Boundary Controllability and Stabilizability of Nonlinear Schrödinger Equations in a Finite Interval

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ABSTRACT

The dissertation focuses on the nonlinear Schrödinger equation

$$iu_t + u_{xx} + \kappa|u|^2u = 0,$$

for the complex-valued function $u = u(x,t)$ with domain $t \geq 0$, $0 \leq x \leq L$, where the parameter κ is any non-zero real number. It is shown that the problem is locally and globally well-posed for appropriate initial data and the solution exponentially decays to zero as $t \to \infty$ under the boundary conditions

$$u(0,t) = \beta u(L,t), \qquad \beta u_x(0,t) - u_x(L,t) = i\alpha u(0,t),$$

where $L > 0$, and α and β are any real numbers satisfying $\alpha\beta < 0$ and $\beta \neq \pm 1$.

Moreover, the numerical study of controllability problem for the nonlinear Schrödinger equations is given. It is proved that the finite-difference scheme for the linear Schrödinger equation is uniformly boundary controllable and the boundary controls converge as the step sizes approach to zero. It is then shown that the discrete version of the nonlinear case is boundary null-controllable by applying the fixed point method. From the new results, some open questions are presented.

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GENERAL AUDIENCE ABSTRACT

The dissertation concerns the solutions of nonlinear Schrodinger (NLS) equation, which arises in many applications of physics and applied mathematics and models the propagation of light waves in fiber optics cables, surface water-waves, Langmuir waves in a hot plasma, oceanic and optical rogue waves, etc. Under certain dissipative boundary conditions, it is shown that for given initial data, the solutions of NLS equation always exist for a finite time, and for small initial data, the solutions exist for all the time and decay exponentially to zero as time goes to infinity. Moreover, by applying a boundary control at one end of the boundary, it is shown using a finite-difference approximation scheme that the linear Schrodinger equation is uniformly controllable. It is proved using fixed point method that the discrete version of the NLS equation is also boundary controllable. The results obtained may be applicable to design boundary controls to eliminate unwanted waves generated by noises as well as create the wave propagation that is important in applications.

Dedication

This thesis is dedicated to my parents.

For their constant support, encouragement and endless love

Acknowledgments

First of all, I would like to express my deepest appreciation to my advisor Dr. Shu-Ming Sun, whose immense knowledge, wisdom, enthusiasm, spirit of adventure in regard to research and patience motivate me to the academic world and support me to complete my Ph.D. study. Without his guidance and persistent help, this dissertation would not have been possible.

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Contents

Chapter 1

Introduction

The Schrödinger equations are partial differential equations (PDEs) arising in quantum mechanics as a model to describe the performance of the quantum state when time changes. Over the past decades, the nonlinear Schrödinger equations played important roles in the fields of physics and applied mathematics. It can be derived as models for many physical phenomena such as propagation of light in fiber optics cables, deep surface water waves, Langmuir waves in a hot plasma, oceanic rogue waves, and optical rogue waves (see [1, 2, 4, 25]). In this dissertation, we study the nonlinear Schrödinger equation

$$iu_t + u_{xx} + \kappa|u|^2 u = 0 \tag{1.0.1}$$

for the complex-valued function $u = u(x,t)$ with domain $t \geq 0$, $0 \leq x \leq L$, where the parameter κ is any non-zero real number. In optics, the function u represents a wave, the term u_{xx} represents the dispersion, and the equation (1.0.1) describes the propagation of the wave in nonlinear fiber optics. For water waves, equation (1.0.1) can model the evolution of the envelope of modulated wave groups.

The local and global well-posedness of the initial-boundary-value problem for (1.0.1) posed either on a half line R^+ or on a bounded interval $(0, L)$ with nonhomogeneous boundary conditions has been discussed in [3]. One type of control problems for the KdV equation with periodic boundary conditions was introduced in [30, 31]. The first part of this thesis uses the methods in [30, 31] to discuss the controllability, stabilizability and well-posedness of equation (1.0.1) with boundary conditions

$$u(0,t) = \beta u(L,t), \qquad \beta u_x(0,t) - u_x(L,t) = i\alpha u(0,t), \tag{1.0.2}$$

where $L > 0$, and α and β are any real numbers satisfying $\alpha\beta < 0$ and $\beta \neq \pm 1$.

Applying the boundary conditions (1.0.2) to (1.0.1) yields

$$\frac{d}{dt}\int_0^L |u(x,t)|^2 dx \leq 0$$

when the nonlinear term in (1.0.1) is absent. Thus, (1.0.2) can be considered as a dissipation mechanism, from which, if there is no nonlinear term in (1.0.1), we have the solution of (1.0.1)-(1.0.2) exponentially decaying to zero as t goes to infinity.

Theorem 1.0.1. *The linear operator A defined by*

$$Au = iu''(x)$$

with the domain

$$\mathscr{D}(A) = \left\{ u \in H^2(0, L) \;\middle|\; u(0) = \beta u(L), \quad \beta u'(0) - u'(L) = i\alpha u(0) \right\},$$

generates a strongly continuous semigroup $S(t), t \geq 0$, of bounded operators on $L^2(0, L)$. In addition, there exist positive numbers ξ and η such that

$$\|S(t)\| \leq \xi e^{-\eta t}, \qquad t \geq 0.$$

We also show that the operator A is a discrete spectral operator and all but a finite number of its eigenvalues λ correspond to one-dimensional projections. The asymptotic form of the eigenvalues can be obtained as

$$\lambda_k = \frac{-4\tau\pi}{L^2} - i\frac{(2k\pi + O(1))^2}{L^2}, \qquad k \to \infty,$$

where

$$\tau = \frac{-\alpha\beta L}{2\pi(\beta^2 + 1)} > 0\,.$$

If we denote A^* as the adjoint of A, then

Proposition 1.0.2. *The operators A, A^* have compact resolvents and their eigenvectors,*

$$\{\phi_k | -\infty < k < +\infty\}, \quad \{\psi_k | -\infty < k < +\infty\},$$

satisfying

$$\psi_j^* \phi_k = \delta_{kj}$$

are complete and form dual Riesz bases for $L^2(0, L)$.

Several types of properties for the Schrödinger operator can be obtained from the above spectral analysis. They are used to derive the existence and uniqueness results of the solution of (1.0.1)-(1.0.2) with initial condition $u(x, 0) = u_0$ through contraction mapping principle. Let us define a Banach space $H^{s,p}_{\alpha,\beta}$ as

$$H^{s,p}_{\alpha,\beta} = \left\{ w = \sum_{k=-\infty}^{\infty} c_k \phi_k ; \sum_{k=-\infty}^{\infty} (1 + |k|^{ps}) |c_k|^p < \infty \right\}$$

with norm

$$\|w\|^p_{s,p} = \|w\|^p_{H^{s,p}_{\alpha,\beta}} = \sum_{k=-\infty}^{\infty} (1 + |k|^{ps}) |c_k|^p$$

for any $s \geq 0$ and $p \geq 1$. When $p = 2$, we denote $H^{s,p}_{\alpha,\beta}$ by $H^s_{\alpha,\beta}$ and its norm as $\|\cdot\|_s$.

Theorem 1.0.3. *Let $\frac{1}{2} < s < 1$.*
(i) Define

$$X_T := C\left(0, T; H^s_{\alpha,\beta}\right) \cap L^\infty\left(0, T; H^s_{\alpha,\beta}\right).$$

There exists a $T = T(\|u_0\|_s) > 0$ such that the IVP (1.0.1)-(1.0.2) with initial condition $u(x, 0) = u_0$ has a unique solution $v \in X_T$ for any $u_0 \in H^s_{\alpha,\beta}$, where $T \to \infty$ as $\|u_0\|_s \to 0$. In addition, there exists a neighborhood U of u_0 in $H^s_{\alpha,\beta}$ such that the map from U to $X_{T'}$,

$$G : u_0 \to v(x, t),$$

is Lipschitz continuous for any $T' < T$.
(ii) Let $s' > \frac{1}{3} + s$ and define

$$X_T := C\left(0, T; L^2\right) \cap L^6\left(0, T; H^s_{\alpha,\beta}\right).$$

There exists a $T = T(\|u_0\|_{s',6}) > 0$ such that the IVP (1.0.1)-(1.0.2) with initial condition $u(x, 0) = u_0$ has a unique solution $v \in X_T$ for any $u_0 \in H^{s',6}_{\alpha,\beta}$, where $T \to +\infty$ as $\sup_{0 \le t \le T} \|u_0\|_{L^2} + \|u_0\|_{s',6} \to 0$. In addition, there exists a neighborhood U of u_0 in $H^{s',6}_{\alpha,\beta}$ such that the map from U to $X_{T'}$,

$$G : u_0 \to v(x, t),$$

is Lipschitz continuous for any $T' < T$.

Both parts of Theorem 1.0.3 are local well-posedness results. We also prove the global well-posedness property that there exists a unique solution in $C\left(0, \infty; H^s_{\alpha,\beta}\right) \cap L^\infty\left(0, \infty; H^s_{\alpha,\beta}\right)$ when $\|u_0\|_s$ is small enough for any $s \in (\frac{1}{2}, 1)$. Moreover, by the Lyapounov's second method, we have that the solution exponentially decays to zero as $t \to \infty$.

In recent years, the numerical approximations of controllability problems for PDEs have attracted a lot of attention. While important progress has been made for the wave and heat equations (cf.[7, 17, 20, 21, 22, 34], e.g.,), there are only a few results for the linear Schrödinger equations. The exact controllability of the linear Schrödinger equation in bounded domains with the Dirichlet boundary conditions was studied in [19]. The exact boundary controllability of (1.0.1) posed on a bounded domain in R^n with either the Dirichlet boundary conditions or the Neumann boundary conditions was discussed in [26]. The local exact controllability and stabilizability of (1.0.1) on a bounded interval, with an internal or boundary control, have been studied in [27]. The results of [27] were extended to any dimension in [28]. Thus, it is natural for us to start the study of numerical approximation of the linear Schrödinger equation with boundary control problems and think whether we can obtain the controls of the linear Schrödinger equation as limits of controls from the numerical approximation schemes. In the second part of this thesis, we use the finite-difference approximation scheme and apply the methods introduced in [33, 34] to prove some new results for the linear and nonlinear Schrödinger equations, and present a number of open questions and future directions of research.

Let us consider the nonlinear Schrödinger equation with boundary control $\nu(t)$ which enters into the system through the boundary at $x = L$:

$$\begin{cases} iu_t + u_{xx} + \kappa|u|^2 u = 0, & x \in (0, L), \ 0 < t < T, \\ u(0,t) = 0, \quad u(L,t) = \nu(t), & 0 < t < T, \\ u(x,0) = u^0(x), & x \in (0, L). \end{cases} \tag{1.0.3}$$

System (1.0.3) is known to be controllable if there exists a control $\nu(t) \in L^2(0,T)$ such that, for all $u^0(x) \in L^2(0,L)$ and $u^1(x) \in L^2(0,L)$, the solution $u(x,t)$ of (1.0.3) satisfies $u(x,T) = u^1(x)$.

We introduce the partition $\{x_j = jh\}_{j=0,\dots,N+1}$ of the interval $(0, L)$ with $x_0 = 0$, $x_{N+1} = L$ and $h = \frac{L}{N+1}$ for an integer $N \in \mathbb{N}$. Then the conservative finite-difference semi-discretization of (1.0.3) can be derived as

$$\begin{cases} v_j' + \dfrac{w_{j+1} - 2w_j + w_{j-1}}{h^2} + \kappa(v_j^2 w_j + w_j^3) = 0, & j = 1, \dots, N, \ 0 < t < T, \\ w_j' - \dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2} - \kappa(w_j^2 v_j + v_j^3) = 0, & j = 1, \dots, N, \ 0 < t < T, \\ v_0(t) = 0, \ w_0(t) = 0, & 0 < t < T, \\ v_{N+1}(t) = a_h(t), \ w_{N+1}(t) = b_h(t), & 0 < t < T, \\ v_j(0) = v_j^0, \ w_j(0) = w_j^0, & j = 0, \dots, N+1, \end{cases} \tag{1.0.4}$$

where $'$ denotes derivative with respect to time t, $v_j(t) + iw_j(t) = u_j(t)$ is the approximation of $u(x,t)$ at the node x_j, $a_h(t)$ and $b_h(t)$ are controls dependent on the size of h. By Hilbert Uniqueness Method (HUM) [16] and Ingham's Theorem [11, 12, 23], we deduce that, in the absence of nonlinear terms, the system (1.0.4) is uniformly controllable in time T and the control of (1.0.3) can be achieved as the limit of $a_h(t) + ib_h(t)$ as $h \to 0$. The controllability of (1.0.4) can be also obtained by the method introduced in [27].

Theorem 1.0.4. *Let $T > 0$. In the absence of nonlinear terms, system (1.0.4) is uniformly controllable as $h \to 0$. More precisely, for any initial states $\{v_j^0\}_{j=1}^N, \{w_j^0\}_{j=1}^N$, final states $\{v_j^1\}_{j=1}^N, \{w_j^1\}_{j=1}^N$ and $h > 0$, there exist controls $a_h(t)$ and $b_h(t) \in L^2(0,T)$ such that the solutions of the system satisfy*

$$v_j(T) = v_j^1, \quad w_j(T) = w_j^1, \quad j = 1, \cdots, N.$$

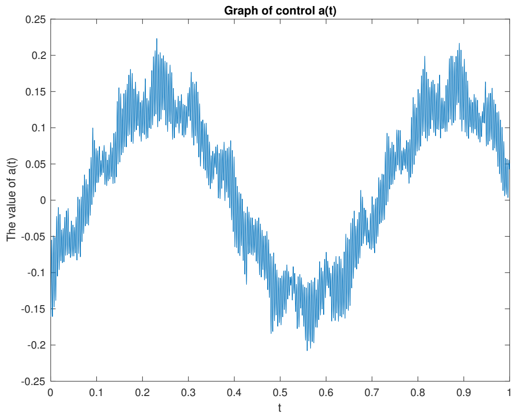
Moreover, there exists a constant $C > 0$, independent of h, such that

$$\|a_h(t)\|_{L^2(0,T)}^2 + \|b_h(t)\|_{L^2(0,T)}^2 \le Ch \sum_{j=1}^N \left(|v_j^0|^2 + |w_j^0|^2 + |v_j^1|^2 + |w_j^1|^2 \right),$$

for all $\{v_j^0\}_{j=1}^N$, $\{w_j^0\}_{j=1}^N$, $\{v_j^1\}_{j=1}^N$, $\{w_j^1\}_{j=1}^N$ and $h > 0$. Finally,

$$a_h(t) \to a(t), \quad b_h(t) \to b(t) \ \text{ in } L^2(0,T) \ \text{ as } h \to 0,$$

where $a(t) + ib(t)$ is a control of (1.0.3) such that $u(x,T) = u^1(x)$, in the absence of the nonlinear term.



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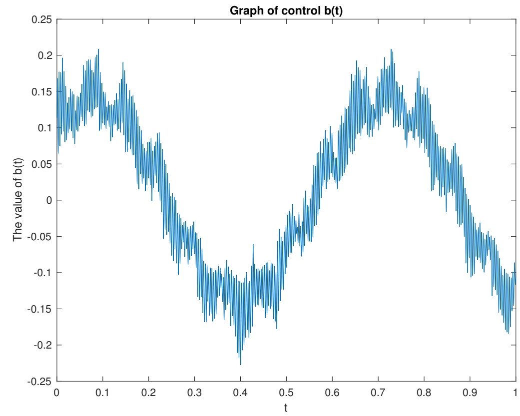

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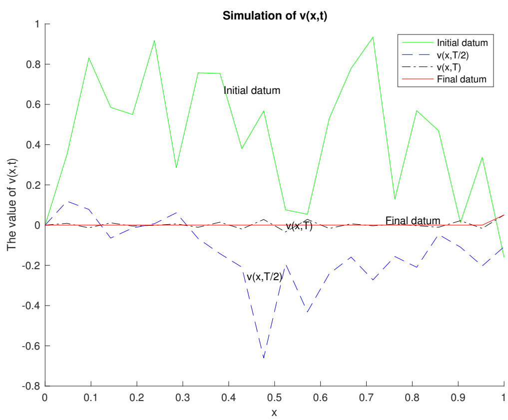

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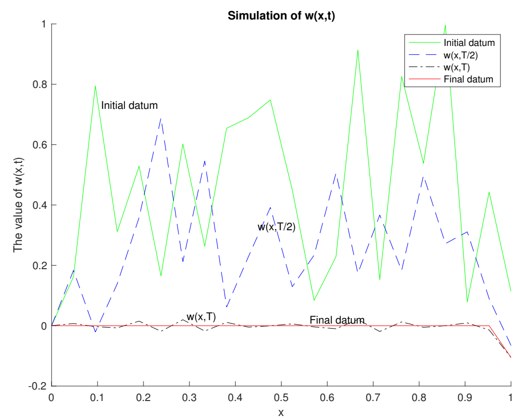


IV

Figure 1.1: Graphs of controls and simulations of solutions for the semi-discrete linear Schrödinger equation. $L = 1, h = \frac{1}{21}$. v_j^0 and w_j^0, $j = 1, \cdots, N$, are random numbers in $(0, 1)$. $v_j^1 = w_j^1 = 0$, $j = 1, \cdots, N$.

Through the algorithm used in the proof of Theorem 1.0.4, in Figure 1.1 and Figure 1.2, we give the graphs of controls $a_h(t)$, $b_h(t)$ and simulate the solutions $\{v_j\}_{j=0}^{N+1}$, $\{w_j\}_{j=0}^{N+1}$ of the semi-discrete Schrödinger system (1.0.4) when $v_j^2 w_j + w_j^3$ and $w_j^2 v_j + v_j^3$ are not present. The end states are null and any given data in Figure 1.1 and Figure 1.2, respectively. From III and IV of those graphs, it is easy to see that the solutions of (1.0.4) go to the given final states as $t \to T$ which implies (1.0.4) is controllable (when $v_j^2 w_j + w_j^3$ and $w_j^2 v_j + v_j^3$ are not present). I and II of the figures are the controls of (1.0.4), as $h \to 0$, which converge to the real and imaginary parts of the control of (1.0.3).

Based on the result of Theorem 1.0.4, we deduce the null controllability of system (1.0.4) by a standard fixed point argument. We introduce any semi-discrete function $Z = Z\,(t) =$

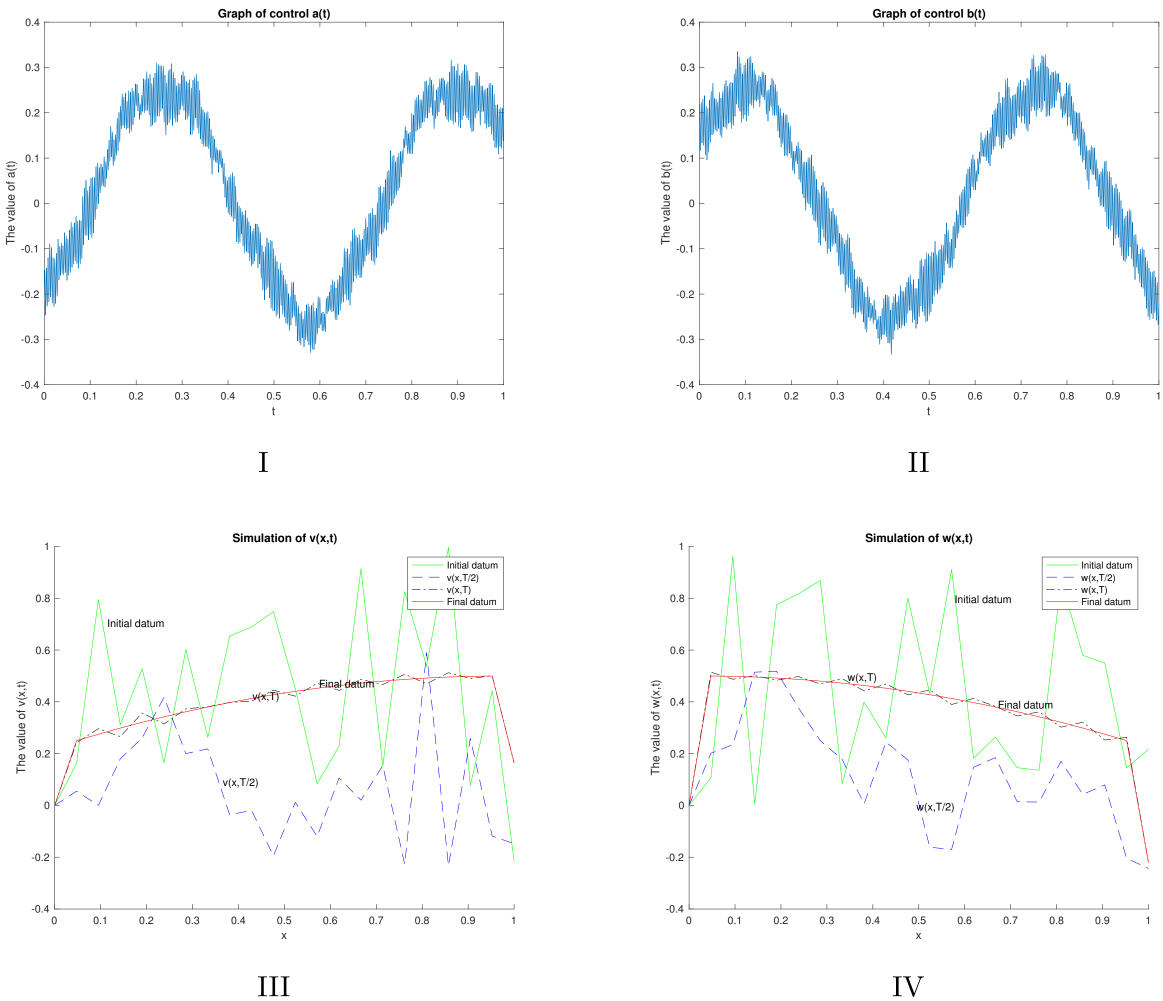


Figure 1.2: Graphs of controls and simulations of solutions for the semi-discrete linear Schrödinger equation. $L = 1, h = \frac{1}{21}$. v_j^0 and w_j^0, $j = 1, \cdots, N$, are random numbers in (0, 1). v_j^1 and w_j^1, $j = 1, \cdots, N$, are any given data.

$(z_0(t), \ldots, z_{N+1}(t)) \in C\left(0, T; \mathbb{C}^{N+2}\right)$ to linearize the nonlinear terms in system (1.0.4), and then convert $v_j^2 w_j + w_j^3$ and $w_j^2 v_j + v_j^3$ to $|z_j|^2 w_j$ and $|z_j|^2 v_j$, respectively. It can be proved that the nonlinear mapping $\Upsilon(Z) = (v_0 + iw_0, v_1 + iw_1, \ldots, v_{N+1} + iw_{N+1})$ has a fixed point which implies $|z_j|^2 w_j = v_j^2 w_j + w_j^3, |z_j|^2 v_j = w_j^2 v_j + v_j^3$ for all $j = 1, \ldots, N$, and, consequently, the solution of the linearized system is the solution of (1.0.4).

Theorem 1.0.5. *The semi-discrete nonlinear Schrödinger system* (1.0.4) *is null controllable. More precisely, if $h > 0$ is fixed, for all initial states $\{v_j^0\}_{j=1}^N, \{w_j^0\}_{j=1}^N \in L^2\left(0, T; R^N\right)$ there exist controls $a_h(t)$ and $b_h(t) \in L^2(0, T)$ such that the solutions of* (1.0.4) *satisfy*

$$v_j(T) = 0, \;\; w_j(T) = 0, \quad j = 1, \ldots, N.$$

In order to observe the tendency of map Υ approaching to its fixed point, we give the graphs of $z_j, j = 0, \ldots, N+1$ at $t = T$ for each iteration. In view of Figure 1.3, it is clear that the fixed point exists.

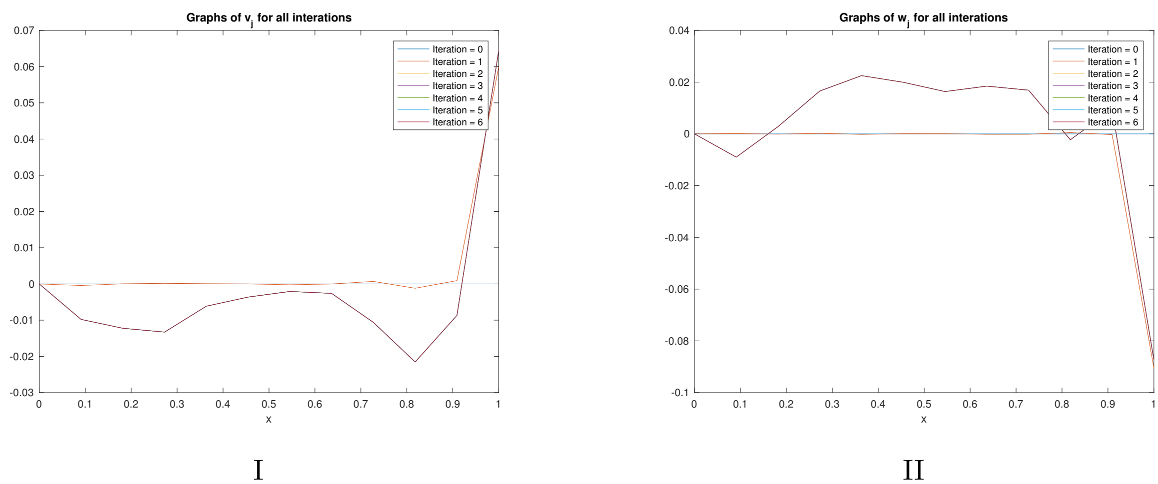
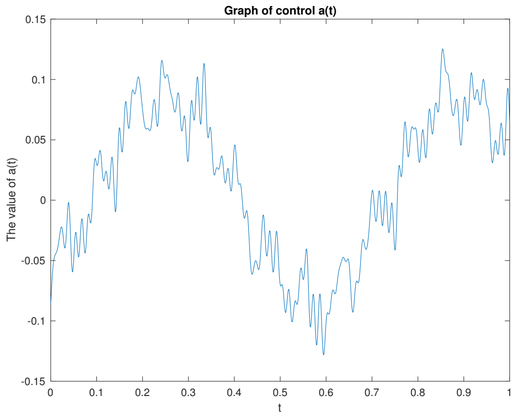


Figure 1.3: Graphs of $z_j, j = 0, 1, \ldots N+1$ for each iteration until the map $\Upsilon(Z)$ reaches a fixed point. I is the real part and II is the imaginary part of z_j. $\kappa = 0.5, L = 1, h = \frac{1}{11}$, $t = T$ and the given initial values of z_j are zeros. After 6 iterations, $\max(|\mathrm{Re}(z_j) - v_j|)$ and $\max(|\mathrm{Im}(z_j) - w_j|) \leq 10^{-5}$, $j = 0, \ldots, N+1$.

After the map $\Upsilon(Z)$ achieves its fixed point, i.e. the number of the iterations is 6 shown in Figure 1.3, the corresponding controls $a_h(t), b_h(t)$ and the simulations of the solutions for the linearized system of (1.0.4) can be obtained as Figure 1.4. Figure 1.4 is also the graph of the controls and solutions of the semi-discrete system (1.0.4). As $h \to 0$, they give the control $\nu(t)$ and solution $u(x,t)$ for the continuous nonlinear Schrödinger system (1.0.3) such that $u(x,T) = 0$ (i.e. system (1.0.3) is null controllable). But in order to prove the convergence, we need to show the uniform null controllability of (1.0.4), i.e. Theorem 1.0.5 holds for any h, which is an open problem that we will study in the future.

The outline of this dissertation is as follows. In Chapter 2, we study the stabilization of the linear Schrödinger equation. In Chapter 3, we analyze the spectral problem of the linear Schrödinger equation which is the preparation of the properties to be proved in Chapter 4. Based on the results of Chapter 4, the local well-posedness of the nonlinear Schrödinger system will be established in Chapter 5. Chapter 6 derives the global existence and exponential decay of the small amplitude solutions. In Chapter 7, we consider the boundary control and numerical approximation of solution for the linear Schrödinger equation using two different methods. Finally, Chapter 8 discusses the controllability of the nonlinear system with boundary control by the fixed point method and gives some open problems related to it.



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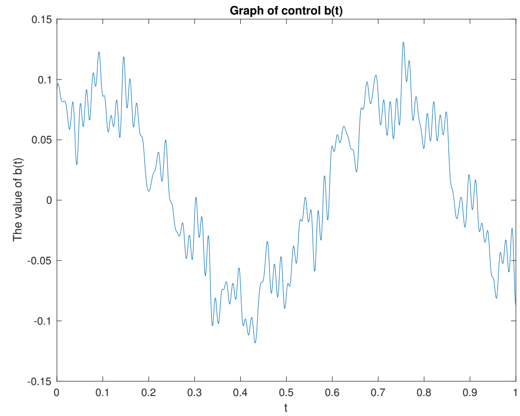

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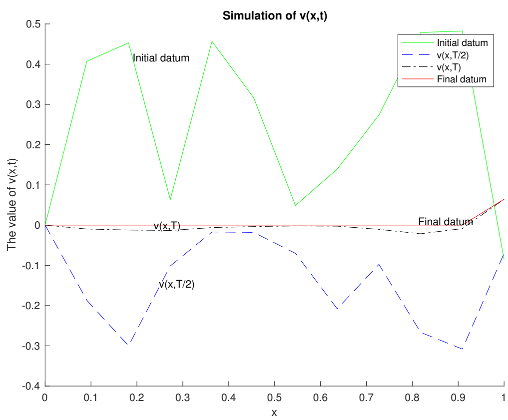

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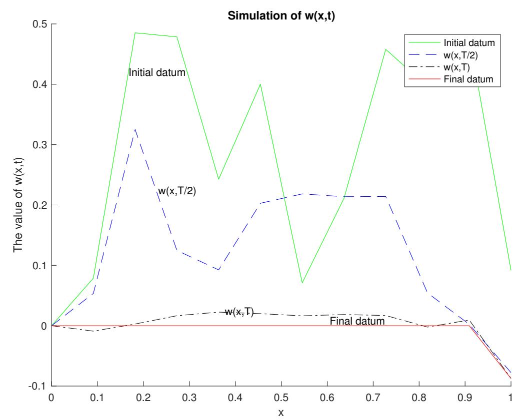


IV

Figure 1.4: Graphs of controls and simulations of solutions for the linearized Schrödinger equation system of (8.0.4) when the map $\Upsilon(Z)$ achieves its fixed point. $\kappa = 0.5, L = 1, h = \frac{1}{11}$. v_j^0 and $w_j^0,\ j = 1, \cdots, N$, are any random numbers in $(0,\ 0.5)$. $v_j^1 = w_j^1 = 0,\ j = 1, \cdots, N$.

Chapter 2

Stability of the Linear Schrödinger Equation with Boundary Dissipation

Before we derive the main results of the nonlinear Schrödinger system (1.0.1)-(1.0.2), it is necessary to study the related linear equation

$$iu_t + u_{xx} = 0 \tag{2.0.1}$$

on the same domain with the same boundary conditions. The conjugate of (2.0.1) is

$$-i\bar{u}_t + \bar{u}_{xx} = 0. \tag{2.0.2}$$

Then (2.0.1) and (2.0.2) give us

$$\begin{aligned} 0 &= (iu_t + u_{xx})\,\bar{u} - (-i\bar{u}_t + \bar{u}_{xx})\,u \\ &= iu_t\bar{u} + u_{xx}\bar{u} + iu\bar{u}_t - u\bar{u}_{xx} \\ &= i\,(u\bar{u})_t + (u_x\bar{u})_x - u_x\bar{u}_x - (u\bar{u}_x)_x + u_x\bar{u}_x \\ &= i\left(|u|^2\right)_t + (u_x\bar{u} - u\bar{u}_x)_x\,. \end{aligned} \tag{2.0.3}$$

From (2.0.3) we have, for appropriately smooth solutions of (2.0.1) and (1.0.2),

$$\begin{aligned} \frac{d}{dt}\int_0^L |u\,(x,t)\,|^2 dx &= \int_0^L \frac{\partial}{\partial t}|u\,(x,t)\,|^2 dx \\ &= \int_0^L i\,(u_x\,(x,t)\,\bar{u}\,(x,t) - u\,(x,t)\,\bar{u}_x\,(x,t))_x\,dx \\ &= i\,(u_x\,(x,t)\,\bar{u}\,(x,t) - u\,(x,t)\,\bar{u}_x\,(x,t))\Big|_0^L \\ &= i\,[u_x\,(L,t)\,\bar{u}\,(L,t) - u\,(L,t)\,\bar{u}_x\,(L,t) - u_x\,(0,t)\,\beta\bar{u}\,(L,t) + \beta u\,(L,t)\,\bar{u}_x\,(0,t)] \\ &= i\bar{u}\,(L,t)\,[u_x\,(L,t) - u_x\,(0,t)\,\beta] + iu\,(L,t)\,[\beta\bar{u}_x\,(0,t) - \bar{u}_x\,(L,t)] \\ &= \alpha\,[u\,(0,t)\,\bar{u}\,(L,t) + u\,(L,t)\,\bar{u}\,(0,t)] \\ &= \alpha\,[\beta u\,(L,t)\,\bar{u}\,(L,t) + u\,(L,t)\,\beta\bar{u}\,(L,t)] \\ &= 2\alpha\beta|u\,(L,t)\,|^2 \le 0, \end{aligned} \tag{2.0.4}$$

which implies that the energy is non-increasing as t increases. Therefore, the boundary conditions (1.0.2) can be considered as a dissipation mechanism for (2.0.1) and it is reasonable to suspect that the solution $u(x,t)$ of (2.0.1) with boundary conditions (1.0.2) goes to zero as $t \to +\infty$. In the rest of this chapter, we will prove this conjecture holds, that means $u(x,t)$ exponentially decays to zero with respect to the norm in $L^2(0.L)$.

