\zeta$
$\xi^2\eta^2 \quad \xi^2\eta\zeta \quad \xi^2\zeta^2$	$\xi^2\eta^2 \quad \xi^2\eta\zeta \quad \xi^2\zeta^2$
$\xi\eta^3 \quad \xi\eta^2\zeta \quad \xi\eta\zeta^2 \quad \xi\zeta^3$	$\xi\eta^3 \quad \xi\eta^2\zeta \quad \xi\eta\zeta^2 \quad \xi\zeta^3$
$\eta^4 \quad \eta^3\zeta \quad \eta^2\zeta^2 \quad \eta\zeta^3 \quad \zeta^4$	$\eta^4 \quad \eta^3\zeta \quad \eta^2\zeta^2 \quad \eta\zeta^3 \quad \zeta^4$

Table 4.2: Monomials in black for $\mathcal{P}_4$ (left) and $\mathcal{U}_3$ (right).

1	1
ξ	ξ
$\eta \quad \zeta$	$\eta \quad \zeta$
ξ^2	ξ^2
$\xi\eta \quad \xi\zeta$	$\xi\eta \quad \xi\zeta$
$\eta^2 \quad \eta\zeta \quad \zeta^2$	$\eta^2 \quad \eta\zeta \quad \zeta^2$
ξ^3	ξ^3
$\xi^2\eta \quad \xi^2\zeta$	$\xi^2\eta \quad \xi^2\zeta$
$\xi\eta^2 \quad \xi\eta\zeta \quad \xi^2\zeta$	$\xi\eta^2 \quad \xi\eta\zeta \quad \xi^2\zeta$
$\eta^3 \quad \eta^2\zeta \quad \eta\zeta^2 \quad \zeta^3$	$\eta^3 \quad \eta^2\zeta \quad \eta\zeta^2 \quad \zeta^3$
ξ^4	ξ^4
$\xi^3\eta \quad \xi^3\zeta$	$\xi^3\eta \quad \xi^3\zeta$
$\xi^2\eta^2 \quad \xi^2\eta\zeta \quad \xi^2\zeta^2$	$\xi^2\eta^2 \quad \xi^2\eta\zeta \quad \xi^2\zeta^2$
$\xi\eta^3 \quad \xi\eta^2\zeta \quad \xi\eta\zeta^2 \quad \xi\zeta^3$	$\xi\eta^3 \quad \xi\eta^2\zeta \quad \xi\eta\zeta^2 \quad \xi\zeta^3$
$\eta^4 \quad \eta^3\zeta \quad \eta^2\zeta^2 \quad \eta\zeta^3 \quad \zeta^4$	$\eta^4 \quad \eta^3\zeta \quad \eta^2\zeta^2 \quad \eta\zeta^3 \quad \zeta^4$

Table 4.3: Monomials in black for $\mathcal{P}_4$ (left) and $\mathcal{M}_3$ (right).

From here on, we use $\mathcal{W}_p$ to denote either $\mathcal{L}_p$, $\mathcal{U}_p$, or $\mathcal{M}_p$ and let $S^{N,p}$ denote the space of piecewise polynomial functions

$$S^{N,p} = \{U, U|_\Delta \in \mathcal{W}_p\}.$$

The discrete DG formulation which consists of determining $U \in S^{N,p}$ such that

$$\begin{aligned}
&\iint_{\Gamma^-} \boldsymbol{a} \cdot \mathbf{n}\tilde{U} V d\sigma + \iint_{\Gamma^+} \boldsymbol{a} \cdot \mathbf{n} U V d\sigma - \iiint_\Delta (\boldsymbol{a} \cdot \nabla V - cV)\, U dxdydz \\
= \;& \iiint_\Delta fV dxdydz,\ \forall\ V \in \mathcal{W}_p. \qquad (4.1.6)
\end{aligned}$$

In order to complete the definition of our DG method we need to select the corrected upwind numerical flux $\tilde{U}$ on Γ^- given by (3.1.5), where the error will be discussed later in section 4.5. Subtracting (4.1.6) from (4.1.5) with $v = V$ leads to the DG orthogonality condition for the local error $\epsilon = u - U$ for all $V \in \mathcal{W}_p$,

$$\iint_{\Gamma^-} \boldsymbol{a} \cdot \mathbf{n}\epsilon^- V d\sigma + \iint_{\Gamma^+} \boldsymbol{a} \cdot \mathbf{n}\epsilon V d\sigma + \iiint_\Delta (-\boldsymbol{a} \cdot \nabla V + cV)\, \epsilon dxdydz = 0. \qquad (4.1.7)$$

We map a physical tetrahedron Δ having vertices $\mathbf{v}_i = (x_i, y_i, z_i)^T$, $1 \le i \le 4$, into the reference tetrahedron $\hat{\Delta}$ with vertices $\hat{\mathbf{v}}_1 = (0,0,0)$, $\hat{\mathbf{v}}_2 = (1,0,0)$, $\hat{\mathbf{v}}_3 = (0,1,0)$, $\hat{\mathbf{v}}_4 = (0,0,1)$, by the standard affine mapping (3.1.7).

Thus, after applying the affine mapping (3.1.7) with $\hat{\epsilon}\,(\xi,\eta,\zeta,h)$ and $\hat{V}\,(\xi,\eta,\zeta)$, the DG orthogonality (4.1.7) becomes

$$\begin{aligned}
&\iint_{\hat{\Gamma}^-} (\mathbf{J}\hat{\boldsymbol{a}}) \cdot \hat{\mathbf{n}}\hat{\epsilon}^- \hat{V} d\hat{\sigma} + \iint_{\hat{\Gamma}^+} (\mathbf{J}\hat{\boldsymbol{a}}) \cdot \hat{\mathbf{n}}\hat{\epsilon}\hat{V} d\hat{\sigma} \\
- &\iiint_{\hat{\Delta}} \left((\mathbf{J}\hat{\boldsymbol{a}}) \cdot \nabla \hat{V} - h\hat{c}\hat{V} \right) \hat{\epsilon} d\xi d\eta d\zeta = 0,\ \forall \hat{V} \in \mathcal{W}_p, \qquad (4.1.8)
\end{aligned}$$

where $\hat{\Gamma}^-$ and $\hat{\Gamma}^+$, respectively, denote the *inflow* ($\mathbf{J}\hat{\boldsymbol{a}} \cdot \hat{\mathbf{n}} < \mathbf{0}$) and *outflow* ($\mathbf{J}\hat{\boldsymbol{a}} \cdot \hat{\mathbf{n}} > \mathbf{0}$) boundaries of $\hat{\Delta}$ with respect to the vector $\mathbf{J}\hat{\boldsymbol{a}}$ and unit normal $\hat{\mathbf{n}}$ on $\hat{\Delta}$. Here $\mathbf{J}$ denotes the Jacobian matrix of the affine mapping (3.1.7) and

$$\begin{aligned}
\hat{\boldsymbol{a}}(\xi,\eta,\zeta,h) &= \boldsymbol{a}(x(\xi,\eta,\zeta,h), y(\xi,\eta,\zeta,h), z(\xi,\eta,\zeta,h)), \\
\hat{c}(\xi,\eta,\zeta,h) &= c(x(\xi,\eta,\zeta,h), y(\xi,\eta,\zeta,h), z(\xi,\eta,\zeta,h)), \\
\hat{\epsilon}(\xi,\eta,\zeta,h) &= \epsilon(x(\xi,\eta,\zeta,h), y(\xi,\eta,\zeta,h), z(\xi,\eta,\zeta,h), h),
\end{aligned}$$

where $\hat{\epsilon}(\xi,\eta,\zeta,h)$ depends explicitly on h since

$$\begin{aligned}
\hat{\epsilon}(\xi,\eta,\zeta,h) = \;& u(x(\xi,\eta,\zeta,h), y(\xi,\eta,\zeta,h), z(\xi,\eta,\zeta,h)) \\
&- U(x(\xi,\eta,\zeta,h), y(\xi,\eta,\zeta,h), z(\xi,\eta,\zeta,h), h),
\end{aligned}$$

and as discussed in section 3.1, this dependance is smooth such that $\hat{\epsilon}$ has derivatives at $h = 0$.

Therefore, the DG orthogonality condition (4.1.8) becomes

$$\iint_{\hat{\Gamma}^-} \tilde{\boldsymbol{a}} \cdot \hat{\mathbf{n}} \hat{\epsilon}^- \hat{V} d\sigma + \iint_{\hat{\Gamma}^+} \tilde{\boldsymbol{a}} \cdot \hat{\mathbf{n}} \hat{\epsilon} \hat{V} d\sigma - \iiint_{\hat{\Delta}} \left(\tilde{\boldsymbol{a}} \cdot \nabla \hat{V} - h \hat{c} \hat{V} \right) \hat{\epsilon} d\xi d\eta d\zeta \;\; = \;\; 0, \quad (4.1.9)$$

for all $\hat{V} \in \mathcal{W}_p$, where $\tilde{\boldsymbol{a}} = \mathbf{J}_0 \hat{\boldsymbol{a}}$, $\mathbf{J}_0$ is independent of h as in (3.1.8).

In the remainder of this chapter we will omit the $\hat{}$ unless needed for clarity. In our error analysis we need Taylor series of the analytic function $\mathbf{J}_0 \boldsymbol{a}(\xi, \eta, \zeta, h)$ about the center of the element Δ. Applying Taylor's theorem we expand $\mathbf{J}_0 \boldsymbol{a}$ as

$$\tilde{\boldsymbol{a}}(\xi, \eta, \zeta, h) = \boldsymbol{a}_0 + \sum_{k=1}^{\infty} h^k \boldsymbol{a}_k(\xi, \eta, \zeta), \quad (4.1.10)$$

where $\boldsymbol{a}_k \in [\mathcal{P}_k]^3$ are obtained by the chain rule as

$$\boldsymbol{a}_k \left(\xi, \eta, \zeta\right) \;\; = \;\; \frac{1}{k!} \mathbf{J}_0 \left. \frac{d^k \boldsymbol{a} \left(x \left(\xi, \eta, \zeta, h\right), y \left(\xi, \eta, \zeta, h\right), z \left(\xi, \eta, \zeta, h\right)\right)}{dh^k} \right|_{h=0},$$

and

$$\boldsymbol{a}_0 = [\alpha, \beta, \gamma]^T = \mathbf{J}_0 \boldsymbol{a} \left(1/4, 1/4, 1/4\right). \quad (4.1.11)$$

Similarly Maclaurin series for c can be written as

$$c(\xi, \eta, \zeta, h) = \sum_{k=0}^{\infty} h^k C_k(\xi, \eta, \zeta), \; C_k(\xi, \eta, \zeta) \in \mathcal{P}_k. \quad (4.1.12)$$

Finally, we write a Maclaurin series of ϵ with respect to h as

$$\epsilon(\xi, \eta, \zeta, h) = \sum_{k=0}^{\infty} Q_k(\xi, \eta, \zeta) h^k, \quad (4.1.13)$$

where $Q_k(\xi, \eta, \zeta) \in \mathcal{P}_k$.

In the next section we will state few preliminary results that will be needed in the superconvergence error analysis.

4.2 Preliminary Results

For the sake of the local error analysis we solve a problem on one element of size h with $\tilde{U}|_{\Gamma^-} = u$, *i.e.*, $\epsilon^- = 0$ while in practice we only need $\epsilon^- = \mathcal{O}\left(h^{p+2}\right)$. We start by stating and proving the following lemma.

Lemma 4.2.1. *Let $\boldsymbol{a}_0 \neq 0$ given in (4.1.11), Δ be the reference tetrahedron, and $\mathbf{n}$ be its unit normal vector. If $Q_k \in \mathcal{P}_k$, $k = 0, 1, \ldots, p$ satisfies*

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_k V d\sigma + \iiint_{\Delta} (-\boldsymbol{a}_0 \cdot \nabla V) Q_k d\xi d\eta d\zeta = 0, \forall V \in \mathcal{W}_p, \tag{4.2.1}$$

then

$$Q_k = 0,\ 0 \leq k \leq p. \tag{4.2.2}$$

Proof. Since $\mathcal{W}_p$ is either $\mathcal{L}_p$, $\mathcal{U}_p$, or $\mathcal{M}_p$, *i.e.* it contains $\mathcal{P}_p$, the proof follows the same line of reasoning as for Lemma 3.2.1. □

In our analysis we need two sets of orthogonal polynomials in $\mathcal{L}^2$ with respect to the weight functions $w = \xi - (1 - \eta - \zeta)$ and $w = \eta$ on the reference tetrahedron.

First, the following lemma gives a set of orthogonal polynomials in $\mathcal{L}^2$ with respect to the weight function $w = \xi - (1 - \eta - \zeta)$.

Lemma 4.2.2. *If $P_k^{s,t}$ denote Jacobi polynomials (2.3), then the polynomials*

$$\tilde{q}_{ij}^m = (1 - \tfrac{\eta}{(1-\zeta)})^m (1-\zeta)^{m+i} P_m^{1,0}(\tfrac{2\xi}{(1-\eta-\zeta)} - 1) P_i^{2m+2,0}(\tfrac{2\eta}{(1-\zeta)} - 1) P_j^{2m+2i+3,0}(2\zeta - 1), \tag{4.2.3}$$

for $0 \leq m, i, j, m+i+j \leq p$, satisfy the orthogonality condition on the reference tetrahedron Δ

$$\iiint_{\Delta} (\xi - (1 - \eta - \zeta))\, \tilde{q}_{ij}^m \tilde{q}_{kl}^n d\xi d\eta d\zeta = c_{ij,kl}^{mn} \delta_{ik} \delta_{jl} \delta_{mn}. \tag{4.2.4}$$

Thus, $\left\{\tilde{q}_{ij}^m, 0 \leq m, i, j, m+i+j \leq p\right\}$ is a basis for $\mathcal{P}_p$.

Proof. Applying the change of variables

$$\hat{\xi} = \frac{2\xi}{1 - \eta - \zeta} - 1,\ \hat{\eta} = \frac{2\eta}{1 - \zeta} - 1,\ \hat{\zeta} = 2\zeta - 1, \tag{4.2.5}$$

the polynomial $\tilde{q}_{ij}^m$ can be written as

$$\tilde{q}_{ij}^m (\xi, \eta, \zeta) \quad = \quad \left(\frac{1-\hat{\eta}}{2}\right)^m \left(\frac{1-\hat{\zeta}}{2}\right)^{m+i} P_m^{1,0}\left(\hat{\xi}\right) P_i^{2m+2,0}(\hat{\eta}) P_j^{2m+2i+3,0}\left(\hat{\zeta}\right).$$

Using the Jacobian of the mapping (4.2.5)

$$\frac{\partial (\xi, \eta, \zeta)}{\partial \left(\hat{\xi}, \hat{\eta}, \hat{\zeta}\right)} = \frac{1}{8} \left(\frac{1-\hat{\eta}}{2}\right) \left(\frac{1-\hat{\zeta}}{2}\right)^2,$$

and noting that

$$\xi - (1-\eta-\zeta) \quad = \quad -\frac{\left(1-\hat{\xi}\right)}{2}\frac{\left(1-\hat{\eta}\right)}{2}\frac{\left(1-\hat{\zeta}\right)}{2},$$

one can write

$$\int\int\int_{\Delta}\left(\xi-(1-\eta-\zeta)\right)\tilde{q}_{ij}^{m}\tilde{q}_{kl}^{n}d\xi d\eta d\zeta = -I_1 I_2 I_3,$$

where

$$\begin{aligned}
I_1 &= \int_{-1}^{1}\tfrac{(1-\hat{\xi})}{2}P_m^{1,0}\left(\hat{\xi}\right)P_n^{1,0}\left(\hat{\xi}\right)d\hat{\xi},\\
I_2 &= \int_{-1}^{1}\left(\tfrac{1-\hat{\eta}}{2}\right)^{m+n+2}P_k^{2m+2,0}\left(\hat{\eta}\right)P_i^{2n+2,0}\left(\hat{\eta}\right)d\hat{\eta},\\
I_3 &= \int_{-1}^{1}\left(\tfrac{1-\hat{\zeta}}{2}\right)^{m+n+i+k+3}P_j^{2m+2i+3,0}\left(\hat{\zeta}\right)P_l^{2n+2k+3,0}\left(\hat{\zeta}\right)d\hat{\zeta}.
\end{aligned}$$

The integral I_1 is zero if $m \neq n$. When $m = n$, I_2 becomes the orthogonality relation for the Jacobi polynomials $P_k^{2m+2,0}\left(\hat{\eta}\right)$ which yields $I_2 = 0$ if $k \neq i$. Finally, I_3 is zero if $j \neq l$, $m = n$ and $k = l$ by the orthogonality of Jacobi polynomials $P_j^{2m+2i+3,0}\left(\hat{\zeta}\right)$ and $P_l^{2m+2i+3,0}\left(\hat{\zeta}\right)$. $\square$

The following lemma gives a set of orthogonal polynomials in $\mathcal{L}^2$ with respect to the weight function $w = \eta$ on the reference tetrahedron.

Lemma 4.2.3. *If $P_k^{s,t}$ denote Jacobi polynomials (2.3), then the polynomials*

$$q_{ij}^{m} = P_m^{1,0}\left(\tfrac{2\xi}{\xi+\eta}-1\right)\left(\tfrac{\xi+\eta}{(1-\zeta)}\right)^m P_i^{2m+2,0}\left(1-\tfrac{2(\xi+\eta)}{(1-\zeta)}\right)(1-\zeta)^{m+i}P_j^{2m+2i+3,0}\left(2\zeta-1\right), \quad (4.2.6)$$

for $0 \leq m, i, j, m+i+j \leq p$, satisfy the orthogonality condition on the reference tetrahedron Δ

$$\int\int\int_{\Delta}\eta q_{ij}^{m}q_{kl}^{n}d\xi d\eta d\zeta = c_{ij,kl}^{mn}\delta_{ik}\delta_{jl}\delta_{mn}. \quad (4.2.7)$$

Thus, $\left\{q_{ij}^{m}, 0 \leq m, i, j, m+i+j \leq p\right\}$ is a basis for $\mathcal{P}_p$.

Proof. The transformation

$$\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix} \longmapsto \begin{pmatrix}\xi\\ w = 1-\xi-\eta-\zeta\\ \zeta\end{pmatrix},$$

maps the reference tetrahedron Δ into itself and the polynomials $\tilde{q}_{i,j}^{m}$ to $q_{i,j}^{m}$ which yields

$$\iiint_{\Delta}\left(\xi-(1-\eta-\zeta)\right)\tilde{q}_{ij}^{m}\tilde{q}_{kl}^{n}d\xi d\eta d\zeta = \iiint_{\Delta}wq_{ij}^{m}q_{kl}^{n}dwd\xi d\zeta.$$

Applying Lemma 4.2.2 completes the proof. $\square$

In the next section we will investigate the local DG error analysis and superconvergence points on elements of *Class I, II, III*, and *Type 1, 2, 3* using the enriched polynomial spaces $\mathcal{L}_p$, $\mathcal{U}_p$, and $\mathcal{M}_p$.

4.3 DG Superconvergence Error Analysis

Now we are ready to state our superconvergence results for the local discretization error using the enriched polynomial spaces $\mathcal{L}_p$ and $\mathcal{U}_p$.
Since $\mathcal{L}_p$ and $\mathcal{U}_p$ contain $\mathcal{P}_p$, the results of Theorems 3.2.1, 3.2.2 and Corollary 3.2.1 still hold. Moreover, the DG solution in the enriched spaces satisfies additional orthogonality conditions that will be used to prove superconvergence. Guided by the work of Adjerid and Baccouch [2] for triangular meshes, we will show several pointwise superconvergence results. For instance, using the space $\mathcal{L}_p$, we show in the following theorem that the leading term Q_{p+1} of the local finite element error is $O(h^{p+2})$ superconvergent on the *outflow* face of elements having one *inflow* and one *outflow* faces.

Theorem 4.3.1. *Let $u \in C^{\infty}(\Omega)$ and $U \in \mathcal{L}_p$, respectively, be the solutions of (3.1.1a), (3.1.1c) and (4.1.6), with $U^-|_{\Gamma^-} = u$. Let Δ be a tetrahedron of Class I and Type 1 (i.e. one inflow and one outflow faces), then the local finite element error can be written as*

$$\epsilon(\xi,\eta,\zeta,h) = \sum_{k=p+1}^{\infty} h^k Q_k(\xi,\eta,\zeta), \tag{4.3.1}$$

and its leading term Q_{p+1} satisfies,

$$\iiint_{\Delta} Q_{p+1} V d\xi d\eta d\zeta \quad = \quad 0,\ \forall\ V \in \mathcal{P}_{p-1}, \tag{4.3.2}$$

and

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+1} V d\sigma \quad = \quad 0,\ \forall\ V \in \mathcal{L}_p. \tag{4.3.3}$$

Furthermore,

$$Q_{p+1}(\xi,0,\zeta) \quad = \quad 0, \text{ on the outflow face}, \tag{4.3.4a}$$

$$Q_{p+1}(\xi,\eta,\zeta) \quad = \quad \eta \sum_{i=0}^{p}\sum_{j=0}^{i} c_{i,j}^{p} q_{i-j,j}^{p-i}(\xi,\eta,\zeta), \text{ on } \Delta, \tag{4.3.4b}$$

where $q_{j,k}^i$ are defined in Lemma 4.2.3.

Proof. Substituting (4.1.10), (4.1.12) and (4.1.13) in (4.1.9) and collecting terms having the same power of h we obtain the following series

$$\sum_{k=0}^{\infty} Z_k h^k = 0,\ \forall\ V \in \mathcal{L}_p, \tag{4.3.5}$$

where

$$Z_0 = \iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_0 V d\sigma - \iiint_{\triangle} \boldsymbol{a}_0 \cdot \nabla V Q_0 d\xi\ d\eta d\zeta, \qquad V \in \mathcal{L}_p,$$

and

$$Z_k = \iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_k V d\sigma - \iiint_{\triangle} [\boldsymbol{a}_0 \cdot \nabla V Q_k - C_0 Q_{k-1} V] d\xi\ d\eta d\zeta, \qquad V \in \mathcal{L}_p,\ k > 0.$$

Thus $Z_k = 0,\ \forall\ V \in \mathcal{L}_p$. By Lemma 4.2.1, $Q_k = 0,\ k = 0, 1, \ldots, p$ which proves (4.3.1).
The leading term Q_{p+1} satisfies

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+1}(\xi, 0, \zeta) V(\xi, 0, \zeta) d\xi d\zeta - \iiint_{\Delta} (\boldsymbol{a}_0 \cdot \nabla V) Q_{p+1} d\xi d\eta d\zeta = 0, \forall\, V \in \mathcal{L}_p. \tag{4.3.6}$$

Since $\mathcal{P}_p \subset \mathcal{L}_p$, (3.2.12) holds which in turn yields (4.3.2). Similarly, (3.2.13) holds which implies (4.3.3) for all $V \in \mathcal{P}_p$. Thus, it remains to show (4.3.3) for all $V \in \mathcal{L}_p \backslash \mathcal{P}_p$ to establish (4.3.3).
Testing against $V(\xi, \eta, \zeta) = \xi^{p+1-j}\zeta^j$, $j = 0, 1, \ldots, p+1$ in (4.3.6), and noting that

$$\boldsymbol{a}_0 \cdot \nabla V = \beta \frac{\partial}{\partial \eta}\left(\xi^{p+1-j}\zeta^j\right) = 0,\ \text{for} j = 0, 1, \ldots, p+1,$$

yields (4.3.3).
Next, we write Q_{p+1} as

$$Q_{p+1}(\xi, \eta, \zeta) = r_{p+1}(\xi, \zeta) + \eta \tilde{r}_p(\xi, \eta, \zeta), \tag{4.3.7}$$

where

$$r_{p+1}(\xi, \zeta) = Q_{p+1}(\xi, 0, \zeta),\ \textit{and}\ \tilde{r}_p \in \mathcal{P}_p.$$

Substituting (4.3.7) in (4.3.3), yields

$$\beta \iint_{\Gamma^+} r_{p+1}(\xi, \zeta) V(\xi, 0, \zeta) d\xi d\zeta = 0, \forall\ V \in \mathcal{L}_p, \tag{4.3.8}$$

which establishes (4.3.4a) and (4.3.4b), since restriction of $\mathcal{L}_p$ to the face $\eta = 0$ is the space of polynomials of degree not exceeding $(p+1)$ in ξ and ζ. □

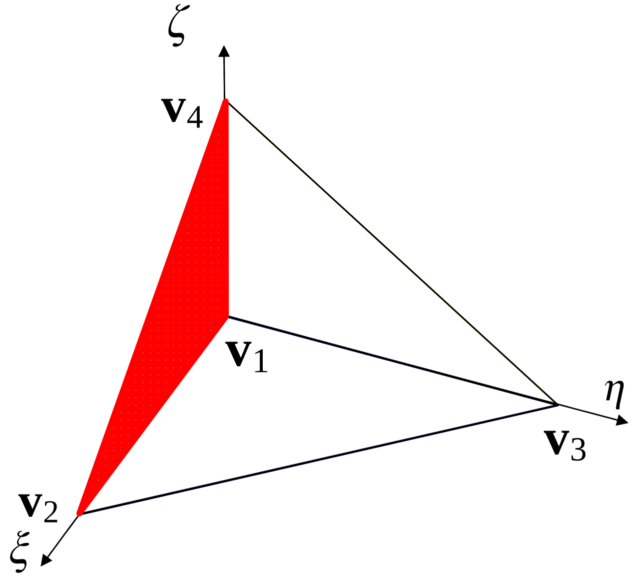


Figure 4.1: Superconvergence points (red) using the space $\mathcal{L}_p$ on elements of *Class I* and *Type 1*.

In Figure 4.1, we illustrate the superconvergence result stated in Theorem 4.3.1.

Noting that, the enriched polynomial space $\mathcal{L}_p$ can be expressed in terms of the two-dimensional orthogonal basis functions $\varphi_j^i(\xi,\zeta)$ (2.3.3) as

$$\mathcal{L}_p = \left\{ q \middle| \, q = \sum_{k=0}^{p}\sum_{i=0}^{k}\sum_{j=0}^{i} c_{i,j}^{k}\eta^{k-i}\varphi_j^{i-j}(\xi,\zeta) \right\} \cup span\left(\left\{\varphi_i^{p+1-i}(\xi,\zeta), i=0,1,\ldots,p+1\right\}\right), \tag{4.3.9}$$

where $\mathcal{L}_p|_{\eta=0} = span\left\{\varphi_j^{i-j}(\xi,\zeta), 0\le j\le i\le p+1\right\}$ is the space of $(p+1)$-degree polynomials in ξ and ζ.
Thus, using an enriched polynomial space $\tilde{\mathcal{W}}_p$ for the DG solution on elements of *Class I* and *Type 1* such that $\mathcal{P}_p \subset \tilde{\mathcal{W}}_p \subset \mathcal{L}_p$, leads to the following superconvergence results.

Corollary 4.3.1. *Let $u\in C^\infty(\Omega)$ and $U\in\tilde{\mathcal{W}}_p$, respectively, be the solutions of (3.1.1a), (3.1.1c) and (4.1.6), with $U^-|_{\Gamma^-}=u$. Let Δ be a tetrahedron of Class I and Type 1 (i.e. one inflow and one outflow faces), then the following results hold:*

(i) If $\tilde{\mathcal{W}}_p = \mathcal{P}_p \cup span\left\{\varphi_j^{p+1-j}(\xi,\zeta), j=0,1,\ldots,i-1,i+1,\ldots,p+1\right\}$ for $i=0,1,\ldots,p+1$, then the leading term Q_{p+1} on the outflow face can be written as

$$Q_{p+1}(\xi,0,\zeta) = C\varphi_i^{p+1-i}(\xi,\zeta), \tag{4.3.10}$$

and the DG error is $O(h^{p+2})$ superconvergent at the zeros of φ_i^{p+1-i}.

(ii) If $\tilde{\mathcal{W}}_p = \mathcal{P}_p \cup span\left\{\varphi_k^{p+1-k}(\xi,\zeta), k=0,1,\ldots,i-1,i+1,\ldots,j-1,j+1,\ldots,p+1\right\}$

for $0 \leq i < j \leq p+1$, then the leading term Q_{p+1} on the outflow face can be written as

$$Q_{p+1}(\xi, 0, \zeta) = C_1 \varphi_i^{p+1-i}(\xi, \zeta) + C_2 \varphi_j^{p+1-j}(\xi, \zeta), \tag{4.3.11}$$

and the DG error is $O\left(h^{p+2}\right)$ superconvergent at the intersection of the zeros of φ_i^{p+1-i} and φ_j^{p+1-j}.

(iii) If $\tilde{\mathcal{W}}_p = \mathcal{P}_p \cup span\left\{\varphi_k^{p+1-k}(\xi, \zeta), k = 2i, 0 \leq i \leq floor(\frac{p+1}{2})\right\}$, then the leading term Q_{p+1} on the outflow face can be written as

$$Q_{p+1}(\xi, 0, \zeta) = \sum_{i=0}^{floor(\frac{p+1}{2})} C_k \varphi_{2i+1}^{p-2i}(\xi, \zeta), \tag{4.3.12}$$

and the DG error is $O\left(h^{p+2}\right)$ superconvergent on the edge $\zeta = 1 - 2\xi$.

Proof. The proof of *(i)* and *(ii)* follows the same line of reasoning as for Theorem 4.3.1. For assertion *(iii),* we note that on the line $\zeta = 1 - 2\xi$ the Legendre polynomials

$$L_i\left(\frac{2\xi}{1-\zeta} - 1\right) = P_{2k+1}^{0,0}(0) = 0, \text{ for } i = 2k+1, k \geq 0,$$

which yields

$$\varphi_i^{p+1-i}(\xi, 1 - 2\xi) = 0, \text{ for } i = 2k+1, k \geq 0.$$

Next, following the same line of reasoning as for Theorem 4.3.1 establishes assertion *(iii).* $\square$

In order to illustrate the superconvergence result stated in Corollary 4.3.1, assertion *(i)*, we plot the zero level curves of φ_0^1, φ_1^0 for $p = 0$, $\varphi_0^2, \varphi_1^1, \varphi_2^0$ for $p = 1$ and $\varphi_0^3, \varphi_1^2, \varphi_2^1, \varphi_3^0$ for $p = 2$, respectively, in Figures 4.2, 4.3 and 4.4. Thus, the DG solution is $O(h^{p+2})$ superconvergent on the zero level curves.

In Corollary 4.3.1, assertion *(ii)* and $p = 0$, we combine in Figure 4.5 the zero level curves of φ_0^1 and φ_1^0. In Figure 4.6 we combine the zero level curves of φ_0^2 and φ_1^1, φ_0^2 and φ_2^0, φ_1^1 and φ_3^0 for $p = 1$. For $p = 2$, we combine the zero level curves of φ_0^3 and φ_1^2, φ_0^3 and φ_2^1, φ_0^3 and φ_3^0, φ_1^2 and φ_2^1, φ_1^2 and φ_3^0, φ_2^1 and φ_3^0 in Figure 4.7. Thus the DG solution is $O(h^{p+2})$ superconvergent at the intersection of the zero level curves of any pair of functions $(\varphi_i^{p+1-i}, \varphi_j^{p+1-j})$ with $i \neq j$.

Next, let ζ_k, $1 \leq k \leq p+1-i$, are the roots of the Jacobi polynomial $P_{p+1-i}^{2i+1,0}$ for $0 \leq i \leq p$, the functions φ_1^p, φ_3^{p-2}, $\varphi_7^{p-2}, \ldots, \varphi_{2floor(\frac{p+1}{2})}^{p+1-floor(\frac{p+1}{2})}$ and φ_i^{p+1-i} with $i \notin \{1, 3, \ldots, 2floor(\frac{p+1}{2})\}$ vanish at the points $\left(\frac{1-\zeta_k}{2}, \zeta_k\right)$, $1 \leq k \leq p+1-i$. Thus, if $\tilde{\mathcal{W}}_p = \mathcal{P}_p \cup span\{\varphi_1^p, \varphi_3^{p-2}, \varphi_7^{p-2}, \ldots, \varphi_{2floor(\frac{p+1}{2})}^{p+1-floor(\frac{p+1}{2})}, \varphi_i^{p+1-i}\}$, the DG solution is $O(h^{p+2})$ superconvergent at the points $\left(\frac{1-\zeta_k}{2}, \zeta_k\right)$ for $1 \leq k \leq p+1-i$.

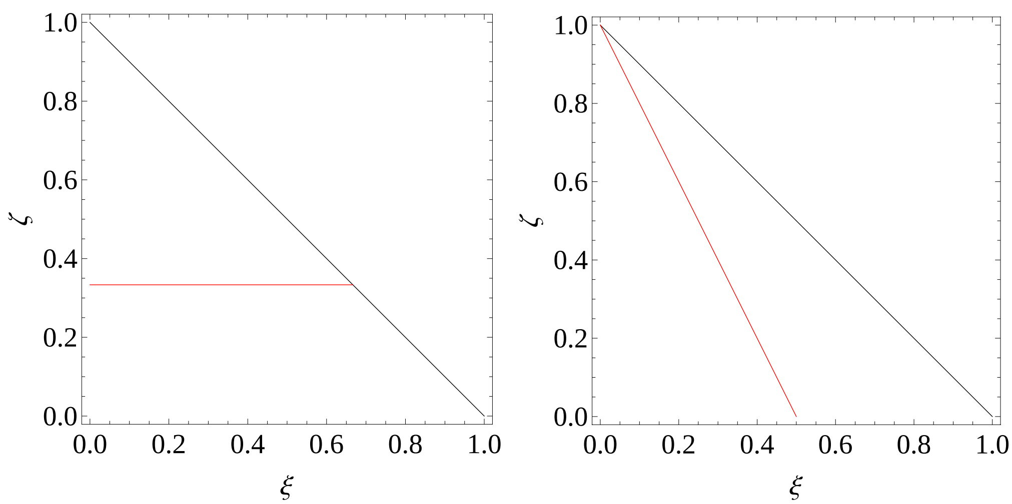


Figure 4.2: Superconvergence points at the zero level curves (red) of φ_0^1 (left) and φ_1^0 (right).

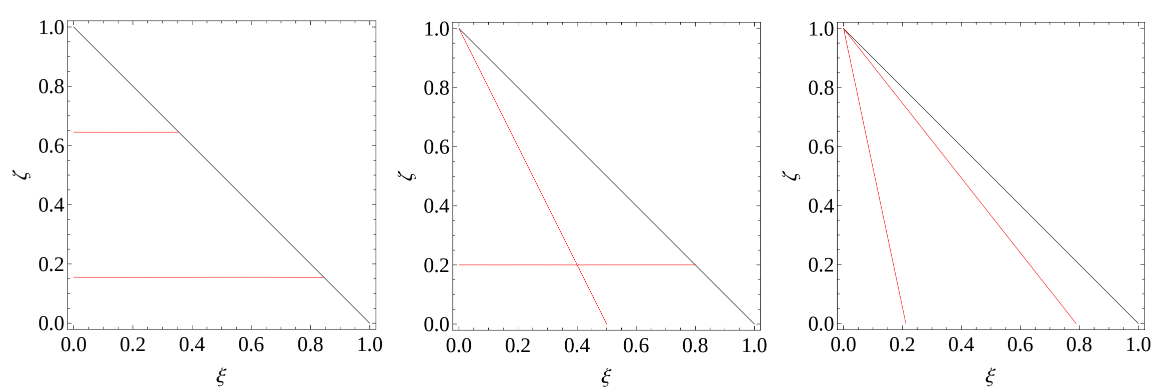


Figure 4.3: Superconvergence points at the zero level curves (red) of φ_0^2 (left), φ_1^1 (center) and φ_2^0 (right).

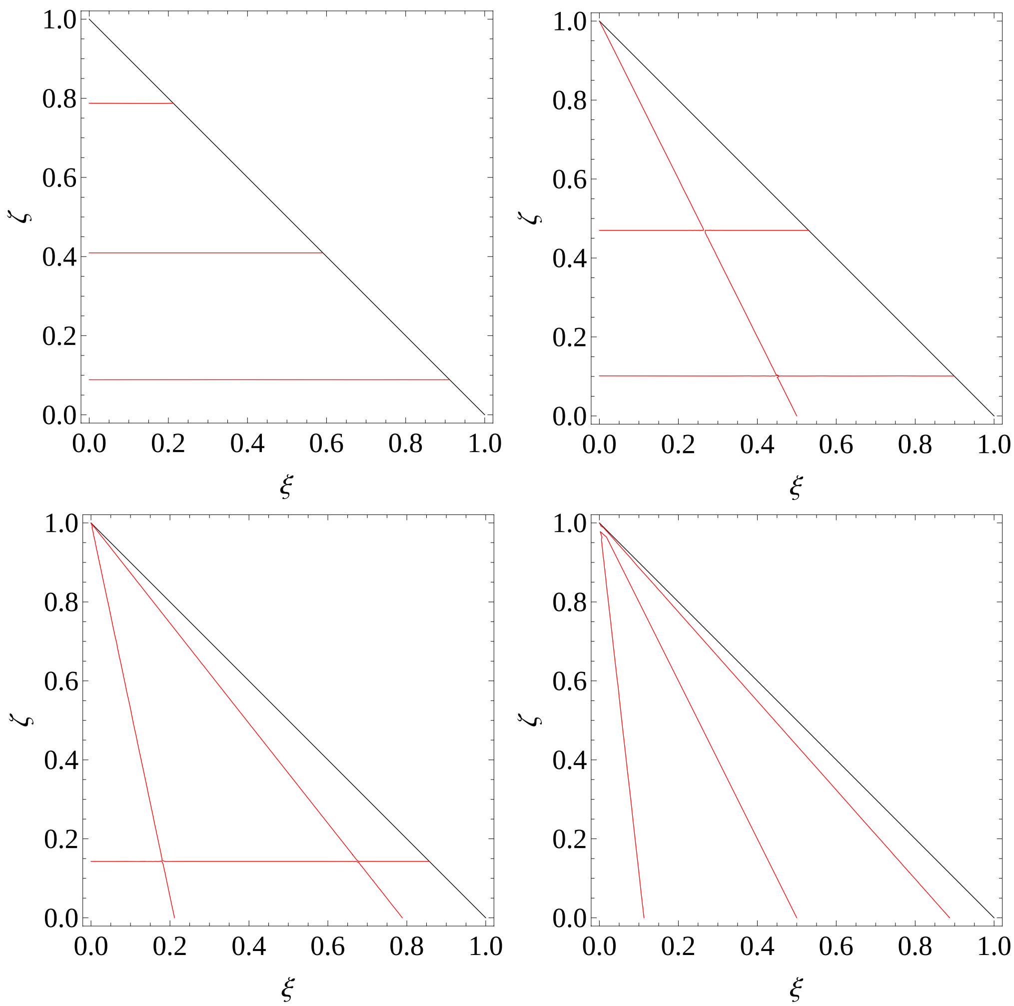


Figure 4.4: Superconvergence points at the zero level curves (red) of φ_0^3 (top-left), φ_1^2 (top-right), φ_2^1 (bottom-left) and φ_3^0 (bottom-right).

