Revisiting the CAPM and the Fama-French Multi-Factor Models:
Modeling Volatility Dynamics in Financial Markets

Michael Michaelides

Dissertation submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Economics, Science

Aris Spanos, Chair

Richard Ashley

Raman Kumar

Kwok Ping Tsang

March 22, 2017
Blacksburg, Virginia

Keywords: CAPM, Fama-French three-factor model, Fama-French five-factor model, factor selection, t-heterogeneity, Gram-Schmidt orthonormal polynomials, statistical adequacy, misspecification testing, Student's t, volatility modeling

Revisiting the CAPM and the Fama-French Multi-Factor Models:
Modeling Volatility Dynamics in Financial Markets

Michael Michaelides

Abstract

The primary objective of this dissertation is to revisit the CAPM and the Fama-French multi-factor models with a view to evaluate the validity of the probabilistic assumptions imposed (directly or indirectly) on the particular data used. By thoroughly testing the assumptions underlying these models, several departures are found and the original linear regression models are respecified. The respecification results in a family of heterogeneous Student's t models which are shown to account for all the statistical regularities in the data. This family of models provides an appropriate basis for revisiting the empirical adequacy of the CAPM and the Fama-French multi-factor models, as well as other models, such as alternative asset pricing models and risk evaluation models. Along the lines of providing a sound basis for reliable inference, the respecified models can serve as a coherent basis for selecting the relevant factors from the set of possible ones. The latter contributes to the enhancement of the substantive adequacy of the CAPM and the multi-factor models.

Revisiting the CAPM and the Fama-French Multi-Factor Models:
Modeling Volatility Dynamics in Financial Markets

Michael Michaelides

General Audience Abstract

The primary objective of this dissertation is to revisit the CAPM and the Fama-French multi-factor models with a view to evaluate the validity of the probabilistic assumptions imposed (directly or indirectly) on the particular data used. By probing for potential departures from the Normality, Linearity, Homoskedasticity, Independence, and t-invariance assumptions, it is shown that the assumptions implicitly imposed on these empirical asset pricing models are inappropriate. In light of these results, the probabilistic assumptions underlying the CAPM and the Fama-French multi-factor models are replaced with the Student's t, Linearity, Heteroskedasticity, Markov Dependence, and t-heterogeneity assumptions. The new probabilistic structure results in a family of heterogeneous Student's t models which are shown to account for all the statistical regularities in the data. This family of models provides an appropriate basis for revisiting the empirical adequacy of the CAPM and the Fama-French multi-factor models, as well as other models, such as alternative asset pricing models and risk evaluation models. Along the lines of providing a sound basis for reliable statistical inference results, the proposed models can serve as a coherent basis for selecting the potential sources of risk from a set of possible ones. The latter contributes to the enhancement of the substantive adequacy of the CAPM and the multi-factor models.

Acknowledgments

First and foremost, I would like to express my sincere gratitude to my advisor Dr. Aris Spanos for his patient guidance, unwavering support, and immense knowledge during my time at Virginia Tech. Dr. Spanos has been a tremendous mentor for me and I will forever remain indebted to him.

Thanks also go to my committee members: Dr. Kwok Ping Tsang, Dr. Richard Ashley, and Dr. Raman Kumar for their useful comments on earlier versions of this dissertation.

I would like to thank the faculty members and the academic support staff of the Department of Economics at Virginia Tech, as well as my fellow graduate students for sharing this adventure with me and making my stay in Blacksburg much more pleasurable.

I am also grateful to my previous teachers Mr. George Tsouris and Dr. Hiroyuki Nakata for helping me realize my potential.

Special thanks to Niraj Poudyal and Didem Beskardes for their valuable assistance with R and Python programming.

I would also like to express my heartfelt thanks to Mrs. Evie Spanos for her immeasurable hospitality and generosity.

I also thank my friends (too many to list here but you know who you are) for their support and friendship.

Last, but certainly not least, I would like to thank my family for believing in me, even when I did not believe in myself. I attribute my education in large part to my father Andreas and my mother Ioanna. I am especially indebted to the efforts made by them. I am also grateful to my brother Damianos, my grandfather Adamos, my grandmother Despina, and my uncle Ploutarchos for their encouragement and the lifetime of unconditional support.

Contents

List of Figures

List of Tables

Chapter 1

1 Introduction

1.1 Overview

The evaluation of financial assets is one of the most fundamental issues faced by individual investors and financial institutions. How does the risk of an investment affects, or should affect, its future return? Why do some financial assets tend to pay higher average returns than others? How are the various types of risk measured? Asset pricing theory is concerned with providing answers to these questions and continues to be one of the most important theoretical underpinnings of portfolio management and investment analysis.

The primary concern of this dissertation is to revisit two of the most important contributions to the empirical study of asset pricing (the Capital Asset Pricing Model and the Fama-French multi-factor models) from an econometric perspective. Yet, much of the discussion is extensive to any empirical testing of asset pricing models, such as the conversion of asset pricing theories into models which are estimable with a given set of data, or empirical studies searching for potential sources of risk. Hence, it is of paramount importance for the reader to keep an eye on the forest and not get distracted by the trees.

This introductory chapter begins by briefly placing, in section 1.2, the construction and empirical implementation of asset pricing models in a historical context. Having discussed in brief the role of factor models in asset pricing, section 1.3 provides a short coverage of the Fama-French multi-factor models. Section 1.4 provides a brief review of previous research, as it relates to regularity patterns exhibited by stock returns data. As to reflect on the serious consequences of ignoring such regularity patterns in asset pricing modeling, section 1.5 goes on to employ a number of Monte-Carlo simulations in an attempt to place in context the importance of these patterns and their role in empirical asset pricing. Finally, section 1.6 serves to illustrate a brief outline of the discussion follows.

1.2 The Development of Asset Pricing Models

The early work of Markowitz (1952) marked a new era in asset pricing theory. In his pioneering paper, Markowitz argued that a risk averse investor who focuses on the mean and variance of the returns of individual assets contained in a portfolio, is motivated by the highest possible profit given the lowest possible risk. By providing a mathematical justification, he showed that an investor wants to maximize the mean and minimize the variance, and therefore, the optimal portfolio is one whose mean-variance combination is in the efficient frontier. Yet, it was Tobin's (1958) separation theorem that extended the logic of efficient frontier using one's attitude towards risk. He demonstrated that an investor who already holds the optimal combination of risky assets and is able to borrow or lend at the risk-free interest rate, can decide whether to borrow or lend, depending on his attitude towards risk. His main conclusion was that the single market portfolio is the efficient frontier that dominates any other combination.

Although the groundwork of Markowitz and Tobin was simple and intuitive, it presented difficulties in the computation of the variance-covariance matrix when the number of assets was very large. The latter limitation, together with the absence of a theory for estimating the correct cost of capital of an investment in the influential work of Modigliani and Miller (1958), provided the primary motivation for Treynor (1962), Sharpe (1964), Lintner (1965a; 1965b), and Mossin (1966), to independently introduce the most popular model in the asset pricing literature; the Capital Asset Pricing Model (CAPM). By assuming the existence of lending and borrowing at a risk-free rate of interest, the CAPM:

$$\mu_i^e - \mu_f = \beta_{im}(\mu_m^e - \mu_f),\ i=1,2,...,k, \tag{1.1}$$

expresses a linear relationship between excess of returns, where μ_i^e is the expected return of asset i, μ_f denotes the return of the risk-free asset, $(\mu_i^e - \mu_f)$ is the expected excess asset return (risk premium) of asset i, μ_m^e is the expected return of the market, $(\mu_m^e - \mu_f)$ is the expected excess market return (market premium), and $\beta_{im} = [Cov(\mu_m, \mu_i)/Var(\mu_m)]$ is the market beta of asset i.

The theoretical underpinnings of the CAPM depended on a number of strong assumptions. First, investors are rational and risk averse, and behave according to a single-period investment horizon. Second, they all select their portfolios according to a mean-variance objective, and behave in a manner as to maximize their economic utility defined in terms of wealth at a specified date in the future. Third, all investors share homogeneous beliefs with respect to the investment opportunities, and use the same estimates of the expectations, variances, and covariances of normally distributed asset returns. Fourth, investors are broadly diversified across a range of assets and can borrow or lend unlimited amounts under the risk-free rate. Fifth, the stock market functions under conditions of perfect competition and equilibrium. Investors, - who all have access to available information - have the right to buy and/or sell any amount of shares, at any period of time, having a very little impact, if any, on the market price. Sixth, investors are not burdened of any transaction costs and at the same time taxation and any other sort of restrictions are similarly absent; for a detailed description of these assumptions, see Bailey (2005, pp. 144-145).

The fact that every assumption is likely to be violated in the real world, motivated researchers to extend the CAPM, in an attempt to improve it by relaxing some of the aforementioned assumptions. For instance, Black (1972) developed a version of the CAPM without the possibility of borrowing and lending at a risk-free rate.

Merton (1973) constructed an intertemporal CAPM (ICAPM) in which investors act as to maximize the economic utility of lifetime consumption and decisions are made in continuous time. Ross (1976) derived the Arbitrage Pricing Theory (APT) which assumes neither risk averse investing nor normally distributed asset returns; for detailed information about extensions to the basic CAPM, see Levy (2012, pp. 156-185).

Besides, the inability of the static (statistical) CAPM to explain the cross-section of average returns, triggered the interest of academics and professionals to identify observed random variables (henceforth factors) that provide additional explanatory power for average returns not captured by the market beta. Harvey et al. (2015) catalogue 250 different factors reported in the literature, not including those reported in unpublished manuscripts. Just to allude to a few of the most widely known empirically motivated factors: Chen et al. (1986) identified several macroeconomic variables (e.g., industrial production growth, unexpected inflation) as significant in affecting stock market returns; Fama and French (1993; 2015) showed that their proposed three-factor and five-factor models capture much of the variation in average returns for portfolios formed on firm characteristics, such as size, book-to-market equity, operating profitability, and investment; Carhart (1997) introduced a four-factor model, adding a momentum factor to enhance the Fama-French three-factor model; Pástor and Stambaugh (2003) found that expected stock returns are related cross-sectionally to the sensitivities of returns to fluctuations in market liquidity.

Did the CAPM developed into a better asset pricing model since its introduction? From the theory perspective, the extensions of CAPM provide a clearer, more refined, and theoretically a more appealing framework which attempts to bridge the gap between theory and data in more elaborated ways. The bridging of the gap between theory and data, however, has a weak link. The validity of the probabilistic assumptions (directly or indirectly) imposed on the data is not thoroughly tested. As a result, the various improvements of the original CAPM have not been subjected to adequate empirical scrutiny. The inclusion of additional factors provides sound empirical improvements to the original model only when they are statistically justified. The latter requires that the statistical tests used to establish their significance are reliable. That is, the actual error probabilities approximate well the nominal (assumed) ones. When one uses a .05 significance level t-test to establish the significance of a new factor, but the actual Type I error probability (due to invalid assumptions for the estimated model) is closer to .9, the inference result is likely to be erroneous. How does one secure the error-reliability of such testing procedures? The estimated model in the context of which the additional factors are tested is statistically adequate. That is, its probabilistic assumptions are valid for the data in question; see the simulation results in Appendix A.

1.3 The Fama-French Multi-Factor Models

The seminal work of Fama and French (1992; 1993) undoubtedly marked a great turning point in the development of factor models. Their work spurred interest in studying cross-sectional return patterns (see Harvey et al., 2015) and has had a substantial impact to the academic literature and professional practice, reflected in citations to these papers and the frequent utilization of their data.

Of course, the work of Fama and French was originally motivated by the poor empirical performance of the CAPM, but what really triggered their interest to study the cross-sectional variation in average stock returns (Fama and French, 1992), and later to introduce the three-factor model (Fama and French, 1993), was the empirical contradictions of the CAPM documented in the existing literature at that time. Notably, the strong negative relation between *size* (market capitalization) and average return (Banz, 1981); the positive relation between earnings-price ratios (E/P) and average return (Basu, 1983); the positive relation between book-to-market equity (B/M) and average return (Rosenberg et al., 1985); and the positive relation between leverage and average return (Bhandari, 1988).

Fama and French (1992) used the cross-sectional regression approach of Fama and MacBeth (1973) to investigate the cross-sectional variation in average stock returns associated with the abovementioned variables. The main findings of the paper were summarized as follows. First, the positive relation between average return and the market beta predicted by the CAPM (see Black et al., 1972; Fama and MacBeth, 1973) disappears during the 1963-1990 period for U.S. stocks. Second, the cross-sectional variation in average stock returns associated with size, B/M, E/P, and leverage is captured by size and B/M. In other words, E/P and leverage are redundant for describing average returns, at least for the data in question.

Supplemental, Fama and French (1993) used the time-series regression approach of Black et al. (1972) to further investigate the explanatory power of size and B/M in capturing the cross-section of average returns. The interpretation of the time-series regressions revealed that portfolios constructed to mimic size and B/M capture common variation in returns, no matter what else is in the time-series regressions. The latter result gave rise to the famous Fama-French three-factor model, which adds two new factors to the CAPM:

$$\underbrace{(R_{it}-R_{ft})=\alpha_i+\beta_i(R_{mt}-R_{ft})}_{\text{CAPM}}+s_i SMB_t+h_i HML_t+\varepsilon_{it},\ i=1,2,...,k,\ t\in\mathbb{N}, \tag{1.2}$$

where R_{it} is the return of asset or portfolio i for period t, R_{ft} is the risk-free return, R_{mt} is the return of the value-weighted market portfolio, SMB_t is the difference between the returns on diversified portfolios of small and big size stocks, HML_t is the difference between the returns on diversified portfolios of high and low B/M stocks, $\alpha_i, \beta_i, s_i, h_i$ refer to the estimated parameters, and ε_{it} is a zero-mean residual.