Let us define a linear operator A by

$$Au = iu''(x) \tag{2.0.5}$$

with the domain

$$\mathscr{D}(A) = \left\{ u \in H^2(0,L) \;\middle|\; u(0) = \beta u(L), \quad \beta u'(0) - u'(L) = i\alpha u(0) \right\}. \tag{2.0.6}$$

Lemma 2.0.1. *The operator A is dissipative and the resolvent $(\lambda I - A)^{-1}$ of A exists and its operator norm is bounded by λ^{-1} for any real $\lambda > 0$.*

Proof. Let Re (u, Au) denote the real part of (u, Au). By definition, A is dissipative if and only if Re $(u, Au) \leq 0$ for every $u \in \mathscr{D}(A)$. From (2.0.4), it is straightward to check that

$$\begin{aligned}
2\text{Re }(u, Au) &= (u, Au) + \overline{(u, Au)} \\
&= \int_0^L u\overline{Au}dx + \int_0^L \bar{u}Audx \\
&= \int_0^L -i\bar{u}''udx + \int_0^L iu''\bar{u}dx \\
&= -i\int_0^L u\bar{u}'' - u''\bar{u}dx \\
&= -i\left[\int_0^L ud\bar{u}' - \int_0^L \bar{u}du'\right] \\
&= -i\left[u\bar{u}'\Big|_0^L - \int_0^L u'\bar{u}'dx - \bar{u}u'\Big|_0^L + \int_0^L \bar{u}'u'dx\right] \\
&= i\left(u'\bar{u} - u\bar{u}'\right)\Big|_0^L \\
&= 2\alpha\beta|u(L)|^2 \leq 0,
\end{aligned} \tag{2.0.7}$$

i.e. A is dissipative. Let $R(\lambda, A) = (\lambda I - A)^{-1}$. Using (2.0.7) we have

$$\begin{aligned}
\|(\lambda I - A)u\|^2 &= \int_0^L |\lambda u - iu''|^2 dx \\
&= \int_0^L \lambda^2|u|^2 + i\lambda\left(u\bar{u}'' - \bar{u}u''\right) + |u''|^2 dx \\
&= \lambda^2\|u\|^2 + \|u''\|^2 - 2\lambda\alpha\beta|u(L)|^2 \\
&\geq \lambda^2\|u\|^2,
\end{aligned} \tag{2.0.8}$$

then

$$\| (\lambda I - A) u\|^2 \geq \lambda^2 \| (\lambda I - A)^{-1} (\lambda I - A) u\|^2,$$

which implies

$$\| (\lambda I - A)^{-1} (\lambda I - A) u\| \leq \frac{\| (\lambda I - A) u\|}{\lambda},$$

i.e. for any f in the range of $\lambda I - A$ and $\lambda > 0$, $\| (\lambda I - A)^{-1} f\| \leq \frac{\|f\|}{\lambda}$. Hence

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}.$$

□

Theorem 2.0.2. *The operator A generates a strongly continuous semigroup $S(t), t \geq 0$, of bounded operators on $L^2(0, L)$.*

Proof. By the Lumer-Phillips theorem [18] and Lemma 2.0.1, A generates a strongly continuous semigroup on $L^2(0, L)$ if the range of $(\lambda I - A)$, denoted by $\Re(\lambda I - A)$, is all of $L^2(0, L)$. Thus, we need to prove that there exists $u \in \mathscr{D}(\lambda I - A)$, the domain of $\lambda I - A$, such that $(\lambda I - A) u = f$ for any $f \in L^2(0, L)$, i.e. we need to find $u \in \mathscr{D}(\lambda I - A)$ satisfying $u'' + i\lambda u = if$. For $\lambda \neq 0$, we denote two square roots of $-i\lambda$ by μ_0 and μ_1, respectively. Denote $u' = z$ and rewrite $u'' + i\lambda u = if$ to a system of first-order differential equations:

$$\begin{cases} u' = z \\ z' = u'' = -i\lambda u + if, \end{cases}$$

which is equivalent to

$$\overrightarrow{u}' = F(\lambda) \overrightarrow{u} + \overrightarrow{\phi}, \tag{2.0.9}$$

where $\overrightarrow{u} = (u, z)^T$, $\overrightarrow{\phi} = (0, if)^T$, and

$$F(\lambda) = \begin{pmatrix} 0 & 1 \\ -i\lambda & 0 \end{pmatrix}.$$

Diagonalize the system with the transformation using $\mu = (\mu_0, \mu_1)$,

$$\overrightarrow{u} = \begin{pmatrix} 1 & 1 \\ \mu_0 & \mu_1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = M(\mu) \overrightarrow{v}, \tag{2.0.10}$$

and plug (2.0.10) into (2.0.9) to obtain

$$\overrightarrow{v}' = \Omega(\mu) \overrightarrow{v} + \overrightarrow{\psi}, \tag{2.0.11}$$

where

$$\Omega(\mu) = M(\mu)^{-1} F(\lambda) M(\mu) = \begin{pmatrix} \mu_0 & 0 \\ 0 & \mu_1 \end{pmatrix}, \qquad \overrightarrow{\psi} = M(\mu)^{-1} \overrightarrow{\phi}. \tag{2.0.12}$$

The solution of (2.0.11) can be written by

$$\overrightarrow{v}(x) = e^{x\Omega(\mu)}\overrightarrow{v}(0) + \int_0^x e^{(x-s)\Omega(\mu)}\overrightarrow{\psi}(s)\,ds. \tag{2.0.13}$$

Then, the boundary conditions (2.0.6) are changed to

$$\overrightarrow{u}(L) = \begin{pmatrix} \frac{1}{\beta} & 0 \\ -i\alpha & \beta \end{pmatrix} \begin{pmatrix} u(0) \\ z(0) \end{pmatrix} = B(\alpha,\beta)\,\overrightarrow{u}(0),$$

or, by the relationship (2.0.10) between $\overrightarrow{u}$ and $\overrightarrow{v}$,

$$\overrightarrow{v}(L) = M(\mu)^{-1} B(\alpha,\beta)\, M(\mu)\,\overrightarrow{v}(0). \tag{2.0.14}$$

Substituting (2.0.13) into (2.0.14) at $x = L$ yields

$$M(\mu)^{-1} B(\alpha,\beta)\, M(\mu)\,\overrightarrow{v}(0) = e^{L\Omega(\mu)}\overrightarrow{v}(0) + \int_0^L e^{(L-s)\Omega(\mu)}\overrightarrow{\psi}(s)\,ds,$$

which implies

$$\left[B(\alpha,\beta)\, M(\mu) - M(\mu)\, e^{L\Omega(\mu)}\right]\overrightarrow{v}(0) = M(\mu)\int_0^L e^{(L-s)\Omega(\mu)}\overrightarrow{\psi}(s)\,ds.$$

To show no positive λ satisfying $\det\left(B(\alpha,\beta)\, M(\mu) - M(\mu)\, e^{L\Omega(\mu)}\right) = 0$, we let $\Phi(\mu,\alpha,\beta) = B(\alpha,\beta)\, M(\mu) - M(\mu)\, e^{L\Omega(\mu)}$ and assume that there exists λ with $\det(\Phi(\mu,\alpha,\beta)) = 0$. Then, any nonzero solution $\overrightarrow{v}(0)$ of $\Phi(\mu,\alpha,\beta)\,\overrightarrow{v}(0) = 0$ gives an eigenfunction

$$\overrightarrow{v}(\mu,\alpha,\beta,x) = e^{x\Omega(\mu)}\overrightarrow{v}(0)$$

for A, which corresponds to the value of λ associated with μ by $-i\lambda = \mu_j^2, j = 0,1$. For this $\overrightarrow{v}(\mu,\alpha,\beta,x)$, there is a nonzero solution u of $u'' + i\lambda u = 0$ (i.e. $\|(\lambda I - A)\,u\| = 0$), which contradicts to (2.0.8). Hence, $(\Phi(\mu,\alpha,\beta))$ is invertible for any $\lambda > 0$ and

$$\overrightarrow{v}(0) = \Phi(\mu,\alpha,\beta)^{-1} M(\mu)\int_0^L e^{(L-s)\Omega(\mu)}\overrightarrow{\psi}(s)\,ds\,.$$

From (2.0.13) we have

$$\overrightarrow{v}(x) = e^{x\Omega(\mu)}\Phi(\mu,\alpha,\beta)^{-1} M(\mu)\int_0^L e^{(L-x)\Omega(\mu)}\overrightarrow{\psi}(x)\,dx + \int_0^x e^{(x-s)\Omega(\mu)}\overrightarrow{\psi}(s)\,ds. \tag{2.0.15}$$

Therefore, for any $f \in L^2(0,L)$, we can find $u = (\lambda I - A)^{-1} f$, which means that the range of $\lambda I - A$ is all functions in $L^2(0,L)$ for any $\lambda > 0$. Thus, A generates a strongly continuous semigroup on $L^2(0,L)$. □

Now, we study the resolvent of A for λ on the imaginary axis.

Lemma 2.0.3. *For any λ on the imaginary axis, $R(\lambda, A) = (\lambda I - A)^{-1}$ exists on $L^2(0, L)$.*

Proof. Let $\lambda = i\omega$ and assume that there is a $\tilde{u}(i\omega, \alpha, \beta, x) = \tilde{u} \in \mathscr{D}(A)$ satisfying $(A - \lambda I)\tilde{u} = (A - i\omega I)\tilde{u} = 0$. By the identity

$$\begin{aligned}(\tilde{u}, A\tilde{u}) + \overline{(\tilde{u}, A\tilde{u})} &= (\tilde{u}, A\tilde{u}) + (A\tilde{u}, \tilde{u}) = (\tilde{u}, \lambda\tilde{u}) + (\lambda\tilde{u}, \tilde{u}) \\ &= \lambda(\tilde{u}, \tilde{u}) + \bar{\lambda}(\tilde{u}, \tilde{u}) = i\omega(\tilde{u}, \tilde{u}) - i\omega(\tilde{u}, \tilde{u}) = 0,\end{aligned}$$

$\tilde{u}(L) = 0$ follows from (2.0.7). Thus, by the boundary conditions (1.0.2), $\tilde{u} \in \mathscr{D}(A)$ satisfies

$$\tilde{u}'' = \omega\tilde{u}, \quad \text{with} \quad \tilde{u}(0) = \tilde{u}(L) = 0, \quad \beta\tilde{u}_x(0) - \tilde{u}_x(L) = 0. \tag{2.0.16}$$

If $\omega \neq 0$, we have

$$\tilde{u} = c_0 e^{\mu_0 x} + c_1 e^{\mu_1 x}, \quad \tilde{u}_x = c_0\mu_0 e^{\mu_0 x} + c_1\mu_1 e^{\mu_1 x},$$

where μ_0 and μ_1 are two different square roots of $-i\lambda = \omega$. Applying the boundary conditions (2.0.16) yields

$$\begin{aligned}&\tilde{u}(0) = c_0 + c_1 = 0 \Rightarrow c_1 = -c_0, \\ &\tilde{u}(L) = c_0 e^{\mu_0 L} + c_1 e^{\mu_1 L} = c_0\left(e^{\mu_0 L} - e^{\mu_1 L}\right) = 0 \Rightarrow c_0 = 0 \Rightarrow c_1 = 0,\end{aligned}$$

which implies that $\tilde{u} = 0$ and $(A - \lambda I)\tilde{u} = 0$ has only trivial solution for $\omega \neq 0$. Similarly, the case for $\omega = 0$ also implies $\tilde{u} = 0$. Indeed, if $\omega = 0$,

$$\tilde{u} = c_0 + c_1 x, \quad \tilde{u}_x = c_1,$$

then

$$\left.\begin{aligned}&\tilde{u}(0) = c_0 = 0 \\ &\tilde{u}(L) = c_0 + c_1 L = 0 \Rightarrow c_1 L = 0\end{aligned}\right\} \Rightarrow c_0 = c_1 = 0.$$

Since A has only discrete spectrum, we conclude that for any λ on the imaginary axis, $R(\lambda, A)$ exists. □

The following is the resolvent estimate for large λ on the imaginary axis.

Lemma 2.0.4. $\|R(i\omega, A)\| = O\left(\omega^{-\frac{1}{2}}\right)$ *as* $|\omega| \to \infty$.

Proof. First, we find the solution u of $(\lambda I - A)u = f$ with boundary conditions (1.0.2) using Green's function. If we define $G(\lambda, x, \zeta)$ by

$$\begin{cases}\lambda G(\lambda, x, \zeta) - iG''(\lambda, x, \zeta) = \delta(x - \zeta), \\ G(\lambda, 0, \zeta) = \beta G(\lambda, L, \zeta), & (2.0.17) \\ \beta G'(\lambda, 0, \zeta) - G'(\lambda, L, \zeta) = i\alpha G(\lambda, 0, \zeta), & (2.0.18)\end{cases}$$

where $\zeta \in [0, L]$ and δ is the Dirac delta function, then the solution u is given by

$$u(\lambda, \alpha, \beta, x) = \int_0^L G(\lambda, x, \zeta) f(\zeta) \, d\zeta. \tag{2.0.19}$$

The Green's function $G(\lambda, x, \zeta)$ can be found as follows. We have G takes the form

$$G(\lambda, x, \zeta) = c_0 e^{\mu_0 (x-\zeta)} + c_1 e^{\mu_1 (x-\zeta)} + H(x-\zeta) \left[\hat{c}_0 e^{\mu_0 (x-\zeta)} + \hat{c}_1 e^{\mu_1 (x-\zeta)} \right], \tag{2.0.20}$$

where the Heaviside function

$$H(x-\zeta) = \begin{cases} 1, & x > \zeta, \\ 0, & x \leq \zeta. \end{cases}$$

From the homogeneous equation $(\lambda I - A) u = 0$, it is obtained that

$$\begin{aligned} G(\lambda, x, \zeta) &= \begin{cases} G_1(\lambda, x, \zeta), & x > \zeta, \\ G_2(\lambda, x, \zeta), & x \leq \zeta, \end{cases} \\ &= \begin{cases} (c_0 + \hat{c}_0) e^{\mu_0 (x-\zeta)} + (c_1 + \hat{c}_1) e^{\mu_1 (x-\zeta)}, & x > \zeta, \\ c_0 e^{\mu_0 (x-\zeta)} + c_1 e^{\mu_1 (x-\zeta)}, & x \leq \zeta, \end{cases} \end{aligned} \tag{2.0.21}$$

and

$$\begin{cases} G_1'(\lambda, x, \zeta) = (c_0 + \hat{c}_0) \mu_0 e^{\mu_0 (x-\zeta)} + (c_1 + \hat{c}_1) \mu_1 e^{\mu_1 (x-\zeta)}, & x > \zeta, \\ G_2'(\lambda, x, \zeta) = c_0 \mu_0 e^{\mu_0 (x-\zeta)} + c_1 \mu_1 e^{\mu_1 (x-\zeta)}, & x \leq \zeta. \end{cases} \tag{2.0.22}$$

The conditions at $x = \zeta$,

$$\begin{aligned} G_1(\lambda, \zeta, \zeta) - G_2(\lambda, \zeta, \zeta) &= 0, \\ G_1'(\lambda, \zeta, \zeta) - G_2'(\lambda, \zeta, \zeta) &= 1, \end{aligned}$$

give us

$$\begin{aligned} & c_0 + \hat{c}_0 + c_1 + \hat{c}_1 - (c_0 + c_1) = \hat{c}_0 + \hat{c}_1 = 0, \\ & (c_0 + \hat{c}_0) \mu_0 + (c_1 + \hat{c}_1) \mu_1 - (c_0 \mu_0 + c_1 \mu_1) = \hat{c}_0 \mu_0 + \hat{c}_1 \mu_1 = 1, \end{aligned}$$

thus it is deduced that

$$\hat{c}_0 = \frac{1}{\mu_0 - \mu_1}, \qquad \hat{c}_1 = \frac{1}{\mu_1 - \mu_0}. \tag{2.0.23}$$

From (2.0.21) we have

$$\begin{aligned} G(\lambda, 0, \zeta) &= c_0 e^{-\mu_0 \zeta} + c_1 e^{-\mu_1 \zeta}, & x \leq \zeta, \\ G(\lambda, L, \zeta) &= (c_0 + \hat{c}_0) e^{\mu_0 (L-\zeta)} + (c_1 + \hat{c}_1) e^{\mu_1 (L-\zeta)}, & x > \zeta, \end{aligned}$$

then the boundary condition (2.0.17) implies

$$c_0 e^{-\mu_0\zeta} + c_1 e^{-\mu_1\zeta} = \beta\left(c_0 + \hat{c}_0\right) e^{\mu_0(L-\zeta)} + \beta\left(c_1 + \hat{c}_1\right) e^{\mu_1(L-\zeta)},$$

hence

$$c_0 e^{-\mu_0\zeta}\left(\frac{1}{\beta} - e^{\mu_0 L}\right) + c_1 e^{-\mu_1\zeta}\left(\frac{1}{\beta} - e^{\mu_1 L}\right) = \hat{c}_0 e^{\mu_0(L-\zeta)} + \hat{c}_1 e^{\mu_1(L-\zeta)}. \tag{2.0.24}$$

Similarly, from (2.0.22)

$$\begin{aligned} G'(\lambda, 0, \zeta) &= c_0\mu_0 e^{-\mu_0\zeta} + c_1\mu_1 e^{-\mu_1\zeta}, && x \le \zeta, \\ G'(\lambda, L, \zeta) &= \left(c_0 + \hat{c}_0\right)\mu_0 e^{\mu_0(L-\zeta)} + \left(c_1 + \hat{c}_1\right)\mu_1 e^{\mu_1(L-\zeta)}, && x > \zeta. \end{aligned}$$

Then the boundary condition (2.0.18) yields

$$\beta c_0\mu_0 e^{-\mu_0\zeta} + \beta c_1\mu_1 e^{-\mu_1\zeta} - \left(c_0 + \hat{c}_0\right)\mu_0 e^{\mu_0(L-\zeta)} - \left(c_1 + \hat{c}_1\right)\mu_1 e^{\mu_1(L-\zeta)} = i\alpha c_0 e^{-\mu_0\zeta} + i\alpha c_1 e^{-\mu_1\zeta},$$

which implies

$$c_0 e^{-\mu_0\zeta}\left(\beta\mu_0 - \mu_0 e^{\mu_0 L} - i\alpha\right) + c_1 e^{-\mu_1\zeta}\left(\beta\mu_1 - \mu_1 e^{\mu_1 L} - i\alpha\right) = \hat{c}_1\mu_1 e^{\mu_1(L-\zeta)} + \hat{c}_0\mu_0 e^{\mu_0(L-\zeta)}. \tag{2.0.25}$$

From the definitions in the proof of Theorem 2.0.2, we have that for $c = \left(c_0, c_1\right)^T, \hat{c} = \left(\hat{c}_0, \hat{c}_1\right)^T$,

$$\begin{aligned} \Phi(\mu, \alpha, \beta) &= B(\alpha, \beta) M(\mu) - M(\mu) e^{L\Omega(\mu)} \\ &= \begin{pmatrix} \frac{1}{\beta} & 0 \\ -i\alpha & \beta \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \mu_0 & \mu_1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ \mu_0 & \mu_1 \end{pmatrix} e^{\begin{pmatrix} \mu_0 & 0 \\ 0 & \mu_1 \end{pmatrix} L} \\ &= \begin{pmatrix} \frac{1}{\beta} & \frac{1}{\beta} \\ -i\alpha + \beta\mu_0 & -i\alpha + \beta\mu_1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ \mu_0 & \mu_1 \end{pmatrix}\begin{pmatrix} e^{\mu_0 L} & 0 \\ 0 & e^{\mu_1 L} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\beta} & \frac{1}{\beta} \\ -i\alpha + \beta\mu_0 & -i\alpha + \beta\mu_1 \end{pmatrix} - \begin{pmatrix} e^{\mu_0 L} & e^{\mu_1 L} \\ \mu_0 e^{\mu_0 L} & \mu_1 e^{\mu_1 L} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\beta} - e^{\mu_0 L} & \frac{1}{\beta} - e^{\mu_1 L} \\ -i\alpha + \beta\mu_0 - \mu_0 e^{\mu_0 L} & -i\alpha + \beta\mu_1 - \mu_1 e^{\mu_1 L} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} &\Phi(\mu, \alpha, \beta) e^{-\Omega(\mu)\zeta} c \\ &= \begin{pmatrix} \frac{1}{\beta} - e^{\mu_0 L} & \frac{1}{\beta} - e^{\mu_1 L} \\ -i\alpha + \beta\mu_0 - \mu_0 e^{\mu_0 L} & -i\alpha + \beta\mu_1 - \mu_1 e^{\mu_1 L} \end{pmatrix}\begin{pmatrix} e^{-\mu_0\zeta} & 0 \\ 0 & e^{-\mu_1\zeta} \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \\ &= \begin{pmatrix} c_0\left(\frac{1}{\beta} - e^{\mu_0 L}\right) e^{-\mu_0\zeta} + c_1\left(\frac{1}{\beta} - e^{\mu_1 L}\right) e^{-\mu_1\zeta} \\ c_0\left(-i\alpha + \beta\mu_0 - \mu_0 e^{\mu_0 L}\right) e^{-\mu_0\zeta} + c_1\left(-i\alpha + \beta\mu_1 - \mu_1 e^{\mu_1 L}\right) e^{-\mu_1\zeta} \end{pmatrix}, \end{aligned}$$

with

$$M(\mu)\, e^{\Omega(\mu)(L-\zeta)}\hat{c} = \begin{pmatrix} 1 & 1 \\ \mu_0 & \mu_1 \end{pmatrix} \begin{pmatrix} e^{(L-\zeta)\mu_0} & 0 \\ 0 & e^{(L-\zeta)\mu_1} \end{pmatrix} \begin{pmatrix} \hat{c}_0 \\ \hat{c}_1 \end{pmatrix} = \begin{pmatrix} \hat{c}_0 e^{(L-\zeta)\mu_0} + \hat{c}_1 e^{(L-\zeta)\mu_1} \\ \hat{c}_0 \mu_0 e^{(L-\zeta)\mu_0} + \hat{c}_1 \mu_1 e^{(L-\zeta)\mu_1} \end{pmatrix}.$$

Thus, if

$$\overrightarrow{p}(\mu,\zeta) = M(\mu)\, e^{\Omega(\mu)(L-\zeta)}\hat{c} \quad \text{and} \quad \overrightarrow{q}(\mu,\zeta) = e^{-\Omega(\mu)\zeta}c, \tag{2.0.26}$$

we can rewrite (2.0.24) and (2.0.25) by

$$\Phi(\mu,\alpha,\beta)\,\overrightarrow{q}(\mu,\zeta) = \overrightarrow{p}(\mu,\zeta).$$

From the Cramer's rule, $\overrightarrow{q}(\mu,\zeta)$ can be solved as follows,

$$\overrightarrow{q}(\mu,\zeta) = \frac{\overrightarrow{r}(\mu,\zeta)}{\det(\Phi(\mu,\alpha,\beta))},$$

where the two components r_0 and r_1 of $\overrightarrow{r}(\mu,\zeta)$ are the determinants of the matrices obtained by replacing the first or second column of $\Phi(\mu,\alpha,\beta)$ by $\overrightarrow{p}(\mu,\zeta)$.

Let us assume $\omega > 0$ and take $\omega = \rho^2$ with $\rho > 0$. Then, the square roots of $-i\lambda = -i(i\omega) = \omega = \rho^2$ are ρ and $-\rho$. Let $\mu_0 = \rho$ and $\mu_1 = -\rho$. Hence, as $\rho \to \infty$,

$$\begin{aligned} r_0(\mu,\zeta) &= \det\begin{pmatrix} \hat{c}_0 e^{(L-\zeta)\mu_0} + \hat{c}_1 e^{(L-\zeta)\mu_1} & \frac{1}{\beta} - e^{\mu_1 L} \\ \hat{c}_0\mu_0 e^{(L-\zeta)\mu_0} + \hat{c}_1\mu_1 e^{(L-\zeta)\mu_1} & -i\alpha + \beta\mu_1 - \mu_1 e^{\mu_1 L} \end{pmatrix} \\ &\approx \hat{c}_0 e^{(L-\zeta)\mu_0} \det\begin{pmatrix} 1 & \frac{1}{\beta} - e^{\mu_1 L} \\ \mu_0 & -i\alpha + \beta\mu_1 - \mu_1 e^{\mu_1 L} \end{pmatrix} \\ &= \hat{c}_0 e^{(L-\zeta)\mu_0}\left(-i\alpha + \beta\mu_1 - \mu_1 e^{\mu_1 L} - \mu_0\frac{1}{\beta} + \mu_0 e^{\mu_1 L}\right) \\ &\approx \hat{c}_0 e^{(L-\zeta)\rho}\left[-i\alpha - \rho\left(\beta + \frac{1}{\beta}\right)\right], \end{aligned} \tag{2.0.27}$$

$$\begin{aligned} r_1(\mu,\zeta) &= \det\begin{pmatrix} \frac{1}{\beta} - e^{\mu_0 L} & \hat{c}_0 e^{(L-\zeta)\mu_0} + \hat{c}_1 e^{(L-\zeta)\mu_1} \\ -i\alpha + \beta\mu_0 - \mu_0 e^{\mu_0 L} & \hat{c}_0\mu_0 e^{(L-\zeta)\mu_0} + \hat{c}_1\mu_1 e^{(L-\zeta)\mu_1} \end{pmatrix} \\ &\approx \hat{c}_0 e^{(L-\zeta)\mu_0} \det\begin{pmatrix} \frac{1}{\beta} - e^{\mu_0 L} & 1 \\ -i\alpha + \beta\mu_0 - \mu_0 e^{\mu_0 L} & \mu_0 \end{pmatrix} \\ &= \hat{c}_0 e^{(L-\zeta)\mu_0}\left(\frac{1}{\beta}\mu_0 - e^{\mu_0 L}\mu_0 + i\alpha - \beta\mu_0 + \mu_0 e^{\mu_0 L}\right) \\ &= \hat{c}_0 e^{(L-\zeta)\rho}\left[i\alpha + \left(\frac{1}{\beta} - \beta\right)\rho\right], \end{aligned} \tag{2.0.28}$$

and

$$\begin{aligned}
&\det\left(\Phi\left(\mu,\alpha,\beta\right)\right)\\
&=\left(\frac{1}{\beta}-e^{\mu_0 L}\right)\left(-i\alpha+\beta\mu_1-\mu_1 e^{\mu_1 L}\right)-\left(\frac{1}{\beta}-e^{\mu_1 L}\right)\left(-i\alpha+\beta\mu_0-\mu_0 e^{\mu_0 L}\right) \qquad (2.0.29)\\
&=\frac{1}{\beta}\left(-2\beta\rho+\rho e^{-\rho L}+\rho e^{\rho L}\right)+i\alpha e^{\mu_0 L}-\beta\mu_1 e^{\mu_0 L}+\mu_1-\mu_0-i\alpha e^{\mu_1 L}+\beta\mu_0 e^{\mu_1 L}\\
&\approx\frac{1}{\beta}\left(-2\beta\rho+\rho e^{\rho L}\right)+i\alpha e^{\rho L}+\beta\rho e^{\rho L}-2\rho\\
&=-4\rho+e^{\rho L}\left(\frac{\rho}{\beta}+\rho\beta+i\alpha\right)\\
&\approx e^{\rho L}\left(\frac{\rho}{\beta}+\rho\beta+i\alpha\right),
\end{aligned}$$

which imply that as $\rho\to\infty$

$$\begin{aligned}
\overrightarrow{q}\left(\mu,\zeta\right)&=\frac{\overrightarrow{r}\left(\mu,\zeta\right)}{\det\left(\Phi\left(\mu,\alpha,\beta\right)\right)}\\
&\approx\hat{c}_0 e^{(L-\zeta)\rho}\begin{pmatrix}\frac{-i\alpha-\rho\left(\beta+\beta^{-1}\right)}{e^{\rho L}(\rho/\beta+\rho\beta+i\alpha)}\\ \frac{i\alpha+\left(\beta^{-1}-\beta\right)\mu_0}{e^{\rho L}(\rho/\beta+\rho\beta+i\alpha)}\end{pmatrix}.
\end{aligned}$$

Let the row vector $\epsilon^*=(1,1)$. Then, from (2.0.20) and (2.0.26),

$$G\left(\lambda,x,\zeta\right)=\epsilon^*\left[e^{\Omega(\mu)x}\overrightarrow{q}\left(\mu,\zeta\right)+H\left(x-\zeta\right)e^{\Omega(\mu)(x-\zeta)}\hat{c}\left(\mu\right)\right].$$

For $x\le\zeta$, i.e. $H\left(x-\zeta\right)=0$,

$$\begin{aligned}
G\left(\lambda,x,\zeta\right)&=\epsilon^*\left[e^{\Omega(\mu)x}\overrightarrow{q}\left(\mu,\zeta\right)\right]\\
&\approx\begin{pmatrix}1&1\end{pmatrix}\left[\begin{pmatrix}e^{\rho x}&0\\0&e^{-\rho x}\end{pmatrix}\begin{pmatrix}\frac{-i\alpha-\rho\left(\beta+\beta^{-1}\right)}{e^{\rho L}(\rho/\beta+\rho\beta+i\alpha)}\\ \frac{i\alpha+\left(\beta^{-1}-\beta\right)\rho}{e^{\rho L}(\rho/\beta+\rho\beta+i\alpha)}\end{pmatrix}\hat{c}_0 e^{(L-\zeta)\rho}\right]\\
&=\left\{\frac{e^{\rho x}\left[-i\alpha-\rho\left(\beta+\beta^{-1}\right)\right]}{e^{\rho L}\left(\rho/\beta+\rho\beta+i\alpha\right)}+\frac{e^{-\rho x}\left[i\alpha+\left(\beta^{-1}-\beta\right)\rho\right]}{e^{\rho L}\left(\rho/\beta+\rho\beta+i\alpha\right)}\right\}\hat{c}_0 e^{(L-\zeta)\rho}\\
&=\hat{c}_0 e^{\rho(x-\zeta)}\frac{-i\alpha-\rho\left(\beta+\beta^{-1}\right)}{\rho/\beta+\rho\beta+i\alpha}+\hat{c}_0\frac{i\alpha+\left(\beta^{-1}-\beta\right)\rho}{e^{\rho(x+\zeta)}\left(\rho/\beta+\rho\beta+i\alpha\right)}\\
&\to-\hat{c}_0 e^{\rho(x-\zeta)}\quad\text{as}\quad\rho\to\infty,
\end{aligned}$$

where $e^{\rho(x-\zeta)}$ is uniformly bounded for $x\le\zeta$. From (2.0.23), $|\hat{c}_0|\approx\rho^{-1}$ as $\rho\to\infty$. Thus, we conclude that for any given $r>0$, there is a constant C_r independent of ρ such that for

$x \leq \zeta$ and $\rho > r$, $|G(\lambda, x, \zeta)| \leq C_r \rho^{-1}$. If $x > \zeta$, i.e. $H(x-\zeta) = 1$,

$$\begin{aligned} G(\lambda, x, \zeta) &= \epsilon^* \left[e^{\Omega(\mu)x} \overrightarrow{q}(\mu, \zeta) + e^{\Omega(\mu)(x-\zeta)} \hat{c}(\mu) \right] \\ &= \hat{c}_0 e^{\rho(x-\zeta)} \frac{-i\alpha - \rho(\beta + \beta^{-1})}{\rho/\beta + \rho\beta + i\alpha} + \hat{c}_0 \frac{i\alpha + (\beta^{-1} - \beta)\rho}{e^{\rho(x+\zeta)}(\rho/\beta + \rho\beta + i\alpha)} + \hat{c}_0 e^{\rho(x-\zeta)} + \hat{c}_1 e^{-\rho(x-\zeta)} \\ &\to -\hat{c}_0 e^{\rho(x-\zeta)} + \hat{c}_0 e^{\rho(x-\zeta)} + \hat{c}_1 e^{-\rho(x-\zeta)} \\ &= \hat{c}_1 e^{-\rho(x-\zeta)} \quad \text{as} \quad \rho \to \infty. \end{aligned}$$

Similar to the case $x \leq \zeta$, $e^{-\rho(x-\zeta)}$ is uniformly bounded for $x > \zeta$ and (2.0.23) implies $|\hat{c}_1| \approx \rho^{-1}$ as $\rho \to \infty$. Thus, we obtain that for any given $r > 0$, there is a constant C_r independent of ρ such that for $x > \zeta$ and $\rho > r$, $|G(\lambda, x, \zeta)| \leq C_r \rho^{-1}$.

When $\omega < 0$, we let $\omega = -\rho^2$ with $\rho > 0$. Thus, the square roots of $-i\lambda = -\rho^2$ are $i\rho$ and $-i\rho$, and $\mu_0 = i\rho$ and $\mu_1 = -i\rho$. From (2.0.23) we have $\hat{c}_1 = -\hat{c}_0$, then (2.0.27)-(2.0.29) give us, as $\rho \to \infty$,

$$\begin{aligned} r_0(\mu, \zeta) &= \left(\hat{c}_0 e^{(L-\zeta)\mu_0} + \hat{c}_1 e^{(L-\zeta)\mu_1} \right) \left(-i\alpha + \beta\mu_1 - \mu_1 e^{\mu_1 L} \right) - \left(\frac{1}{\beta} - e^{\mu_1 L} \right) \left(\hat{c}_0 \mu_0 e^{(L-\zeta)\mu_0} + \hat{c}_1 \mu_1 e^{(L-\zeta)\mu_1} \right) \\ &= \hat{c}_0 (i\alpha + \beta\mu_0) \left(e^{-(L-\zeta)\mu_0} - e^{(L-\zeta)\mu_0} \right) - \frac{1}{\beta} \hat{c}_0 \mu_0 \left(e^{-(L-\zeta)\mu_0} + e^{(L-\zeta)\mu_0} \right) + 2\hat{c}_0 \mu_0 e^{-\zeta\mu_0} \\ &\approx \hat{c}_0 \beta\mu_0 \left(e^{-(L-\zeta)\mu_0} - e^{(L-\zeta)\mu_0} \right) - \frac{1}{\beta} \hat{c}_0 \mu_0 \left(e^{-(L-\zeta)\mu_0} + e^{(L-\zeta)\mu_0} \right) + 2\hat{c}_0 \mu_0 e^{-\zeta\mu_0}, \end{aligned}$$

$$\begin{aligned} r_1(\mu, \zeta) &= \left(\frac{1}{\beta} - e^{\mu_0 L} \right) \left(\hat{c}_0 \mu_0 e^{(L-\zeta)\mu_0} + \hat{c}_1 \mu_1 e^{(L-\zeta)\mu_1} \right) - \left(\hat{c}_0 e^{(L-\zeta)\mu_0} + \hat{c}_1 e^{(L-\zeta)\mu_1} \right) \left(-i\alpha + \beta\mu_0 - \mu_0 e^{\mu_0 L} \right) \\ &= \hat{c}_0 (i\alpha - \beta\mu_0) \left(e^{(L-\zeta)\mu_0} - e^{-(L-\zeta)\mu_0} \right) + \frac{1}{\beta} \hat{c}_0 \mu_0 \left(e^{-(L-\zeta)\mu_0} + e^{(L-\zeta)\mu_0} \right) - 2\hat{c}_0 \mu_0 e^{\zeta\mu_0} \\ &\approx -\hat{c}_0 \beta\mu_0 \left(e^{(L-\zeta)\mu_0} - e^{-(L-\zeta)\mu_0} \right) + \frac{1}{\beta} \hat{c}_0 \mu_0 \left(e^{-(L-\zeta)\mu_0} + e^{(L-\zeta)\mu_0} \right) - 2\hat{c}_0 \mu_0 e^{\zeta\mu_0}, \end{aligned}$$

$$\begin{aligned} \det(\Phi(\mu, \alpha, \beta)) &= -4\mu_0 + \left(\frac{1}{\beta} + \beta \right) \mu_0 \left(e^{\mu_0 L} + e^{-\mu_0 L} \right) + i\alpha \left(e^{\mu_0 L} - e^{-\mu_0 L} \right) \\ &\approx -4\mu_0 + \left(\frac{1}{\beta} + \beta \right) \mu_0 \left(e^{\mu_0 L} + e^{-\mu_0 L} \right). \end{aligned}$$

Thus,

$$\begin{aligned} |G(\lambda, x, \zeta)| &= \left| \epsilon^* \left[e^{\Omega(\mu)x} \overrightarrow{q}(\mu, \zeta) + H(x-\zeta) e^{\Omega(\mu)(x-\zeta)} \hat{c}(\mu) \right] \right| \\ &= \left| e^{\mu_0 x} r_0 / \det[\Phi(\mu, \alpha, \beta)] + e^{\mu_1 x} r_1 / \det[\Phi(\mu, \alpha, \beta)] + H(x-\zeta) \left(\hat{c}_0 e^{\mu_0 (x-\zeta)} + \hat{c}_1 e^{\mu_1 (x-\zeta)} \right) \right| \\ &\leq |\hat{c}_0| \left(\frac{|2\beta i \sin[(L-\zeta)\rho] + 2\beta^{-1} \cos[(L-\zeta)\rho]| + 2}{|-2 + (\beta^{-1} + \beta)\cos(\rho L)|} + 2 \right) \\ &\leq C_r |\hat{c}_0| \approx C_r \rho^{-1} \end{aligned}$$

as $\rho \to \infty$, for some constant C_r independent of ρ.

From above discussion, it is found that $|G(\lambda, x, \zeta)| \leq C_r \rho^{-1}$ for $\rho > r$. Because of $\omega = \rho^2$, (2.0.19) yields $\|R(i\omega, A)\| = O\left(\omega^{-\frac{1}{2}}\right)$, as $|\omega| \to \infty$. □

Corollary 2.0.5. *The semigroup $S(t)$ generated by the operator A is also a strongly differentiable semigroup on $L^2(0, L)$.*

Proof. From Lemma 2.0.4 we have

$$\begin{aligned}
&\lim_{|\omega|\to\infty} \sup \log|\omega| \|R(i\omega, A)\| \\
&= \lim_{|\omega|\to\infty} \sup \log|\omega| O\left(\omega^{-\frac{1}{2}}\right) \\
&= \lim_{|\omega|\to\infty} \frac{\sup \log|\omega|}{O\left(\omega^{\frac{1}{2}}\right)} \\
&= \lim_{|\omega|\to\infty} \frac{\sup |\omega|^{-1}}{O\left(\frac{1}{2}\omega^{-\frac{1}{2}}\right)} \\
&= \lim_{|\omega|\to\infty} \sup \frac{O\left(2\omega^{\frac{1}{2}}\right)}{|\omega|} \\
&= 0.
\end{aligned}$$

Then, $S(t)$ is a strongly differentiable semigroup by Cor.4.10 in [24]. □

The following gives the exponential decay of the semi-group for the linear Schrodinger operator (2.0.5) with boundary conditions given by (2.0.6).