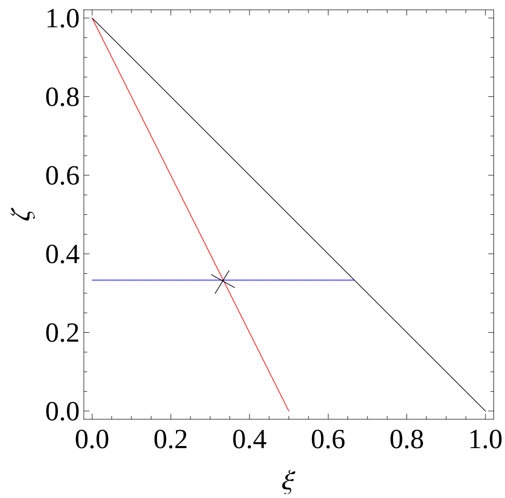
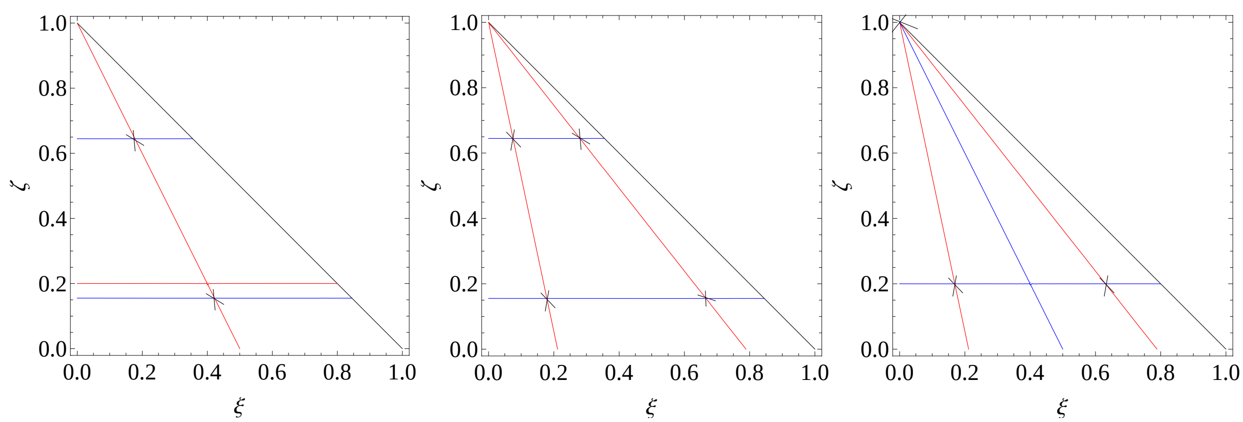


Figure 4.5: Superconvergence points at the intersection of the zero level curves (marked by x) of φ_0^1 (blue) and φ_1^0 (red).

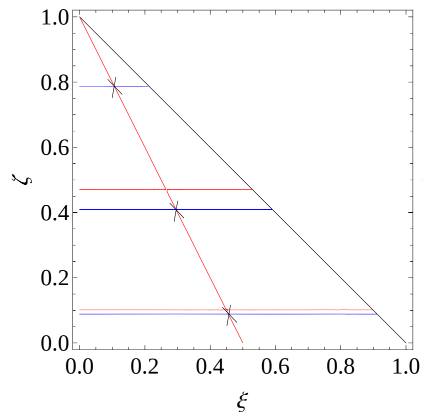


(a) Zero level curves of φ_0^2 (blue) and φ_1^1 (red)

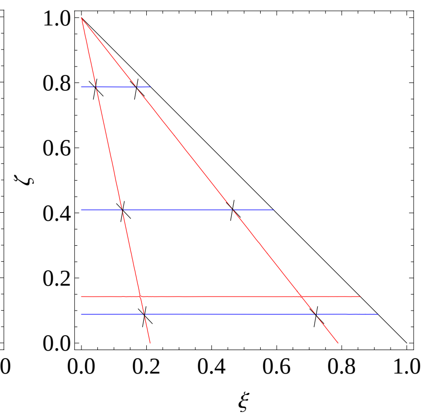
(b) Zero level curves of φ_0^2 (blue) and φ_2^0 (red)

(c) Zero level curves of φ_1^1 (blue) and φ_2^0 (red)

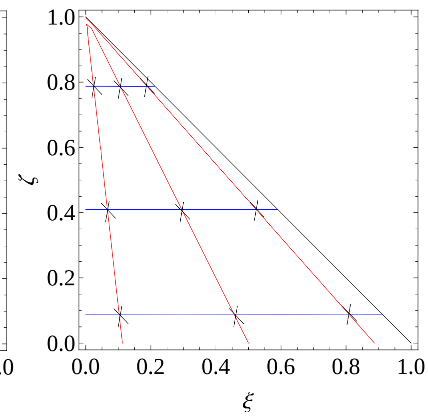
Figure 4.6: Superconvergence points (marked by x) for DG solutions as in Corollary 4.3.1 assertion *(ii)*.



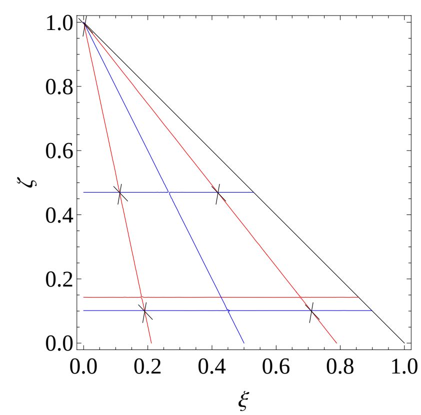
(a) Zeros intersection of φ_0^3 (blue) and φ_1^2 (red)



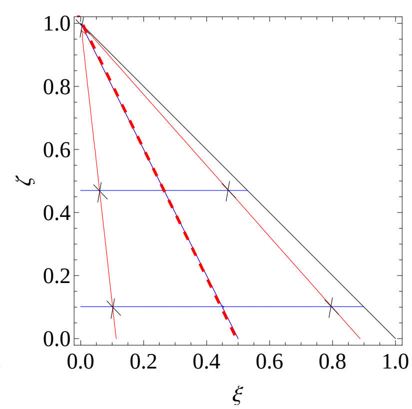
(b) Zeros intersection of φ_0^3 (blue) and φ_2^1 (red)



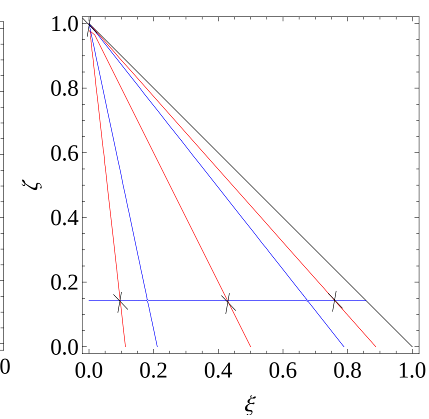
(c) Zeros intersection of φ_0^3 (blue) and φ_3^0 (red)



(d) Zeros intersection of φ_1^2 (blue) and φ_2^1(red)



(e) Zeros intersection of φ_1^2 (blue) and φ_3^0(red)



(f) Zeros intersection of φ_2^1 (blue) and φ_3^0(red)

Figure 4.7: Superconvergence points (marked by x and the dashed line) by combining φ_i^{3-i} and φ_j^{3-j}, for $0 \leq i, j \leq 3,\ i \neq j$.

For general tetrahedral elements, we need to consider more enriched polynomial spaces to show superconvergence results.

Before we prove our next pointwise superconvergence results, we note that Γ_4 can be parameterized as

$$\begin{aligned}
\Gamma_4 &= \{(\xi,\eta,\zeta), \zeta = 1-\xi-\eta, 0 \le \eta \le 1-\xi, 0 \le \xi \le 1\}, \\
&= \{(\xi,\eta,\zeta), \eta = 1-\xi-\zeta, 0 \le \zeta \le 1-\xi, 0 \le \xi \le 1\}, \\
&= \{(\xi,\eta,\zeta), \xi = 1-\eta-\zeta, 0 \le \zeta \le 1-\eta, 0 \le \eta \le 1\},
\end{aligned}$$

then for a given function $g(\xi,\eta,\zeta)$, we have

$$\begin{aligned}
\frac{1}{\sqrt{3}}\iint_{\Gamma_4} g\,d\sigma &= \int_0^1\int_0^{1-\xi} g(\xi,\eta,1-\xi-\eta)\,d\eta d\xi = \int_0^1\int_0^{1-\eta} g(\xi,\eta,1-\xi-\eta)\,d\xi d\eta, \\
&= \int_0^1\int_0^{1-\xi} g(\xi,1-\xi-\zeta,\zeta)\,d\zeta d\xi = \int_0^1\int_0^{1-\zeta} g(\xi,1-\xi-\zeta,)\,d\xi d\zeta, \\
&= \int_0^1\int_0^{1-\eta} g(1-\eta-\zeta,\eta,\zeta)\,d\zeta d\eta = \int_0^1\int_0^{1-\zeta} g(1-\eta-\zeta,\eta,\zeta)\,d\eta d\zeta. \quad (4.3.13)
\end{aligned}$$

Let $\mathbf{v}_i$, $i=1,2,3,4$ be the vertices of the reference tetrahedron defined in section 3.1, and $\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4$, respectively, denote the faces $\eta=0,\ \zeta=0,\ \xi=0,\ 1-\xi-\eta-\zeta=0$ of the reference element.

In the next theorem, we prove that the local discretization error is $O(h^{p+2})$ superconvergent on the edges $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$ of the *inflow* face using the space $\mathcal{U}_p$ on elements of *Type 1*.

Theorem 4.3.2. *Let $u \in C^\infty(\Omega)$ and $U \in \mathcal{U}_p$, respectively, be the solutions of (3.1.1a), (3.1.1c) and (4.1.6), with $U^-|_{\Gamma^-} = u$. Let Δ be a tetrahedron of Type 1, with Γ_4 being the inflow face on the reference element. Then the local finite element error can be written as (4.3.1), and its leading term Q_{p+1} satisfies*

$$\iiint_\Delta (\xi+\eta+\zeta-1)\,(\boldsymbol{a}_0\cdot\nabla Q_{p+1})\,V\,d\xi d\eta d\zeta = 0, \forall\ V \in \mathcal{P}_{p-1}, \tag{4.3.14}$$

and

$$\begin{aligned}
Q_{p+1}(\xi,1-\xi-\zeta,\zeta) - \tfrac{1}{(\alpha+\beta+\gamma)}\int_0^{1-\xi-\zeta} \boldsymbol{a}_0\cdot\nabla Q_{p+1}\,d\eta &= 0, && (4.3.15a)\\
Q_{p+1}(\xi,\eta,1-\xi-\eta) - \tfrac{1}{(\alpha+\beta+\gamma)}\int_0^{1-\xi-\eta} \boldsymbol{a}_0\cdot\nabla Q_{p+1}\,d\zeta &= 0, && (4.3.15b)\\
Q_{p+1}(1-\eta-\zeta,\eta,\zeta) - \tfrac{1}{(\alpha+\beta+\gamma)}\int_0^{1-\eta-\zeta} \boldsymbol{a}_0\cdot\nabla Q_{p+1}\,d\xi &= 0. && (4.3.15c)
\end{aligned}$$

Furthermore, the leading term Q_{p+1} is zero on the edges $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$

$$Q_{p+1}|_{\mathbf{v}_2\mathbf{v}_3} = Q_{p+1}|_{\mathbf{v}_3\mathbf{v}_4} = Q_{p+1}|_{\mathbf{v}_2\mathbf{v}_4} = 0. \tag{4.3.16}$$

Proof. Applying Lemma 4.2.1 yields (4.3.1). Now, substituting (4.1.10), (4.1.12) and (4.3.1) into (4.1.9) yields the following orthogonality condition for the leading term Q_{p+1}

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+1} V d\sigma - \iiint_{\Delta} \left(\boldsymbol{a}_0 \cdot \nabla V\right) Q_{p+1} d\xi d\eta d\zeta = 0, \forall\ V \in \mathcal{U}_p. \tag{4.3.17}$$

Applying Stokes' theorem we write

$$-\iint_{\Gamma^-} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+1} V d\sigma + \iiint_{\Delta} \left(\boldsymbol{a}_0 \cdot \nabla Q_{p+1}\right) V d\xi d\eta d\zeta = 0, \forall\ V \in \mathcal{U}_p. \tag{4.3.18}$$

Testing against $V = (\xi + \eta + \zeta - 1)\tilde{V}$ in (4.3.18), where $\tilde{V} \in \mathcal{P}_{p-1} \cup span\left\{\xi^p, \eta^p, \zeta^p\right\}$, and noting that $V|_{\Gamma^-} = 0$ leads to (4.3.14).
Testing against $V = \xi^i \zeta^j$ in (4.3.18) for $0 \leq i, j, i + j \leq p + 1$ yields

$$\int_0^1 \int_0^{1-\zeta} Q_{p+1}\left(\xi, 1-\xi-\zeta, \zeta\right) \xi^i \zeta^j d\xi d\zeta - \tfrac{1}{(\alpha+\beta+\gamma)} \iiint_{\Delta} \left(\boldsymbol{a}_0 \cdot \nabla Q_{p+1}\right) \xi^i \zeta^j d\xi d\eta d\zeta = 0, \tag{4.3.19}$$

which, in turn, can be written as

$$\int_0^1 \int_0^{1-\zeta} \left\{ Q_{p+1}\left(\xi, 1-\xi-\zeta, \zeta\right) - \tfrac{1}{(\alpha+\beta+\gamma)} \int_0^{1-\xi-\zeta} \left(\boldsymbol{a}_0 \cdot \nabla Q_{p+1}\right) d\eta \right\} \xi^i \zeta^j d\xi d\zeta = 0. \tag{4.3.20}$$

and leads to (4.3.15a).
Next, testing against $V = \xi^i \eta^j$ in (4.3.18) for $0 \leq i, j, i + j \leq p + 1$ yields

$$\int_0^1 \int_0^{1-\xi} Q_{p+1}\left(\xi, \eta, 1-\xi-\eta\right) \xi^i \eta^j d\eta d\xi - \tfrac{1}{(\alpha+\beta+\gamma)} \iiint_{\Delta} \left(\boldsymbol{a}_0 \cdot \nabla Q_{p+1}\right) \xi^i \eta^j d\eta d\xi d\zeta = 0, \tag{4.3.21}$$

which, in turn, can be written as

$$\int_0^1 \int_0^{1-\xi} \left\{ Q_{p+1}\left(\xi, \eta, 1-\xi-\eta\right) - \tfrac{1}{(\alpha+\beta+\gamma)} \int_0^{1-\xi-\zeta} \left(\boldsymbol{a}_0 \cdot \nabla Q_{p+1}\right) d\zeta \right\} \xi^i \eta^j d\eta d\xi = 0. \tag{4.3.22}$$

and leads to (4.3.15b).
Similarly, testing against $V = \eta^i \zeta^j$ in (4.3.18) for $0 \leq i, j, i + j \leq p + 1$ yields

$$\int_0^1 \int_0^{1-\zeta} Q_{p+1}\left(1-\eta-\zeta, \eta, \zeta\right) \eta^i \zeta^j d\eta d\zeta - \tfrac{1}{(\alpha+\beta+\gamma)} \iiint_{\Delta} \left(\boldsymbol{a}_0 \cdot \nabla Q_{p+1}\right) \eta^i \zeta^j d\xi d\eta d\zeta = 0, \tag{4.3.23}$$

which, in turn, can be written as

$$\int_0^1 \int_0^{1-\zeta} \left\{ Q_{p+1}\left(1-\eta-\zeta, \eta, \zeta\right) - \tfrac{1}{(\alpha+\beta+\gamma)} \int_0^{1-\eta-\zeta} \left(\boldsymbol{a}_0 \cdot \nabla Q_{p+1}\right) d\xi \right\} \eta^i \zeta^j d\eta d\zeta = 0. \tag{4.3.24}$$

this leads to (4.3.15c), and (4.3.16) follows directly from (4.3.15). $\square$

Similarly, we have the following superconvergence results for elements of *Class I* and *Type 1* (*i.e.*, one *inflow* and one *outflow* face), using the space $\mathcal{U}_p$.

Theorem 4.3.3. *Let $u \in C^\infty(\Omega)$ and $U \in \mathcal{U}_p$, respectively, be the solutions of (3.1.1a), (3.1.1c) and (4.1.6), with $U^-|_{\Gamma^-} = u$. Let Δ be a tetrahedron of Class I and Type 1 (i.e., $\alpha = \gamma = 0$ and $\beta < 0$), then the local finite element error can be written as (4.3.1) and its leading term Q_{p+1} satisfies,*

$$\iiint_\Delta Q_{p+1} V d\xi d\eta d\zeta = 0,\ \forall\ V \in \mathcal{P}_p \backslash\, span\,(S)\,. \tag{4.3.25}$$

where $S = \left\{\xi^{i+1}\eta^j\zeta^{k+1}, 0 \le i,j,k \le p,\ \text{and}\ i+j+k = p-2\right\}$,
and Q_{p+1} can be written as

$$Q_{p+1}(\xi,\eta,\zeta) = \eta \sum_{i=1}^{\frac{1}{6}p\left(p^2-1\right)} C_i \psi_i(\xi,\eta,\zeta)\,, \tag{4.3.26}$$

with $\left\{\psi_i, i = 1,2,\ldots,\frac{1}{6}p\left(p^2-1\right)\right\}$ being an orthogonal basis for $span\,(S)$ with respect to the weight function η.
Furthermore, the leading term Q_{p+1} is zero on the outflow face $\eta = 0$, and on the edges $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, i.e.,

$$Q_{p+1}|_{\Gamma^+} = Q_{p+1}|_{\mathbf{v}_2\mathbf{v}_3} = Q_{p+1}|_{\mathbf{v}_3\mathbf{v}_4} = Q_{p+1}|_{\mathbf{v}_2\mathbf{v}_4} = 0. \tag{4.3.27}$$

Proof. From Theorem 3.2.2, it suffices to show (4.1.12) for all $V \in \mathcal{U}_p \backslash \mathcal{P}_p$.
Since an element of *Class I* and *Type 1* is a particular case of *Type 1*, (4.3.17) holds and yields the following orthogonality condition for the leading term Q_{p+1}

$$\iiint_\Delta \eta q_p \frac{\partial V}{\partial \eta} d\xi d\eta d\zeta = 0, \forall\ V \in \mathcal{U}_p,\ \text{and}\ q_p \in \mathcal{P}_p, \tag{4.3.28}$$

where we used the fact that $\alpha = \gamma = 0$ and $Q_{p+1}(\xi, 0, \zeta) = 0$ (from (4.3.4a)).
Noting that $\frac{\partial V}{\partial \eta} \in \mathcal{P}_p \backslash\, span\,(S)$ establishes (4.3.25).
Next, let $\left\{\psi_i, i = 1,2,\ldots,\frac{1}{6}p\left(p^2-1\right)\right\}$ be an orthogonal basis for the polynomial space $span\,(S)$ with respect to the weight function η, and using Gram-Schmidt algorithm to construct orthogonal basis $\left\{\tilde{\psi}_i, i = 1,2,\ldots,\frac{1}{2}p\left(8+9p+3p^2\right)\right\}$ for the polynomial space $\mathcal{P}_p \backslash\, span\,(S)$, such that

$$\iiint_\Delta \eta \tilde{\psi}_i \psi_j d\xi d\eta d\zeta = 0.$$

Thus, the orthogonality conditions (4.3.25) leads to (4.3.26).
Combining (4.3.4a) and (4.3.16) establishes (4.3.27). □

In the next theorem, we state and prove new superconvergence results for the local discretization error using the space $\mathcal{U}_p$ on elements of *Type 2*.

Theorem 4.3.4. *Let $u \in C^{\infty}(\Omega)$ and $U \in \mathcal{U}_p$, respectively, be the solutions of (3.1.1a), (3.1.1c) and (4.1.6), with $U^-|_{\Gamma^-} = u$. Let Δ be a tetrahedron of Type 2, with Γ_2 and Γ_4 being the inflow faces on the reference element. Then the local finite element error can be written as (4.3.1), with its leading term Q_{p+1} satisfying*

$$\iiint_{\Delta} \zeta(\xi+\eta+\zeta-1)(\boldsymbol{a}_0 \cdot \nabla Q_{p+1}) V d\xi d\eta d\zeta = 0, \forall\, V \in \mathcal{P}_{p-2} \cup span\left(\{\eta^{p-1}, \zeta^{p-1}\}\right), \tag{4.3.29}$$

and

$$(\alpha+\beta+\gamma) Q_{p+1}(\xi,\eta,1-\xi-\eta) + \gamma Q_{p+1}(\xi,\eta,0) - \int_0^{1-\xi-\eta} (\boldsymbol{a}_0 \cdot \nabla Q_{p+1}) d\zeta \;=\; 0, \tag{4.3.30}$$

Furthermore, the leading term Q_{p+1} is zero on the edge $\mathbf{v}_2\mathbf{v}_3$

$$Q_{p+1}|_{\mathbf{v}_2\mathbf{v}_3} = 0. \tag{4.3.31}$$

Proof. Applying Lemma 4.2.1 we obtain (4.3.1). Substituting (4.1.10), (4.1.12) and (4.3.1) into (4.1.9), and applying Stokes' theorem yields (4.3.18), with $\Gamma^- = \Gamma_2 \cup \Gamma_4$.
Testing against $V = \zeta(\xi+\eta+\zeta-1)\tilde{V} \in \mathcal{U}_p$ for all $\tilde{V} \in \mathcal{P}_{p-2} \cup span(\{\eta^{p-1}, \zeta^{p-1}\})$ in (4.3.18), and noting that $V|_{\Gamma^-} = 0$ leads to (4.3.29).
Testing against $V = \xi^i\eta^j \in \mathcal{U}_p$ in (4.3.18) for $0 \le i, j, i+j \le p+1$ yields

$$\begin{aligned} &-(\alpha+\beta+\gamma)\int_0^1\int_0^{1-\xi} Q_{p+1}(\xi,\eta,1-\xi-\eta)\,\xi^i\eta^j d\eta d\xi - \gamma\int_0^1\int_0^{1-\xi} Q_{p+1}(\xi,\eta,0)\,\xi^i\eta^j d\eta d\xi \\ &+\int_0^1\int_0^{1-\xi}\int_0^{1-\xi-\eta} (\boldsymbol{a}_0 \cdot \nabla Q_{p+1})\,\xi^i\eta^j d\zeta d\eta d\xi \;=0, \end{aligned} \tag{4.3.32}$$

which can be written as

$$\begin{aligned} \int_0^1\int_0^{1-\xi} \Big\{ &(\alpha+\beta+\gamma) Q_{p+1}(\xi,\eta,1-\xi-\eta) + \gamma Q_{p+1}(\xi,\eta,0) \\ &- \int_0^{1-\xi-\eta} (\boldsymbol{a}_0 \cdot \nabla Q_{p+1}) d\zeta \Big\} \eta^i\zeta^j d\eta d\xi \;= 0. \end{aligned} \tag{4.3.33}$$

Thus, we obtain (4.3.30).
Next, substituting $\eta = 1-\xi$ in (4.3.30) and noting that $\boldsymbol{a}_0 \cdot \mathbf{n} < 0$ on the *inflow* faces (i.e. $(\alpha+\beta+\gamma)+\gamma < 0$), yields (4.3.31). $\square$

In the next theorem, using the space $\mathcal{U}_p$ on elements of *Class I* and *Type 2*, we show that the local discretization error is $O(h^{p+2})$ superconvergent on the edges $\mathbf{v}_2\mathbf{v}_3$ and $\mathbf{v}_3\mathbf{v}_4$.

Theorem 4.3.5. *Let $u \in C^{\infty}(\Omega)$ and $U \in \mathcal{U}_p$, respectively, be the solutions of (3.1.1a), (3.1.1c) and (4.1.6), with $U^-|_{\Gamma^-} = u$. Let Δ be a tetrahedron of Class I and Type 2 , with*

Γ_2 and Γ_4 are the inflow faces on the reference element and $\alpha = 0$. Then the local finite element error can be written as (4.3.1), and its leading term Q_{p+1} satisfies (4.3.29), and

$$(\beta+\gamma)\, Q_{p+1}(\xi,\eta,1-\xi-\eta) + \gamma Q_{p+1}(\xi,\eta,0) - \int_0^{1-\xi-\eta} (\boldsymbol{a}_0 \cdot \nabla Q_{p+1})\, d\zeta = 0, \quad (4.3.34a)$$

$$\int_0^1 \int_0^{1-\eta} \eta^j Q_{p+1}(\xi,\eta,0)\, d\xi d\eta = 0, \ \textit{for } 0 \le j \le p+1, \quad (4.3.34b)$$

$$-(\beta+\gamma)\, Q_{p+1}(1-\zeta-\eta,\eta,\zeta) + \int_0^{1-\zeta-\eta} (\boldsymbol{a}_0 \cdot \nabla Q_{p+1})\, d\xi = 0. \quad (4.3.34c)$$

Furthermore, the leading term Q_{p+1} is zero on the edges $\mathbf{v}_2\mathbf{v}_3$ and $\mathbf{v}_3\mathbf{v}_4$

$$Q_{p+1}|_{\mathbf{v_2 v_3}} = Q_{p+1}|_{\mathbf{v_3 v_4}} = 0. \quad (4.3.35)$$

Proof. Equation (4.3.34a) follows from (4.3.30), where $\alpha = 0$.
Letting $\alpha = 0$ in (4.3.17), and testing against $V = 1$ and $V = \eta^j$, for $1 \le j \le p+1$ yields

$$\int_0^1 \int_0^{1-\xi} Q_{p+1}(\xi,0,\zeta)\, d\zeta d\xi = 0, \quad (4.3.36)$$

and

$$\iiint_\Delta \eta^{j-1} Q_{p+1} d\xi d\eta d\zeta = 0. \quad (4.3.37)$$

Now, testing against $V = 1$ in (4.3.18), we obtain

$$(\beta+\gamma) \int_0^1 \int_0^{1-\xi} Q_{p+1}(\xi,\eta,1-\xi-\eta)\, d\eta d\xi + \gamma \int_0^1 \int_0^{1-\xi} Q_{p+1}(\xi,\eta,0)\, d\eta d\xi$$
$$- \iiint_\Delta \left(\beta \frac{\partial Q_{p+1}}{\partial \eta} + \gamma \frac{\partial Q_{p+1}}{\partial \zeta} \right) d\xi d\eta d\zeta = 0, \quad (4.3.38)$$

where we used the fact that $\alpha = 0$.
The first integral can be written as

$$\beta \int_0^1 \int_0^{1-\xi} Q_{p+1}(\xi,1-\xi-\zeta,\zeta)\, d\zeta d\xi + \gamma \int_0^1 \int_0^{1-\xi} Q_{p+1}(\xi,\eta,1-\xi-\eta)\, d\eta d\xi. \quad (4.3.39)$$

The third integral can be written as

$$\beta \int_0^1 \int_0^{1-\xi} \left(\int_0^{1-\xi-\zeta} \frac{\partial Q_{p+1}}{\partial \eta} d\eta \right) d\zeta d\xi + \gamma \int_0^1 \int_0^{1-\xi} \left(\int_0^{1-\xi-\eta} \frac{\partial Q_{p+1}}{\partial \zeta} d\zeta \right) d\eta d\xi, \quad (4.3.40)$$

which after integration, leads to

$$\beta \int_0^1 \int_0^{1-\xi} \{Q_{p+1}(\xi,1-\xi-\zeta,\zeta) - Q_{p+1}(\xi,0,\zeta)\}\, d\zeta d\xi +$$
$$\gamma \int_0^1 \int_0^{1-\xi} \{Q_{p+1}(\xi,\eta,1-\xi-\eta) - Q_{p+1}(\xi,\eta,0)\}\, d\eta d\xi. \quad (4.3.41)$$

Substituting (4.3.39) and (4.3.41) in (4.3.38) yields

$$\beta \int_0^1 \int_0^{1-\xi} Q_{p+1}(\xi, 0, \zeta)\, d\zeta d\xi + 2\gamma \int_0^1 \int_0^{1-\eta} Q_{p+1}(\xi, \eta, 0)\, d\xi d\eta = 0. \tag{4.3.42}$$

Using (4.3.36) leads to

$$\int_0^1 \int_0^{1-\eta} Q_{p+1}(\xi, \eta, 0)\, d\xi d\eta = 0. \tag{4.3.43}$$

Testing against $V = \eta^j, 1 \leq j \leq p+1$ in (4.3.18), we obtain

$$(\beta + \gamma) \int_0^1 \int_0^{1-\xi} \eta^j Q_{p+1}(\xi, \eta, 1-\xi-\eta)\, d\eta d\xi + \gamma \int_0^1 \int_0^{1-\xi} \eta^j Q_{p+1}(\xi, \eta, 0)\, d\eta d\xi$$
$$- \iiint_\Delta \left(\beta \frac{\partial Q_{p+1}}{\partial \eta} + \gamma \frac{\partial Q_{p+1}}{\partial \zeta}\right) \eta^j d\xi d\eta d\zeta = 0. \tag{4.3.44}$$

The first integral in (4.3.44) can be written as

$$\beta \int_0^1 \int_0^{1-\xi} (1-\xi-\zeta)^j\, Q_{p+1}(\xi, 1-\xi-\zeta, \zeta)\, d\zeta d\xi$$
$$+\gamma \int_0^1 \int_0^{1-\xi} \eta^j Q_{p+1}(\xi, \eta, 1-\xi-\eta)\, d\eta d\xi, \tag{4.3.45}$$

while the third integral in (4.3.44) can be written as

$$\beta \int_0^1 \int_0^{1-\xi} \left(\int_0^{1-\xi-\zeta} \eta^j \frac{\partial Q_{p+1}}{\partial \eta} d\eta\right) d\zeta d\xi + \gamma \int_0^1 \int_0^{1-\xi} \left(\eta^j \int_0^{1-\xi-\eta} \frac{\partial Q_{p+1}}{\partial \zeta} d\zeta\right) d\eta d\xi. \tag{4.3.46}$$

We integrate by parts the first term and integrate the second term with respect to ζ, to obtain

$$\beta \int_0^1 \int_0^{1-\xi} (1-\xi-\zeta)^j\, Q_{p+1}(\xi, 1-\xi-\zeta, \zeta)\, d\zeta d\xi - j\beta \iiint_\Delta \eta^{j-1} Q_{p+1} d\xi d\eta d\zeta$$
$$+\gamma \int_0^1 \int_0^{1-\xi} \eta^j \left\{Q_{p+1}(\xi, \eta, 1-\xi-\eta) - Q_{p+1}(\xi, \eta, 0)\right\} d\eta d\xi. \tag{4.3.47}$$

Substituting (4.3.45) and (4.3.47) in (4.3.44) and using (4.3.37), we obtain

$$2\gamma \int_0^1 \int_0^{1-\xi} \eta^j Q_{p+1}(\xi, \eta, 0)\, d\eta d\xi = 0. \tag{4.3.48}$$

Combining (4.3.43) with (4.3.48) yields (4.3.34b).
Testing against $V = \zeta^i \eta^j \in \mathcal{U}_p$ in (4.3.18), for $1 \leq i \leq p+1$ and $0 \leq j, i+j \leq p+1$, leads to

$$\int_0^1 \int_0^{1-\zeta} \left\{-(\beta+\gamma)\, Q_{p+1}(1-\eta-\zeta, \eta, \zeta) + \int_0^{1-\zeta-\eta} (\boldsymbol{a}_0 \cdot \nabla Q_{p+1})\, d\xi\right\} \zeta^i \eta^j d\eta d\zeta = 0. \tag{4.3.49}$$

Testing against $V = \eta^j$, $0 \leq j \leq p+1$, and using (4.3.34b), we obtain

$$\int_0^1 \int_0^{1-\zeta} \left\{ -(\beta+\gamma) Q_{p+1}(1-\eta-\zeta,\eta,\zeta) + \int_0^{1-\eta-\zeta} (\boldsymbol{a}_0 \cdot \nabla Q_{p+1}) \, d\xi \right\} \eta^j d\eta d\zeta = 0. \quad (4.3.50)$$

Combining (4.3.49) and (4.3.50), yields (4.3.34c).
Substituting $\eta = 1-\xi$ in (4.3.34a), $\zeta = 1-\eta$ in (4.3.34c), and noting that $\boldsymbol{a}_0 \cdot \mathbf{n} < 0$ on the *inflow* faces, we obtain (4.3.35) which completes the proof. □

For an element of *Type 3* with the space $\mathcal{U}_p$, we show in the following theorem that the local finite element error is $O(h^{p+2})$ superconvergent on the edges $\mathbf{v}_2\mathbf{v}_3$ and $\mathbf{v}_3\mathbf{v}_4$.