The work of Fama and French (1992; 1993) has been criticized over time from various perspectives. For instance, one key criticism is that the model is motivated by a purely empirical conception and the selection of the factors is ad hoc; see Bailey (2005, p. 186) for a general discussion on this ad hoc selection. While the ICAPM is often interpreted as providing a justification for the use of factor models, the size and

B/M are not themselves state variables as assumed by the ICAPM. Instead, these constructed factors allow in explaining expected returns of state variables without identifying them; see Fama (1996). Yet, it is important to note that also the five-factor model discussed below, as well as the majority of the factor models in the literature are subject to similar and other criticisms; see for e.g., Black (1993), Kothari et al. (1995), for criticisms centered on data snooping and survivor bias.

More recently, Fama and French (2015) introduced a five-factor asset pricing model, adding operating profitability (OP) and investment (Inv) factors to augment the three-factor model. By employing a time-series regression approach, they showed that the five-factor model explains between 71% and 94% of the cross-section variance of expected returns for the size, B/M, OP, and Inv portfolios examined. Besides, the results suggest that the portfolio constructed to mimic B/M is redundant for describing average returns when mimicking portfolios for OP and Inv are included in the model, at least for the data in question. The Fama-French five-factor model takes the following form:

$$\underbrace{\underbrace{(R_{it}-R_{ft})=\alpha_i+\beta_i(R_{mt}-R_{ft})}_{\text{CAPM}}+s_iSMB_t+h_iHML_t}_{\text{Fama-French three-factor model}}+r_iRMW_t+c_iCMA_t+\varepsilon_{it}, \quad i=1,2,...,k, \;\; t\in\mathbb{N}, \tag{1.3}$$

where RMW_t is the difference between the returns on diversified portfolios of robust and weak profitability stocks, CMA_t is the difference between the returns on diversified portfolios of low (conservative) and high (aggressive) investment firms, and r_i, c_i, refer to the estimated parameters; more information about the Fama-French factors will be discussed later as needed.

1.4 Non-Normality and Heterogeneity

The mean-variance framework pioneered by Markowitz (1952) was build upon the assumption that future asset returns are normally distributed. This conventional approach was defensive to the conception of previous studies (e.g., Bachelier, 1900; Osborne, 1959), whose arguments were based on misinterpretations of the Central Limit Theorem. Over the years, however, several studies provided enough evidence against Normality and suggested that the peak of the return distribution is much higher, and the tails are much thicker than for the Normal distribution. Some of the best-known alternatives to the Normal distribution in modeling stock returns include the stable Paretian family of distributions (Mandelbrot, 1963; Fama, 1965); compound Poisson process (Press, 1967); Student's t distribution (Praetz, 1972; Blattberg and Gonedes, 1974); Log-normal distribution (Clark, 1973); family of Exponential Power distributions (Box and Tiao, 1973); Logistic distribution (Smith, 1981); and generalized Beta distribution of the second kind (Bookstaber and McDonald, 1987); for additional evidence against Normality in both the stock returns and market model residuals, see also Blume (1968), Fama et al. (1969), Teichmoeller (1971), Officer (1972), Fama (1976), Westerfield (1977), Hagerman (1978), Affleck-Graves and McDonald (1989), and Zhou (1993).

Besides, evidence of heteroskedasticity in both the stock returns and market model residuals has been documented by various authors, including Praetz (1969), Miller and Scholes (1972), Morgan (1976), Belkaoui (1977), Brown (1977), Bey and Pinches (1980), Barone-Adesi and Talwar (1983), and Schwert and Seguin (1990). Indeed, a number of studies presented evidence in favor of homoskedasticity (e.g., Martin and Klemkosky, 1975; Fama et al., 1969; Brenner and Smidt, 1977) which is questionable because homoskedasticity, in turn, provides evidence in favor of Normality since within the elliptically symmetric family of distributions, the Normal is the only distribution which is homoskedastic; see Kelker (1970).

Although most attention of the literature has been focused on the analysis of the distribution of stock returns, a number of studies presented results concerning the dependence and time-heterogeneity (henceforth t-heterogeneity or heterogeneity) assumptions. Indeed, these results have been few and somewhat mixed. For instance, numerous studies (e.g., Granger and Morgenstern, 1963; King, 1966; Blume, 1968;

Fama et al., 1969) indicated that stock returns and market model residuals are serially independent, while other studies have documented the presence of different forms of temporal dependence, such as serial correlation or long-term dependence; see for e.g., Young (1971), Greene and Fielitz (1977), and Perry (1982). Likewise, a number of studies (e.g., Jensen, 1969; Beaver et al., 1970; Blume, 1971) presented results in favor of mean homogeneity, whereas others found evidence supporting mean heterogeneity; see for e.g., Gonedes (1973), Meyers (1973), and Blume (1975).

In addition, numerous studies have investigated mean heterogeneity in the form of seasonal patterns in stock returns. For instance, significant focus has been paid on the so-called January effect (see Rozeff and Kinney, 1976) where stock returns are higher in January than in the rest of the year. Although not as significant as the January effect, a large number of other seasonal patterns have been examined, including the weekend effect (French, 1980), turn-of-the-month effect (Ariel, 1987), and the Halloween effect (Bouman and Jacobsen, 2002).

As a matter of fact, there is, perhaps, no study pertaining to the investigation and testing of variance heterogeneity in stock returns. Having said that, it is worth mentioning that variance heterogeneity is often wrongly misinterpreted as heteroskedasticity. Thus, one of the primary aims of the present study is to delineate the difference between the two in the context of linear regression models. In point of fact, variance heterogeneity is the case where the stock returns are characterized by a non-identically distributed process and on that account the conditional variance varies with the time index. In contrast, heteroskedasticity is the case where the conditional variance varies with the conditioning variables. Indeed, this delineation will play a crucial role later, when modeling portfolio return volatility dynamics.

1.5 Statistical Adequacy and its Role in Asset Pricing

As will be discussed in great detail in the subsequent chapters, the success of empirical asset pricing depends crucially on how adequately the regularity patterns discussed above, are accounted for by asset pricing models. As it will be argued, in the general context of Linear Regression (LR) models, the regularities in the data take the form of testable probabilistic assumptions. Indeed, in empirical asset pricing, more often than not, the set of probabilistic assumptions forming an asset pricing model is a parameterization of the Normal, Independent, and Identically Distributed (NIID) process underlying a given set of data. In this section, the term 'NIID' will be used exclusively for simplicity. Yet, in chapter 3 a statistical model will be specified in terms of a complete, internally consistent, and testable list of model assumptions.

As of now, this section will serve as an early attempt to bring out the serious consequences that may arise when certain regularity patterns are ignored in asset pricing modeling. To bring out these serious consequences a number of Monte-Carlo simulation designs are employed. These simulations are designed in a way to comprise departures from the NIID assumptions. The latter will allow us to compare inference results of statistically adequate models, and models which are rendered misspecified when certain departures from the NIID assumptions are ignored. The simulation designs, as well as the simulation results in terms of the means and Standard Errors (SEs) of the sampling distributions of the Maximum Likelihood Estimation (MLE) estimators, and their t-tests are presented and discussed in Appendix A. The broad conclusion and teaching lessons that one may take from these results are briefly discussed below.

The simulation results in tables A.1–A.2 reveal that when the modeler ignores the presence of departures from the IID assumptions, the sample statistics may become highly misleading. The means and SEs of the sampling distribution of the MLE estimators are often shifted to the right or to the left, and the t-tests for their significance often erroneously reject the true null hypothesis, even 100% of the time. Moreover, it is often insufficiently appreciated that when the data exhibit mean heterogeneity (non-ID process), the traditional estimator:

$$R^2=1-[\sum_{t=1}^n \widehat{u}^2]/[\sum_{t=1}^n (y_t-\overline{y})^2],\ \ \overline{y}=\tfrac{1}{n}\sum_{t=1}^n y_t, \tag{1.4}$$

printed out by almost all computer packages, is an inconsistent estimator of the $\mathcal{R}^2$. Hence, goodness-of-fit measures are likely to be inappropriate when the estimated model is statistically misspecified.

The simulation results in table A.3 suggest that when the modeler ignores the presence of departures from Normality, the *SE*s of the sampling distributions of the MLE estimators are often shifted to the right or to the left, and the t-tests for their significance often incorrectly reject the true null hypothesis when the null hypothesis is true 5% of the time. In this particular simulation design, the two distributions employed (Normal and Student's t) are members of the elliptically symmetric family of distributions. Thereby, the statistical Generating Mechanism (GM) of both the adequate and misspecified models takes the same form in (A.4), and the respective mean estimates are accurate.[1] Besides, the two models differ in respect of their skedastic functions. The Normal has a homoskedastic skedastic function $[Var(y_t|\mathbf{X}_t{=}\mathbf{x}_t){=}\sigma^2]$, in contrast to the heteroskedastic function $\left[Var(y_t|\sigma(\mathbf{X}_t)){=}\left(\frac{\nu\sigma^2}{\nu+l-2}\right)\left(1+\frac{1}{\nu}(\mathbf{X}_t-\boldsymbol{\mu}_x)^\top\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_t-\boldsymbol{\mu}_x)\right)\right]$ of Student's t; see chapter 4 for a more detailed discussion.

The simulation results teach us two very important and related lessons. The first lesson is that ignoring the problem of statistical misspecification has devastating effects on the reliability of estimation and the testing associated with the LR models. As shown, all statistical inference results are rendered untrustworthy when the estimated model is statistically misspecified. The second lesson is that from the misspecification perspective the IID are more crucial assumptions compared to the distribution assumption. The simulation results reveal that departures from the distribution assumption may distort the *SE*s of the sampling distributions of the MLE estimators, whereas departures from the IID assumptions can distort both the means and *SE*s. Irrespective of this aspect, one may argue that the empirical asset pricing literature has focused exclusively on the departures from the distributional assumptions with

[1] It is important to note that this might not be the case if the true and estimated distributions come from different family of distributions. For example, if the true is a non-linear distribution (e.g., Exponential), but the estimated distribution is the Normal, then both the mean and the *SE*s of the sampling distributions of the MLE estimators are expected to be highly inaccurate.

the adoption of the Generalized Method of Moments (GMM) framework (Hansen, 1982), or Generalized Autoregressive Conditional Heteroskedasticity (GARCH) type models (Engle, 1982; Bollerslev, 1986). Indeed, one could argue that little to no attention has been paid on the dependence and t-heterogeneity assumptions. In the subsequent chapters, these teaching lessons will be brought together in an attempt to specify a family of models that account for all the regularity patterns exhibited by portfolio returns.

1.6 What's Next

A brief overview will help set the stage of what will follows.

Chapter 2 will address the serious issues raised when modeling t-heterogeneity in the general context of LR models. In this context, particular attention is paid on the modeling of different forms of heterogeneity exhibited by financial data. As a matter of fact, heterogeneity has been the least developed probabilistic assumption in the sense of devising concepts to account for different patterns of heterogeneity in financial data. Given that the results of this chapter will play a crucial role in all the subsequent chapters, the chapter is considered the basis of this dissertation.

In chapter 3, the primary objective is to revisit the CAPM and the Fama-French three-factor model with a view to evaluate their empirical adequacy vis-à-vis the Fama and French (1993) data. By employing thorough Mis-Specification (M-S) testing, the chapter shows that both the two models are empirically invalidated for the data in question. In light of these M-S testing results, chapter 4 will proceed to respecify the statistical models underlying the CAPM and the Fama-French three-factor model. The respecification results in the heterogeneous Student's t/ Heteroskedastic Vector Autoregressive (VAR) model and the reparameterized heterogeneous Student's t/ Heteroskedastic Dynamic LR model. The probabilistic structure of these models is rich enough to account for all the statistical systematic information of the data in hand.

Chapter 5 uses the data in Fama and French (2015) to test for the significance of the Fama-French factors on the basis of a statistically adequate model. The chapter argues that the reliability of these significance tests depends crucially on whether the original CAPM is statistically adequate. As it is shown, the value factor HML is not redundant as conjectured by Fama and French (2015), whereas the redundant factor for describing average returns is the investment factor CMA. In addition, the lingering heterogeneity in the data indicates that there are still missing relevant factors. At this stage, the reader is advised to keep an eye on the forest and not get too distracted by the trees. Seeing the wider picture, the justifications in this chapter propose a factor selection procedure. The latter procedure has the advantage of providing a coherent basis for selecting the relevant factors from the set of possible ones.

Chapter 6 revisits the risk measures of Value-at-Risk (VaR) and Expected Shortfall (ES) by paying particular attention on the probabilistic assumptions underlying the internal models used by banks to measure market risk capital requirements. In an attempt to assess the performance of these internal models an empirical example is employed. The empirical results are compared to the results of the proposed heterogeneous Student's t/ Heteroskedastic Autoregressive (AR) model which is shown to have better predictive ability, especially in periods of high volatility. The chapter argues that seeing things from a purely probabilistic perspective, the proposal of the Basel Committee of Banking Supervision (BCBS) to replace the risk measure of VaR with ES will not necessarily improve the market risk forecasting process. As the chapter contends, BCBS should instead severely restrict the scope of internal modeling to models which are likely to pass some kind of statistical adequacy test.