Theorem 2.0.6. *There exists positive numbers ξ and η such that*

$$\|S(t)\| \leq \xi e^{-\eta t}, \qquad t \geq 0.$$

Proof. By Lemmas 2.0.3 and 2.0.4, it is obtained that for all λ on the imaginary axis, $R(\lambda, A)$ exists and is uniformly bounded for large λ. Thus, to derive the uniform exponential decay property of $S(t)$ using the result by Huang [10], we only need to prove that $R(\lambda, A)$ is bounded as $\lambda \to 0$. From (2.0.10), (2.0.12) and (2.0.15) it is deduced that

$$\begin{aligned}
\overrightarrow{u} &= M(\mu)\left[e^{x\Omega(\mu)}\Phi(\mu,\alpha,\beta)^{-1} M(\mu)\int_0^L e^{(L-x)\Omega(\mu)}\psi(x)\,dx + \int_0^x e^{(x-s)\Omega(\mu)}\overrightarrow{\psi}(s)\,ds\right] \\
&= M(\mu)\, e^{x\Omega(\mu)}\Phi(\mu,\alpha,\beta)^{-1}\int_0^L M(\mu)\, e^{(L-x)\Omega(\mu)} M(\mu)^{-1}\overrightarrow{\phi}(x)\,dx + \int_0^x M(\mu)\, e^{(x-s)\Omega(\mu)} M(\mu)^{-1}\overrightarrow{\phi}(s)\,ds \\
&= M(\mu)\, e^{x\Omega(\mu)} M(\mu)^{-1} M(\mu)\,\Phi(\mu,\alpha,\beta)^{-1}\int_0^L e^{(L-x)F(\lambda)}\overrightarrow{\phi}(x)\,dx + \int_0^x e^{(x-s)F(\lambda)}\overrightarrow{\phi}(s)\,ds \\
&= e^{xF(\lambda)} M(\mu)\,\Phi(\mu,\alpha,\beta)^{-1}\int_0^L e^{(L-x)F(\lambda)}\overrightarrow{\phi}(x)\,dx + \int_0^x e^{(x-s)F(\lambda)}\overrightarrow{\phi}(s)\,ds,
\end{aligned}$$

where

$$\begin{aligned} M(\mu)\Phi(\mu,\alpha,\beta)^{-1} &= \left[\Phi(\mu,\alpha,\beta)M(\mu)^{-1}\right]^{-1} \\ &= \left[\left(B(\alpha,\beta)M(\mu)-M(\mu)e^{L\Omega(\mu)}\right)M(\mu)^{-1}\right]^{-1} \\ &= \left[B(\alpha,\beta)M(\mu)M(\mu)^{-1}-M(\mu)e^{L\Omega(\mu)}M(\mu)^{-1}\right]^{-1} \\ &= \left[B(\alpha,\beta)-e^{LF(\lambda)}\right]^{-1}. \end{aligned} \tag{2.0.30}$$

Using Taylor expansion of $e^{LF(\lambda)}$ for small λ, we find

$$\begin{aligned} B(\alpha,\beta)-e^{LF(\lambda)} &= B(\alpha,\beta)-\left[I+LF(\lambda)+\frac{1}{2}L^2F(\lambda)^2+\frac{1}{6}L^3F(\lambda)^3+F(\lambda)G(\lambda)\right] \\ &= B(\alpha,\beta)-I-LF(\lambda)+\frac{1}{2}i\lambda L^2 I+\frac{1}{6}i\lambda L^3F(\lambda)+\lambda^2G(\lambda) \\ &= \begin{pmatrix} \frac{1}{\beta}-1+\frac{1}{2}i\lambda L^2 & \frac{1}{6}i\lambda L^3-L \\ \frac{1}{6}\lambda L\left(6i+\lambda L^2\right)-\alpha i & \beta-1+\frac{1}{2}i\lambda L^2 \end{pmatrix}+O\left(|\lambda|^2\right) \\ &= \begin{pmatrix} \frac{1}{\beta}-1 & -L \\ -\alpha i & \beta-1 \end{pmatrix}+O\left(|\lambda|^2\right) \end{aligned}$$

and

$$\det\begin{pmatrix} \frac{1}{\beta}-1 & -L \\ -\alpha i & \beta-1 \end{pmatrix}=\left(\frac{1}{\beta}-1\right)(\beta-1)-i\alpha L\neq 0.$$

Therefore, the matrix (2.0.30) is uniformly bounded in a neighborhood of $\lambda=0$ and $R(\lambda,A)$ is bounded as $\lambda\to 0$. □

Chapter 3

Spectral Analysis of Linear Schrödinger Operator and Function Spaces

In this chapter, we will discuss the spectral properties of A in (2.0.5)-(2.0.6) together with its semi-group $S(t)$. Since

$$\begin{aligned}
\int_0^L i\bar{u}''v dx &= -i\int_0^L v d\bar{u}' \\
&= -i\left(v\bar{u}'\Big|_0^L - \int_0^L \bar{u}' dv\right) \\
&= -i\left(v\bar{u}'\Big|_0^L - \int_0^L \bar{u}' v' dx\right) \\
&= -i\left(v\bar{u}'\Big|_0^L - \int_0^L v' d\bar{u}\right) \\
&= -i\left(v\bar{u}'\Big|_0^L - v'\bar{u}\Big|_0^L + \int_0^L \bar{u}v'' dx\right) \\
&= -i\left(v\bar{u}' - v'\bar{u}\right)\Big|_0^L + \int_0^L \bar{u}\left(-iv''\right) dx,
\end{aligned}$$

where

$$\begin{aligned}
\left(v\bar{u}' - v'\bar{u}\right)\Big|_0^L &= v(L)\,\bar{u}'(L) - v'(L)\,\bar{u}(L) - \left[v(0)\,\bar{u}'(0) - v'(0)\,\bar{u}(0)\right] \\
&= v(L)\left[\beta\bar{u}'(0) + i\alpha\bar{u}(0)\right] - v'(L)\,\bar{u}(L) - v(0)\,\bar{u}'(0) + v'(0)\,\beta\bar{u}(L) \\
&= \left[v(L)\,\beta - v(0)\right]\bar{u}'(0) + i\alpha\beta\bar{u}(L)\,v(L) - v'(L)\,\bar{u}(L) + v'(0)\,\beta\bar{u}(L) \\
&= \left[v(L)\,\beta - v(0)\right]\bar{u}'(0) + \left[i\alpha\beta v(L) - v'(L) + v'(0)\,\beta\right]\bar{u}(L),
\end{aligned}$$

it is obtained that the adjoint operator A^* of A in $L^2 = L^2(0, L)$ is

$$\left(A^*v\right)(x) = -iv''(x)$$

with domain

$$\mathscr{D}(A^*) = \left\{ v \in H^2(0,L) \;\middle|\; v(0) = \beta v(L), \quad \beta v'(0) - v'(L) = -i\alpha v(0) \right\}. \tag{3.0.1}$$

Proposition 3.0.1. *The operator A is a discrete spectral operator, all but a finite number of whose eigenvalues λ correspond to one-dimensional projections $E(\lambda;T)$.*

Proof. The results can be obtained from Theorem 8 of Dunford's book [5] on p.2334, if the hypotheses are verified. Using the notations there, we let

$$B_1(u) = u(0) - \beta u(L), \qquad B_2(u) = \beta u'(0) - u'(L) - i\alpha u(0) .$$

Also, $p = m_1 + m_2 = 0 + 1 = 1$ and $n = 2$ with $n = 2\nu$ and $\nu = 1$. $\omega_0 = 1, \omega_\nu = \omega_1 = -1$ and $\sigma_k(x,\mu) = e^{i\mu\omega_k x}$ with $k = 0, 1$. Then, $\sigma_0'(x,\mu) = i\mu e^{i\mu x}, \sigma_1'(x,\mu) = -i\mu e^{-i\mu x}$. Moreover,

$$M_{ik}(\mu) = B_i(\sigma_k(\mu)), \qquad i = 1,2, \quad k = 0,1,$$

and

$$\begin{aligned}
N(\mu) &= \det\begin{pmatrix} B_1(\sigma_0) & B_1(\sigma_1) \\ B_2(\sigma_0) & B_2(\sigma_1) \end{pmatrix} \\
&= \det\begin{pmatrix} \sigma_0(0) - \beta\sigma_0(L) & \sigma_1(0) - \beta\sigma_1(L) \\ \beta\sigma_0'(0) - \sigma_0'(L) - i\alpha\sigma_0(0) & \beta\sigma_1'(0) - \sigma_1'(L) - i\alpha\sigma_1(0) \end{pmatrix} \\
&= \det\begin{pmatrix} 1 - \beta e^{i\mu L} & 1 - \beta e^{-i\mu L} \\ i\mu\beta - i\mu e^{i\mu L} - i\alpha & -i\mu\beta + i\mu e^{-i\mu L} - i\alpha \end{pmatrix} \\
&= \left(1 - \beta e^{i\mu L}\right)\left(-i\mu\beta + i\mu e^{-i\mu L} - i\alpha\right) - \left(1 - \beta e^{-i\mu L}\right)\left(i\mu\beta - i\mu e^{i\mu L} - i\alpha\right) \\
&= -4i\mu\beta + i\mu e^{-i\mu L} + i\mu\beta^2 e^{i\mu L} + i\alpha\beta e^{i\mu L} + i\mu e^{i\mu L} + i\mu\beta^2 e^{-i\mu L} - i\alpha\beta e^{-i\mu L} \\
&= e^{i\mu L}\left[i\mu\left(\beta^2+1\right) + i\alpha\beta\right] + e^{-i\mu L}\left[i\mu\left(\beta^2+1\right) - i\alpha\beta\right] - 4i\mu\beta.
\end{aligned}$$

Thus, the coefficients are

$$\Pi_1(\mu) = i\mu\left(\beta^2+1\right) + i\alpha\beta, \quad \Pi_2(\mu) = i\mu\left(\beta^2+1\right) - i\alpha\beta, \quad \Pi_3(\mu) = -4i\mu\beta,$$

with the leading order parts in terms of μ as

$$a_p = i\left(\beta^2+1\right), \quad b_p = i\left(\beta^2+1\right), \quad c_p = -4i\beta .$$

It is obtained that $\kappa = a_p e^{i\alpha} = -b_p e^{-i\alpha}$. If $e^{i\alpha} = s$, then $s = \pm i$ and $\kappa = \mp(\beta^2+1)$, which implies that

$$\theta = \frac{c_p}{2i\kappa} = \frac{-4i\beta}{\mp 2i\left(\beta^2+1\right)} = \pm\frac{2\beta}{\beta^2+1} \neq \pm 1.$$

Hence, the hypotheses in Theorem 8 of Dunford's book [5] on p.2334 are satisfied. □

In the following, for $\psi \in L^2$, we denote ψ^* as the corresponding adjoint vector of ψ in the Hilbert space L^2, i.e., for any ϕ in L^2,

$$\psi^*\phi = (\phi, \psi)_{L^2} .$$

Proposition 3.0.2. *The operators A, A^* with the corresponding domains have compact resolvents and their eigenvectors,*

$$\{\phi_k| - \infty < k < +\infty\}, \quad \{\psi_k| - \infty < k < +\infty\},$$

satisfying

$$\psi_j^*\phi_k = \delta_{kj}$$

are complete and form dual Riesz bases in $L^2[0, L]$. The eigenvalues λ of A satisfy $Re\,\lambda \leq -\gamma < 0$, for some constant $\gamma > 0$, and have the asymptotic form

$$\lambda_k = \frac{-4\tau\pi}{L^2} - i\frac{(2k\pi + O(1))^2}{L^2}, \qquad k \to \infty, \tag{3.0.2}$$

where

$$\tau = \frac{-\alpha\beta L}{2\pi(\beta^2 + 1)} > 0 .$$

Proof. By Lemmas 2.0.1 and 2.0.3, it is obvious that any λ with $\mathrm{Re}\,\lambda \geq 0$ is not an eigenvalue of operator A or A^*. Thus, the eigenvalues must satisfy $\mathrm{Re}\,\lambda < 0$. For $\mathrm{Im}\,\lambda > 0$, the eigenfunction ϕ satisfies

$$\phi'' + i\lambda\phi = 0, \qquad \phi(0) = \beta\phi(L), \qquad \beta\phi'(0) - \phi'(L) = i\alpha\phi(0) . \tag{3.0.3}$$

The general solution of (3.0.3) is

$$\phi(x) = c_0 e^{\mu_0 x} + c_1 e^{\mu_1 x}. \tag{3.0.4}$$

Substituting (3.0.4) into the boundary conditions in (3.0.3), we have

$$\begin{aligned}
&c_0 + c_1 = \beta c_0 e^{\mu_0 L} + \beta c_1 e^{\mu_1 L},\\
&\beta\mu_0 c_0 + \beta\mu_1 c_1 - \left(\mu_0 c_0 e^{\mu_0 L} + \mu_1 c_1 e^{\mu_1 L}\right) = i\alpha(c_0 + c_1),
\end{aligned}$$

which give a system of equations

$$\begin{pmatrix} 1 - \beta e^{\mu_0 L} & 1 - \beta e^{\mu_1 L} \\ \beta\mu_0 - \mu_0 e^{\mu_0 L} - i\alpha & \beta\mu_1 - \mu_1 e^{\mu_1 L} - i\alpha \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = 0. \tag{3.0.5}$$

Setting the determinant of the above matrix equal to zero gives us

$$\begin{aligned}
&\left(1 - \beta e^{\mu_0 L}\right)\left(\beta\mu_1 - \mu_1 e^{\mu_1 L} - i\alpha\right) - \left(1 - \beta e^{\mu_1 L}\right)\left(\beta\mu_0 - \mu_0 e^{\mu_0 L} - i\alpha\right)\\
&\quad = \beta\mu_1 - \mu_1 e^{\mu_1 L} - i\alpha - \left(\beta^2 e^{\mu_0 L}\mu_1 - \beta e^{\mu_0 L}\mu_1 e^{\mu_1 L} - e^{\mu_0 L} i\alpha\beta\right) - \left(\beta\mu_0 - \mu_0 e^{\mu_0 L} - i\alpha\right)\\
&\qquad + \beta^2 e^{\mu_1 L}\mu_0 - \beta e^{\mu_1 L}\mu_0 e^{\mu_0 L} - e^{\mu_1 L} i\alpha\beta\\
&\quad = \beta\mu_1 - \mu_1 e^{\mu_1 L} - i\alpha - \beta^2 e^{\mu_0 L}\mu_1 + \beta e^{\mu_0 L}\mu_1 e^{\mu_1 L} + e^{\mu_0 L} i\alpha\beta - \beta\mu_0 + \mu_0 e^{\mu_0 L} + i\alpha\\
&\qquad + \beta^2 e^{\mu_1 L}\mu_0 - \beta e^{\mu_1 L}\mu_0 e^{\mu_0 L} - e^{\mu_1 L} i\alpha\beta
\end{aligned}$$

$$
\begin{aligned}
&= -2\beta\mu_0 + \mu_0 e^{-\mu_0 L} + \beta^2 e^{\mu_0 L}\mu_0 + e^{\mu_0 L} i\alpha\beta - 2\beta\mu_0 + \mu_0 e^{\mu_0 L} + \beta^2 e^{-\mu_0 L}\mu_0 - e^{-\mu_0 L} i\alpha\beta \\
&= -4\beta\mu_0 + e^{\mu_0 L}\left(\beta^2\mu_0 + i\alpha\beta + \mu_0\right) + e^{-\mu_0 L}\left(\beta^2\mu_0 + \mu_0 - i\alpha\beta\right) = 0. \qquad (3.0.6)
\end{aligned}
$$

If the real part of $\mu_0 \to \infty$, (3.0.6) implies $(\beta^2 + 1) = 0$, which does not hold. Thus, the real part of μ_0 must be bounded. Dividing (3.0.6) by μ_0 gives

$$
\begin{aligned}
0 &= \frac{-4\beta\mu_0}{\mu_0} + e^{\mu_0 L}\frac{\beta^2\mu_0 + i\alpha\beta + \mu_0}{\mu_0} + e^{-\mu_0 L}\frac{\beta^2\mu_0 + \mu_0 - i\alpha\beta}{\mu_0} \\
&= -4\beta + e^{\mu_0 L}\left[\left(\beta^2 + 1\right) + \frac{i\alpha\beta}{\mu_0}\right] + e^{-\mu_0 L}\left[\left(\beta^2 + 1\right) - \frac{i\alpha\beta}{\mu_0}\right] \\
&= -4\beta + e^{\mu_0 L}\left(\beta^2 + 1\right) + e^{-\mu_0 L}\left(\beta^2 + 1\right) + O(1/\mu_0) \quad \text{as} \quad |\mu_0| \to \infty. \qquad (3.0.7)
\end{aligned}
$$

Hence,

$$
4\beta = \left(e^{\mu_0 L} + e^{-\mu_0 L}\right)\left(\beta^2 + 1\right) + O\left(1/\mu_0\right) \Rightarrow \cosh(\mu_0 L) = \frac{2\beta}{\left(\beta^2 + 1\right)} + O\left(1/\mu_0\right).
$$

If $\mu_0 = a + bi$, then

$$
\cosh(aL)\cos(bL) = \frac{2\beta}{\beta^2 + 1} + O(1/\mu_0), \quad \sinh(aL)\sin(bL) = O(1/\mu_0).
$$

Since $\cosh x \geq 1$ and $|2\beta/(\beta^2 + 1)| < 1$ for $\beta \neq \pm 1$, $|\cos(bL)|$ cannot approach to one for large b or $\sin(bL)$ cannot go to zero. Thus, $\sinh(aL) = O(1/\mu_0), \cosh(aL) = O(1)$, and

$$
\cos(bL) = \frac{2\beta}{\beta^2 + 1} + O(1/\mu_0), \qquad \sin(bL) = \pm\frac{|\beta^2 - 1|}{\beta^2 + 1} + O(1/\mu_0).
$$

If $\beta > 0$, let $\theta = \sin^{-1}\frac{|\beta^2 - 1|}{\beta^2 + 1}$, and if $\beta < 0$, let $\theta = \pi - \sin^{-1}\frac{|\beta^2 - 1|}{\beta^2 + 1}$. Therefore, $bL = 2k\pi + \theta + O(1/\mu_0)$ and $\mu_0 L = i(2k\pi + \theta + O(1/\mu_0)) = i(2k\pi + \theta + \varepsilon_k)$. (3.0.6) gives

$$
\begin{aligned}
0 = &-4\beta i(2k\pi + \theta + \varepsilon_k)/L + (\beta^2 + 1)\left(e^{i(2k\pi + \theta + \varepsilon_k)} + e^{-i(2k\pi + \theta + \varepsilon_k)}\right) i(2k\pi + \theta + \varepsilon_k)/L \\
&+ i\alpha\beta\left(e^{i(2k\pi + \theta + \varepsilon_k)} - e^{-i(2k\pi + \theta + \varepsilon_k)}\right) \\
= &\; i\left((2k\pi + \theta + \varepsilon_k)/L\right)\left(-4\beta + 2(\beta^2 + 1)\cos(\theta + \varepsilon_k)\right) - 2\alpha\beta\sin(\theta + \varepsilon_k).
\end{aligned}
$$

Since

$$
\cos(\theta + \varepsilon_k) = \cos\theta\cos\varepsilon_k - \sin\theta\sin\varepsilon_k = \cos\theta - \varepsilon_k\sin\theta + O(\varepsilon_k^2)
$$

and $\sin(\theta + \varepsilon_k) = \sin\theta + O(\varepsilon_k)$, it is obtained that

$$
2(\beta^2 + 1)\varepsilon_k\sin\theta = -(\alpha\beta L/ik\pi)\sin\theta + O(1/k^2),
$$

where $\cos\theta = 2\beta/(\beta^2 + 1)$ has been used. Thus,

$$
\varepsilon_k = -\left((\alpha\beta L)/(2ik\pi(\beta^2 + 1))\right) + O(1/k^2) \quad \text{or} \quad \mu_0 L = i(2k\pi + \theta) + (\tau/k) + O(1/k^2),
$$

where $\tau = (-\alpha\beta L)/(2(\beta^2+1)\pi) > 0$. Hence,

$$\begin{aligned}\lambda_k = i\mu_0^2 &= i\left(\frac{\tau}{k} + i(2k\pi+\theta)\right)^2 L^{-2} + O(1/k)\\ &= i\left(4\tau i\pi - (2k\pi+\theta)^2\right)L^{-2} + O(1/k)\\ &= -\frac{4\tau\pi}{L^2} - \frac{i(2k\pi+\theta)^2}{L^2} + O\left(\frac{1}{k}\right).\end{aligned}$$

Using Rouché's theorem, the one to one relationship between the eigenvalues λ_k and the indices k with $k = 0, \pm1, \pm2, \ldots$ can be established. Therefore, there is a $\gamma > 0$ such that $\operatorname{Re}\lambda_k \le -\gamma < 0$. A similar argument gives that $\overline{\lambda_k}$, the complex conjugate of λ_k, is the eigenvalue of adjoint operator A^*. We see now that the eigenfunctions of A corresponding to λ_k take the form

$$\phi_k(x) = c_{0,k}e^{\mu_{0,k}x} + c_{1,k}e^{\mu_{1,k}x}.$$

(3.0.5) implies that

$$c_1 = -\frac{1-\beta e^{\mu_0 L}}{1-\beta e^{\mu_1 L}}c_0$$

and $c_{1,k}$ is uniformly bounded relative to $c_{0,k}$ as $|k| \to \infty$. The eigenfunctions $\psi_k(x)$ of adjoint operator A^* take the form

$$\psi_k(x) = \overline{\phi_k(x)}, \quad -\infty < k < \infty,$$

then

$$(\psi_k, \phi_j) = \int_0^L \psi_k\bar{\phi}_j dx = \int_0^L \bar{\phi}_k\bar{\phi}_j dx = \overline{\int_0^L \phi_k\phi_j dx}.$$

By the boundary conditions (2.0.6), it is obtained that

$$\begin{aligned}\int_0^L \phi_k''\phi_j dx &= \int_0^L \phi_j d\phi_k' = \phi_j\phi_k'|_0^L - \int_0^L \phi_k'\phi_j' dx\\ &= \phi_j\phi_k'|_0^L - \int_0^L \phi_j' d\phi_k = \phi_j\phi_k'|_0^L - \phi_j'\phi_k|_0^L + \int_0^L \phi_j''\phi_k dx\\ &= \phi_j(L)\phi_k'(L) - \phi_j(0)\phi_k'(0) - \phi_j'(L)\phi_k(L) + \phi_j'(0)\phi_k(0) + \int_0^L \phi_j''\phi_k dx\\ &= \phi_j(L)\phi_k'(L) - \beta\phi_j(L)\phi_k'(0) - \phi_j'(L)\phi_k(L) + \phi_j'(0)\beta\phi_k(L) + \int_0^L \phi_j''\phi_k dx\\ &= \phi_j(L)\left[\phi_k'(L) - \beta\phi_k'(0)\right] - \phi_k(L)\left[\phi_j'(L) - \beta\phi_j'(0)\right] + \int_0^L \phi_j''\phi_k dx\\ &= \phi_j(L)\left[-i\alpha\phi_k(0)\right] + \phi_k(L)\, i\alpha\phi_j(0) + \int_0^L \phi_j''\phi_k dx\\ &= -i\alpha\beta\phi_k(L)\phi_j(L) + i\alpha\phi_k(L)\beta\phi_j(L) + \int_0^L \phi_j''\phi_k dx\\ &= \int_0^L \phi_j''\phi_k dx.\end{aligned}$$

Hence,

$$\int_0^L \phi_k \phi_j dx = \frac{i}{\lambda_k} \int_0^L \phi_k'' \phi_j dx = \frac{i}{\lambda_k} \int_0^L \phi_k \phi_j'' dx = \frac{i}{\lambda_k} \int_0^L \phi_k \frac{\lambda_j}{i} \phi_j dx = \frac{\lambda_j}{\lambda_k} \int_0^L \phi_k \phi_j dx,$$

which implies

$$\left(1 - \frac{\lambda_j}{\lambda_k}\right) \int_0^L \phi_k \phi_j dx = 0.$$

When $j \neq k$ that implies $\lambda_k \neq \lambda_j$, we have that $\int_0^L \phi_k \phi_j dx = 0$ and $(\psi_k, \phi_j) = 0$. When $j = k$, an appropriate choice of the coefficients c_0 and c_1 makes $(\psi_k, \phi_j) = 1$. Therefore, we arrive at the formula

$$\psi_j^* \phi_k \equiv (\psi_k, \phi_j)_{L^2} = \delta_{kj} = \begin{cases} 0, & j \neq k, \\ 1, & j = k, \end{cases} \quad -\infty < k, j < \infty. \tag{3.0.8}$$

Now, we show that $\{\psi_k\}$ and $\{\phi_k\}$ form Riesz bases. Based on the Carleson theory [9], since both A and A^* are discrete spectral operators, the $\{\phi_k\}$ and $\{\psi_k\}$ sequences have the uniform l^2-convergence property, i.e., for any square-summable complex coefficient sequence $\{f_k\} \in l^2$ or $\{g_j\} \in l^2$, there is some number $D > 0$ independent of $\{f_k\}$ and $\{g_j\}$ such that

$$\left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 \leq D^2 \sum_{k=-\infty}^{\infty} |f_k|^2, \tag{3.0.9}$$

$$\left\| \sum_{j=-\infty}^{\infty} g_j \psi_j \right\|_{L^2}^2 \leq D^2 \sum_{j=-\infty}^{\infty} |g_j|^2. \tag{3.0.10}$$

Replacing g_j in inequality (3.0.10) by f_j,

$$\left\| \sum_{j=-\infty}^{\infty} f_j \psi_j \right\|^2 \leq D^2 \sum_{j=-\infty}^{\infty} |f_j|^2,$$

and applying (3.0.8) we have

$$\begin{aligned} \left(\sum_{k=-\infty}^{\infty} |f_k|^2 \right)^2 &= \left(\sum_{k=-\infty}^{\infty} f_k \phi_k, \sum_{j=-\infty}^{\infty} f_j \psi_j \right)_{L^2}^2 \\ &\leq \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 \left\| \sum_{j=-\infty}^{\infty} f_j \psi_j \right\|_{L^2}^2 \\ &\leq \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 D^2 \sum_{j=-\infty}^{\infty} |f_j|^2, \end{aligned}$$

which implies

$$D^{-2}\sum_{k=-\infty}^{\infty}|f_k|^2 \le \left\|\sum_{k=-\infty}^{\infty} f_k\phi_k\right\|_{L^2}^2. \tag{3.0.11}$$

A similar argument gives

$$D^{-2}\sum_{j=-\infty}^{\infty}|g_j|^2 \le \left\|\sum_{j=-\infty}^{\infty} g_j\psi_j\right\|_{L^2}^2. \tag{3.0.12}$$

From (3.0.11) and (3.0.12), it is shown that the sequences $\{\phi_k\}$ and $\{\psi_k\}$ also have the uniform l^2-independent property. Thus, from Proposition 3.0.1, we showed that both of $\{\phi_k\}$ and $\{\psi_k\}$ are complete in $L^2(0,L)$, which yields that $\{\phi_k\}$ and $\{\psi_k\}$ are two Riesz bases in L^2. □

Next, we derive the relations between Sobolev norms and the norms obtained from those Riesz bases. It has been shown that the eigenfunctions $\{\phi_k\}$ and $\{\psi_k\}$ possess the uniform l^2-convergence property (3.0.9)-(3.0.10) and l^2-independent property (3.0.11)-(3.0.12), we will extend them to the functions $\left\{\frac{\phi_k^{(n)}}{k^n}\right\}$ and $\left\{\frac{\psi_k^{(n)}}{k^n}\right\}$, $n\ge 1$.

Proposition 3.0.3. *There exists some number D_n such that*

$$\left\|\sum_{k=-\infty}^{\infty} f_k\phi_k^{(n)}\right\|_{L^2}^2 = \left\|\sum_{k=-\infty,k\neq 0}^{\infty} k^n f_k\frac{\phi_k^{(n)}}{k^n}\right\|_{L^2} \le D_n^2\sum_{k=-\infty}^{\infty}|k^n f_k|^2$$

for any complex sequence $\{f_k\}\in l_n^2$, where

$$l_n^2 = \{a_k : \sum_{k=-\infty}^{\infty}|k^n a_k|^2<\infty\},\quad n\ge 1.$$

Proof. In the proof of Proposition 3.0.2, it is known that Re $(\mu_{0,k})\approx\left(\frac{\tau}{kL}\right)$ is bounded, which gives that $\left|c_{0,k}e^{Re(\mu_{0,k})x}\right|, \left|c_{1,k}e^{Re(\mu_{1,k})x}\right|$ are uniformly bounded by some constant b_k with respect to k. Thus, by the Ingham-Komornik result in [13] and the forms of $\operatorname{Im}\mu_{0,k}, \operatorname{Im}\mu_{1,k}$ derived from Proposition 3.0.2, we have

$$\begin{aligned}
\left\|\sum_{k=-\infty}^{\infty} f_k\phi_k^{(n)}\right\|_{L^2}^2 &= \left\|\sum_{k=-\infty}^{\infty} f_k\left(c_{0,k}\mu_{0,k}^n e^{\mu_{0,k}x} + c_{1,k}\mu_{0,k}^n e^{\mu_{1,k}x}\right)\right\|_{L^2}^2\\
&\le \left(\left\|\sum_{k=-\infty}^{\infty} f_k k^n c_{0,k}e^{\operatorname{Re}(\mu_{0,k})x}e^{\operatorname{Im}(\mu_{0,k})x}\right\|_{L^2} + \left\|\sum_{k=-\infty}^{\infty} f_k k^n c_{1,k}e^{\operatorname{Re}(\mu_{1,k})x}e^{\operatorname{Im}(\mu_{1,k})x}\right\|_{L^2}\right)^2\\
&\le \left(\left\|\sum_{k=-\infty}^{\infty} f_k k^n b_k e^{\operatorname{Im}(\mu_{0,k})x}\right\|_{L^2} + \left\|\sum_{k=-\infty}^{\infty} f_k k^n b_k e^{\operatorname{Im}(\mu_{1,k})x}\right\|_{L^2}\right)^2\\
&\le D_n^2\sum_{k=-\infty}^{\infty}|k^n f_k|^2.
\end{aligned}$$

A similar proof works for

$$\left\| \sum_{j=-\infty}^{\infty} g_j \psi_j^{(n)} \right\|_{L^2}^2 = \left\| \sum_{j=-\infty, j\neq 0}^{\infty} j^n g_j \frac{\psi_j^{(n)}}{j^n} \right\| \leq D_n^2 \sum_{j=-\infty}^{\infty} |j^n g_j|^2, \quad -\infty < k < \infty,$$

for any $\{g_j\} \in l_n^2$. Thus, $\left\{\frac{\phi_k^{(n)}}{k^n}\right\}_{k=\infty}^{\infty}$ and $\left\{\frac{\psi_k^{(n)}}{k^n}\right\}_{k=\infty}^{\infty}$ are uniformly l_n^2-convergent in $L^2(0, L)$. □

Proposition 3.0.4. $\left\{\frac{\phi_k^{(n)}}{k^n}\right\}_{k=-\infty}^{\infty}$ *is also uniform* l_n^2*-independent in* $L^2(0, L)$ *for* $n \geq 1$*, i.e. there exists* $\tilde{D}_n$ *such that for any sequence of complex numbers* $\{f_k\} \in l_n^2$*,*

$$\left\| \sum_{k=-\infty, k\neq 0}^{\infty} k^n f_k \frac{\phi_k^{(n)}}{k^n} \right\|_{L^2}^2 + \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 \geq \tilde{D}_n^2 \sum_{k=-\infty}^{\infty} |k^n f_k|^2.$$

Proof. The case $n = 0$ was proved in Proposition 3.0.2. For $n = 1$, from the boundary conditions (3.0.1), we have an identity,

$$\begin{aligned}
-\left(\phi_k', \psi_j'\right) &= \int_0^L \phi_k'(x)\, \bar{\psi}_j'(x)\, dx = \int_0^L \phi_k'(x)\, d\bar{\psi}_j(x) \\
&= \phi_k'(x)\, \bar{\psi}_j(x)\, |_0^L - \int_0^L \phi_k''(x)\, \bar{\psi}_j(x)\, dx \\
&= \phi_k'(L)\, \bar{\psi}_j(L) - \phi_k'(0)\, \bar{\psi}_j(0) - \int_0^L \phi_k''(x)\, \bar{\psi}_j(x)\, dx \\
&= [\beta\phi_k'(0) - i\alpha\phi_k(0)]\, \bar{\psi}_j(L) - \phi_k'(0)\, \beta\bar{\psi}_j(L) - \int_0^L \phi_k''(x)\, \bar{\psi}_j(x)\, dx \\
&= -i\alpha\phi_k(0)\, \bar{\psi}_j(L) + i\int_0^L i\phi_k''(x)\, \bar{\psi}_j(x)\, dx \\
&= -i\alpha\phi_k(0)\, \bar{\psi}_j(L) + i\left(i\phi_k'', \psi_j\right) \\
&= -i\alpha\phi_k(0)\, \bar{\psi}_j(L) + i\left(A\phi_k, \psi_j\right) \\
&= -i\alpha\phi_k(0)\, \bar{\psi}_j(L) + i\left(\lambda_k\phi_k, \psi_j\right) \\
&= -i\alpha\phi_k(0)\, \bar{\psi}_j(L) + i\lambda_k\delta_{k,j},
\end{aligned}$$

which implies that

$$\begin{aligned}
\delta_{k,j} &= \frac{i\left(\phi_k', \psi_j'\right) + \alpha\phi_k(0)\, \bar{\psi}_j(L)}{\lambda_k} \\
&= i\frac{kj}{\lambda_k}\left(\frac{\phi_k'}{k}, \frac{\psi_j'}{j}\right) + \alpha\frac{kj}{\lambda_k}\frac{\phi_k(0)}{k}\frac{\bar{\psi}_j(L)}{j} \\
&= i\frac{j^2}{\lambda_j}\left(\frac{\phi_k'}{k}, \frac{\psi_j'}{j}\right) + \alpha\frac{j^2}{\lambda_j}\frac{\phi_k(0)}{k}\frac{\bar{\psi}_j(L)}{j}.
\end{aligned}$$

From (3.0.2) we have the magnitude of λ_k is proportional to k^2 as $k \to \infty$. Then, by the Sobolev embedding theorem, it is obtained that

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} |kf_k|^2 &= \left(\sum_{k=-\infty}^{\infty} kf_k\phi_k, \sum_{j=-\infty}^{\infty} jf_j\psi_j \right) = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} kjf_k\bar{f}_j\delta_{k,j} \\
&= \sum_{k=-\infty,k\neq 0}^{\infty} \sum_{j=-\infty,j\neq 0}^{\infty} kjf_k\bar{f}_j \left[i\frac{j^2}{\lambda_j} \left(\frac{\phi_k'}{k}, \frac{\psi_j'}{j} \right) + \alpha \frac{j^2}{\lambda_j} \frac{\phi_k(0)}{k} \frac{\bar{\psi}_j(L)}{j} \right] \\
&= \sum_{k=-\infty,k\neq 0}^{\infty} \sum_{j=-\infty,j\neq 0}^{\infty} kjf_k\bar{f}_j i\frac{j^2}{\lambda_j} \left(\frac{\phi_k'}{k}, \frac{\psi_j'}{j} \right) + \sum_{k=-\infty,k\neq 0}^{\infty} \sum_{j=-\infty,j\neq 0}^{\infty} kjf_k\bar{f}_j \alpha \frac{j^2}{\lambda_j} \frac{\phi_k(0)}{k} \frac{\bar{\psi}_j(L)}{j} \\
&= \left(i \sum_{k=-\infty,k\neq 0}^{\infty} kf_k\frac{\phi_k'}{k}, \sum_{j=-\infty,j\neq 0}^{\infty} jf_j\frac{j^2}{\bar{\lambda}_j}\frac{\psi_j'}{j} \right) + \sum_{k=-\infty,k\neq 0}^{\infty} kf_k\frac{\phi_k(0)}{k} \sum_{j=-\infty,j\neq 0}^{\infty} j\bar{f}_j\alpha\frac{j^2}{\lambda_j}\frac{\bar{\psi}_j(L)}{j} \\
&\le \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k' \right\|_{L^2} \left\| \sum_{j=-\infty}^{\infty} f_j\frac{j^2}{\bar{\lambda}_j}\psi_j' \right\|_{L^2} + |\alpha| \left| \sum_{k=-\infty}^{\infty} f_k\phi_k(0) \right| \left| \sum_{j=-\infty}^{\infty} \bar{f}_j\frac{j^2}{\lambda_j}\bar{\psi}_j(L) \right| \\
&\le \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k' \right\|_{L^2} \left\| \sum_{j=-\infty}^{\infty} f_j\frac{j^2}{\bar{\lambda}_j}\psi_j' \right\|_{L^2} \\
&\quad + 2|\alpha| \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k \right\|_{L^2}^{\frac{1}{2}} \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k' \right\|_{L^2}^{\frac{1}{2}} \left\| \sum_{j=-\infty}^{\infty} f_j\psi_j\frac{j^2}{\lambda_j} \right\|_{L^2}^{\frac{1}{2}} \left\| \sum_{j=-\infty}^{\infty} f_j\psi_j'\frac{j^2}{\lambda_j} \right\|_{L^2}^{\frac{1}{2}} \\
&\le \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k' \right\|_{L^2} \left\| \sum_{j=-\infty}^{\infty} f_j\frac{j^2}{\bar{\lambda}_j}\psi_j' \right\|_{L^2} \\
&\quad + |\alpha| \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k \right\|_{L^2} \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k' \right\|_{L^2} + |\alpha| \left\| \sum_{j=-\infty}^{\infty} f_j\psi_j\frac{j^2}{\lambda_j} \right\|_{L^2} \left\| \sum_{j=-\infty}^{\infty} f_j\psi_j'\frac{j^2}{\lambda_j} \right\|_{L^2} \\
&\le D_1 \left(\sum_{k=-\infty}^{\infty} |kf_k|^2 \right)^{\frac{1}{2}} \left\{ \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k' \right\|_{L^2} + |\alpha| \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k \right\|_{L^2} + |\alpha| \left\| \sum_{j=-\infty}^{\infty} f_j\psi_j\frac{j^2}{\lambda_j} \right\|_{L^2} \right\}
\end{aligned}
$$

which implies

$$
\begin{aligned}
\frac{1}{D_1^2} \sum_{k=-\infty}^{\infty} |kf_k|^2 &\le \left(\left\| \sum_{k=-\infty}^{\infty} f_k\phi_k' \right\|_{L^2} + |\alpha| \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k \right\|_{L^2} + |\alpha| \left\| \sum_{j=-\infty}^{\infty} f_j\psi_j\frac{j^2}{\lambda_j} \right\|_{L^2} \right)^2 \\
&\le (1+2|\alpha|) \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k' \right\|_{L^2}^2 + |\alpha|(1+2|\alpha|) \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k \right\|_{L^2}^2 + |\alpha|(1+2|\alpha|) \left\| \sum_{j=-\infty}^{\infty} f_j\psi_j\frac{j^2}{\lambda_j} \right\|_{L^2}^2 \\
&\le (1+2|\alpha|) \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k' \right\|_{L^2}^2 + |\alpha|(1+2|\alpha|) \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k \right\|_{L^2}^2 + |\alpha|(1+2|\alpha|) D^2 \sum_{k=-\infty}^{\infty} |f_k|^2,
\end{aligned}
$$

and

$$\frac{1}{D_1^2}\sum_{k=-\infty}^{\infty}|kf_k|^2-|\alpha|\left(1+2|\alpha|\right)D^2\sum_{k=-\infty}^{\infty}|f_k|^2\leq\left(1+2|\alpha|\right)\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k'\right\|_{L^2}^2+|\alpha|\left(1+2|\alpha|\right)\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k\right\|_{L^2}^2.$$