Theorem 4.3.6. *Let $u \in C^\infty(\Omega)$ and $U \in \mathcal{U}_p$, respectively, be the solutions of (3.1.1a), (3.1.1c) and (4.1.6), with $U^-|_{\Gamma^-} = u$. Let Δ be a tetrahedron of Type 3, with Γ_2, Γ_3 and Γ_4 being the inflow faces on the reference element. Then the local finite element error can be written as (4.3.1) and its leading term Q_{p+1} satisfies*

$$\iiint_\Delta \xi\zeta(\xi+\eta+\zeta-1)(\boldsymbol{a}_0 \cdot \nabla Q_{p+1}) V d\xi d\eta d\zeta = 0, \forall\ V \in \mathcal{P}_{p-3}, \quad (4.3.51)$$

and

$$\int_0^1 \left\{ \int_0^{1-\eta} \alpha Q_{p+1}(0,\eta,\zeta) \, d\zeta + \gamma \int_0^{1-\eta} Q_{p+1}(\xi,\eta,0) \, d\xi \right\} \eta^j d\eta = 0, \ \textit{for } 0 \leq j \leq p+1, \quad (4.3.52a)$$

$$(\alpha+\beta) Q_{p+1}(\xi,\eta,1-\xi-\eta) + \gamma Q_{p+1}(\xi,\eta,0) - \int_0^{1-\xi-\eta} \left(\alpha \frac{\partial Q_{p+1}}{\partial \xi} + \beta \frac{\partial Q_{p+1}}{\partial \eta} \right) d\zeta = 0, \quad (4.3.52b)$$

$$(\beta+\gamma) Q_{p+1}(1-\eta-\zeta,\eta,\zeta) + \alpha Q_{p+1}(0,\eta,\zeta) - \int_0^{1-\eta-\zeta} \left(\beta \frac{\partial Q_{p+1}}{\partial \eta} + \gamma \frac{\partial Q_{p+1}}{\partial \zeta} \right) d\xi = 0. \quad (4.3.52c)$$

Furthermore, the leading term Q_{p+1} is zero on the edges $\mathbf{v}_2\mathbf{v}_3$ and $\mathbf{v}_3\mathbf{v}_4$

$$Q_{p+1}|_{\mathbf{v}_2\mathbf{v}_3} = Q_{p+1}|_{\mathbf{v}_3\mathbf{v}_4} = 0. \quad (4.3.53)$$

Proof. Applying Lemma 4.2.1 we obtain (4.3.1). Now, substituting (4.1.10), (4.1.12) and (4.3.1) into (4.1.9), and applying Stokes' theorem yields (4.3.18), with $\Gamma^- = \Gamma_1 \cup \Gamma_2 \cup \Gamma_4$.
Testing against $V = (\eta - (1-\xi-\zeta))\xi\zeta\tilde{V} \in \mathcal{U}_p$ for $\tilde{V} \in \mathcal{P}_{p-3}$ in (4.3.18), and noting that $V|_{\Gamma^-} = 0$ leads to (4.3.51).
Testing against $V = 1$ in (4.3.17) yields

$$\int_0^1 \int_0^{1-\xi} Q_{p+1}(\xi,0,\zeta) \, d\zeta d\xi = 0. \quad (4.3.54)$$

while testing against $V = \eta^j, 1 \leq j \leq p+1$ in (4.3.17) leads to

$$\iiint_\Delta \eta^{j-1} Q_{p+1} d\xi d\eta d\zeta = 0. \quad (4.3.55)$$

Now, testing against $V = 1$ in (4.3.18), we obtain

$$
\begin{aligned}
(\alpha+\beta+\gamma)\int_0^1\int_0^{1-\xi} Q_{p+1}\left(\xi,\eta,1-\xi-\eta\right)d\eta d\xi \\
+\alpha\int_0^1\int_0^{1-\eta} Q_{p+1}\left(0,\eta,\zeta\right)d\zeta d\eta+\gamma\int_0^1\int_0^{1-\xi} Q_{p+1}\left(\xi,\eta,0\right)d\eta d\xi \\
-\iiint_\Delta\left(\alpha\frac{\partial Q_{p+1}}{\partial\xi}+\beta\frac{\partial Q_{p+1}}{\partial\eta}+\gamma\frac{\partial Q_{p+1}}{\partial\zeta}\right)d\xi d\eta d\zeta=0.
\end{aligned}
\tag{4.3.56}
$$

Using (4.3.13), the first integral can be written as

$$
\begin{aligned}
\alpha\int_0^1\int_0^{1-\eta} Q_{p+1}\left(1-\eta-\zeta,\eta,\zeta\right)d\zeta d\eta+\beta\int_0^1\int_0^{1-\xi} Q_{p+1}\left(\xi,1-\xi-\zeta,\zeta\right)d\zeta d\xi \\
+\gamma\int_0^1\int_0^{1-\xi} Q_{p+1}\left(\xi,\eta,1-\xi-\eta\right)d\eta d\xi,
\end{aligned}
\tag{4.3.57}
$$

while the fourth integral in (4.3.56) can be written as

$$
\begin{aligned}
\alpha\int_0^1\int_0^{1-\eta}\left(\int_0^{1-\eta-\zeta}\frac{\partial Q_{p+1}}{\partial\xi}d\xi\right)d\zeta d\eta+\beta\int_0^1\int_0^{1-\xi}\left(\int_0^{1-\xi-\zeta}\frac{\partial Q_{p+1}}{\partial\eta}d\eta\right)d\zeta d\xi \\
+\gamma\int_0^1\int_0^{1-\xi}\left(\int_0^{1-\xi-\eta}\frac{\partial Q_{p+1}}{\partial\zeta}d\zeta\right)d\eta d\xi,
\end{aligned}
\tag{4.3.58}
$$

which, after integration, leads to

$$
\begin{aligned}
\alpha\int_0^1\int_0^{1-\eta}\left\{Q_{p+1}\left(1-\eta-\zeta,\eta,\zeta\right)-Q_{p+1}\left(0,\eta,\zeta\right)\right\}d\zeta d\eta+ \\
\beta\int_0^1\int_0^{1-\xi}\left\{Q_{p+1}\left(\xi,1-\xi-\zeta,\zeta\right)-Q_{p+1}\left(\xi,0,\zeta\right)\right\}d\zeta d\xi+ \\
\gamma\int_0^1\int_0^{1-\xi}\left\{Q_{p+1}\left(\xi,\eta,1-\xi-\eta\right)-Q_{p+1}\left(\xi,\eta,0\right)\right\}d\eta d\xi.
\end{aligned}
\tag{4.3.59}
$$

Substituting (4.3.57) and (4.3.59) in (4.3.56) yields

$$
\begin{aligned}
\beta\int_0^1\int_0^{1-\xi} Q_{p+1}\left(\xi,0,\zeta\right)d\zeta d\xi+2\alpha\int_0^1\int_0^{1-\eta} Q_{p+1}\left(0,\eta,\zeta\right)d\zeta d\eta \\
+2\gamma\int_0^1\int_0^{1-\eta} Q_{p+1}\left(\xi,\eta,0\right)d\xi d\eta=0.
\end{aligned}
\tag{4.3.60}
$$

Using (4.3.54), (4.3.60) becomes

$$
\int_0^1\left(\alpha\int_0^{1-\eta} Q_{p+1}\left(0,\eta,\zeta\right)d\zeta+\gamma\int_0^{1-\eta} Q_{p+1}\left(\xi,\eta,0\right)d\xi\right)d\eta=0. \tag{4.3.61}
$$

Next, testing against $V = \eta^j, 1 \leq j \leq p+1$ in (4.3.18), we obtain

$$(\alpha+\beta+\gamma)\int_0^1\int_0^{1-\xi}\eta^j Q_{p+1}(\xi,\eta,1-\xi-\eta)\,d\eta d\xi$$
$$+\alpha\int_0^1\int_0^{1-\eta}\eta^j Q_{p+1}(0,\eta,\zeta)\,d\zeta d\eta+\gamma\int_0^1\int_0^{1-\xi}\eta^j Q_{p+1}(\xi,\eta,0)\,d\eta d\xi$$
$$-\iiint_\Delta\left(\alpha\frac{\partial Q_{p+1}}{\partial\xi}+\beta\frac{\partial Q_{p+1}}{\partial\eta}+\gamma\frac{\partial Q_{p+1}}{\partial\zeta}\right)\eta^j\,d\xi d\eta d\zeta=0. \tag{4.3.62}$$

Using (4.3.13), the first integral can be written as

$$\alpha\int_0^1\int_0^{1-\eta}\eta^j Q_{p+1}(1-\eta-\zeta,\eta,\zeta)\,d\zeta d\eta+\beta\int_0^1\int_0^{1-\xi}(1-\xi-\zeta)^j Q_{p+1}(\xi,1-\xi-\zeta,\zeta)\,d\zeta d\xi$$
$$+\gamma\int_0^1\int_0^{1-\xi}\eta^j Q_{p+1}(\xi,\eta,\eta,1-\xi-\eta)\,d\eta d\xi. \tag{4.3.63}$$

The fourth integral in (4.3.62) can be written as

$$\alpha\int_0^1\int_0^{1-\eta}\left(\eta^j\int_0^{1-\eta-\zeta}\frac{\partial Q_{p+1}}{\partial\xi}d\xi\right)d\zeta d\eta+\beta\int_0^1\int_0^{1-\xi}\left(\int_0^{1-\xi-\zeta}\eta^j\frac{\partial Q_{p+1}}{\partial\eta}d\eta\right)d\zeta d\xi$$
$$+\gamma\int_0^1\int_0^{1-\xi}\left(\eta^j\int_0^{1-\xi-\eta}\frac{\partial Q_{p+1}}{\partial\zeta}d\zeta\right)d\eta d\xi.$$

We integrate the second integral by parts, and integrate the first and the third integrals, respectively, with respect to ξ and ζ, to obtain

$$\alpha\int_0^1\int_0^{1-\eta}\eta^j\left\{Q_{p+1}(1-\eta-\zeta,\eta,\zeta)-Q_{p+1}(0,\eta,\zeta)\right\}d\zeta d\eta$$
$$+\gamma\int_0^1\int_0^{1-\xi}\eta^j\left\{Q_{p+1}(\xi,\eta,1-\xi-\eta)-Q_{p+1}(\xi,\eta,0)\right\}d\eta d\xi$$
$$+\beta\int_0^1\int_0^{1-\xi}\left\{(1-\xi-\zeta)^j Q_{p+1}(\xi,1-\xi-\zeta,\zeta)\right\}d\zeta d\xi$$
$$-j\beta\iiint_\Delta\eta^{j-1}Q_{p+1}d\xi d\eta d\zeta. \tag{4.3.64}$$

Substituting (4.3.63) and (4.3.64) in (4.3.62) and using (4.3.55), we obtain

$$\int_0^1\eta^j\left(\alpha\int_0^{1-\eta}Q_{p+1}(0,\eta,\zeta)\,d\zeta+\gamma\int_0^{1-\eta}Q_{p+1}(\xi,\eta,0)\,d\xi\right)d\eta=0. \tag{4.3.65}$$

Combining (4.3.61) with (4.3.65) yields (4.3.52a).
Next, in order to prove (4.3.52b), we need to show

$$(\alpha+\beta)\int_0^1\int_0^{1-\xi}\left\{Q_{p+1}(\xi,\eta,1-\xi-\eta)\,d\eta d\xi+\gamma Q_{p+1}(\xi,\eta,0)\right.$$
$$\left.-\int_0^{1-\xi-\zeta}\left(\alpha\frac{\partial Q_{p+1}}{\partial\xi}+\beta\frac{\partial Q_{p+1}}{\partial\eta}\right)d\zeta\right\}\xi^i\eta^j\,d\eta d\xi=0.\text{ for }0\leq i,j\leq p+1. \tag{4.3.66}$$

Testing against $V=\eta^j, 0\le j\le p+1$ in (4.3.18), and using (4.3.52a), we obtain

$$(\alpha+\beta+\gamma)\int_0^1\int_0^{1-\xi}\eta^j Q_{p+1}\left(\xi,\eta,1-\xi-\eta\right)d\eta d\xi$$
$$-\iiint_\Delta\left(\alpha\frac{\partial Q_{p+1}}{\partial\xi}+\beta\frac{\partial Q_{p+1}}{\partial\eta}+\gamma\frac{\partial Q_{p+1}}{\partial\zeta}\right)\eta^j d\xi d\eta d\zeta\ =0. \tag{4.3.67}$$

Integrating $\frac{\partial Q_{p+1}}{\partial\zeta}$ with respect to ζ, leads to (4.3.66) for $i=0$ and $0\le j\le p+1$.
Testing against $V=\xi^i\eta^j$ for $1\le i,j,i+j\le p+1$ in (4.3.17), to obtain

$$\alpha\int_0^1\int_0^{1-\eta}\left\{\int_0^{1-\eta-\zeta}i\xi^{i-1}\eta^j Q_{p+1}d\xi\right\}d\zeta d\eta+\beta\int_0^1\int_0^{1-\xi}\left\{\int_0^{1-\xi-\zeta}j\xi^i\eta^{j-1}Q_{p+1}d\eta\right\}d\zeta d\xi=0. \tag{4.3.68}$$

Integrating by parts the first and the second integrals, respectively, with respect to ξ and η yields

$$\alpha\int_0^1\int_0^{1-\eta}\left(1-\eta-\zeta\right)^i\eta^j Q_{p+1}\left(1-\eta-\zeta,\eta,\zeta\right)d\zeta d\eta$$
$$+\beta\int_0^1\int_0^{1-\xi}\left(1-\xi-\zeta\right)^j\xi^i Q_{p+1}\left(\xi,1-\xi-\zeta,\zeta\right)d\zeta d\xi$$
$$-\iiint_\Delta\left(\alpha\frac{\partial Q_{p+1}}{\partial\xi}+\beta\frac{\partial Q_{p+1}}{\partial\eta}\right)\xi^i\eta^j d\zeta d\eta d\xi=0. \tag{4.3.69}$$

Using (4.3.13), (4.3.69) becomes

$$\int_0^1\int_0^{1-\xi}\Bigg\{\left(\alpha+\beta\right)Q_{p+1}\left(\xi,\eta,1-\xi-\eta\right)$$
$$-\int_0^{1-\xi-\eta}\left(\alpha\frac{\partial Q_{p+1}}{\partial\xi}+\beta\frac{\partial Q_{p+1}}{\partial\eta}\right)d\zeta\Bigg\}\xi^i\eta^j d\eta d\xi=0,\text{ for }1\le i,j,i+j\le p+1. \tag{4.3.70}$$

Noting that the orthogonality conditions (4.3.70) are the same as (4.3.66), except the integral with the coefficient γ is missing. Next we show that this missing integral is zero for $1\le i,j,i+j\le p+1$.
Testing against $V=\xi^i\eta^j$, for $1\le i,j,i+j\le p+1$ in (4.3.18), yields

$$\int_0^1\int_0^{1-\xi}\Bigg\{\left(\alpha+\beta+\gamma\right)Q_{p+1}\left(\xi,\eta,1-\xi-\eta\right)+\gamma Q_{p+1}\left(\xi,\eta,0\right)$$
$$-\int_0^{1-\xi-\eta}\left(\alpha\frac{\partial Q_{p+1}}{\partial\xi}+\beta\frac{\partial Q_{p+1}}{\partial\eta}+\gamma\frac{\partial Q_{p+1}}{\partial\zeta}\right)d\zeta\Bigg\}\xi^i\eta^j d\eta d\xi\ =0. \tag{4.3.71}$$

Integrating $\frac{\partial Q_{p+1}}{\partial\zeta}$ in the third integral with respect to ζ, leads to

$$\int_0^1\int_0^{1-\xi}\Big\{\left(\alpha+\beta+\gamma\right)Q_{p+1}\left(\xi,\eta,1-\xi-\eta\right)+2\gamma Q_{p+1}\left(\xi,\eta,0\right)$$
$$-\int_0^{1-\xi-\eta}\left(\alpha\frac{\partial Q_{p+1}}{\partial\xi}+\beta\frac{\partial Q_{p+1}}{\partial\eta}\right)d\zeta\Bigg\}\xi^i\eta^j d\eta d\xi\ =0. \tag{4.3.72}$$

Subtracting (4.3.72) from (4.3.70), we obtain

$$2\gamma\int_0^1\int_0^{1-\xi} Q_{p+1}(\xi,\eta,0)\,\xi^i\eta^j d\eta d\xi = 0,\ \text{for } 1\le i,j,i+j\le p+1. \tag{4.3.73}$$

Combining (4.3.73) and (4.3.70), leads to (4.3.66) for $1\le i,j,i+j\le p+1$.
Next, testing against $V=\xi^i, 1\le i\le p+1$ in (4.3.17), yields

$$\beta\int_0^1\int_0^{1-\xi}\xi^i Q_{p+1}(\xi,0,\zeta)\,d\zeta d\xi - \alpha\int_0^1\int_0^{1-\eta}\left(\int_0^{1-\eta-\zeta} i\xi^{i-1}Q_{p+1}d\xi\right)d\zeta d\eta = 0. \tag{4.3.74}$$

Using (4.3.13), the first integral can be written as

$$\int_0^1\int_0^{1-\xi}\xi^i Q_{p+1}(\xi,\eta,1-\xi-\eta)\,d\zeta d\xi - \int_0^1\int_0^{1-\xi}\int_0^{1-\xi-\eta}\xi^i\frac{\partial Q_{p+1}}{\partial\eta}d\zeta d\eta d\xi, \tag{4.3.75}$$

while integrating by parts the second integral in (4.3.74) with respect to ξ and using (4.3.13), leads to

$$\int_0^1\int_0^{1-\xi}\xi^i Q_{p+1}(\xi,\eta,1-\xi-\eta)\,d\eta d\xi - \int_0^1\int_0^{1-\xi}\int_0^{1-\xi-\eta}\xi^i\frac{\partial Q_{p+1}}{\partial\xi}d\zeta d\eta d\xi. \tag{4.3.76}$$

Substituting (4.3.75) and (4.3.76) in (4.3.74), yields

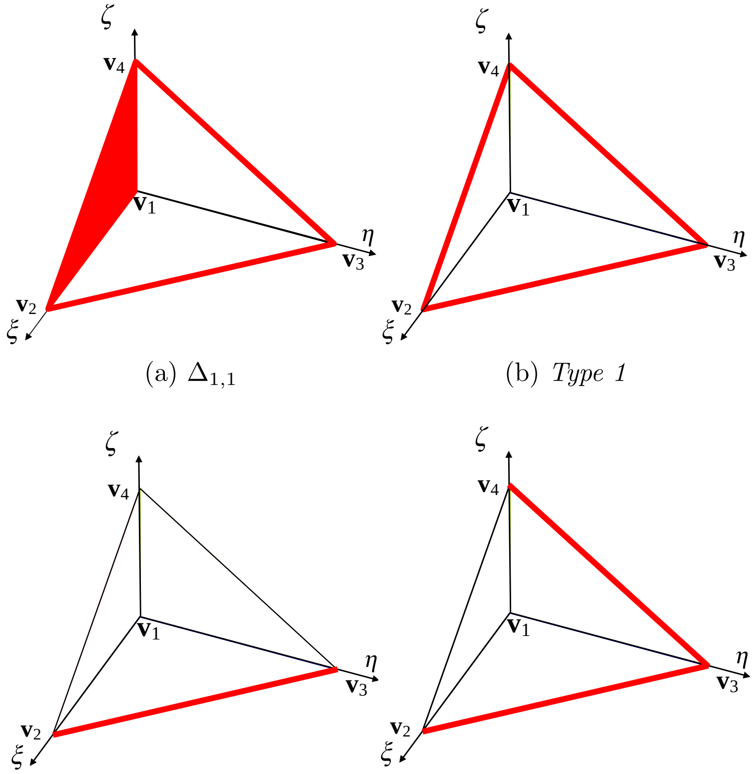
$$\begin{aligned}&\int_0^1\int_0^{1-\xi}\{(\alpha+\beta)\,Q_{p+1}(\xi,\eta,1-\xi-\eta)\\&-\int_0^{1-\xi-\eta}\left(\alpha\frac{\partial Q_{p+1}}{\partial\xi}+\beta\frac{\partial Q_{p+1}}{\partial\eta}\right)d\zeta\Big\}\,\xi^i d\eta d\xi = 0, \text{for } 1\le i\le p+1.\end{aligned} \tag{4.3.77}$$

Noting that the orthogonality conditions (4.3.77) are the same as (4.3.66) with $j=0$, except for the integral with the coefficient γ is missing. Next we show that this integral is zero for $1\le i\le p+1$.
Testing against $V=\xi^i$, for $1\le i\le p+1$ in (4.3.18), yields

$$\begin{aligned}&\int_0^1\int_0^{1-\xi}\Big\{(\alpha+\beta+\gamma)\,Q_{p+1}(\xi,\eta,1-\xi-\eta)+\gamma Q_{p+1}(\xi,\eta,0)\\&-\int_0^{1-\xi-\eta}\left(\alpha\frac{\partial Q_{p+1}}{\partial\xi}+\beta\frac{\partial Q_{p+1}}{\partial\eta}+\gamma\frac{\partial Q_{p+1}}{\partial\zeta}\right)d\zeta\Big\}\,\xi^i d\eta d\xi = 0.\end{aligned}$$

Integrating $\frac{\partial Q_{p+1}}{\partial\zeta}$ in the third integral with respect to ζ, leads to

$$\begin{aligned}&\int_0^1\int_0^{1-\xi}\{(\alpha+\beta)\,Q_{p+1}(\xi,\eta,1-\xi-\eta)+2\gamma Q_{p+1}(\xi,\eta,0)\\&-\int_0^{1-\xi-\eta}\left(\alpha\frac{\partial Q_{p+1}}{\partial\xi}+\beta\frac{\partial Q_{p+1}}{\partial\eta}\right)d\zeta\Big\}\,\xi^i d\eta d\xi = 0.\end{aligned} \tag{4.3.78}$$



(a) $\Delta_{1,1}$ (b) *Type 1*

(c) *Type 2* (d) $\Delta_{1,2}$ and *Type 3*

Figure 4.8: Superconvergence points (red) using the space $\mathcal{U}_p$ for each type of element.

Subtracting (4.3.77) from (4.3.78), we obtain

$$2\gamma \int_0^1 \int_0^{1-\xi} Q_{p+1}(\xi, \eta, 0)\, \xi^i d\eta d\xi = 0, \text{ for } 1 \le i, j, i+j \le p+1. \tag{4.3.79}$$

Combining (4.3.79) and (4.3.77) leads to (4.3.66) for $1 \le i \le p+1$ and $j = 0$, which completes the proof of (4.3.66).
Now, using (4.3.66) yields (4.3.52b).
The proof of (4.3.52c), follow the same line of reasoning as for (4.3.52b).
Finally, substituting $\zeta = 1-\eta$, $\eta = 1-\xi$ in (4.3.52a) and (4.3.52b), respectively, and noting that $\boldsymbol{a}_0 \cdot \mathbf{n} < 0$ on the *inflow* faces, yields (4.3.53) which completes the proof. □

We summarize the new superconvergence results in Table 4.4 and Figure 4.8 for all types of elements and the space $\mathcal{U}_p$.

Next, we consider an element of *Class I* and show that using the space $\mathcal{M}_p$ leads to a more efficient *a posteriori* error estimation proceeder.

	Type 1			*Type 2*		*Type 3*
	$\Delta_{1,1}$	$\Delta_{2,1}$	$\Delta_{3,1}$	$\Delta_{1,2}$	$\Delta_{2,2}$	$\Delta_{1,3}$
$\Gamma_1 : \eta = 0$	✓	×	×	×	×	×
$\Gamma_2 : \zeta = 0$	×	×	×	×	×	×
$\Gamma_3 : \xi = 0$	×	×	×	×	×	×
$\Gamma_4 : \zeta + \xi + \eta = 1$	×	×	×	×	×	×
$Edge_1 : \mathbf{v}_1\mathbf{v}_2$	✓	×	×	×	×	×
$Edge_2 : \mathbf{v}_1\mathbf{v}_4$	✓	×	×	×	×	×
$Edge_3 : \mathbf{v}_2\mathbf{v}_4$	✓	✓	✓	×	×	×
$Edge_4 : \mathbf{v}_1\mathbf{v}_3$	×	×	×	×	×	×
$Edge_5 : \mathbf{v}_3\mathbf{v}_4$	✓	✓	✓	✓	×	✓
$Edge_6 : \mathbf{v}_2\mathbf{v}_3$	✓	✓	✓	✓	✓	✓
$\mathbf{v}_1 = (0,0,0)$	✓	×	×	×	×	×
$\mathbf{v}_2 = (1,0,0)$	✓	✓	✓	✓	✓	✓
$\mathbf{v}_3 = (0,1,0)$	✓	✓	✓	✓	✓	✓
$\mathbf{v}_4 = (0,0,1)$	✓	✓	✓	✓	×	✓

Table 4.4: Superconvergence points using space $\mathcal{U}_p$.

Theorem 4.3.7. *Let $u \in C^\infty(\Omega)$ and $U \in \mathcal{M}_p$, respectively, be the solutions of (3.1.1a), (3.1.1c) and (4.1.6), with $U^-|_{\Gamma^-} = u$. Let Δ be a tetrahedron of Class I, then the local finite element error can be written as (4.3.1) and its leading term Q_{p+1} satisfies the following orthogonality conditions,*

$$\iiint_\Delta Q_{p+1} V d\xi d\eta d\zeta \quad = \quad 0, \ \forall \ V \in \mathcal{P}_p, \tag{4.3.80}$$

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+1} V d\sigma \quad = \quad 0, \ \forall \ V \in \mathcal{M}_p. \tag{4.3.81}$$

Furthermore,

$$Q_{p+1}(\xi, 0, \zeta) \quad = \quad \sum_{i=0}^{p+1} C_i^{p+1} \varphi_i^{p+1-i}(\xi, \zeta), \text{ on the outflow face,} \tag{4.3.82a}$$

$$Q_{p+1}(\xi, \eta, \zeta) \quad = \quad Q_{p+1}(\xi, 0, \zeta) + \eta \sum_{i=0}^{p} \sum_{j=0}^{i} C_{i,j}^p q_{i-j,j}^{p-i}(\xi, \eta, \zeta), \text{ on } \Delta. \tag{4.3.82b}$$

where $q_{j,k}^i$ are defined in Lemma 4.2.3.

Proof. Applying Lemma 4.2.1 leads to (4.3.1). From Theorem 3.2.2, it suffices to show the orthogonality conditions (4.3.80) for $V \in \mathcal{P}_p \backslash \mathcal{P}_{p-1}$.

Substituting (4.1.10), (4.1.12) and (4.3.1) into (4.1.9) yields the following orthogonality condition for the leading term Q_{p+1}

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} Q_{p+1}(\xi, 0, \zeta) V(\xi, 0, \zeta)\, d\xi d\zeta - \iiint_{\Delta} (\boldsymbol{a}_0 \cdot \nabla V) Q_{p+1} d\xi d\eta d\zeta = 0, \forall\, V \in \mathcal{M}_p. \tag{4.3.83}$$

Testing against $V(\xi, \eta, \zeta) = \eta\left(\xi^i \eta^j \zeta^k\right)$ in (4.3.83), $0 \leq i, j, k \leq p-1, i+j+k = p$ and noting that $V(\xi, 0, \zeta) = 0$ and

$$span\left\{\boldsymbol{a}_0 \cdot \nabla\left(\xi^i \eta^{j+1} \zeta^k\right), 0 \leq i, j, k \leq p, i+j+k = p\right\} = \mathcal{P}_p \backslash \mathcal{P}_{p-1},$$

yield (4.3.80) for $V \in \mathcal{P}_p \backslash \mathcal{P}_{p-1}$.
Using (4.3.80) leads to (4.3.81).
On the *outflow* face, (4.3.82a) follows directly from (4.3.81) and (3.2.25b). Writing Q_{p+1} as (4.3.7) and substituting into (4.3.80) yields

$$\iiint_{\Delta} r_{p+1}(\xi, \zeta) V d\xi d\eta d\zeta + \iiint_{\Delta} \eta \tilde{r}_p(\xi, \eta, \zeta) V d\xi d\eta d\zeta = 0, \forall\, V \in \mathcal{P}_p. \tag{4.3.84}$$

Testing against $V = \xi^i \eta^j \zeta^k$ for $0 \leq i, j, k, i+j+k \leq p-1$, the first integral can be written as

$$\int_0^1 \int_0^{1-\zeta} \xi^i \zeta^k r_{p+1}(\xi, \zeta) \int_0^{1-\xi-\zeta} \eta^j d\eta d\xi d\zeta = \int_0^1 \int_0^{1-\zeta} (1-\xi-\zeta)^{j+1} \xi^i \zeta^k r_{p+1}(\xi, \zeta) d\xi d\zeta. \tag{4.3.85}$$

Combining, (4.3.85), (4.3.81) and (4.3.84) yields

$$\iiint_{\Delta} \eta \tilde{r}_p(\xi, \eta, \zeta) V d\xi d\eta d\zeta = 0, \forall\, V \in \mathcal{P}_{p-1}. \tag{4.3.86}$$

Using Lemma 4.2.3 we write

$$\tilde{r}_p(\xi, \eta, \zeta) = \sum_{n=0}^{p} \sum_{i=0}^{n} \sum_{j=0}^{i} C_{i,j}^n q_{i-j,j}^{n-i}(\xi, \eta, \zeta). \tag{4.3.87}$$

By the orthogonality (4.3.86) and (4.2.7) we obtain (4.3.82b). □

4.4 Factoring the Leading Term of the DG Error

In this section we introduce Groebner bases, which will alow us to factor the leading term Q_{p+1} of the local finite element error. First, in the following subsection we review basic algebraic geometry results [34].

4.4.1 Basic Algebraic Geometry and Notations

Let $f_1, f_2, \ldots, f_s$ be polynomials in $\mathbb{C}[x_1, x_2, \ldots, x_n]$, which denotes the set of all polynomials in $x_1, x_2, \ldots, x_n$ with coefficients in $\mathbb{C}$. Then the set

$$\mathbf{V}\left(f_1, f_2, \ldots, f_s\right) = \left\{(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n : f_i\left(a_1, a_2, \ldots, a_n\right) = 0 \text{ for all } 1 \leq i \leq s\right\},$$

is called the affine variety defined by $f_1, f_2, \ldots, f_s$.

Definition 4.4.1. *A subset* $\mathbf{I} \subset \mathbb{C}\left[x_1, x_2, \ldots, x_n\right]$ *is an ideal if it satisfies:*

(i) $0 \in \mathbf{I}$.

(ii) If $f, g \in \mathbf{I}$, *then* $f + g \in \mathbf{I}$

(iii) If $f \in \mathbf{I}$ *and* $h \in \mathbb{C}\left[x_1, x_2, \ldots, x_n\right]$, *then* $hf \in \mathbf{I}$.

If $f_1, f_2, \ldots, f_s$ are polynomials in $\mathbb{C}[x_1, x_2, \ldots, x_n]$, then the set

$$\langle f_1, f_2, \ldots, f_s\rangle = \left\{\sum_{i=1}^{s} h_i f_i : h_1, h_2, \ldots, h_s \in \mathbb{C}\left[x_1, x_2, \ldots, x_n\right]\right\},$$

is an ideal of $\mathbb{C}\left[x_1, x_2, \ldots, x_n\right]$ and is called the ideal generated by $f_1, f_2, \ldots, f_s$ [34].

Definition 4.4.2. *Let* $\mathbf{V} \subset \mathbb{C}^n$ *be an affine variety. Then we set*

$$\mathbf{I}\left(\mathbf{V}\right) = \left\{f \in \mathbb{C}\left[x_1, x_2, \ldots, x_n\right] : f\left(a_1, a_2, \ldots, a_n\right) = 0 \textit{ for all } \left(a_1, a_2, \ldots, a_n\right) \in \mathbf{V}\right\}.$$

The observation is that $\mathbf{I}\left(\mathbf{V}\right)$ is an ideal and is called the ideal of $\mathbf{V}$ [34].

Next, we define the affine variety by an ideal $\mathbf{I}\left(\mathbf{V}\right) \subset \mathbb{C}\left[x_1, x_2, \ldots, x_n\right]$.

Definition 4.4.3. *Let* $\mathbf{I}\left(\mathbf{V}\right) \subset \mathbb{C}\left[x_1, x_2, \ldots, x_n\right]$ *be an ideal. We will denote by* $\mathbf{V}\left(\mathbf{I}\right)$ *the set*

$$\mathbf{V}\left(\mathbf{I}\right) = \left\{(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n : f\left(a_1, a_2, \ldots, a_n\right) = 0 \textit{ for all } f \in \mathbf{I}\right\}.$$

The following proposition holds.

Proposition 4.4.1. $\mathbf{V}\left(\mathbf{I}\right)$ *is an affine variety. In particular, if* $\mathbf{I} = \langle f_1, f_2, \ldots, f_s\rangle$, *then*

$$\mathbf{V}\left(\mathbf{I}\right) = \mathbf{V}\left(f_1, f_2, \ldots, f_s\right).$$

Proof. Consult [34]. □

The operation on varieties corresponds to the operation of intersection on ideals and is given by the following result.

Theorem 4.4.1. *If* $\mathbf{I}$ *and* $\mathbf{J}$ *are ideals in* $\mathbb{C}[x_1, x_2, \ldots, x_n]$*, then* $\mathbf{V}(\mathbf{I} \cap \mathbf{J}) = \mathbf{V}(\mathbf{I}) \cup \mathbf{V}(\mathbf{J})$*.*

Proof. Consult [34]. □

Definition 4.4.4. *Let* $\mathbf{I} \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$ *be an ideal.*
(i) The radical of $\mathbf{I}$*, denoted* $\sqrt{\mathbf{I}}$*, is the set*

$$\{f : f^m \in \mathbf{I} \text{ for some integer } m \geq 1\}.$$

(ii) $\mathbf{I}$ *is prime if whenever* $f, g \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ *and* $fg \in \mathbf{I}$*, then either* $f \in \mathbf{I}$ *or* $g \in \mathbf{I}$*.*

Thus we have the following proposition.

Proposition 4.4.2. *Let* $\mathbf{I}$*,* $\mathbf{J}$ *be any ideals in* $\mathbb{C}[x_1, x_2, \ldots, x_n]$*, then to following holds:*
(i) $\sqrt{\mathbf{I} \cap \mathbf{J}} = \sqrt{\mathbf{I}} \cap \sqrt{\mathbf{J}}$*.*
(ii) If $\mathbf{I}$ *is a prime ideal then* $\sqrt{\mathbf{I}} = \mathbf{I}$*.*

Proof. Consult [34]. □

4.4.2 Groebner Bases

In order to introduce Groebner bases, we will use the following terminology.

Definition 4.4.5. *A monomial ordering* $>$ *on* $\mathbb{C}[x_1, x_2, \ldots, x_n]$ *is any relation on the set of monomials* $\mathbf{x}^\alpha, \alpha \in \mathbb{Z}^n_{\geq 0}$*, satisfying:*
(i) $>$ *is a total (or linear) ordering on* $\mathbb{Z}^n_{\geq 0}$*.*
(ii) If $\alpha > \beta$ *and* $\gamma \in \mathbb{Z}^n_{\geq 0}$*, then* $\alpha + \gamma > \beta + \gamma$*.*
(iii) $>$ *is a well-ordering on* $\mathbb{Z}^n_{\geq 0}$*, i.e. every nonempty subset of* $\mathbb{Z}^n_{\geq 0}$ *has a smallest element under* $>$ *.*

Definition 4.4.6. *Let* $f = \sum_\alpha a_\alpha \mathbf{x}^\alpha$ *be a nonzero polynomial in* $\mathbb{C}[x_1, x_2, \ldots, x_n]$ *and let* $>$ *be a monomial order. The leading term of* f *is*

$$LT(f) = a_{multideg(f)} \cdot \mathbf{x}^{multideg(f)},$$

where $multideg(f) = \max\left(\alpha \in \mathbb{Z}^n_{\geq 0} \text{ such that } a_\alpha \neq 0\right)$*, (the maximum is taken with respect to* $>$*).*

Definition 4.4.7. *Let* $\mathbf{I} \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$ *be an ideal other than* $\{0\}$*.*
(i) We denote by $LT(\mathbf{I})$ *the set of leading terms of elements of* $\mathbf{I}$*. Thus,*

$$LT(\mathbf{I}) = \{c\mathbf{x}^\alpha : \text{ there exists } f \in \mathbf{I} \text{ with } LT(\mathbf{f}) = c\mathbf{x}^\alpha\}$$

(ii) We denote by $\langle LT(\mathbf{I}) \rangle$ *the ideal generated by the elements of* $LT(\mathbf{I})$*.*

Thus, a Groebner basis for an ideal $\mathbf{I}$ is defined as follows.

Definition 4.4.8. *For a given monomial order. A finite subset $G = \{g_1, g_2, \ldots, g_s\}$ of an ideal $\mathbf{I}$ is said to be a Groebner basis (or standard basis) if*

$$\langle LT(g_1), LT(g_2), \ldots, LT(g_s)\rangle = \langle LT(\mathbf{I})\rangle .$$

For a given monomial order, every ideal $\mathbf{I} \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$ other than $\{0\}$ has a Groebner basis [34].
The next proposition states that the remainder is uniquely determined when we divide by a Groebner basis.

Proposition 4.4.3. *Let $G = \{g_1, g_2, \ldots, g_s\}$ be a Groebner basis for an ideal $\mathbf{I} \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$ and let $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$. Then there is a unique $r \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ with the following two properties:*
(i) No term of r is divisible by any of $LT(g_1), LT(g_2), \ldots, LT(g_s)$.
(ii) There is $g \in \mathbf{I}$ such that $f = g + r$.

Proof. Consult [34]. □

4.4.3 Factoring Polynomials in $\mathbb{C}[\xi, \eta, \zeta]$

Before stating our main result on factoring the leading DG error term we state and prove several lemmas.

Lemma 4.4.1. *Let $Q_k(\xi, \eta, \zeta)$ be a polynomial of degree $k \geq 3$ that is identically zero at these locations: (i) $\eta = 0$, (ii) $\xi + \eta + \zeta = 1$ and $\xi = 0$, (iii) $\xi + \eta + \zeta = 1$ and $\zeta = 0$. Then $Q_k(\xi, \eta, \zeta)$ can be factored as*

$$\begin{aligned} Q_k(\xi, \eta, \zeta) &= \eta\left((1 - \xi - \eta - \zeta) q_r(\xi, \eta, \zeta) + \xi\zeta q_s(\xi, \eta, \zeta)\right), && \text{(4.4.1a)} \\ &= \eta\left((\xi + \eta + \zeta - 1) R_{r'} + \zeta(\eta + \zeta - 1) R_{s'}\right), && \text{(4.4.1b)} \end{aligned}$$

where q_r, $R_{r'}$, q_s, and $R_{s'}$, respectively, are polynomials of degree $r, r' \leq k-2$, and $s, s' \leq k-3$.

Proof. Noting that $Q_k = 0$ at points of the form *(i)*, *(ii)*, *(iii)* is equivalent to $Q_k = 0$ on the set

$$\mathbf{V}(\langle \eta \rangle) \cup \mathbf{V}(\langle \xi + \eta + \zeta - 1, \xi \rangle) \cup \mathbf{V}(\langle \xi + \eta + \zeta - 1, \zeta \rangle), \tag{4.4.2}$$

which, in turn, is equivalent to

$$Q_k \in \mathbf{I}\left(\mathbf{V}(\langle \eta \rangle) \cup \mathbf{V}(\langle \xi + \eta + \zeta - 1, \xi \rangle) \cup \mathbf{V}(\langle \xi + \eta + \zeta - 1, \zeta \rangle)\right), \tag{4.4.3}$$

Using Theorem 4.4.1, leads to

$$Q_k \in \mathbf{I}\left(\mathbf{V}(\langle \eta \rangle \cap \langle \xi + \eta + \zeta - 1, \xi \rangle \cap \langle \xi + \eta + \zeta - 1, \zeta \rangle)\right). \tag{4.4.4}$$

Next, by the strong Nullstellensatz Theorem [34], we have

$$Q_k \in \sqrt{\langle\eta\rangle \cap \langle\xi+\eta+\zeta-1,\xi\rangle \cap \langle\xi+\eta+\zeta-1,\zeta\rangle}, \tag{4.4.5}$$

which by Proposition 4.4.2, yields

$$Q_k \in \sqrt{\langle\eta\rangle} \cap \sqrt{\langle\xi+\eta+\zeta-1,\xi\rangle} \cap \sqrt{\langle\xi+\eta+\zeta-1,\zeta\rangle}. \tag{4.4.6}$$

Since the ideals $\langle\eta\rangle$, $\langle\xi+\eta+\zeta-1,\xi\rangle$ and $\langle\xi+\eta+\zeta-1,\zeta\rangle$ are all prime, so equal to their own radical, we have

$$Q_k \in \langle\eta\rangle \cap \langle\xi+\eta+\zeta-1,\xi\rangle \cap \langle\xi+\eta+\zeta-1,\zeta\rangle. \tag{4.4.7}$$

Let $J = \langle\eta\rangle \cap \langle\xi+\eta+\zeta-1,\xi\rangle \cap \langle\xi+\eta+\zeta-1,\zeta\rangle$, an ideal of $\mathbb{C}[\xi,\eta,\zeta]$. To test whether $Q_k \in J$, we need a Groebner basis for J

$$\{\eta(\xi+\eta+\zeta-1), \eta\zeta(\eta+\zeta-1)\},$$

computed with Mathematica.
Now, $Q_k \in J$ if and only if Q_k can be written as

$$Q_k(\xi,\eta,\zeta) = \eta(\xi+\eta+\zeta-1)R_{r'} + \eta\zeta(\eta+\zeta-1)R_{s'},$$

for some polynomials $R_{r'}$ and $R_{s'}$. Using Mathematica, we can show that the ideal $K = \langle\eta(\xi+\eta+\zeta-1), \eta\xi\zeta\rangle$, has the same Groebner basis as J. Thus $K = J$, which completes the proof. □

Lemma 4.4.2. *Let $Q_k(\xi,\eta,\zeta)$ be a polynomial of degree $k \geq 3$ that is identically zero at these locations: (i) $\xi+\eta+\zeta = 1$ and $\xi = 0$, (ii) $\xi+\eta+\zeta = 1$ and $\zeta = 0$. Then $Q_k(\xi,\eta,\zeta)$ can be factored as*

$$\begin{aligned} Q_k(\xi,\eta,\zeta) &= (1-\xi-\eta-\zeta)\,q_r(\xi,\eta,\zeta) + \xi\zeta q_s(\xi,\eta,\zeta), && (4.4.8a) \\ &= (\xi+\eta+\zeta-1)R_{r'} + \zeta(\eta+\zeta-1)R_{s'}, && (4.4.8b) \end{aligned}$$

where q_r, $R_{r'}$, q_s, and $R_{s'}$, respectively, are polynomials of degree $r, r' \leq k-1$, and $s, s' \leq k-2$.