Appendix A: Monte-Carlo Simulations

I. Normal Identically Distributed/non-Independent

The Monte-Carlo simulation design used is the following:

$$\begin{pmatrix} y_t \\ x_{1t} \\ x_{2t} \\ y_{t\text{-}1} \\ x_{1t\text{-}1} \\ x_{2t\text{-}1} \end{pmatrix} \backsim \mathsf{N}\left(\begin{pmatrix} 2.5 \\ 1.9 \\ 0.8 \\ 2.5 \\ 1.9 \\ 0.8 \end{pmatrix} \begin{pmatrix} 1.80 & .780 & -.650 & .810 & .615 & -.220 \\ .780 & 1.10 & .200 & .615 & .710 & .160 \\ -.650 & .200 & 1.00 & -.220 & .160 & .740 \\ .810 & .615 & -.220 & 1.80 & .780 & -.650 \\ .615 & .710 & .160 & .780 & 1.10 & .200 \\ -.220 & .160 & .740 & -.650 & .200 & 1.00 \end{pmatrix} \right)$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} 1.10 & .200 & .615 & .710 & .160 \\ .200 & 1.00 & -.220 & .160 & .740 \\ .615 & -.220 & 1.80 & .780 & -.650 \\ .710 & .160 & .780 & 1.10 & .200 \\ .160 & .740 & -.650 & .200 & 1.00 \end{pmatrix}^{-1} \begin{pmatrix} .780 \\ -.650 \\ .810 \\ .615 \\ -.220 \end{pmatrix} = \begin{pmatrix} .650 \\ -1.580 \\ .685 \\ -.364 \\ 1.363 \end{pmatrix}$$

$$\sigma^2 = 1.8 - \begin{pmatrix} .780 & -.650 & .810 & .615 & -.220 \end{pmatrix} \begin{pmatrix} .650 \\ -1.580 \\ .685 \\ -.364 \\ 1.363 \end{pmatrix} = .235$$

$$\mathcal{R}^2 = 1 - \tfrac{\sigma^2}{\sigma_{11}} = 1 - \tfrac{.235}{1.8} = .869$$

$$\alpha_0 = 2.5 - .650\,(1.9) + 1.580\,(0.8) - .685(2.5) + .364\,(1.9) - 1.363\,(0.8) = .418$$

The statistical GM takes the following form:

$$y_t = .418 + .650x_{1t} - 1.580x_{2t} + .685y_{t-1} - .364x_{1t-1} + 1.363x_{2t-1} + \sqrt{.235}\epsilon_t,\ \ t \in \mathbb{N}, \tag{A.1}$$

where $\epsilon_t \backsim \mathsf{N}(0,1)$ denotes pseudo-random numbers.

The results in table A.1 refer to two scenarios: (a) statistically adequate case where the simulation results are based on estimating the correct model in (A.1), and (b) statistically misspecified case where the modeler ignored the presence of lags and estimated $y_t = \alpha_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + \sigma\epsilon_t$.

Table A.1 - Simulation I: Normal Identically Distributed/non-Independent								
Replications:	(a) **Adequate model**				(b) **Misspecified model**			
N=10000	$y_t=\alpha_0+\sum_{i=1}^{2}\beta_i x_{it}+\sum_{i=1}^{3}\gamma_i z_{it-1}+\varepsilon_{1t}$				$y_t=\alpha_0+\sum_{i=1}^{2}\beta_i x_{it}+\varepsilon_{2t}$			
Estimators	n=100		n=250		n=100		n=250	
of Parameters	Mean	*SE*	Mean	*SE*	Mean	*SE*	Mean	*SE*
$\widehat{\alpha}_0$ $[\alpha_0^*$=0.418$]$	0.421	0.151	0.417	0.094	1.522	0.166	1.525	0.104
$\widehat{\beta}_1$ $[\beta_1^*$=0.650$]$	0.650	0.064	0.650	0.040	0.860	0.076	0.859	0.048
$\widehat{\beta}_2$ $[\beta_2^*$=-1.580$]$	-1.580	0.085	-1.580	0.053	-0.821	0.080	-0.821	0.050
$\widehat{\gamma}_1$ $[\gamma_1^*$=0.685$]$	0.684	0.075	0.685	0.047	-	-	-	-
$\widehat{\gamma}_2$ $[\gamma_2^*$=-0.364$]$	-0.364	0.085	-0.363	0.053	-	-	-	-
$\widehat{\gamma}_3$ $[\gamma_3^*$=1.363$]$	1.362	0.119	1.363	0.073	-	-	-	-
$\widehat{\sigma}^2$ $[\sigma_*^2$=0.235$]$	0.233	0.034	0.235	0.021	0.590	0.085	0.593	0.053
R^2 $[R_*^2$=0.869$]$	0.874	0.025	0.871	0.016	0.671	0.055	0.669	0.034
t-statistics	Mean	α=0.05	Mean	α=0.05	Mean	α=0.05	Mean	α=0.05
$\tau_{\alpha_0}=\frac{(\widehat{\alpha}_0-\alpha_0^*)}{\hat{\sigma}_{\alpha_0}}$	0.021	0.048	-0.006	0.051	6.711	1.000	10.680	1.000
$\tau_{\beta_1}=\frac{(\widehat{\beta}_1-\beta_1^*)}{\hat{\sigma}_{\beta_1}}$	0.006	0.049	0.009	0.050	2.785	0.777	4.400	0.990
$\tau_{\beta_2}=\frac{(\widehat{\beta}_2-\beta_2^*)}{\hat{\sigma}_{\beta_2}}$	-0.002	0.048	0.003	0.049	9.602	1.000	15.238	1.000
$\tau_{\gamma_1}=\frac{(\widehat{\gamma}_1-\gamma_1^*)}{\hat{\sigma}_{\gamma_1}}$	-0.017	0.052	-0.024	0.049	-	-	-	-
$\tau_{\gamma_2}=\frac{(\widehat{\gamma}_2-\gamma_2^*)}{\hat{\sigma}_{\gamma_2}}$	0.006	0.050	0.025	0.048	-	-	-	-
$\tau_{\gamma_3}=\frac{(\widehat{\gamma}_3-\gamma_3^*)}{\hat{\sigma}_{\gamma_3}}$	-0.010	0.047	-0.014	0.050	-	-	-	-

(a) Statistically adequate models: (i) The point estimates are *highly accurate* because they come from *consistent* estimators of the corresponding parameters, (ii) the empirical Type I error probabilities associated with the t-tests are very close to the nominal error probabilities (α=0.05), and (iii) the accuracy of the point estimates and the empirical Type I error probabilities improves as the sample size (n) increases.

(b) Statistically misspecified models: (i) The point estimates are *highly inaccurate* because they stem from *inconsistent* estimators of the corresponding parameters, (ii) there are substantial discrepancies between the empirical Type I error probabilities and the nominal error probabilities (α=0.05), and (iii) the inaccuracy of the point estimates and the empirical Type I error probabilities worsens as n increases.

II. Normal Independent/non-Identically Distributed

The Monte-Carlo simulation design used is the following:

$$\begin{pmatrix} y_t \\ x_{1t} \\ x_{2t} \end{pmatrix} \backsim \mathsf{N}\left(\begin{pmatrix} 2.5+.13t \\ 1.9+.09t \\ 0.8+.11t \end{pmatrix}\begin{pmatrix} 1.80 & .780 & -.650 \\ .780 & 1.10 & .200 \\ -.650 & .200 & 1.00 \end{pmatrix}\right)$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1.10 & .200 \\ .200 & 1.00 \end{pmatrix}^{-1}\begin{pmatrix} .780 \\ -.650 \end{pmatrix} = \begin{pmatrix} .858 \\ -.822 \end{pmatrix}$$

$$\sigma^2{=}1.8{-}\begin{pmatrix} .780 & -.650 \end{pmatrix}\begin{pmatrix} .858 \\ -.822 \end{pmatrix}{=}.596$$

$$\mathcal{R}^2{=}1{-}\frac{\sigma^2}{\sigma_{11}}{=}1{-}\frac{.596}{1.8}{=}.669$$

$$\alpha_0(t){=}2.5+.13t{-}.858\,(1.9{+}.09t)\,{+}.822\,(0.8{+}.11t)\,{=}1.527{+}.143t$$

The statistical GM takes the following form:

$$y_t{=}1.527+.143t+.858x_{1t}-.822x_{2t}+\sqrt{.596}\epsilon_t,\ \ t{\in}\mathbb{N}, \tag{A.2}$$

where $\epsilon_t \backsim \mathsf{N}(0,1)$ denotes pseudo-random numbers.

The results in table A.2 refer to two scenarios: (a) statistically adequate case where the simulation results are based on estimating the correct model in (A.2), and (b) statistically misspecified case where the modeler ignored the presence of trends and estimated $y_t{=}\alpha_0{+}\beta_1 x_{1t}{+}\beta_2 x_{2t}{+}\sigma\epsilon_t$.

(a) Statistically adequate models: (i) The point estimates are *highly accurate* because they come from *consistent* estimators of the corresponding parameters, (ii) the empirical Type I error probabilities associated with the t-tests are very close to the nominal error probabilities (α=0.05), and (iii) the accuracy of the point estimates and the empirical Type I error probabilities improves as n increases.

(b) Statistically misspecified models: (i) The point estimates are *highly inaccurate* because they stem from *inconsistent* estimators of the corresponding parameters, (ii) there are substantial discrepancies between the empirical Type I error probabilities and the nominal error probabilities (α=0.05), and (iii) the inaccuracy of the point estimates and the empirical Type I error probabilities worsens as n increases.

Table A.2 - Simulation II: Normal Independent/non-Identically Distributed								
Replications:	(a) **Adequate model**				(b) **Misspecified model**			
N=10000	y_t=α_0+$\delta_1 t+\sum_{i=1}^{2}\beta_i x_i$+$\varepsilon_{1t}$				y_t=$\alpha_0+\sum_{i=1}^{2}\beta_i x_i$+$\varepsilon_{2t}$			
Estimators	n=100		n=250		n=100		n=250	
of Parameters	Mean	*SE*	Mean	*SE*	Mean	*SE*	Mean	*SE*
$\widehat{\alpha}_0$ [α_0^*=1.527]	1.529	0.215	1.527	0.135	0.378	0.339	0.016	0.204
$\widehat{\delta}_1$ [δ_1^*=0.143]	0.143	0.011	0.143	0.006	-	-	-	-
$\widehat{\beta}_1$ [β_1^*=0.858]	0.858	0.077	0.858	0.048	1.380	0.113	1.410	0.072
$\widehat{\beta}_2$ [β_2^*=-0.822]	-0.822	0.080	-0.822	0.050	-0.034	0.095	0.014	0.059
$\hat{\sigma}^2$ [σ_*^2=0.596]	0.591	0.085	0.593	0.054	1.745	0.251	1.820	0.162
R^2 [R_*^2=0.669]	0.674	0.054	0.671	0.034	0.892	0.016	0.980	0.002
t-statistics	Mean	α=0.05	Mean	α=0.05	Mean	α=0.05	Mean	α=0.05
$\tau_{\alpha_0}=\frac{(\widehat{\alpha}_0-\alpha_0^*)}{\hat{\sigma}_{\alpha_0}}$	0.016	0.051	0.008	0.050	-3.400	0.932	-7.414	1.000
$\tau_{\delta_1}=\frac{(\widehat{\delta}_1-\delta_1^*)}{\hat{\sigma}_{\delta_1}}$	0.016	0.049	0.020	0.050	-	-	-	-
$\tau_{\beta_1}=\frac{(\widehat{\beta}_1-\beta_1^*)}{\hat{\sigma}_{\beta_1}}$	-0.009	0.047	-0.012	0.049	4.645	0.994	7.692	1.000
$\tau_{\beta_2}=\frac{(\widehat{\beta}_2-\beta_2^*)}{\hat{\sigma}_{\beta_2}}$	-0.001	0.049	0.005	0.050	8.349	1.000	14.214	1.000

Of particular interest in this context is the estimator of $\mathcal{R}^2$:

$$R^2{=}1-[\textstyle\sum_{t=1}^{n}\widehat{u}^2]/[\sum_{t=1}^{n}(y_t-2.5-.13t)^2], \tag{A.3}$$

which is highly precise when the LR model is statistically adequate, but highly inaccurate – close to one – when it is misspecified. It is often insufficiently appreciated that when the data exhibit mean heterogeneity, the traditional estimator of R^2 printed out by almost all computer packages, is an *inconsistent* estimator of the $\mathcal{R}^2$.

III. Non-Normal Independent Identically Distributed

The Monte-Carlo simulation design used is the following:

$$\begin{pmatrix} y_t \\ x_{1t} \\ x_{2t} \end{pmatrix} \backsim \mathsf{St}\left(\begin{pmatrix} 2.5 \\ 1.9 \\ 0.8 \end{pmatrix}\begin{pmatrix} 1.80 & .780 & -.650 \\ .780 & 1.10 & .200 \\ -.650 & .200 & 1.00 \end{pmatrix}\right)$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1.10 & .200 \\ .200 & 1.00 \end{pmatrix}^{-1}\begin{pmatrix} .780 \\ -.650 \end{pmatrix} = \begin{pmatrix} .858 \\ -.822 \end{pmatrix}$$

$$\sigma^2{=}1.8{-}\begin{pmatrix} .780 & -.650 \end{pmatrix}\begin{pmatrix} .858 \\ -.822 \end{pmatrix}{=}.596,$$

$$\mathcal{R}^2{=}1{-}\tfrac{\sigma^2}{\sigma_{11}}{=}1{-}\tfrac{.596}{1.8}{=}.669$$

$$\alpha_0{=}2.5{-}.858\,(1.9)+.822\,(0.8)={1.527}$$

The statistical GM takes the following form:

$$y_t{=}1.527 + .858x_{1t} - .822x_{2t} + \sqrt{Var(y_t|\sigma(x_{1t},x_{2t})}\epsilon_t,\ \ t{\in}\mathbb{N} \qquad \text{(A.4)}$$

where $\epsilon_t \backsim \mathsf{St}(0,1;\nu{=}4)$ denotes pseudo-random numbers and 'St' stands for the Student's t distribution with ν degrees of freedom.