Using (3.0.11) we have

$$\begin{aligned}\frac{1}{D_1^2}\sum_{k=-\infty}^{\infty}|kf_k|^2-|\alpha|\left(1+2|\alpha|\right)D^4\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k\right\|_{L^2}^2&\\ \leq\left(1+2|\alpha|\right)\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k'\right\|_{L^2}^2+|\alpha|\left(1+2|\alpha|\right)\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k\right\|_{L^2}^2&\\ \Rightarrow\frac{1}{D_1^2}\sum_{k=-\infty}^{\infty}|kf_k|^2\leq\left(1+2|\alpha|\right)\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k'\right\|_{L^2}^2+|\alpha|\left(1+2|\alpha|\right)\left(1+D^4\right)\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k\right\|_{L^2}^2,&\end{aligned}$$

thus

$$\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k'\right\|_{L^2}^2+\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k\right\|_{L^2}^2\geq\tilde{D}_1^2\sum_{k=-\infty}^{\infty}|kf_k|^2.$$

For $n=2$, since

$$i\phi''=\lambda\phi\ \Rightarrow\ \sum_{k=-\infty}^{\infty}f_ki\phi_k''=\sum_{k=-\infty}^{\infty}f_k\lambda_k\phi_k,$$

it is deduced that

$$\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k''\right\|_{L^2}^2=\left\|\sum_{k=-\infty}^{\infty}\lambda_kf_k\phi_k\right\|_{L^2}^2\geq\frac{1}{D_2^2}\sum_{k=-\infty}^{\infty}|\lambda_kf_k|^2=\frac{1}{D_2^2}\sum_{k=-\infty}^{\infty}|k^2f_k|^2,$$

and

$$\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k''\right\|_{L^2}^2+\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k\right\|_{L^2}^2\geq\tilde{D}_2^2\sum_{k=-\infty}^{\infty}|k^2f_k|^2.$$

For $n=3$,

$$i\phi'''=\lambda\phi'\ \Rightarrow\ \sum_{k=-\infty}^{\infty}f_ki\phi_k'''=\sum_{k=-\infty}^{\infty}f_k\lambda_k\phi_k'\ \Rightarrow\ \left\|\sum_{k=-\infty}^{\infty}f_k\phi_k'''\right\|_{L^2}^2=\left\|\sum_{k=-\infty}^{\infty}\lambda_kf_k\phi_k'\right\|_{L^2}^2.$$

Then,

$$\begin{aligned}\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k'''\right\|_{L^2}^2+\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k\right\|_{L^2}^2&=\left\|\sum_{k=-\infty}^{\infty}\lambda_kf_k\phi_k'\right\|_{L^2}^2+\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k\right\|_{L^2}^2\\ &\geq\tilde{D}_1^2\sum_{k=-\infty}^{\infty}|kf_k\lambda_k|^2-\left\|\sum_{k=-\infty}^{\infty}f_k\lambda_k\phi_k\right\|_{L^2}^2+\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k\right\|_{L^2}^2\\ &\geq\tilde{D}_1^2\sum_{k=-\infty}^{\infty}|kf_k\lambda_k|^2-D^2\sum_{k=-\infty}^{\infty}|f_k\lambda_k|^2+\left\|\sum_{k=-\infty}^{\infty}f_k\phi_k\right\|_{L^2}^2\end{aligned}$$

$$= \tilde{D}_1^2 \sum_{k=-\infty}^{\infty} |k f_k \lambda_k|^2 - D^2 \sum_{k=-\infty}^{\infty} |f_k k^2|^2 + \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2$$
$$\geq \tilde{D}_1^2 \sum_{k=-\infty}^{\infty} |k f_k \lambda_k|^2 + D^2 \left[-\frac{1}{\tilde{D}_2^2} \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k'' \right\|_{L^2}^2 - \frac{1}{\tilde{D}_2^2} \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 \right] + \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2$$
$$= \tilde{D}_1^2 \sum_{k=-\infty}^{\infty} |k f_k \lambda_k|^2 - \frac{D^2}{\tilde{D}_2^2} \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k'' \right\|_{L^2}^2 - \frac{D^2}{\tilde{D}_2^2} \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 + \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2,$$

which implies

$$\left\| \sum_{k=-\infty}^{\infty} f_k \phi_k''' \right\|_{L^2}^2 + \frac{D^2}{\tilde{D}_2^2} \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k'' \right\|_{L^2}^2 + \frac{D^2}{\tilde{D}_2^2} \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 \geq \tilde{D}_1^2 \sum_{k=-\infty}^{\infty} |k f_k \lambda_k|^2.$$

Hence, by the Sobolev interpolation inequality

$$c(\epsilon) \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 + \epsilon \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k''' \right\|_{L^2}^2 \geq \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k'' \right\|_{L^2}^2$$

for any small $\epsilon > 0$, it is obtained that

$$\left\| \sum_{k=-\infty}^{\infty} f_k \phi_k''' \right\|_{L^2}^2 + c(\epsilon) \frac{D^2}{\tilde{D}_2^2} \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 + \epsilon \frac{D^2}{\tilde{D}_2^2} \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k''' \right\|_{L^2}^2 + \frac{D^2}{\tilde{D}_2^2} \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 \geq \tilde{D}_1^2 \sum_{k=-\infty}^{\infty} |k f_k \lambda_k|^2$$
$$\Rightarrow \left(1 + \epsilon \frac{D^2}{\tilde{D}_2^2}\right) \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k''' \right\|_{L^2}^2 + (1 + c(\epsilon)) \frac{D^2}{\tilde{D}_2^2} \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 \geq \tilde{D}_1^2 \sum_{k=-\infty}^{\infty} |k f_k \lambda_k|^2$$
$$\Rightarrow \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k''' \right\|_{L^2}^2 + \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 \geq \tilde{D}_3^2 \sum_{k=-\infty}^{\infty} |k^3 f_k|^2.$$

The cases for $n \geq 4$ can be handled similarly. The same proof works for $\{\psi_k^{(n)}\}$ with $n \geq 1$, thus we also have

$$\left\| \sum_{j=-\infty, j\neq 0}^{\infty} j^n g_j \frac{\psi_j^{(n)}}{j^n} \right\|_{L^2}^2 + \left\| \sum_{j=-\infty}^{\infty} g_j \psi_j \right\|_{L^2}^2 \geq \tilde{D}_n^2 \sum_{j=-\infty}^{\infty} |j^n g_j|^2.$$

□

Now, we can define the spaces to be used for the nonlinear problem.

Definition 3.0.5. *We define the Hilbert space*

$$H_{\alpha,\beta}^n = \left\{ w \in H^n[0, L] \;\middle|\; w^{(2j)}(0) = \beta w^{(2j)}(L),\ \beta w^{(2j+1)}(0) - w^{(2j+1)}(L) = i\alpha w^{(2j)}(0) \right\}$$

with $2j + 1 \leq n - 1$. The norm in $H_{\alpha,\beta}^n$ is $\| \, \|_n$, inherited from $H^n[0, L]$.

Corollary 3.0.6. *Let* $w = \sum_{k=-\infty}^{\infty} c_k\phi_k \in L^2$, *then* $w \in H^n_{\alpha,\beta}$ *if and only if* $\sum_{k=-\infty}^{\infty} |k^n c_k|^2 < \infty$. *In addition,*

$$\|w\|^2_{H^n_{\alpha,\beta}} \approx \sum_{k=-\infty}^{\infty} \left[\left(1+|k|^{2n}\right)|c_k|^2\right]. \tag{3.0.13}$$

Proof. It is easy to get the sufficient and necessary conditions from Propositions 3.0.3 and 3.0.4. In order to prove (3.0.13), we need to find two positive numbers denoted by e_n, E_n (may depend on α, β) such that, uniformly for $w \in H^n_{\alpha,\beta}$,

$$e_n^2 \sum_{k=-\infty}^{\infty} \left[|c_k|^2 + |k^n c_k|^2\right] \le \|w\|^2_{H^n_{\alpha,\beta}} \le E_n^2 \sum_{k=-\infty}^{\infty} \left(|c_k|^2 + |k^n c_k|^2\right).$$

Since $\|f^{(m)}\|^2_{L^2} \le c(\epsilon)\|f\|^2_{L^2} + \epsilon\|f^{(n)}\|^2_{L^2}$, for $m = 1, 2, 3, \cdots, n-1$, Propositions 3.0.2 and 3.0.3 imply that

$$\begin{aligned}
\|w\|^2_{H^n_{\alpha,\beta}} &= \sum_{j=0}^{n} \|w^{(j)}\|^2_{L^2} = \sum_{j=0}^{n} \left\| \sum_{k=-\infty}^{\infty} c_k \phi_k^{(j)} \right\|^2_{L^2} \\
&\le c_1 \left\| \sum_{k=-\infty}^{\infty} c_k\phi_k \right\|^2_{L^2} + c_2 \left\| \sum_{k=-\infty}^{\infty} c_k \phi_k^{(n)} \right\|^2_{L^2} \\
&\le E_n^2 \sum_{k=-\infty}^{\infty} |c_k|^2 + E_n^2 \sum_{k=-\infty}^{\infty} |k^n c_k|^2.
\end{aligned}$$

From Propositions 3.0.2 and 3.0.4, we have

$$\begin{aligned}
2\left\| \sum_{k=-\infty}^{\infty} f_k \phi_k^{(n)} \right\|^2_{L^2} + 2\left\| \sum_{k=-\infty}^{\infty} f_k\phi_k \right\|^2_{L^2} &\ge \tilde{D}_n^2 \sum_{k=-\infty}^{\infty} |k^n f_k|^2 + \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k \right\|^2_{L^2} \\
&\ge \tilde{D}_n^2 \sum_{k=-\infty}^{\infty} |k^n f_k|^2 + D^{-2} \sum_{k=-\infty}^{\infty} |f_k|^2,
\end{aligned}$$

which implies

$$\left\| \sum_{k=-\infty}^{\infty} f_k \phi_k^{(n)} \right\|^2_{L^2} + \left\| \sum_{k=-\infty}^{\infty} f_k\phi_k \right\|^2_{L^2} \ge e_n^2 \sum_{k=-\infty}^{\infty} \left(|k^n f_k|^2 + |f_k|^2\right).$$

Thus,

$$\begin{aligned}
\|w\|^2_{H^n_{\alpha,\beta}} &= \sum_{j=0}^{n} \left\| \sum_{k=-\infty}^{\infty} c_k \phi_k^{(j)} \right\|^2_{L^2} \ge \left\| \sum_{k=-\infty}^{\infty} c_k\phi_k \right\|^2_{L^2} + \left\| \sum_{k=-\infty}^{\infty} c_k \phi_k^{(n)} \right\|^2_{L^2} \\
&\ge e_n^2 \sum_{k=-\infty}^{\infty} \left(|c_k|^2 + |k^n c_k|^2\right).
\end{aligned}$$

The proof is completed. $\square$

Let $\{\phi_k(x)\}$ be the eigenvectors of operator A defined in Proposition 3.0.2. By Corollary 3.0.6, we define a class of Banach spaces

$$H^{s,p}_{\alpha,\beta} = \left\{ w = \sum_{k=-\infty}^{\infty} c_k \phi_k ; \sum_{k=-\infty}^{\infty} \left(1 + |k|^{ps}\right) |c_k|^p < \infty \right\}$$

with norm

$$\|w\|^p_{H^{s,p}_{\alpha,\beta}} = \sum_{k=-\infty}^{\infty} \left(1 + |k|^{ps}\right) |c_k|^p$$

for any $s \geq 0$ and $p \geq 1$. This space will be used to study some important properties of the Schrödinger operator discussed. Because of the space ℓ^p is continuously imbedded into ℓ^q, denoted as $\ell^p \hookrightarrow \ell^q$, for any $q > p \geq 2$, it is easy to show that

$$\begin{aligned} H^{s,p}_{\alpha,\beta} &\hookrightarrow H^{s',p}_{\alpha,\beta}, \quad s' < s, \\ H^{s,p}_{\alpha,\beta} &\hookrightarrow H^{s,q}_{\alpha,\beta}, \quad q > p \geq 2\,. \end{aligned}$$

If $p = 2$, we denote $H^{s,p}_{\alpha,\beta}$ by $H^s_{\alpha,\beta}$. $H^s_{\alpha,\beta}$ is an interpolation space when s is not an integer, hence $H^s_{\alpha,\beta}$ is a subspace of H^s for any $s \geq 0$. In the next three chapters, we denote $\|\cdot\|_s$ as the norm of $H^s_{\alpha,\beta}$. Moreover, if $s = n$ is an integer and $p = 2$, the space $H^{s,p}_{\alpha,\beta}$ is same as $H^n_{\alpha,\beta}$ defined by Definition 3.0.5.

Chapter 4

Properties of Semi-groups Generated by Linear Schrödinger Operators

From the discussion of Chapter 3, if

$$P_k = \phi_k \psi_k^* : \; L^2 \to L^2, \; -\infty < k < \infty,$$

where $\{\phi_k\}$ and $\{\psi_k\}$ are the eigenvectors of operators A and its adjoint A^*, then we can obtain that the resolution of the identity associated with the operator A is

$$I = \sum_{k=-\infty}^{\infty} P_k,$$

which is strongly convergent in $\mathscr{L}(L^2, L^2)$. The corresponding strongly convergent semi-group generated by A is

$$S(t) = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} \phi_k \psi_k^* = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} P_k.$$

Hence, the solution of the nonhomogeneous problem

$$\begin{cases} u_t - iu_{xx} = f, \qquad x \in (0, L), \; t \geq 0, \\ u(x, 0) = u_0(x), \\ u(0, t) = \beta u(L, t), \; \beta u_x(0, t) - u_x(L, t) = i\alpha u(0, t), \qquad L > 0, \end{cases} \tag{4.0.1}$$

is given by

$$u(t) = S(t) u_0(x) + \int_0^t S(t - \tau) f(\cdot, \tau) \, d\tau.$$

Notice that

$$u(t) = S(t) u_0(x)$$

is the solution of the initial value problem

$$u_t = Au, \quad u(x,0) = u_0(x), \quad u \in \mathscr{D}(A),$$

and $u(t) = \int_0^t S(t-\tau) f(\cdot,\tau)\, d\tau$ is the solution of the system (4.0.1) with $u_0(x) = 0$.

From Proposition 3.0.2 we have there exists some positive number $\gamma > 0$ such that $\operatorname{Re}\lambda_k \leq -\gamma < 0$. We will use this fact to prove the following propositions in this chapter.

Proposition 4.0.1. *For any given $s \geq 0$ and $T > 0$. If $w_0 \in H^s_{\alpha,\beta}$,*

$$\|S(t) u_0\|_s \leq e^{-\gamma t}\|u_0\|_s, \quad t \geq 0. \tag{4.0.2}$$

$$\int_0^T \|S(t) u_0\|_s^2 dt \leq (2\gamma)^{-1} \|u_0\|_s^2.$$

Proof. If $u_0 = \sum\limits_{k=-\infty}^{\infty} c_k \phi_k$, then

$$S(t) u_0 = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} c_k \phi_k. \tag{4.0.3}$$

Since $\operatorname{Re}\lambda_k \leq -\gamma < 0$, the definition of $H^s_{\alpha,\beta}$ gives

$$\|u_0\|_s^2 = \sum_{k=-\infty}^{\infty} |c_k|^2 \left(1 + |k|^{2s}\right),$$

and

$$\begin{aligned}
\|S(t) u_0\|_s^2 &= \sum_{k=-\infty}^{\infty} |c_k|^2 \left(1 + |k|^{2s}\right) |e^{\lambda_k t}|^2 \\
&= \sum_{k=-\infty}^{\infty} |c_k|^2 \left(1 + |k|^{2s}\right) e^{2\operatorname{Re}\lambda_k t} \\
&\leq e^{-2\gamma t} \sum_{k=-\infty}^{\infty} |c_k|^2 \left(1 + |k|^{2s}\right) \\
&= e^{-2\gamma t}\|u_0\|_s^2.
\end{aligned}$$

From (4.0.2) we have

$$\begin{aligned}
\int_0^T \|S(t) u_0\|_s^2 dt &\leq \int_0^T e^{-2\gamma t}\|u_0\|_s^2 dt \\
&= \|u_0\|_s^2 \frac{1}{2\gamma}\left(1 - e^{-2\gamma T}\right) \\
&\leq \frac{1}{2\gamma}\|u_0\|_s^2.
\end{aligned}$$

□

The following propositions will be used for the estimates of solutions corresponding to the nonhomogeneous terms.

Proposition 4.0.2. *For any given $s \geq 0$ and $T > 0$. If $f \in L^2\left(0, T; H^s_{\alpha,\beta}\right)$,*

$$\sup_{0\leq t\leq T}\left\|\int_0^t S\left(t-\tau\right) f\left(x,\tau\right) d\tau\right\|_s \leq (2\gamma)^{-1/2}\left(\int_0^T \|f\left(x,\tau\right)\|_s^2 d\tau\right)^{\frac{1}{2}}.$$

Proof. Since

$$f\left(x,t\right) = \sum_{k=-\infty}^{\infty} f_k\left(t\right)\phi_k\left(x\right), \tag{4.0.4}$$

we have

$$\begin{aligned}
\sup_{0\leq t\leq T}\left\|\int_0^t S\left(t-\tau\right) f\left(x,\tau\right) d\tau\right\|_s^2 &= \sup_{0\leq t\leq T}\left\|\int_0^t \sum_{k=-\infty}^{\infty} e^{\lambda_k(t-\tau)} f_k\left(\tau\right)\phi_k\left(x\right) d\tau\right\|_s^2 \\
&= \sup_{0\leq t\leq T}\left\|\sum_{k=-\infty}^{\infty}\left(\int_0^t e^{\lambda_k(t-\tau)} f_k\left(\tau\right) d\tau \phi_k\left(x\right)\right)\right\|_s^2 \\
&= \sup_{0\leq t\leq T}\sum_{k=-\infty}^{\infty}\left(1+|k|^{2s}\right)\left|\int_0^t e^{\lambda_k(t-\tau)} f_k\left(\tau\right) d\tau\right|^2 \\
&\leq \sum_{k=-\infty}^{\infty}\sup_{0\leq t\leq T}\left(\int_0^t |e^{\lambda_k(t-\tau)}||f_k\left(\tau\right)| d\tau\right)^2\left(1+|k|^{2s}\right) \\
&\leq \sum_{k=-\infty}^{\infty}\sup_{0\leq t\leq T}\left(\int_0^t |e^{\lambda_k(t-\tau)}|^2 d\tau \int_0^t |f_k\left(\tau\right)|^2 d\tau\right)\left(1+|k|^{2s}\right) \\
&= \sum_{k=-\infty}^{\infty}\sup_{0\leq t\leq T}\left(\int_0^t e^{2\mathrm{Re}\,\lambda_k(t-\tau)} d\tau \int_0^t |f_k\left(\tau\right)|^2 d\tau\right)\left(1+|k|^{2s}\right) \\
&\leq \sum_{k=-\infty}^{\infty}\sup_{0\leq t\leq T}\left(\int_0^t e^{-2\gamma(t-\tau)} d\tau \int_0^t |f_k\left(\tau\right)|^2 d\tau\right)\left(1+|k|^{2s}\right) \\
&= \sum_{k=-\infty}^{\infty}\sup_{0\leq t\leq T}\frac{1}{2\gamma}\left(1-e^{-2\gamma t}\right)\int_0^t |f_k\left(\tau\right)|^2 d\tau\left(1+|k|^{2s}\right) \\
&\leq \sum_{k=-\infty}^{\infty}\sup_{0\leq t\leq T}\frac{1}{2\gamma}\int_0^t |f_k\left(\tau\right)|^2 d\tau\left(1+|k|^{2s}\right) \\
&\leq \frac{1}{2\gamma}\sum_{k=-\infty}^{\infty}\int_0^T |f_k\left(\tau\right)|^2 d\tau\left(1+|k|^{2s}\right) \\
&= \frac{1}{2\gamma}\int_0^T\sum_{k=-\infty}^{\infty}|f_k\left(\tau\right)|^2\left(1+|k|^{2s}\right) d\tau = (2\gamma)^{-1/2}\int_0^T \|f\left(x,\tau\right)\|_s^2 d\tau.
\end{aligned}$$

□

Proposition 4.0.3. *If $f \in L^{\infty}\left(0, \infty; H^s_{\alpha,\beta}\right)$, then there exists a number $B_s > 0$ such that*

$$\sup_{0 \le t < \infty} \left\| \int_0^t S(t-\tau) f(x,\tau)\, d\tau \right\|_s \le B_s \sup_{0 \le t < \infty} \| f(x,t) \|_s$$

for any $s \ge 0$.

Proof. Define $\hat{t} = \max\{t-1, 0\}$. Then,

$$\begin{aligned} \int_0^t S(t-\tau) f(\tau)\, d\tau &= \int_{\hat{t}}^t S(t-\tau) f(\tau)\, d\tau + \int_0^{\hat{t}} S(t-\tau) f(\tau)\, d\tau \\ &= \hat{h}(\cdot, t) + h(\cdot, t). \end{aligned} \tag{4.0.5}$$

By Proposition 4.0.2 and $t - \hat{t} \le 1$, there is a $B > 0$ such that

$$\begin{aligned} \|\hat{h}(\cdot,t)\|_s^2 &\le B^2 \int_{\hat{t}}^t \|f(\cdot,\tau)\|_s^2 d\tau \\ &\le B^2 \sup_{0 \le t \le \infty} \|f(\cdot,t)\|_s^2 \int_{\hat{t}}^t 1 d\tau \\ &= B^2 \sup_{0 \le t \le \infty} \|f(\cdot,t)\|_s^2 \left(t - \hat{t}\right) \\ &\le B^2 \sup_{0 \le t \le \infty} \|f(\cdot,t)\|_s^2. \end{aligned} \tag{4.0.6}$$

If $t \le 1$, i.e. $\hat{t} = 0$, we have $h(\cdot,t) = \int_0^{\hat{t}} S(t-\tau) f(\cdot,\tau)\, d\tau = 0$. Thus, from (4.0.5)-(4.0.6) the proof is completed. If $t \ge 1$, i.e. $\hat{t} = t-1$, by (4.0.4),

$$\begin{aligned} h(x,t) &= \sum_{j=-\infty}^{\infty} \int_0^{t-1} S(t-\tau) f(x,\tau)\, d\tau \\ &= \sum_{j=-\infty}^{\infty} \int_0^{t-1} e^{\lambda_j (t-\tau)} f_j(\tau)\, d\tau \phi_j(x). \end{aligned}$$

When $\tau \in [0, t-1]$, it is obtained that

$$\begin{aligned} \sum_{j=-\infty}^{\infty} \left(1 + |j|^{2s}\right) \left| e^{\lambda_j (t-\tau)} f_j^2(\tau) \right| &= \sum_{j=-\infty}^{\infty} \left(1 + |j|^{2s}\right) \left| e^{\lambda_j (t-\tau)} \right| |f_j(\tau)|^2 \\ &= \sum_{j=-\infty}^{\infty} \left(1 + |j|^{2s}\right) e^{\mathrm{Re}\, \lambda_j (t-\tau)} |f_j(\tau)|^2 \\ &\le e^{-\gamma (t-\tau)} \sum_{j=-\infty}^{\infty} \left(1 + |j|^{2s}\right) |f_j(\tau)|^2 \\ &= e^{-\gamma (t-\tau)} \|f(x,\tau)\|_s^2 \end{aligned}$$

which implies

$$\begin{aligned}
\|h(x,\tau)\|_s^2 &= \left\| \sum_{j=-\infty}^{\infty} \int_0^{t-1} e^{\lambda_j(t-\tau)} f_j(\tau)\, d\tau \phi_j(x) \right\|_s^2 \\
&= \sum_{j=-\infty}^{\infty} \left(1+|j|^{2s}\right) \left| \int_0^{t-1} e^{\lambda_j(t-\tau)} f_j(\tau)\, d\tau \right|^2 \\
&\le \sum_{j=-\infty}^{\infty} \left(1+|j|^{2s}\right) \int_0^{t-1} \left|e^{\lambda_j(t-\tau)}\right| d\tau \int_0^{t-1} \left|e^{\lambda_j(t-\tau)}\right| |f_j(\tau)|^2\, d\tau \\
&= \int_0^{t-1} e^{\mathrm{Re}\,\lambda_j(t-\tau)} d\tau \int_0^{t-1} \sum_{j=-\infty}^{\infty} \left(1+|j|^{2s}\right) \left|e^{\lambda_j(t-\tau)} f_j^2(\tau)\right| d\tau \\
&\le \int_0^{t-1} e^{-\gamma(t-\tau)} d\tau \int_0^{t-1} e^{-\gamma(t-\tau)} \|f(x,\tau)\|_s^2 d\tau \\
&\le \left(\int_0^{t-1} e^{-\gamma(t-\tau)} d\tau \right)^2 \sup_{0\le\tau<\infty} \|f(x,\tau)\|_s^2 \\
&\le \gamma^{-2} e^{-2\gamma} \sup_{0\le t<\infty} \|f(x,t)\|_s^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sup_{0\le t<\infty} \left\| \int_0^t S(t-\tau) f(x,\tau)\, d\tau \right\|_s &= \sup_{0\le t<\infty} \left\| \hat{h}(x,t) + h(x,t) \right\|_s \\
&\le \sup_{0\le t<\infty} \left\| \hat{h}(x,t) \right\|_s + \sup_{0\le t<\infty} \|h(x,t)\|_s \\
&\le B \sup_{0\le t<\infty} \|f(x,t)\|_s + \gamma^{-1} e^{-\gamma} \sup_{0\le t<\infty} \|f(x,t)\|_s \\
&= B_s \sup_{0\le t<\infty} \|f(x,t)\|_s
\end{aligned}$$

with $B_s = B + \gamma^{-1} e^{-\gamma}$. □

Proposition 4.0.4. *Assume that $s \ge 0$ and $s' > s + \frac{1}{3}$. If $u_0 \in H_{\alpha,\beta}^{s',6}$, then there is a constant $B_{s'} > 0$ such that, for any $T > 0$,*

$$\int_0^T \|S(t) u_0\|_s^6 dt \le B_{s'} \|u_0\|_{s',6}^6,$$

where $B_{s'} \to +\infty, s' \to s + \frac{1}{3}$.

Proof. According to $S(t)u_0$ which has already been defined in (4.0.3), it is obtained that

$$\begin{aligned}
\|S(t) u_0\|_s^6 &= \left(\sum_{k=-\infty}^{\infty} \left(1+|k|^{2s}\right) |e^{\lambda_k t} c_k|^2 \right)^3 \\
&= \left(\sum_{k=-\infty}^{\infty} \left(1+|k|^{2s}\right) |e^{\lambda_k t} c_k|^2 \left(1+|k|^{\epsilon}\right) \left(1+|k|^{\epsilon}\right)^{-1} \right)^3
\end{aligned}$$

$$\leq \left[\left(\sum_{k=-\infty}^{\infty} \left(1+|k|^{2s}\right)^3 |e^{\lambda_k t} c_k|^6 \left(1+|k|^{\epsilon}\right)^3 \right)^{\frac{1}{3}} \left(\sum_{k=-\infty}^{\infty} \left(1+|k|^{\epsilon}\right)^{-\frac{3}{2}} \right)^{\frac{2}{3}} \right]^3$$
$$= C_{s'} \sum_{k=-\infty}^{\infty} \left(1+|k|^{2s}\right)^3 e^{6\mathrm{Re}\,\lambda_k t} |c_k|^6 \left(1+|k|^{\epsilon}\right)^3,$$

where

$$C_{s'} = \left(\sum_{k=-\infty}^{\infty} \left(1+|k|^{\epsilon}\right)^{-\frac{3}{2}} \right)^2 \tag{4.0.7}$$

with $\epsilon = 2\left(s'-s\right) > \frac{2}{3}$ and $C_{s'} \to +\infty$ as $\epsilon \to \frac{2}{3}$. Since

$$\int_0^T e^{6\mathrm{Re}\,\lambda_k t} dt = -\frac{1}{6\mathrm{Re}\,\lambda_k} \left(1 - e^{6\mathrm{Re}\,\lambda_k T}\right) \leq \frac{1}{6\gamma},$$

$$\begin{aligned} \int_0^T \|S(t)\,u_0\|_s^6 dt &= \int_0^T C_{s'} \sum_{k=-\infty}^{\infty} e^{6\mathrm{Re}\,\lambda_k t} |c_k|^6 \left(1+|k|^{2s}\right)^3 \left(1+|k|^{\epsilon}\right)^3 dt \\ &\leq \frac{C_{s'}}{6\gamma} \sum_{k=-\infty}^{\infty} |c_k|^6 \left(1+|k|^{2s}\right)^3 \left(1+|k|^{\epsilon}\right)^3 \\ &\leq B_{s'} \sum_{k=-\infty}^{\infty} |c_k|^6 \left(1+|k|^{6(s+\epsilon/2)}\right) \\ &= B_{s'} \|u_0\|_{s',6}^6, \end{aligned}$$

where $s' = s + \frac{\epsilon}{2} > s + \frac{1}{3}$. Then, from (4.0.7), we have $B_{s'} \to +\infty$ as $s' \to s + \frac{1}{3}$. □

Chapter 5

Local Well-Posedness of the Nonlinear Problem

In this chapter, we consider the local well-posedness of the IVP for the nonlinear Schrödinger equation

$$\begin{cases} iu_t + u_{xx} + \kappa|u|^2 u = 0, \qquad 0 < x < L, t \geq 0 \\ u(x,0) = u_0(x), \\ u(0,t) = \beta u(L,t), \ \beta u_x(0,t) - u_x(L,t) = i\alpha u(0,t), \end{cases} \tag{5.0.1}$$

where $u = u(x,t)$ is complex-valued function, parameter κ is a non-zero real number, and α and β are any real numbers satisfying $\alpha\beta < 0$ and $\beta \neq \pm 1$. By comparing to (4.0.1), the inhomogeneous term f is $i\kappa|u|^2 u$ and the solution of (5.0.1) is

$$u(x,t) = S(t)\,u_0 + \int_0^t S(t-\tau)\left(i\kappa|u|^2 u\right)(x,\tau)\,d\tau. \tag{5.0.2}$$

We will study the solution of (5.0.2) using a fixed-point theorem for the mapping $F : v(x,t) \to u(x,t)$ defined by

$$u(x,t) = (Fv)(x,t) := S(t)\,u_0 + \int_0^t S(t-\tau)\left(i\kappa|v|^2 v\right)(x,\tau)\,d\tau, \tag{5.0.3}$$

here u_0 will be considered as a fixed parameter.

The following lemma gives that the Sobolev norm in H^s for $s > 1/2$ is an algebra.

Lemma 5.0.1. *Let $s > \frac{1}{2}$. There exists some constant $C > 0$ such that $\|fg\|_s \leq C\|f\|_s\|g\|_s$ for any functions f and g in H^s.*

Proof. Since

$$\begin{aligned} \|D^s(fg)\|_s &\leq c'\|D^s f\|_{L2}\|g\|_\infty + c'\|D^s g\|_{L2}\|f\|_\infty \\ &\leq c'\|f\|_s\|g\|_\infty + c'\|g\|_s\|f\|_\infty \\ &\leq 2c'\|f\|_s\|g\|_s, \end{aligned}$$

for some positive number c', and

$$\begin{aligned}\|fg\|_{L^2} &= \left(\int |fg|^2 dx\right)^{\frac{1}{2}} = \left(\int |f|^2|g|^2 dx\right)^{\frac{1}{2}} \\ &\le \left(\sup |f|^2\right)^{\frac{1}{2}} \left(\int |g|^2 dx\right)^{\frac{1}{2}} \\ &\le \|f\|_s \|g\|_s,\end{aligned}$$

we have

$$\|fg\|_s^2 = \|fg\|_{L^2}^2 + \|D^s (fg)\|_{L^2}^2 \le \|f\|_s^2\|g\|_s^2 + 4c'^2\|f\|_s^2\|g\|_s^2 = \left(1+4c'^2\right)\|f\|_s^2\|g\|_s^2.$$

Therefore,

$$\|fg\|_s \le C\|f\|_s\|g\|_s.$$

□

Remark 5.0.2. *From Lemma 5.0.1 we have, if $s > \frac{1}{2}$, there exists some constant $c > 0$ such that*

$$\left\|i|v|^2 v\right\|_s = \left\||v|^2 v\right\|_s = \|\bar{v}vv\|_s \le c\,\|\bar{v}\|_s\,\|v\|_s\,\|v\|_s = c\,\|v\|_s^3$$

for any function $v \in H^s$.

Now, we prove that the IVP of (5.0.1) is well-posed in the space $H^s_{\alpha,\beta}$ for $\frac{1}{2} < s < 1$.

Theorem 5.0.3. *Let $\frac{1}{2} < s < 1$. We define*

$$X_T := C\left(0, T; H^s_{\alpha,\beta}\right) \cap L^\infty\left(0, T; H^s_{\alpha,\beta}\right).$$

There exists a $T = T\left(\|u_0\|_s\right) > 0$ such that the IVP (5.0.1) has a unique solution $v \in X_T$ for any $u_0 \in H^s_{\alpha,\beta}$, where $T \to \infty$ as $\|u_0\|_s \to 0$. In addition, there exists a neighborhood U of u_0 in $H^s_{\alpha,\beta}$ such that the map from U to $X_{T'}$,

$$G : u_0 \to v\left(x,t\right),$$

is Lipschitz continuous for any $T' < T$.

Proof. Define

$$Y_{T,b} = \left\{ v \in X_T \;\middle|\; \sup_{0\le t\le T} \|v\left(x,t\right)\|_s \le b \right\},$$

here $b > 0$ and $T > 0$ will be chosen appropriately such that the map defined by (5.0.3) is a contraction from $Y_{T,b}$ to $Y_{T,b}$.

Applying Propositions 4.0.1 and 4.0.2 to (5.0.3) yields

$$\begin{aligned}
\sup_{0\le t\le T}\|Fv\|_s &= \sup_{0\le t\le T}\left\|S(t)\,u_0+\int_0^t S(t-\tau)\left(i\kappa|v|^2v\right)(x,\tau)\,d\tau\right\|_s\\
&\le \sup_{0\le t\le T}\|S(t)\,u_0\|_s+\sup_{0\le t\le T}\left\|\int_0^t S(t-\tau)\left(i\kappa|v|^2v\right)(x,\tau)\,d\tau\right\|_s\\
&\le c\,\|u_0\|_s+c\left(\int_0^T\left\|i\kappa|v|^2v\right\|_s^2 d\tau\right)^{\frac{1}{2}}\\
&\le c\,\|u_0\|_s+cT^{\frac{1}{2}}\left(\sup_{0\le t\le T}\left\|i\kappa|v|^2v\right\|_s^2\right)^{\frac{1}{2}}\\
&= c\,\|u_0\|_s+cT^{\frac{1}{2}}\sup_{0\le t\le T}\left\|i\kappa|v|^2v\right\|_s\\
&\le c\,\|u_0\|_s+c|\kappa|T^{\frac{1}{2}}\sup_{0\le t\le T}\|v\|_s^3 \qquad (5.0.4)
\end{aligned}$$

for some positive number c independent of T and v. Choose

$$b=2c\,\|u_0\|_s \qquad (5.0.5)$$

and $T>0$ such that

$$c|\kappa|T^{\frac{1}{2}}b^3\le 3c|\kappa|T^{\frac{1}{2}}b^3\le\frac{b}{2}\ \Rightarrow\ 3c|\kappa|T^{\frac{1}{2}}b^2\le\frac{1}{2}. \qquad (5.0.6)$$

Then, by (5.0.4)-(5.0.6) and the definition of $Y_{T,b}$,

$$\sup_{0\le t\le T}\|Fv\|_s\le\frac{b}{2}+\frac{b}{2}=b.$$

Thus, F is a map defined on $Y_{T,b}$. Next, we show that F is a contraction on $Y_{T,b}$. For any $v_1,v_2\in Y_{T,b}$, let $v=v_1-v_2$. Then, from Proposition 4.0.2 and (5.0.6),

$$\begin{aligned}
&\sup_{0\le t\le T}\|Fv_1-Fv_2\|_s\\
&\quad=\sup_{0\le t\le T}\left\|\int_0^t S(t-\tau)\left(i\kappa|v_1|^2v_1\right)(x,\tau)\,d\tau-\int_0^t S(t-\tau)\left(i\kappa|v_2|^2v_2\right)(x,\tau)\,d\tau\right\|_s\\
&\quad=\sup_{0\le t\le T}\left\|\int_0^t S(t-\tau)\,i\kappa\left(|v_1|^2v_1-|v_2|^2v_2\right)(x,\tau)\,d\tau\right\|_s\\
&\quad\le c\left(\int_0^T\left\|i\kappa\left(|v_1|^2v_1-|v_2|^2v_2\right)\right\|_s^2 d\tau\right)^{\frac{1}{2}}\\
&\quad\le T^{\frac{1}{2}}c\left(\sup_{0\le t\le T}\left\|i\kappa\left(|v_1|^2v_1-|v_2|^2v_2\right)\right\|_s^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= T^{\frac{1}{2}} c|\kappa| \sup_{0\le t\le T} \left\| |v_1|^2 v_1 - |v_2|^2 v_2 \right\|_s \\
&= T^{\frac{1}{2}} c|\kappa| \sup_{0\le t\le T} \left\| |v_1|^2 v + |v_2|^2 v + v_1 v_2 \bar{v} \right\|_s \\
&\le T^{\frac{1}{2}} c|\kappa| \sup_{0\le t\le T} \left(\left\| |v_1|^2 v \right\|_s + \left\| |v_2|^2 v \right\|_s + \|v_1 v_2 \bar{v}\|_s \right) \\
&\le T^{\frac{1}{2}} c|\kappa| \sup_{0\le t\le T} \left(\|v_1\|_s^2 \|v\|_s + \|v_2\|_s^2 \|v\|_s + \|v_1\|_s \|v_2\|_s \|v\|_s \right) \\
&\le T^{\frac{1}{2}} c|\kappa| \left[\left(\sup_{0\le t\le T} \|v_1\|_s \right)^2 + \left(\sup_{0\le t\le T} \|v_2\|_s \right)^2 + \sup_{0\le t\le T} \|v_1\|_s \sup_{0\le t\le T} \|v_2\|_s \right] \sup_{0\le t\le T} \|v\|_s \\
&\le 3T^{\frac{1}{2}} c|\kappa| b^2 \sup_{0\le t\le T} \|v\|_s \\
&\le \frac{1}{2} \sup_{0\le t\le T} \|v_1 - v_2\|_s .
\end{aligned}
$$

Therefore, the map defined by (5.0.3) has a unique fixed point which is the desired solution of the IVP (5.0.1). Using (5.0.5)-(5.0.6), we have

$$
3cT^{\frac{1}{2}} b^2 |\kappa| \le \frac{1}{2} \Rightarrow T \le \left(6cb^2|\kappa|\right)^{-2} = \left(24|\kappa| c^3 \|u_0\|_s^2\right)^{-2} .
$$

Thus, T can be chosen as $(24|\kappa|c^3\|u_0\|_s^2)^{-2}$ which goes to ∞ as $\|u_0\|_s \to 0$.