Proof. Noting that $Q_k = 0$ at points of the form *(i), (ii)* is equivalent to $Q_k = 0$ on the set

$$\mathbf{V}(\langle\xi+\eta+\zeta-1,\xi\rangle) \cup \mathbf{V}(\langle\xi+\eta+\zeta-1,\zeta\rangle), \tag{4.4.9}$$

which, in turn, is equivalent to

$$Q_k \in \mathbf{I}(\mathbf{V}(\langle\xi+\eta+\zeta-1,\xi\rangle) \cup \mathbf{V}(\langle\xi+\eta+\zeta-1,\zeta\rangle)). \tag{4.4.10}$$

Using Theorem 4.4.1, leads to

$$Q_k \in \mathbf{I}(\mathbf{V}(\langle\xi+\eta+\zeta-1,\xi\rangle \cap \langle\xi+\eta+\zeta-1,\zeta\rangle)). \tag{4.4.11}$$

Next, by the strong Nullstellensatz Theorem [34], we have

$$Q_k \in \sqrt{\langle \xi+\eta+\zeta-1, \xi\rangle \cap \langle \xi+\eta+\zeta-1, \zeta\rangle}, \tag{4.4.12}$$

which by Proposition 4.4.2, yields

$$Q_k \in \sqrt{\langle \xi+\eta+\zeta-1, \xi\rangle} \cap \sqrt{\langle \xi+\eta+\zeta-1, \zeta\rangle}. \tag{4.4.13}$$

Since the ideals $\langle \xi+\eta+\zeta-1, \xi\rangle$ and $\langle \xi+\eta+\zeta-1, \zeta\rangle$ are all prime, so equal to their own radical, we have

$$Q_k \in \langle \xi+\eta+\zeta-1, \xi\rangle \cap \langle \xi+\eta+\zeta-1, \zeta\rangle\,. \tag{4.4.14}$$

Let $J = \langle \eta\rangle \cap \langle \xi+\eta+\zeta-1, \zeta\rangle \cap \langle \xi+\eta+\zeta-1, \xi\rangle$, an ideal of $\mathbb{C}\left[\xi, \eta, \zeta\right]$. To test whether $Q_k \in J$, we need a Groebner basis for J

$$\left\{\left(\xi+\eta+\zeta-1\right), \zeta\left(\eta+\zeta-1\right)\right\},$$

computed with Mathematica.
Now, $Q_k \in J$ if and only if Q_k can be written as

$$Q_k\left(\xi, \eta, \zeta\right) = \left(\xi+\eta+\zeta-1\right) R_{r'} + \zeta\left(\eta+\zeta-1\right) R_{s'},$$

for some polynomials $R_{r'}$ and $R_{s'}$. Using Mathematica, we can show that the ideal $K = \langle \xi+\eta+\zeta-1, \xi\zeta\rangle$, has the same Groebner basis as J. Thus $K = J$, which completes the proof. $\square$

Lemma 4.4.3. *Let $Q_k\left(\xi, \eta, \zeta\right)$ be a polynomial of degree $k \geq 3$ that is identically zero at these locations: (i) $\xi+\eta+\zeta = 1$ and $\xi = 0$, (ii) $\xi+\eta+\zeta = 1$ and $\eta = 0$, (iii) $\xi+\eta+\zeta = 1$ and $\zeta = 0$. Then $Q_k\left(\xi, \eta, \zeta\right)$ can be factored as*

$$\begin{aligned} Q_k\left(\xi, \eta, \zeta\right) &= \left(1-\xi-\eta-\zeta\right) q_r\left(\xi, \eta, \zeta\right) + \eta\xi\zeta q_s\left(\xi, \eta, \zeta\right), && (4.4.15a)\\ &= \left(\xi+\eta+\zeta-1\right) R_{r'} + \zeta\eta\left(\eta+\zeta-1\right) R_{s'}, && (4.4.15b)\end{aligned}$$

where q_r, $R_{r'}$, q_s, and $R_{s'}$, respectively, are polynomials of degree $r, r' \leq k-1$, and $s, s' \leq k-3$.

Proof. Noting that $Q_k = 0$ at points of the form *(i), (ii)* is equivalent to $Q_k = 0$ on the set

$$\mathbf{V}\left(\langle \xi+\eta+\zeta-1, \xi\rangle\right) \cup \mathbf{V}\left(\langle \xi+\eta+\zeta-1, \eta\rangle\right) \cup \mathbf{V}\left(\langle \xi+\eta+\zeta-1, \zeta\rangle\right), \tag{4.4.16}$$

which, in turn, is equivalent to

$$Q_k \in \mathbf{I}\left(\mathbf{V}\left(\langle \xi+\eta+\zeta-1, \xi\rangle\right) \cup \mathbf{V}\left(\langle \xi+\eta+\zeta-1, \eta\rangle\right) \cup \mathbf{V}\left(\langle \xi+\eta+\zeta-1, \zeta\rangle\right)\right), \tag{4.4.17}$$

Using Theorem 4.4.1, leads to

$$Q_k \in \mathbf{I}\left(\mathbf{V}\left(\langle \xi+\eta+\zeta-1, \xi\rangle \cap \langle \xi+\eta+\zeta-1, \eta\rangle \cap \langle \xi+\eta+\zeta-1, \zeta\rangle\right)\right). \tag{4.4.18}$$

Next, by the strong Nullstellensatz Theorem [34], we have

$$Q_k \in \sqrt{\langle \xi+\eta+\zeta-1, \xi\rangle \cap \langle \xi+\eta+\zeta-1, \eta\rangle \cap \langle \xi+\eta+\zeta-1, \zeta\rangle}, \tag{4.4.19}$$

which by Proposition 4.4.2, yields

$$Q_k \in \sqrt{\langle \xi+\eta+\zeta-1, \xi\rangle} \cap \sqrt{\langle \xi+\eta+\zeta-1, \eta\rangle} \cap \sqrt{\langle \xi+\eta+\zeta-1, \zeta\rangle}. \tag{4.4.20}$$

Since the ideals $\langle \xi+\eta+\zeta-1, \xi\rangle$, $\langle \xi+\eta+\zeta-1, \eta\rangle$ and $\langle \xi+\eta+\zeta-1, \zeta\rangle$ are all prime, so equal to their own radical, we have

$$Q_k \in \langle \xi+\eta+\zeta-1, \xi\rangle \cap \langle \xi+\eta+\zeta-1, \eta\rangle \cap \langle \xi+\eta+\zeta-1, \zeta\rangle\,. \tag{4.4.21}$$

Let $J = \langle \xi+\eta+\zeta-1, \xi\rangle \cap\cap \langle \xi+\eta+\zeta-1, \eta\rangle \cap \langle \xi+\eta+\zeta-1, \zeta\rangle$, an ideal of $\mathbb{C}[\xi,\eta,\zeta]$. To test whether $Q_k \in J$, we need a Groebner basis for J

$$\{(\xi+\eta+\zeta-1)\,, \zeta\eta\,(\eta+\zeta-1)\}\,, \tag{4.4.22}$$

computed with Mathematica.
Now, $Q_k \in J$ if and only if Q_k can be written as

$$Q_k(\xi,\eta,\zeta) = (\xi+\eta+\zeta-1)\,R_{r'} + \eta\zeta\,(\eta+\zeta-1)\,R_{s'}, \tag{4.4.23}$$

for some polynomials $R_{r'}$ and $R_{s'}$. Using Mathematica, we can show that the ideal $K = \langle \xi+\eta+\zeta-1, \eta\xi\zeta\rangle$, has the same Groebner basis as J. Thus $K = J$, which completes the proof. □

Lemma 4.4.4. *Let $Q_k(\xi,\eta,\zeta)$ be a polynomial of degree $k \geq 3$ that is identically zero at these locations: (i) $\xi+\eta+\zeta = 1$ and $\zeta = 0$. Then $Q_k(\xi,\eta,\zeta)$ can be factored as*

$$\begin{aligned} Q_k(\xi,\eta,\zeta) &= (1-\xi-\eta-\zeta)\,q_r(\xi,\eta,\zeta) + \zeta q_s(\xi,\eta,\zeta)\,, && (4.4.24a)\\ &= (\xi+\eta-1)\,R_{r'} + \zeta R_{s'}, && (4.4.24b)\end{aligned}$$

where q_r, $R_{r'}$, q_s, and $R_{s'}$, respectively, are polynomials of degree $r, r' \leq k-1$, and $s, s' \leq k-1$.

Proof. Noting that $Q_k = 0$ at points of the form *(i)* is equivalent to $Q_k = 0$ on the set

$$\mathbf{V}\left(\langle \xi+\eta+\zeta-1, \zeta\rangle\right), \tag{4.4.25}$$

which in turn is equivalent to

$$Q_k \in \mathbf{I}\left(\mathbf{V}\left(\langle \xi+\eta+\zeta-1, \zeta\rangle\right)\right). \tag{4.4.26}$$

Next, by the strong Nullstellensatz Theorem [34], we have

$$Q_k \in \sqrt{\langle \xi+\eta+\zeta-1, \zeta\rangle}. \tag{4.4.27}$$

Since the ideal $\langle \xi+\eta+\zeta-1,\zeta\rangle$ is prime, so equal to his own radical, we have

$$Q_k \in \langle \xi+\eta+\zeta-1,\zeta\rangle\,. \tag{4.4.28}$$

Let $J=\langle \xi+\eta+\zeta-1,\zeta\rangle$, an ideal of $\mathbb{C}\left[\xi,\eta,\zeta\right]$. To test whether $Q_k\in J$, we need a Groebner basis for J

$$\left\{\xi+\eta-1,\zeta\right\}. \tag{4.4.29}$$

computed with Mathematica.
Now, $Q_k\in J$ if and only if Q_k can be written as

$$Q_k\left(\xi,\eta,\zeta\right)=\left(\xi+\eta-1\right)R_{r'}+\zeta R_{s'}, \tag{4.4.30}$$

for some polynomials $R_{r'}$ and $R_{s'}$. Using Mathematica, we can show that the ideal $K=\langle \xi+\eta+\zeta-1,\zeta\rangle$, has the same Groebner basis as J. Thus $K=J$, which completes the proof. □

As a direct consequence of the last four lemmas, we have the following factoring of the leading term Q_{p+1} of the local finite element error.

Theorem 4.4.2. *Under the same assumptions of Theorem 4.3.2, the leading term of the discretization error Q_{p+1} can be factored as*

$$\begin{aligned} Q_{p+1}\left(\xi,\eta,\zeta\right) &= \left(1-\xi-\eta-\zeta\right)q_r\left(\xi,\eta,\zeta\right)+\eta\xi\zeta q_s\left(\xi,\eta,\zeta\right), && \text{(4.4.31a)}\\ &= \left(\xi+\eta+\zeta-1\right)R_{r'}+\zeta\eta\left(\eta+\zeta-1\right)R_{s'}, && \text{(4.4.31b)}\end{aligned}$$

where q_r, $R_{r'}$, q_s, and $R_{s'}$, respectively, are polynomials of degree $r,r'\leq p$,and $s,s'\leq p-2$. Furthermore, if Δ is a tetrahedron of Class I and Type 1 (i.e. $\alpha=\gamma=0$ and $\beta<0$), then Q_{p+1} can be factored as

$$\begin{aligned} Q_{p+1}\left(\xi,\eta,\zeta\right) &= \eta\left(\left(1-\xi-\eta-\zeta\right)\tilde{q}_r\left(\xi,\eta,\zeta\right)+\xi\zeta\tilde{q}_s\left(\xi,\eta,\zeta\right)\right), && \text{(4.4.32a)}\\ &= \eta\Big(\left(\xi+\eta+\zeta-1\right)\tilde{R}_{r'}+\zeta\left(\eta+\zeta-1\right)\tilde{R}_{s'}\Big), && \text{(4.4.32b)}\end{aligned}$$

where $\tilde{q}_r$, $\tilde{R}_{r'}$, $\tilde{q}_s$, and $\tilde{R}_{s'}$, respectively, are polynomials of degree $r,r'\leq p-1$,and $s,s'\leq p-2$.

Proof. Using (4.3.16), one can verify that Q_{p+1} is zero at points of the form *(i), (ii), (iii)* of the Lemma 4.4.3, which in turn leads to (4.4.31).
Next, from (4.3.4a) and (4.3.16), $Q_{p+1}=0$ at points of the form *(i), (ii), (iii)* of the Lemma 4.4.1. Thus, (4.4.32) follows from Lemma 4.4.1. □

In the next theorem, for elements of *Type 2* we show the following factoring for the leading term of the discretization error.

Theorem 4.4.3. *Under the same assumptions of Theorem 4.4.2, the leading term of the discretization error Q_{p+1} can be factored as*

$$\begin{aligned} Q_{p+1}(\xi,\eta,\zeta) &= (1-\xi-\eta-\zeta)\, q_r(\xi,\eta,\zeta) + \zeta q_s(\xi,\eta,\zeta), && (4.4.33a)\\ &= (\xi+\eta-1)\, R_{r'} + \zeta R_{s'}, && (4.4.33b) \end{aligned}$$

where q_r, $R_{r'}$, q_s, and $R_{s'}$, respectively, are polynomials of degree $r, r', s, s' \leq p$. Furthermore, if Δ is a tetrahedron of Class I and Type 2 (i.e. $\Delta_{1,2}$), Q_{p+1} can be factored as

$$\begin{aligned} Q_{p+1}(\xi,\eta,\zeta) &= (1-\xi-\eta-\zeta)\, \tilde{q}_r(\xi,\eta,\zeta) + \xi\zeta \tilde{q}_s(\xi,\eta,\zeta), && (4.4.34a)\\ &= (\xi+\eta+\zeta-1)\, \tilde{R}_{r'} + \zeta(\eta+\zeta-1)\, \tilde{R}_{s'}, && (4.4.34b) \end{aligned}$$

where $\tilde{q}_r$, $\tilde{R}_{r'}$, $\tilde{q}_s$, and $\tilde{R}_{s'}$, respectively, are polynomials of degree $r, r' \leq p$, and $s, s' \leq p-1$.

Proof. Equation (4.4.33) is obtained by combining Theorem 4.4.2 and Lemma 4.4.4. Next, Theorem 4.3.5 and Lemma 4.4.2 lead to (4.4.34). □

On elements of *Type 3* and using the space $\mathcal{U}_p$ the leading term of the discretization error Q_{p+1} have the following factorization.

Theorem 4.4.4. *Under the same assumptions of Theorem 4.3.6, the leading term of the discretization error $Q_{p+1}(\xi,\eta,\zeta)$ can be factored as*

$$\begin{aligned} Q_{p+1}(\xi,\eta,\zeta) &= (1-\xi-\eta-\zeta)\, q_r(\xi,\eta,\zeta) + \xi\zeta q_s(\xi,\eta,\zeta), && (4.4.35a)\\ &= (\xi+\eta+\zeta-1)\, R_{r'} + \zeta(\eta+\zeta-1)\, R_{s'}, && (4.4.35b) \end{aligned}$$

where q_r, $R_{r'}$, q_s, and $R_{s'}$, respectively, are polynomials of degree $r, r' \leq p$, and $s, s' \leq p-1$.

Proof. Equation (4.4.35) is obtained by combining Theorem 4.3.6 and Lemma 4.4.2. □

4.5 Basis Functions for DG Error Estimation

In order to compute efficient *a posteriori* error estimates for the leading error term Q_{p+1}, on all elements using the spaces $\mathcal{L}_p$, $\mathcal{U}_p$ and $\mathcal{M}_p$ for the DG solution, we construct optimal finite element spaces for the error and solve a local problem on each element. Next we construct basis functions for the error by approximating the true error $u-U$ by its leading term E as

$$(u-U)(x,y,z,h) \approx E(\xi,\eta,\zeta,h) = Q_{p+1}(\xi,\eta,\zeta)\, h^{p+1} = \sum_{i=0}^{p+1}\sum_{j=0}^{i}\sum_{k=0}^{j} c^i_{j,k}\varphi^{i-j}_{j-k,k}(\xi,\eta,\zeta),$$

and applying the orthogonality conditions

$$\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} E V d\sigma - \iiint_{\Delta} \left(\boldsymbol{a}_0 \cdot \nabla V\right) E d\xi d\eta d\zeta = 0, \forall\, V \in \mathcal{W}_p. \tag{4.5.1}$$

Testing against $V = \varphi^{q-r}_{r-s,s}$ in (4.5.1) with $0 \le s \le r \le q \le p$, yields

$$\sum_{i=0}^{p+1}\sum_{j=0}^{i}\sum_{k=0}^{j} c^i_{j,k} \left[\iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n} \varphi^{i-j}_{j-k,k}\varphi^{q-r}_{r-s,s} d\sigma - \iiint_{\Delta} \left(\boldsymbol{a}_0 \cdot \nabla\varphi^{q-r}_{r-s,s}\right) \varphi^{i-j}_{j-k,k} d\xi d\eta d\zeta\right] = 0. \tag{4.5.2}$$

Let $m = \dim \mathcal{W}_p$, $n = \dim \mathcal{P}_{p+1}$. Thus, if

$$\mathbf{C} = \left(\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{p+1}\right) \in \mathbb{R}^n,$$

and

$$\boldsymbol{\Phi} = \left(\Phi_1, \Phi_2, \ldots, \Phi_n\right)^T = \left(\phi_1, \phi_2, \ldots, \phi_{p+1}\right)^T,$$

where

$$\mathbf{c}_l = \left(c^l_{0,0}, c^l_{1,0}, c^l_{1,1}, \ldots, c^l_{l,l}\right),$$

and

$$\phi_l = \left(\varphi^l_{0,0}, \varphi^{l-1}_{1,0}, \varphi^{l-1}_{0,1}, \ldots, \varphi^0_{0,l}\right),$$

the orthogonality conditions (4.5.2) may be written in a matrix form $\mathbf{AC} = 0$ where

$$a_{ij} = \iint_{\Gamma^+} \boldsymbol{a}_0 \cdot \mathbf{n}\Phi_i\Phi_j d\sigma - \iiint_{\Delta} \left(\boldsymbol{a}_0 \cdot \nabla\Phi_i\right)\Phi_j d\xi d\eta d\zeta,\ j = 1, 2, \ldots, n,\ i = 1, 2, , \ldots, m.$$

Without loss of generality we assume $\boldsymbol{a}_0 = (\alpha, \beta, \gamma)^T$ such that $\beta \neq 0$ and let $\lambda = \frac{\alpha}{\beta}$, $\mu = \frac{\gamma}{\beta}$. We now define the finite element space

$$\mathcal{E} = \left\{\mathbf{C}^T\boldsymbol{\Phi}, \mathbf{C} \in \mathcal{N}\left(\mathbf{A}\right)\right\},$$

for the error and state the following lemma.

Lemma 4.5.1. *The polynomial space $\mathcal{E}$ is isomorphic to the null space $\mathcal{N}(\mathbf{A})$, and the leading term Q_{p+1} can be written as*

$$E = Q_{p+1}h^{p+1} = \sum_{i=1}^{n-m} d_i\chi_i,$$

where $\chi_i = \mathbf{C}_i^T\boldsymbol{\Phi}$, $i = 1, 2, \ldots, n-m$, with $\{\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_{n-m}\}$ being a basis of $\mathcal{N}(\mathbf{A})$. Furthermore, for element of Class I and Type 1, the basis $\{\chi_1, \chi_2, \ldots, \chi_{n-m}\}$ is independent of λ and μ.

Proof. The proof follows the line of reasoning used in Lemma 3.3.1. □

Remark 4.5.1. *Lemma 4.5.1 gives an optimal basis for the leading error term Q_{p+1} and reduces the number of degrees of freedom from $n = \dfrac{(p+2)(p+3)(p+4)}{6}$ to $m = \dfrac{p(p-1)}{2}$ for the space $\mathcal{U}_p$. This technique leads to a more efficient error estimation procedure.*

In the following section we construct the error basis functions χ_i.

4.5.1 Basis Functions for Element of *Class I, II* and *III*

We use the space $\mathcal{U}_p$ and follow Lemma 4.5.1 to find a basis of $\mathcal{N}(\boldsymbol{A})$ and construct basis functions for $\mathcal{E}$. For instance, a basis of $\mathcal{N}(\boldsymbol{A})$ for $p = 2$ on a reference element of *Class I* and *Type 1* is given as

$$\boldsymbol{C} = \left(-\frac{7}{60}, -\frac{1}{9}, -\frac{4}{27}, \frac{1}{108}, 0, -\frac{2}{15}, \frac{1}{45}, 0, -\frac{1}{27}, \frac{2}{135}, 0, 1, 0, \frac{4}{3}, \frac{8}{3}, 0, \frac{1}{3}, \frac{4}{3}, 1, 0\right)^T,$$

$$\boldsymbol{\Phi} = (\varphi^0_{0,0}, \varphi^1_{0,0}, \varphi^0_{1,0}, \varphi^0_{0,1}, \ldots, \varphi^0_{0,3})^T,$$

and

$$\chi = \boldsymbol{C}^T\boldsymbol{\Phi} = \frac{1}{3}\eta\left(3\zeta^2 + 4\zeta(\eta + 2\xi - 1) + \eta^2 + \eta(4\xi - 2) + 3\xi^2 - 4\xi + 1\right).$$

Error basis functions for $p = 3, 4$ are computed using *Mathematica* and are shown in Table 4.5.

Similarly, error basis functions for elements of *Class I* and *Type 2* are shown in Table 4.6, while basis functions on elements of *Class I* and *Type 3* are shown in Table 4.7.

Error basis functions on a reference tetrahedron of *Class II* ant *Type 1* are obtained following the previous lemma and using space $\mathcal{U}_p$, *i.e.*, we compute a basis for $\mathcal{N}(\boldsymbol{A})$ which depends on μ and construct the basis functions for $\mathcal{E}$ shown in Table 4.8.

Error basis functions for elements of *Class II* and *Type 2* are shown in Table 4.9.

Similarly, error basis functions for $p = 2$ on a reference tetrahedron of *Class III* are shown in Table 4.10. Error functions for higher-degree polynomials are computed using *Mathematica* and are not presented here.

Remark 4.5.2. *Using the enriched space $\mathcal{U}_2$, and applying Lemma 4.5.1 show that the dimension of the error space $\mathcal{E}$ is one. Thus, we have several interior superconvergence surfaces for each class and type of elements. For instance, for elements of Class I and Type 1, the DG solution is $O(h^{p+2})$ superconvergent on the surfaces given by the following equations*

$$\eta = \sqrt{\zeta^2 + \xi^2} - 2\zeta - 2\xi + 1,$$

and

$$\eta = -\sqrt{\zeta^2 + \xi^2} - 2\zeta - 2\xi + 1,$$

$p=3$	$\chi_1 = -\frac{1}{6}\eta\begin{pmatrix} +\zeta\left(9\eta^2+2\eta(18\xi-5)+27\xi^2-20\xi+1\right) \\ +3\eta^3+\eta^2(18\xi-7)+\eta\left(27\xi^2-28\xi+5\right) \\ -6\zeta^3+6\zeta^2-21\xi^2++12\xi^3 10\xi-1 \end{pmatrix}$ $\chi_2 = \frac{1}{27}\eta(9\eta-2)\begin{pmatrix} 3\zeta^2+4\zeta(\eta+2\xi-1)+\eta^2 \\ +\eta(4\xi-2)+3\xi^2-4\xi+1 \end{pmatrix}$ $\chi_3 = \frac{1}{27}\eta\begin{pmatrix} +\zeta\left(4\eta(9\xi-1)+54\xi^2-44\xi+4\right) \\ +\eta^2(9\xi-1)+\eta\left(27\xi^2-22\xi+2\right)+18\xi^3 \\ +3\zeta^2(9\xi-1)-30\xi^2+13\xi-1 \end{pmatrix}$
$p=4$	$\chi_1 = \frac{1}{165}\eta\left(55\eta^2-30\eta+3\right)\begin{pmatrix} 3\zeta^2+4\zeta(\eta+2\xi-1)+\eta^2 \\ +\eta(4\xi-2)+3\xi^2-4\xi+1 \end{pmatrix}$ $\chi_2 = \frac{1}{55}\eta\begin{pmatrix} 33\eta^4+4\eta^3(66\xi-23)+6\eta^2\left(99\xi^2-92\xi+15\right) \\ +55\zeta^4-80\zeta^3+36\zeta^2+12\eta\left(44\xi^3-69\xi^2+30\xi-3\right) \\ +8\zeta\begin{pmatrix} 11\eta^3+6\eta^2(11\xi-3)+9\eta\left(11\xi^2-8\xi+1\right) \\ +44\xi^3-54\xi^2+18\xi-2 \end{pmatrix} \\ +165\xi^4-368\xi^3+270\xi^2-72\xi+5 \end{pmatrix}$ $\chi_3 = -\frac{1}{110}\eta\begin{pmatrix} \zeta^3(20-110\eta)+\eta\left(220\xi^3-495\xi^2+306\xi-43\right) \\ +3\zeta\begin{pmatrix} 20\eta^2(11\xi-4)+3\eta\left(55\xi^2-60\xi+9\right) \\ +55\eta^3+-30\xi^2+24\xi-2 \end{pmatrix} \\ +5\eta^3(66\xi-29)+3\eta^2\left(165\xi^2-200\xi+43\right) \\ +18\zeta^2(5\eta-1)+55\eta^4-40\xi^3+72\xi^2-36\xi+4 \end{pmatrix}$ $\chi_4 = -\frac{1}{220}\eta\begin{pmatrix} \zeta^3(20-220\xi)+3\eta^2\left(165\xi^2-110\xi+8\right) \\ +6\zeta\begin{pmatrix} \eta^2(55\xi-5)+3\eta\left(55\xi^2-30\xi+2\right)+110\xi^3 \\ -120\xi^2+27\xi-1 \end{pmatrix} \\ +6\eta\left(110\xi^3-150\xi^2+51\xi-3\right)+275\xi^4-580\xi^3 \\ +18\zeta^2(10\xi-1)+10\eta^3(11\xi-1)+387\xi^2-86\xi+4 \end{pmatrix}$ $\chi_5 = \frac{1}{165}\eta\begin{pmatrix} 3\zeta^2(\eta(55\xi-5)-10\xi+1)+\eta^3(55\xi-5)-20\xi^3+33\xi^2 \\ +\zeta\left(20\eta^2(11\xi-1)+6\eta\left(55\xi^2-50\xi+4\right)-60\xi^2+48\xi-4\right) \\ +\eta^2\left(165\xi^2-140\xi+11\right)+\eta\left(110\xi^3-210\xi^2+99\xi-7\right) \\ -14\xi+1 \end{pmatrix}$ $\chi_6 = \frac{1}{495}\eta\begin{pmatrix} 9\zeta^2\left(55\xi^2-20\xi+1\right)+275\xi^4-560\xi^3+354\xi^2-72\xi+3 \\ +4\zeta\left(3\eta\left(55\xi^2-20\xi+1\right)+220\xi^3-255\xi^2+66\xi-3\right) \\ +3\eta^2\left(55\xi^2-20\xi+1\right)+2\eta\left(220\xi^3-255\xi^2+66\xi-3\right) \end{pmatrix}$

Table 4.5: Error basis functions for elements of *Class I* and *Type 1* using the space $\mathcal{U}_p$.

$p=2$	$\chi_1 = \left(\begin{array}{l} 3\eta\left(\zeta^2(6\mu-3)+2\zeta(3\mu-2)(2\xi-1)+(\mu-1)\left(3\xi^2-4\xi+1\right)\right) \\ +\mu\left(10\zeta^3+18\zeta^2(2\xi-1)+9\zeta\left(3\xi^2-4\xi+1\right)+(\xi-1)^2(4\xi-1)\right) \\ +3\eta^2(\zeta(3\mu-4)+(\mu-2)(2\xi-1))+\eta^3(\mu-3) \end{array} \right)$
$p=3$	$\chi_1 = \left(\begin{array}{l} -12\eta \left(\begin{array}{l} (\xi-1)\left(\begin{array}{l}\mu^2\left(176\xi^2-115\xi+9\right)-36\xi \\ +54\xi^2-7\mu\left(34\xi^2-23\xi+2\right)+3\end{array}\right) \\ +\zeta\left(\begin{array}{l}\mu^2\left(804\xi^2-652\xi+58\right)\\ +6\left(27\xi^2-22\xi+2\right)\\ +\mu\left(-789\xi^2+646\xi-60\right)\end{array}\right) \\ +3\zeta^2\left(20\mu^2-16\mu+3\right)(9\xi-1)\end{array}\right) \\ +\mu \left(\begin{array}{l} 105\zeta^4(\mu+1)-40\zeta^3(\mu(78\xi-4)-39\xi+9) \\ -36\zeta^2\left(5\mu\left(51\xi^2-40\xi+3\right)-96\xi^2+86\xi-11\right) \\ -12\zeta(\xi-1)\left(\begin{array}{l}\mu\left(524\xi^2-340\xi+26\right)-178\xi^2\\ +128\xi-13\end{array}\right) \\ -(\xi-1)^2\left(\begin{array}{l}\mu\left(885\xi^2-502\xi+37\right)\\ -3\left(95\xi^2-58\xi+5\right)\end{array}\right)\end{array}\right) \\ -6\eta^2\left(\begin{array}{c} 2\zeta\left(6\mu^2(41\xi-3)+\mu(61-465\xi)+12(9\xi-1)\right) \\ +7\mu^2\left(33\xi^2-24\xi+1\right)+\mu\left(-660\xi^2+558\xi-59\right) \\ +6\left(27\xi^2-22\xi+2\right)\end{array}\right) \\ +63\eta^4\mu(\mu+1)+4\eta^3\left(\begin{array}{c}\mu^2(42\zeta-24\xi-23)\\ +\mu(42\zeta+363\xi-66)-81\xi+9\end{array}\right) \end{array}\right)$
	$\chi_2 = \left(\begin{array}{l} 2\eta\left(\begin{array}{l} 14\zeta^3(\mu+1)+9\zeta^2(\mu(58\xi-8)-23\xi+1) \\ +\zeta\left(\mu\left(738\xi^2-606\xi+57\right)-477\xi^2+384\xi-33\right) \\ +(\xi-1)\left(\mu\left(158\xi^2-103\xi+8\right)-166\xi^2+113\xi-10\right) \end{array}\right) \\ +3\mu\left(\begin{array}{l} 18\zeta\left(18\xi^3-30\xi^2+13\xi-1\right)+(\xi-1)^2\left(45\xi^2-26\xi+2\right) \\ +20\zeta^3(9\xi-1)+18\zeta^2\left(27\xi^2-22\xi+2\right) \end{array}\right) \\ +9\eta^2\left(\begin{array}{l} 2\zeta(\mu(23\xi-1)-40\xi+6)+3(7\mu-20)\xi^2 \\ +(52-14\mu)\xi-6 \end{array}\right) \\ -6\eta^3(7\zeta(\mu+1)+\mu(\xi-4)+37\xi-8)-14\eta^4(\mu+1) \end{array} \right)$
	$\chi_3 = \left(\begin{array}{l} 3\eta^2\left(\begin{array}{l} 21\zeta^2(\mu+1)-4\zeta(9\mu\xi+6\mu-54\xi+13) \\ -21\mu\xi^2+7\mu+141\xi^2-132\xi+19 \end{array}\right) \\ -6\eta\left(\begin{array}{l} +3\zeta^2(\mu(34\xi-3)-20\xi+3) \\ +2\zeta\left(\mu\left(75\xi^2-58\xi+4\right)-60\xi^2+52\xi-6\right) \\ +(\xi-1)\left(32\mu\xi^2-19\mu\xi+\mu-40\xi^2+29\xi-3\right) \end{array}\right) \\ -2\mu\left(\begin{array}{l} 20\zeta^3(9\xi-1)+18\zeta^2\left(27\xi^2-22\xi+2\right) \\ +18\zeta\left(18\xi^3-30\xi^2+13\xi-1\right)+(\xi-1)^2\left(45\xi^2-26\xi+2\right) \end{array}\right) \\ +4\eta^3(21\zeta(\mu+1)+15\mu\xi-11\mu+51\xi-15)+21\eta^4(\mu+1) \end{array} \right)$

Table 4.6: Error basis functions for elements of *Class I* and *Type 2* using the space $\mathcal{U}_p$.

$p=2$	$\chi = \begin{pmatrix} 4\zeta^3(5\mu-1)+9\zeta^2(\eta(4\mu-3)+\mu(8\xi-4)-3\xi+1) \\ +(\eta+\xi-1)\begin{pmatrix} \eta^2(2\mu-7)+2\eta(\mu(5\xi-2)-13\xi+4) \\ +2\mu\left(4\xi^2-5\xi+1\right)-10\xi^2+8\xi-1 \end{pmatrix} \\ +6\zeta\begin{pmatrix} \eta^2(3\mu-5)+2\eta(\mu(6\xi-3)-7\xi+3) \\ +3\mu\left(3\xi^2-4\xi+1\right)-6\xi^2+6\xi-1 \end{pmatrix} \end{pmatrix}$

Table 4.7: Error basis functions for elements of *Class I* and *Type 3* using the space $\mathcal{U}_p$, with $\lambda=-\frac{1}{2}$.

$p=2$	$\chi = \begin{pmatrix} 3\eta\begin{pmatrix} \zeta^2(9\lambda+3)+4\zeta\left(2\lambda^2\xi+6\lambda\xi-3\lambda+2\xi-1\right) \\ +(\xi-1)\left(4\lambda^2\xi+9\lambda\xi-3\lambda+3\xi-1\right) \end{pmatrix} \\ +\lambda(\zeta+\xi-1)\left(8\zeta^2+\zeta(9\lambda\xi+19\xi-10)+(\xi-1)(3\lambda\xi+5\xi-2)\right) \\ +3\eta^2\left(\zeta(8\lambda+4)+3\lambda^2\xi+9\lambda\xi-4\lambda+4\xi-2\right)+\eta^3(5\lambda+3) \end{pmatrix}$

Table 4.8: Error basis functions for elements of *Class II* and *Type 1* using the space $\mathcal{U}_2$.

$p=2$	$\chi = \begin{pmatrix} 4\zeta^3\left(160\mu^2+105\mu+36\right) \\ +3\zeta^2\begin{pmatrix} 320\mu^2(\xi-1)-12\mu(\xi+17)+27(7\xi-4) \\ +\eta\left(320\mu^2+96\mu+270\right) \end{pmatrix} \\ +6\zeta\begin{pmatrix} (\xi-1)\left(64\mu^2(\xi-1)-3\mu(23\xi+13)+90\xi-36\right) \\ +4\eta\left(32\mu^2(\xi-1)-3\mu(7\xi+2)+99\xi-45\right) \\ +\eta^2\left(64\mu^2-15\mu+144\right) \end{pmatrix} \\ +(\eta+\xi-1)\begin{pmatrix} +(\xi-1)\begin{pmatrix} 32\mu^2(\xi-1)-18\mu(5\xi+1) \\ +9(13\xi-4) \end{pmatrix} \\ +\eta\begin{pmatrix} 64\mu^2(\xi-1)-18\mu(7\xi-1) \\ +9(89\xi-26) \end{pmatrix} \\ +2\eta^2\left(16\mu^2-18\mu+99\right) \end{pmatrix} \end{pmatrix}$

Table 4.9: Error basis functions for elements of *Class II* and *Type 2* using the space $\mathcal{U}_p$, with $\lambda=\frac{1}{2}$

$p=2$	$\chi=$	$\left(\begin{array}{l} \zeta^3\left(24\mu^3+96\mu^2+98\mu+48\right) \\ +3\zeta^2\left(\begin{array}{c} 2\eta\left(16\mu^3+76\mu^2+96\mu+45\right)+16\mu^3(2\xi-1) \\ +4\mu^2(37\xi-18)+24\mu(7\xi-3)+9(7\xi-4) \end{array}\right) \\ +6\zeta\left(\begin{array}{l} \eta^2\left(12\mu^3+72\mu^2+101\mu+48\right) \\ +4\eta\left(\begin{array}{c} \mu^3(8\xi-4)+24\mu^2(2\xi-1) \\ +\mu(70\xi-31)+33\xi-15 \end{array}\right) \\ +4\mu^3\left(3\xi^2-4\xi+1\right)+24\mu^2\left(3\xi^2-4\xi+1\right) \\ +\mu\left(89\xi^2-112\xi+23\right)+6\left(5\xi^2-7\xi+2\right) \end{array}\right) \\ +(\eta+\xi-1)\left(\begin{array}{l} \eta\left(12\mu^2(19\xi-8)+4\mu(118\xi-37)+267\xi-78\right) \\ +(\xi-1)\left(12\mu^2(7\xi-2)+4\mu(32\xi-5)+39\xi-12\right) \\ +2\eta^2\left(36\mu^2+64\mu+33\right) \end{array}\right) \end{array}\right)$

Table 4.10: Error basis functions for elements of *Class III* and *Type 1* using the space $\mathcal{U}_p$, with $\lambda=\frac{1}{2}$.