The results in table A.3 refer to two scenarios: (a) statistically adequate case where the simulation results are based on estimating the correct (Student's t) model in (A.4), and (b) statistically misspecified case where the modeler ignored the departures from Normality and estimated $y_t{=}\alpha_0{+}\beta_1x_{1t}{+}\beta_2x_{2t}{+}\sigma\epsilon_t,\ \ \epsilon_t\backsim\mathsf{N}(0,1)$. The probabilistic assumptions of the statistically adequate and misspecified models differ in one important respect: the former has a heteroskedastic conditional variance $\left[Var(y_t|\sigma(\mathbf{X}_t))\overset{\mathbf{St}}{=}\left(\frac{\nu\sigma^2}{\nu+l-2}\right)\left(1{+}\frac{1}{\nu}(\mathbf{X}_t{-}\boldsymbol{\mu}_x)^\top\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{X}_t{-}\boldsymbol{\mu}_x)\right)\right]$, whereas the conditional variance of the latter is homoskedastic $\left[Var(y_t|\mathbf{X}_t{=}\mathbf{x}_t)\overset{\mathbf{N}}{=}\sigma^2\right]$.

(a) Statistically adequate model: (i) The point estimates are *highly accurate* because they come from *consistent* estimators of the corresponding parameters, (ii) the empirical Type I error probabilities associated with the t-tests are very close to the nominal error probabilities ($\alpha{=}0.05$), and (iii) the accuracy of the point estimates and the empirical Type I error probabilities improves as n increases.

(b) Statistically misspecified model: (i) The mean estimates are *highly accurate* and their accuracy improves as n increases, but (ii) the *SE* estimates are *highly inaccurate* because they come from *inconsistent* estimators of the corresponding parameters, (iii) there are substantial discrepancies between the empirical Type I error probabilities and the nominal error probabilities (α=0.05), and (iv) the inaccuracy of the *SE* estimates and the empirical Type I error probabilities worsens as n increases.

Table A.3 - Simulation III: Non-Normal (Student's t) Independent Identically Distributed								
Replications:	(a) **Adequate model**				(b) **Misspecified model**			
N=10000	$y_t=\alpha_0+\sum_{i=1}^{2}\beta_i x_i+\varepsilon_{1t},\ \varepsilon_{1t}\backsim\mathsf{StIID}$				$y_t=\alpha_0+\sum_{i=1}^{2}\beta_i x_i+\varepsilon_{2t},\ \varepsilon_{2t}\backsim\mathsf{NIID}$			
Estimators	n=100		n=250		n=100		n=250	
of Parameters	Mean	*SE*	Mean	*SE*	Mean	*SE*	Mean	*SE*
$\widehat{\alpha}_0$ $[\alpha_0^*$=1.527$]$	1.529	0.186	1.527	0.117	1.529	0.181	1.527	0.115
$\widehat{\beta}_1$ $[\beta_1^*$=0.858$]$	0.857	0.085	0.858	0.054	0.856	0.076	0.858	0.048
$\widehat{\beta}_2$ $[\beta_2^*$=-0.822$]$	-0.821	0.089	-0.822	0.056	-0.819	0.080	-0.822	0.050
$\hat{\sigma}^2$ $[\sigma_*^2$=0.596$]$	0.582	0.105	0.591	0.067	1.127	0.339	2.929	0.712
R^2 $[R_*^2$=0.669$]$	0.677	0.059	0.672	0.038	0.674	0.087	0.673	0.062
t-statistics	Mean	α=0.05	Mean	α=0.05	Mean	α=0.05	Mean	α=0.05
$\tau_{\alpha_0}=\frac{(\widehat{\alpha}_0-\alpha_0^*)}{\hat{\sigma}_{\alpha_0}}$	0.008	0.019	-0.001	0.016	0.012	0.163	-0.007	0.192
$\tau_{\beta_1}=\frac{(\widehat{\beta}_1-\beta_1^*)}{\hat{\sigma}_{\beta_1}}$	-0.016	0.021	0.001	0.017	-0.026	0.208	0.022	0.255
$\tau_{\beta_2}=\frac{(\widehat{\beta}_2-\beta_2^*)}{\hat{\sigma}_{\beta_2}}$	0.015	0.019	0.002	0.015	0.039	0.208	-0.019	0.243

It is important to note that in the case of the Student's t, the t-tests are only approximations. Thus, to confirm the reliability of the empirical Type I error probabilities, the exact F-test:

$$F(\mathbf{St})=\frac{\left(\mathbf{I}\widehat{\beta}-\boldsymbol{\beta}^*\right)^\top\left[\mathbf{I}(\mathbf{X}^\top\widehat{\Omega}\mathbf{X})^{-\mathbf{1}}\mathbf{I}^\top\right]^{-1}\left(\mathbf{I}\widehat{\beta}-\boldsymbol{\beta}^*\right)}{l\widehat{\sigma}^2}\sim\mathbf{F}(l,n-l-1), \tag{A.5}$$

for the joint significance of $\widehat{\beta}_1$ and $\widehat{\beta}_2$ is used, where $\mathbf{I}$:=diag(1,1,...,1), $\mathbf{X}:=(x_{1t},x_{2t})$, $\widehat{\beta}=(\mathbf{X}^\top\widehat{\Omega}\mathbf{X})^{-\mathbf{1}}\mathbf{X}^\top\widehat{\Omega}\mathbf{y}$, $\widehat{\Omega}$:=diag$(\widehat{\omega}_1,\widehat{\omega}_2,...,\widehat{\omega}_n)$, $\widehat{\sigma}^2=\left(\frac{\nu+l}{\nu}\right)\left(\frac{\widehat{\mathbf{u}}^\top\widehat{\Omega}^{-1}\widehat{\mathbf{u}}}{n}\right)$, $\boldsymbol{\beta}^*$ denotes the true parameters of $(.858,-.822)$, and l refers to the number of regressors.

Chapter 2

2 On Modeling Heterogeneity in Linear Models using Orthogonal Polynomials

2.1 Overview

The grouping of assets into portfolios in asset pricing modeling is viewed as a way to reduce the errors-in-variables problem in such data, and thus improve the precision of the estimated betas; see Blume (1970), Friend and Blume (1970), Black et al. (1972). There is, however, a potential side-effect when the original data exhibit different forms of heterogeneity. This means that averaging individual such stock returns could generate complicated forms of heterogeneity for the portfolio. The latter is especially problematic for portfolios composed of a very large number of stocks.

The aims of this chapter are (a) to consider the problems and issues raised when modeling heterogeneity in the general context of linear models using trend polynomials; and (b) to address the serious problem of modeling different forms of heterogeneity exhibited by financial data. Much of the discussion addresses the practical difficulties when using ordinary trend polynomials by exploring different forms of polynomials whose degree could be potentially extended to greater than 4 without the *near-collinearity* (near-multicollinearity) problems. The chapter begins, however, with a brief background on modeling heterogeneity in linear models (section 2.1) by placing the problem in a historical context. Section 2.2 discusses the problem of near-collinearity in the context of a linear model that stems from the ill-conditioning of the regressor data matrix ($\mathbf{X}^\intercal\mathbf{X}$). Section 2.3 evaluates a number of different ways one could deal with the near-collinearity problem. These include (scaled) ordinary polynomials, the most widely used continuous and discrete orthogonal polynomials, as well as the Gram-Schmidt polynomials. As it is shown, one can adequately address the ill-conditioning problem by combining (a) continuous polynomials which are orthogonal with respect to a weight function with (b) scaling the time ordering (t=$1, 2, ..., n$) to lie within $[-1, 1]$. The well-conditioning of the observed regressor data matrix ($\mathbf{X}^\intercal\mathbf{X}$) improves when the orthogonal polynomials also take values within the interval $[-1, 1]$. In addition, an alternative way to model complex forms of heterogeneity is the usage of the orthonormal Gram-Schmidt polynomials. Finally section 2.4 illustrates these results using an empirical example.

2.2 A Brief History of Heterogeneity Modeling

Beginning in the late 19th century, the modeling of economic time-series was primarily based on a decomposition (see Morgan, 1990) of the form:

$$y_t = \text{trend} + \text{seasonal} + \text{cycles} + \text{noise}, \; t{=}1, 2, ..., n, ... \tag{2.1}$$

This was motivated by the fact that even a casual inspection of a t-plot of most economic time-series reveals three distinct regularity patterns: a trend, different cycles, and a certain degree of jaggedness around the trend and cycles that was viewed as random noise. When the cycles seem very regular, are often interpreted as seasonality, but when they are irregular are viewed as business cycles.

The introduction of correlation and regression analysis by Galton in the 1880s (see Stigler, 1986) provided the key tools for modeling time-series data using (2.1). It soon became apparent, however, that correlation and regression analysis could give rise to 'dubious' results: apparent statistical associations that after closer scrutiny of the data are rendered 'spurious'; see Hooker (1901) and Yule (1903). This led practitioners to use several ad hoc methods to remove the trend component with a view to address the problem with statistically spurious results. The most popular was to first remove the trend in some way, usually using deterministic trend polynomials up to order 3 (see Moore, 1914), and then apply correlation/regression to the detrended series. An alternative procedure, suggested by Hooker (1905), was to first difference the data series and then apply correlation/regression using the first differences. It was not, however, obvious the extend to which these methods address the spuriousness of the ensuing statistical results. To this day, this remains a problem that has not been adequately addressed, despite the extraordinary developments in statistics and econometrics since then. Indeed, the problem of spurious regression results resurfaced in Granger and Newbold (1974) using simulated data based on unit root processes. Despite the elucidating technical account proposed by Phillips (1986) to explain their simulation results, the real problem of addressing the spuriousness remained unresolved. As argued in Spanos (1990b) the real problem is one of statistical misspecification and the key to its resolution is to respecify the original model with a view to secure its statistical adequacy; that is, the probabilistic assumptions comprising the assumed statistical model are valid for the particular data.

In empirical modeling the statistical systematic information comes in the form of chance regularity patterns exhibited by the data. When evaluated in terms of learning from data about phenomena of interest, the success of such modeling depends crucially on how adequately these patterns are accounted for by the prespecified statistical model that comprises probabilistic assumptions from three broad categories: *Distribution*, *Dependence*, and *Heterogeneity*; see Spanos (1999). One can make a strong case that the heterogeneity category has been the least developed in the sense of devising concepts to account for different patterns of heterogeneity in economic and financial data. There are several reasons for this neglect, including the following.

First, over the last quarter century (see Choi, 2015) the focus on unit roots has created the impression that the scope of unit-root heterogeneity is broad enough to account for the majority of economic and financial time-series. In fact unit root type heterogeneity is of rather limited scope. Focusing on the first two moments for simplicity, a stochastic process $\{Z_t,\ t\in\mathbb{N}:=(1,2,...,n,...)\}$ is t-heterogeneous when:

$$E(Z_t)=\mu(t),\ Cov(Z_t,Z_s)=E[Z_t-\mu(t)][Z_s-\mu(s)]=v(t,s),\ \text{for all } t,s\in\mathbb{N}, \tag{2.2}$$

where $(\mu(t),v(t,s),\ t,s\in\mathbb{N})$ are arbitrary functions of t and s. Unit root heterogeneity represents a very simple (linear) form of these arbitrary functions:

$$E(Z_t)=\mu\cdot t,\ Var(Z_t)=\sigma(0)\cdot t,\ Cov(Z_t,Z_s)=\sigma(0)\min(t,s),\ \text{for all } t,s\in\mathbb{N}. \tag{2.3}$$

Due to its highly restrictive form, one would expect only a small fraction of t-heterogeneity in economic and financial time-series data to be adequately modeled using this form of heterogeneity. Similarly, the debate concerning the modeling of t-heterogeneity using deterministic trends (polynomials) vs. differences (stochastic trends) is rather misleading because the two are not incompatible; see Andreou and Spanos (2003). This is because, even after the data are differenced, there is often lingering t-heterogeneity that can be accounted for by a trend polynomial. Moreover, trend polynomials can play a very crucial role in Mis-Specification (M-S) testing because they provide a generic way to account for different forms of t-heterogeneity; see Phillips (2005).

Second, accounting for mean t-heterogeneity using generic trend polynomials in Linear Regression (LR) and related models is hampered by practical difficulties when the degree of the polynomial is greater than 4 or so. It is well known that such polynomials are likely to give rise to near-collinearity problems; see Greene (2012). Complicated forms of t-heterogeneity can easily arise in cases where several data series with heterogeneity are aggregated; such as the aggregation of individual stocks into portfolios. When one's data exhibit mean t-heterogeneity ($\mu(t)$) but it is not accounted for, all the sample second and higher central moments will be inconsistent estimators of the corresponding distribution moments, leading to inconsistent estimators for the regression coefficients.

2.3 Linear Models and Trend Polynomials

2.3.1 Near-Collinearity in Numerical Analysis

From a numerical analysis perspective, the real issue associated with near-collinearity concerns the potential instabilities (wobbliness) of the numerical values of $\mathbf{b}=\mathbf{Z}^{-1}\mathbf{y}$ to small changes in the data $(\mathbf{Z},\mathbf{y})$, when solving the linear system $\mathbf{Z}\mathbf{b}=\mathbf{y}$ for $\mathbf{b}$. From this perspective, the volatility stems from the ill-conditioning of $\mathbf{Z}$; see Gautschi and Inglese (1988), Gautschi (1990), Tyrtyshinikov (1994).