It is obvious that there exists a neighborhood U of u_0 in $H^s_{\alpha,\beta}$ such that the map G from U to $X_{T'}$ is well-defined for any $T' < T$. For any $u_1, u_2 \in U$, let $v_1 = Gu_1, v_2 = Gu_2$, then Proposition 4.0.1 and the contraction property of F yield

$$
\begin{aligned}
\sup_{0\le t\le T'} \|v_1 - v_2\|_s &= \sup_{0\le t\le T'} \left\| S(t)(u_1 - u_2) + \int_0^t S(t-\tau)\, i\kappa \left(|v_1|^2 v_1 - |v_2|^2 v_2 \right)(x,\tau)\, d\tau \right\|_s \\
&\le \sup_{0\le t\le T'} \|S(t)(u_1 - u_2)\|_s + \sup_{0\le t\le T'} \left\| \int_0^t S(t-\tau)\, i\kappa \left(|v_1|^2 v_1 - |v_2|^2 v_2 \right)(x,\tau)\, d\tau \right\|_s \\
&\le c\, \|u_1 - u_2\|_s + \rho \sup_{0\le t\le T'} \|v_1 - v_2\|_s ,
\end{aligned}
$$

where $\rho = 3\,(T')^{\frac{1}{2}}\, cb^2|\kappa| \le \frac{1}{2}$, which implies

$$
\sup_{0\le t\le T'} \|v_1 - v_2\|_s \le \frac{c}{1-\rho} \|u_1 - u_2\|_s .
$$

Thus, G is Lipschitz continuous from U to $X_{T'}$. □

The following theorem gives a local well-posedness of the nonlinear problem in another space.

Theorem 5.0.4. *Let $\frac{1}{2} < s < 1$ and $s' > \frac{1}{3} + s$. We define*

$$X_T := C\left(0, T; L^2\right) \cap L^6\left(0, T; H^s_{\alpha,\beta}\right).$$

There exists a $T = T\left(\|u_0\|_{s',6}\right) > 0$ such that the IVP (5.0.1) has a unique solution $v \in X_T$ for any $u_0 \in H^{s',6}_{\alpha,\beta}$, where $T \to +\infty$ as $\sup\limits_{0\le t\le T} \|u_0\|_{L^2} + \|u_0\|_{s',6} \to 0$. In addition, there exists a neighborhood U of u_0 in $H^{s',6}_{\alpha,\beta}$ such that the map from U to $X_{T'}$,

$$G : u_0 \to v\left(x, t\right),$$

is Lipschitz continuous for any $T' < T$.

Proof. Similar with the proof of Theorem 5.0.3, here we define

$$Y_{T,b} = \left\{ v \in X_T \;\middle|\; \|v\|_{X_T} = \sup_{0\le t\le T} \|v\left(x,t\right)\|_{L^2} + \left(\int_0^T \|v\left(x,t\right)\|_s^6 dt\right)^{\frac{1}{6}} < b \right\}.$$

From Proposition 4.0.1, for some constant $c > 0$,

$$\begin{aligned}
\sup_{0\le t\le T} \|Fv\|_{L^2} &= \sup_{0\le t\le T} \left\| S\left(t\right)u_0 + \int_0^t S\left(t-\tau\right)\left(i\kappa|v|^2 v\right)\left(x,\tau\right) d\tau \right\|_{L^2} \\
&\le \sup_{0\le t\le T} \|S\left(t\right)u_0\|_{L^2} + \sup_{0\le t\le T} \left\| \int_0^t S\left(t-\tau\right)\left(i\kappa|v|^2 v\right)\left(x,\tau\right) d\tau \right\|_{L^2} \\
&\le \sup_{0\le t\le T} \|u_0\|_{L^2} + \sup_{0\le t\le T} \left\| \int_0^t S\left(t-\tau\right)\left(i\kappa|v|^2 v\right)\left(x,\tau\right) d\tau \right\|_{L^2} \\
&\le \sup_{0\le t\le T} \|u_0\|_{L^2} + \int_0^T \left\| S\left(t-\tau\right)\left(i\kappa|v|^2 v\right)\left(x,\tau\right) \right\|_{L^2} d\tau \\
&\le \sup_{0\le t\le T} \|u_0\|_{L^2} + \int_0^T \left\| i\kappa|v|^2 v \right\|_{L^2} d\tau \\
&\le \sup_{0\le t\le T} \|u_0\|_{L^2} + c|\kappa| \int_0^T \|v\|_s^3 \, d\tau \\
&\le \sup_{0\le t\le T} \|u_0\|_{L^2} + c|\kappa| \left(\int_0^T 1 dt\right)^{\frac{1}{2}} \left(\int_0^T \left(\|v\|_s^3\right)^2 d\tau\right)^{\frac{1}{2}} \\
&= \sup_{0\le t\le T} \|u_0\|_{L^2} + c|\kappa| T^{\frac{1}{2}} \left(\int_0^T \|v\|_s^6 \, dt\right)^{\frac{1}{2}}.
\end{aligned}$$

Using Propositions 4.0.2 and 4.0.4, it is obtained that

$$\begin{aligned}
\left(\int_0^T \|Fv\|_s^6 \, dt\right)^{\frac{1}{6}} &= \left(\int_0^t \left\| S\left(t\right)u_0 + \int_0^t S\left(t-\tau\right)\left(i\kappa|v|^2 v\right)\left(x,\tau\right) d\tau \right\|_s^6 dt\right)^{\frac{1}{6}} \\
&\le \left(\int_0^T \|S\left(t\right)u_0\|_s^6 \, dt\right)^{\frac{1}{6}} + \left(\int_0^T \left\| \int_0^t S\left(t-\tau\right)\left(i\kappa|v|^2 v\right)\left(x,\tau\right) d\tau \right\|_s^6 dt\right)^{\frac{1}{6}}
\end{aligned}$$

$$\leq c\left\|u_0\right\|_{s',6} + T^{\frac{1}{6}} \sup_{0\leq t\leq T}\left\|\int_0^t S\left(t-\tau\right)\left(i\kappa|v|^2 v\right)(x,\tau)\,d\tau\right\|_s$$
$$\leq c\left\|u_0\right\|_{s',6} + T^{\frac{1}{6}} c\left(\int_0^T \left\|i\kappa|v|^2 v\right\|_s^2 dt\right)^{\frac{1}{2}} \leq c\left\|u_0\right\|_{s',6} + T^{\frac{1}{6}} c|\kappa|\left(\int_0^T \left\|v\right\|_s^6 dt\right)^{\frac{1}{2}}.$$

Thus,

$$\sup_{0\leq t\leq T}\left\|Fv\right\|_{L^2} + \left(\int_0^T \left\|Fv\right\|_s^6 dt\right)^{\frac{1}{6}}$$
$$\leq \sup_{0\leq t\leq T}\left\|u_0\right\|_{L^2} + cT^{\frac{1}{2}}|\kappa|\left(\int_0^T \left\|v\right\|_s^6 dt\right)^{\frac{1}{2}} + c\left\|u_0\right\|_{s',6} + T^{\frac{1}{6}} c|\kappa|\left(\int_0^T \left\|v\right\|_s^6 dt\right)^{\frac{1}{2}}$$
$$\leq c\left(\sup_{0\leq t\leq T}\left\|u_0\right\|_{L^2} + \left\|u_0\right\|_{s',6}\right) + c|\kappa|\left(T^{\frac{1}{2}} + T^{\frac{1}{6}}\right)\left(\int_0^T \left\|v\right\|_s^6 dt\right)^{\frac{1}{2}}$$
$$\leq b$$

if we choose

$$b = 2c\left(\sup_{0\leq t\leq T}\left\|u_0\right\|_{L^2} + \left\|u_0\right\|_{s',6}\right) \tag{5.0.7}$$

and T such that

$$c|\kappa|\left(T^{\frac{1}{2}} + T^{\frac{1}{6}}\right) b^3 \leq \frac{b}{2}. \tag{5.0.8}$$

Therefore, $Fv \in Y_{T,b}$, i.e. F is a map from $Y_{T,b}$ into $Y_{T,b}$. Let $v = v_1 - v_2$, $\tilde{v} = \max\{\|v_1\|_s, \|v_2\|_s\}$ for any $v_1, v_2 \in Y_{T,b}$. Then, by Propositions 4.0.1 and 4.0.2,

$$\sup_{0\leq t\leq T}\left\|Fv_1 - Fv_2\right\|_{L^2}$$
$$= \sup_{0\leq t\leq T}\left\|\int_0^t S\left(t-\tau\right) i\kappa\left(|v_1|^2 v + |v_2|^2 v + v_1 v_2 \bar{v}\right)(x,\tau)\,d\tau\right\|_{L^2}$$
$$\leq \sup_{0\leq t\leq T}\int_0^t \left\|S\left(t-\tau\right) i\kappa\left(|v_1|^2 v + |v_2|^2 v + v_1 v_2 \bar{v}\right)\right\|_{L^2} d\tau$$
$$\leq \int_0^T \left\|S\left(t-\tau\right) i\kappa\left(|v_1|^2 v + |v_2|^2 v + v_1 v_2 \bar{v}\right)\right\|_{L^2} d\tau$$
$$\leq \int_0^T \left\|i\kappa\left(|v_1|^2 v + |v_2|^2 v + v_1 v_2 \bar{v}\right)\right\|_{L^2} d\tau$$
$$\leq c|\kappa|\int_0^T \left\|v_1\right\|_s^2\left\|v\right\|_s + \left\|v_2\right\|_s^2\left\|v\right\|_s + \left\|v_1\right\|_s\left\|v_2\right\|_s\left\|v\right\|_s d\tau$$
$$\leq c|\kappa|\int_0^T 3\tilde{v}^2\left\|v\right\|_s d\tau \leq c|\kappa|\left(\int_0^T 1^2 dt\right)^{\frac{1}{2}}\left(\int_0^T \left(\tilde{v}^2\right)^3 dt\right)^{\frac{1}{3}}\left(\int_0^T \left\|v\right\|_s^6 dt\right)^{\frac{1}{6}}$$
$$= c|\kappa| T^{\frac{1}{2}}\left(\int_0^T \tilde{v}^6 dt\right)^{\frac{1}{3}}\left(\int_0^T \left\|v\right\|_s^6 dt\right)^{\frac{1}{6}}$$
$$\leq c|\kappa| T^{\frac{1}{2}} b^2\left(\int_0^T \left\|v\right\|_s^6 dt\right)^{\frac{1}{6}} \tag{5.0.9}$$

and

$$\begin{aligned}
&\left(\int_0^T \|Fv_1 - Fv_2\|_s^6 \, dt\right)^{\frac{1}{6}} \\
&\quad = \left(\int_0^T \left\|\int_0^t S(t-\tau)\, i\kappa \left(|v_1|^2 v + |v_2|^2 v + v_1 v_2 \bar{v}\right) d\tau\right\|_s^6 dt\right)^{\frac{1}{6}} \\
&\quad \leq |\kappa| \left\{\int_0^T \left[B_s \left(\int_0^T \left\||v_1|^2 v + |v_2|^2 v + v_1 v_2 \bar{v}\right\|_s^2 d\tau\right)^{\frac{1}{2}}\right]^6 dt\right\}^{\frac{1}{6}} \\
&\quad = c|\kappa| T^{\frac{1}{6}} \left(\int_0^T \left\||v_1|^2 v + |v_2|^2 v + v_1 v_2 \bar{v}\right\|_s^2 d\tau\right)^{\frac{1}{2}} \\
&\quad \leq c|\kappa| T^{\frac{1}{6}} \left(\int_0^T \|v_1\|_s^4 \|v\|_s^2 + \|v_2\|_s^4 \|v\|_s^2 + \|v_1\|_s^2 \|v_2\|_s^2 \|v\|_s^2 \, d\tau\right)^{\frac{1}{2}} \\
&\quad \leq c|\kappa| T^{\frac{1}{6}} \left(3 \int_0^T \|v\|_s^2 \tilde{v}^4 d\tau\right)^{\frac{1}{2}} \leq c|\kappa| T^{\frac{1}{6}} \left\{\left[\int_0^T \left(\|v\|_s^2\right)^3 dt\right]^{\frac{1}{3}} \left[\int_0^T \left(\tilde{v}^4\right)^{\frac{3}{2}} dt\right]^{\frac{2}{3}}\right\}^{\frac{1}{2}} \\
&\quad = c|\kappa| T^{\frac{1}{6}} \left(\int_0^T \|v\|_s^6 \, dt\right)^{\frac{1}{6}} \left(\int_0^T \tilde{v}^6 dt\right)^{\frac{1}{3}} \leq c T^{\frac{1}{6}} b^2 |\kappa| \left(\int_0^T \|v\|_s^6 \, dt\right)^{\frac{1}{6}} \qquad (5.0.10)
\end{aligned}$$

for some positive constant c. Thus, using (5.0.8), we have

$$\begin{aligned}
\|Fv_1 - Fv_2\|_{X_T} &= \sup_{0\leq t\leq T} \|Fv_1 - Fv_2\|_{L^2} + \left(\int_0^T \|Fv_1 - Fv_2\|_s^6 \, dt\right)^{\frac{1}{6}} \\
&\leq c T^{\frac{1}{2}} b^2 |\kappa| \left(\int_0^T \|v\|_s^6 \, dt\right)^{\frac{1}{6}} + c T^{\frac{1}{6}} |\kappa| \left(\int_0^T \|v\|_s^6 \, dt\right)^{\frac{1}{6}} b^2 \\
&= c\left(T^{\frac{1}{2}} + T^{\frac{1}{6}}\right) b^2 |\kappa| \left(\int_0^T \|v\|_s^6 \, dt\right)^{\frac{1}{6}} \\
&\leq \frac{1}{2} \|v_1 - v_2\|_{X_T} < \|v_1 - v_2\|_{X_T}
\end{aligned}$$

which implies F is a contraction from $Y_{T,b}$ to $Y_{T,b}$. From (5.0.7) - (5.0.8),

$$T^{\frac{1}{2}} + T^{\frac{1}{6}} \leq \frac{1}{2cb^2|\kappa|} = \frac{1}{8c^3|\kappa|} \left(\sup_{0\leq t\leq T} \|u_0\|_{L^2} + \|u_0\|_{s',6}\right)^{-2}$$

where $T \to +\infty$ as $\sup_{0\leq t\leq T} \|u_0\|_{L^2} + \|u_0\|_{s',6} \to 0$.

For any $w_1, w_2 \in U$, let $u_1 = Gw_1, u_2 = Gw_2$ and $u = u_1 - u_2$, then

$$\begin{aligned}
u &= S(t)(w_1 - w_2) + \int_0^t S(t-\tau)\, i\kappa \left(|u_1|^2 u_1 - |u_2|^2 u_2\right)(x,\tau)\, d\tau \\
&= S(t)(w_1 - w_2) + Fu_1 - Fu_2.
\end{aligned}$$

By Propositions 4.0.1 and 4.0.4, and (5.0.9) - (5.0.10) with T substituted by T',

$$\begin{aligned}\sup_{0\le t\le T'}\|u\|_{L^2} &= \sup_{0\le t\le T'}\|S(t)(w_1-w_2)+Fu_1-Fu_2\|_{L^2}\\ &\le \sup_{0\le t\le T'}\|S(t)(w_1-w_2)\|_{L^2}+\sup_{0\le t\le T'}\|Fu_1-Fu_2\|_{L^2}\\ &\le c\|w_1-w_2\|_{L^2}+cT'^{\frac12}b^2|\kappa|\left(\int_0^{T'}\|u_1-u_2\|_s^6\,dt\right)^{\frac16}\end{aligned}$$

and

$$\begin{aligned}\left(\int_0^{T'}\|u\|_s^6\,dt\right)^{\frac16} &= \left(\int_0^{T'}\|S(t)(w_1-w_2)+Fu_1-Fu_2\|_s^6\,dt\right)^{\frac16}\\ &\le \left(\int_0^{T'}\|S(t)(w_1-w_2)\|_s^6\right)^{\frac16}+\left(\int_0^{T'}\|Fu_1-Fu_2\|_s^6\,dt\right)^{\frac16}\\ &\le c\|w_1-w_2\|_{s',6}+cT'^{\frac16}|\kappa|\left(\int_0^{T'}\|u_1-u_2\|_s^6\,dt\right)^{\frac16}b^2,\end{aligned}$$

for some positive constant c, which give us that

$$\|u\|_{X_{T'}}=\sup_{0\le t\le T'}\|u\|_{L^2}+\left(\int_0^{T'}\|u\|_s^6\,dt\right)^{\frac16}\le c\left(\|w_1-w_2\|_{L^2}+\|w_1-w_2\|_{s',6}\right)+\rho\|u\|_{X_{T'}}$$

with $\rho=cb^2\left(T'^{\frac12}+T'^{\frac16}\right)|\kappa|\le\frac12<1$. Hence,

$$\begin{aligned}(1-\rho)\|u\|_{X_{T'}} &\le c\left(\|w_1-w_2\|_{L^2}+\|w_1-w_2\|_{s',6}\right)\\ \|u\|_{X_{T'}} &\le \frac{c}{1-\rho}\left(\|w_1-w_2\|_{L^2}+\|w_1-w_2\|_{s',6}\right).\end{aligned}$$

Therefore, the map G is Lipschitz continuous from U to $X_{T'}$. □

From the proof of Theorem 5.0.3, we have the solution $u(x,t)$ of the IVP (5.0.1) belongs to $H^s_{\alpha,\beta}$ if its initial state $u_0\in H^s_{\alpha,\beta}$ for $\frac12<s<1$. Since $S(t)u_0$ is the solution of the IVP in the absence of the nonlinear term, From Proposition 4.0.1, it is easy to conclude that $S(t)u_0\in H^s_{\alpha,\beta}$ for any $s>0$ and $u_0\in H^s_{\alpha,\beta}$. Now, we consider the regularity of solutions for the nonlinear problem (5.0.1), i.e. whether the solution $u(\cdot,t)\in H^n_{\alpha,\beta}$ if its initial state $u_0\in H^n_{\alpha,\beta}$ for any given positive integer $n\ge1$. The answer is negative, since u is defined in a special space $H^n_{\alpha,\beta}$ which is a Hilbert space inherited from Sobolev space H^n with boundary conditions (1.0.2), which implies that

$$u\in H^n_{\alpha,\beta}\quad\not\Rightarrow\quad u_x\in H^{n-1}_{\alpha,\beta}\quad n\ge1.$$

Thus, we cannot apply the properties established in Chapter 4 directly to obtain the existence of the solution in the space $H^n_{\alpha,\beta}$, $n\ge1$, as Theorem 5.0.3. However, it is easy to see that $\partial_t u$ and $\partial_x^2 u$ are in the same space H^n and $\partial_t u$ satisfies the boundary conditions in (5.0.1). Therefore, we may be able to establish the regularity of the solution by obtaining the regularity of $\partial_t u$ first.

To this end, consider the estimates for $\dot{f}(\cdot,t) \equiv \partial_t f(\cdot,t)$.

Lemma 5.0.5. *For $s \geq 0$, assume that $f \in C\left[0,T;H^s_{\alpha,\beta}\right]$ and $\partial_t f(x,t) \in L^2\left[0,T;H^s_{\alpha,\beta}\right]$. Then,*

$$\partial_t\left(\int_0^t S(t-\tau)\, f(x,\tau)\, d\tau\right) = S(t)\, f(x,0) + \int_0^t S(t-\tau)\, \dot{f}(x,\tau)\, d\tau$$

satisfying

$$\sup_{0\leq t\leq T}\left\|\partial_t \int_0^t S(t-\tau)\, f(x,\tau)\, d\tau\right\|_s \leq c\| f(x,0)\|_s + c\left(\int_0^T \|\dot{f}(x,\tau)\|_s^2 dt\right)^{\frac{1}{2}}$$

for some constant $c > 0$ (independent of T). Moreover,

$$\sup_{0\leq t<\infty}\left\|\partial_t \int_0^t S(t-\tau)\, f(x,\tau)\, d\tau\right\|_s \leq c\| f(x,0)\|_s + c\sup_{0\leq t<\infty}\|\dot{f}(x,\tau)\|_s.$$

Proof. Consider the initial boundary value problem

$$\begin{cases} u_t - iu_{xx} = f, & 0<x<L,\ t\geq 0 \\ u(x,0) = 0, \\ u(0,t) = \beta u(L,t), \\ \beta u_x(0,t) - u_x(L,t) = i\alpha u(0,t), \end{cases}$$

which has a solution

$$u = \int_0^t S(t-\tau)\, f(x,\tau)\, d\tau. \tag{5.0.11}$$

Let $v = \partial_t u$. Then,

$$v(x,t) = \frac{\partial}{\partial t}\int_0^t S(t-\tau)\, f(x,\tau)\, d\tau = S(t-t)\, f(x,t) = S(0)\, f(x,t) = f(x,t)$$

which implies $v(x,0) = f(x,0)$. Also,

$$\begin{cases} \partial_t v - iv_{xx} = \dot{f}, & 0<x<L, t\geq 0, \\ v(x,0) = f(x,0), \\ v(0,t) = \beta v(L,t), \\ \beta v_x(0,t) - v_x(L,t) = i\alpha v(0,t), \end{cases}$$

has a solution

$$v(x,t) = S(t)\, f(x,0) + \int_0^t S(t-\tau)\, \dot{f}(x,\tau)\, d\tau. \tag{5.0.12}$$

From (5.0.11) and (5.0.12), we can get

$$\frac{\partial}{\partial t}u = v \ \Rightarrow \ \frac{\partial}{\partial t}\int_0^t S(t-\tau) f(x,\tau)\, d\tau = S(t) f(x,0) + \int_0^t S(t-\tau)\dot{f}(x,\tau)\, d\tau.$$

Then, by Propositions 4.0.1 and 4.0.2,

$$\begin{aligned}\sup_{0\le t\le T}\left\|\partial_t \int_0^t S(t-\tau) f(x,\tau)\, d\tau\right\|_s &= \sup_{0\le t\le T}\left\|S(t) f(x,0) + \int_0^t S(t-\tau)\dot{f}(x,\tau)\, d\tau\right\|_s \\ &\le \sup_{0\le t\le T}\|S(t) f(x,0)\|_s + \sup_{0\le t\le T}\left\|\int_0^t S(t-\tau)\dot{f}(x,\tau)\, d\tau\right\|_s \\ &\le c\|f(x,0)\|_s + c\left(\int_0^T \left\|\dot{f}(x,\tau)\right\|_s^2 d\tau\right)^{\frac{1}{2}},\end{aligned}$$

and by Propositions 4.0.1 and 4.0.3,

$$\begin{aligned}\sup_{0\le t<\infty}\left\|\partial_t \int_0^t S(t-\tau) f(x,\tau)\, d\tau\right\|_s &= \sup_{0\le t<\infty}\left\|S(t) f(x,0) + \int_0^t S(t-\tau)\dot{f}(x,\tau)\, d\tau\right\|_s \\ &\le \sup_{0\le t<\infty}\|S(t) f(x,0)\|_s + \sup_{0\le t<\infty}\left\|\int_0^t S(t-\tau)\dot{f}(x,\tau)\, d\tau\right\|_s \\ &\le c\|f(x,0)\|_s + c\sup_{0\le t<\infty}\left\|\dot{f}(x,\tau)\right\|_s.\end{aligned}$$

The proof of Lemma 5.0.5 is completed. □

Theorem 5.0.6. *Let $\frac{1}{2} < s < 1$. We define*

$$X = \left\{\phi \in H^{s+2}\cap H^2_{\alpha,\beta} \;\middle|\; i\phi_{xx} + i|\phi|^2\phi \in H^s_{\alpha,\beta}\right\}$$

and a Banach space Y_T as

$$Y_T = \left\{v \in C^1\left(0,T; H^s_{\alpha,\beta}\right) \;\middle|\; \sup_{0\le t\le T}\|v(t)\|_s < \infty,\ \sup_{0\le t\le T}\|\dot{v}(t)\|_s < \infty\right\}$$

with the norm

$$\|v\|_{Y_T} := \left(\sup_{0\le t\le T}\|v\|_s^2 + \sup_{0\le t\le T}\|\dot{v}(t)\|_s^2\right)^{\frac{1}{2}} \simeq \sup_{0\le t\le T}\|v\|_s + \sup_{0\le t\le T}\|\dot{v}\|_s.$$

There exists a $T = T(\|u_0\|_X) > 0$ such that the IVP (5.0.1) has a unique solution $v \in Y_T$ for any $u_0 \in X$. In addition, there exists a neighborhood U of u_0 in X such that the map from U to $Y_{T'}$,

$$G : u_0 \to v(x,t),$$

is Lipschitz continuous for any $0 < T' < T$.

Proof. Let

$$Y_{T,b} = \left\{ v \in Y_T \;\middle|\; \|v\|_{Y_T} \leq b, v(x,0) = u_0(x) \right\}$$

for some $T > 0$ and $b > 0$ to be determined. First, we want to show that the map F defined by (5.0.3) is well defined on $Y_{T,b}$, i.e. for any $v \in Y_{T,b}$, $\|Fv\|_{Y_T} \leq b$. According to the proof of Theorem 5.0.3, from Propositions 4.0.1 and 4.0.2, we have for some constant $c > 0$,

$$\begin{aligned}\sup_{0\leq t\leq T} \|Fv\|_s &= \sup_{0\leq t\leq T} \left\| S(t)\,u_0 + \int_0^t S(t-\tau)\left(i\kappa|v|^2 v\right)(x,\tau)\,d\tau \right\|_s \\ &\leq c\|u_0\|_s + cT^{\frac{1}{2}}|\kappa| \sup_{0\leq t\leq T} \|v\|_s^3. \end{aligned} \tag{5.0.13}$$

We know that $u = S(t)\,u_0$ is the solution of

$$\begin{cases} u_t - iu_{xx} = 0, \\ u(x,0) = u_0(x), \\ u(0,t) = \beta u(L,t), \\ \beta u_x(0,t) - u_x(L,t) = i\alpha u(0,t). \end{cases}$$

Let $z = \partial_t u$, which implies that $z = iu_{xx}$ and $z(0,x) = iu_{xx}(0,x)$. Thus, z is the solution of

$$\begin{cases} z_t - iz_{xx} = 0, \\ z(x,0) = iu_{xx}(x,0), \\ z(0,t) = \beta z(L,t), \\ \beta z_x(0,t) - z_x(L,t) = i\alpha z(0,t). \end{cases}$$

Therefore,

$$\frac{\partial}{\partial t} S(t)\,u_0 = \frac{\partial}{\partial t} u = z = S(t)\,iu_{xx}(x,0) = S(t)\,iu_0''(x).$$

By Lemma 5.0.5 and $v(x,0) = u_0(x)$,

$$\frac{\partial}{\partial t}\int_0^t S(t-\tau)\left(i\kappa|v|^2 v\right)(x,\tau)\,d\tau = S(t)\,i\kappa|u_0|^2 u_0 + \int_0^t S(t-\tau)\frac{\partial}{\partial t}\left(i\kappa|v|^2 v\right)d\tau. \tag{5.0.14}$$

Then, Propositions 4.0.1 and 4.0.2 give

$$\begin{aligned}\sup_{0\leq t\leq T} \|\partial_t F v\|_s &= \sup_{0\leq t\leq T} \left\| \frac{\partial}{\partial t} S(t)\,u_0 + \frac{\partial}{\partial t}\int_0^t S(t-\tau)\left(i\kappa|v|^2 v\right)(x,\tau)\,d\tau \right\|_s \\ &= \sup_{0\leq t\leq T} \left\| S(t)\,iu_0'' + S(t)\,i|u_0|^2 u_0 + \int_0^t S(t-\tau)\frac{\partial}{\partial t}\left(i\kappa|v|^2 v\right)d\tau \right\|_s \\ &\leq \sup_{0\leq t\leq T} \left\| S(t)\,i\left(u_0'' + \kappa|u_0|^2 u_0\right)\right\|_s + \sup_{0\leq t\leq T} \left\| \int_0^t S(t-\tau)\frac{\partial}{\partial t}\left(i\kappa|v|^2 v\right)d\tau \right\|_s \\ &\leq c\left(|\kappa|\|u_0\|_s^3 + \|u_0\|_{s+2}\right) + c\left(\int_0^T \left\| \frac{\partial}{\partial t} i\kappa|v|^2 v\right\|_s^2 d\tau\right)^{\frac{1}{2}} \\ &\leq c\left(|\kappa|\|u_0\|_s^3 + \|u_0\|_{s+2}\right) + cT^{\frac{1}{2}}|\kappa| \sup_{0\leq t\leq T} \left\|\partial_t\left(i|v|^2 v\right)\right\|_s \end{aligned}$$

$$
\begin{aligned}
&\leq c\left(|\kappa|\|u_0\|_s^3+\|u_0\|_{s+2}\right)+cT^{\frac{1}{2}}|\kappa|\sup_{0\leq t\leq T}\left\|i\left(2v\dot{v}\bar{v}+v^2\dot{\bar{v}}\right)\right\|_s\\
&\leq c\left(|\kappa|\|u_0\|_s^3+\|u_0\|_{s+2}\right)+cT^{\frac{1}{2}}|\kappa|\left(2\sup_{0\leq t\leq T}\|v\|_s^2\|\dot{v}\|_s+\sup_{0\leq t\leq T}\|v\|_s^2\|\dot{\bar{v}}\|_s\right)\\
&=c\left(|\kappa|\|u_0\|_s^3+\|u_0\|_{s+2}\right)+3cT^{\frac{1}{2}}|\kappa|\sup_{0\leq t\leq T}\|v\|_s^2\|\dot{v}\|_s\\
&\leq c\left(|\kappa|\|u_0\|_s^3+\|u_0\|_{s+2}\right)+cT^{\frac{1}{2}}|\kappa|\left(\sup_{0\leq t\leq T}\|v\|_s^4+\sup_{0\leq t\leq T}\|\dot{v}\|_s^2\right).
\end{aligned}
\tag{5.0.15}
$$

Combing (5.0.13) and (5.0.15), it is obtained that

$$
\begin{aligned}
\|Fv\|_{Y_T}&\leq\sup_{0\leq t\leq T}\|Fv\|_s+\sup_{0\leq t\leq T}\|\partial_tFv\|_s\\
&\leq c\left(\|u_0\|_s+|\kappa|\|u_0\|_s^3+\|u_0\|_{s+2}\right)+cT^{\frac{1}{2}}|\kappa|\left(\sup_{0\leq t\leq T}\|v\|_s^3+\sup_{0\leq t\leq T}\|v\|_s^4+\sup_{0\leq t\leq T}\|\dot{v}\|_s^2\right)\\
&\leq c\left(\|u_0\|_s+|\kappa|\|u_0\|_s^3+\|u_0\|_{s+2}\right)+cT^{\frac{1}{2}}|\kappa|\left(b^3+b^4+b^2\right).
\end{aligned}
$$

If we choose

$$c\left(\|u_0\|_s+|\kappa|\|u_0\|_s^3+\|u_0\|_{s+2}\right)=\frac{b}{2}$$

and $T>0$ such that

$$c|\kappa|T^{\frac{1}{2}}\left(b^3+b^4+b^2\right)\leq\frac{b}{2},$$

then $\|Fv\|_{Y_T}\leq b$.

Define $v=v_1-v_2$ for any $v_1,v_2\in Y_{T,b}$. Then,

$$Fv_1-Fv_2=\int_0^t S\left(t-\tau\right)i\kappa\left(|v_1|^2v_1-|v_2|^2v_2\right)(x,\tau)\,d\tau.$$

Similar to the proof of Theorem 5.0.3, we have

$$\sup_{0\leq t\leq T}\|Fv_1-Fv_2\|_s\leq cT^{\frac{1}{2}}|\kappa|\left(\sup_{0\leq t\leq T}\|v_1\|_s^2+\sup_{0\leq t\leq T}\|v_2\|_s^2\right)\sup_{0\leq t\leq T}\|v\|_s.$$

By $v_1\left(x,0\right)=v_2\left(x,0\right)=u_0\left(x\right)$ and (5.0.14),

$$
\begin{aligned}
&\sup_{0\leq t\leq T}\left\|\frac{\partial}{\partial t}\left(Fv_1-Fv_2\right)\right\|_s\\
&\quad=\sup_{0\leq t\leq T}\left\|\int_0^t S\left(t-\tau\right)\frac{\partial}{\partial t}\left(i|v_1|^2v_1-i|v_2|^2v_2\right)\kappa d\tau\right\|_s\\
&\quad=\sup_{0\leq t\leq T}\left\|\int_0^t S\left(t-\tau\right)\frac{\partial}{\partial t}i\left(|v_1|^2v+|v_2|^2v+v_1v_2\bar{v}\right)\kappa d\tau\right\|_s\\
&\quad\leq c|\kappa|\left(\int_0^T\left\|\frac{\partial}{\partial t}\left(|v_1|^2v+|v_2|^2v+v_1v_2\bar{v}\right)\right\|_s^2d\tau\right)^{\frac{1}{2}}\\
&\quad\leq cT^{\frac{1}{2}}|\kappa|\sup_{0\leq t\leq T}\left\|\frac{\partial}{\partial t}\left(|v_1|^2v+|v_2|^2v+v_1v_2\bar{v}\right)\right\|_s
\end{aligned}
$$

$$
\begin{aligned}
&\le cT^{\frac{1}{2}}|\kappa| \sup_{0\le t\le T} \|\dot{v}_1\bar{v}_1 v + v_1\dot{\bar{v}}_1 v + v_1\bar{v}_1\dot{v} + \dot{v}_2\bar{v}_2 v + v_2\dot{\bar{v}}_2 v + v_2\bar{v}_2\dot{v} + \dot{v}_1 v_2\bar{v} + v_1\dot{v}_2\bar{v} + v_1 v_2\dot{\bar{v}}\|_s \\
&\le cT^{\frac{1}{2}}|\kappa| \sup_{0\le t\le T} \left(2\|\dot{v}_1\|_s\|v_1\|_s + 2\|\dot{v}_2\|_s\|v_2\|_s + \|\dot{v}_1\|_s\|v_2\|_s + \|v_1\|_s\|\dot{v}_2\|_s\right)\|v\|_s \\
&\quad + cT^{\frac{1}{2}}|\kappa| \sup_{0\le t\le T} \left(\|v_1\|_s^2 + \|v_2\|_s^2 + \|v_1\|_s\|v_2\|_s\right)\|\dot{v}\|_s.
\end{aligned}
$$

Hence,

$$
\|Fv_1 - Fv_2\|_{Y_T} \le cT^{\frac{1}{2}} b^2 |\kappa| \left(\sup_{0\le t\le T} \|v\|_s + \sup_{0\le t\le T} \|\dot{v}\|_s \right) \le \frac{1}{2}\|v\|_{Y_T}.
$$

Therefore, F is a contraction on $Y_{T,b}$. As a consequence, the map F has a fixed point $v \in Y_T$ which is the unique solution of the IVP (5.0.1). Since $u_t \in C\left(0,T; H^s_{\alpha,\beta}\right)$ and $u_t = iu_{xx} + i|u|^2u$, we have $u \in C\left(0,T; H^{s+2}\right)$. Moreover, the boundary conditions require $u \in C\left(0,T; H^2_{\alpha,\beta}\right)$. Thus, $u \in C\left(0,T; H^{s+2}\cap H^2_{\alpha,\beta}\right)$.

To prove the map G is Lipschitz continuous, let $u_1 = Gw_1, u_2 = Gw_2$, for any $w_1, w_1 \in U$, and $u = u_1 - u_2$. Then,

$$
\begin{aligned}
\|u\|_{Y_T} &= \left\|S(t)(w_1 - w_2) + Fu_1 - Fu_2\right\|_{Y_T} \\
&\le \left\|S(t)(w_1 - w_2)\right\|_{Y_T} + \left\|Fu_1 - Fu_2\right\|_{Y_T} \\
&= \sup_{0\le t\le T} \|S(t)(w_1 - w_2)\|_s + \sup_{0\le t\le T} \left\|\frac{\partial}{\partial t} S(t)(w_1 - w_2)\right\|_s + \|Fu_1 - Fu_2\|_{Y_T} \\
&= \sup_{0\le t\le T} \|S(t)(w_1 - w_2)\|_s + \sup_{0\le t\le T} \left\|S(t)\, i\partial_x^2 (w_1 - w_2)\right\|_s + \|Fu_1 - Fu_2\|_{Y_T} \\
&\le c\left(\|w_1 - w_2\|_s + \|w_1 - w_2\|_{s+2}\right) + \rho\|u_1 - u_2\|_{Y_T}.
\end{aligned}
$$

Thus,

$$
\|u_1 - u_2\|_{Y_T} \le c\left(\|w_1 - w_2\|_s + \|w_1 - w_2\|_{s+2}\right) + \rho\|u_1 - u_2\|_{Y_T}
$$

which implies

$$
\|u_1 - u_2\|_{Y_T} \le \frac{c}{1-\rho}\left(\|w_1 - w_2\|_s + \|w_1 - w_2\|_{s+2}\right)
$$

with $\rho = cT^{\frac{1}{2}} b^2 |\kappa| < 1$. The proof is completed. □

Using the method in the proof of Theorem 5.0.6, we can obtain the following property. Define a series of differential operators $\{P_k\}, k = 0, \ldots, n$ as

$$
\begin{cases}
P_0(\phi) = \phi, \\
P_1(\phi) = i\dfrac{\partial^2 \phi}{\partial x^2} + i\kappa|\phi|^2\phi, \\
\quad \cdots \\
P_n(\phi) = i\dfrac{\partial^2 P_{n-1}}{\partial x^2} + i\kappa|P_{n-1}|^2 P_{n-1},
\end{cases}
\tag{5.0.16}
$$

for any $\phi \in H^{2n-\frac{1}{2}}$.