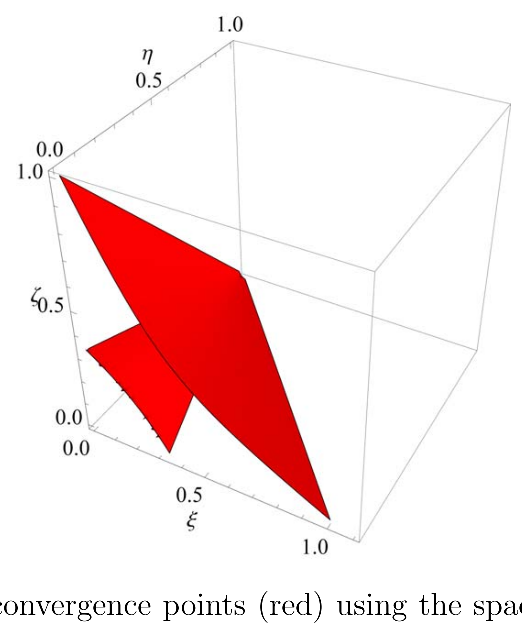
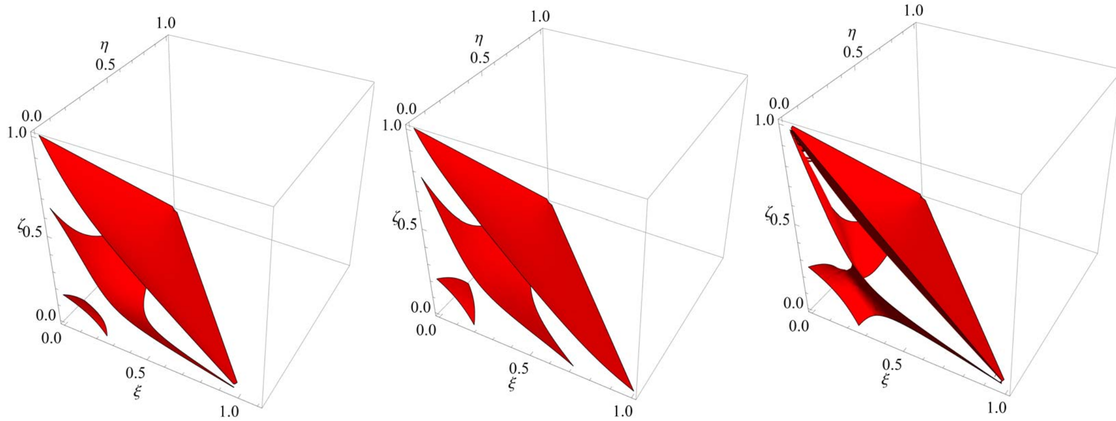
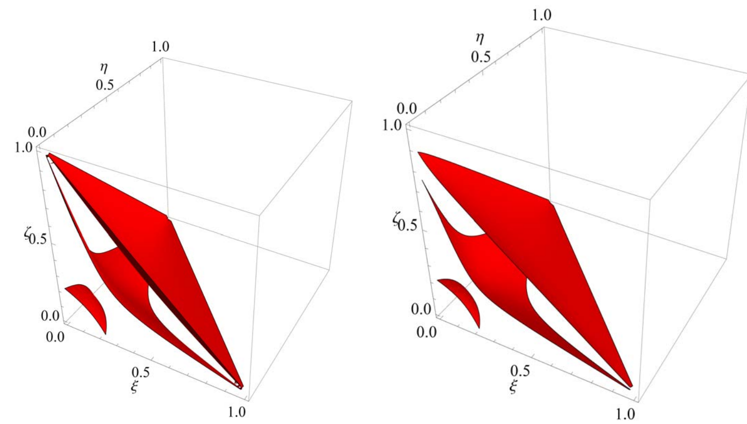


Figure 4.9: Interior superconvergence points (red) using the space $\mathcal{U}_2$ for elements of *Class I* and *Type 1.*



(a) $\Delta_{1,2}$: $\lambda = 0$ and $\mu = -\frac{1}{2}$ (b) $\Delta_{1,3}$: $\lambda = -\frac{1}{2}$ and $\mu = -\frac{1}{3}$ (c) $\Delta_{2,1}$: $\lambda = 1$ and $\mu = 0$



(d) $\Delta_{2,2}$: $\lambda = \frac{1}{2}$ and $\mu = -1$ (e) $\Delta_{3,1}$: $\lambda = \frac{1}{2}$ and $\mu = 1$

Figure 4.10: Interior superconvergence points (red) using the space $\mathcal{U}_2$.

addition to the outflow face $\eta = 0$, and illustrated in Figure 4.9.

On other classes and types of elements, the DG solution is $O(h^{p+2})$ superconvergent on the zero level surfaces of the corresponding error basis functions given in Tables 4.6, 4.7, 4.8, 4.9 and 4.10 where the interior superconvergent surfaces depend in λ and μ. Instances of interior superconvergent surfaces are shown in Figure 4.10.

4.5.2 Error Estimation Procedure

Integrating (3.1.4) by parts shows that the DG solution U on a physical element Δ satisfies

$$\iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}\left(\tilde{U}-U\right)V dS + \iiint_{\Delta} (\boldsymbol{a}\cdot\nabla U + cU)\, V dxdydz = \iiint_{\Delta} fV dxdydz,\ \forall V \in \mathcal{W}_p, \tag{4.5.5}$$

where the numerical flux $\tilde{U}$ is given by (3.1.5a) or (3.1.5b). In order to estimate the finite element error $e = u - U$ on Δ we assume that the leading term of e exhibits the same asymptotic behavior as the local error on Δ. Thus, the error e on Δ is approximated by

$$E\,(x,y,z) = \sum_{i=1}^{K} d_i\chi_i(\xi(x,y,z),\eta(x,y,z),\zeta(x,y,z)), \tag{4.5.6a}$$

which satisfies the weak finite element problem

$$\begin{aligned}&\iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}\left(E^- - E\right)V d\sigma + \iiint_{\Delta} (\boldsymbol{a}\cdot\nabla E)\, V dxdydz\\ = &\iiint_{\Delta} rV dxdydz - \iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}\left(U^- - U\right)V d\sigma,\ \forall\ V \in \mathcal{E},\end{aligned} \tag{4.5.6b}$$

where $r = (f - \boldsymbol{a}\cdot\nabla U - cU)$ is the interior residual.

This local weak formulation can be approximated by

$$\iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}\left(E^- - E\right)V d\sigma = -\iint_{\Gamma^-} \boldsymbol{a}\cdot\mathbf{n}\left(U^- - U\right)V d\sigma,\ \forall\ V \in \mathcal{E}. \tag{4.5.7}$$

The accuracy of *a posteriori* error estimates is measured by the ratio of the error estimate over the true error. The local and global effectivity indices are given by (3.3.11) and (3.3.12).

The following are the main steps of our modified DG method with error estimation.

1. Create a mesh for the domain Ω
2. Find the set $\mathcal{Z}^0$ of elements whose *inflow* faces are on the domain *inflow* boundary $\partial\Omega^-$.
3. For $k = 1, 2, \ldots$, find the set $\mathcal{Z}^k$ of all elements not in $\mathcal{Z}^{k-1}$ whose *inflow* faces are either on $\partial\Omega^-$ or are shared by an element from $\mathcal{Z}^{k-1}$.
4. For $k = 0, 1, \ldots$, find the DG solution and error estimate:
   (a) Compute the DG solution U on each element in $\mathcal{Z}^k$ by solving the DG finite element problem (4.5.5) with boundary conditions (3.1.5a) or (3.1.5b), respectively, for standard and the modified DG method.
   (b) Compute the error estimate E on each element of $\mathcal{Z}^k$ by solving (4.5.6).

4.6 Computational Examples

We solve several linear hyperbolic problems on uniform and general unstructured tetrahedral meshes and test our *a posteriori* error estimation for the standard and modified DG methods. In order to test the robustness of our procedures we use the following three families of meshes.

1. A family of uniform meshes obtained by partitioning the domain $[0,1]^3$ into n^3 cubes, $n = 1, 2, \ldots, 10$ and subdividing each cube into five tetrahedra [40]. The resulting meshes have $N = 5n^3 = 5, 40, \ldots, 5000$ tetrahedral elements with diameter $h_{max} = \frac{\sqrt{2}}{n}$.
2. A family of uniform meshes obtained by partitioning the domain $[0,1]^3$ into n^3 cubes, $n = 1, 2, \ldots, 20$ and subdividing each cube into six tetrahedra [40]. These meshes have $N = 6n^3 = 6, 48, \ldots, 41154$ tetrahedral elements with diameter $h_{max} = \frac{\sqrt{2}}{n}$.
3. A family of unstructured meshes generated by COMSOL software [33] with maximum mesh size $h_{\max} = 1/n,\ n = 1, 2, \ldots, 10$ which yields ten unstructured meshes having $N = 24, 476,\ 2121{,}5846,$ tetrahedral elements. See Figure 3.3 for two typical meshes having 953 and 8713 elements generated by *COMSOL*.

In the following examples we compute L^∞ error norm $||e||_{\infty,\Xi}$, where Ξ is either the face Γ_1 or the edges $\mathbf{v}_2\mathbf{v}_3,\ \mathbf{v}_2\mathbf{v}_4,\ \mathbf{v}_3\mathbf{v}_4$, as follows

$$\begin{aligned}
\|e\|_{\infty,\Gamma_1} &= \max_{1\le j\le N} \|e\|^j_{\infty,\Gamma_1}, & \|e\|^j_{\infty,\Gamma_1} &= \max_{1\le i\le M} \left| e\left(\xi_i, 0, \zeta_i\right)\right|_{\Delta_j},\\
\|e\|_{\infty,\mathbf{v}_2\mathbf{v}_3} &= \max_{1\le j\le N} \|e\|^j_{\infty,\mathbf{v}_2\mathbf{v}_3}, & \|e\|^j_{\infty,\mathbf{v}_2\mathbf{v}_3} &= \max_{1\le i\le M} \left| e\left(\xi_i, 1-\xi_i, 0\right)\right|_{\Delta_j},\\
\|e\|_{\infty,\mathbf{v}_2\mathbf{v}_4} &= \max_{1\le j\le N} \|e\|^j_{\infty,\mathbf{v}_2\mathbf{v}_4}, & \|e\|^j_{\infty,\mathbf{v}_2\mathbf{v}_4} &= \max_{1\le i\le M} \left| e\left(\xi_i, 0, 1-\xi_i\right)\right|_{\Delta_j},\\
\|e\|_{\infty,\mathbf{v}_3\mathbf{v}_4} &= \max_{1\le j\le N} \|e\|^j_{\infty,\mathbf{v}_3\mathbf{v}_4}, & \|e\|^j_{\infty,\mathbf{v}_3\mathbf{v}_4} &= \max_{1\le i\le M} \left| e\left(0, \eta_i, 1-\eta_i\right)\right|_{\Delta_j},
\end{aligned}$$

and

$$\|e\|_{\infty,\Gamma^*} = \begin{cases}
\max\limits_{1\le j\le N} \left\{\|e\|^j_{\infty,\Gamma_1}, \|e\|^j_{\infty,\mathbf{v}_2\mathbf{v}_3}, \|e\|^j_{\infty,\mathbf{v}_2\mathbf{v}_4}, \|e\|^j_{\infty,\mathbf{v}_3\mathbf{v}_4}\right\}, & \text{if } \Delta_j \text{ of } \textit{Class I} \text{ and } \textit{Type I},\\
\max\limits_{1\le j\le N} \left\{\|e\|^j_{\infty,\mathbf{v}_2\mathbf{v}_3}, \|e\|^j_{\infty,\mathbf{v}_2\mathbf{v}_4}, \|e\|^j_{\infty,\mathbf{v}_3\mathbf{v}_4}\right\}, & \text{if } \Delta_j \text{ of } \textit{Type I},\\
\max\limits_{1\le j\le N} \left\{\|e\|^j_{\infty,\mathbf{v}_2\mathbf{v}_3}\right\}, & \text{if } \Delta_j \text{ of } \textit{Type II},\\
\max\limits_{1\le j\le N} \left\{\|e\|^j_{\infty,\mathbf{v}_2\mathbf{v}_3}, \|e\|^j_{\infty,\mathbf{v}_3\mathbf{v}_4}\right\}, & \text{if } \Delta_j \text{ of } \textit{Type III},
\end{cases}$$

where N the number of elements on the mesh and (ξ_i, η_i, ζ_i), $1 \le i \le M$ are the quadrature points presented in Table 2.3 for $p \le 3$, or the quadrature points in [43] for $p = 4$.

Example 4.6.1.

Let us consider the following linear hyperbolic problem

$$-19u_x = -19e^{x+y+z},\ (x,y,z) \in \Omega = [0,1]^3\,, \tag{4.6.1a}$$

subject to the boundary conditions on the *inflow* boundary Γ^- such that the exact solution is given by

$$u\,(x,y,z) = e^{x+y+z}. \tag{4.6.1b}$$

We solve (4.6.1) with the exact *inflow* boundary condition, $U^- = u$, using the standard and the modified DG methods on uniform meshes having $N = 6n^3$, $n = 10, 11, \ldots, 19$ tetrahedral elements for $p = 1,2,3$. Since all elements of the mesh are of *Class I* and *Type 1*, then the leading error term is zero on the *outflow* face for the space $\mathcal{L}_p$, both the standard and the modified DG methods give the same results. We present L^∞ error $||e||_{\infty,\Gamma^+}$ on the *outflow* face, the L^2 errors $||e||_{2,\Omega}$ and $||e_c||_{2,\Omega} = ||u - U - E||_{2,\Omega}$, their orders of convergence, the maximum and minimum element effectivity indices, and the global effectivity indices in Table 4.11. We observe that the error estimates are accurate and asymptotically exact under mesh refinement for the standard and the modified DG methods, which confirm the pointwise superconvergence results on elements of *Class I* and *Type 1* using $\mathcal{L}_p$.

Next, we solve (4.6.1) using the modified DG method on uniform meshes having $N = 6n^3$, $n = 1, 2, \ldots, 11$ tetrahedral elements for $p = 2,3,4$ and using the space $\mathcal{U}_p$. We present L^∞ error $||e||_{\infty,\Gamma^*}$, the L^2 errors $||e||_{2,\Omega}$ and $||e_c||_{2,\Omega} = ||u-U-E||_{2,\Omega}$, their orders of convergence, the maximum and minimum element effectivity indices, and the global effectivity indices in Table 4.12.
Let $\mathcal{O}_{\infty,\Xi}$ denote the order of convergence for the error, where Ξ is either the face Γ_1 or the edges $\mathbf{v}_2\mathbf{v}_3,\ \mathbf{v}_2\mathbf{v}_4,\ \mathbf{v}_3\mathbf{v}_4$. In Table 4.13, we present the order of convergence $\mathcal{O}_{\infty,\Gamma_1}$ and $\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_3}$, $\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_4}$, $\mathcal{O}_{\infty,\mathbf{v}_3\mathbf{v}_4}$, respectively, on the face Γ_1 and the edges $\mathbf{v}_2\mathbf{v}_3, \mathbf{v}_2\mathbf{v}_4, \mathbf{v}_3\mathbf{v}_4$.

We observe that the effectivity indices for the DG method converge to unity under mesh refinement, and the DG solution is $O(h^{p+2})$ superconvergent to the true solution on the *outflow* face Γ_1 and on the three edges $\mathbf{v}_2\mathbf{v}_3,\ \mathbf{v}_2\mathbf{v}_4,\ \mathbf{v}_3\mathbf{v}_4$ of the *inflow* face.

Example 4.6.2.

Let us consider the following linear hyperbolic problem

$$-3u_x - 7u_y + 13u_z = 3e^{x+y+z},\ (x,y,z) \in \Omega = [0,1]^3\,, \tag{4.6.2a}$$

subject to the boundary conditions on the *inflow* boundary Γ^- such that the exact solution is given by

$$u\,(x,y,z) = e^{x+y+z}. \tag{4.6.2b}$$

We solve (4.6.2) with the exact *inflow* boundary condition $U^- = u$, the modified DG methods on uniform meshes having $N = 6n^3$, $n = 1, 2, \ldots, 10$ tetrahedral elements using the spaces

p=1									
N	$\|e\|_{2,\Omega}$	order	$\|e\|_{\infty,\Gamma_1}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
6	6.1083e-01	-	8.2607e-01	-	1.6375e-01	-	1.0152	1.0858	1.0427
48	1.6605e-01	1.8792	1.9579e-01	2.0770	2.2986e-02	2.8327	1.0049	1.0508	1.0301
162	7.5107e-02	1.9567	7.6039e-02	2.3326	6.9908e-03	2.9356	1.0029	1.0354	1.0222
384	4.2532e-02	1.9767	3.6806e-02	2.5222	2.9803e-03	2.9636	1.0021	1.0270	1.0174
750	2.7311e-02	1.9850	2.0479e-02	2.6273	1.5343e-03	2.9756	1.0016	1.0219	1.0143
1296	1.9003e-02	1.9893	1.2530e-02	2.6944	8.9079e-04	2.9821	1.0013	1.0185	1.0122
2058	1.3979e-02	1.9919	8.2122e-03	2.7409	5.6218e-04	2.9860	1.0011	1.0160	1.0106
3072	1.0712e-02	1.9935	5.6693e-03	2.7751	3.7719e-04	2.9886	1.0010	1.0141	1.0093
4374	8.4691e-03	1.9947	4.0760e-03	2.8013	2.6521e-04	2.9904	1.0009	1.0126	1.0083
6000	6.8632e-03	1.9955	3.0276e-03	2.8221	1.9351e-04	2.9918	1.0008	1.0114	1.0076
7986	5.6742e-03	1.9961	2.3099e-03	2.8389	1.4548e-04	2.9928	1.0007	1.0104	1.0069
10368	4.7693e-03	1.9966	1.8021e-03	2.8528	1.1212e-04	2.9936	1.0007	1.0096	1.0064
13182	4.0647e-03	1.9970	1.4329e-03	2.8645	8.8227e-05	2.9943	1.0006	1.0089	1.0059
16464	3.5055e-03	1.9973	1.1580e-03	2.8744	7.0667e-05	2.9948	1.0006	1.0083	1.0055
20250	3.0542e-03	1.9976	9.4912e-04	2.8830	5.7473e-05	2.9953	1.0005	1.0077	1.0051
24576	2.6847e-03	1.9978	7.8759e-04	2.8905	4.7370e-05	2.9957	1.0005	1.0073	1.0048
29478	2.3785e-03	1.9980	6.6073e-04	2.8971	3.9502e-05	2.9960	1.0005	1.0068	1.0045
34992	2.1218e-03	1.9981	5.5971e-04	2.9030	3.3285e-05	2.9963	1.0004	1.0065	1.0043
41154	1.9045e-03	1.9983	4.7827e-04	2.9082	2.8306e-05	2.9965	1.0004	1.0061	1.0041
p=2									
6	1.3908e-01	-	1.1144e-01	-	2.5449e-02	-	1.0224	1.0797	1.0512
48	1.9500e-02	2.8343	1.5002e-02	2.8930	1.8066e-03	3.8163	1.0091	1.0428	1.0297
162	5.9212e-03	2.9395	3.8635e-03	3.3459	3.6696e-04	3.9311	1.0056	1.0295	1.0207
384	2.5218e-03	2.9671	1.3982e-03	3.5329	1.1739e-04	3.9617	1.0041	1.0227	1.0159
750	1.2973e-03	2.9786	6.2114e-04	3.6362	4.8357e-05	3.9747	1.0032	1.0184	1.0129
1296	7.5287e-04	2.9847	3.1628e-04	3.7019	2.3399e-05	3.9816	1.0026	1.0155	1.0108
2058	4.7497e-04	2.9883	1.7750e-04	3.7474	1.2658e-05	3.9857	1.0022	1.0133	1.0093
3072	3.1859e-04	2.9906	1.0713e-04	3.7809	7.4312e-06	3.9885	1.0019	1.0117	1.0082
4374	2.2396e-04	2.9922	6.8426e-05	3.8065	4.6445e-06	3.9904	1.0017	1.0105	1.0073
6000	1.6338e-04	2.9934	4.5721e-05	3.8267	3.0499e-06	3.9918	1.0015	1.0094	1.0066
7986	1.2282e-04	2.9943	3.1699e-05	3.8431	2.0845e-06	3.9929	1.0014	1.0086	1.0060
10368	9.4640e-05	2.9950	2.2662e-05	3.8567	1.4726e-06	3.9937	1.0013	1.0079	1.0055
13182	7.4464e-05	2.9956	1.6628e-05	3.8681	1.0697e-06	3.9944	1.0012	1.0073	1.0051
16464	5.9637e-05	2.9960	1.2475e-05	3.8778	7.9555e-07	3.9949	1.0011	1.0068	1.0047
20250	4.8499e-05	2.9964	9.5411e-06	3.8862	6.0388e-07	3.9954	1.0010	1.0063	1.0044
24576	3.9971e-05	2.9967	7.4211e-06	3.8935	4.6661e-07	3.9958	1.0009	1.0060	1.0042
29478	3.3330e-05	2.9970	5.8585e-06	3.8999	3.6622e-07	3.9961	1.0009	1.0056	1.0039
34992	2.8082e-05	2.9972	4.6864e-06	3.9056	2.9143e-07	3.9964	1.0008	1.0053	1.0037
41154	2.3881e-05	2.9974	3.7932e-06	3.9107	2.3480e-07	3.9966	1.0008	1.0050	1.0035
p=3									
6	2.3664e-02	-	1.0298e-02	-	2.5804e-03	-	1.0122	1.1056	1.0599
48	1.6983e-03	3.8005	6.8399e-04	3.9122	9.4132e-05	4.7768	1.0035	1.0580	1.0319
162	3.4568e-04	3.9261	1.1672e-04	4.3608	1.2825e-05	4.9161	1.0018	1.0397	1.0217
384	1.1066e-04	3.9593	3.1576e-05	4.5446	3.0846e-06	4.9532	1.0011	1.0302	1.0164
750	4.5598e-05	3.9733	1.1198e-05	4.6456	1.0178e-06	4.9690	1.0008	1.0243	1.0132
1296	2.2067e-05	3.9807	4.7448e-06	4.7098	4.1071e-07	4.9774	1.0006	1.0204	1.0110
2058	1.1939e-05	3.9851	2.2800e-06	4.7543	1.9054e-07	4.9824	1.0005	1.0175	1.0095
3072	7.0095e-06	3.9880	1.2032e-06	4.7869	9.7914e-08	4.9858	1.0004	1.0154	1.0083
4374	4.3811e-06	3.9901	6.8265e-07	4.8118	5.4412e-08	4.9881	1.0003	1.0137	1.0074
6000	2.8770e-06	3.9915	4.1032e-07	4.8315	3.2164e-08	4.9898	1.0003	1.0123	1.0067
7986	1.9664e-06	3.9927	2.5851e-07	4.8475	1.9988e-08	4.9911	1.0002	1.0112	1.0061
10368	1.3892e-06	3.9935	1.6935e-07	4.8607	1.2946e-08	4.9922	1.0002	1.0103	1.0056
13182	1.0091e-06	3.9942	1.1467e-07	4.8718	8.6808e-09	4.9930	1.0002	1.0095	1.0051
16464	7.5049e-07	3.9948	7.9862e-08	4.8812	5.9957e-09	4.9937	1.0002	1.0088	1.0048
20250	5.6968e-07	3.9953	5.6995e-08	4.8894	4.2481e-09	4.9942	1.0002	1.0083	1.0045
24576	4.4019e-07	3.9957	4.1552e-08	4.8965	3.0775e-09	4.9947	1.0001	1.0078	1.0042
29478	3.4548e-07	3.9960	3.0868e-08	4.9027	2.2734e-09	4.9952	1.0001	1.0073	1.0039
34992	2.7493e-07	3.9963	2.3317e-08	4.9083	1.7088e-09	4.9954	1.0001	1.0069	1.0037
41154	2.2150e-07	3.9966	1.7877e-08	4.9132	1.3043e-09	4.9958	1.0001	1.0065	1.0035

Table 4.11: L^2 errors and effectivity indices for problem (4.6.1) using the standard and modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{L}_p$, $p = 1, 2, 3$.

p=2									
N	$\|e\|_{2,\Omega}$	order	$\|e\|_{\infty,\Gamma^*}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
6	4.9668e-02	-	2.4679e-02	-	5.0118e-01	-	0.6274	0.9190	0.8368
48	6.9806e-03	2.8309	5.6507e-02	3.1488	1.7462e-03	3.8210	0.8349	0.9790	0.9529
162	2.1540e-03	2.8999	1.3717e-02	3.4917	3.5412e-04	3.9351	0.9045	0.9894	0.9764
384	9.2719e-04	2.9301	4.8199e-03	3.6355	1.1318e-04	3.9649	0.9357	0.9932	0.9852
750	4.8041e-04	2.9466	2.1037e-03	3.7154	4.6596e-05	3.9773	0.9526	0.9951	0.9895
1296	2.8021e-04	2.9568	1.0586e-03	3.7665	2.2537e-05	3.9838	0.9630	0.9962	0.9920
2058	1.7745e-04	2.9638	5.8915e-04	3.8019	1.2188e-05	3.9876	0.9698	0.9969	0.9936
3072	1.1937e-04	2.9689	3.5337e-04	3.8280	7.1541e-06	3.9901	0.9747	0.9974	0.9946
4374	8.4108e-05	2.9727	2.2459e-04	3.8480	4.4705e-06	3.9919	0.9782	0.9978	0.9954
6000	6.1472e-05	2.9757	1.4948e-04	3.8638	2.9352e-06	3.9931	0.9810	0.9980	0.9960
7986	4.6281e-05	2.9781	1.0331e-04	3.8767	2.0059e-06	3.9941	0.9831	0.9983	0.9965
p=3									
6	1.0330e-02	-	2.2676e-03	-	8.2930e-02	-	0.8113	1.0009	0.9682
48	7.8372e-04	3.7204	4.6920e-03	4.1436	8.0152e-05	4.8223	0.9108	1.0129	0.9904
162	1.6275e-04	3.8766	7.5859e-04	4.4940	1.0811e-05	4.9409	0.9423	1.0113	0.9947
384	5.2635e-05	3.9239	1.9974e-04	4.6385	2.5877e-06	4.9701	0.9575	1.0095	0.9964
750	2.1822e-05	3.9458	6.9698e-05	4.7183	8.5137e-07	4.9818	0.9664	1.0080	0.9973
1296	1.0604e-05	3.9582	2.9214e-05	4.7692	3.4291e-07	4.9877	0.9722	1.0070	0.9978
2058	5.7539e-06	3.9661	1.3930e-05	4.8043	1.5887e-07	4.9911	0.9764	1.0061	0.9982
3072	3.3857e-06	3.9715	7.3087e-06	4.8303	8.1561e-08	4.9932	0.9794	1.0055	0.9984
4374	2.1198e-06	3.9755	4.1282e-06	4.8498	4.5289e-08	4.9947	0.9818	1.0049	0.9986
6000	1.3939e-06	3.9785	2.4723e-06	4.8661	2.6755e-08	4.9955	0.9836	1.0045	0.9988
p=4									
6	1.7291e-03	-	3.0548e-04	-	7.2794e-03	-	0.8908	0.9833	0.9673
48	6.4882e-05	4.7361	2.1363e-04	5.0906	5.8286e-06	5.7118	0.9530	0.9942	0.9881
162	8.9309e-06	4.8908	2.3302e-05	5.4646	5.3211e-07	5.9035	0.9712	0.9967	0.9931
384	2.1595e-06	4.9347	4.6310e-06	5.6165	9.6075e-08	5.9501	0.9795	0.9977	0.9953
750	7.1488e-07	4.9544	1.2976e-06	5.7017	2.5376e-08	5.9662	0.9839	0.9984	0.9964
1296	2.8912e-07	4.9653	4.5364e-07	5.7643	8.5465e-09	5.9690	0.9864	0.9989	0.9971
2058	1.3436e-07	4.9711	1.8439e-07	5.8401	4.1520e-09	4.6833	0.9808	0.9994	0.9972
3072	6.9134e-08	4.9763	8.5564e-08	5.7498	2.1395e-09	4.9653	0.9790	1.0007	0.9979
4374	3.8469e-08	4.9769	4.3803e-08	5.6847	1.5483e-09	2.7456	0.9409	1.0030	0.9978

Table 4.12: L^2 errors and effectivity indices for problem (4.6.1) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$.

p=2					
N	# elements $\Delta_{1,1}$	$\mathcal{O}_{\infty,\Gamma_1}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_3}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_4}$	$\mathcal{O}_{\infty,\mathbf{v}_3\mathbf{v}_4}$
48	48	2.8930	3.1488	3.1734	3.1755
162	162	3.3459	3.4917	3.5062	3.5064
384	384	3.5329	3.6355	3.6458	3.6457
750	750	3.6362	3.7154	3.7235	3.7233
1296	1296	3.7019	3.7665	3.7731	3.7728
2058	2058	3.7474	3.8019	3.8075	3.8073
3072	3072	3.7809	3.8280	3.8329	3.8326
4374	4374	3.8065	3.8480	3.8523	3.8521
6000	6000	3.8267	3.8638	3.8677	3.8675
7986	7986	3.8431	3.8767	3.8802	3.8800
p=3					
48	48	3.9122	4.1436	4.1903	4.1639
162	162	4.3608	4.4940	4.5197	4.5054
384	384	4.5446	4.6385	4.6564	4.6465
750	750	4.6457	4.7183	4.7321	4.7245
1296	1296	4.7098	4.7692	4.7803	4.7742
2058	2058	4.7543	4.8043	4.8137	4.8086
3072	3072	4.7867	4.8303	4.8384	4.8340
4374	4374	4.8125	4.8498	4.8569	4.8531
6000	6000	4.8307	4.8661	4.8723	4.8689
p=4					
48	48	4.9279	5.0282	5.1077	5.0906
162	162	5.3699	5.4295	5.4737	5.4646
384	384	5.5534	5.5908	5.6213	5.6165
750	750	5.6505	5.6804	5.7087	5.7017
1296	1296	5.6910	5.7588	5.7626	5.7643
2058	2058	5.6801	5.8259	5.8356	5.8401
3072	3072	6.0729	5.7381	5.7099	5.7498
4374	4374	5.2478	5.5077	5.6834	5.6847

Table 4.13: Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.1) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$.

$\mathcal{U}_p$ for $p = 2, 3, 4$. We present L^∞ error $||e||_{\infty,\Gamma^*}$ on the set Γ^* of all superconvergence points, the L^2 errors $||e||$ and $||e_c|| = ||u - U - E||$, their orders of convergence, maximum and minimum element effectivity indices, and global effectivity indices in Table 4.14. Moreover we present the order of convergence of the maximum errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3, \mathbf{v}_2\mathbf{v}_4, \mathbf{v}_3\mathbf{v}_4$ in Tables 4.15, 4.16, 4.17, 4.18, 4.19, 4.20, respectively, for elements meshes of *Type 1, 2, 3*, and *Class I, II, III*.

We observe that the effectivity indices for the modified DG method converge to unity under mesh refinement, and the DG solution is $O(h^{p+2})$ convergent to the true solution for elements of *Type 1* on the edges $\mathbf{v}_2\mathbf{v}_3, \mathbf{v}_2\mathbf{v}_4, \mathbf{v}_3\mathbf{v}_4$, for all meshes and polynomial degrees. Similarly, we observe $O(h^{p+2})$ superconvergence for elements of *Type 2* on edge $\mathbf{v}_2\mathbf{v}_3$, and on edges $\mathbf{v}_2\mathbf{v}_3, \mathbf{v}_2\mathbf{v}_4$ for elements of *Type 3*.

Next, we solve (4.6.2) using the modified DG method on unstructured tetrahedral meshes having $N = 24, 476, 2121$,5846 elements using the spaces $\mathcal{U}_p$ for $p = 2, 3, 4$. The L^∞ error $||e||_{\infty,\Gamma^*}$, the L^2 errors $||e||$ and $||e_c||$, their orders of convergence, maximum and minimum element effectivity indices, and global effectivity indices presented in Table 4.21 show that the method DG method exhibits optimal convergence rates and the L^∞ error on the set of all superconvergence points is $O(h^{p+2})$ convergent to the true solution. Moreover, the effectivity indices converge to unity under mesh refinement.

We also solve problem (4.6.2) using the modified DG method and the approximated weak formulation for the error (4.5.7) on uniform meshes having $N = 5n^3$ elements for the enriched finite element spaces $\mathcal{U}_p$ with $p = 2, 3$. We present the L^∞ error $||e||_{\infty,\Gamma^*}$, the L^2 errors $||e||$ and $||e_c||$, their orders of convergence, maximum and minimum element effectivity indices, and global effectivity indices in Table 4.22. We again observe that the error estimates are accurate and asymptotically exact under mesh refinement.

Example 4.6.3.

Let us consider the following linear hyperbolic problem

$$-u_x - 2u_y + 3u_z = -\frac{-x - 2y + 3z}{\sqrt{1 + x^2 + y^2 + z^2}}, (x, y, z) \in \Omega = [0, 1]^3\,, \tag{4.6.3a}$$

subject to the boundary conditions on the *inflow* boundary Γ^- such that the exact solution is given by

$$u\,(x, y, z) = \frac{1}{\sqrt{1 + x^2 + y^2 + z^2}}. \tag{4.6.3b}$$

We solve (4.6.3) with the exact *inflow* boundary condition, $U^- = u$, the modified DG methods on uniform meshes having $N = 6n^3$, $n = 1, 2, \ldots, 10$ tetrahedral elements for $p = 2, 3, 4$. We present L^∞ error $||e||_{\infty,\Gamma^*}$ on the set Γ^* of all superconvergence points, the L^2 errors $||e||$ and $||e_c|| = ||u - U - E||$, their orders of convergence, maximum and minimum element effectivity indices, and global effectivity indices in Table 4.23. Moreover we present

N	$\|e\|_{2,\Omega}$	order	$\|e\|_{\infty,\Gamma^*}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
p=2									
6	4.3725e-02	-	2.8642e-02	-	4.3741e-01	-	0.3751	0.8870	0.7433
48	5.8355e-03	2.9055	5.1102e-02	3.0975	2.1251e-03	3.7525	0.5268	1.0574	0.9043
162	1.7927e-03	2.9108	1.2296e-02	3.5134	4.3979e-04	3.8852	0.6929	1.0550	0.9443
384	7.7238e-04	2.9269	4.3299e-03	3.6280	1.4203e-04	3.9288	0.7796	1.0442	0.9615
750	4.0042e-04	2.9441	1.8896e-03	3.7157	5.8804e-05	3.9520	0.8332	1.0355	0.9713
1296	2.3367e-04	2.9541	9.5099e-04	3.7661	2.8560e-05	3.9611	0.8678	1.0296	0.9773
2058	1.4806e-04	2.9601	5.2928e-04	3.8014	1.5500e-05	3.9646	0.8915	1.0254	0.9810
3072	9.9659e-05	2.9646	3.1747e-04	3.8277	9.1244e-06	3.9685	0.9086	1.0222	0.9837
4374	7.0250e-05	2.9689	2.0178e-04	3.8477	5.7119e-06	3.9767	0.9213	1.0197	0.9858
6000	5.1361e-05	2.9725	1.3431e-04	3.8636	3.7553e-06	3.9805	0.9311	1.0177	0.9875
7986	3.8681e-05	2.9749	9.2821e-05	3.8764	2.5697e-06	3.9807	0.9389	1.0161	0.9887
p=3									
6	7.3060e-03	-	3.2978e-03	-	7.1616e-02	-	0.5682	1.0821	0.9471
48	5.5455e-04	3.7197	3.9308e-03	4.1874	1.2706e-04	4.6980	0.7868	1.1703	0.9941
162	1.1593e-04	3.8601	6.3825e-04	4.4834	1.7393e-05	4.9045	0.8717	1.1053	1.0000
384	3.7672e-05	3.9075	1.6731e-04	4.6540	4.2177e-06	4.9248	0.9120	1.0758	1.0018
750	1.5665e-05	3.9324	5.8336e-05	4.7218	1.3914e-06	4.9696	0.9345	1.0591	1.0022
1296	7.6286e-06	3.9464	2.4424e-05	4.7754	5.6292e-07	4.9635	0.9484	1.0484	1.0023
2058	4.1460e-06	3.9556	1.1889e-05	4.6705	2.6111e-07	4.9835	0.9577	1.0410	1.0021
3072	2.4426e-06	3.9621	6.4144e-06	4.6210	1.3437e-07	4.9753	0.9644	1.0356	1.0020
4374	1.5308e-06	3.9672	3.7034e-06	4.6635	7.4662e-08	4.9888	0.9693	1.0314	1.0019
p=4									
6	1.2089e-03	-	4.1217e-04	-	8.0347e-03	-	0.7311	0.9935	0.9338
48	4.4790e-05	4.7544	2.1003e-04	5.2576	8.4628e-06	5.6060	0.8560	1.0809	0.9849
162	6.1527e-06	4.8958	2.3005e-05	5.4542	7.8401e-07	5.8674	0.9178	1.0550	0.9949
384	1.4872e-06	4.9359	4.5090e-06	5.6648	1.4269e-07	5.9224	0.9446	1.0410	0.9986
750	4.9239e-07	4.9538	1.2564e-06	5.7264	3.7892e-08	5.9420	0.9590	1.0326	1.0001
1296	1.9919e-07	4.9640	4.3894e-07	5.7681	1.2820e-08	5.9441	0.9676	1.0271	1.0009
2058	9.2576e-08	4.9705	1.7919e-07	5.8121	5.1115e-09	5.9650	0.9731	1.0245	1.0011
3072	4.7644e-08	4.9746	8.2426e-08	5.8153	2.3384e-09	5.8566	0.9767	1.0231	1.0012

Table 4.14: L^2 errors and effectivity indices for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$.

p=2					
N	# elements *Type 1*	$\mathcal{O}_{\infty,\Gamma_1}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_3}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_4}$	$\mathcal{O}_{\infty,\mathbf{v}_3\mathbf{v}_4}$
48	8	2.8038	3.1786	2.8480	3.1387
162	27	2.7686	3.5281	3.2898	3.5064
384	64	2.7311	3.6569	3.5156	3.6413
750	125	2.5240	3.7322	3.6143	3.7208
1296	216	2.6151	3.7799	3.6851	3.7707
2058	343	2.6767	3.8132	3.7322	3.8055
3072	512	2.7212	3.8377	3.7673	3.8311
4374	729	2.7549	3.8566	3.7943	3.8508
6000	1000	2.7813	3.8715	3.8156	3.8663
7986	1331	2.8026	3.8835	3.8329	3.8789
p=3					
48	8	3.2622	4.1515	3.6292	4.1341
162	27	3.5747	4.4979	4.3120	4.4831
384	64	3.6536	4.6451	4.4985	4.6355
750	125	3.6587	4.7225	4.6082	4.7146
1296	216	3.7212	4.7732	4.6780	4.7668
2058	343	3.7642	4.8078	4.7268	4.8023
3072	512	3.7957	4.8334	4.7626	4.8285
4374	729	3.8197	4.8522	4.7899	4.8489
p=4					
48	8	4.2546	5.1705	5.0286	5.0473
162	27	4.5999	5.4989	5.2932	5.2648
384	64	4.5428	5.6466	5.5156	5.5605
750	125	4.5756	5.7198	5.3741	5.5960
1296	216	4.6433	5.8292	5.7317	5.5686
2058	343	4.7259	5.7325	5.3861	6.0339
3072	512	4.7514	5.5983	5.2591	5.5711