A widely used measure to quantify the degree of ill-conditioning of any matrix $\mathbf{A}$ is the condition number:

$$\kappa(\mathbf{A})=\|\mathbf{A}\|\cdot\|\mathbf{A}^{-1}\|, \tag{2.4}$$

where $\|\cdot\|$ denotes a matrix norm. The most widely used norms for a matrix $\mathbf{A}$ are the Euclidean $\|\mathbf{A}\|_p$, for p=2, and the Frobenius $\|\mathbf{A}\|_F$ defined by:

$$\|\mathbf{A}\|_p=\tfrac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p},\ p=1,2,...,\infty,\quad \|\mathbf{A}\|_F=\sqrt{\textstyle\sum_{i=1}^{n}\sum_{j=1}^{p}|a_{ij}|^2}. \tag{2.5}$$

It turns out that both of these norms are directly related to the spectrum (eigenvalues) of $\mathbf{A}$. In particular, it can be shown that (Golub and Van Loan, 2013):

$$\|\mathbf{A}\|_2=\sqrt{\lambda_1(\mathbf{A}^\top\mathbf{A})},\ \|\mathbf{A}\|_F=\sqrt{\text{trace}(\mathbf{A}^\top\mathbf{A})}=\sqrt{\textstyle\sum_{i=1}^{p}\lambda_i^2}, \tag{2.6}$$

where $\lambda_1\geq\lambda_2\geq...\geq\lambda_p$, are the eigenvalues of the positive-semidefinite matrix $(\mathbf{A}^\top\mathbf{A})$, and the two norms are related via:

$$\|\mathbf{A}\|_2\leq\|\mathbf{A}\|_F\leq\sqrt{p}\,\|\mathbf{A}\|_2\,. \tag{2.7}$$

The eigenvalues stem from the Singular Value Decomposition (SVD) of $\mathbf{A}$:

$$\mathbf{U}^\top\mathbf{A}\mathbf{V}=\mathbf{D}=\text{diag}\left(\lambda_1,\lambda_2,\ldots,\lambda_p\right), \tag{2.8}$$

where $\mathbf{U}$ and $\mathbf{V}$ are $(n\times n)$ and $(p\times p)$ matrices, respectively.

In relation to these results it is important to note that the eigenvalues λ_i, i=1, 2, .., p, of any data matrix $\mathbf{X}$, rank($\mathbf{X}$)=p, are directly related to those of $(\mathbf{X}^\intercal\mathbf{X})$, which are

$\lambda_i^2,\ i{=}1,2,..,p$. Hence, the Euclidean and Frobenius condition numbers for $(\mathbf{X}^\intercal\mathbf{X})$ take the forms:

$$\begin{aligned}
&\kappa_2(\mathbf{X}^\intercal\mathbf{X})=\|(\mathbf{X}^\intercal\mathbf{X})\|_2{\cdot}\|(\mathbf{X}^\intercal\mathbf{X})^{-1}\|_2=\tfrac{\lambda_{\max}}{\lambda_{\min}},\\
&\kappa_F(\mathbf{X}^\intercal\mathbf{X})=\|(\mathbf{X}^\intercal\mathbf{X})\|_F{\cdot}\|(\mathbf{X}^\intercal\mathbf{X})^{-1}\|_F=\left(\left(\textstyle\sum_{i=1}^p\lambda_i^2\right)\left(\sum_{i=1}^p\lambda_i^{-2}\right)\right).
\end{aligned}\tag{2.9}$$

2.3.2 Polynomial Regression Models

Consider the Polynomial Linear Regression (PLR) model:

$$y_t{=}\beta_0{+}\textstyle\sum_{k=1}^p\beta_k x_t^k{+}u_t,\ t{\in}\mathbb{N}.\tag{2.10}$$

Using the following matrix notation:

$$\underbrace{\begin{pmatrix}y_1\\y_2\\y_3\\\vdots\\y_n\end{pmatrix}}_{\mathbf{y}}=\underbrace{\begin{pmatrix}1 & x_1 & x_1^2 & \cdots & x_1^p\\1 & x_2 & x_2^2 & \cdots & x_2^p\\1 & x_3 & x_3^2 & \cdots & x_3^p\\\vdots & \vdots & \vdots & \ddots & \vdots\\1 & x_n & x_n^2 & \cdots & x_n^p\end{pmatrix}}_{\mathbf{X}}\underbrace{\begin{pmatrix}\beta_0\\\beta_1\\\beta_2\\\vdots\\\beta_p\end{pmatrix}}_{\boldsymbol{\beta}}+\underbrace{\begin{pmatrix}u_1\\u_2\\u_3\\\vdots\\u_n\end{pmatrix}}_{\mathbf{u}},\tag{2.11}$$

reveals that $\mathbf{X}$, for $n\geq p$, is a particular form of a Vandermonde matrix:

$$\mathbf{Z}_n{=}[z_{ij}],\ z_{ij}{:=}c_i^{j-1},\ i{=}1,2,...,n,\ j{=}1,2,...,p,\tag{2.12}$$

and the $(\mathbf{X}^\intercal\mathbf{X})$ matrix is a real positive definite Hankel matrix $\mathbf{H}{=}[h_{i,j}]_{i,j=1}^p$, where $h_{i,j}{=}h_{i+1,j-1}{=}h_{i+j-2}$, which is known to be ill-conditioned even for low degree polynomials p; see Tyrtyshinikov (1994). In particular, for any $p>4$, $(\mathbf{X}^\intercal\mathbf{X})$ is ill-conditioned and the problem becomes worse as the sample size (n) increases. The instability of $(\mathbf{X}^\intercal\mathbf{X})$ is likely to render volatile the estimators:

$$\widehat{\boldsymbol{\beta}}{=}\left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}\mathbf{X}^\intercal\mathbf{y},\quad s^2{=}\tfrac{1}{n-p}\mathbf{u}^\intercal\mathbf{u},\tag{2.13}$$

of $(\boldsymbol{\beta},\sigma^2)$, and give rise to misleading t-statistics and p-values since $\widehat{Cov(\widehat{\boldsymbol{\beta}})}{=}s^2\left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}$.

Example. To illustrate the ill-conditioning of a system of least-squares estimates and the induced instability in the estimated coefficients, consider the following simplistic but suggestive example with artificial data:

$$\mathbf{X}{:=}\begin{pmatrix} 1.00 & 4.00 \\ 2.01 & 8.00 \\ 4.02 & 16.00 \end{pmatrix},\ \mathbf{y}{:=}\begin{pmatrix} 4.0 \\ 2.2 \\ 4.4 \end{pmatrix},\ \ (\mathbf{X}^{\intercal}\mathbf{X})=\begin{pmatrix} 21.2005 & 84.4 \\ 84.4 & 336.0 \end{pmatrix}, \tag{2.14}$$

giving rise to $\kappa_F(\mathbf{X}^{\intercal}\mathbf{X})$=15949024.65$\simeq\kappa_2(\mathbf{X}^{\intercal}\mathbf{X})$=15949022.65, indicating that $(\mathbf{X}^{\intercal}\mathbf{X})$ is ill-conditioned. Note that the two condition numbers are almost identical, which turns out to be the rule rather than the exception. Hence, the Frobenius condition number $\kappa_F(\mathbf{X}^{\intercal}\mathbf{X})$ will be used in what follows since it involves all the eigenvalues.

The OLS estimator $\widehat{\boldsymbol{\beta}}{=}(\mathbf{X}^{\intercal}\mathbf{X})^{-1}\mathbf{X}^{\intercal}\mathbf{y}$ yields:

$$\begin{pmatrix} \widehat{\beta}_1 \\ \widehat{\beta}_2 \end{pmatrix}=\begin{pmatrix} 42000.0 & -10550.0 \\ -10550.0 & 2650.0625 \end{pmatrix}\begin{pmatrix} 26.11 \\ 104.0 \end{pmatrix}=\begin{pmatrix} -580.0 \\ 146.0 \end{pmatrix}. \tag{2.15}$$

Change 1: Changing the first element of $\mathbf{X}$ from 1.00 to 1.01. The impact on the OLS estimates is that they switch signs:

$$\begin{pmatrix} \widehat{\beta}_1 \\ \widehat{\beta}_2 \end{pmatrix}=\begin{pmatrix} 42000.0 & -10555.0 \\ -10555.0 & 2652.575 \end{pmatrix}\begin{pmatrix} 26.15 \\ 104.0 \end{pmatrix}=\begin{pmatrix} 580.0 \\ -145.45 \end{pmatrix}. \tag{2.16}$$

Change 2: Changing the first element of $\mathbf{X}$ from 1.00 to 1.02. The result is that the OLS estimates change magnitudes dramatically:

$$\begin{pmatrix} \widehat{\beta}_1 \\ \widehat{\beta}_2 \end{pmatrix}=\begin{pmatrix} 4666.667 & -1173.333 \\ -1173.333 & 295.0125 \end{pmatrix}\begin{pmatrix} 26.19 \\ 104.0 \end{pmatrix}=\begin{pmatrix} 193.3 \\ -48.3 \end{pmatrix}, \tag{2.17}$$

and so does the $(\mathbf{X}^{\intercal}\mathbf{X})^{-1}$, which affects the significance of (β_1, β_2):

$$\begin{array}{rl} \text{Original } SE\text{s:} & \sqrt{Var(\widehat{\beta}_1)}{=}204.9\sigma,\quad \sqrt{Var(\widehat{\beta}_2)}{=}51.5\sigma. \\ \text{Change 2 } SE\text{s:} & \sqrt{Var(\widehat{\beta}_1)}{=}68.3\sigma,\quad \sqrt{Var(\widehat{\beta}_2)}{=}17.2\sigma. \end{array}$$

Change 3: Changing the first element of $\mathbf{y}$ from 4.0 to 4.5. The impact on the OLS estimates is that their magnitudes change significantly:

$$\begin{pmatrix} \widehat{\beta}_1 \\ \widehat{\beta}_2 \end{pmatrix} = \begin{pmatrix} 42000.0 & -10550.0 \\ -10550.0 & 2650.0625 \end{pmatrix} \begin{pmatrix} 26.61 \\ 106.0 \end{pmatrix} = \begin{pmatrix} -680.0 \\ 171.13 \end{pmatrix}. \tag{2.18}$$

2.3.3 Ill-conditioning vs. High Correlation

As argued in Spanos and McGuirk (2002), contrary to conventional wisdom (see Greene, 2012), the problem of near-collinearity boils down to two rather different issues which are often conflated.

(a) A numerical issue (the regressor data matrix $(\mathbf{X}^\intercal\mathbf{X})$ is ill-conditioned) which gives rise to *erratic volatility*. The latter is only detectable using numerical analysis measures, such as the norm condition number.

(b) A statistical issue (high correlation among regressors) which, under certain conditions, gives rise to *systematic volatility*. Genuine high correlation between regressors is neither necessary nor sufficient for near-collinearity.

A key problem in this literature has been to relate the numerical to the statistical measures. The numerical analysis perspective, which revolves around $(\mathbf{X}^\intercal\mathbf{X})$ is ill-conditioned, is often conflated with the statistical perspective that revolves around the sample correlations among the regressors. In many cases the way $(\mathbf{X}^\intercal\mathbf{X})$ is transformed into sample correlations undermines the reliability of the statistical measures used to detect near-collinearity. Often this transformation implicitly assumes constant means for all the variables involved since the estimators of the second moments are based on $(\overline{y},\ \overline{x}_1, ..., \overline{x}_p)$, which is often an inappropriate assumption. For instance, in the case of (2.14) it is clear that the values of the regressors are trending, and thus the evaluation of their sample correlation using $Corr(x_{1t}, x_{2t})=\frac{84.4}{\sqrt{(21.2005)(336)}}=.9999$ is statistically spurious, thus highly misleading. Trending means will also undermine other measures of near-collinearity, such as the Variance Inflation Factors (VIFs); $VIF_i=[1/(1-R^2_{[i]})],\ i=2,3,...,k,$ where $R^2_{[i]}$ is the estimated squared multiple correlation coefficient of the auxiliary regression of x_{it} on all the other regressors $\mathbf{x}_{(i)t}$:

$$x_{it}=\alpha_0+\boldsymbol{\alpha}_{(i)}^{\intercal}\mathbf{x}_{(i)t}+\eta_{it},\ \ \eta_{it}\backsim\mathsf{NIID}(0,\sigma_i^2),\ \ i=2,3,...,p. \tag{2.19}$$

Worse, in the case where the regressors include trend polynomials terms $(t,t^2,...,t^p)$ a constant mean assumption is false by definition.

In summary, moving from numerical measures of ill-conditioning to statistical measures based on sample correlations is not as straightforward as often presumed. Trending means will undermine the reliability of all statistical measures. To avoid this misspecification problem, one can use deterministic trend polynomials ($\mathbf{D}$) to account for such mean heterogeneity. That, however, calls for separating $\mathbf{D}$ from the other regressors ($\mathbf{X}_1$), say:

$$\mathbf{y}=\mathbf{D}\boldsymbol{\delta}+\mathbf{X}_1\boldsymbol{\beta}_1+\mathbf{u}, \tag{2.20}$$

and eliminating the trends terms $\mathbf{D}$ using the Frisch-Waugh result (see Pesaran, 2015, pp. 42-43), before the statistical measures are evaluated. This amounts to detrending the data before the correlations among the genuine regressors are evaluated.

2.4 Trend Polynomials and Ill-conditioning

2.4.1 Ordinary Trend Polynomials

A common case where the near-collinearity problem exists is when fitting ordinary trend polynomials using least-squares. The use of these polynomials offers a generic way to capture a variety of different forms of t-heterogeneity. In this context, the modeler can construct a data matrix whose column space consists of $(t^0,t,t^2,t^3,...,t^p)$, where $t{:=}(1,2,...,n)$. The latter is a Vandermonde matrix by definition.