Theorem 5.0.7. *Let $n \geq 1$ and $\frac{1}{2} < s < 1$. If*

$$P_k\left(u_0\right) \in H^2_{\alpha,\beta} \cap H^{2(n-k)+s}, \quad k = 0, \ldots, n-1,$$

and

$$P_n\left(u_0\right) \in H^s_{\alpha,\beta},$$

for any $u_0 \in H^2_{\alpha,\beta} \cap H^{2n+s}$, then there exists a $T = T\left(\|u_0\|_{2(n-k)+s}\right) > 0$ such that (5.0.1) has a unique solution

$$u \in C\left(0, T; H^2_{\alpha,\beta} \cap H^{2n+s}\right).$$

In addition,

$$\partial_t^k u \in C\left(0, T; H^2_{\alpha,\beta} \cap H^{2(n-k)+s}\right), \quad k = 0, \ldots, n-1,$$

and

$$\partial_t^n u \in C\left(0, T; H^s_{\alpha,\beta}\right).$$

Chapter 6

Global Results and Exponential Decay of Small Amplitude Solutions

In Chapter 5, we only proved the local existence of the unique solution u of (5.0.1), because u exists in a finite time interval $[0,T)$ with T depends on the size of initial value u_0. In this chapter, we discuss the global well-posedness i.e. whether T can be infinite.

Theorem 6.0.1. *Let $\frac{1}{2} < s < 1$. For $u_0 \in H^s_{\alpha,\beta}$, either the IVP (5.0.1) has a unique solution $u \in C\left(0,\infty; H^s_{\alpha,\beta}\right)$ or there exists a finite T^* (called the lifespan of the solution) such that $u \in C\left(0,T^*; H^s_{\alpha,\beta}\right)$ and $\lim\limits_{t\to T^*} \|u(x,t)\|_s = \infty$.*

Proof. Following Theorem 5.0.3, for any given $u_0 \in H^s_{\alpha,\beta}$, there exists a $T = T(\|u_0\|_s) > 0$ such that (5.0.1) has a unique solution u on $(0,T)$. We can extend the time interval of existence for the solution as long as $\|u\|_s$ is bounded. Thus, either u exists for all time $t > 0$ or $\|u\|_s$ blows up at some $T^* > 0$. □

Theorem 6.0.2. *Let $\frac{1}{2} < s < 1$. Define*

$$X_\infty := C\left(0,\infty; H^s_{\alpha,\beta}\right) \cap L^\infty\left(0,\infty; H^s_{\alpha,\beta}\right).$$

There exist positive numbers $\wp$ and b such that for any $u_0 \in H^s_{\alpha,\beta}$ with $\|u_0\|_s \le \wp$, the IVP (5.0.1) has a unique solution $v \in X_\infty$ satisfying $\sup\limits_{0\le t<\infty} \|v(x,t)\|_s \le b$.

Proof. Define

$$Y_{\infty,b} = \left\{ v \in X_\infty \,\middle|\, \sup_{0\le t<\infty} \|v(x,t)\|_s \le b \right\}$$

with $b > 0$ to be determined. Consider the map F defined by (5.0.3). For any $v \in Y_{\infty,b}$, Propositions 4.0.1 and 4.0.3 imply that

$$\begin{aligned}
\sup_{0\le t<\infty}\|Fv\|_s &= \sup_{0\le t<\infty}\left\|S(t)\,u_0+\int_0^t S(t-\tau)\left(i\kappa|v|^2v\right)(x,\tau)\,d\tau\right\|_s \\
&\le \sup_{0\le t<\infty}\|S(t)\,u_0\|_s+\sup_{0\le t<\infty}\left\|\int_0^t S(t-\tau)\left(i\kappa|v|^2v\right)(x,\tau)\,d\tau\right\|_s \\
&\le c\,\|u_0\|_s+c|\kappa|\sup_{0\le t<\infty}\left\|i|v|^2v\right\|_s \\
&\le c\,\|u_0\|_s+c|\kappa|\left(\sup_{0\le t<\infty}\|v\|_s\right)^3.
\end{aligned}$$

Choosing $b>0$ and $\wp>0$ such that

$$cb^3|\kappa|\le\frac{1}{2}b\quad\text{and}\quad c\wp\le\frac{1}{2}b, \tag{6.0.1}$$

we have that $\sup\limits_{0\le t<\infty}\|Fv\|_s\le b$ if $\|u_0\|_s\le\wp$. Thus, F is a mapping defined on $Y_{\infty,b}$.

Let $v=v_1-v_2$ for any $v_1,v_2\in Y_{\infty,b}$, Proposition 4.0.3 and (6.0.1) yield

$$\begin{aligned}
\sup_{0\le t<\infty}\|Fv_1-Fv_2\|_s &= \sup_{0\le t<\infty}\left\|\int_0^t S(t-\tau)\,i\kappa\left(|v_1|^2v_1-|v_2|^2v_2\right)(x,\tau)\,d\tau\right\|_s \\
&= \sup_{0\le t<\infty}\left\|\int_0^t S(t-\tau)\,i\kappa\left(|v_1|^2v+|v_2|^2v+v_1v_2\bar{v}\right)(x,\tau)\,d\tau\right\|_s \\
&\le c|\kappa|\sup_{0\le t<\infty}\left\||v_1|^2v+|v_2|^2v+v_1v_2\bar{v}\right\|_s \\
&\le cb^2|\kappa|\sup_{0\le t<\infty}\|v\|_s \\
&\le \frac{1}{2}\sup_{0\le t<\infty}\|v_1-v_2\|_s.
\end{aligned}$$

Thus, the contraction property of the map F is obtained, which gives a desired solution. The proof is completed. □

Similar to Theorem 5.0.7, we also have the general result for any $t>0$, that is

Theorem 6.0.3. *Let* $\frac{1}{2}<s<1$. *Assume the differential operators* $\{P_k\}$ *defined by* (5.0.16) *satisfies the hypothesis in Theorem 5.0.7, then for* $n\ge 1$ *be given, there exists a* $\wp(n)>0$ *such that* (5.0.1) *has a unique solution*

$$u\in C\left(0,\infty;H^2_{\alpha,\beta}\cap H^{2n+s}\right)\cap L^\infty\left(0,\infty;H^{2n+s}\right),$$

if $u_0\in H^2_{\alpha,\beta}\cap H^{2n+s}$ *with* $\|u_0\|_{2n+s}\le\wp(n)$. *Moreover,*

$$\partial_t^k u\in C\left(0,\infty;H^2_{\alpha,\beta}\cap H^{2(n-k)+s}\right),\quad k=0,\dots,n-1,$$

and

$$\partial_t^n u\in C\left(0,\infty;H^s_{\alpha,\beta}\right)\cap L^\infty\left(0,\infty;H^s_{\alpha,\beta}\right).$$

We proved that when the size of the initial value u_0 is small enough, the solution of (5.0.1) exists for all time $t > 0$. Next we will use the Lyapounov's second method [15, 29, 32] to show the small amplitude solution exponentially decays to zero as $t \to \infty$.

Theorem 6.0.4. *Let $\frac{1}{2} < s < 1$. There exists a $\eta > 0$ such that the unique solution of* (5.0.1) *satisfies*

$$\|u(x,t)\|_{L^2} \le c_1 e^{-c_2 t}\|u_0\|_{L^2}, \quad t \ge 0,$$

for any $u_0 \in H^s_{\alpha,\beta}$ with $\|u_0\|_s < \eta$, where c_1 and c_2 are positive constants (independent of u_0).

Proof. Define the operator $Y : L^2 \to L^2$ as

$$Y = \sum_{k=-\infty}^{\infty} Y_k, \qquad Y_k = \psi_k \psi_k^*,$$

where ψ_k is the normalized eigenvector of A^* (introduced in Chapter 3) and Y is a strongly convergent series. Let $w \in L^2$ and $w = \sum_{j=-\infty}^{\infty} c_j\phi_j$, applying (3.0.8) and (3.0.9), we have

$$\begin{aligned} w^*Yw &= \left(\sum_{j=-\infty}^{\infty} c_j\phi_j\right)^* \sum_{k=-\infty}^{\infty} \psi_k\psi_k^* \left(\sum_{j=-\infty}^{\infty} c_j\phi_j\right) = \sum_{j=-\infty}^{\infty} c_j^*\phi_j^*\psi_j\psi_j^* c_j\phi_j \\ &= \sum_{j=-\infty}^{\infty} |c_j|^2 \ge c\left\|\sum_{j=-\infty}^{\infty} c_j\phi_j\right\|_{L^2}^2 = c\|w\|_{L^2}^2, \end{aligned} \tag{6.0.2}$$

for some constant $c > 0$. Hence, Y is bounded and positive defined on L^2. Define another operator $S : L^2 \to L^2$ by

$$S = \sum_{k=-\infty}^{\infty} \zeta_k Y_k, \quad \zeta_k = -\frac{1}{2\operatorname{Re}\lambda_k} \ge \frac{1}{2\gamma} > 0$$

where $-\gamma \ge \operatorname{Re}\lambda_k \ge -c_0$ with $c_0, \gamma > 0$. Hence, it is obtained that

$$\begin{aligned} A^*S + SA + Y &= \sum_{k=-\infty}^{\infty} \left(A^*\zeta_k Y_k + \zeta_k Y_k A + Y_k\right) \\ &= \sum_{k=-\infty}^{\infty} \left(\zeta_k A^*\psi_k\psi_k^* + \zeta_k\psi_k\psi_k^* A + \psi_k\psi_k^*\right) \\ &= \sum_{k=-\infty}^{\infty} \left[\zeta_k\left(A^*\psi_k\right)\psi_k^* + \zeta_k\psi_k\left(A^*\psi_k\right)^* + \psi_k\psi_k^*\right] \\ &= \sum_{k=-\infty}^{\infty} \left\{\zeta_k\left[\left(\lambda_k^*\psi_k\right)\psi_k^* + \psi_k\left(\lambda_k^*\psi_k\right)^*\right] + \psi_k\psi_k^*\right\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{\infty} \left\{\zeta_k \left[\lambda_k^* \left(\psi_k\psi_k^*\right) + \left(\psi_k\psi_k^*\right)\lambda_k\right] + \psi_k\psi_k^*\right\} \\
&= \sum_{k=-\infty}^{\infty} \left[\zeta_k \left(\lambda_k^* + \lambda_k\right) + 1\right] \psi_k\psi_k^* \\
&= \sum_{k=-\infty}^{\infty} \left[\zeta_k \left(2Re\lambda_k\right) + 1\right] \psi_k\psi_k^* \\
&= 0,
\end{aligned}
$$

which gives

$$
\begin{aligned}
\frac{d}{dt}\left(u^*Su\right) &= \left(\frac{d}{dt}u\right)^* Su + u^*S\left(\frac{d}{dt}u\right) \\
&= \left(Au + |u|^2u\right)^* Su + u^*S\left(Au + |u|^2u\right) \\
&= u^*A^*Su + u^*SAu + \left(|u|^2u\right)^* Su + u^*S|u|^2u \\
&= u^*\left(A^*S + SA\right)u + \left(|u|^2u\right)^* Su + u^*S|u|^2u \\
&= -u^*Yu + \left(|u|^2u\right)^* Su + u^*S|u|^2u.
\end{aligned}
$$

Let $w = |u|^2u$. For any constant $d > 0$,

$$
\begin{aligned}
&w^*Su + u^*Sw - d^2u^*Su - \frac{1}{d^2}w^*Sw \\
&\quad = -\left(du^* - \frac{1}{d}w^*\right)\left(dSu - \frac{1}{d}Sw\right) \\
&\quad = -\left(du^* - \frac{1}{d}w^*\right) S \left(du - \frac{1}{d}w\right) \\
&\quad \leq 0,
\end{aligned}
$$

and

$$
\left(|u|^2u\right)^* Su + u^*S\left(|u|^2u\right) \leq d^2u^*Su + \frac{1}{d^2}\left(|u|^2u\right)^* S\left(|u|^2u\right).
$$

Choose $d > 0$ so that in the sense of quadratic forms on L^2,

$$
d^2S \leq \frac{1}{2}Y.
$$

From Theorem 6.0.2 we have the unique solution u of (5.0.1) satisfies that $\sup_{0\leq t<\infty} \|u\|_s \leq b$ if $\|u_0\|_s \leq \wp$. Using (6.0.2), for some constant $c > 0$, we have

$$
\begin{aligned}
\frac{1}{d^2}\left(|u|^2u\right)^* S\left(|u|^2u\right) &= \frac{1}{d^2}\left(|u|^2u\right)^* \sum_{k=-\infty}^{\infty} \zeta_k\psi_k\psi_k^*\left(|u|^2u\right) \\
&= \frac{1}{d^2}\sum_{k=-\infty}^{\infty} \zeta_k\left(|u|^2u\right)^*\psi_k\psi_k^*\left(|u|^2u\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2c_0d^2}\sum_{k=-\infty}^{\infty}\left|\int_0^L |u|^2u\bar{\psi}_k dx\right|^2\\
&\leq \frac{D^2}{2c_0d^2}\left\||u|^2u\right\|_{L^2}^2\\
&\leq \frac{cD^2}{2c_0d^2}\left(\sup_{0\leq t<\infty}\left\||u|^2\right\|_{L^2}\cdot\|u\|_{L^2}\right)^2\\
&\leq \frac{cD^2}{2c_0d^2}\left(\sup_{0\leq t<\infty}\|u\|_s^2\cdot\|u\|_{L^2}\right)^2\\
&\leq \frac{cD^2b^4}{2c_0d^2}\|u\|_{L^2}^2\\
&\leq \frac{1}{4}u^*Yu
\end{aligned}
$$

where b is given in Theorem 6.0.2 and satisfies $cD^2b^4\left(2c_0d^2\right)^{-1}\leq\frac{1}{4}$. Then,

$$
\begin{aligned}
\frac{d}{dt}\left(u^*Su\right) &= -u^*Yu+\left(|u|^2u\right)^*Su+u^*S|u|^2u\\
&\leq -u^*Yu+d^2u^*Su+\frac{1}{d^2}\left(|u|^2u\right)^*S\left(|u|^2u\right)\\
&\leq -u^*Yu+d^2u^*Su+\frac{1}{4}u^*Yu\\
&\leq -u^*Yu+\frac{3}{4}u^*Yu\\
&= -\frac{1}{4}u^*Yu.
\end{aligned}
$$

Since $\frac{d}{dt}\left(u^*Iu\right)\leq 0$ where I is the identity map on L^2. By the Gronwall's inequality,

$$
\begin{aligned}
&\frac{d}{dt}\left[u^*\left(I+S\right)u\right]\leq\frac{d}{dt}\left(u^*Su\right)\leq-\frac{1}{4}u^*Yu\\
&\Rightarrow u^*\left(I+S\right)u\leq-\int_0^t\frac{1}{4}u^*Yuds+u_0^*\left(I+S\right)u_0,\\
&\Rightarrow u^*Iu+u^*Su\leq-\int_0^t\frac{1}{4}u^*Yuds+u_0^*Iu_0+u_0^*Su_0,\\
&\Rightarrow u^*Su\leq-\int_0^t\frac{1}{4}u^*Yuds+u_0^*Iu_0+u_0^*Su_0,\\
&\Rightarrow \|u\left(\cdot,t\right)\|_{L^2}^2\leq C_1\|u_0\|_{L^2}^2-\int_0^t C_2\|u\left(\cdot,s\right)\|_{L^2}^2ds,\\
&\Rightarrow \|u\left(\cdot,t\right)\|_{L^2}^2\leq C_1\|u_0\|_{L^2}^2e^{-C_2t},
\end{aligned}
$$

where C_1, C_2 are positive constants. The proof is completed. □

Chapter 7

Control and Numerical Approximation of Linear Schrödinger Equation

Starting from this chapter, we will discuss the boundary control and numerical approximation problems of Schrödinger equations. Here we consider the $1-d$ linear Schrödinger equation with boundary control:

$$\begin{cases} iu_t + u_{xx} = 0, & x \in (0, L)\,,\ 0 < t < T, \\ u\,(0, t) = 0, \quad u\,(L, t) = \nu\,(t)\,, & 0 < t < T, \\ u\,(x, 0) = u^0\,(x)\,, & x \in (0, L)\,, \end{cases} \tag{7.0.1}$$

where $\nu\,(t)$ is the control which enters into the system through the boundary at $x = L$. The energy of solution of system (7.0.1) is given by

$$E\,(t) = \frac{1}{2} \int_0^L |u\,(x, t)\,|^2 dx \tag{7.0.2}$$

when $\nu\,(t) \equiv 0$. By (2.0.3),

$$\frac{d}{dt} \int_0^L |u\,(x, t)\,|^2 dx = i\,(u_x\,(x, t)\,\bar{u}\,(x, t) - u\,(x, t)\,\bar{u}_x\,(x, t))\,|_0^L = 0.$$

Thus, the energy is conserved along time, i.e.

$$E\,(t) = E\,(0)\,, \quad \forall\ 0 < t < T.$$

7.1 Hilbert Uniqueness Method (HUM)

Let $u(x,t) = v(x,t) + iw(x,t), \nu(t) = a(t) + ib(t)$, where all of $v(x,t), w(x,t), a(t), b(t)$ are real functions. Then, the system (7.0.1) can be rewritten as

$$\begin{cases} iu_t + u_{xx} = -(w_t - v_{xx}) + (v_t + w_{xx})\, i, & x \in (0,L),\ 0 < t < T, \\ v(0,t) + iw(0,t) = 0, & 0 < t < T, \\ v(L,t) + iw(L,t) = a(t) + ib(t), & 0 < t < T, \\ v(x,0) + iw(x,0) = v^0(x) + iw^0(x), & x \in (0,L), \end{cases}$$

which is equivalent to

$$\begin{cases} v_t + w_{xx} = 0, & x \in (0,L),\ 0 < t < T, \\ w_t - v_{xx} = 0, & x \in (0,L),\ 0 < t < T, \\ v(0,t) = 0,\ w(0,t) = 0, & 0 < t < T, \\ v(L,t) = a(t),\ w(L,t) = b(t), & 0 < t < T, \\ v(x,0) = v^0(x),\ w(x,0) = w^0(x), & x \in (0,L). \end{cases} \tag{7.1.1}$$

We introduce the partition $\{x_j = jh\}_{j=0,\ldots,N+1}$ of the interval $(0,L)$ with $x_0 = 0$, $x_{N+1} = L$ and $h = \frac{L}{N+1}$ for an integer $N \in \mathbb{N}$, and consider the finite-difference space semi-discretization of (7.1.1):

$$\begin{cases} v_j' + \dfrac{w_{j+1} - 2w_j + w_{j-1}}{h^2} = 0, & j = 1,2,\ldots,N,\ 0 < t < T, \\ w_j' - \dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2} = 0, & j = 1,2,\ldots,N,\ 0 < t < T, \\ v_0(t) = 0,\ w_0(t) = 0, & 0 < t < T, \\ v_{N+1}(t) = a(t),\ w_{N+1}(t) = b(t), & 0 < t < T, \\ v_j(0) = v_j^0,\ w_j(0) = w_j^0, & j = 0,\ldots,N+1, \end{cases} \tag{7.1.2}$$

where $'$ denotes derivative with respect to time t. The system (7.1.2) has $2N$ linear differential equations with $2N$ unknowns $v_1(t), \cdots, v_N(t), w_1(t), \cdots, w_N(t)$, where $v_j(t)$ and $w_j(t)$ are approximations of $v(x,t)$ and $w(x,t)$ for (7.1.1) at the node x_j.

Since (7.1.2) depends on the size of h, we will denote the controls of it as $a_h(t)$ and $b_h(t)$. Corresponding to the energy of continuous version (7.0.2), the energy of system (7.1.2) with $a_h(t) = b_h(t) \equiv 0$ takes the form

$$E_h(t) = \frac{h}{2} \sum_{j=0}^{N+1} \left(v_j^2(t) + w_j^2(t) \right).$$

It is easy to see that the energy E_h satisfies

$$E_h(t) = E_h(0), \quad \forall\, 0 < t < T,$$

i.e. it is also conserved along time for the solutions of (7.1.2).

The goal of this section is to analyze whether the system (7.1.2) is controllable and the controls of it converge to those of (7.1.1). We rewrite the system (7.1.2) as

$$\begin{cases} Y' + AY = BH, \quad 0 \leq t \leq T, \\ Y(0) = Y^0, \end{cases} \tag{7.1.3}$$

where $Y = (y_{-N}(t), \ldots, y_{-1}(t), y_1(t), \ldots, y_N(t))^T = (v_1(t), \ldots, v_N(t), w_1(t), \ldots, w_N(t))^T$ represents the vector whose elements are the $2N$ unknowns of (7.1.2), $H(t) = (b_h(t), a_h(t))^T$ is the 2-dimensional control,

$$A = \begin{pmatrix} \mathbf{0} & A_1 \\ -A_1 & \mathbf{0} \end{pmatrix} \text{ with } A_1 = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & & & \mathbf{0} \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ \mathbf{0} & & & 1 & -2 \end{pmatrix}_{N \times N}$$

is the $2N \times 2N$ matrix containing the real coefficients of (7.1.2) which determines the dynamics of (7.1.3), and the matrix

$$B = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ -\frac{1}{h^2} & 0 \\ 0 & \vdots \\ \vdots & 0 \\ 0 & \frac{1}{h^2} \end{pmatrix}_{2N \times 2}\,,$$

whose $N1^{th}$ and $(2N)2^{th}$ elements are $-\frac{1}{h^2}$ and $\frac{1}{h^2}$, respectively, models the way controls $a_h(t)$ and $b_h(t)$ act on it. Consider the adjoint system of (7.1.3)

$$\begin{cases} -\Phi' + A^* \Phi = 0, \quad 0 \leq t \leq T, \\ \Phi(T) = \Phi^0, \end{cases} \tag{7.1.4}$$

which is equivalent to

$$\begin{cases} \xi_j' + \dfrac{\eta_{j+1} - 2\eta_j + \eta_{j-1}}{h^2} = 0, & j = 1, 2, \ldots, N, \ 0 < t < T, \\ \eta_j' - \dfrac{\xi_{j+1} - 2\xi_j + \xi_{j-1}}{h^2} = 0, & j = 1, 2, \ldots, N, \ 0 < t < T, \\ \xi_0(t) = 0, \ \eta_0(t) = 0, & 0 < t < T, \\ \xi_{N+1}(t) = 0, \ \eta_{N+1}(t) = 0, & 0 < t < T, \\ \xi_j(T) = \xi_j^0, \ \eta_j(T) = \eta_j^0, & j = 0, 1, \ldots, N+1, \end{cases} \tag{7.1.5}$$

with $\Phi = (\phi_{-N}, \ldots, \phi_{-1}, \phi_1, \ldots, \phi_N)^T = (\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N)^T$. It is easy to get that $A^* = -A = A^T$. Now, we analyze the spectral problem associated with the adjoint system (7.1.4):

$$-A\Phi = \lambda\Phi. \tag{7.1.6}$$

We known that the eigenvalues of the following matrix [6]

$$M = \begin{pmatrix} p-r & 2q & r & & & \\ 2q & p & 2q & r & & \\ r & 2q & p & 2q & r & \\ & & & & & \\ & & & & & r \\ & & & & p & 2q \\ & & & r & 2q & p-r \end{pmatrix}_{n\times n}$$

are

$$\begin{aligned} &(p-2r) - \frac{1}{r}\left(q^2 - (q - 2r\cos k\theta)^2\right) \\ &= p - 2r - 4\left(q\cos k\theta - r\cos^2 k\theta\right), \quad \theta = \frac{\pi}{n+1} \\ &= p - 4q\cos k\theta + 2r\cos 2k\theta, \quad k = 1, 2, \ldots, n. \end{aligned} \tag{7.1.7}$$

Since the eigenvalues of (7.1.6) satisfying $A_1^2 - \lambda^2 I = 0$ and

$$A_1^2 = \frac{1}{h^4}\begin{pmatrix} 5 & -4 & 1 & & & \\ -4 & 6 & -4 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ & & & & & \\ & & & & & 1 \\ & & & & 6 & -4 \\ & & & 1 & -4 & 5 \end{pmatrix}_{N\times N},$$

from formula (7.1.7), it is obtained that

$$\begin{aligned} -\lambda_k^2 &= \frac{1}{h^4}\left(6 + 8\cos\frac{k\pi h}{L} + 2\cos\frac{2k\pi h}{L}\right) \\ &= \frac{1}{h^4}\left(4 + 8\cos\frac{k\pi h}{L} + 4\cos^2\frac{k\pi h}{L}\right) \\ &= \frac{4}{h^4}\left(1 + \cos\frac{k\pi h}{L}\right)^2 \\ &= \frac{16}{h^4}\cos^4\left(\frac{k\pi h}{2L}\right) \\ &= \frac{16}{h^4}\sin^4\left(\frac{\pi}{2} - \frac{k\pi h}{2L}\right) \\ &= \frac{16}{h^4}\sin^4\left(\frac{(N+1-k)\pi h}{2L}\right), \end{aligned}$$

where $N+1-k = 1,2,...,N$ for $k = 1,2,\ldots,N$. Thus,

$$-\lambda_k^2 = \frac{16}{h^4}\sin^4\left(\frac{k\pi h}{2L}\right), \quad k = 1,2,\ldots,N.$$

Then, the $2N$ eigenvalues, $\lambda_{-N},\ldots,\lambda_{-1},\lambda_1,\ldots,\lambda_N$, of (7.1.6) are $-i\mu_N(h),\cdots,-i\mu_1(h),i\mu_1(h),\cdots,i\mu_N(h)$ with

$$\mu_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2L}\right), \quad k = 1,2,\ldots,N,$$

which is increasing as k increases. The corresponding eigenvectors are

$$\begin{aligned}\phi_k(h) &= (\phi_{k,-N},\ldots,\phi_{k,-1},\phi_{k,1},\ldots,\phi_{k,N})^T \\ &= \begin{cases} (\tau_{k,1}(h),\cdots,\tau_{k,N}(h),i\tau_{k,1}(h),\cdots,i\tau_{k,N}(h))^T, & k = 1,2,\ldots,N, \\ -(i\tau_{k,1}(h),\cdots,i\tau_{k,N}(h),\tau_{k,1}(h),\cdots,\tau_{k,N}(h))^T, & k = -N,\ldots,-1, \end{cases}\end{aligned}$$

where

$$\tau_{k,j} = \sin\left(\frac{j\pi hk}{L}\right), \quad j = 1,2,\ldots,N. \tag{7.1.8}$$

Lemma 7.1.1.

$$\mu_{k+1}(h) - \mu_k(h) \geq \frac{3\pi}{L^2}$$

for any $k = 1,2,\cdots,N-1$.

Proof.

$$\begin{aligned}\mu_{k+1} - \mu_k &= \frac{4}{h^2}\left[\sin^2\left(\frac{(k+1)\pi h}{2L}\right) - \sin^2\left(\frac{k\pi h}{2L}\right)\right] \\ &= \frac{2}{h^2}\left[\cos\frac{k\pi h}{L} - \cos\frac{(k+1)\pi h}{L}\right] \\ &= -\frac{2}{h^2}\left[\cos\frac{(k+1)\pi h}{L} - \cos\frac{k\pi h}{L}\right] \\ &= \frac{2\pi}{hL}\sin(\zeta)\end{aligned}$$

for some $\zeta \in \left[\frac{k\pi h}{L},\frac{(k+1)\pi h}{L}\right]$, which implies $\mu_{k+1}-\mu_k \geq \frac{2\pi}{hL}\sin\left(\frac{\pi h}{L}\right)$. Since $\frac{2\pi}{hL}\sin\left(\frac{\pi h}{L}\right)$ is the decreasing function of h on the domain $\left(0,\frac{L}{3}\right]$, we can get the smallest value, which is $\frac{3\pi}{L^2}$ by setting $h = \frac{L}{3}$. Thus Lemma 7.1.1 holds. □

Proposition 7.1.2. $\sum_{j=1}^{N}\tau_{l,j}\tau_{m,j} = 0$, *where $l,m = -N,\cdots,N$ with $l,m \neq 0$ and $l < m$.*

Proof. From (7.1.8),

$$
\begin{aligned}
\sum_{j=1}^{N}\tau_{l,j}\tau_{m,j} &= \sum_{j=1,l<m}^{N} \sin\left(\frac{j\pi lh}{L}\right)\sin\left(\frac{j\pi mh}{L}\right)\\
&= \frac{1}{2}\sum_{j=1,l<m}^{N}\left[\cos\left(j\frac{\pi h}{L}(l-m)\right)-\cos\left(j\frac{\pi h}{L}(l+m)\right)\right]\\
&= \frac{1}{2}\sum_{j=1,l<m}^{N}\left[\frac{e^{j\frac{\pi h}{L}(l-m)i}+e^{-j\frac{\pi h}{L}(l-m)i}}{2}-\frac{e^{j\frac{\pi h}{L}(l+m)i}+e^{-j\frac{\pi h}{L}(l+m)i}}{2}\right]\\
&= \frac{1}{4}\left[\sum_{j=1,l<m}^{N}e^{j\frac{\pi h}{L}(l-m)i}+\sum_{j=1,l<m}^{N}e^{-j\frac{\pi h}{L}(l-m)i}-\sum_{j=1,l<m}^{N}e^{j\frac{\pi h}{L}(l+m)i}-\sum_{j=1,l<m}^{N}e^{-j\frac{\pi h}{L}(l+m)i}\right]\\
&= \frac{1}{4}\left[\frac{1-\left(e^{\frac{\pi h}{L}(l-m)i}\right)^{N+1}}{1-e^{\frac{\pi h}{L}(l-m)i}}+\frac{1-\left(e^{-\frac{\pi h}{L}(l-m)i}\right)^{N+1}}{1-e^{-\frac{\pi h}{L}(l-m)i}}-\frac{1-\left(e^{\frac{\pi h}{L}(l+m)i}\right)^{N+1}}{1-e^{\frac{\pi h}{L}(l+m)i}}-\frac{1-\left(e^{-\frac{\pi h}{L}(l+m)i}\right)^{N+1}}{1-e^{-\frac{\pi h}{L}(l+m)i}}\right]\\
&= \frac{1}{4}\left[\frac{1-e^{\pi(l-m)i}}{1-e^{\frac{\pi h}{L}(l-m)i}}+\frac{1-e^{-\pi(l-m)i}}{1-e^{-\frac{\pi h}{L}(l-m)i}}-\frac{1-e^{\pi(l+m)i}}{1-e^{\frac{\pi h}{L}(l+m)i}}-\frac{1-e^{-\pi(l+m)i}}{1-e^{-\frac{\pi h}{L}(l+m)i}}\right]. \qquad (7.1.9)
\end{aligned}
$$

If both of l and m are even or odd integers, it is obvious that (7.1.9) equals zero. If one of l and m is even integer and the other one is odd integer,

$$
\begin{aligned}
\sum_{j=1}^{N}\tau_{l,j}\tau_{m,j} &= \frac{1}{4}\left[\frac{2}{1-e^{\frac{\pi h}{L}(l-m)i}}+\frac{2}{1-e^{-\frac{\pi h}{L}(l-m)i}}-\frac{2}{1-e^{\frac{\pi h}{L}(l+m)i}}-\frac{2}{1-e^{-\frac{\pi h}{L}(l+m)i}}\right]\\
&= \frac{1}{2}\left[\frac{1}{1-e^{\frac{\pi h}{L}(l-m)i}}+\frac{1}{1-e^{-\frac{\pi h}{L}(l-m)i}}-\frac{1}{1-e^{\frac{\pi h}{L}(l+m)i}}-\frac{1}{1-e^{-\frac{\pi h}{L}(l+m)i}}\right]\\
&= \frac{1}{2}\left[\frac{2-e^{\frac{\pi h}{L}(l-m)i}-e^{-\frac{\pi h}{L}(l-m)i}}{2-e^{\frac{\pi h}{L}(l-m)i}-e^{-\frac{\pi h}{L}(l-m)i}}-\frac{2-e^{\frac{\pi h}{L}(l+m)i}-e^{-\frac{\pi h}{L}(l+m)i}}{2-e^{\frac{\pi h}{L}(l+m)i}-e^{-\frac{\pi h}{L}(l+m)i}}\right]\\
&= \frac{1}{2}(1-1)\\
&= 0.
\end{aligned}
$$

Therefore, $\sum_{j=1}^{N}\tau_{l,j}\tau_{m,j}=0$ for any $l,m=-N,\cdots,N$ with $l,m\neq 0$ and $l<m$. ☐

Proposition 7.1.3. $h\sum_{j=1}^{N}|\tau_{k,j}|^2 = h\sum_{j=1}^{N}\sin^2\left(\frac{j\pi hk}{L}\right)=\frac{L}{2}$ *for any* $k=1,2,\cdots,N$.

Proof. We know that

$$1+x+x^2+\cdots+x^n=\frac{1-x^{n+1}}{1-x}.$$

Then,

$$\begin{aligned}
h\sum_{j=1}^{N}|\tau_{k,j}|^2 &= h\sum_{j=1}^{N}\sin^2\left(\frac{j\pi hk}{L}\right) = h\sum_{j=1}^{N}\frac{1-\cos\left(\frac{2j\pi hk}{L}\right)}{2} \\
&= \frac{h}{2}\sum_{j=1}^{N}1-\frac{h}{2}\sum_{j=1}^{N}\cos\left(\frac{2j\pi hk}{L}\right) \\
&= \frac{h}{2}N-\frac{h}{2}\sum_{j=1}^{N}\frac{e^{2j\pi hki/L}+e^{-2j\pi hki/L}}{2} \\
&= \frac{h}{2}N-\frac{h}{4}\sum_{j=1}^{N}e^{2j\pi hki/L}-\frac{h}{4}\sum_{j=1}^{N}e^{-2j\pi hki/L} \\
&= \frac{h}{2}N+\frac{h}{2}-\frac{h}{4}\left(\frac{1-e^{2\pi h(N+1)ki/L}}{1-e^{2\pi hki/L}}+\frac{1-e^{-2\pi h(N+1)ki/L}}{1-e^{-2\pi hki/L}}\right) \\
&= \frac{h}{2}\left(N+1\right)-\frac{h}{4}\left(\frac{1-e^{2\pi ki}}{1-e^{2\pi hki/L}}+\frac{1-e^{-2\pi ki}}{1-e^{-2\pi hki/L}}\right) \\
&= \frac{L}{2}.
\end{aligned}$$

$\square$

Theorem 7.1.4. *If $N>1$, there exists a constant $C=C\left(T\right)>0$ (independent of h) such that*

$$h\sum_{j=-N,j\neq 0}^{N}|\phi_j\left(T\right)|^2\leq C\int_0^T\left|\frac{\phi_{-1}}{h}\right|^2+\left|\frac{\phi_N}{h}\right|^2dt \tag{7.1.10}$$

for all solution Φ of (7.1.4).