Table 4.15: Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, on elements of *Type 1*.

p=2					
N	# elements *Type 2*	$\mathcal{O}_{\infty,\Gamma_1}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_3}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_4}$	$\mathcal{O}_{\infty,\mathbf{v}_3\mathbf{v}_4}$
48	32	1.7344	2.8676	2.9837	2.8932
162	108	2.1228	3.1570	2.5026	3.0829
384	256	2.3491	3.4564	2.5135	2.4875
750	500	2.4993	3.5681	2.6226	2.5860
1296	864	2.5891	3.6439	2.6908	2.6621
2058	1372	2.6518	3.6979	2.7383	2.7144
3072	2048	2.6978	3.7373	2.7731	2.7527
4374	2916	2.7331	3.7676	2.7997	2.7818
6000	4000	2.7609	3.7917	2.8208	2.8048
7986	5324	2.7835	3.8112	2.8378	2.8235
p=3					
48	32	2.9155	3.4827	3.2554	3.4953
162	108	3.3157	4.1872	3.3617	3.4136
384	256	3.5173	4.4331	3.5483	3.5829
750	500	3.6221	4.5547	3.6489	3.6436
1296	864	3.6904	4.6333	3.7130	3.7078
2058	1372	3.7374	4.6895	3.7572	3.7530
3072	2048	3.7721	4.7303	3.7896	3.7860
4374	2916	3.7987	4.7615	3.8143	3.8112
p=4					
48	32	3.9234	4.6502	3.9048	4.0522
162	108	4.3828	5.1121	4.3838	4.4029
384	256	4.5607	5.3602	4.5583	4.5751
750	500	4.6596	5.5441	4.6581	4.6701
1296	864	4.7209	5.5713	4.7207	4.7283
2058	1372	4.7649	5.7484	4.7640	4.7731
3072	2048	4.7974	5.8304	4.7962	4.8017

Table 4.16: Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Type 2*.

p=2					
N	# elements *Type 3*	$\mathcal{O}_{\infty,\Gamma_1}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_3}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_4}$	$\mathcal{O}_{\infty,\mathbf{v}_3\mathbf{v}_4}$
48	8	1.6071	0.5911	3.0047	3.0975
162	27	2.0373	2.9660	2.9971	3.5134
384	64	2.2840	3.3913	2.4811	3.6280
750	125	2.4512	3.4886	2.5983	3.7157
1296	216	2.5509	3.5897	2.6724	3.7661
2058	343	2.6200	3.6518	2.7173	3.8014
3072	512	2.6707	3.6978	2.7537	3.8277
4374	729	2.7094	3.7332	2.7827	3.8477
6000	1000	2.7399	3.7612	2.8056	3.8636
7986	1331	2.7647	3.7838	2.8241	3.8764
p=3					
48	8	2.9578	0.9111	3.5044	4.1874
162	27	3.3802	4.1691	3.3688	4.4834
384	64	3.5660	4.3269	3.5492	4.6540
750	125	3.6340	4.5120	3.6512	4.7218
1296	216	3.6821	4.5891	3.7153	4.7754
2058	343	3.7304	4.6563	3.7596	4.6705
3072	512	3.7660	4.7006	3.7919	4.6210
4374	729	3.7933	4.7369	3.8165	4.6635
p=4					
48	8	4.0845	1.2256	3.9979	5.2576
162	27	4.4537	5.2896	4.3934	5.4542
384	64	4.6133	5.2910	4.5660	5.6648
750	125	4.6977	5.5072	4.6640	5.7264
1296	216	4.6327	5.6259	4.7249	5.7681
2058	343	4.6776	5.5535	4.7665	5.8121
3072	512	4.7240	5.5632	4.7965	5.8153

Table 4.17: Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Type 3*.

p=2					
N	# elements *Class I*	$\mathcal{O}_{\infty,\Gamma_1}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_3}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_4}$	$\mathcal{O}_{\infty,\mathbf{v}_3\mathbf{v}_4}$
48	8	1.6071	0.5911	3.0047	3.0975
162	27	2.0373	2.9660	2.9971	3.5134
384	64	2.2840	3.3913	2.4811	3.6280
750	125	2.4512	3.4886	2.5983	3.7157
1296	216	2.5509	3.5897	2.6724	3.7661
2058	343	2.6200	3.6518	2.7173	3.8014
3072	512	2.6707	3.6978	2.7537	3.8277
4374	729	2.7094	3.7332	2.7827	3.8477
6000	1000	2.7399	3.7612	2.8056	3.8636
7986	1331	2.7647	3.7838	2.8241	3.8764
p=3					
48	8	2.9578	0.9111	3.5044	4.1874
162	27	3.3802	4.1691	3.3688	4.4834
384	64	3.5660	4.3269	3.5492	4.6540
750	125	3.6340	4.5120	3.6512	4.7218
1296	216	3.6821	4.5891	3.7153	4.7754
2058	343	3.7304	4.6563	3.7596	4.6705
3072	512	3.7660	4.7006	3.7919	4.6210
4374	729	3.7933	4.7369	3.8165	4.6635
p=4					
48	8	4.0845	1.2256	3.9979	5.2576
162	27	4.4537	5.2896	4.3934	5.4542
384	64	4.6133	5.2910	4.5660	5.6648
750	125	4.6977	5.5072	4.6640	5.7264
1296	216	4.6327	5.6259	4.7249	5.7681
2058	343	4.6776	5.5535	4.7665	5.8121
3072	512	4.7240	5.5632	4.7965	5.8153

Table 4.18: Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Class I*.

p=2					
N	# elements *Class II*	$\mathcal{O}_{\infty,\Gamma_1}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_3}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_4}$	$\mathcal{O}_{\infty,\mathbf{v}_3\mathbf{v}_4}$
48	32	1.7344	2.8676	2.9837	2.8932
162	108	2.1228	3.1570	2.5026	3.0829
384	256	2.3491	3.4564	2.5135	2.4875
750	500	2.4993	3.5681	2.6226	2.5860
1296	864	2.5891	3.6439	2.6908	2.6621
2058	1372	2.6518	3.6979	2.7383	2.7144
3072	2048	2.6978	3.7373	2.7731	2.7527
4374	2916	2.7331	3.7676	2.7997	2.7818
6000	4000	2.7609	3.7917	2.8208	2.8048
7986	5324	2.7835	3.8112	2.8378	2.8235
p=3					
48	32	2.9155	3.4827	3.2554	3.4953
162	108	3.3157	4.1872	3.3617	3.4136
384	256	3.5173	4.4331	3.5483	3.5829
750	500	3.6221	4.5547	3.6489	3.6436
1296	864	3.6904	4.6333	3.7130	3.7078
2058	1372	3.7374	4.6895	3.7572	3.7530
3072	2048	3.7721	4.7303	3.7896	3.7860
4374	2916	3.7987	4.7615	3.8143	3.8112
p=4					
48	32	3.9234	4.6502	3.9048	4.0522
162	108	4.3828	5.1121	4.3838	4.4029
384	256	4.5607	5.3602	4.5583	4.5751
750	500	4.6596	5.5441	4.6581	4.6701
1296	864	4.7209	5.5713	4.7207	4.7283
2058	1372	4.7649	5.7484	4.7640	4.7731
3072	2048	4.7974	5.8304	4.7962	4.8017

Table 4.19: Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Class II*.

p=2					
N	# elements *Class III*	$\mathcal{O}_{\infty,\Gamma_1}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_3}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_4}$	$\mathcal{O}_{\infty,\mathbf{v}_3\mathbf{v}_4}$
48	8	2.8038	3.1786	2.8480	3.1387
162	27	2.7686	3.5281	3.2898	3.5064
384	64	2.7311	3.6569	3.5156	3.6413
750	125	2.5240	3.7322	3.6143	3.7208
1296	216	2.6151	3.7799	3.6851	3.7707
2058	343	2.6767	3.8132	3.7322	3.8055
3072	512	2.7212	3.8377	3.7673	3.8311
4374	729	2.7549	3.8566	3.7943	3.8508
6000	1000	2.7813	3.8715	3.8156	3.8663
7986	1331	2.8026	3.8835	3.8329	3.8789
p=3					
48	8	3.2622	4.1515	3.6292	4.1341
162	27	3.5747	4.4979	4.3120	4.4831
384	64	3.6536	4.6451	4.4985	4.6355
750	125	3.6587	4.7225	4.6082	4.7146
1296	216	3.7212	4.7732	4.6780	4.7668
2058	343	3.7642	4.8078	4.7268	4.8023
3072	512	3.7957	4.8334	4.7626	4.8285
4374	729	3.8197	4.8522	4.7899	4.8489
p=4					
48	8	4.2546	5.1705	5.0286	5.0473
162	27	4.5999	5.4989	5.2932	5.2648
384	64	4.5428	5.6466	5.5156	5.5605
750	125	4.5756	5.7198	5.3741	5.5960
1296	216	4.6433	5.8292	5.7317	5.5686
2058	343	4.7259	5.7325	5.3861	6.0339
3072	512	4.7514	5.5983	5.2591	5.5711

Table 4.20: Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.2) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Class III*.

p=2									
N	$\|e\|_{2,\Omega}$	order	$\|e\|_{\infty,\Gamma^*}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
24	4.1814e-03	-	1.0273e-03	-	1.5152e-02	-	0.0023	1.0344	1.0056
476	2.0470e-04	4.3524	1.2220e-03	3.6322	4.3198e-05	4.5718	0.0009	1.0972	0.9690
2121	4.5968e-05	3.6837	2.1566e-04	4.2778	5.3303e-06	5.1605	0.0000	1.1582	0.9919
5846	1.8354e-05	3.1914	6.2261e-05	4.3186	1.5819e-06	4.2227	0.0007	1.2630	0.9956
p=3									
24	2.7200e-04	-	4.5241e-05	-	8.2571e-04	-	0.0030	1.0163	1.0032
476	6.7252e-06	5.3379	4.1565e-05	4.3122	1.0089e-06	5.4868	0.0023	1.1442	0.9925
2121	8.8561e-07	5.0000	4.0882e-06	5.7197	8.1894e-08	6.1934	0.0002	1.1976	0.9995
5846	2.6704e-07	4.1673	8.9224e-07	5.2910	1.8262e-08	5.2161	0.0035	1.2248	0.9994

Table 4.21: L^2 errors and effectivity indices for problem (4.6.2) using the modified DG method on unstructured meshes having N elements for the spaces $\mathcal{U}_p$, $p = 2, 3$.

p=2									
N	$\lVert e\rVert_{2,\Omega}$	order	$\lVert e\rVert_{\infty,\Gamma^*}$	order	$\lVert e_c\rVert_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
5	2.0498e-002	-	1.6645e-002	-	3.9521e-001	-	0.0000	0.7618	0.3468
40	4.3270e-003	2.2441	2.4822e-002	3.9930	9.2981e-004	4.1620	0.0000	1.0282	0.9954
135	1.2824e-003	2.9994	9.6183e-003	2.3382	2.1373e-004	3.6261	0.0000	1.0280	0.9957
320	5.8326e-004	2.7385	2.5880e-003	4.5634	7.1796e-005	3.7920	0.0000	1.0249	1.0004
625	3.0035e-004	2.9743	1.4353e-003	2.6417	3.1431e-005	3.7019	0.0000	1.0218	1.0001
1080	1.7787e-004	2.8736	6.0712e-004	4.7193	1.5947e-005	3.7216	0.0000	1.0192	1.0005
1715	1.1247e-004	2.9731	3.9712e-004	2.7537	9.0136e-006	3.7011	0.0000	1.0170	1.0002
2560	7.6148e-005	2.9210	2.0940e-004	4.7929	5.5100e-006	3.6858	0.0000	1.0167	1.0001
3645	5.3635e-005	2.9756	1.5035e-004	2.8123	3.5747e-006	3.6735	0.0000	1.0167	0.9999
5000	3.9332e-005	2.9438	9.0330e-005	4.8359	2.4353e-006	3.6429	0.0000	1.0164	0.9998
p=3									
5	2.6776e-003	-	1.8089e-003	-	5.3071e-002	-	0.0098	0.6553	0.4775
40	2.8585e-004	3.2276	1.6517e-003	5.0059	4.8552e-005	5.2195	0.0001	1.0604	0.9994
135	5.6892e-005	3.9814	4.2358e-004	3.3561	7.4270e-006	4.6305	0.0000	1.0504	0.9976
320	1.9447e-005	3.7315	8.5459e-005	5.5641	1.8304e-006	4.8684	0.0000	1.0412	1.0013
625	8.0233e-006	3.9675	3.7755e-005	3.6610	6.3632e-007	4.7351	0.0000	1.0427	1.0007
1080	3.9628e-006	3.8689	1.3328e-005	5.7112	2.6536e-007	4.7972	0.0000	1.0376	1.0009
1715	2.1493e-006	3.9691	7.4459e-006	3.7766	1.2736e-007	4.7620	0.0000	1.0360	1.0005
2560	1.2738e-006	3.9176	3.4423e-006	5.7779	6.7236e-008	4.7839	0.0000	1.0326	1.0005
3645	7.9778e-007	3.9728	2.1901e-006	3.8391	3.8339e-008	4.7693	0.0000	1.0303	1.0003

Table 4.22: L^2 errors and effectivity indices for problem (4.6.2) using the modified DG method and the approximated weak formulation for the error (4.5.7) on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3$.

the order of convergence of the maximum errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3, \mathbf{v}_2\mathbf{v}_4, \mathbf{v}_3\mathbf{v}_4$ in Table 4.24, 4.25, 4.26, 4.27, 4.28, 4.29, respectively, for elements meshes of *Type 1, 2, 3*, and *Class I, II, III.*

We again observe that the error estimates are accurate and asymptotically exact under mesh refinement. The DG solution is $O(h^{p+2})$ superconvergent, respectively, on the face Γ_1 and on the edges $\mathbf{v}_2\mathbf{v}_3, \mathbf{v}_2\mathbf{v}_4, \mathbf{v}_3\mathbf{v}_4$, with respect to each type of elements.

Conclusion

We investigated higher-order discontinuous Galerkin methods for three-dimensional scalar first-order hyperbolic problems on tetrahedral meshes. We construct simple, efficient and asymptotically correct *a posteriori* error estimates for discontinuous finite element solutions where we explicitly write the basis functions for the error spaces corresponding to the enriched finite element spaces $\mathcal{L}_p$, $\mathcal{U}_p$ and $\mathcal{M}_p$. We showed that the discretization error on tetrahedral elements having one *inflow* and one *outflow* faces, is $O(h^{p+2})$ superconvergent on the *outflow* face and on the three edges of the *inflow* face. On tetrahedral elements with one *inflow* face, the discretization error is $O(h^{p+2})$ superconvergent on the three edges of the *inflow* face. Furthermore, we show that, on tetrahedral elements with two *inflow* faces, the DG solution is $O(h^{p+2})$ superconvergent on the edge shared by two of the *inflow* faces. On elements with two *inflow* and one *outflow* faces and on elements with three *inflow* faces, the DG

p=2									
N	$\|e\|_{2,\Omega}$	order	$\|e\|_{\infty,\Gamma^*}$	order	$\|e_c\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
6	2.4735e-03	-	1.5738e-03	-	1.3768e-02	-	0.2194	1.0386	0.8477
48	2.7216e-04	3.1840	1.1629e-03	3.5655	8.3903e-05	4.2294	0.1256	1.1875	0.9564
162	7.4772e-05	3.1863	3.0457e-04	3.3044	1.8777e-05	3.6922	0.0125	1.2010	0.9711
384	3.0502e-05	3.1169	1.6570e-04	2.1160	6.1565e-06	3.8762	0.0063	1.1721	0.9822
750	1.5329e-05	3.0834	8.3577e-05	3.0670	2.5563e-06	3.9390	0.0023	1.1615	0.9886
1296	8.7671e-06	3.0645	4.4234e-05	3.4899	1.2434e-06	3.9529	0.0074	1.1641	0.9925
2058	5.4769e-06	3.0520	2.5029e-05	3.6940	6.7503e-07	3.9626	0.0001	1.1417	0.9948
3072	3.6480e-06	3.0432	1.5293e-05	3.6895	3.9720e-07	3.9715	0.0069	1.1533	0.9964
4374	2.5509e-06	3.0370	9.9825e-06	3.6214	2.4870e-07	3.9751	0.0083	1.1380	0.9975
4374	2.5509e-06	3.0370	9.9825e-06	3.6214	2.4870e-07	3.9751	0.0083	1.1380	0.9975
6000	1.8533e-06	3.0322	6.7659e-06	3.6916	1.6353e-07	3.9794	0.0043	1.1446	0.9982
7986	1.3887e-06	3.0284	4.7351e-06	3.7444	1.1187e-07	3.9830	0.0015	1.1344	0.9988
p=3									
6	6.2358e-04	-	1.4087e-04	-	1.3506e-03	-	0.7993	0.9595	0.9195
48	2.9621e-05	4.3959	4.4939e-04	1.5875	1.4040e-05	3.3267	0.3408	1.2056	0.8767
162	5.8470e-06	4.0018	1.0197e-04	3.6581	1.9929e-06	4.8150	0.2043	1.1569	0.9420
384	1.8183e-06	4.0602	2.6299e-05	4.7105	4.8401e-07	4.9195	0.3089	1.1362	0.9658
750	7.3480e-07	4.0603	8.3676e-06	5.1320	1.6135e-07	4.9229	0.2655	1.2249	0.9774
1296	3.5100e-07	4.0523	3.7414e-06	4.4148	6.5636e-08	4.9334	0.2900	1.2195	0.9841
2058	1.8816e-07	4.0446	1.8192e-06	4.6776	3.0630e-08	4.9441	0.3491	1.2301	0.9881
3072	1.0973e-07	4.0384	1.0288e-06	4.2688	1.5811e-08	4.9524	0.3510	1.2708	0.9908
4374	6.8234e-08	4.0337	5.9625e-07	4.6311	8.8164e-09	4.9589	0.3279	1.2136	0.9927
4374	6.8234e-08	4.0337	5.9625e-07	4.6311	8.8164e-09	4.9589	0.3279	1.2136	0.9927
6000	4.4628e-08	4.0298	3.5625e-07	4.8883	5.2257e-09	4.9641	0.3554	1.3424	0.9940
p=4									
6	5.3701e-05	-	5.6911e-05	-	7.9804e-04	-	0.4708	2.8159	1.0038
48	5.8464e-06	3.1993	7.9629e-05	3.3251	2.5218e-06	4.4962	0.2592	1.0893	0.9275
162	7.3096e-07	5.1280	7.3985e-06	5.8602	2.0676e-07	6.1686	0.2593	1.1472	0.9627
384	1.6816e-07	5.1079	1.7894e-06	4.9339	3.9132e-08	5.7864	0.1469	1.1369	0.9722
750	5.4166e-08	5.0768	5.0553e-07	5.6645	1.0521e-08	5.8864	0.3816	1.1303	0.9791
1296	2.1529e-08	5.0606	1.7402e-07	5.8493	3.5660e-09	5.9344	0.3759	1.1208	0.9838
2058	9.8844e-09	5.0499	7.5324e-08	5.4321	1.4250e-09	5.9506	0.2787	1.1061	0.9869
3072	5.0415e-09	5.0419	4.0444e-08	4.6572	6.4787e-010	5.9030	0.2786	1.0961	0.9889
3072	5.0415e-09	5.0419	4.0444e-08	4.6572	6.4787e-010	5.9030	0.2786	1.0961	0.9889
4374	2.7858e-09	5.0361	2.2601e-08	4.9404	3.2364e-010	5.8927	0.2985	1.0924	0.9906

Table 4.23: L^2 errors and effectivity indices for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$.

p=2					
N	# elements *Type 1*	$\mathcal{O}_{\infty,\Gamma_1}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_3}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_4}$	$\mathcal{O}_{\infty,\mathbf{v}_3\mathbf{v}_4}$
48	8	3.4430	3.1683	4.0553	3.4137
162	27	2.8944	2.4924	3.0712	3.2065
384	64	2.7682	2.6081	2.6665	2.2831
750	125	3.1165	3.3267	3.0842	2.9691
1296	216	2.8962	3.6390	3.4168	3.3795
2058	343	3.0662	3.7934	3.5912	3.5853
3072	512	2.9374	3.5489	3.6959	3.7029
4374	729	2.9652	3.6107	3.7642	3.7763
6000	1000	2.9394	3.6882	3.8115	3.8253
7986	1331	3.0572	3.7435	3.8458	3.8596
p=3					
48	8	3.1053	2.2665	2.5603	1.2503
162	27	4.5408	4.1517	3.8251	3.8397
384	64	3.0953	4.9929	4.0125	4.8957
750	125	4.1133	4.6288	4.4182	4.8621
1296	216	4.2979	4.5334	3.9853	4.6517
2058	343	3.9505	4.9049	4.4351	5.0449
3072	512	3.7233	4.7514	4.7568	4.9668
4374	729	4.0327	4.8269	4.5890	4.8649
6000	1000	4.1534	5.0524	4.7063	5.0935
p=4					
48	8	4.4292	3.3993	4.0635	3.6656
162	27	4.5236	6.0145	5.6065	5.6031
384	64	5.1156	5.8164	5.5710	5.7308
750	125	4.6649	5.4384	5.2582	5.5015
1296	216	4.8958	5.5246	5.6540	5.2230
2058	343	5.1161	6.0758	5.4420	4.9686
3072	512	5.0657	5.6314	5.6420	5.2820
4374	729	4.9090	5.7679	5.4660	5.3536

Table 4.24: Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Type 1*.

p=2					
N	# elements *Type 2*	$\mathcal{O}_{\infty,\Gamma_1}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_3}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_4}$	$\mathcal{O}_{\infty,\mathbf{v}_3\mathbf{v}_4}$
48	32	2.6057	3.7098	2.1379	2.0808
162	108	3.1190	3.3943	2.9222	3.1629
384	256	2.5210	2.5768	2.7938	2.7929
750	500	3.0506	3.1927	2.8613	2.9291
1296	864	2.9520	3.4637	3.2294	2.9797
2058	1372	2.8823	3.6136	2.7929	3.0253
3072	2048	2.9869	3.7070	3.0552	2.8965
4374	2916	3.0120	3.7697	3.0168	2.9858
6000	4000	2.9484	3.8141	3.0010	3.0490
7986	5324	2.9213	3.8055	2.9481	2.9298
p=3					
48	32	3.8542	1.6037	3.2575	2.9677
162	108	3.3701	4.3875	3.6915	4.1122
384	256	2.7637	4.4651	4.2316	3.6029
750	500	3.2985	4.3451	2.7411	2.9948
1296	864	3.5339	4.8434	3.2061	3.3909
2058	1372	3.6641	4.9210	3.4406	3.5899
3072	2048	3.7452	4.5244	3.5804	3.7051
4374	2916	3.7998	4.8286	3.6722	3.7779
6000	4000	3.8383	4.8602	3.7362	3.8267
p=4					
48	32	2.4795	4.3030	2.6719	2.5720
162	108	4.6449	4.5141	3.7830	3.9014
384	256	5.3034	5.9114	4.8477	4.9176
750	500	4.6855	5.0987	5.1547	5.1153
1296	864	4.7800	5.8849	4.5784	4.7337
2058	1372	5.1274	5.5732	4.8942	4.8845
3072	2048	5.0064	5.8405	5.0535	4.9783
4374	2916	4.8428	5.2071	4.8912	4.8810

Table 4.25: Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Type 2*.

N	# elements *Type 3*	$\mathcal{O}_{\infty,\Gamma_1}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_3}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_4}$	$\mathcal{O}_{\infty,\mathbf{v}_3\mathbf{v}_4}$
p=2					
48	8	2.6304	2.8259	2.4932	3.5655
162	27	3.0914	2.2084	3.1917	3.3728
384	64	2.5952	2.3846	2.6377	2.0196
750	125	3.0692	3.0315	2.9669	3.0670
1296	216	2.9660	3.3312	3.0695	3.4899
2058	343	2.9292	3.5043	2.9378	3.6940
3072	512	2.8824	3.6169	2.9507	3.6895
4374	729	3.1135	3.6954	2.9488	3.6214
6000	1000	2.8496	3.7523	3.0968	3.6916
7986	1331	3.0070	3.7950	2.8638	3.7444
p=3					
48	8	3.7949	-1.0274	2.7253	1.5875
162	27	2.6918	4.0440	3.9633	3.6581
384	64	3.0112	5.2511	4.9571	4.7105
750	125	3.4643	4.7253	2.9786	5.1320
1296	216	3.6691	4.9518	3.3158	4.4148
2058	343	3.7784	4.9654	3.4938	4.6776
3072	512	3.8426	4.9467	3.6058	4.2688
4374	729	3.8830	5.2297	3.6833	4.6311
6000	1000	3.9099	4.6461	3.7397	4.8883
p=4					
48	8	2.6617	3.2635	2.7777	3.3251
162	27	4.9018	5.3379	3.9718	5.8602
384	64	5.3616	5.1854	4.9615	4.9339
750	125	4.5536	4.9503	4.9173	5.6645
1296	216	4.8814	6.0312	4.9361	5.8493
2058	343	5.0664	5.4866	4.9411	5.4321
3072	512	4.8772	4.2412	4.8348	4.6572
4374	729	5.0133	4.6284	5.0458	4.9404

Table 4.26: Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Type 3*.

p=2					
N	# elements *Class I*	$\mathcal{O}_{\infty,\Gamma_1}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_3}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_4}$	$\mathcal{O}_{\infty,\mathbf{v}_3\mathbf{v}_4}$
48	8	2.6304	2.8259	2.4932	3.5655
162	27	3.0914	2.2084	3.1917	3.3728
384	64	2.5952	2.3846	2.6377	2.0196
750	125	3.0692	3.0315	2.9669	3.0670
1296	216	2.9660	3.3312	3.0695	3.4899
2058	343	2.9292	3.5043	2.9378	3.6940
3072	512	2.8824	3.6169	2.9507	3.6895
4374	729	3.1135	3.6954	2.9488	3.6214
6000	1000	2.8496	3.7523	3.0968	3.6916
7986	1331	3.0070	3.7950	2.8638	3.7444
p=3					
48	8	3.7949	-1.0274	2.7253	1.5875
162	27	2.6918	4.0440	3.9633	3.6581
384	64	3.0112	5.2511	4.9571	4.7105
750	125	3.4643	4.7253	2.9786	5.1320
1296	216	3.6691	4.9518	3.3158	4.4148
2058	343	3.7784	4.9654	3.4938	4.6776
3072	512	3.8426	4.9467	3.6058	4.2688
4374	729	3.8830	5.2297	3.6833	4.6311
6000	1000	3.9099	4.6461	3.7397	4.8883
p=4					
48	8	2.6617	3.2635	2.7777	3.3251
162	27	4.9018	5.3379	3.9718	5.8602
384	64	5.3616	5.1854	4.9615	4.9339
750	125	4.5536	4.9503	4.9173	5.6645
1296	216	4.8814	6.0312	4.9361	5.8493
2058	343	5.0664	5.4866	4.9411	5.4321
3072	512	4.8772	4.2412	4.8348	4.6572
4374	729	5.0133	4.6284	5.0458	4.9404

Table 4.27: Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Class I*.

p=2					
N	# elements *Class II*	$\mathcal{O}_{\infty,\Gamma_1}$	$\mathcal{O}_{\infty,\mathbf{v_2v_3}}$	$\mathcal{O}_{\infty,\mathbf{v_2v_4}}$	$\mathcal{O}_{\infty,\mathbf{v_3v_4}}$
48	32	2.6057	3.7098	2.1379	2.0808
162	108	3.1190	3.3943	2.9222	3.1629
384	256	2.5210	2.5768	2.7938	2.7929
750	500	3.0506	3.1927	2.8613	2.9291
1296	864	2.9520	3.4637	3.2294	2.9797
2058	1372	2.8823	3.6136	2.7929	3.0253
3072	2048	2.9869	3.7070	3.0552	2.8965
4374	2916	3.0120	3.7697	3.0168	2.9858
6000	4000	2.9484	3.8141	3.0010	3.0490
7986	5324	2.9213	3.8055	2.9481	2.9298
p=3					
48	32	3.8542	1.6037	3.2575	2.9677
162	108	3.3701	4.3875	3.6915	4.1122
384	256	2.7637	4.4651	4.2316	3.6029
750	500	3.2985	4.3451	2.7411	2.9948
1296	864	3.5339	4.8434	3.2061	3.3909
2058	1372	3.6641	4.9210	3.4406	3.5899
3072	2048	3.7452	4.5244	3.5804	3.7051
4374	2916	3.7998	4.8286	3.6722	3.7779
6000	4000	3.8383	4.8602	3.7362	3.8267
p=4					
48	32	2.4795	4.3030	2.6719	2.5720
162	108	4.6449	4.5141	3.7830	3.9014
384	256	5.3034	5.9114	4.8477	4.9176
750	500	4.6855	5.0987	5.1547	5.1153
1296	864	4.7800	5.8849	4.5784	4.7337
2058	1372	5.1274	5.5732	4.8942	4.8845
3072	2048	5.0064	5.8405	5.0535	4.9783
4374	2916	4.8428	5.2071	4.8912	4.8810

Table 4.28: Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Class II*.

p=2					
N	# elements *Class III*	$\mathcal{O}_{\infty,\Gamma_1}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_3}$	$\mathcal{O}_{\infty,\mathbf{v}_2\mathbf{v}_4}$	$\mathcal{O}_{\infty,\mathbf{v}_3\mathbf{v}_4}$
48	8	3.4430	3.1683	4.0553	3.4137
162	27	2.8944	2.4924	3.0712	3.2065
384	64	2.7682	2.6081	2.6665	2.2831
750	125	3.1165	3.3267	3.0842	2.9691
1296	216	2.8962	3.6390	3.4168	3.3795
2058	343	3.0662	3.7934	3.5912	3.5853
3072	512	2.9374	3.5489	3.6959	3.7029
4374	729	2.9652	3.6107	3.7642	3.7763
6000	1000	2.9394	3.6882	3.8115	3.8253
7986	1331	3.0572	3.7435	3.8458	3.8596
p=3					
48	8	3.1053	2.2665	2.5603	1.2503
162	27	4.5408	4.1517	3.8251	3.8397
384	64	3.0953	4.9929	4.0125	4.8957
750	125	4.1133	4.6288	4.4182	4.8621
1296	216	4.2979	4.5334	3.9853	4.6517
2058	343	3.9505	4.9049	4.4351	5.0449
3072	512	3.7233	4.7514	4.7568	4.9668
4374	729	4.0327	4.8269	4.5890	4.8649
6000	1000	4.1534	5.0524	4.7063	5.0935
p=4					
48	8	4.4292	3.3993	4.0635	3.6656
162	27	4.5236	6.0145	5.6065	5.6031
384	64	5.1156	5.8164	5.5710	5.7308
750	125	4.6649	5.4384	5.2582	5.5015
1296	216	4.8958	5.5246	5.6540	5.2230
2058	343	5.1161	6.0758	5.4420	4.9686
3072	512	5.0657	5.6314	5.6420	5.2820
4374	729	4.9090	5.7679	5.4660	5.3536

Table 4.29: Orders of convergence for the errors on Γ_1, $\mathbf{v}_2\mathbf{v}_3$, $\mathbf{v}_2\mathbf{v}_4$, and $\mathbf{v}_3\mathbf{v}_4$, for problem (4.6.3) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{U}_p$, $p = 2, 3, 4$, and elements of *Class III*.

solution is $O(h^{p+2})$ superconvergent on two edges of the *inflow* faces. Moreover, enriched polynomial spaces lead to a simpler *a posteriori* error estimation procedure. Finally, the pointwise superconvergence results for the leading term of the discretization error on each elements are tested on several linear problems and tetrahedral meshes for smooth solutions.

Chapter 5

The Discontinuous Galerkin Method for Nonlinear Problems

The most challenging hyperbolic partial differential equations are nonlinear, because they develop propagating discontinuities known as shocks. In this chapter we extend our error analysis to three-dimensional nonlinear hyperbolic scalar problems.

5.1 Nonlinear Hyperbolic Problems

We consider nonlinear scalar hyperbolic problems of the form

$$\nabla \cdot \mathbf{F}\left(u\right) = f\left(x, y, z\right), \ \left(x, y, z\right) \in \Omega, \tag{5.1.1a}$$

subject to the boundary conditions

$$u\left(x, y, z\right) = h_0\left(x, y, z\right), \ \left(x, y, z\right) \in \partial\Omega^-, \tag{5.1.1b}$$

where $\mathbf{F} : \ \mathbb{R} \longrightarrow \ \mathbb{R}^3$, $u \ : \ \mathbb{R}^3 \longrightarrow \ \mathbb{R}$, f and h_0 are analytic functions, such that

$$\mathbf{F}'\left(u\right) \neq 0. \tag{5.1.1c}$$

We further assume that $\mathbf{F}(u)$ is such that the boundary $\partial\Omega$ can be split into *inflow* $\partial\Omega^-$, *outflow* $\partial\Omega^+$ and characteristic $\partial\Omega^0$ boundaries using $\boldsymbol{a}(u) = \mathbf{F}'(u)$, where, $\partial\Omega^-$, $\partial\Omega^+$ and $\partial\Omega^0$ are defined as follows:

$$\begin{aligned}
\partial\Omega^- &= \left\{(x, y, z) \in \partial\Omega, \mathbf{F}'\left(u\right) \cdot \mathbf{n} < 0\right\}, && (5.1.2a)\\
\partial\Omega^+ &= \left\{(x, y, z) \in \partial\Omega, \mathbf{F}'\left(u\right) \cdot \mathbf{n} > 0\right\}, && (5.1.2b)\\
\partial\Omega^0 &= \left\{(x, y, z) \in \partial\Omega, \mathbf{F}'\left(u\right) \cdot \mathbf{n} = 0\right\}, && (5.1.2c)
\end{aligned}$$

with the boundary of Ω, $\partial\Omega = \partial\Omega^- \cup \partial\Omega^+ \cup \partial\Omega^0$ and $\mathbf{n}$ is the outward unit normal to $\partial\Omega$.

We obtain a weak DG formulation of (5.1.1a) by multiplying (5.1.1a) by a test function v, integrating over an arbitrary element Δ, and applying Stokes' theorem as

$$\iint\limits_{\Gamma^-} \mathbf{n}\cdot\mathbf{F}(u)\,v d\sigma + \iint\limits_{\Gamma^+} \mathbf{n}\cdot\mathbf{F}(u)\,v d\sigma + \iiint\limits_{\Delta} \mathbf{F}(u)\cdot\nabla v dxdydz = \iiint\limits_{\Delta} f v dxdydz. \quad (5.1.3)$$

The discrete DG method consists of finding $U \in \mathcal{W}_p$ on an element Δ such that for all $V \in \mathcal{W}_p$ we have

$$\iint\limits_{\Gamma^-} \mathbf{n}\cdot\mathbf{F}\left(\tilde{U}\right) V d\sigma + \iint\limits_{\Gamma^+} \mathbf{n}\cdot\mathbf{F}(U)\,V d\sigma + \iiint\limits_{\Delta} \mathbf{F}(U)\cdot\nabla V dxdydz = \iiint\limits_{\Delta} f V dxdydz, \quad (5.1.4)$$

where $\tilde{U}$ is given by (3.1.5b) and $\mathcal{W}_p$ is either $\mathcal{P}_p$, $\mathcal{L}_p$, $\mathcal{U}_p$, or $\mathcal{M}_p$.
Subtract (5.1.4) from (5.1.3) with $\tilde{U} = u$ and $v = V$ to obtain the DG orthogonality condition for the local error, for all $V \in \mathcal{W}_p$,

$$\iint_{\Gamma^+} \mathbf{n}\cdot(\mathbf{F}(u) - \mathbf{F}(U))\,V d\sigma - \iiint_{\Delta} (\mathbf{F}(u) - \mathbf{F}(U))\,.\nabla V dxdydz = 0. \quad (5.1.5)$$

We map a physical tetrahedron Δ having vertices $\mathbf{v}_i = (x_i, y_i, z_i)$, $1 \le i \le 4$, into the reference tetrahedron $\hat{\Delta}$ with vertices $\hat{\mathbf{v}}_1 = (0,0,0)$, $\hat{\mathbf{v}}_2 = (1,0,0)$, $\hat{\mathbf{v}}_3 = (0,1,0)$, $\hat{\mathbf{v}}_4 = (0,0,1)$, by the standard affine mapping (3.1.7) and as illustrated in Figures 3.1 and 3.2. Thus, the DG orthogonality (5.1.5) becomes

$$\iint_{\hat{\Gamma}^+} \hat{\mathbf{n}}\cdot\left(\mathbf{J}_0\left(\mathbf{F}(\hat{u}) - \mathbf{F}\left(\hat{U}\right)\right)\right)\hat{V} d\hat{\sigma} - \iiint_{\hat{\Delta}} \left(\mathbf{J}_0\left(\mathbf{F}(\hat{u}) - \mathbf{F}\left(\hat{U}\right)\right)\right).\nabla\hat{V} d\xi d\eta d\zeta = 0, \quad (5.1.6)$$

for all $V \in \mathcal{W}_p$.
In the remainder of this chapter we will omit the $\hat{}$ unless needed for clarity. In the following theorem we derive orthogonality condition for the leading term of the local DG error for (5.1.1a) having smooth solutions.

Theorem 5.1.1. *Let $u \in C^\infty(\Omega)$ and $U \in \mathcal{W}_p$, respectively, be the solutions of (5.1.1a), (5.1.1b) and (5.1.4), with $\tilde{U}|_{\Gamma^-} = u$. Then the local finite element error can be written as*

$$\epsilon(\xi,\eta,\zeta,h) = \sum_{k=p+1}^{\infty} h^k Q_k(\xi,\eta,\zeta)\,. \quad (5.1.7)$$

Furthermore, the leading term satisfies

$$\iint_{\Gamma^+} \mathbf{d}\cdot\mathbf{n} Q_{p+1} V d\sigma + \iiint_{\Delta} \mathbf{d}\cdot\nabla Q_{p+1} V d\xi d\eta d\zeta = 0,\ \forall\ V \in \mathcal{W}_p, \quad (5.1.8)$$

where

$$\mathbf{d} = \mathbf{J}_0\mathbf{F}'(u^*)\,, \quad (5.1.9)$$

and u^* is the average of the values of u at the centers of inflow faces of Δ.