Table 2.1 - Ordinary Trend Polynomials: $\kappa_F\left(\mathbf{X}^{\intercal}\mathbf{X}\right)$

n	p=3	p=5	p=7
n=100	2.57×10^{12}	4.27×10^{20}	7.05×10^{28}
n=250	5.84×10^{14}	3.44×10^{24}	1.89×10^{34}

The results in table 2.1 show that the Frobenius norm condition numbers grow exponentially with both the sample size (n) and the degree of the polynomial (p). In point of fact, the ill-conditioning appears to be particularly severe for n=100 and p=3 since $\kappa_F\,(\mathbf{X}^\intercal\mathbf{X})$=$2.57\times10^{12}$. A closer look at the diagonal values of the $(\mathbf{X}^\intercal\mathbf{X})$ matrix indicates that there are large differences among those values.

In light of that, one might conjecture that a way to reduce the near-collinearity problem will be to change $(t^0, t, t^2, t^3, ..., t^p)$ using monotonic transformations:

(i) scale their range by the sample size n,

(ii) use logarithmic transformations,

(iii) scale their range to lie in the interval $[-1, 1]$ (Seber and Lee, 2003, p. 166):

$$x_i^*=\tfrac{(2x_i-x_{[n]}-x_{[1]})}{(x_{[n]}-x_{[1]})},\ \ i=1,2,...,n, \tag{2.21}$$

where $x_{[n]}$=max$(x_1, x_2, ..., x_n)$, $x_{[1]}$=min$(x_1, x_2, ..., x_n)$.

Table 2.2 - Scaled Ordinary Trend Polynomials: $\kappa_F\,(\mathbf{X}^\intercal\mathbf{X})$					
Scaling	Range	n	p=3	p=5	p=7
x_i^*	$[-1, 1]$	n=100	72.07	1.99×10^3	5.86×10^4
		n=250	73.71	2.06×10^3	6.10×10^4
$\frac{x_i}{n}$	$(0, 1]$	n=100	1.61×10^4	1.59×10^7	1.67×10^{10}
		n=250	1.58×10^4	1.54×10^7	1.59×10^{10}
$\ln(x_i)$	$[0, \infty)$	n=100	1.01×10^6	7.74×10^9	1.83×10^{14}
		n=250	4.88×10^6	4.60×10^{10}	1.18×10^{15}
$\ln(\frac{x_i}{n})$	$(-\infty, 0]$	n=100	1.16×10^4	1.02×10^8	1.50×10^{12}
		n=250	1.31×10^4	1.50×10^8	2.93×10^{12}

The results in table 2.2 suggest that the transformed data matrix $\mathbf{X}$ has a substantially reduced norm condition number. For example, the norm condition number for n=250 and p=7 is reduced by $\times10^{30}$ when the range of t:=$(1, 2, ..., n)$ is scaled to lie within $[-1, 1]$. In spite of that, the transformed data matrices remain Vandermonde in form, showing signs of $(\mathbf{X}^\intercal\mathbf{X})$ being ill-conditioned. This indicates that the scaling has a significant effect on tempering the increase in n, since the norm condition numbers remain relatively stable as the sample size increases, but does little to curtail the effect of increasing p.

In addition, it is interesting to examine the magnitude of the determinants of the $(\mathbf{X}^\intercal\mathbf{X})$ matrix presented in table 2.3. The determinants of the (unscaled) ordinary polynomials suggest that $(\mathbf{X}^\intercal\mathbf{X})$ is not close to being singular, but the determinants of the scaled by n polynomials approximate zero for higher degree of polynomials.

Table 2.3 - Determinants: $\det(\mathbf{X}^\intercal\mathbf{X})$ when scaled by n

Scaling	n	p=3	p=5	p=7
x_i	n=100	1.65×10^{25}	5.31×10^{54}	2.65×10^{95}
	n=250	3.85×10^{31}	1.13×10^{69}	8.00×10^{120}
$\frac{x_i}{n}$	n=100	16.50	5.31×10^{-6}	2.65×10^{-17}
	n=250	645.67	1.31×10^{-3}	4.15×10^{-14}

These results suggest that $\det(\mathbf{X}^\intercal\mathbf{X})\simeq0$ is not a good measure of the near singularity of $(\mathbf{X}^\intercal\mathbf{X})$. For instance, for $\mathbf{A}_n=\text{diag}(10^{-1},...,10^{-1})$, $\det(\mathbf{A}_n)=10^{-n}$, but $\kappa_2(\mathbf{A}_n)=1$; see Golub and Van Loan (2013, p. 89).

2.4.2 Orthogonal Polynomials

A common way to reduce near-collinearity is to replace the ordinary with orthogonal polynomials, $\{\phi_k(.),\ k=0,1,2,...\}$; see Seber and Lee (2003, p. 166). It is important, however, to distinguish between continuous and discrete polynomials.

Definition 2.1 *A sequence of continuous polynomials $\{\varphi_k(x),k=0,1,2,...\}$ is orthogonal with respect to a continuous weight function $w(x)\geq0$ on the interval (a,b) if:*

$$\int_a^b w(x)\varphi_i(x)\varphi_j(x)dx=c_i\delta_{ij},\ \ \delta_{ij}=\begin{cases}0 & i\neq j\\ 1 & i=j\end{cases},\ \ i,j=0,1,2,..., \tag{2.22}$$

where $c_i=\int_a^b w(x)\left[\varphi_i(x)\right]^2dx$, and δ_{ij} is the Kronecker delta.

Definition 2.2 *A sequence of discrete polynomials* $\{\wp_k(t), k{=}0, 1, 2, ..., N\}$ *is called orthogonal with respect to a positive weight* w_t *on the countable set* $\mathbb{T}$*:*

$$\sum\nolimits_{t\in\mathbb{T}} w_t \wp_i(t) \wp_j(t){=}c_i \delta_{ij}, \;\; i, j{=}0, 1, 2, ..., N, \tag{2.23}$$

where $c_i{=}\sum\nolimits_{t\in\mathbb{T}} w_t \left[\wp_i(t)\right]^2$ *, and* δ_{ij} *denotes the Kronecker delta.*

For these polynomials the $\mathbf{X}$ matrix in (2.11) takes the form of a Vandermonde-like matrix whose column space consists of the p-th degree of orthogonal polynomials:

$$\mathbf{X}{=}([\phi_i(x_t)]_{i,t}), \;\; x{\in} (a, b)\,, \;\; i{=}0, 1, ..., p, \;\; t{=}1, 2, ..., n, \tag{2.24}$$

where x_t is replaced with t, for discrete polynomials.

To evaluate the ill-conditioning of various continuous and discrete orthogonal polynomials (see Appendix B), we report their norm condition numbers for different sample sizes (n) and degree of polynomials (p). Table 2.4 reports the lowest condition numbers of the polynomials among the monotonic transformations (i)-(ii) for $t{:=}(1, 2, ..., n)$ - scaled by the sample size and logarithmic transformations. For comparison purposes, table 2.5 reports the condition numbers for the same polynomials using the scaling in (iii) - equally spaced ordering on the interval $[-1, 1]$. The scaling for continuous polynomials ensures that the interval of orthogonality is pertinent. In addition, lower bounds for the condition numbers are calculated for polynomials whose parameters can take different values. Note that the Chebyshev, Legendre, and Gegenbauer are special cases of the Jacobi polynomials.

Surprisingly, the norm condition numbers in table 2.4 are similar to the ones of the scaled ordinary polynomials in table 2.2. Specifically, the norm condition numbers grow exponentially with the degree of the polynomial, but remain stable with respect to increases to the sample size. In a nutshell, all the orthogonal polynomials show signs of $(\mathbf{X}^\intercal\mathbf{X})$ being ill-conditioned, whereas the only polynomials that indicate marginal improvements are the Jacobi. These results indicate that orthogonality itself is not sufficient enough to overcome the near-collinearity problem.

Table 2.4 - Scaled Orthogonal Polynomials: $\kappa_F\left(\mathbf{X}^\intercal\mathbf{X}\right)$					
Polynomials	Scale	n	p=3	p=5	p=7
Jacobi $[\alpha;\beta]$	$\frac{x_i}{n}$	n=100	10.16[.95; 7.63]	90.7[1.49; 11.65]	9.00×10^2[2.02; 16.16]
		n=250	10.09[.97; 7.62]	89.7[1.52; 11.63]	8.86×10^2[2.07; 16.13]
Gegenbauer $[\lambda]$	$\frac{x_i}{n}$	n=100	7.33×10^3 [.82]	6.09×10^6 [.79]	5.74×10^9 [.78]
		n=250	7.12×10^3 [.82]	5.77×10^6 [.80]	5.29×10^9 [.79]
Chebyshev - 1st	$\frac{x_i}{n}$	n=100	1.61×10^4	1.76×10^7	2.01×10^{10}
		n=250	1.59×10^4	1.71×10^7	1.93×10^{10}
Chebyshev - 2nd	$\frac{x_i}{n}$	n=100	8.04×10^3	7.23×10^6	7.39×10^9
		n=250	7.76×10^3	6.75×10^6	6.60×10^9
Legendre	$\frac{x_i}{n}$	n=100	1.15×10^4	1.04×10^7	1.04×10^{10}
		n=250	1.13×10^4	1.01×10^7	9.94×10^9
Hermite	$\frac{x_i}{n}$	n=100	1.17×10^5	1.47×10^{10}	1.15×10^{16}
		n=250	1.16×10^5	1.44×10^{10}	1.11×10^{16}
Laguerre	$\ln(x_i)$	n=100	3.32×10^5	3.08×10^{10}	1.35×10^{16}
		n=250	2.31×10^5	1.29×10^{10}	2.89×10^{15}
Charlier $[\alpha]$	–	n=100	3.19×10^2 [52.92]	3.82×10^3 [49.25]	4.60×10^4 [46.94]
		n=250	3.16×10^2 [132.72]	4.21×10^3 [123.83]	7.03×10^4 [119.61]
Chebyshev $[N]$	$\ln(t)$	n=100	2.15×10^4 [6]	2.47×10^7 [6]	7.58×10^{16} [8]
		n=250	9.36×10^4 [7]	1.69×10^8 [7]	6.16×10^{14} [8]
Krawtchouk $[\theta;N]$	$\ln(t)$	n=100	25.86 [.87; 4]	6.00×10^3 [.68; 5]	3.11×10^7 [.44; 7]
		n=250	15.62 [.85; 5]	1.44×10^3 [.87; 5]	1.62×10^6 [.58; 7]
Meixner $[\beta;c]$	–	n=100	35.32[2.69; .91]	7.47×10^2[4.34; .85]	2.17×10^4[5.08; .81]
		n=250	29.76[2.97; .96]	4.76×10^2[5.10; .93]	8.97×10^3[6.06; .91]
NOTE: the parameter values corresponding to the lower bounds are given in square brackets.					

In contrast, the results in table 2.5 point out that the norm condition numbers of some orthogonal polynomials are significantly reduced when the scaling in (iii) is applied. Of particular interest is the fact that the continuous polynomials whose interval of orthogonality is $[-1,1]$, including the Jacobi, Gegenbauer, Chebyshev, and Legendre, appear to be well-conditioned. On the other hand, all the discrete polynomials, as well as continuous polynomials whose orthogonality is outside the interval

$[-1,1]$, like the Hermite, show signs of ill-conditioning. Given these results, the main conclusion seems to be that one way to reduce the near-collinearity problem when fitting trend polynomials using least-squares is to combine orthogonality and scaling of the time ordering. That is, the practitioner should use an equally spaced transformation of the original time ordering, $t \in \mathbb{N} := (1, 2, ..., n, ...)$, on the interval $[-1,1]$, in conjunction with continuous orthogonal polynomials whose interval of orthogonality is $[-1,1]$.

Table 2.5 - Orthogonal Polynomials over $[-1,1]$: $\kappa_F\left(\mathbf{X}^\intercal\mathbf{X}\right)$

Polynomials	n	p=3	p=5	p=7
Jacobi $[\alpha;\beta]$	n=100	4.59 [.31; .31]	7.85 [.29; .29]	11.75 [.27; .27]
	n=250	4.54 [.32; .32]	7.59 [.31; .31]	11.05 [.29; .29]
Gegenbauer $[\lambda]$	n=100	4.59 [.81]	7.85 [.79]	11.75 [.77]
	n=250	4.54 [.82]	7.59 [.81]	11.05 [.79]
Chebyshev - 1st	n=100	8.06	13.01	17.63
	n=250	8.41	14.21	20.36
Chebyshev - 2nd	n=100	5.49	10.39	17.24
	n=250	5.33	9.65	15.11
Legendre	n=100	9.51	17.32	26.43
	n=250	9.75	17.96	27.56
Hermite	n=100	1.93×10^2	8.26×10^5	2.26×10^{10}
	n=250	2.02×10^2	8.87×10^5	2.44×10^{10}
Laguerre	Range of values of t outside the interval $[-1,1]$.			
Charlier $[\alpha]$	n=100	3.02×10^4 [.86]	1.30×10^{10} [1.54]	2.35×10^{16} [4.34]
	n=250	3.10×10^4 [.86]	1.36×10^{10} [1.53]	1.74×10^{16} [4.47]
Chebyshev $[N]$	n=100	2.17×10^7 [4]	1.14×10^{17} [6]	8.23×10^{28} [8]
	n=250	2.21×10^7 [4]	1.18×10^{17} [6]	6.02×10^{27} [8]
Krawtchouk $[\theta;N]$	n=100	3.61×10^3 [.01; 3]	7.95×10^7 [.01; 5]	8.81×10^{12} [.01; 7]
	n=250	3.73×10^3 [.01; 3]	8.35×10^7 [.01; 5]	9.38×10^{12} [.01; 7]
Meixner $[\beta;c]$	n=100	1.98×10^3 [.58; .06]	2.17×10^7 [.09; .01]	2.57×10^{12} [.01; .01]
	n=250	2.02×10^3 [.57; .06]	2.25×10^7 [.09; .01]	2.71×10^{12} [.01; .01]

NOTE: the parameter values corresponding to the lower bounds are given in square brackets.