Proof. The solution of (7.1.4) can be developed in Fourier series as

$$\phi_j\left(h,t\right)=\sum_{k=-N,k\neq 0}^{N}c_k e^{-\lambda_k(h)(T-t)}\phi_{k,j}\left(h\right).$$

In view of Lemma 7.1.1 and according to Ingham's inequality [11, 12, 23], there exist some constants $C_1,C_2>0$ such that

$$\begin{aligned}
\int_0^T\left(\left|\frac{\phi_{-1}}{h}\right|^2+\left|\frac{\phi_N}{h}\right|^2\right)dt &= \int_0^T\frac{1}{h^2}\left(\left|\sum_{k=-N,k\neq 0}^{N}c_k e^{-\lambda_k(T-t)}\phi_{k,-1}\left(h\right)\right|^2+\left|\sum_{k=-N,k\neq 0}^{N}c_k e^{-\lambda_k(T-t)}\phi_{k,N}\left(h\right)\right|^2\right)dt \\
&\geq \frac{C_1}{h^2}\sum_{k=-N,k\neq 0}^{N}|c_k|^2|\phi_{k,-1}|^2+\frac{C_2}{h^2}\sum_{k=-N,k\neq 0}^{N}|c_k|^2|\phi_{k,N}|^2 \\
&= \frac{C_1}{h^2}\sum_{k=-N,k\neq 0}^{N}|c_k|^2\sin^2\left(\frac{Nk\pi h}{L}\right)+\frac{C_2}{h^2}\sum_{k=-N,k\neq 0}^{N}|c_k|^2\sin^2\left(\frac{k\pi hN}{L}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{C}{h^2} \sum_{k=-N,k\neq 0}^{N} |c_k|^2 \sin^2\left(\frac{k\pi hN}{L}\right) \\
&= \frac{C}{h^2} \sum_{k=-N,k\neq 0}^{N} |c_k|^2 \sin^2\left(\frac{k\pi h\,(N+1) - k\pi h}{L}\right) \\
&= \frac{C}{h^2} \sum_{k=-N,k\neq 0}^{N} |c_k|^2 \sin^2\left(\frac{k\pi L - k\pi h}{L}\right) \\
&= \frac{C}{h^2} \sum_{k=-N,k\neq 0}^{N} |c_k|^2 \sin^2\left(k\pi - \frac{k\pi h}{L}\right) \\
&= \frac{C}{h^2} \sum_{k=-N,k\neq 0}^{N} |c_k|^2 \sin^2\left(\frac{k\pi h}{L}\right) \\
&\geq \sin^2\left(\frac{\pi h}{L}\right) \frac{C}{h^2} \sum_{k=-N,k\neq 0}^{N} |c_k|^2 \\
&\geq \frac{C}{L^2} \sum_{k=-N,k\neq 0}^{N} |c_k|^2
\end{aligned}
$$

for $N > 1$, where C is a positive constant (independent of h). Then, by Propositions 7.1.2 and 7.1.3, it is obtained that

$$
\begin{aligned}
h \sum_{j=-N,j\neq 0}^{N} |\phi_j\,(T)\,|^2 &= h \sum_{j=-N,j\neq 0}^{N} \left| \sum_{k=-N,k\neq 0}^{N} c_k \phi_{k,j}\,(h) \right|^2 \\
&= h \sum_{j=-N,j\neq 0}^{N} \left(\sum_{k=-N,k\neq 0}^{N} c_k \phi_{k,j}\,(h) \right) \left(\sum_{k=-N,k\neq 0}^{N} \bar{c}_k \bar{\phi}_{k,j}\,(h) \right) \\
&\leq h \sum_{j=-N,j\neq 0}^{N} \left[\sum_{k=-N,k\neq 0}^{N} |c_k|^2 |\phi_{k,j}|^2 + \sum_{\substack{l,m=-N \\ l,m\neq 0, l<m}}^{N} \left(\bar{c}_l c_m \bar{\phi}_{l,j} \phi_{m,j} + c_l \bar{c}_m \phi_{l,j} \bar{\phi}_{m,j} \right) \right] \\
&= h \sum_{j=-N,j\neq 0}^{N} \sum_{k=-N,k\neq 0}^{N} |c_k|^2 |\phi_{k,j}|^2 + h \sum_{j=-N,j\neq 0}^{N} \sum_{\substack{l,m=-N \\ l,m\neq 0, l<m}}^{N} \left(\bar{c}_l c_m \bar{\phi}_{l,j} \phi_{m,j} + c_l \bar{c}_m \phi_{l,j} \bar{\phi}_{m,j} \right) \\
&= h \sum_{k=-N,k\neq 0}^{N} |c_k|^2 \sum_{j=-N,j\neq 0}^{N} |\phi_{k,j}|^2 + h \sum_{\substack{l,m=-N \\ l,m\neq 0, l<m}}^{N} \bar{c}_l c_m \sum_{j=-N,j\neq 0}^{N} \bar{\phi}_{l,j} \phi_{m,j} \\
&\quad + h \sum_{\substack{l,m=-N \\ l,m\neq 0, l<m}}^{N} c_l \bar{c}_m \sum_{j=-N,j\neq 0}^{N} \phi_{l,j} \bar{\phi}_{m,j}
\end{aligned}
$$

$$= 2 \sum_{k=-N,k\neq 0}^{N} |c_k|^2 h \sum_{j=1}^{N} |\phi_{k,j}|^2 + 2h \sum_{\substack{l,m=-N \\ l,m\neq 0, l<m}}^{N} \bar{c}_l c_m \sum_{j=1}^{N} \bar{\phi}_{l,j}\phi_{m,j}$$

$$+ 2h \sum_{\substack{l,m=-N \\ l,m\neq 0, l<m}}^{N} c_l \bar{c}_m \sum_{j=1}^{N} \phi_{l,j}\bar{\phi}_{m,j}$$

$$= L \sum_{k=-N,k\neq 0}^{N} |c_k|^2$$

$$\leq C \int_0^T \left|\frac{\phi_{-1}}{h}\right|^2 + \left|\frac{\phi_N}{h}\right|^2 dt,$$

where C is a positive constant (independent of h). Thus (7.1.10) holds. □

From Theorem 7.1.4 we can say the adjoint system (7.1.4) is observable in time T.

Theorem 7.1.5. *System* (7.1.3) *is uniformly controllable in time T, i.e. for any initial datum $Y^0 \in R^{2N}$ and final datum $Y^1 \in R^{2N}$, there exists a control $H(t) \in \left(L^2(0,T)\right)^2$ such that the solution Y of* (7.1.3) *satisfies*

$$Y(T) = Y^1.$$

Proof. Define the quadratic functional $\mathcal{J} : R^{2N} \to R$:

$$\mathcal{J}\left(\Phi^0\right) = \frac{h^2}{2}\int_0^T |B^*\Phi(t)|^2 dt - h < Y^1, \Phi^0 > + h < Y^0, \Phi(0) >,$$

where $B^* = B^T$ is the transpose of matrix B defined in (7.1.3). Here and in the sequel $\cdot$ or $< \cdot, \cdot >$ represents the scalar product in the Euclidean space. The general solution of the adjoint system (7.1.4) is

$$\Phi(t) = L(t)\Phi^0 \text{ with } L(t) = e^{A^*(t-T)}.$$

It is straightforward to get that

$$\frac{h^2}{2}\int_0^T |B^*\Phi(t)|^2 dt = \frac{1}{2}\int_0^T \left|\frac{\phi_{-1}}{h}\right|^2 + \left|\frac{\phi_N}{h}\right|^2 dt.$$

From the proof of Theorem 7.1.4 we have

$$h \sum_{j=-N,j\neq 0}^{N} |\phi_j(0)|^2 \leq C \int_0^T \left|\frac{\phi_{-1}}{h}\right|^2 + \left|\frac{\phi_N}{h}\right|^2 dt$$

as well, for some positive constant C (independent of h). Therefore, the property

$$\lim_{|\Phi^0|\to\infty} \mathcal{J}\left(\Phi^0\right) = \infty$$

holds, which gives us the functional $\mathcal{J}$ is coercive. It is easy to check that the functional $\mathcal{J}$ which is defined in a finite dimensional Euclidean space is also continuous, quadratic. Since for any $\Phi_1^0, \Phi_2^0 \in R^{2N}$ and $\epsilon \in [0,1]$, $\mathcal{J}\left(\epsilon\Phi_1^0 + (1-\epsilon)\Phi_2^0\right) \leq \epsilon\mathcal{J}(\Phi_1^0) + (1-\epsilon)\mathcal{J}(\Phi_2^0)$. $\mathcal{J}$ is also convex. Thus, through the Direct Method of the Calculus of Variations (DMCV), we can obtain that the functional $\mathcal{J}$ has a minimizer, denoted by $\hat{\Phi}^0$, such that $D\mathcal{J}\left(\hat{\Phi}^0\right) = 0$. Let $\hat{\Phi}$ be the solution of (7.1.4) with $\Phi(T) = \hat{\Phi}^0$. Then,

$$\mathcal{J}\left(\hat{\Phi}^0\right) \leq \mathcal{J}\left(\Phi^0\right)$$

for any $\Phi^0 \in R^{2N}$, which implies

$$\mathcal{J}\left(\hat{\Phi}^0\right) \leq \mathcal{J}\left(\hat{\Phi}^0 + \varepsilon\Gamma\right)$$

for any $\Gamma \in R^{2N}, \varepsilon \in R$. Define $F : (-\infty, +\infty) \to R$ by

$$\begin{aligned} F(\varepsilon) &= \mathcal{J}\left(\hat{\Phi}^0 + \varepsilon\Gamma\right) \\ &= \frac{h^2}{2}\int_0^T |B^*\left(L(t)\left(\hat{\Phi}^0 + \varepsilon\Gamma\right)\right)|^2 dt - h < Y^1, \hat{\Phi}^0 + \varepsilon\Gamma > +h < Y^0, L(0)\left(\hat{\Phi}^0 + \varepsilon\Gamma\right) > . \end{aligned}$$

Since $F(0) = \mathcal{J}\left(\hat{\Phi}^0\right) \leq \mathcal{J}\left(\hat{\Phi}^0 + \varepsilon\Gamma\right) = F(\varepsilon)$ for any $\varepsilon \in (-\infty, +\infty)$, $F(\varepsilon)$ has a minimum on $(-\infty, +\infty)$ at $\varepsilon = 0$. By Leibniz's Formula, taking the derivative of $F(\varepsilon)$ with respect to ε and setting ε to 0 yields

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}F(\varepsilon)\mid_{\varepsilon=0} \\ &= \int_0^T < hB^*\left(L(t)\hat{\Phi}^0\right), hB^*\left(L(t)\Gamma\right) > dt - h < Y^1, \Gamma > +h < Y^0, L(0)\Gamma > . \end{aligned}$$

Replacing Γ by Φ^0 gives

$$\begin{aligned} 0 &= \int_0^T < hB^*\left(L(t)\hat{\Phi}^0\right), hB^*\left(L(t)\Phi^0\right) > dt - h < Y^1, \Phi^0 > +h < Y^0, L(0)\Phi^0 > \\ &= \int_0^T hB^*\hat{\Phi} \cdot hB^*\Phi dt - h < Y^1, \Phi^0 > +h < Y^0, \Phi(0) > \end{aligned}$$

for all $\Phi^0 \in R^{2N}$. In other words,

$$\int_0^T BH \cdot \Phi dt =< Y^1, \Phi^0 > - < Y^0, \Phi(0) >, \tag{7.1.11}$$

if the control $H(t)$ is chosen as

$$H(t) = hB^*\hat{\Phi}. \tag{7.1.12}$$

Next, we multiply the state equation (7.1.3) by any solution Φ of the adjoint system (7.1.4). Then

$$\begin{aligned} \int_0^T BH \cdot \Phi dt &= \int_0^T \left[Y'(t) + AY(t)\right] \cdot \Phi dt \\ &= \int_0^T Y \cdot \left[-\Phi' + A^*\Phi\right] dt + < Y, \Phi >|_0^T \\ &=< Y(T), \Phi^0 > - < Y^0, \Phi(0) > . \end{aligned} \tag{7.1.13}$$

Combining (7.1.11) and (7.1.13) yields

$$< Y(T) - Y^1, \Phi^0 >= 0$$

for all $\Phi^0 \in R^{2N}$. Thus, the control condition $Y(T) = Y^1$ is achieved. The constant C in (7.1.10) is independent of h, and hence (7.1.3) is uniformly controllable. □

From Theorem 7.1.5 we have, for $T > 0$, system (7.1.2) is uniformly controllable as $h \to 0$, i.e. for any initial states $\{v_j^0\}_{j=1}^N, \{w_j^0\}_{j=1}^N$, final states $\{v_j^1\}_{j=1}^N, \{w_j^1\}_{j=1}^N$ and $h > 0$, there exist controls $a_h(t)$ and $b_h(t) \in L^2(0,T)$ such that the solutions of system (7.1.2) satisfy

$$v_j(T) = v_j^1, \quad w_j(T) = w_j^1, \quad j = 1, 2, \cdots, N.$$

Remark 7.1.6. *The controllability of the finite-difference scheme is false for the wave equation [11], because the semi-discrete system presents spurious high frequency oscillations. This problem can be solved by using the two-grid scheme [8, 34]. However, our Schrödinger equation system can damp out the spurious modes which is similar to the case of heat equation.*

In view of the observability inequality (7.1.10), we can obtain the uniform bound on the controls of (7.1.2).

Corollary 7.1.7. *There exists a constant $C > 0$, independent of h, such that*

$$\|H(t)\|_{L^2(0,T)}^2 \le Ch\left(|Y^0|^2 + |Y^1|^2\right), \tag{7.1.14}$$

i.e.

$$\|a_h(t)\|_{L^2(0,T)}^2 + \|b_h(t)\|_{L^2(0,T)}^2 \le Ch\sum_{j=1}^N\left(|v_j^0|^2 + |w_j^0|^2 + |v_j^1|^2 + |w_j^1|^2\right),$$

for all $\{v_j^0\}_{j=1}^N, \{w_j^0\}_{j=1}^N, \{v_j^1\}_{j=1}^N, \{w_j^1\}_{j=1}^N$ and $h > 0$.

Proof. Since the functional $\mathcal{J}\left(\hat{\Phi}^0\right) \le 0$ with $\hat{\Phi}^0$ is the minimizer of $\mathcal{J}(\Phi^0)$,

$$\begin{aligned}\frac{h^2}{2}\int_0^T |B^*\hat{\Phi}(t)|^2 dt &\le |h < Y^1, \hat{\Phi}^0 > -h < Y^0, \hat{\Phi}(0) > | \\ &\le \sqrt{h\left(|\hat{\Phi}^0|^2 + |\hat{\Phi}(0)|^2\right)}\sqrt{h\left(|Y^1|^2 + |Y^0|^2\right)} \\ &\le \sqrt{Ch^2\int_0^T |B^*\hat{\Phi}|^2 dt}\sqrt{h\left(|Y^1|^2 + |Y^0|^2\right)},\end{aligned}$$

which implies

$$h^2\int_0^T |B^*\hat{\Phi}(t)|^2 dt \le 4Ch\left(|Y^1|^2 + |Y^0|^2\right). \tag{7.1.15}$$

By (7.1.15) we can obtain (7.1.14), because (7.1.12) gives us

$$h^2\int_0^T |B^*\hat{\Phi}(t)|^2 dt = \|H(t)\|_{L^2(0,T)}^2.$$

□

Corollary 7.1.8. *The controls $a_h(t)$ and $b_h(t)$ of (7.1.2) are such that*

$$a_h(t) \to a(t), \quad b_h(t) \to b(t) \quad in\ L^2(0,T) \quad as\ h \to 0, \tag{7.1.16}$$

where $a(t)$ and $b(t)$ are controls such that the solutions of (7.1.1) satisfy

$$v(x,T) = v^1(x), \qquad w(x,T) = w^1(x).$$

Proof. Let $H^s(0,L)$ be the discrete version of Sobolev space with $s > 0$. By the interpolation arguments, Corollary 7.1.7 also holds for $H(t) \in H^s(0,T)$ and $Y^0, Y^1 \in H^s(0,L)$, i.e. $\|H(t)\|^2_{H^s(0,T)} \le C\left(\|Y^0\|^2_{H^s(0,L)} + \|Y^1\|^2_{H^s(0,L)}\right)$ for some constant C independent of h. Thus, the controls $a_h(t)$ and $b_h(t)$ are uniformly bounded and, by the well-posedness of (7.1.3), the solution Y is also uniformly bounded in $L^\infty(0,T,H^s(0,L))$. Then, $a_h(t), b_h(t) \rightharpoonup a(t), b(t)$ weakly in $H^s(0,T)$ and $Y \rightharpoonup y$ weakly in $L^2(0,T,H^s(0,L))$, where $a(t), b(t)$ and $y = (v(x,t), w(x,t))$ are controls and solutions of the continuous Schrödinger system (7.1.2). Consequently, Corollary (7.1.8) holds. □

Following the proof of Theorem 7.1.5, we can set an algorithm to find the minimizer $\hat{\Phi}^0$ and the corresponding control $H(t)$ of (7.1.3). To be consistent with the index of matrix, we redefine Y and Φ as $Y = (y_1,\dots,y_{2N})^T = (v_1,\dots,v_N,w_1,\dots,w_N)^T$ and $\Phi = (\phi_1,\dots,\phi_{2N})^T = (\xi_1,\dots,\xi_N,\eta_1,\dots,\eta_N)^T$, respectively. Then

$$\begin{aligned}\mathcal{J}\left(\Phi^0\right) &= \frac{1}{2}\int_0^T \left|\frac{\phi_N}{h}\right|^2 + \left|\frac{\phi_{2N}}{h}\right|^2 dt - h < Y^1, \Phi^0 > + h < Y^0, \Phi(0) > \\ &= \frac{1}{2h^2}\int_0^T \left(\sum_{i=1}^{2N}\phi_i^0 L_{Ni}(t)\right)^2 + \left(\sum_{i=1}^{2N}\phi_i^0 L_{2Ni}(t)\right)^2 dt - h\sum_{i=1}^{2N} y_i^1\phi_i^0 + h\sum_{i=1}^{2N} y_i^0\left(\sum_{j=1}^{2N}\phi_j^0 L_{ij}(0)\right).\end{aligned}$$

Taking the derivative of $\mathcal{J}(\Phi^0)$ with respect to ϕ_k^0 for any $k = 1,\dots,2N$ yields

$$\frac{\partial \mathcal{J}\left(\Phi^0\right)}{\partial \phi_k^0} = \frac{1}{h^2}\int_0^T \left(\sum_{i=1}^{2N}\phi_i^0 L_{Ni}(t)\right) L_{Nk} + \left(\sum_{i=1}^{2N}\phi_i^0 L_{2Ni}(t)\right) L_{2Nk} dt - hy_k^1 + h\sum_{i=1}^{2N} y_i^0 L_{ik}(0).$$

Let $\frac{\partial \mathcal{J}\left(\Phi^0\right)}{\partial \phi_k^0} = 0$ for any $k = 1,\dots,2N$. Then,

$$\Xi \cdot \Phi^0 = \Omega, \tag{7.1.17}$$

where the ij^{th} $(i,j = 1,\dots,2N)$ element of matrix Ξ is

$$\frac{1}{h^2}\int_0^T L_{Ni}(t) L_{Nj}(t) + L_{2Ni}(t) L_{2Nj}(t)\, dt,$$

and the k^{th} $(k = 1,\dots,2N)$ element of vector Ω is

$$hy_k^1 - h\sum_{i=1}^{2N} y_i^0 L_{ik}(0).$$

The solution Φ^0 of system (7.1.17) is the minimizer $\hat{\Phi}^0$ of $\mathcal{J}\left(\Phi^0\right)$ and the control of system (7.1.3) is

$$H\left(t\right) = hB^*\left(L\left(t\right)\hat{\Phi}^0\right). \tag{7.1.18}$$

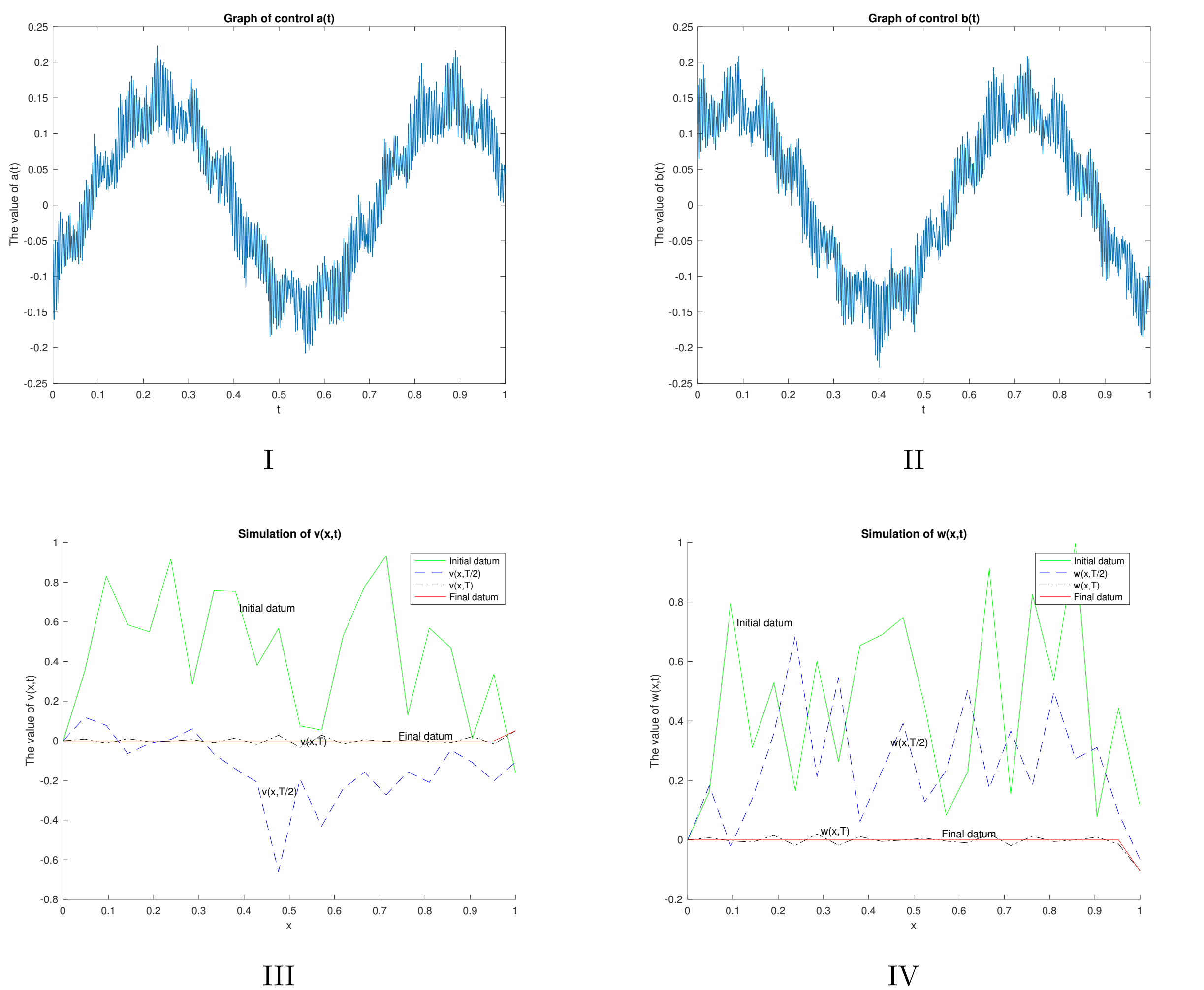


Figure 7.1: Graphs of controls and simulations of solutions for the semi-discrete linear Schrödinger system (7.1.2). $L = 1$, $h = \frac{1}{21}$. v_j^0 and w_j^0, $j = 1, \cdots, N$, are random numbers in (0, 1). $v_j^1 = w_j^1 = 0$, $j = 1, \cdots, N$.

Applying (7.1.18) to (7.1.3) and using the Euler method, the solutions of (7.1.2) can be simulated. The graphs of controls and solutions of (7.1.2) are shown in Figure 7.1 and 7.2 (see appendix A for the MATLAB code). The end states are null and any given data in Figure 7.1 and Figure 7.2, respectively. From III and IV of those graphs, it is easy to see that the solutions of (7.1.2) go to the given final states as $t \to T$ which implies (7.1.2) is controllable. I and II of the figures are the controls of (7.1.2) which converge to the controls of (7.1.1) as $h \to 0$.

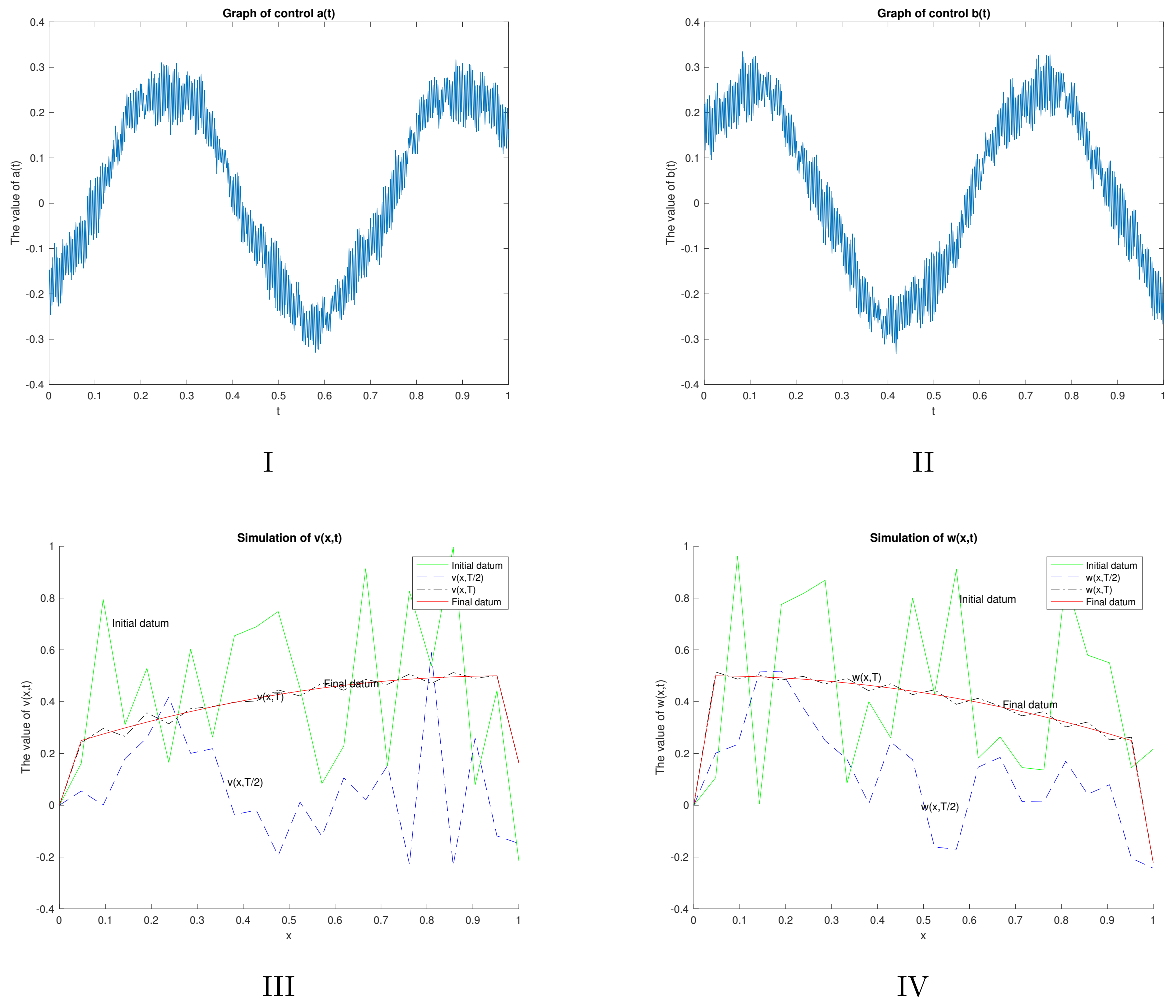


Figure 7.2: Graphs of controls and simulations of solutions for the semi-discrete linear Schrödinger system (7.1.2). $L = 1$, $h = \frac{1}{21}$. v_j^0 and w_j^0, $j = 1, \cdots, N$, are random numbers in $(0, 1)$. v_j^1 and w_j^1, $j = 1, \cdots, N$, are any given data.

7.2 Transformation Method

By the same finite-difference scheme mentioned in the previous section, we divide the interval $(0, L)$ into $N+1$ sub-intervals with $h = \frac{L}{N+1}$ and $x_j = jh, j = 0, 1, \ldots, N, N+1$. Then the semi-discretized version of (7.0.1) is

$$\begin{cases} iu_j' + \dfrac{u_{j+1} + u_{j-1} - 2u_j}{h^2} = 0, & j = 1, \ldots, N, \ 0 < t < T, \\ u_0(t) = 0, \quad u_{N+1}(t) = \nu(t), & 0 < t < T, \\ u_j(0) = u_j^0, & 0 < x < L, \end{cases} \tag{7.2.1}$$

here u_j is the complex function in term of t, which approximates the value of $u(x,t)$ at node x_j. Comparing (7.2.1) with system (7.1.2), we have $u_j = v_j + iw_j$. In this section we will prove (7.2.1) is controllable using the Fourier expansion of its solution.

The associated eigenvalue problem of (7.2.1) is

$$-\frac{\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j}{h^2} = \lambda\varphi_j, \qquad j = 1, \ldots, N, \tag{7.2.2}$$

with $\varphi_0 = 0, \varphi_{N+1} = 0$, thus

$$-\frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & \mathbf{0} \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ \mathbf{0} & & & 1 & -2 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \vdots \\ \vdots \\ \varphi_N \end{pmatrix} = \lambda\begin{pmatrix} \varphi_1 \\ \vdots \\ \vdots \\ \varphi_N \end{pmatrix}.$$

According to formula in [14] we have

$$\begin{aligned} -\lambda h^2 &= -2 - 2\cos\left(\frac{k\pi}{N+1}\right) = -2\left[1 + \cos\left(\frac{k\pi}{N+1}\right)\right] \\ &= -4\cos^2\left(\frac{k\pi}{2(N+1)}\right) = -4\cos^2\left(\frac{k\pi h}{2L}\right) \\ &= -4\sin^2\left(\frac{\pi}{2} - \frac{k\pi h}{2L}\right) \\ &= -4\sin^2\left(\frac{(N+1-k)\pi h}{2L}\right) \\ &= -4\sin^2\left(\frac{k\pi h}{2L}\right) \end{aligned}$$

which implies that there are N eigenvalues

$$0 < \lambda_1(h) < \lambda_2(h) < \cdots < \lambda_N(h)$$

with

$$\lambda_k(h) = \frac{4}{h^2}\sin^2\left(\frac{\pi k h}{2L}\right).$$

From (7.2.2) we have the corresponding eigenvectors,

$$\varphi^k = \left(\varphi_1^k, \cdots, \varphi_N^k\right)^T; \qquad \varphi_j^k = \sin\left(\frac{jkh\pi}{L}\right),\ k,j = 1,2,\ldots,N.$$

The inner product in the space C^N is $h\sum_{j=1}^{N} a_j \bar{b}_j$. If we let $U(t) = (u_1, \cdots, u_N)^T$, then $U(t) = \sum_{k=1}^{N} d_k(t)\varphi^k$ where $d_k(t) = h\sum_{j=1}^{N} u_j(t)\bar{\varphi}_j^k$.

Now, we multiply (7.2.1) by $h\bar{\varphi}_j^k$ and sum these up to get

$$\begin{aligned}
0 &= ih\sum_{j=1}^{N} u_j'\bar{\varphi}_j^k + \frac{1}{h}\sum_{j=1}^{N}\left(u_{j+1} + u_{j-1} - 2u_j\right)\bar{\varphi}_j^k \\
&= ih\sum_{j=1}^{N} u_j'\bar{\varphi}_j^k + \frac{1}{h}\left(\sum_{j=1}^{N} u_j\left(\bar{\varphi}_{j+1}^k + \bar{\varphi}_{j-1}^k - 2\bar{\varphi}_j^k\right) + u_{N+1}\bar{\varphi}_N^k + u_0\bar{\varphi}_1^k\right) \\
&= ih\sum_{j=1}^{N} u_j'\bar{\varphi}_j^k + \frac{1}{h}\left(\sum_{j=1}^{N} u_j h^2\left(-\lambda_k\right)\bar{\varphi}_j^k + u_{N+1}\bar{\varphi}_N^k + u_0\bar{\varphi}_1^k\right) \\
&= ih\sum_{j=1}^{N} u_j'\bar{\varphi}_j^k + \frac{1}{h}\left(\sum_{j=1}^{N} u_j h^2\left(-\lambda_k\right)\bar{\varphi}_j^k + \nu(t)\bar{\varphi}_N^k\right) \\
&= id_k'(t) - \lambda_k d_k(t) + \nu(t)\bar{\varphi}_N^k/h
\end{aligned}$$

which gives

$$id_k'(t) - \lambda_k d_k(t) = -\nu(t)\bar{\varphi}_N^k/h. \tag{7.2.3}$$

Multiplying (7.2.3) by $e^{i\lambda_k t}$ we have

$$e^{i\lambda_k t}id_k'(t) - e^{i\lambda_k t}\lambda_k d_k(t) = -e^{i\lambda_k t}\nu(t)\bar{\varphi}_N^k/h$$

which implies

$$\left[e^{i\lambda_k t}d_k(t)\right]' = -e^{i\lambda_k t}\nu(t)\bar{\varphi}_N^k/h,$$

and

$$e^{i\lambda_k t}d_k(t) = C + \frac{i}{h}\int_0^t e^{i\lambda_k s}\nu(s)\bar{\varphi}_N^k ds.$$

When $t = 0$ we get $C = d_k(0)$. Thus,

$$e^{i\lambda_k t}d_k(t) = d_k(0) + \frac{i}{h}\int_0^t e^{i\lambda_k s}\nu(s)\bar{\varphi}_N^k ds.$$

Then, for any given initial and end conditions $U(0) = U^0 = \sum_{k=1}^{N} d_k(0)\bar{\varphi}^k$ and $U(T) = U^1 = \sum_{k=1}^{N} d_k(T)\bar{\varphi}^k$, the control $\nu(t)$ must be determined by

$$\int_0^T e^{i\lambda_k s}\nu(s)\,ds = \frac{h}{i\bar{\varphi}_N^k}\left(e^{i\lambda_k T}d_k(T) - d_k(0)\right). \tag{7.2.4}$$

Since $e^{i\lambda_k s}, k = 1, 2, \ldots, N$, form a Riesz basis on the span of these functions over $(0, T)$, there exists a dual basis $p_k(t), k = 1, 2, \ldots, N$. Let

$$\nu(t) = \sum_{k=1}^{N} \nu_k p_k(t), \qquad \nu_k, k = 1, 2, \ldots, N, \text{ are constants.} \tag{7.2.5}$$

Then, there exists constants C_1 and C_2 such that

$$C_1 \sum_{k=1}^{N} |\nu_k|^2 \le \|\nu(t)\|^2_{L^2(0,T)} \le C_2 \sum_{k=1}^{N} |\nu_k|^2. \tag{7.2.6}$$

By (7.2.4) and (7.2.5) we have

$$< e^{i\lambda_k s}, \nu >=< e^{i\lambda_k s}, \sum_{k=1}^{N} \nu_k p_k(t) >= \nu_k = \frac{h}{i\bar{\varphi}_1^k}\left(e^{i\lambda_k T}d_k(T) - d_k(0)\right). \tag{7.2.7}$$

To get the estimates of ν_k, we need to analyze $|\frac{h}{i\bar{\varphi}_N^k}|$ using the form of φ_N^k. Since

$$\begin{aligned}\left|\sin\left(\frac{Nkh\pi}{L}\right)\right| &= \left|\sin\left(\frac{(N+1)kh\pi - kh\pi}{L}\right)\right| \\ &= \left|\sin\left(k\pi - \frac{kh\pi}{L}\right)\right| \\ &= \left|\sin\left(\frac{kh\pi}{L}\right)\right| \\ &= \sin\left(\frac{kh\pi}{L}\right).\end{aligned}$$

If $1 \le k \le [N/2]$, then there exists some fixed constant C' such that $\sin\left(\frac{kh\pi}{L}\right) \ge C'\frac{kh\pi}{L}$ which implies $\left|\frac{h}{i\bar{\varphi}_1^k}\right| \le \frac{hl}{Ckh\pi} = l/(Ck\pi) \le C$ for some fixed constant C. If $[N/2]+1 \le k \le N$, similarly $\left|\frac{h}{i\bar{\varphi}_1^k}\right| \le C$. Thus, from (7.2.7) we obtain that

$$|\nu_k|^2 \le C\left(|d_k(T)|^2 + |d_k(0)|^2\right).$$

Combining (7.2.6) and (7.2.7) we have

$$\|\nu\left(t\right)\|_{L^2(0,T)}^2 \leq C \sum_{k=1}^{N} \left(|d_k\left(T\right)|^2 + |d_k\left(0\right)|^2\right) \leq Ch\left(|U^0|^2 + |U^1|^2\right),$$

for all U^0, U^1 and $h > 0$. The continuous dependence of $\nu\left(t\right)$ in terms of $u^0\left(x\right), u\left(x, T\right)$ can be obtained similarly. This is also true for $U^0, U^1 \in H^s\left(0, L\right)$ and $\nu \in H^s\left(0, T\right)$ with $s \geq 0$. Here $H^s\left(0, L\right)$ is denoted as the discrete version of Sobolev space.

Therefore, for (7.2.1), if U^0, U^1 are given in $H^s\left(0, L\right)$, then there is a control $\nu\left(t\right) \in H^s\left(0, T\right)$. We can denote a bounded map P with $\nu = P\left(U^0, U^1\right)$ from $H^s\left(0, L\right) \times H^s\left(0, L\right)$ to $H^s\left(0, T\right)$.

Chapter 8

Null-Controllability for the Nonlinear Schrödinger Equation

In this chapter we are going to discuss the $1-d$ semilinear Schrödinger equation with boundary control:

$$\begin{cases} iu_t + u_{xx} + \kappa|u|^2 u = 0, & x \in (0,L),\ 0 < t < T, \\ u(0,t) = 0, \quad u(L,t) = \nu(t), & 0 < t < T, \\ u(x,0) = u^0(x), & x \in (0,L). \end{cases} \tag{8.0.1}$$

Similar with the linear case, we split (8.0.1) into real and imaginary parts, i.e.

$$\begin{aligned} iu_t + u_{xx} + |u|^2 u &= i(v+wi)_t + (v+wi)_{xx} + \kappa|v+wi|^2(v+wi) \\ &= iv_t - w_t + v_{xx} + iw_{xx} + \kappa(v^2+w^2)(v+wi) \\ &= -w_t + v_{xx} + \kappa v^3 + \kappa w^2 v + (v_t + w_{xx} + \kappa v^2 w + \kappa w^3)i. \end{aligned}$$

Then, the system (8.0.1) is equivalent to

$$\begin{cases} v_t + w_{xx} + \kappa(v^2 w + w^3) = 0, & x \in (0,L),\ 0 < t < T, \\ w_t - v_{xx} - \kappa(w^2 v + v^3) = 0, & x \in (0,L),\ 0 < t < T, \\ v(0,t) = 0,\ w(0,t) = 0, & 0 < t < T, \\ v(L,t) = a(t),\ w(L,t) = b(t), & 0 < t < T, \\ v(x,0) = v^0(x),\ w(x,0) = w^0(x), & x \in (0,L). \end{cases} \tag{8.0.2}$$

Consider the conservative finite-difference semi-discretization of (8.0.2) as follows:

$$\begin{cases} v_j' + \dfrac{w_{j+1} - 2w_j + w_{j-1}}{h^2} + \kappa(v_j^2 w_j + w_j^3) = 0, & j = 1, \dots, N, \ 0 < t < T, \\ w_j' - \dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2} - \kappa(w_j^2 v_j + v_j^3) = 0, & j = 1, \dots, N, \ 0 < t < T, \\ v_0(t) = 0, \ w_0(t) = 0, & 0 < t < T, \\ v_{N+1}(t) = a(t), \ w_{N+1}(t) = b(t), & 0 < t < T, \\ v_j(0) = v_j^0, \ w_j(0) = w_j^0, & j = 0, \dots, N+1. \end{cases} \tag{8.0.3}$$

For the linear case, we analyzed the controls of $L^2(0,T)$-norm which are given by the Hilbert Uniqueness Method (HUM) [16]. In this chapter, we will discuss the null-controllability of (8.0.3) through fixed point argument [35]. Since (8.0.3) depends on h, we shall also denote the controls of it as $a_h(t)$ and $b_h(t)$.