Proof. The proof follows the same line of reasoning as in [15]. $\square$

Thus, the results of chapter 3 stated in Theorems 3.2.1, 3.2.2, 3.2.3, 3.2.4 and their corollaries for the space $\mathcal{P}_p$, and the results of chapter 4 stated in Theorems 4.3.1, 4.3.2, 4.3.3, 4.4.2, 4.3.5, 4.3.6, 4.3.7, 4.4.2, 4.4.3, 4.4.4, and their corollaries for the spaces $\mathcal{L}_p$, $\mathcal{U}_p$, and $\mathcal{M}_p$, still hold for nonlinear scalar problems.

5.2 Error Estimation Procedure

The discrete DG method consists of finding $U \in \mathcal{P}_p$ on an element Δ such that

$$\iint\limits_{\Gamma^-} \mathbf{n} \cdot \left(\mathbf{F}\left(\tilde{U}\right) - \mathbf{F}\left(U\right)\right) V d\sigma + \iiint\limits_{\Delta} \left(\mathbf{F}'\left(U\right) \cdot \nabla U\right) V dxdydz$$
$$= \iiint\limits_{\Delta} fV dxdydz, \ \forall\ V \in \mathcal{W}_p. \tag{5.2.1}$$

We estimate the error by finding E of the form (3.3.9a) solution of the linearized weak problem

$$\iint\limits_{\Gamma^-} \mathbf{n} \cdot \mathbf{F}'\left(U\right)\left(E^- - E\right) V d\sigma + \iiint\limits_{\Delta} \mathbf{F}'\left(U\right) \cdot \nabla\left(U + E\right) V dxdydz$$
$$= \iiint\limits_{\Delta} fV dxdydz - \iint\limits_{\Gamma^-} \mathbf{n} \cdot \mathbf{F}'\left(U\right)\left(U^- - U\right) V d\sigma, \ \forall\ V \in \mathcal{E}, \tag{5.2.2}$$

where $\mathcal{E}$ is the finite element space for the error defined in section 3.3 for $\mathcal{P}_p$ and section 4.5 for $\mathcal{U}_p$. However, in the numerical examples we use the space $\mathcal{P}_p$.

Remarks:

1. The *inflow* solution $\tilde{U}$ is given by (3.1.5b)
2. We first find U using Newton's iteration by solving the nonlinear problem (5.2.1) and then solve the linear problem (5.2.2) to find the error estimate E.
3. We follow the steps of the modified DG algorithm for linear problems given in section 3.3 to find the DG solution and error on all elements.
4. We utilize the error basis functions χ_i for linear problems shown in Tables 3.2, 3.3, and 3.4 with $(\alpha, \beta, \gamma)^T = \mathbf{J}_0\mathbf{F}'(U^*)$ where U^* is the average of the values of U^- at the centers of *inflow* faces.

The accuracy of *a posteriori* error estimates is measured by the ratio of the error estimate over the true error. The local and global effectivity indices are given by (3.3.11) and (3.3.12).

5.3 Computational Examples

In the following examples we use the exact boundary conditions at the *inflow* boundary and solve several nonlinear hyperbolic problems on uniform tetrahedral meshes, and test our modified DG method and *a posteriori* error estimation strategy for smooth and discontinuous solutions.

We use the space $\mathcal{P}_p$ on a family of uniform meshes obtained by partitioning the domain $[0,1]^3$ into n^3 cubes, $n = 1, 2, \ldots, 14$ and subdividing each cube into five tetrahedra [40]. The resulting meshes have $N = 5n^3 = 5, 40, \ldots, 13720$ tetrahedral elements with diameter $h_{max} = \frac{\sqrt{2}}{n}$.

Example 5.3.1.

We consider the two-dimensional inviscid Burgers' equation

$$u_t + uu_x + uu_y = f(x,y,t),\ (x,y,t) \in \Omega = [0,1]^3\,, \tag{5.3.1a}$$

and select f and the *inflow* boundary conditions such that the exact solution is given by

$$u\,(x,y,t) = 1 + \frac{1}{2}\sin\left(x + y + t\right). \tag{5.3.1b}$$

We solve (5.3.1) with the exact *inflow* boundary condition, $U^- = u$, using the space-time modified DG method on structured tetrahedral meshes having $N =$ 5, 40, 135, 320, 625, 1080, 1715, 2560, 3645, 5000, 6655, 8640, 10985, 13720 elements with the spaces $\mathcal{P}_p$, $p = 0, 1, 2$. The L^2 errors, orders of convergence and effectivity indices shown in Table 5.1, indicate that the modified DG method yields optimal convergence rates. Furthermore, the effectivity indices for the modified DG are close to unity and converge to one under mesh refinement. We also observe that the error estimate leads to $O\left(h^{p+2}\right)$ convergence of the corrected DG solution.

Example 5.3.2.

Next, we consider the homogeneous inviscid Burgers' equation

$$u_t + uu_x + uu_y = 0,\ (x,y,t) \in \Omega = [0,1]^3\,, \tag{5.3.2a}$$

subject to the initial conditions

$$u\,(x,y,0) = 1 + \frac{1}{2}\sin\left(\pi(x + y + 1/2)\right). \tag{5.3.2b}$$

N	$\|e\|_{2,\Omega}$	order	$\|e_{cr}\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
p=0							
5	7.2747e-02	-	4.0382e-02	-	0.8856	1.1489	1.0714
40	4.8896e-02	0.5732	1.0560e-02	1.9352	0.8477	1.1571	1.0016
135	3.3112e-02	0.9613	5.0110e-03	1.8384	0.0801	1.1711	1.0097
320	2.5358e-02	0.9274	2.8915e-03	1.9113	0.3326	1.8542	1.0080
625	2.0367e-02	0.9823	1.9130e-03	1.8514	0.1396	1.5012	1.0075
1080	1.7057e-02	0.9728	1.3787e-03	1.7966	0.1411	1.4226	1.0066
1715	1.4641e-02	0.9906	1.0569e-03	1.7243	0.4408	1.3008	1.0061
2560	1.2836e-02	0.9855	8.4625e-04	1.6644	0.3775	1.4842	1.0057
3645	1.1417e-02	0.9941	7.0141e-04	1.5938	0.3009	2.4717	1.0053
5000	1.0286e-02	0.9909	5.9648e-04	1.5381	0.1660	1.6640	1.0050
6655	9.3541e-03	0.9959	5.1812e-04	1.4775	0.1511	1.4954	1.0048
8640	8.5793e-03	0.9937	4.5748e-04	1.4306	0.1445	1.4220	1.0046
10985	7.9213e-03	0.9970	4.0959e-04	1.3816	0.1408	1.8185	1.0045
13720	7.3581e-03	0.9953	3.7076e-04	1.3440	0.0873	2.6471	1.0043
p=1							
5	4.0691e-02	-	3.5262e-03	-	0.9430	1.1378	0.9579
40	1.0880e-02	1.9030	1.1422e-03	1.6263	0.9309	1.1055	1.0290
135	4.8542e-03	1.9906	3.1471e-04	3.1792	0.8683	1.1129	1.0166
320	2.7331e-03	1.9967	1.3918e-04	2.8361	0.8340	1.1063	1.0113
625	1.7494e-03	1.9994	7.1652e-05	2.9753	0.8968	1.0983	1.0081
1080	1.2153e-03	1.9979	4.1422e-05	3.0058	0.8923	1.0915	1.0061
1715	8.9290e-04	1.9998	2.5977e-05	3.0269	0.9163	1.0858	1.0049
2560	6.8373e-04	1.9989	1.7370e-05	3.0139	0.9105	1.0812	1.0040
3645	5.4023e-04	2.0000	1.2181e-05	3.0127	0.9278	1.0774	1.0034
5000	4.3762e-04	1.9993	8.8754e-06	3.0053	0.9236	1.0743	1.0029
6655	3.6166e-04	2.0001	6.6662e-06	3.0032	0.9382	1.0716	1.0026
8640	3.0391e-04	1.9996	5.1355e-06	2.9980	0.9315	1.0693	1.0023
10985	2.5895e-04	2.0001	4.0404e-06	2.9966	0.9442	1.0674	1.0021
13720	2.2328e-04	1.9997	3.2369e-06	2.9917	0.9379	1.0657	1.0019
p=2							
5	2.1707e-03	-	7.0465e-04	-	1.0699	1.1125	1.0905
40	4.1651e-04	2.3817	5.0503e-05	3.8025	0.8644	1.0494	1.0110
135	1.2148e-04	3.0389	1.2557e-05	3.4326	0.3881	1.0558	1.0102
320	5.2136e-05	2.9404	4.1295e-06	3.8657	0.6155	1.1316	1.0077
625	2.6737e-05	2.9928	1.7372e-06	3.8804	0.2510	1.1469	1.0062
1080	1.5530e-05	2.9796	8.5697e-07	3.8756	0.1122	1.1123	1.0050
1715	9.7873e-06	2.9952	4.6504e-07	3.9655	0.7552	1.0650	1.0043
2560	6.5657e-06	2.9897	2.7463e-07	3.9445	0.6185	1.1375	1.0037
3645	4.6130e-06	2.9970	1.7230e-07	3.9579	0.6679	1.2512	1.0032
5000	3.3650e-06	2.9939	1.1364e-07	3.9503	0.6348	1.1534	1.0028
6655	2.5287e-06	2.9980	7.7956e-08	3.9544	0.3966	1.0937	1.0025
8640	1.9484e-06	2.9960	5.5293e-08	3.9477	0.5151	1.0569	1.0023
10985	1.5326e-06	2.9986	4.0296e-08	3.9527	0.3589	1.1138	1.0021
13720	1.2274e-06	2.9972	3.0083e-08	3.9443	0.4803	1.1981	1.0019

Table 5.1: L^2 errors, their orders and effectivity indices for problem (5.3.1) using the modified DG method on uniform meshes having $N = 6n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0, 1, 2$.

This is subjected to *inflow* boundary conditions $u(0,y,t)$ and $u(x,0,t)$ assuming a periodic true solution.

First we solve (5.3.2) for $0 \leq t \leq 0.2$ on uniform tetrahedral meshes having $N = 5n^3 =$ 5,40,135,320,625,...,16875 elements with the spaces $\mathcal{P}_p$, $p = 0,1$ and show the L^2 errors, orders and effectivity indices in Table 5.2 for $\mathcal{P}_p$, $p = 0,1$. We observe that the effectivity indices are close to unity under mesh refinement. The modified DG solution exhibits optimal $O(h^{p+1})$ convergence rates, however, $U + E$ is not $O(h^{p+2})$.

p=0							
N	$\|e\|_{2,\Omega}$	order	$\|e_{cr}\|_{2,\Omega}$	order	$\theta_{\Delta,\min}$	$\theta_{\Delta,\max}$	θ
5	1.4816e-01	-	7.4264e-02	-	0.6630	1.0469	0.7765
40	8.6731e-02	0.7726	3.8110e-02	0.9625	0.1503	1.5470	0.7996
135	5.9875e-02	0.9139	2.0264e-02	1.5577	0.4499	1.8889	0.8750
320	4.5261e-02	0.9727	1.2997e-02	1.5438	0.1675	2.5686	0.8892
625	3.6105e-02	1.0129	9.2082e-03	1.5445	0.4139	2.0871	0.9071
1080	2.9944e-02	1.0262	6.9381e-03	1.5525	0.1393	2.7602	0.9208
1715	2.5564e-02	1.0258	5.4822e-03	1.5279	0.0247	2.1614	0.9318
2560	2.2310e-02	1.0197	4.4808e-03	1.5105	0.0153	2.8448	0.9403
3645	1.9803e-02	1.0120	3.7557e-03	1.4987	0.0406	2.2482	0.9470
5000	1.7813e-02	1.0052	3.2098e-03	1.4909	0.0575	2.8859	0.9522
6655	1.6193e-02	1.0003	2.7864e-03	1.4842	0.0171	2.2957	0.9563
8640	1.4847e-02	0.9972	2.4510e-03	1.4740	0.0337	2.9087	0.9597
10985	1.3710e-02	0.9959	2.1807e-03	1.4597	0.0505	2.3260	0.962
13720	1.2735e-02	0.9957	1.9598e-03	1.4415	0.0635	2.9225	0.9647
16875	1.1889e-02	0.9963	1.7769e-03	1.4200	0.0737	2.3464	0.9667
p=1							
5	7.1928e-02	-	5.2996e-02	-	0.5146	0.7117	0.5239
40	3.1054e-02	1.2118	1.4968e-02	1.8240	0.4417	1.8059	0.7662
135	1.5454e-02	1.7212	6.8287e-03	1.9356	0.3728	1.6145	0.7816
320	9.0408e-03	1.8635	3.6777e-03	2.1511	0.1960	1.5647	0.8329
625	5.8697e-03	1.9357	2.1897e-03	2.3237	0.2270	1.6356	0.8737
1080	4.1301e-03	1.9279	1.4001e-03	2.4530	0.3223	1.5525	0.9062
1715	3.0821e-03	1.8988	9.4211e-04	2.5701	0.2281	1.4513	0.9295
2560	2.3996e-03	1.8745	6.5802e-04	2.6877	0.3133	1.3648	0.9457
3645	1.9248e-03	1.8721	4.7853e-04	2.7042	0.3847	1.3467	0.9559
5000	1.5768e-03	1.8929	3.6071e-04	2.6826	0.3576	1.3529	0.9630
6655	1.3118e-03	1.9300	2.8113e-04	2.6151	0.3408	1.3300	0.9681
8640	1.1052e-03	1.9701	2.2581e-04	2.5184	0.2012	1.2993	0.9720
10985	9.4113e-04	2.0073	1.8623e-04	2.4075	0.4688	1.2682	0.9750
13720	8.0950e-04	2.0331	1.5700e-04	2.3039	0.4823	1.2396	0.9774
16875	7.0282e-04	2.0483	1.3465e-04	2.2256	0.4656	1.2258	0.9795

Table 5.2: L^2 errors, their orders and effectivity indices for problem (5.3.2) using the modified DG method on uniform meshes having $N = 5n^3$ elements for the spaces $\mathcal{P}_p$, $p = 0,1$.

Next, we integrate until $t = 1$ where the solution develops shock discontinuities with a first shock forming and leaving the domain and a second shock entering the domain. We use a uniform mesh having $N = 13720$ tetrahedral elements with the spaces $\mathcal{P}_p$, $p = 0,1$. We plot the true solution, L^2 local true errors and local effectivity indices at $t = 0.5, 1.0$ in Figures 5.1 and 5.2, respectively. These numerical results indicate that, before the shock forms, the error estimates are accurate and converge to the true errors under mesh refinement. After the shock forms the effectivity indices are still close to one in regions away from the

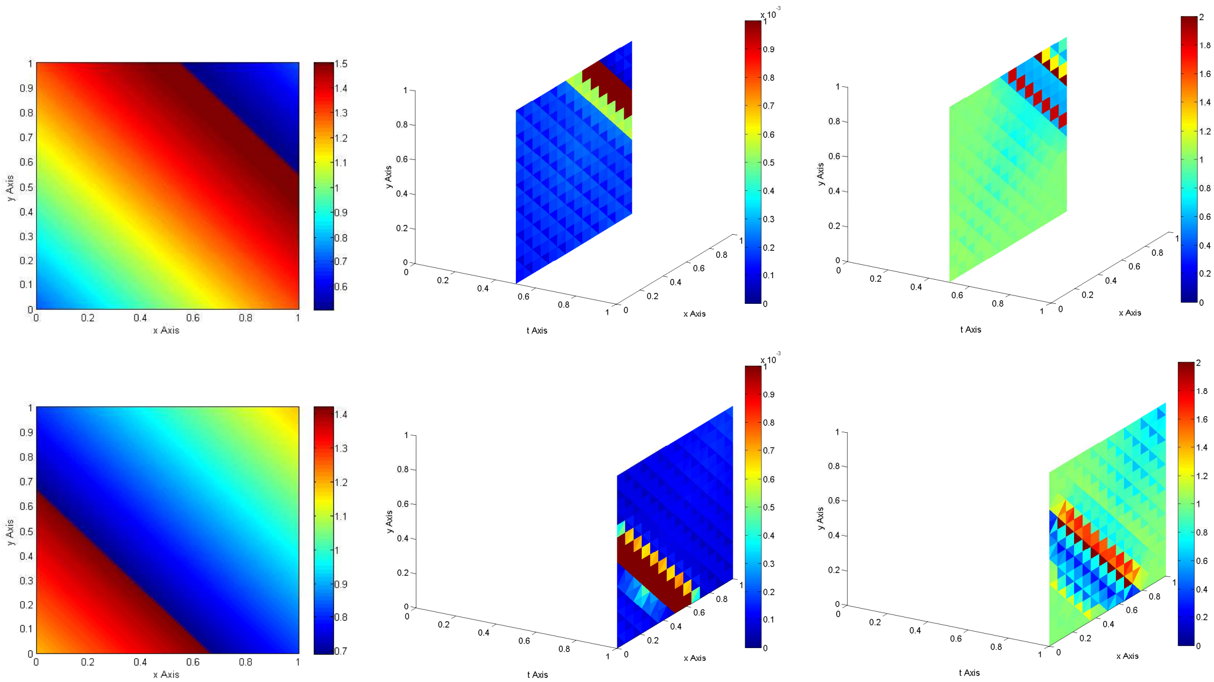


Figure 5.1: True solution (left), L^2 true local errors (center) and local effectivity indices (right) for (5.3.2) on a uniform mesh having N=13720 tetrahedra for the space $\mathcal{P}_0$ at $t = 0.5$ (top) and $t = 1$ (bottom).

discontinuity. By comparing the plots of Figure 5.2 to those of Figure 5.3 on a coarse mesh with $N = 5000$ elements and $\mathcal{P}_1$, we notice an improvement of the effectivity indices under mesh refinement. Applying a local adaptive mesh refinement algorithm which refines elements near the discontinuity should further improve the accuracy of both the *a posteriori* error estimate and the solution.

5.4 Conclusion

In this chapter we have extended the error analysis for nonlinear hyperbolic problems. The numerical results indicate that, before the shock forms, the error estimates are accurate and converge to the true errors under mesh refinement, and the error estimate leads to $O\left(h^{p+2}\right)$ convergence of the corrected DG solution. After the shock forms the effectivity indices are still close to one in regions away from the discontinuity. Applying a local adaptive mesh refinement algorithm which refines elements near the discontinuity should further improve the accuracy of both the *a posteriori* error estimate and the solution.

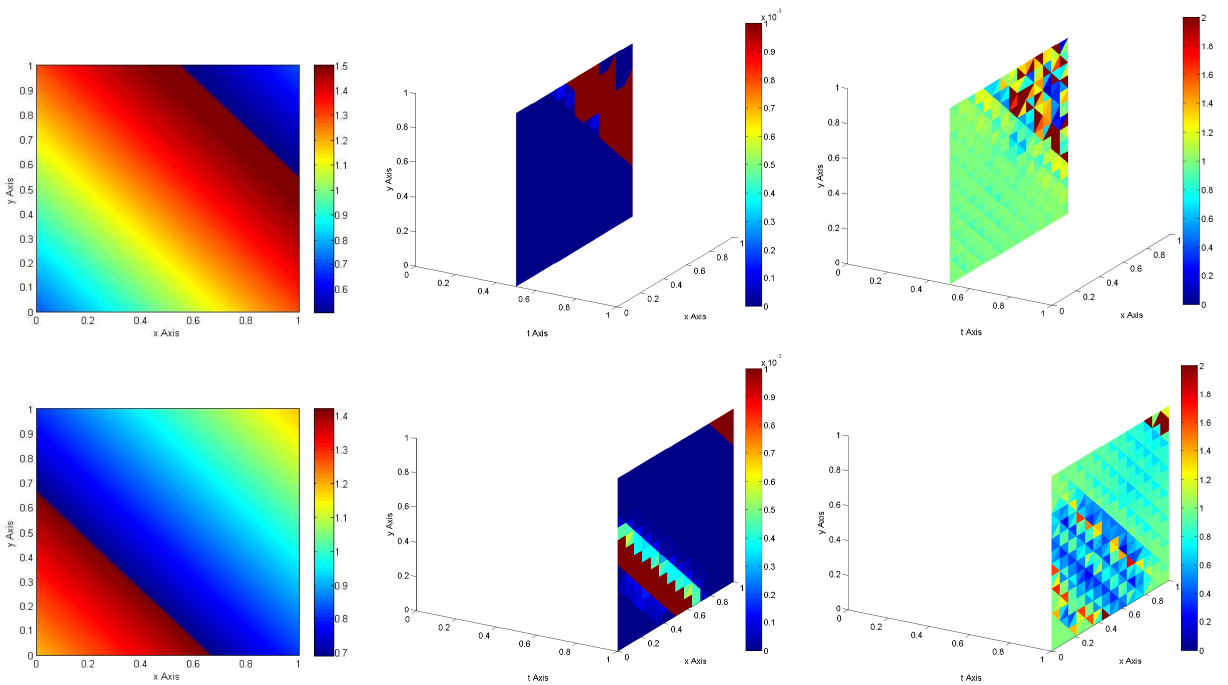


Figure 5.2: True solution (left), L^2 true local errors (center) and local effectivity indices (right) for (5.3.2) using a uniform mesh having N=13720 tetrahedra for the space $\mathcal{P}_1$ at $t = 0.5$ (top) and $t = 1$ (bottom).

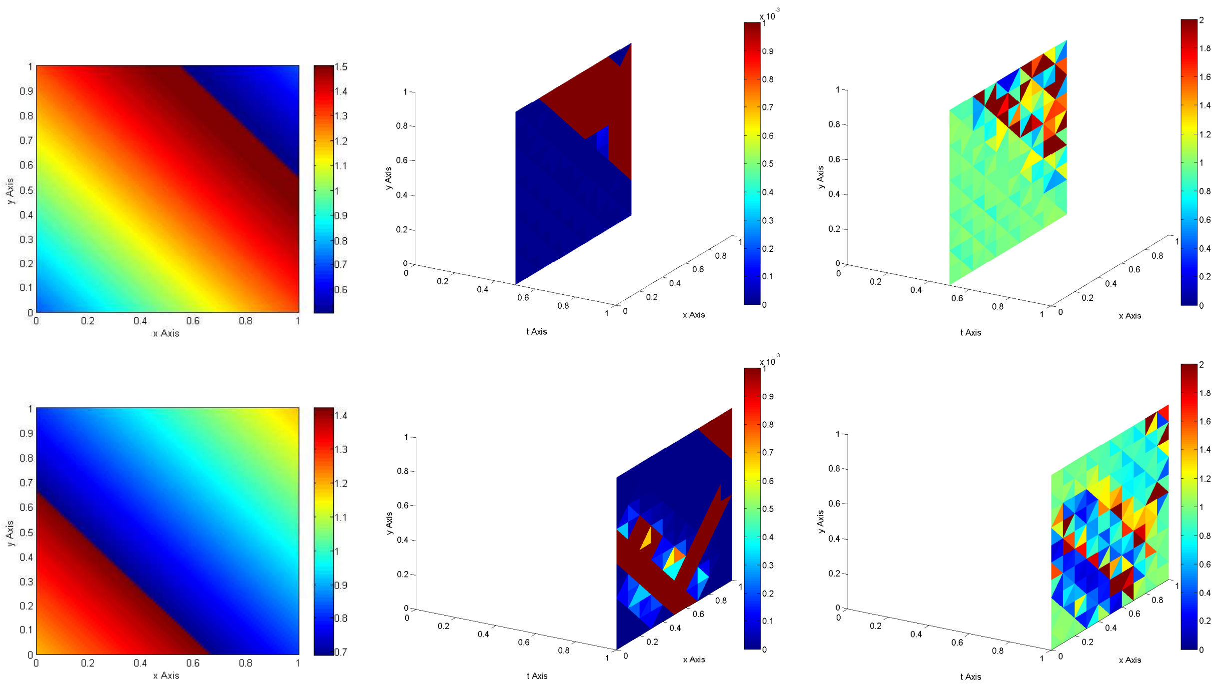


Figure 5.3: True solution (left), L^2 true local errors (center) and local effectivity indices (right) for (5.3.2) using a uniform mesh having N=5000 tetrahedra for the space $\mathcal{P}_1$ at $t = 0.5$ (top) and $t = 1$ (bottom).

Chapter 6

A Posteriori Error Estimation for Linear Hyperbolic systems

We present an *a posteriori* error analysis for the discontinuous Galerkin method applied to first-order linear symmetric hyperbolic systems of partial differential equations with smooth solutions. We perform a local error analysis by writing the local error as a series and showing that its leading term is $O\left(h^{p+1}\right)$. This can be used to solve relatively small local problems to compute efficient and asymptotically exact estimates of the finite element error.

6.1 DG Formulation and Local Error Analysis

In this section we introduce a Runge–Kutta discontinuous Galerkin (RKDG) method applied to first-order symmetric linear hyperbolic systems in multiple space dimensions. Let $\mathbf{u}(t,x,y,z) : [0,T] \times \mathbb{R}^3 \longrightarrow \mathbb{R}^m$ be the true solution of the linear hyperbolic system

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y + \mathbf{C}\mathbf{u}_z = \mathbf{f}(t,x,y,z),\ 0 \le t \le T,\ (x,y,z) \in \Omega = (0,1)^3, \tag{6.1.1a}$$

with symmetric real constant coefficient matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in $\mathbb{R}^{m\times m}$, and subject to the initial and boundary conditions

$$\begin{aligned} \mathbf{u}(0,x,y,z) &= \mathbf{u}_0(x,y,z), & \text{(6.1.1b)}\\ \mathbf{M}^{-}\mathbf{u}(t,x,y,z) &= \mathbf{M}^{-}\mathbf{u}_B(t,x,y,z),\ (x,y,z) \in \partial\Omega, 0 < t < T, & \text{(6.1.1c)} \end{aligned}$$

where the boundary of Ω is denoted by $\partial\Omega$ and $\mathbf{n} = (n_1,n_2,n_3)^T$ denotes the unit outward normal on $\partial\Omega$, $\mathbf{u}_t$, $\mathbf{u}_x$, $\mathbf{u}_y$ and $\mathbf{u}_z$ respectively, denote the partial derivatives of $\mathbf{u}$ with respect to t, x, y and z.

Let Γ_i and $\mathbf{n}_i = (n_{i,1}, n_{i,2}, n_{i,3})^T$, respectively, denote the faces of Δ and the unit outward normal to Γ_i for $i = 1,2,3,4$.

We define $\mathbf{M}_i^{\pm}$ for $i=1,2,3,4$ by setting

$$\begin{aligned}
\mathbf{M}_i &= n_{i,1}\mathbf{A}+n_{i,2}\mathbf{B}+n_{i,3}\mathbf{C} \\
&= \mathbf{P}diag\left(\lambda_1^i,\lambda_2^i,\ldots,\lambda_m^i\right)\mathbf{P}^T \\
&= \mathbf{M}_i^+ + \mathbf{M}_i^-,
\end{aligned} \tag{6.1.2a}$$

with

$$\begin{aligned}
\mathbf{M}_i^+ &= \mathbf{P}diag\left(\max\left(\lambda_1^i,0\right),\max\left(\lambda_2^i,0\right),\ldots,\max\left(\lambda_m^i,0\right)\right)\mathbf{P}^T, \\
\mathbf{M}_i^- &= \mathbf{P}diag\left(\min\left(\lambda_1^i,0\right),\min\left(\lambda_2^i,0\right),\ldots,\min\left(\lambda_m^i,0\right)\right)\mathbf{P}^T,
\end{aligned} \tag{6.1.2b}$$

where $\lambda_1^i,\lambda_2^i,\ldots,\lambda_m^i\in\mathbb{R}$.

We select $\mathbf{f}$, the initial condition $\mathbf{u}_0$ and the boundary condition $\mathbf{u}_1$ such that the exact solution $\mathbf{u}(t,x,y,z)\in\left(C^2[0,T]\times C^\infty(\Omega)\right)^m$.
In order to obtain the weak discontinuous Galerkin formulation, we partition the domain Ω into a regular mesh having N tetrahedral elements Δ_j, $j=1,2,\ldots,N$ and assume, for simplicity, that this can be done without error. In the remainder of this chapter we omit the element index and refer to an arbitrary element by Δ whenever confusion in unlikely.
Let us multiply (6.1.1a) by a test function $\mathbf{v}$, integrate over an arbitrary element Δ, and apply Stokes' theorem to write

$$\begin{aligned}
&-\iiint_\Delta\left(-\mathbf{v}^T\mathbf{u}_t+\mathbf{v}_x^T\mathbf{A}\mathbf{u}+\mathbf{v}_y^T\mathbf{B}\mathbf{u}+\mathbf{v}_z^T\mathbf{C}\mathbf{u}\right)dxdydz \\
&+\sum_{i=1}^4\iint_{\Gamma_i}\mathbf{v}^T\left(\mathbf{M}_i^++\mathbf{M}_i^-\right)\mathbf{u}d\sigma=\iiint_\Delta\mathbf{v}^T\mathbf{f}dxdydz,
\end{aligned} \tag{6.1.3}$$

where we used (6.1.2).

Next we approximate $\mathbf{u}(t,x,y,z)$ by a piecewise polynomial function $\mathbf{U}(t,x,y,z)$ whose restriction to Δ is in $[\mathcal{P}_p]^m$ consisting of complete polynomials of degree p defined in (2.3.1). The discrete DG formulation consists of determining $\mathbf{U}\in\left[S^{N,p}\right]^m$ such that

$$\begin{aligned}
&-\iiint_\Delta\left(-\mathbf{V}^T\mathbf{U}_t+\mathbf{V}_x^T\mathbf{A}\mathbf{U}+\mathbf{V}_y^T\mathbf{B}\mathbf{U}+\mathbf{V}_z^T\mathbf{C}\mathbf{U}\right)dxdydz \\
&+\sum_{i=1}^4\iint_{\Gamma_i}\mathbf{V}^T\left(\mathbf{M}_i^+\mathbf{U}+\mathbf{M}_i^-\mathbf{U}^-\right)d\sigma\;=\;\iiint_\Delta\mathbf{V}^T\mathbf{f}dxdydz,\;\forall\;\mathbf{V}\in[\mathcal{P}_p]^m,
\end{aligned} \tag{6.1.4}$$

where $\mathbf{U}^-$ is defined as

$$\mathbf{U}^-=\lim_{\varepsilon\longrightarrow 0}U\left(t,\mathbf{x}+\varepsilon\mathbf{n}\right)\;\text{ for }\mathbf{x}\in\Gamma_i.$$

For simplicity, we only consider the approximation (6.1.4) on an element Δ, such that $\mathbf{U}^-=\mathbf{u}_B$ on Γ_i, $i=1,2,3,4$.

Next, subtracting (6.1.4) from (6.1.3) with $\mathbf{v} = \mathbf{V}$ to obtain the DG orthogonality condition for the local error $\epsilon = \mathbf{u} - \mathbf{U}$

$$-\iiint_{\Delta} \left(-\mathbf{V}^T \epsilon_t + \mathbf{V}_x^T \mathbf{A}\epsilon + \mathbf{V}_y^T \mathbf{B}\epsilon + \mathbf{V}_z^T \mathbf{C}\epsilon\right) dxdydz$$
$$+\sum_{i=1}^{4} \iint_{\Gamma_i} \mathbf{V}^T \mathbf{M}_i^+ \epsilon d\sigma = 0, \forall\, \mathbf{V} \in [\mathcal{P}_p]^m . \tag{6.1.5}$$

We map a physical tetrahedron Δ having vertices $\mathbf{v}_i = (x_i, y_i, z_i)$, $i = 1, 2, 3, 4$ into the reference tetrahedron having vertices $\hat{\mathbf{v}}_1 = (0, 0, 0)$, $\hat{\mathbf{v}}_2 = (1, 0, 0)$, $\hat{\mathbf{v}}_3 = (0, 1, 0)$, and $\hat{\mathbf{v}}_4 = (0, 0, 1)$ by the standard affine mapping (3.1.7).
Letting $\tau = T^{-1}t$ the DG orthogonality condition for the local error $\hat{\epsilon}\,(\tau, \xi, \eta, \zeta, h)$ becomes

$$\sum_{i=1}^{4} \iint_{\hat{\Gamma}_i} \mathbf{V}^T \mathbf{M}_i^+ \hat{\epsilon} d\hat{\sigma} - \iiint_{\hat{\Delta}} \left(-\frac{h}{T}\mathbf{V}^T \hat{\epsilon}_t + \mathbf{V}_\xi^T \mathbf{A}\hat{\epsilon} + \mathbf{V}_\eta^T \mathbf{B}\hat{\epsilon} + \mathbf{V}_\zeta^T \mathbf{C}\hat{\epsilon}\right) d\xi d\eta d\zeta = 0, \forall\, \mathbf{V} \in [\mathcal{P}_p]^m . \tag{6.1.6}$$

If the exact solution is analytic, we can write the local error as a Maclaurin series

$$\hat{\epsilon}\,(\tau, \xi, \eta, \zeta, h) = \sum_{k=0}^{\infty} \mathbf{Q}_k\,(\tau, \xi, \eta, \zeta)\, h^k, \tag{6.1.7}$$

where $\mathbf{Q}_k(\tau, .) \in [\mathcal{P}_k]^m$.

In the remainder of this chapter we will omit the ˆ unless needed for clarity.
Before stating our main results we need the following preliminary lemma.

Lemma 6.1.1. *If $\mathbf{Q}_k \in [\mathcal{P}_k]^m$ for $k \geq 0$, satisfies*

$$\sum_{i=1}^{4} \iint_{\Gamma_i} \mathbf{V}^T \mathbf{M}_i^+ \mathbf{Q}_k d\sigma - \iiint_{\Delta} \left(\mathbf{V}_\xi^T \mathbf{A} + \mathbf{V}_\eta^T \mathbf{B} + \mathbf{V}_\zeta^T \mathbf{C}\right) \mathbf{Q}_k d\xi d\eta d\zeta = 0, \forall\, \mathbf{V} \in [\mathcal{P}_k]^m , \tag{6.1.8}$$

then

$$\mathbf{Q}_k \in \bar{\mathcal{P}}_k, \tag{6.1.9a}$$

where

$$\bar{\mathcal{P}}_k = \{\mathbf{V} \in \mathcal{P}_k : \mathbf{A}\mathbf{V}_\xi + \mathbf{B}\mathbf{V}_\eta + \mathbf{C}\mathbf{V}_\zeta = 0 \text{ on } \Delta, \mathbf{M}_i\mathbf{V} = 0 \text{ on } \Gamma_i, 1 \leq i \leq 4\}. \tag{6.1.9b}$$

Proof. Using Stokes' theorem we write (6.1.8) as

$$-\sum_{i=1}^{4} \iint_{\Gamma_i} \mathbf{V}^T \mathbf{M}_i^- \mathbf{Q}_k d\sigma + \iiint_{\Delta} \mathbf{V}^T \left(\mathbf{A}\mathbf{Q}_{k,\xi} + \mathbf{B}\mathbf{Q}_{k,\eta} + \mathbf{C}\mathbf{Q}_{k,\zeta}\right) d\xi d\eta d\zeta = 0, \forall\, \mathbf{V} \in [\mathcal{P}_p]^m . \tag{6.1.10}$$

Adding (6.1.8) to (6.1.10) and testing against $\mathbf{V} = \mathbf{Q}_k$, we note that by the symmetry of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, the double integrals on Δ cancel out. Thus, $\mathbf{Q}_k$ satisfies

$$\sum_{i=1}^{4} \left(\iint_{\Gamma_i} \mathbf{Q}_k^T \left(\mathbf{M}_i^+ - \mathbf{M}_i^- \right) \mathbf{Q}_k d\sigma \right) d\sigma = 0. \tag{6.1.11}$$

Since $\left(\mathbf{M}_i^+ - \mathbf{M}_i^-\right)$ is symmetric positive semi-definite it admits a Cholesky factorization $\left(\mathbf{M}_i^+ - \mathbf{M}_i^-\right) = \mathbf{L}_i^T \mathbf{L}_i$. Thus, (6.1.11) can be written as

$$\sum_{i=1}^{4} \left(\iint_{\Gamma_i} \|\mathbf{L}_i \mathbf{Q}_k\|_2^2 \, d\sigma \right) d\sigma = 0.$$

Thus, $\mathbf{L}_i \mathbf{Q}_k = 0$ on Γ_i for $i = 1, 2, 3, 4$ which yields

$$\mathbf{L}_i^T \mathbf{L}_i \mathbf{Q}_k = \left(\mathbf{M}_i^+ - \mathbf{M}_i^- \right) \mathbf{Q}_k = 0 \text{ on } \Gamma_i, \;\; i = 1, 2, 3, 4. \tag{6.1.12}$$

Since $\mathcal{N}\left(\mathbf{M}_i^+ - \mathbf{M}_i^-\right) = \mathcal{N}\left(\mathbf{M}_i^+ + \mathbf{M}_i^-\right)$, (6.1.12) leads to

$$\left(\mathbf{M}_i^+ + \mathbf{M}_i^- \right) \mathbf{Q}_k = 0 \text{ on } \Gamma_i, \;\; i = 1, 2, 3, 4. \tag{6.1.13}$$

Thus, one can show that

$$\mathbf{M}_i^{\pm} \mathbf{Q}_k = 0 \text{ on } \Gamma_i, i = 1, 2, 3, 4. \tag{6.1.14}$$

Testing against $\mathbf{V} = \mathbf{A}\mathbf{Q}_{k,\xi} + \mathbf{B}\mathbf{Q}_{k,\eta} + \mathbf{C}\mathbf{Q}_{k,\zeta}$ in (6.1.10) and combining the resulting equation with (6.1.14) lead to

$$\iiint\limits_{\Delta} \left\| \mathbf{A}\mathbf{Q}_{k,\xi} + \mathbf{B}\mathbf{Q}_{k,\eta} + \mathbf{C}\mathbf{Q}_{k,\zeta} \right\|_2^2 d\xi d\eta d\zeta = 0, \tag{6.1.15}$$

which, in turn, yields

$$\mathbf{A}\mathbf{Q}_{k,\xi} + \mathbf{B}\mathbf{Q}_{k,\eta} + \mathbf{C}\mathbf{Q}_{k,\zeta} = 0, \text{ on } \Delta. \tag{6.1.16}$$

Combining (6.1.13) and (6.1.16) proves the lemma. $\square$

We obtain the following expression for the local DG error.