2.4.3 From Orthogonal to Orthonormal Polynomials

A more direct way to avoid near-collinearity when using trend polynomials is to use the Gram-Schmidt (G-S) discrete polynomials by transforming the sequence $\{t^k, k{=}0,1,2,...\}$ to recursively generate:

$$\widetilde{v}_0{=}1, \quad \widetilde{v}_k{=}t_k{-}\sum\nolimits_{l=0}^{k-1}\eta_l\widetilde{v}_l, \;\; \eta_l{=}\tfrac{(t_k,\widetilde{v}_l)}{(\widetilde{v}_l,\widetilde{v}_l)}, \;\; v_k{=}\tfrac{\widetilde{v}_k}{\|\widetilde{v}_k\|}, \;\; k{=}0,1,2,..., \tag{2.25}$$

rendering the polynomial v_k orthogonormal which satisfies: $v_k v_l{=}\delta_{kl}$, where δ_{kl} denotes the Kronecker delta, for $l{=}1,2,...,k$.

Evaluating the norm condition numbers for the orthogonal G-S polynomials in table 2.6 suggests that orthogonality by itself does not overcome the $(\mathbf{X}^\intercal\mathbf{X})$ ill-conditioning problem since the $\kappa_F\,(\mathbf{X}^\intercal\mathbf{X})$ numbers are similar to those of the unscaled ordinary polynomials in table 2.1. But when the G-S are normalized, $\kappa_F\,(\mathbf{X}^\intercal\mathbf{X}){=}p{+}1$.

Table 2.6 - Gram-Schmidt (G-S) polynomials: $\kappa_F\,(\mathbf{X}^\intercal\mathbf{X})$

G-S	n	$p{=}3$	$p{=}5$	$p{=}7$
Orthogonal	$n{=}100$	3.57×10^{8}	1.42×10^{14}	5.58×10^{19}
	$n{=}250$	8.72×10^{10}	1.36×10^{18}	2.10×10^{25}
Orthonormal	$n{=}100$	4	6	8
	$n{=}250$	4	6	8

Caution is advisable when using the G-S orthonormalization for $p{>}7$. To avoid the sensitivity of the orthogonalization process to rounding errors, one is advised to use the modified G-S algorithm; see Trefethen and Bau (1997, pp. 57-58).

The G-S orthonormal polynomials can be very effective in estimation and M-S testing because they are generated for a specific sample size n. However, in cases where the modeling calls for explicit functional forms for the polynomials, such as forecasting and policy simulations, one is better of by choosing among the continuous orthogonal polynomials defined over the interval $[-1,1]$, shown in table 2.5.

2.5 Empirical Example: Aggregate Portfolios

This empirical example illustrates how one can capture the t-heterogeneity in asset pricing modeling using continuous orthogonal polynomials defined over the interval $[-1,1]$, or G-S orthonormal polynomials. The data used are the Fama and French (2015) data for the period of July 1963 to December 2015. The portfolio employed is the (smallest size/lowest profitability) which includes all NYSE, AMEX, and NASDAQ stocks whose market capitalization and operating profitability are respectively within the bottom quintile of NYSE stocks. Given that most AMEX and NASDAQ stocks are smaller in market capitalization than the NYSE stocks, this portfolio, on average, contains approximately 1058 stocks, many more than other portfolios constructed on similar characteristics.

To account for all the statistical regularities exhibited by the observed data, the following heterogeneous Student's t/ Heteroskedastic Dynamic LR model of the CAPM is estimated:

$$\begin{gathered}
y_t = \alpha + \sum_{i=1}^{12} \delta_{1i} t^i + \sum_{j=2}^{12} \delta_{2j} d_{jt} + \beta_1 x_{1t} + \beta_2 x_{2t} + \boldsymbol{\gamma}_1^\top \mathbf{Z}_{t-1} + \varepsilon_t, \\
(\varepsilon_t | \mathbf{Z}_{t-1}) \backsim \mathsf{St}(0, \sigma^2(t)),\ \sigma^2(t) = \gamma_0 + [\mathbf{W}_t - \boldsymbol{\mu}(t)]^\top \mathbf{Q}^{-1} [\mathbf{W}_t - \boldsymbol{\mu}(t)],
\end{gathered} \tag{2.26}$$

where y_t is the return of the (smallest size/lowest profitability) portfolio for period t; $x_{1t}=R_{mt}$ is the return on the value-weighted market portfolio; $x_{2t}=R_{ft}$ is the return on the risk-free asset; $t^i:=(t, t^2, ..., t^{12})$, denotes the polynomials of {Ordinary over $[-1,1]$, Meixner by n, Hermite over $[-1,1]$, Jacobi over $[-1,1]$, G-S orthonormal}; $d_{jt}:=(d_{2t}, d_{3t}, ..., d_{12t})$ are the monthly dummy variables for the months of February through December; $\mathbf{Z}_t:=(y_t, \mathbf{X}_t)$, $\mathbf{X}_t:=(x_{1t}, x_{2t})$, $\mathbf{W}_t:=(\mathbf{X}_t, \mathbf{Z}_{t-1})$, $\boldsymbol{\mu}(t)=E(\mathbf{W}_t)$, $\mathbf{Q}=Cov(\mathbf{W}_t)$. Note that $\sigma^2(t)$ is both heteroskedastic and heterogeneous, stemming from the Multivariate Student's t distribution; see Spanos (1994).

Table 2.7 presents the results from estimating the same overall model in (2.26) by including the different types of trend polynomials. The inflated estimated coefficients and standard errors for the cases of ordinary over $[-1, 1]$, Meixner scaled by n, and Hermite over $[-1, 1]$ polynomials indicate the presence of serious near-collinearity. It is increasingly difficult for the modeler to assess the relative contribution of each polynomial in terms of significance since the coefficients of interest are not statistically significant. The picture becomes much clearer when the G-S orthonormal polynomials or the Jacobi over $[-1, 1]$ polynomials are used to capture the t-heterogeneity in the data. As can be seen from table 2.7, the absence of near-collinearity for these polynomials provides more reliable evidence for the significance of the higher order trends up to order 12.

These results confirm that neither the scaling of the time ordering nor the orthogonality property alone are sufficient enough to overcome the near-collinearity problem when fitting trend polynomials in the context of linear models. Yet, the most efficient way to adequately address the near-collinearity problem is to use the G-S orthonormal polynomials, or to combine continuous orthogonal polynomials whose interval of orthogonality is $[-1, 1]$, with scaling the time ordering to lie within the same interval.

The importance of this result stems from the fact that establishing the statistical adequacy of the estimated model, secures the reliability of inferences based on it. In the case of the empirical multi-factor models, like the Fama-French models (see Fama and French, 1993; 2015), these polynomials secure the statistical adequacy of the estimated models which can provide a sound basis for evaluating the significance tests of the additional factors. Capturing the t-heterogeneity in this data, gives rise to trustworthy results when posing substantive questions of interest to the data, including which additional factors are significant or not. In contrast, when the t-heterogeneity in the data is not captured, the statistical significance of the added factors is likely to be untrustworthy.

The statistical model specified in (2.26), as well as the role that such polynomials may play in asset pricing modeling, will be discussed at greater length in the chapters to follow.

Table 2.7 - Empirical Example: Modeling Heterogeneity in Portfolio Returns

Polyn.	Ordinary	Meixner [β=9.02; c=.91]	Hermite	Jacobi [α=.29; β=.29]	G-S orthon.
c	2.108[.001] (0.636)	3.756[.000] (0.637)	-13.620[.844] (69.155)	3.098[.000] (0.538)	3.101[.000] (0.522)
t	-2.804[.157] (1.980)	-0.967[.047] (0.485)	57.022[.929] (639.717)	-0.577[.001] (0.166)	-12.696[.000] (3.555)
t^2	32.352[.008] (12.233)	-0.878[.205] (0.691)	- 91.199 [.929] (1017.265)	0.231[.199] (0.180)	3.134[.392] (3.660)
t^3	-14.478[.627] (29.815)	5.474[.000] (1.515)	42.768 [.987] (2645.153)	0.049[.714] (0.134)	1.058[.657] (2.379)
t^4	-187.952[.125] (122.399)	8.345[.000] (1.958)	- 41.418 [.990] (3155.453)	-0.393[.001] (0.122)	-7.050[.001] (2.142)
t^5	157.738[.330] (161.963)	0.567[.835] (2.724)	-115.527[.978] (4173.752)	-0.432[.004] (0.151)	-4.093[.098] (2.473)
t^6	467.668[.338] (488.179)	-10.053[.005] (3.571)	106.080[.978] (3854.339)	0.424[.003] (0.141)	4.671[.039] (2.264)
t^7	-363.972[.331] (374.098)	-11.637[.006] (4.191)	- 38.292 [.979] (1450.451)	0.588[.000] (0.153)	11.727[.000] (2.458)
t^8	-606.016[.510] (918.897)	-6.738[.098] (4.061)	13.097[NaN] (NaN)	-0.705[.000] (0.154)	-11.304[.000] (2.389)
t^9	296.251[.440] (383.424)	4.189[.539] (6.820)	202.963[NaN] (NaN)	0.589[.000] (0.142)	7.569[.001] (2.207)
t^{10}	423.236[.604] (815.902)	1.128[.837] (5.471)	-160.951[NaN] (NaN)	-0.281[.095] (0.168)	-1.454[.550] (2.429)
t^{11}	-71.542[.619] (143.884)	6.819[.118] (4.352)	138.265[NaN] (NaN)	-0.445[.001] (0.137)	-7.134[.001] (2.217)
t^{12}	-130.003[.637] (275.466)	-2.961[.730] (8.569)	-94.774[NaN] (NaN)	0.626[.000] (0.145)	9.956[.000] (2.301)
d_2	-2.457[.000] (0.467)	-2.512[.000] (0.464)	-2.405[.000] (0.459)	-2.462[.000] (0.471)	-2.458[.000] (0.471)
d_3	-2.469[.000] (0.446)	-2.328[.000] (0.441)	-2.394[.000] (0.440)	-2.478[.000] (0.445)	-2.473[.000] (0.445)
d_4	-3.815[.000] (0.483)	-3.767[.000] (0.477)	-3.868[.000] (0.478)	-3.932[.000] (0.482)	-3.930[.000] (0.481)
d_5	-2.533[.000] (0.477)	-2.596[.000] (0.472)	-2.528[.000] (0.468)	-2.607[.000] (0.473)	-2.603[.000] (0.473)
d_6	-2.635[.000] (0.489)	-2.715[.000] (0.484)	-2.484[.000] (0.482)	-2.757[.000] (0.485)	-2.753[.000] (0.485)
d_7	-3.334[.000] (0.501)	-3.273[.000] (0.493)	-3.259[.000] (0.491)	-3.535[.000] (0.500)	-3.533[.000] (0.500)
d_8	-3.234[.000] (0.483)	-3.302[.000] (0.483)	-3.106[.000] (0.467)	-3.569[.000] (0.485)	-3.559[.000] (0.485)
d_9	-2.296[.000] (0.493)	-2.249[.000] (0.484)	-2.254[.000] (0.477)	-2.522[.000] (0.480)	-2.522[.000] (0.479)
d_{10}	-4.150[.000] (0.512)	-4.278[.000] (0.509)	-4.191[.000] (0.512)	-4.281[.000] (0.505)	-4.277[.000] (0.505)
d_{11}	-3.586[.000] (0.493)	-3.654[.000] (0.488)	-3.605[.000] (0.485)	-3.814[.000] (0.485)	-3.811[.000] (0.485)
d_{12}	-3.174[.000] (0.473)	-3.319[.000] (0.470)	-3.143[.000] (0.462)	-3.344[.000] (0.474)	-3.337[.000] (0.474)
x_{1t}	1.254[.000] (0.029)	1.240[.000] (0.029)	1.259[.000] (0.029)	1.249[.000] (0.028)	1.249[.000] (0.028)
x_{2t}	-5.587[.091] (3.305)	-5.657[.087] (3.298)	-4.733[.154] (3.317)	-4.967[.124] (3.225)	-4.957[.125] (3.229)
y_{t-1}	0.103[.019] (0.044)	0.096[.029] (0.044)	0.128[.002] (0.042)	0.078[.070] (0.043)	0.079[.067] (0.043)
x_{1t-1}	0.174[.010] (0.067)	0.187[.005] (0.067)	0.154[.021] (0.066)	0.203[.002] (0.065)	0.203[.002] (0.065)
x_{2t-1}	2.590[.448] (3.414)	2.856[.403] (3.412)	1.887[.581] (3.421)	2.686[.421] (3.333)	2.683[.422] (3.336)
$\kappa\left(\mathbf{X}^{\intercal}\mathbf{X}\right)_F$	3.37×10^8	1.16×10^7	1.12×10^{15}	20.07	13

2.6 Summary and Conclusions

The chapter evaluated a number of different ways one can address the problem of ill-conditioning in the context of linear models of the form $\mathbf{y}=\mathbf{X}\boldsymbol{\beta}+\mathbf{u}$ when $\mathbf{X}$ includes ordinary trend polynomials based on the ordering $(t=1,2,...,n)$ of degree higher than 4. The problem of ill-conditioning can be mitigated using a twofold strategy. The first is to orthogonalize the terms $t, t^2, ..., t^p$. This led us to the evaluation of several continuous and discrete orthogonal polynomials, such as the Chebyshev, Hermite, Jacobi, Laguerre, the Krawtchouk, Charlier and Meixner. The second is a scaling of the trend ordering $(t=1,2,...,n)$. This led us to evaluate different forms of scaling and concluded that the best is the one which yields approximately equally spaced discrete values over the interval $[-1,1]$. The ill-conditioning can be effectively circumvented by combining the orthogonality with this particular scaling.