Theorem 8.0.1. *The semi-discrete nonlinear Schrödinger system* (8.0.3) *is null controllable. More precisely, if $h > 0$ is fixed, for all initial states $\{v_j^0\}_{j=1}^N, \{w_j^0\}_{j=1}^N \in L^2(0,T;R^N)$ there exists controls $a_h(t)$ and $b_h(t) \in L^2(0,T)$ such that the solutions of* (8.0.3) *satisfy*

$$v_j(T) = 0, \ w_j(T) = 0, \quad j = 1, \dots, N.$$

Proof. Let us introduce a semi-discrete function $Z = Z(t) = (z_0(t), \dots, z_{N+1}(t)) \in$ C([0,T]; $\mathbb{C}^{N+2}$) and linearize (8.0.3) to

$$\begin{cases} v_j' + \dfrac{w_{j+1} - 2w_j + w_{j-1}}{h^2} + \kappa|z_j|^2 w_j = 0, & j = 1, \dots, N, \ 0 < t < T, \\ w_j' - \dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2} - \kappa|z_j|^2 v_j = 0, & j = 1, \dots, N, \ 0 < t < T, \\ v_0(t) = 0, \ w_0(t) = 0, & 0 < t < T, \\ v_{N+1}(t) = a_h(t), \ w_{N+1}(t) = b_h(t), & 0 < t < T, \\ v_j(0) = v_j^0, \ w_j(0) = w_j^0, & j = 0, \dots, N+1. \end{cases} \tag{8.0.4}$$

On the basis of the HUM controls for the linearized system (8.0.4), Theorem 8.0.1 can be obtained by the standard fixed point argument [35]. To be more precise, for each fixed h, following the same idea used to prove the controllability of the linear Schrödinger equation in Chapter 7, we can find the control of minimal $L^2(0,T)$-norm $H_Z(t) = (b_h(t), a_h(t))$. We use H_Z to represent the 2-dimensional control of (8.0.4), in order to underline the fact that the controls depend on Z. The control is unique, which gives that the solution $Y_Z = (v_0, v_1, \dots, v_{N+1}, w_0, w_1, \dots, w_{N+1})^T$ of system (8.0.4) is uniquely determined. Thus the nonlinear map $\Upsilon(Z) = (v_0 + iw_0, v_1 + iw_1, \dots, v_{N+1} + iw_{N+1})$ is well defined. If the map Υ has a fixed point, i.e. $Z = Y_Z^T$, we can have $|z_j|^2 w_j = v_j^2 w_j + w_j^3$, $|z_j|^2 v_j = w_j^2 v_j + v_j^3$ for all $j = 1, \dots, N$. Consequently, Y_Z is the solution of (8.0.3) and the controls can be achieved as well.

Let us start from the adjoint system of (8.0.4):

$$\begin{cases} p_j' + \dfrac{q_{j+1} - 2q_j + q_{j-1}}{h^2} + \kappa|z_j|^2 q_j = 0, & j = 1, \dots, N,\ 0 < t < T, \\ q_j' - \dfrac{p_{j+1} - 2p_j + p_{j-1}}{h^2} - \kappa|z_j|^2 p_j = 0, & j = 1, \dots, N,\ 0 < t < T, \\ p_0(t) = q_0(t) = 0, & 0 < t < T, \\ p_{N+1}(t) = q_{N+1}(t) = 0, & 0 < t < T, \\ p_j(T) = p_j^0,\ q_j(T) = q_j^0, & j = 1, \dots, N. \end{cases} \tag{8.0.5}$$

In Chapter 7, it was proved that, when $Z = \mathbf{0}$, the observability inequality (7.1.10) holds for all solutions of (8.0.5) and, consequently, (8.0.4) is controllable. For the perturbed case, if we can also have that there exists at least one constant $C > 0$ such that the solutions of (8.0.5) satisfy

$$h|\Psi(T)|^2 \leq C \int_0^T \left|\frac{p_N}{h}\right|^2 + \left|\frac{q_N}{h}\right|^2 dt, \tag{8.0.6}$$

where $\Psi = (\psi_1, \psi_2, \dots, \psi_{2N})^T = (p_1, \dots, p_N, q_1, \dots, q_N)^T$, for any $Z \in C([0,T]; \mathbb{C}^{N+2})$, then the controllability of (8.0.4) will be obtained.

Since (8.0.5) is a finite-dimensional system when h is fixed, we can obtain the following unique continuation property for all Z:

$$\Psi(t) \equiv 0 \ \text{ if } \ p_N(t) = q_N(t) = 0,\ 0 < t < T. \tag{8.0.7}$$

Indeed, if we rewrite the equations in (8.0.5) for $j = N$ to

$$\begin{aligned} \frac{q_{N-1}}{h^2} &= -p_N' - \frac{q_{N+1} - 2q_N}{h^2} - \kappa|z_N|^2 q_N = 0, \\ \frac{p_{N-1}}{h^2} &= q_N' - \frac{p_{N+1} - 2p_N}{h^2} - \kappa|z_N|^2 p_N = 0, \end{aligned}$$

then the boundary conditions $p_{N+1} = q_{N+1} = 0$ and the fact $p_N = q_N = 0$ can help us to find out that $p_{N-1} = q_{N-1} = 0$. Repeating this argument and by induction we deduce $\Phi \equiv 0$. Next, we will prove the observability inequality (8.0.6) holds for any solution Ψ of (8.0.5) by contradiction. Assume (8.0.6) fails, i.e. for any constant C,

$$h|\Psi(T)|^2 > C \int_0^T \left|\frac{p_N}{h}\right|^2 + \left|\frac{q_N}{h}\right|^2 dt.$$

If we pick $C = n$, then

$$h|\Psi(T)|^2/n > \int_0^T \left|\frac{p_N}{h}\right|^2 + \left|\frac{q_N}{h}\right|^2 dt. \tag{8.0.8}$$

The left hand side of (8.0.8) will go to zero as $n \to \infty$ which implies $p_N = q_N = 0$. Then, from (8.0.7) we have $\Psi \equiv 0$ which is a contradiction. Thus, (8.0.6) holds and (8.0.4) is controllable. Note that the constant C in (8.0.6) depends on h.

It is proved that the solution Y_Z of (8.0.4) is bounded by some constant C, which depends on the mesh-size h, the norm of the given initial data and the time of control T, but is independent of Z. Then, the map Υ is compact from $L^2(0,T;\mathbb{C}^{N+2})$ into itself which allows us to apply the Schauder's fixed point theorem to deduce the existence of its fixed point. At the end, we can conclude that the system (8.0.3) is controllable for any fixed $h > 0$.

For the completeness, we will find the controls of (8.0.4) by the HUM. Similar with the proof of Theorem 7.1.5, let us introduce the continuous, quadratic and convex function $\mathcal{J}_h$ as

$$\mathcal{J}_h(\Psi^0) = \frac{1}{2}\int_0^T \left|\frac{\psi_N}{h}\right|^2 + \left|\frac{\psi_{2N}}{h}\right|^2 dt + h < \tilde{Y}(0), \Psi(0) >,$$

where $\tilde{Y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{2N})^T = (v_1, v_2, \ldots, v_N, w_1, w_2, \ldots, w_N)^T$ is the solution of (8.0.4). The observability inequality (8.0.6) yields $\mathcal{J}_h$ is also coercivity. Thus, the minimizer $\hat{\Psi}$ exists and satisfies

$$D\mathcal{J}_h(\hat{\Psi}^0) = 0.$$

Then,

$$\begin{aligned} 0 &=< D\mathcal{J}_h(\hat{\Psi}^0), \Psi^0 > \\ &= \left(\frac{\hat{\psi}_N}{h}, \frac{\psi_N}{h}\right)_{L^2(0,T)} + \left(\frac{\hat{\psi}_{2N}}{h}, \frac{\psi_{2N}}{h}\right)_{L^2(0,T)} + h\sum_{j=1}^{2N}[\tilde{y}_j(0)\psi_j(0)] \\ &= \int_0^T a_h(t)\frac{\psi_{2N}}{h}dt - \int_0^T b_h(t)\frac{\psi_N}{h}dt + h\sum_{j=1}^{2N}[\tilde{y}_j(0)\psi_j(0)] \end{aligned} \tag{8.0.9}$$

if we choose the controls

$$a_h(t) = \frac{\hat{\psi}_{2N}}{h}, \qquad b_h(t) = -\frac{\hat{\psi}_N}{h}. \tag{8.0.10}$$

On the other hand, using (8.0.10) as controls in (8.0.4), from (8.0.4) and (8.0.5) we deduce that

$$0 = \int_0^T \sum_{i=1}^N p_j v_j' dt + \int_0^T \sum_{i=1}^N \frac{w_{j+1} - 2w_j + w_{j-1}}{h^2} p_j dt + \kappa \int_0^T \sum_{i=1}^N |z_j|^2 w_j p_j dt$$

$$\begin{aligned} &= \sum_{i=1}^N p_j v_j\Big|_0^T - \int_0^T \sum_{i=1}^N p_j' v_j dt + \int_0^T \sum_{i=1}^N \frac{p_{j+1} - 2p_j + p_{j-1}}{h^2} w_j dt \\ &\quad + \int_0^T \frac{w_{N+1}p_N + w_0 p_0}{h^2} dt + \kappa \int_0^T \sum_{i=1}^N |z_j|^2 w_j p_j dt \\ &= \sum_{i=1}^N p_j v_j\Big|_0^T - \int_0^T \sum_{i=1}^N p_j' v_j dt + \int_0^T \sum_{i=1}^N q_j' w_j dt + \frac{1}{h^2}\int_0^T b_h(t) p_N dt. \end{aligned} \tag{8.0.11}$$

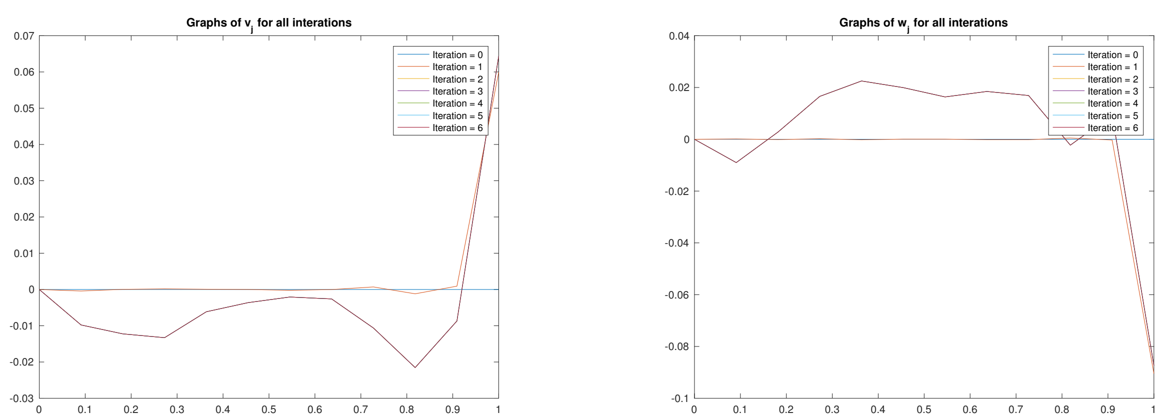
Similarly,

$$0 = \sum_{i=1}^{N} q_j w_j \Big|_0^T - \int_0^T \sum_{i=1}^{N} q_j' w_j dt + \int_0^T \sum_{i=1}^{N} p_j' v_j dt - \frac{1}{h^2} \int_0^T a_h(t) q_N dt. \tag{8.0.12}$$

Then, (8.0.9), (8.0.11) and (8.0.12) imply the solutions of (8.0.4) satisfy

$$\begin{aligned}
0 &= h \sum_{i=1}^{N} p_j v_j \Big|_0^T + h \sum_{i=1}^{N} q_j w_j \Big|_0^T + \frac{1}{h} \int_0^T b_h(t) p_N dt - \frac{1}{h} \int_0^T a_h(t) q_N dt \\
&= \int_0^T b_h(t) \frac{p_N}{h} dt - \int_0^T a_h(t) \frac{q_N}{h} dt + h \sum_{i=1}^{N} p_j v_j \Big|_0^T + h \sum_{i=1}^{N} q_j w_j \Big|_0^T \\
&= \int_0^T b_h(t) \frac{\psi_N}{h} dt - \int_0^T a_h(t) \frac{\psi_{2N}}{h} dt + h \sum_{j=1}^{2N} [\tilde{y}_j(T)\psi_j(T) - \tilde{y}_j(0)\psi_j(0)] \\
&= h \sum_{j=1}^{2N} [\tilde{y}_j(0)\psi_j(0)] + h \sum_{j=1}^{2N} [\tilde{y}_j(T)\psi_j(T) - \tilde{y}_j(0)\psi_j(0)] \\
&= h \sum_{j=1}^{2N} [\tilde{y}_j(T)\psi_j(T)]
\end{aligned}$$

which gives us $\tilde{Y}(T) = 0$. Thus, (8.0.4) is null controllable and the controls are functions in (8.0.10). □



I II

Figure 8.1: Graphs of $z_j, j = 0, 1, \ldots N+1$ for each iteration until the map $\Upsilon(Z)$ achieves a fixed point. I is the real part and II is the imaginary part of z_j. $\kappa = 0.5, L = 1, h = \frac{1}{11}$, $t = T$ and the given initial values of z_j are zeros. After 6 iterations, $\max(|\mathrm{Re}(z_j) - v_j|)$ and $\max(|\mathrm{Im}(z_j) - w_j|) \leq 10^{-5}$, $j = 0, \ldots, N+1$.

In order to find the minimizer $\hat{\Psi}^0$, we take the derivative of $\mathcal{J}(\Psi^0)$ with respect to ψ_k^0 for any $k = 1, \ldots, 2N$. Then

$$\begin{aligned}\mathcal{J}(\Psi^0) &= \frac{1}{2}\int_0^T \left|\frac{\psi_N}{h}\right|^2 + \left|\frac{\psi_{2N}}{h}\right|^2 dt + h < \tilde{Y}^0, \Psi(0) > \\ &= \frac{1}{2h^2}\int_0^T \left(\sum_{i=1}^{2N}\psi_i^0 \tilde{L}_{Ni}(t)\right)^2 + \left(\sum_{i=1}^{2N}\psi_i^0 \tilde{L}_{2Ni}(t)\right)^2 dt + h\sum_{i=1}^{2N}\tilde{y}_i^0\left(\sum_{j=1}^{2N}\psi_j^0 \tilde{L}_{ij}(0)\right)\end{aligned}$$

and

$$\frac{\partial \mathcal{J}(\Psi^0)}{\partial \phi_k^0} = \frac{1}{h^2}\int_0^T \left(\sum_{i=1}^{2N}\psi_i^0 \tilde{L}_{Ni}(t)\right)\tilde{L}_{Nk} + \left(\sum_{i=1}^{2N}\psi_i^0 \tilde{L}_{2Ni}(t)\right)\tilde{L}_{2Nk} dt + h\sum_{i=1}^{2N}\tilde{y}_i^0 \tilde{L}_{ik}(0).$$

where the matrix $\tilde{L} = e^{A^*(t-T)+\int_t^T D(s)ds}$ is the solution of (8.0.4) with

$$D(t) = \kappa \begin{pmatrix} \mathbf{0} & \begin{matrix} |z_1(t)|^2 & & \\ & \ddots & \\ & & |z_N|(t)^2 \end{matrix} \\ \begin{matrix} |z_1(t)|^2 & & \\ & \ddots & \\ & & |z_N(t)|^2 \end{matrix} & \mathbf{0} \end{pmatrix}_{2N\times 2N}.$$

Letting $\frac{\partial \mathcal{J}(\Phi^0)}{\partial \phi_k^0} = 0$ for any $k = 1, \ldots, 2N$, we have

$$\tilde{\Xi} \cdot \Psi^0 = \tilde{\Omega}, \tag{8.0.13}$$

where the $ij^{th}(i, j = 1, \ldots, 2N)$ element of matrix $\tilde{\Xi}$ is

$$\frac{1}{h^2}\int_0^T \tilde{L}_{Ni}(t)\tilde{L}_{Nj}(t) + \tilde{L}_{2Ni}(t)\tilde{L}_{2Nj}(t)dt,$$

and the $k^{th}(k = 1, \ldots, 2N)$ element of vector $\tilde{\Omega}$ is

$$-h\sum_{i=1}^{2N}\tilde{y}_i^0 \tilde{L}_{ik}(0),$$

Then, the solution Ψ^0 of system (8.0.13) is the minimizer $\hat{\Psi}^0$ of $\mathcal{J}(\Psi^0)$. (see appendix B for the MATLAB code)

We draw the graphs of $z_j, j = 0, \ldots, N+1$ at $t = T$ for each iteration, in order to observe the tendency of map Υ approaching to its fixed point. In view of Figure 8.1, it is clear that the fixed point exists. After the map $\Upsilon(Z)$ achieves its fixed point, i.e. the number of the iterations is 6 shown in Figure 8.1, the corresponding controls $a_h(t), b_h(t)$ and the simulations of the solutions for (8.0.4) can be obtained as Figure 8.2. Figure 8.2 is also the graph of the

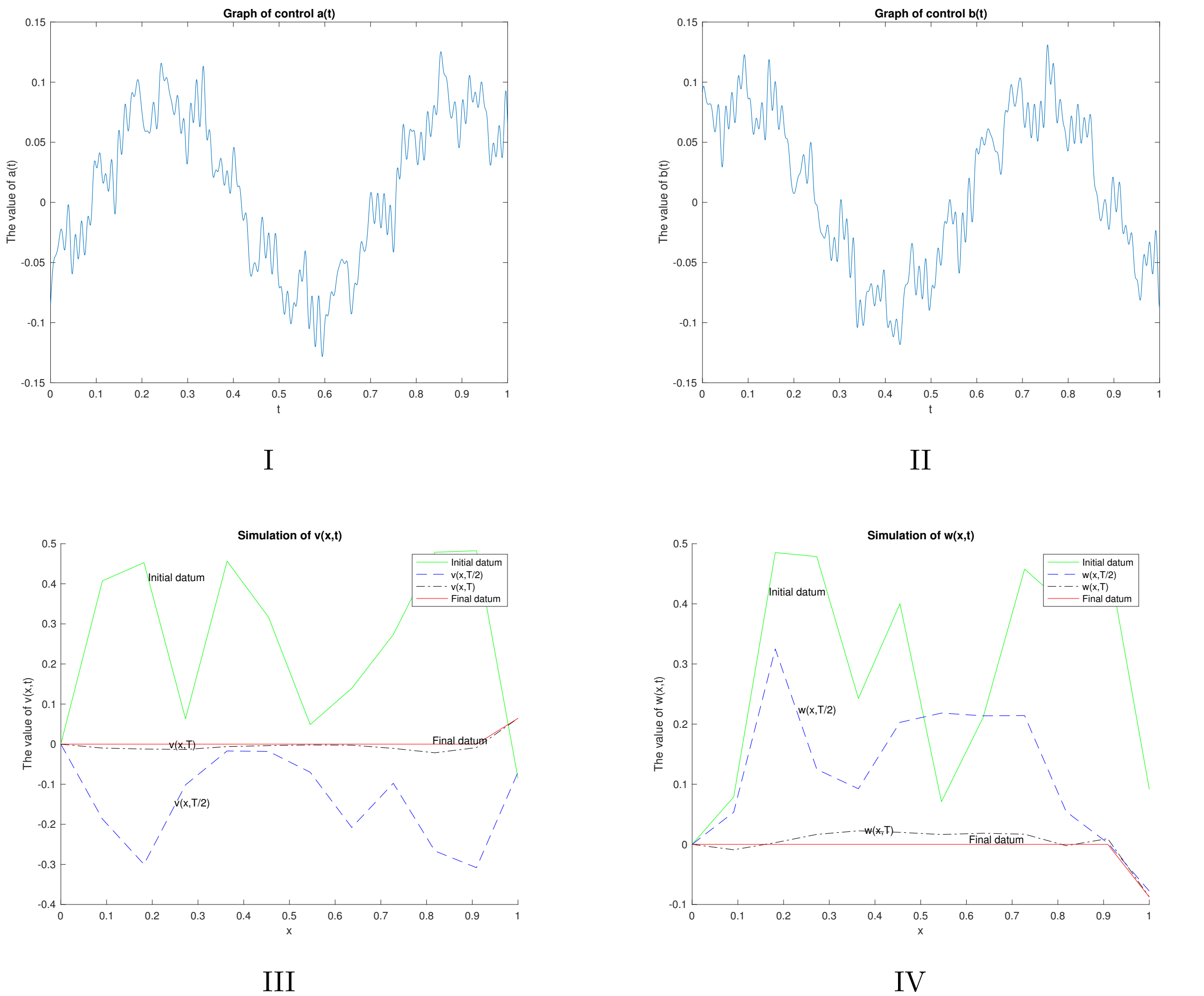


Figure 8.2: Graphs of controls and simulations of solutions for the linearized Schrödinger equation system (8.0.4) when the map $\Upsilon(Z)$ achieves its fixed point. $\kappa = 0.5, L = 1, h = \frac{1}{11}$. v_j^0 and $w_j^0,\ j = 1, \cdots, N$, are any random numbers in $(0, 0.5)$. $v_j^1 = w_j^1 = 0,\ j = 1, \cdots, N$.

controls and solutions of (8.0.3). As $h \to 0$, they approach to the controls $a(t)$ and $b(t)$, and solutions $v(x,t)$ and $w(x,t)$ for the continuous nonlinear Schrödinger system (8.0.2). But in order to prove the convergence, we need to show the uniformly null controllable of (8.0.3), i.e. Theorem 8.0.1 holds for any h, which is an open problem we will study in the future.

The main idea is to prove that the observability inequality (7.1.10) holds for (8.0.5), with a constant C which is independent of h. We can decompose the solution Ψ of (8.0.5) as $\Psi = \Phi + \Sigma$, where Φ solves the unperturbed system (7.1.4) with the same initial data as Ψ itself, i.e. $\Phi^0 = \Psi^0$. The remainder $\Sigma = (\sigma_{-N}, \ldots, \sigma_{-1}, \sigma_1, \ldots, \sigma_N)^T = (\varsigma_1, \ldots, \varsigma_N, \varrho_1, \ldots, \varrho_N)^T$

solves

$$\begin{cases} \varsigma_j' + \dfrac{\varrho_{j+1} - 2\varrho_j + \varrho_{j-1}}{h^2} = -|z_j|^2 q_j, & j = 1, \ldots, N,\ 0 < t < T, \\ \varrho_j' - \dfrac{\varsigma_{j+1} - 2\varsigma_j + \varsigma_{j-1}}{h^2} = |z_j|^2 p_j, & j = 1, \ldots, N,\ 0 < t < T, \\ \varsigma_0(t) = \varrho_0(t) = 0, & 0 < t < T, \\ \varsigma_{N+1}(t) = \varrho_{N+1}(t) = 0, & 0 < t < T, \\ \varsigma_j(T) = \varrho_j(T) = 0, & j = 1, \ldots, N. \end{cases} \tag{8.0.14}$$

Φ satisfying (7.1.10) has been proved in Chapter 7. We only need to consider the observability inequality for (8.0.14), which is not easy.

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Appendix A

MATLAB Code for Numerical Approximation of Linear Schrödinger Equation

```
format long
L = 1;
Np = 20; %number of interior points
Nx = 2*Np;
h = L/(Np+1); %width of subinterval on (0,1), x_0=0, x_(N+1)=L
x_interval = 0:h:L; %all of the points on x-axis

%%define matrix A
PA = zeros(Nx, Nx);
for i = 1:Np-1;
    PA(i, Np+i) = -2;
    PA(i+1, Np+i) = 1;
    PA(i, Np+i+1) = 1;
    PA(Np+i, i) = 2;
    PA(Np+i+1, i) = -1;
    PA(Np+i, i+1) = -1;
end
PA(Np, Nx) = -2;
PA(Nx, Np) = 2;
A = PA/(h^2);

%%initial and end states
Y0 = rand(Nx,1)';
Y1 = zeros(1,Nx); %null end states
```

```
%given end states
%x = linspace(0,1,Nx);
%Y1 = -(x-0.5).^2+0.5;

%%time
T = 1;
dt = 0.0000001;   % the step size of time
t = 0:dt:T;
Nt = length(t);

%%the right hand side Omega of equation 7.1.17
L_zero = expm(-A'*T);   %L when t=0, which is expm(-A'*T)
Omega = zeros(Nx,1); %we need Omega to be vector
for i = 1:Nx
    Omega(i) = h*(Y1(i)-sum(Y0'.*L_zero(:,i)));
end

%%the left hand side Xi of equation 7.1.17
a = L_zero(Np,:);
b = L_zero(Nx,:);
L_T = eye(Nx);
a1 = L_T(Np,:);
b1 = L_T(Nx,:);
lhs_sum = zeros(Nx,Nx);
for j = 1:Nt
    L = expm(A'*(t(j)-T)); % L for one fixed time
    L_N = L(Np,:);
    L_2N = L(Nx,:);
    lhs_element = (L_N'*L_N)+(L_2N'*L_2N);
    lhs_sum = lhs_element + lhs_sum;
end
lhs_integral = (dt)*(lhs_sum-(a'*a+b'*b)/2-(a1'*a1+b1'*b1)/2);
lhs = lhs_integral/(h^2);

%%minimizer varphi0
varphi0 = lhs\Omega;

%%define Matrix B
B = zeros(Nx, 2);
B(Np,1) = -1/(h^2);
B(Nx,2) = 1/(h^2);
```

```
%%control H(t) = h*B_star*(L*varphi0)
for i = 1:Nt
    control(:,i) = h*B*(expm(A'*(t(i)-T))*varphi0);
end

%%the values of controls at t=0,T/2 and T
Gcontrol(1,:) = control(:,1)';
Gcontrol(2,:) = control(:,(Nt-1)/2+1)';
Gcontrol(3,:) = control(:,Nt)';

%%container to draw graphs
Gv = zeros(4,Np+2);
Gw = zeros(4,Np+2);

%%solve Y by forward Euler method
Y_start = Y0';
for i = 1:(Nt-1)
    Y = Y_start+dt*(-A*Y_start+B*control(:,i));
    if i == (Nt-1)/2
        Gv(2,2:Np+1) = Y(1:Np)'
        Gw(2,2:Np+1) = Y(Np+1:2*Np)'
    end
    Y_start = Y;
end

%%draw the graphs of controls and solutions
Gv(1:3,Np+2) = Gcontrol(:,2);
Gw(1:3,Np+2) = Gcontrol(:,1);
Gv(1,2:Np+1) = Y0(1:Np);
Gw(1,2:Np+1) = Y0(Np+1:2*Np);
Gv(3,2:Np+1) = Y(1:Np)';
Gw(3,2:Np+1) = Y(Np+1:2*Np)';
Gv(4,2:Np+1) = Y1(1:Np);
Gw(4,2:Np+1) = Y1(Np+1:2*Np);
Gv(4,Np+2) = Gcontrol(3,2);
Gw(4,Np+2) = Gcontrol(3,1);

%graphs of controls a(t) and b(t)
figure
plot(t,control(2,:))
title('Graph of control a(t)')
xlabel(' t ')
```

```
ylabel('The value of a(t)')
figure
plot(t,control(1,:))
title('Graph of control b(t)')
xlabel(' t ')
ylabel('The value of b(t)')

%graphs of v and w
figure; hold on
x = x_interval;
v1 = Gv(1,:);
v2 = Gv(2,:);
v3 = Gv(3,:);
v4 = Gv(4,:);
plot(x, v1,'-g', x,v2,'--b',x,v3,'-.k', x,v4,'-r')
legend('Initial datum','v(x,T/2)','v(x,T)','Final datum')
title('Simulation of v(x,t)')
xlabel(' x ')
ylabel('The value of v(x,t)')
gtext({'Initial datum';'v(x,T/2)';'v(x,T)';'Final datum'})
figure; hold on
x = x_interval;
w1 = Gw(1,:);
w2 = Gw(2,:);
w3 = Gw(3,:);
w4 = Gw(4,:);
plot(x, w1,'-g', x,w2,'--b',x,w3,'-.k', x,w4,'-r')
legend('Initial datum','w(x,T/2)','w(x,T)','Final datum')
title('Simulation of w(x,t)')
xlabel(' x ')
ylabel('The value of w(x,t)')
gtext({'Initial datum';'w(x,T/2)';'w(x,T)';'Final datum'})
```

Appendix B

MATLAB Code for Numerical Approximation of Nonlinear Schrödinger Equation

```
format long
L = 1;
Np = 10; %number of interior points
Nx = 2*Np;
h = L/(Np+1); %width of subinterval on (0,1), x_0=0, x_(Np+1)=L
x_interval = 0:h:L; %all of the points on x-axis
kappa = 0.5;

%%define matrix A
PA = zeros(Nx, Nx);
for i = 1:Np-1;
    PA(i, Np+i) = -2;
    PA(i+1, Np+i) = 1;
    PA(i, Np+i+1) = 1;
    PA(Np+i, i) = 2;
    PA(Np+i+1, i) = -1;
    PA(Np+i, i+1) = -1;
end
PA(Np, Nx) = -2;
PA(Nx, Np) = 2;
A = PA/(h^2);

%%initial and end states
Y0 = rand(Nx,1)'./ 2;
```

```
Y1 = zeros(1,Nx);

%%define the coefficient matrix of |z_j|^2
matrixcz = zeros(Nx,Nx);
for i = 1:Np;
    matrixcz(i, Np+i) = 1;
    matrixcz(Np+i, i) = -1;
end

%%define time
T = 1;
dt = 0.0000001;  % the step size of time
t = 0:dt:T;
Nt = length(t);

%%find fixed point
Z_full = zeros(Nx, Nt); %intial data of z_j
YCompare = Z_full;
maxerror = 1e-5; %error boundary of z_j and u_j
error_fixpoint = 1;
iteration = 0; %intial data for number of iterations
while maxerror < error_fixpoint
    %find the vector Omega on the right have side of equation 8.0.13
    D_sum = zeros(Nx, Nx);
    for i = 1:Nt
        D = kappa*diag(Z_full(:,i))*matrixcz;
        D_sum = (D)'+D_sum;
    end
    Dp_sum = D_sum;
    D_zero = (kappa*diag(Z_full(:,1))*matrixcz)';
    D_T = (kappa*diag(Z_full(:,Nt))*matrixcz)';
    Dp_T = D_T;
    L_zero = expm(-A'*T-dt*(D_sum-D_zero/2-D_T/2));
    Omega = zeros(Nx,1);  %we need Omega to be vector
    for i = 1:Nx
        Omega(i) = h*(-sum(Y0'.*L_zero(:,i))); %zero end status
    end

    %find the matrix Xi on the left hand side of equation 8.0.13
    a = L_zero(Np,:);
    b = L_zero(Nx,:);
```

```
L_T = eye(Nx);
a1 = L_T(Np,:);
b1 = L_T(Nx,:);
lhs_sum = zeros(Nx,Nx);
for j = 1:Nt
    D_j = (kappa*diag(Z_full(:,j))*matrixcz)';
    integral_sum = dt*(D_sum-D_j/2-D_T/2);
    L = expm(A'*(t(j)-T)-integral_sum);  %L for one fixed time
    L_N = L(Np,:);
    L_2N = L(Nx,:);% temporary matrix
    lhs_element = (L_N'*L_N)+(L_2N'*L_2N);
    lhs_sum = lhs_element + lhs_sum;
    D_sum = D_sum - D_j;
end
lhs_integral = (dt)*(lhs_sum-(a'*a+b'*b)/2-(a1'*a1+b1'*b1)/2);
lhs = lhs_integral/(h^2);

%solve the minimizer
varphi0 = lhs \ Omega

%define Matrix B
B = zeros(Nx, 2);
B(Np,1) = -1/(h^2);
B(Nx,2) = 1/(h^2);

%control H(t)=h*B_star*(L*varphi0)
for i = 1:Nt
    Dp_i = (kappa*diag(Z_full(:,i))*matrixcz)';
    integralp_sum = dt*(Dp_sum-Dp_i/-Dp_T/2);
    Lp = expm(A'*(t(i)-T)-integralp_sum);
    control(:,i) = h * B' * Lp * varphi0;
    Dp_sum = Dp_sum - Dp_i;
end

%simulate the solution Y by backward Euler method
Y(:,1) = Y0';
for i = 1:(Nt-1)
    Y(:,i+1) = ((eye(Nx)+A*dt)^(-1))*(Y(:,i)+B*control(:,i+1)*dt);
end

error_fixpoint = norm(Y-YCompare,Inf);
YCompare = Y;
```

```
    for i = 1:Np,
        Z_full(i,:) = Y(i,:).^2+Y(i+Np,:).^2;
        Z_full(Np+i,:) = Y(i,:).^2+Y(i+Np,:).^2;
    end

    G_V(:,iteration+1) = Y(1:Np, Nt);
    G_W(:,iteration+1) = Y(1+Np:Nx, Nt);
    G_VC(1,iteration+1) = control(2,Nt);
    G_WC(1,iteration+1) = control(1,Nt);

    controla_Z(:,iteration+1) = control(2,:)';
    controlb_Z(:,iteration+1) = control(1,:)';

    iteration = iteration + 1
end

%%draw graphs
G_Vtotal = zeros(Np+2, iteration+1);
G_Wtotal = zeros(Np+2, iteration+1);
G_Vtotal(2:Np+1, 2:iteration+1) = G_V(:,1:iteration);
G_Wtotal(2:Np+1, 2:iteration+1) = G_W(:,1:iteration);
G_Vtotal(Np+2, 2:iteration+1) = G_VC(1,1:iteration);
G_Wtotal(Np+2, 2:iteration+1) = G_WC(1,1:iteration);

Gcontrol(1,:) = control(:,1)';
Gcontrol(2,:) = control(:,(Nt-1)/2+1)';
Gcontrol(3,:) = control(:,Nt)';
Gv = zeros(4,Np+2);
Gw = zeros(4,Np+2);
Gv(2,2:Np+1) = Y(1:Np,(Nt-1)/2+1)';
Gw(2,2:Np+1) = Y(Np+1:2*Np,(Nt-1)/2+1)';
Gv(1:3,Np+2) = Gcontrol(:,2);
Gw(1:3,Np+2) = Gcontrol(:,1);
Gv(1,2:Np+1) = Y0(1:Np);
Gw(1,2:Np+1) = Y0(Np+1:2*Np);
Gv(3,2:Np+1) = Y(1:Np,Nt)';
Gw(3,2:Np+1) = Y(Np+1:2*Np,Nt)';
Gv(4,Np+2) = Gcontrol(3,2);
Gw(4,Np+2) = Gcontrol(3,1);

%Graphs of v_j for all interations
```

```
figure;
for i = 1:iteration+1
    plot(x_interval, G_Vtotal(:,i))
    hold on
    legendInfo{i} = ['Iteration = ' num2str(i-1)];
end
legend(legendInfo)
hold off
title('Graphs of v_j for all interations')
xlabel(' x ')

%Graphs of w_j for all interations
figure;
for i = 1:iteration+1
    plot(x_interval, G_Wtotal(:,i))
    hold on
    legendInfo{i} = ['Iteration = ' num2str(i-1)];
end
legend(legendInfo)
hold off
title('Graphs of w_j for all interations')
xlabel(' x ')

%graph of controls a(t) and b(t)
figure
plot(t,control(2,:))
title('Graph of control a(t)')
xlabel(' t ')
ylabel('The value of a(t)')
figure
plot(t,control(1,:))
title('Graph of control b(t)')
xlabel(' t ')
ylabel('The value of b(t)')

%graph of v and w
figure; hold on
    x = x_interval;
    v1 = Gv(1,:);
    v2 = Gv(2,:);
    v3 = Gv(3,:);
    v4 = Gv(4,:);
```

```
    plot(x, v1,'-g', x,v2,'--b',x,v3,'-.k', x,v4,'-r')
    legend('Initial datum','v(x,T/2)','v(x,T)','Final datum')
    hold off
    title('Simulation of v(x,t)')
    xlabel(' x ')
    ylabel('The value of v(x,t)')
    gtext({'Initial datum';'v(x,T/2)';'v(x,T)';'Final datum'})
figure; hold on
    x = x_interval;
    w1 = Gw(1,:);
    w2 = Gw(2,:);
    w3 = Gw(3,:);
    w4 = Gw(4,:);
    plot(x, w1,'-g', x,w2,'--b',x,w3,'-.k', x,w4,'-r')
    legend('Initial datum','w(x,T/2)','w(x,T)','Final datum')
    hold off
    title('Simulation of w(x,t)')
    xlabel(' x ')
    ylabel('The value of w(x,t)')
    gtext({'Initial datum';'w(x,T/2)';'w(x,T)';'Final datum'})
```