Theorem 6.1.1. *Let* $\mathbf{u} \in [C^\infty(\Omega)]^m$ *and* $\mathbf{U} \in [\mathcal{P}_p]^m$*, respectively, be the solutions of (6.1.1) and (6.1.4), with*

$$\begin{aligned} \epsilon(0, \xi, \eta, \zeta, h) &= O\left(h^{p+2}\right), \\ \epsilon^-(\tau, \xi, \eta, \zeta, h) &= O\left(h^{p+2}\right), \;\; 0 \le \tau \le 1. \end{aligned}$$

Then the local finite element error can be written as

$$\epsilon(\tau, \xi, \eta, \zeta, h) = \sum_{k=p+1}^{\infty} h^k \mathbf{Q}_k(\tau, \xi, \eta, \zeta), \tag{6.1.17}$$

where

$$\sum_{i=1}^{4} \iint_{\Gamma_i} \mathbf{M}_i^+ \mathbf{Q}_{p+1} d\sigma = 0. \tag{6.1.18}$$

Furthermore, the local error satisfies

$$\sum_{i=1}^{4} \iint_{\Gamma_i} \mathbf{M}_i^+ \epsilon d\sigma = O\left(h^{p+2}\right). \tag{6.1.19}$$

Proof. The orthogonality condition (6.1.6) for the local error can be written for all $\mathbf{V} \in [\mathcal{P}_p]^m$ and $0 < \tau < 1$ as

$$\begin{aligned} \frac{h}{T} \iiint_{\Delta} \mathbf{V}^T \epsilon_\tau d\xi d\eta d\zeta = & \iiint_{\Delta} \left(\mathbf{V}_\xi^T \mathbf{A} + \mathbf{V}_\eta^T \mathbf{B} + \mathbf{V}_\zeta^T \mathbf{C}\right) \epsilon d\xi d\eta d\zeta \\ & - \sum_{i=1}^{4} \iint_{\Gamma_i} \left(\mathbf{V}^T \mathbf{M}_i^+ \epsilon + \mathbf{V}^T \mathbf{M}_i^- \epsilon^-\right) d\sigma. \end{aligned} \tag{6.1.20}$$

Following Adjerid and Weinhart [54], we define the orthogonal complement of $\bar{\mathcal{P}}_p$ in $[L^2(\Delta)]^m$ by

$$\bar{\mathcal{P}}_p{}^{\perp} = \left\{ \mathbf{W} \in \left[L^2(\Delta)\right]^m : \iiint_{\Delta} \mathbf{V}^T \mathbf{W} d\xi d\eta d\zeta = \mathbf{0},\ \forall\ \mathbf{V} \in \bar{\mathcal{P}}_p \right\}. \tag{6.1.21}$$

Since $\mathcal{P}_p$ is a subspace of $[L^2(\Delta)]^m$, we can split ϵ by

$$\epsilon = \bar{\epsilon} + \bar{\epsilon}^{\perp},\ \bar{\epsilon} \in \bar{\mathcal{P}}_p,\ \bar{\epsilon}^{\perp} \in \bar{\mathcal{P}}_p^{\perp}. \tag{6.1.22}$$

First, we will show that

$$\bar{\epsilon}(\tau, \xi, \eta, \zeta, h) = O\left(h^{p+2}\right),\ (\xi, \eta, \zeta) \in \Delta,\ 0 \le \tau \le 1. \tag{6.1.23}$$

Since $\bar{\mathcal{P}}_p$ is a finite dimensional vector space and $\bar{\epsilon} \in \bar{\mathcal{P}}_p$, we have for $(\xi, \eta, \zeta) \in \Delta,\ 0 \le \tau \le 1$,

$$\bar{\epsilon}_\tau(\tau, \xi, \eta, h) = \lim_{\varepsilon \longrightarrow 0} \frac{\bar{\epsilon}(\tau + \varepsilon, \xi, \eta, \zeta, h) - \bar{\epsilon}(\tau, \xi, \eta, \zeta, h)}{\varepsilon} \in \bar{\mathcal{P}}_p, \tag{6.1.24}$$

$$\bar{\epsilon}_\tau{}^{\perp}(\tau, \xi, \eta, h) = \lim_{\varepsilon \longrightarrow 0} \frac{\bar{\epsilon}^{\perp}(\tau + \varepsilon, \xi, \eta, \zeta, h) - \bar{\epsilon}^{\perp}(\tau, \xi, \eta, \zeta, h)}{\varepsilon} \in \bar{\mathcal{P}}_p{}^{\perp}. \tag{6.1.25}$$

By the definition of $\bar{\mathcal{P}}_p$ and the symmetry of $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{M}_i$ and $\mathbf{M}_i^{\pm},\ 1 \le i \le 4$, (6.1.20) yields

$$\frac{h}{T} \iiint_{\Delta} \mathbf{V}^T \bar{\epsilon}_\tau d\xi d\eta d\zeta = 0,\ \forall\ \mathbf{V} \in \bar{\mathcal{P}}_p,\ 0 \le \tau \le 1. \tag{6.1.26}$$

Thus $\bar{\epsilon}_\tau \in \bar{\mathcal{P}}_p{}^{\perp}$, which combined with (6.1.22) and (6.1.25) yields

$$\bar{\epsilon}_\tau = \epsilon_\tau - \bar{\epsilon}^{\perp} \in \bar{\mathcal{P}}_p{}^{\perp},\ 0 \le \tau \le 1. \tag{6.1.27}$$

Combining (6.1.24) and (6.1.27) we infer that

$$\bar{\epsilon}_\tau\left(\tau,\xi,\eta,\zeta,h\right)=0,\ 0\le\tau\le 1. \tag{6.1.28}$$

Consequently, we have

$$\bar{\epsilon}\left(\tau,\xi,\eta,\zeta,h\right)=\bar{\epsilon}\left(0,\xi,\eta,\zeta,h\right)=O\left(h^{p+2}\right),\ 0\le\tau\le 1. \tag{6.1.29}$$

In the remainder of the proof, we investigate the asymptotic behavior of $\bar{\epsilon}^{\perp}$. We write the Maclaurin series of ϵ with respect to the mesh parameter h as

$$\epsilon\left(\tau,\xi,\eta,\zeta,h\right)=\sum_{k=0}^{p+1}\mathbf{Q}_k\left(\tau,\xi,\eta,\zeta\right)h^k+O\left(h^{p+2}\right),\ (\xi,\eta,\zeta)\in\Delta,\ 0\le\tau\le 1, \tag{6.1.30}$$

where

$$\mathbf{Q}_k\left(\tau,\xi,\eta,\zeta\right)=\left.\frac{d^k\epsilon\left(\tau,\xi,\eta,\zeta\right)}{dh^k}\right|_{h=0}. \tag{6.1.31}$$

We write the Maclaurin series of $\epsilon^{\perp}$ with respect to the mesh parameter h as

$$\epsilon^{\perp}\left(\tau,\xi,\eta,\zeta,h\right)=\sum_{k=0}^{\infty}\tilde{\mathbf{Q}}_k\left(\tau,\xi,\eta,\zeta\right)h^k,\ \tilde{\mathbf{Q}}_k\in\bar{\mathcal{P}}_k^{\perp},\ (\xi,\eta,\zeta)\in\Delta,\ 0\le\tau\le 1. \tag{6.1.32}$$

Subtracting (6.1.31) from (6.1.32) and equating all terms having the same power h yields

$$\mathbf{Q}_k=\tilde{\mathbf{Q}}_k\in\bar{\mathcal{P}}_k^{\perp},\ 0\le k\le p+1, \tag{6.1.33}$$

where we used (6.1.22) and (6.1.29).

Substituting (6.1.30) in (6.1.20) yields

$$\begin{aligned}\sum_{k=0}^{p+1}h^k\Bigg(\frac{h}{T}\iiint_\Delta\mathbf{V}^T\mathbf{Q}_{k,\tau}d\xi d\eta d\zeta&-\iiint_\Delta\left(\mathbf{V}_\xi^T\mathbf{A}\mathbf{Q}_k+\mathbf{V}_\eta^T\mathbf{B}\mathbf{Q}_k+\mathbf{V}_\zeta^T\mathbf{C}\mathbf{Q}_k\right)d\xi d\eta d\zeta\\+\sum_{i=1}^{4}\iint_{\Gamma_i}\mathbf{V}^T\mathbf{M}_i^+\mathbf{Q}_k d\sigma\Bigg)&=O\left(h^{p+2}\right),\ \forall\ \mathbf{V}\in\left[\mathcal{P}_p\right]^m,\end{aligned} \tag{6.1.34}$$

where we used the fact that the boundary conditions satisfy

$$\epsilon^{-}\left(\tau,\xi,\eta,\zeta\right)=O\left(h^{p+2}\right).$$

Now assume $T=O(1)$ and set to zero all terms in (6.1.34) having the same power of h. The $O(1)$ term $\mathbf{Q}_0$ satisfies the orthogonality condition

$$\sum_{i=1}^{4}\iint_{\Gamma_i}\mathbf{V}^T\mathbf{M}_i^+\mathbf{Q}_0d\sigma-\iiint_\Delta\left(\mathbf{V}_\xi^T\mathbf{A}\mathbf{Q}_0+\mathbf{V}_\eta^T\mathbf{B}\mathbf{Q}_0\right)d\xi d\eta d\zeta=0,\ \forall\ \mathbf{V}\in\left[\mathcal{P}_p\right]^m. \tag{6.1.35}$$

By Lemma 6.1.1 $\mathbf{Q}_0 \in \bar{\mathcal{P}}_0$ which combined with (6.1.33) shows that $\mathbf{Q}_0 = 0$ on Δ.
Assume that $\mathbf{Q}_j = 0$ for $0 \le j \le k-1$, where $k \le p$. Thus, the $O\left(h^k\right)$ term is written as

$$\sum_{i=1}^{4} \iint_{\Gamma_i} \mathbf{V}^T \mathbf{M}_i^+ \mathbf{Q}_k d\sigma - \iiint_{\Delta} \left(\mathbf{V}_\xi^T \mathbf{A} \mathbf{Q}_k + \mathbf{V}_\eta^T \mathbf{B} \mathbf{Q}_k + \mathbf{V}_\zeta^T \mathbf{C} \mathbf{Q}_k\right) d\xi d\eta d\zeta = 0, \ \forall \ \mathbf{V} \in \left[\mathcal{P}_p\right]^m. \tag{6.1.36}$$

By Lemma 6.1.1 $\mathbf{Q}_k \in \bar{\mathcal{P}}_k$, which combined with (6.1.33) shows that $\mathbf{Q}_k = 0$ on Δ for $0 \le k \le p$.
The leading term $\mathbf{Q}_{p+1}$ satisfies

$$\begin{aligned} -\iiint_{\Delta} \left(\mathbf{V}_\xi^T \mathbf{A} \mathbf{Q}_{p+1} + \mathbf{V}_\eta^T \mathbf{B} \mathbf{Q}_{p+1} + \mathbf{V}_\zeta^T \mathbf{C} \mathbf{Q}_{p+1}\right) d\xi d\eta d\zeta \\ \sum_{i=1}^{4} \iint_{\Gamma_i} \mathbf{V}^T \mathbf{M}_i^+ \mathbf{Q}_{p+1} d\sigma = 0, \forall \ \mathbf{V} \in \left[\mathcal{P}_p\right]^m. \end{aligned} \tag{6.1.37}$$

Let $\mathbf{e}_j$, $j = 1, 2, \ldots, m$ be the canonical vectors in $\mathbb{R}^m$ and test against $\mathbf{V} = \mathbf{e}_j$, $j = 1, 2, \ldots, m$ yields (6.1.18). While (6.1.19) follows from (6.1.17) and (6.1.18). □

6.2 *A Posteriori* Error Estimation

In this section we present an error estimation procedure by first constructing bases functions for the leading term of the DG error and stating a weak problem on each element to compute error estimates. In order to construct efficient and asymptotically exact *a posteriori* error estimates for the leading term $\mathbf{Q}_{p+1}$ on each element, we assume that the behavior of the local error holds on all elements, *i.e.*, we write

$$(\mathbf{u} - \mathbf{U})(t, x, y, z, h) \approx \mathbf{E}(\tau, \xi, \eta, \zeta, h) = \mathbf{Q}_{p+1}(\tau, \xi, \eta, \zeta) h^{p+1} = \sum_{k=1}^{n} \mathbf{c}_k\left(\tau\right) \Phi_k(\xi, \eta, \zeta),$$

where $\mathbf{Q}_{p+1}(\tau, .) \in \left[\mathcal{P}_{p+1}\right]^m$, $\mathbf{c}_k\left(\tau\right) \in \mathbb{R}^m$, $n = \dim\left(\mathcal{P}_{p+1}\right)$ and

$$\mathbf{\Phi} = \left(\Phi_1, \Phi_2, ..., \Phi_n\right)^T = \left(\phi_0, \phi_1, \ldots, \phi_{p+1}\right)^T,$$

with

$$\phi_l = \left(\varphi_{0,0}^0, \varphi_{0,0}^1, ..., \varphi_{0,l}^0\right).$$

The leading term $\mathbf{E}$ satisfies the following orthogonality conditions on the reference element

$$\sum_{i=1}^{4} \iint_{\Gamma_i} \mathbf{V}^T \mathbf{M}_i^+ \mathbf{E} d\sigma - \iiint_{\Delta} \left(\mathbf{V}_\xi^T \mathbf{A} + \mathbf{V}_\eta^T \mathbf{B} + \mathbf{V}_\zeta^T \mathbf{C}\right) \mathbf{E} d\xi d\eta d\zeta = 0, \ \forall \ \mathbf{V} \in \left[\mathcal{P}_{p+1}\right]^m. \tag{6.2.1}$$

Testing against $\mathbf{V}_j{=}\mathbf{e}_j\Phi_l$, $j=1,2,\dots,m$, $l=1,2,\dots,\tilde{n}=\dim(\mathcal{P}_p)$ yields

$$\sum_{i=1}^{4}\iint_{\Gamma_i}\Phi_l\mathbf{e}_j^T\mathbf{M}_i^+\mathbf{E}d\sigma-\iiint_{\Delta}\left(\Phi_{l,\xi}\mathbf{e}_j^T\mathbf{A}+\Phi_{l,\eta}\mathbf{e}_j^T\mathbf{B}+\Phi_{l,\zeta}\mathbf{e}_j^T\mathbf{C}\right)\mathbf{E}d\xi d\eta d\zeta=0. \tag{6.2.2}$$

Next, we define the tensor product

$$K\otimes\mathcal{A}=\begin{bmatrix} K_{11}\mathcal{A} & \cdots & K_{1n}\mathcal{A}\\ \vdots & \ddots & \vdots\\ K_{\tilde{n}1}\mathcal{A} & \cdots & K_{\tilde{n}n}\mathcal{A}\end{bmatrix},$$

for $K\in R^{\tilde{n}\times n}$ and $\mathcal{A}\in R^{m\times m}$.

Thus, if $\tilde{\mathbf{c}}=\left(\mathbf{c}_1^T,\mathbf{c}_2^T,\dots,\mathbf{c}_n^T\right)^T\in\mathbb{R}^{nm}$, the orthogonality condition (6.2.2) may be written in a matrix form $\tilde{\mathbf{A}}\tilde{\mathbf{c}}=0$ where $\tilde{\mathbf{A}}\in\mathbb{R}^{(\tilde{n}\ m)\times(n\ m)}$ and

$$\tilde{\mathbf{A}}=\left[\left(\sum_{i=1}^{4}D^i\otimes\mathbf{M}_i^+\right)-\left(K^1\otimes\mathbf{A}{+}K^2\otimes\mathbf{B}{+}K^3\otimes\mathbf{C}\right)\right],$$

with

$$D_{lk}^i=\iint_{\Gamma_i}\Phi_l\Phi_k d\sigma,$$
$$K_{lk}^1=\iiint_{\Delta}\Phi_{l,\xi}\Phi_k d\xi d\eta d\zeta,$$
$$K_{lk}^2=\iiint_{\Delta}\Phi_{l,\zeta}\Phi_k d\xi d\eta d\zeta,$$
$$K_{lk}^3=\iiint_{\Delta}\Phi_{l,\zeta}\Phi_k d\xi d\eta d\zeta.$$

We now define the finite element space

$$\mathcal{E}=\left\{\sum_{k=1}^{n}\tilde{\mathbf{c}}_k\Phi_k,\ \tilde{\mathbf{c}}\in\mathcal{N}\left(\tilde{\mathbf{A}}\right)\right\},$$

for the error and state the following lemma.

Lemma 6.2.1. *The polynomial space $\mathcal{E}$ is isomorphic to the null space $\mathcal{N}\left(\tilde{\mathbf{A}}\right)$, and the leading term $\mathbf{Q}_{p+1}$ may be written as*

$$\mathbf{E}=\mathbf{Q}_{p+1}h^{p+1}=\sum_{i=1}^{n-\tilde{n}}d_i\chi_i,$$

where $\chi_i=\sum_{k=1}^{n}\tilde{\mathbf{c}}_k^i\Phi_k$, $i=1,2,\dots,n-\tilde{n}$, with $\{\tilde{\mathbf{c}}^1,\tilde{\mathbf{c}}^2,\dots,\tilde{\mathbf{c}}^{n-\tilde{n}}\}$ being a basis of $\mathcal{N}(\tilde{\mathbf{A}})$.

Proof. The proof follows the same line of reasoning used in Lemma 3.3.1. □

6.3 Solution and Error Estimation Procedure

Integrating (6.1.4) by parts shows that the DG solution U on a physical element Δ satisfies

$$
\begin{aligned}
&+\iiint_{\Delta} \mathbf{V}^{T}\left(\mathbf{U}_{t}+\mathbf{A}\mathbf{U}_{x}+\mathbf{B}\mathbf{U}_{y}+\mathbf{C}\mathbf{U}_{z}\right) dxdydz \\
&+\sum_{i=1}^{4}\iint_{\Gamma_i} \mathbf{V}^{T}\mathbf{M}_{i}^{-}\left(\tilde{\mathbf{U}}-\mathbf{U}\right) d\sigma \;=\; \iiint_{\Delta} \mathbf{V}^{T}\mathbf{f} dxdydz,\ \forall\ \mathbf{V}\in\left[\mathcal{P}_p\right]^{m}, \qquad (6.3.1)
\end{aligned}
$$

where the numerical flux

$$\tilde{\mathbf{U}} = \begin{cases} \mathbf{u}, & \text{if } \Gamma_i \subset \partial\Omega \\ \mathbf{U}^{-}+\mathbf{E}^{-} & \textit{otherwise} \end{cases},$$

for the modified DG method, subjected to initial boundary condition $\mathbf{U}(0,x,y,z)=\mathbf{u}_B(x,y,z)$.

Next, we write the DG solution as

$$\mathbf{U}=\sum_{i=1}^{\tilde{n}} \mathbf{c}_i\left(t\right)\Phi_i\left(\xi,\eta,\zeta\right),$$

thus, the weak formulation (6.3.1) yields the ode system

$$\dot{\mathbf{c}}=\mathbf{F}\left(\mathbf{c},\mathbf{E},t\right).$$

Next, we select a time integration strategy. For example, using explicit Euler scheme we write

$$\frac{\mathbf{c}^{n+1}-\mathbf{c}^{n}}{t_{n+1}-t_n}=\mathbf{F}\left(\mathbf{c}^{n},\mathbf{E}^{n},t_n\right).$$

Alternatively, we may use TVB Runge-Kutta integration scheme [29].
The initial values $\mathbf{c}^0$ is obtained by solving the following problem using the DG method

$$\mathbf{A}\left(\mathbf{u_0}\right)_x+\mathbf{B}\left(\mathbf{u_0}\right)_y+\mathbf{C}\left(\mathbf{u_0}\right)_z=\mathbf{g}\left(x,y,z\right),\quad (x,y,z)\in\Delta \qquad (6.3.2a)$$

$$\mathbf{M}^{-}\mathbf{u}\left(0,x,y,z\right)=\mathbf{M}^{-}\mathbf{u}_B\left(0,x,y,z\right),\ (x,y,z)\in\partial\Omega, \qquad (6.3.2b)$$

and then computing the error estimation.

In order to estimate the finite element error $\mathbf{e}=\mathbf{u}-\mathbf{U}$ on Δ we assume that the leading term of $\mathbf{e}$ exhibits the same asymptotic behavior as the local error on Δ. Thus, the error $\mathbf{e}$ on Δ is approximated by

$$\mathbf{E}\left(t,x,y,z\right)=\sum_{i=1}^{N} d_i(t)\chi_i(\xi(x,y,z),\eta(x,y,z),\zeta(x,y,z)), \qquad (6.3.3a)$$

and satisfies the weak finite element problem

$$\sum_{i=1}^{4} \iint_{\Gamma_i} \mathbf{V}^T \mathbf{M}_i^- \left(\mathbf{E}^- - \mathbf{E}\right) d\sigma + \iiint_{\Delta} \mathbf{V}^T \left(\mathbf{A}\mathbf{E}_x + \mathbf{B}\mathbf{E}_y + \mathbf{C}\mathbf{E}_z\right) dxdydz$$
$$= \iiint_{\Delta} \mathbf{V}^T \mathbf{r} dxdydz - \sum_{i=1}^{4} \iint_{\Gamma_i} \mathbf{V}^T \mathbf{M}_i^- \left(\mathbf{U}^- - \mathbf{U}\right) d\sigma, \ \forall \ \mathbf{V} \in \mathcal{E}, \qquad (6.3.3b)$$

where $\mathbf{r} = (\mathbf{f} - (\mathbf{U}_t + \mathbf{A}\mathbf{U}_x + \mathbf{B}\mathbf{U}_y + \mathbf{C}\mathbf{U}_z))$ is the interior residual.

6.4 Conclusion

In this chapter we have extended the error analysis to first-order linear symmetric hyperbolic systems of partial differential equations with smooth solutions. We performed a local error analysis where we showed that the discretization error is $O\left(h^{p+1}\right)$ and its leading term belongs to a smaller polynomial space.

Chapter 7

Conclusions and Future Work

7.1 Contributions

Discontinuous Galerkin (DG) methods for solving partial differential equations, became popular among computational scientists because they provide an attractive approach to address problems having discontinuities, such as those arising in hyperbolic conservation laws. In the traditional finite element, the approximated solution is forced to be continuous across element boundaries. However, the discontinuous Glerkin method allow to the DG solution to be discontinuous across element boundaries, which leads to a simple communication pattern between elements sharing a common face that makes them useful for parallel computation, and simplifies h- and p-refinement techniques. However, for DG methods to be used in an adaptive framework one needs a posteriori error estimates to guide adaptivity and stop the refinement process.

In this dissertation we presented a simple and efficient *a posteriori* error estimation procedure for a discontinuous finite element method applied to first-order hyperbolic problems on structured and unstructured tetrahedral meshes. First we grouped elements of the mesh into *Class I, II* and *III*, respectively, if it has one, two and three *outflow* faces. Moreover, elements are said to be of *Type* 1, 2 and 3, respectively, the sets of elements having one, two and three *inflow* faces. We presented a local error analysis on an arbitrary tetrahedron by constructing a family of similar tetrahedra with size h and having the same center. This family of tetrahedra is such that as $h \to 0$ the limit is the common center. Assuming we compute a *p-degree* DG approximation of a smooth solution, we expanded the local error as a power series with respect to h and prove that the leading term of the DG error is a $O(h^{p+1})$ polynomial of degree *p+1*. We further observed that the leading term of the error satisfies a DG orthogonality condition which simplifies the form of leading term of the error. For instance, on a tetrahedron of *Class I*, the leading term may be written in terms of orthogonal polynomials of degrees p and *p+1* only. We further simplified the leading term and expressed it in terms of an optimal set of polynomials which are used to estimate the error. Similarly, optimal error basis functions are derived on elements of *Class II* and *III*. Moreover, on the *outflow* face of an arbitrary element of *Class I* the local error is $\mathcal{O}(h^{2p+2})$ on average. Numerical computations showed that the implicit *a posteriori* error estimation procedure yields accurate estimates for linear and nonlinear problems with smooth solutions. Furthermore, we showed the performance of our error estimates on problems with discontinuous solutions.

We investigated the pointwise superconvergence properties of the DG method, using enriched polynomial spaces. We have performed a numerical and theoretical study of the existence of pointwise superconvergent results for DG methods. We have studied the effect of enriched finite element spaces on the superconvergence properties of DG solutions on each class and type of tetrahedral elements. Several superconvergence phenomena have been discovered.

We showed that using the space $\mathcal{L}_p$ the discretization error on tetrahedral elements of *Class I* and *Type 1* is $O(h^{p+2})$ superconvergent on the *outflow* face. For the enriched space $\mathcal{U}_p$, the discretization error on tetrahedral elements of *Type 1* is $O(h^{p+2})$ superconvergent on the three edges of the *inflow* face, while on elements of *Class I* and *Type 1* the DG solution is $O(h^{p+2})$ superconvergent on the *outflow* face in addition to the three edges of the *inflow* face. Furthermore, we showed that, on tetrahedral elements of *Type 2*, the DG solution is $O(h^{p+2})$ superconvergent on the edge shared by two of the *inflow* faces. On elements of *Class I* and *Type 2* and on elements of *Type 3*, the DG solution is $O(h^{p+2})$ superconvergent on two edges of the *inflow* faces. Moreover, we showed that using the enriched space $\mathcal{M}_p$ we obtain a simpler *a posteriori* error estimation procedure. The superconvergence results on each elements are tested on several linear problems and tetrahedral meshes for smooth solutions.

Finally, we extended our error analysis to the discontinuous Galerkin method applied to linear three-dimensional hyperbolic systems of conservation laws with smooth solutions. We performed a local error analysis by writing the local error as a series and showing that its leading term is $O\left(h^{p+1}\right)$. We further simplify the leading term and express it in terms of an optimal set of polynomials which can be used to estimate the error.

7.2 Future Work

In fact, in this work we did not prove the asymptotic exactness of our global *a posteriori* error estimates and global pointwise superconvergence for the discretization error. However, the numerical results suggest that global *a posteriori* error estimates are asymptotically exact, and the pointwise superconvergence are globally maintained for smooth solutions. Thus, a main point of research in the near future will be to establish global superconvergence and a global error analysis.

Time-dependent problem play an important role in applied mathematics and many other areas of science. We plan to extend our error analysis to time-dependent problems governed by hyperbolic equations and provide numerical study to show the effectiveness of the error estimate and to validate the theory.

Furthermore, we plan to develop an *hp*-adaptive framework of the modified DG method applied to hyperbolic systems and provide numerical computation to test our *a posteriori* error estimates on general unstructured tetrahedral meshes for both smooth and discontinuous solutions. We extend the error analysis to nonlinear hyperbolic systems and study effect of several numerical fluxes such as LaxFriedrichs, Godunov and Roe fluxes.

We note that our error analysis does not apply near discontinuities and we are not able to construct asymptotically correct error estimates for discontinuous solutions except in special cases. We expect to obtain more accurate estimates by investigating the impact of the strategies of essentially nonoscillatory (ENO) and weighted ENO (WENO) limiters on our error estimator.

Bibliography

[1] S. Adjerid. Asymptotically exact *a posteriori* error estimates for one-dimensional convection-diffusion problems. submitted, 2011.

[2] S. Adjerid and M. Baccouch. The discontinuous Galerkin method for two-dimensional hyperbolic problems. Part I: Superconvergence error analysis. *Journal of Scientific Computing*, 33:75–113, 2007.

[3] S. Adjerid and M. Baccouch. The discontinuous Galerkin method for two-dimensional hyperbolic problems. Part II: *A Posteriori* error estimation. *Journal of Scientifc Computing*, 38:15–49, 2009.

[4] S. Adjerid and M. Baccouch. *A Posteriori* error analysis of the discontinuous Galerkin method for two-dimensional hyperbolic problems on unstructured meshes. *Computer Methods in Applied Mechanics and Engineering*, 200:162–177, 2011.

[5] S. Adjerid, B. Belguendouz, and J.E. Flaherty. *A posteriori* finite element error estimation for diffusive systems. *SIAM Journal on Scientific Computing*, 21:728–746, 1999.

[6] S. Adjerid, K.D. Devine, J.E. Flaherty, and L. Krivodonova. *A posteriori* error estimation for discontinuous Galerkin solutions of hyperbolic problems. *Computer Methods in Applied Mechanics and Engineering*, 191:1097–1112, 2002.

[7] S. Adjerid and A. Klauser. Superconvergence of discontinuous finite element solutions for convection-diffusion problems. *Journal of Scientific Computing*, 21-22:5–24, 2005.

[8] S. Adjerid and T. C. Massey. *A posteriori* discontinuous finite element error estimation for two-dimensional hyperbolic problems. *Computer Methods in Applied Mechanics and Engineering*, 191:5877–5897, 2002.

[9] S. Adjerid and T. C. Massey. Superconvergence of discontinuous finite element solutions for nonlinear hyperbolic problems. *Computer Methods in Applied Mechanics and Engineering*, 195:3331–3346, 2006.

[10] S. Adjerid and T. Weinhart. Discontinuous Galerkin error estimation for linear symmetric hyperbolic systems. *Computer Methods in Applied Mechanics and Engineering*, 198:3113–3129, 2009.

[11] S. Adjerid and T. Weinhart. Asymptotically exact discontinuous Galerkin error estimates for linear symmetric hyperbolic systems. submitted, 2011.

[12] S. Adjerid and T. Weinhart. Discontinuous Galerkin error estimation for linear symmetrizable hyperbolic systems. *Mathematics of Computation*, 80:1335–1367, 2011.

[13] D.N. Arnold, F. Brezzi, B. Cockburn, and D. Marini. Discontinuous Galerkin methods for elliptic problems. In B. Cockburn, G. Karniadakis, and C.-W. Shu, editors, *Discontinuous Galerkin Methods: Theory, Computation and Applications*, volume 11 of *Lecture Notes in Computational Science and Engineering*, pages 89–102, Berlin, 2000. Springer.

[14] M. Baccouch. *Superconvergence and A Posteriori Error Estimation for the Discontinuous Galerkin Method Applied to Hyperbolic Problems on Triangular Meshes*. PhD thesis, Virginia Polytechnic Institute and State University, 2008.

[15] M. Baccouch and S. Adjerid. Discontinuous Galerkin error estimation for hyperbolic problems on unstructured triangular meshes. *Computer Methods in Applied Mechanics and Engineering*, 200:162–177, 2011.

[16] F. Bassi and S. Rebay. A high-order accurate discontinuous finite element method for the numerical solution of the compressible navier-stokes equations. *J. Comp. Phys.*, 131:267–279, 1997.

[17] C. E. Baumann and J. T. Oden. A discontinuous hp finite element method for convection-diffusion problems. *Computer Methods in Applied Mechanics and Engineering*, 175:311–341, 1999.

[18] R. Biswas, K. Devine, and J.E. Flaherty. Parallel adaptive finite element methods for conservation laws. *Applied Numerical Mathematics*, 14:255–284, 1994.

[19] K. Bottcher and R. Rannacher. Adaptive error control in solving ordinary differential equations by the discontinuous Galerkin method. Tech. report, University of Heidelberg, 1996.

[20] F. Brezzi, M. Manzini, D. Marini, P. Pietra, and A. Russo. Discontinuous finite elements for diffusion problems. Atti Convegno in onore Di F. Brioschi (Milano 1997), Instituto Lombardo, Accademia di Scienze e Lettere, 1999.

[21] J. Burkardt. http://people.sc.fsu.edu/ jburkardt/.

[22] G. Chavent and B. Cockburn. The local projection p0-p1-discontinuous galerkin method for scalar conservation laws. *Modl. Math. Anal. Numr*, 23:565592, 1989.

[23] G. Chavent and G. Salzano. A finite-element method for the 1-d water flooding problem with gravity. *J. Comput. Phys*, 45:307344, 1982.

[24] B. Cockburn. Discontinuous Galerkin methods for convection dominated problems. In T. Barth and H. Deconink, editors, *High-order Methods for Computational Physics*, volume 9 of *Lecture Notes in Computational Science and Engineering*, pages 69–224, Berlin, 1999. Springer.

[25] B. Cockburn and J. Guzman. Error estimates for the runge-kutta discontinuous galerkin method for the transport equation with discontinuous initial data. *SIAM J. Numer. Anal.*, 46:1364–1398, March 2008.

[26] B. Cockburn, S. Hou, and C.-W. Shu. The Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: the multidimensional case. *Mathematics of Computation*, 54:545–581, 1990.

[27] B. Cockburn, S.Y. Lin, and C.-W. Shu. TVB Runge-Kutta local projection discontinuous Galerkin methods of scalar conservation laws III: One dimensional systems. *Journal of Computational Physics*, 84:90–113, 1989.

[28] B. Cockburn, M. Luskin, C.-W. Shu, and E. Süli. Enhanced accuracy by post-processing for finite element methods for hyperbolic equations. *Mathematics of Computation*, 72:577–606, 2003.

[29] B. Cockburn and C.-W. Shu. TVB Runge-Kutta local projection discontinuous Galerkin methods for scalar conservation laws II: General framework. *Mathematics of Computation*, 52:411–435, 1989.

[30] B. Cockburn and C.-W. Shu. The Runge-Kutta local projection P^1-discontinuous-Galerkin finite element method for scalar conservation laws. *Mathematical Modelling and Numerical Analysis*, 25:337–361, 1991.

[31] B. Cockburn and C.-W. Shu. The local discontinuous Galerkin finite element method for convection-diffusion systems. *SIAM Journal on Numerical Analysis*, 35:2240–2463, 1998.

[32] B. Cockburn and C.-W. Shu. The Runge-Kutta discontinuous Galerkin finite element method for conservation laws V: Multidimensional systems. *Journal of Computational Physics*, 141:199–224, 1998.

[33] COMSOL Multiphysics. Comsol, Inc.. Partial differential equation package. http://www.comsol.com, 1999-2010.

[34] D. Cox, J. Little, and D. O'Shea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. 2007.

[35] C.N. Dawson. Godunov-mixed methods for advection-diffusion equations in one space dimension. *SIAM Journal on Numerical Analysis*, 30:1282–1309, 1991.

[36] C.N. Dawson. Godunov-mixed methods for advection-diffusion equations in multidimensions. *SIAM Journal on Numerical Analysis*, 30:1315–1332, 1993.

[37] K.D. Devine and J.E. Flaherty. Parallel adaptive hp-refinement techniques for conservation laws. *Computer Methods in Applied Mechanics and Engineering*, 20:367–386, 1996.

[38] K. Ericksson and C. Johnson. Adaptive finite element methods for parabolic problems I: A linear model problem. *SIAM Journal on Numerical Analysis*, 28:12–23, 1991.

[39] K. Ericksson and C. Johnson. Adaptive finite element methods for parabolic problems II: Optimal error estimates in $L_\infty L_2$ and $L_\infty L_\infty$. *SIAM Journal on Numerical Analysis*, 32:706–740, 1995.

[40] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*. Springer, New York, 2004.

[41] D. Estep. *A posteriori* error bounds and global error control for approximation of ordinary differential equations. *SIAM Journal on Numerical Analysis*, 32:1–48, 1995.

[42] J. Flaherty. Finite element analysis lecture notes. *Rensselaer Polytechnic Institute.*

[43] A. Grundmann and M. Moeller. Invariant integration formulas for the n-simplex by combinatorial methods,. *SIAM Journal on Numerical Analysis*, 15:282–290, 1978.

[44] C.-W. Shu J. Ryan and H. Atkins. Extension of a post-processing technique for the discontinuous Galerkin method for hyperbolic equations with application to an aeroacoustic problem. *SIAM J. Sci. Comput.*, 26:821–843, 2005.

[45] C. Johnson. Error estimates and adaptive time-step control for a class of one-step methods for stiff ordinary differential equations. *SIAM Journal on Numerical Analysis*, 25:908–926, 1988.

[46] G.E. Karniadakis and S. J. Sherwin. *Spectral/hp Element Methods for CFD*. Oxford University Press, New York, 1999.

[47] P. Keast. Moderate-degree tetrahedral quadrature formulas. *Computer Methods in Applied Mechanics and Engineering*, 55:339–348, 1986.

[48] L. Krivodonova and J.E. Flaherty. Error estimation for discontinuous Galerkin solutions of multidimensional hyperbolic problems. *Advances in Computational Mathematics*, pages 57–71, 2003.

[49] P. Lesaint and P. Raviart. On a finite element method for solving the neutron transport equations. In C. de Boor, editor, *Mathematical Aspects of Finite Elements in Partial Differential Equations*, pages 89–145, New York, 1974. Academic Press.

[50] Q. Lin and A.H. Zhou. Convergence of the discontinuous Galerkin methods for a scalar hyperbolic equation. *Acta Math. Sci.*, 13:207–210, 1993.

[51] T.E. Peterson. A note on the convergence of the discontinuous Galerkin method for a scalar hyperbolic equation. *SIAM J. Numer. Anal.*, 28:133–140, 1991.

[52] J. Pitkaranta and C. Johnson. An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation. *Mathematics of Computation*, 46:1–26, 1986.

[53] W.H. Reed and T.R. Hill. Triangular mesh methods for the neutron transport equation. Technical Report LA-UR-73-479, Los Alamos Scientific Laboratory, Los Alamos, 1973.

[54] T. Weinhart S. Adjerid. Asymptotically exact discontinuous galerkin error estimates for linear symmetric hyperbolic systems, submitted for publication.

[55] J.K. Ryan S. Curtin, R.M. Kirby and C.-W. Shu. Post-processing for the discontinuous Galerkin method over non-uniform meshes. *SIAM J. Sci. Comput.*, 30:272289, 2007.

[56] B. van Leer. Towards the ultimate conservation difference scheme IV. A new approach of numerical convection. *Journal of Computational Physics*, 23:276–299, 1977.

[57] T. Weinhart. *A Posteriori Error Analysis of the Discontinuous Galerkin Method for Linear Hyperbolic Systems of Conservation Laws.* PhD thesis, Virginia Polytechnic Institute and State University, 2009.