The discussion in the chapter can be summarized by a few recommendations to the practitioner. First, the simplest way to avoid serious near-collinearity problems when fitting polynomials of order $p<7$, is to re-scale the original time ordering using (2.21) to confine its range of values within $[-1,1]$. Second, the best choice for polynomials of order $p\geq 7$ is the modified G-S orthonormal polynomials; they are particularly useful for estimation and M-S testing. Third, for forecasting and simulation purposes, however, the practitioner might prefer an explicit functional form for the orthogonal polynomials, and not a sequence of values for a fixed n, as given by the G-S polynomials. For such purposes the recommendation is to choose among certain orthogonal polynomials, such as the Jacobi, specified over the interval $[-1,1]$. Fourth, to avoid misleading statistical measures of near-collinearity, it is important to treat these polynomials differently from the other regressors and apply the Frisch-Waugh result before statistical measures are evaluated. This amounts to detrending the data before the correlations among the genuine regressors are evaluated.

In addition, it is important to emphasize that the main arguments in this chapter hold more generally for cross-section data. In modeling such data there is a misleading impression that dependence and/or heterogeneity are irrelevant because we know how to select 'random samples' from populations of individual units such as people, households, firms, cities, states, countries, etc.; see Wooldridge (2012). As a matter of fact cross-section data are vulnerable to departures from dependence and/or heterogeneity because they often have several orderings of potential interest, not just one (time) as in time-series data. The only difference between the two is the measurement scale of the relevant ordering(s). For time-series data, the time ordering is measured on an interval scale, but for cross-section data the orderings might vary from ratio scale (e.g., geographical position, age, size) to nominal scale (e.g., gender, religion); see Spanos (1999). For nominal scale orderings the trend polynomials need to be replaced with more flexible functions such as splines; see Ruppert et al. (2003).

The results of this chapter will play a very crucial role in what follows. The G-S orthonormal trend polynomials will be used to 'capture' any mean and variance heterogeneity in the portfolio data. Indeed, these data exhibit complicated forms of heterogeneity and the G-S orthonormal polynomials can be very effective in M-S testing (chapter 3) and estimation/respecification (chapter 4). Moreover, these polynomials will play the most crucial role in the factor selection procedure proposed (chapter 5), as well as in market risk forecasting (chapter 6).

Appendix B: Orthogonal Polynomials

This Appendix presents the definitions and recurrence relations of all the continuous and discrete orthogonal polynomials investigated. For most of the formulae below, as well as a detailed discussion on these polynomials, see Gautschi (2004) and Dunkl and Xu (2014).

Definition 2.3 *The Jacobi polynomials* $P_k^{(\alpha,\beta)}(x)$*, for parameters* $\alpha, \beta > -1$*, are a set of orthogonal polynomials with weight function* $w(x;\alpha,\beta)=(1-x)^\alpha(1+x)^\beta$ *on* $x\in[-1,1]$.

$$P_k^{(\alpha,\beta)}(x)=\frac{(-1)^k}{2^k k!}(1-x)^{-\alpha}(1+x)^{-\beta}\left(\frac{d}{dx}\right)^k\left[(1-x)^{\alpha+k}(1+x)^{\beta+k}\right],\ k\geq 0. \tag{B.1}$$

The recurrence relation of the Jacobi polynomials is:

$$\begin{aligned}
&2(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta)P_{k+1}^{(\alpha,\beta)}(x)=\\
&=[(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)(2k+\alpha+\beta)x+\alpha^2-\beta^2]P_k^{(\alpha,\beta)}(x)-\\
&-2(k+\alpha)(k+\beta)(2k+\alpha+\beta+2)P_{k-1}^{(\alpha,\beta)}(x),\quad k\geq 1,\\
&P_0^{(\alpha,\beta)}(x)=1,\quad P_1^{(\alpha,\beta)}(x)=\tfrac{1}{2}(\alpha+\beta+2)x+\tfrac{1}{2}(\alpha-\beta).
\end{aligned} \tag{B.2}$$

Definition 2.4 *The Gegenbauer polynomials* $C_k^{(\lambda)}(x)$*, for parameter* $\lambda > -\frac{1}{2}$*, are a set of orthogonal polynomials with weight function* $w(x;\lambda)=(1-x^2)^{\lambda-\frac{1}{2}}$ *on* $x\in[-1,1]$.

$$C_k^{(\lambda)}(x)=\frac{(-1)^k}{2^k(\lambda+\frac{1}{2})_k}(1-x^2)^{\frac{1}{2}-\lambda}\left(\frac{d}{dx}\right)^k(1-x^2)^{\lambda+k-\frac{1}{2}},\ k\geq 0. \tag{B.3}$$

The recurrence relation of the Gegenbauer polynomials is:

$$\begin{gathered}
(k+1)C_{k+1}^{(\lambda)}(x)=2(k+\lambda)xC_k^{(\lambda)}(x)-(k+2\lambda-1)C_{k-1}^{(\lambda)}(x),\quad k\geq 1,\\
C_0^{(\lambda)}(x)=1,\quad C_1^{(\lambda)}(x)=2\lambda x.
\end{gathered} \tag{B.4}$$

The Gegenbauer polynomials can be viewed as a special case of the Jacobi polynomials with $\alpha{=}\beta{=}\lambda - \frac{1}{2}$.

Definition 2.5 *The Chebyshev polynomials of the first kind $T_k(x)$ are a set of orthogonal polynomials with weight function $w(x){=}(1-x^2)^{-\frac{1}{2}}$ on $x{\in}[-1,1]$.*

$$T_k(x){=}\tfrac{(-1)^k 2^k k!}{(2k)!}(1-x^2)^{\frac{1}{2}}\left(\tfrac{d}{dx}\right)^k(1-x^2)^{k-\frac{1}{2}},\ \ k{\geq}0. \tag{B.5}$$

Definition 2.6 *The Chebyshev polynomials of the second kind $U_k(x)$ are a set of orthogonal polynomials with weight function $w(x){=}(1-x^2)^{\frac{1}{2}}$ on $x{\in}[-1,1]$.*

$$U_k(x){=}\tfrac{(-1)^k 2^k (k+1)!}{(2k+1)!}(1-x^2)^{-\frac{1}{2}}\left(\tfrac{d}{dx}\right)^k(1-x^2)^{k+\frac{1}{2}},\ \ k{\geq}0. \tag{B.6}$$

The recurrence relation of the Chebyshev polynomials is:

$$\begin{gathered} y_{k+1}{=}2xy_k{-}y_{k-1}, \quad k{\geq}1, \\ y_0{=}1, \quad y_1(T_k(x)){=}x, \quad y_1(U_k(x)){=}2x. \end{gathered} \tag{B.7}$$

The Chebyshev polynomials of the first and second kind can be viewed as special cases of the Jacobi polynomials with $\alpha{=}\beta{=}{-}\frac{1}{2}$ and $\alpha{=}\beta{=}\frac{1}{2}$, respectively. Further, the Chebyshev polynomials of the first and second kind can also be viewed as special cases of the Gegenbauer polynomials with $\lambda{=}0$ and $\lambda{=}1$, respectively.

Definition 2.7 *The Legendre polynomials $P_k(x)$ are a set of orthogonal polynomials with weight function $w(x){=}1$ on $x{\in}[-1,1]$.*

$$P_k(x){=}\tfrac{(-1)^k}{2^k k!}\left(\tfrac{d}{dx}\right)^k(1-x^2)^k,\ \ k{\geq}0. \tag{B.8}$$

The recurrence relation of the Legendre polynomials is:

$$\begin{gathered} (k{+}1)P_{k+1}(x){=}(2k{+}1)xP_k(x){-}kP_{k-1}(x), \quad k{\geq}1, \\ P_0(x){=}1, \quad P_1(x){=}x. \end{gathered} \tag{B.9}$$

The Legendre polynomials can be viewed as a special case of the Jacobi polynomials with $\alpha{=}\beta{=}0$. Further, the Legendre polynomials can also be viewed as a special case of the Gegenbauer polynomials with $\lambda{=}\frac{1}{2}$.

Definition 2.8 *The Hermite polynomials $H_k(x)$ are a set of orthogonal polynomials with weight function $w(x){=}e^{-x^2}$ on $x{\in}\mathbb{R}$.*

$$H_k(x){=}(-1)^k e^{x^2}\left(\tfrac{d}{dx}\right)^k e^{-x^2},\ k{\geq}0. \tag{B.10}$$

The recurrence relation of the Hermite polynomials is:

$$\begin{gathered} H_{k+1}(x){=}2xH_k(x){-}2kH_{k-1}(x),\quad k{\geq}1, \\ H_0(x){=}1,\quad H_1(x){=}2x. \end{gathered} \tag{B.11}$$

Definition 2.9 *The Laguerre polynomials $L_k^{(\alpha)}(x)$, for parameter $a{>}-1$, are a set of orthogonal polynomials with weight function $w(x;a){=}e^{-x}x^{\alpha}$ on $x{\in}\mathbb{R}_+$.*

$$L_k^{(\alpha)}(x){=}\tfrac{e^x x^{-\alpha}}{k!}\left(\tfrac{d}{dx}\right)^k\left(e^{-x}x^{k+\alpha}\right),\ k{\geq}0. \tag{B.12}$$

The recurrence relation of the Laguerre polynomials is:

$$\begin{gathered} (k{+}1)L_{k+1}^{(\alpha)}(x){=}(2k{+}\alpha{+}1{-}x)L_k^{(\alpha)}(x){-}(k{+}\alpha)L_{k-1}^{(\alpha)}(x),\quad k{\geq}1, \\ L_0^{(\alpha)}(x){=}1,\quad L_1^{(\alpha)}(x){=}\alpha{+}1{-}x. \end{gathered} \tag{B.13}$$

The classical Laguerre polynomials corresponds to $\alpha{=}0$.

Definition 2.10 *The Discrete Chebyshev polynomials (also known as Gram polynomials) $D_k^{(N)}(t)$, for parameter $N{\in}\mathbb{N}$, are a set of orthogonal polynomials with weight function $w(t){=}1$ on $t{=}0,1,...,N{-}1$.*

$$D_k^{(N)}(t){=}\Delta^k\tfrac{(t-k+1)_k(t-N-k+1)_k}{k!},\ k{=}0,1,2,...,N{-}1. \tag{B.14}$$

The recurrence relation of the Discrete Chebyshev polynomials is:

$$(k+1)D_{k+1}^{(N)}(t)=2(2k+1)(t-\tfrac{1}{2}(N-1))D_k^{(N)}(t)-t(N^2-t^2)D_{k-1}^{(N)}(t),\ \ k=0,1,2,...,N-1,$$
$$D_0^{(N)}(t)=1,\quad D_1^{(N)}(t)=2t-N+1. \tag{B.15}$$

Definition 2.11 *The Charlier polynomials $C_k^{(\alpha)}(t)$, for parameter $\alpha>0$, are a set of orthogonal polynomials with weight function $w(t;\alpha)=\frac{e^{-\alpha}\alpha^t}{t!}$ on $t\in\mathbb{N}_0$.*

$$C_k^{(\alpha)}(t)=\tfrac{t!}{\alpha^t}\,\triangle^k\left[\tfrac{\alpha^{t-k}}{(t-k)!}\right],\ \ k\geq 0. \tag{B.16}$$

The recurrence relation of the Charlier polynomials is:

$$\alpha C_{k+1}^{(\alpha)}(t)=(k+\alpha-t)C_k^{(\alpha)}(t)-kC_{k-1}^{(\alpha)}(t),\ \ k\geq 0,$$
$$C_0^{(\alpha)}(t)=1,\quad C_1^{(\alpha)}(t)=-\tfrac{1}{\alpha}t. \tag{B.17}$$

Definition 2.12 *The Krawtchouk polynomials $K_k^{(\theta,N)}(t)$, for parameters $0<\theta<1$ and $N\in\mathbb{N}$, are a set of orthogonal polynomials with weight function $w(t;\theta,N)=\begin{pmatrix}N\\t\end{pmatrix}\theta^t(1-\theta)^{N-t}$ on $t=0,1,...,N$.*

$$K_k^{(\theta,N)}(t)=\tfrac{(-1)^k t!(N-t)!}{k!\theta^t(1-\theta)^{N-t}}\,\triangle^k\left[\tfrac{\theta^t(1-\theta)^{N-t+k}}{(t-k)!(N-t)!}\right],\ \ k=0,1,2,...,N. \tag{B.18}$$

The recurrence relation of the Krawtchouk polynomials is:

$$(k+1)K_{k+1}^{(\theta,N)}(t)=[t-(k+\theta(N-2k))]K_k^{(\theta,N)}(t)-(N-k+1)\theta(1-\theta)K_{k-1}^{(\theta,N)}(t),\ \ k=0,1,2,...,N,$$
$$K_0^{(\theta,N)}(t)=1,\quad K_1^{(\theta,N)}(t)=t-\theta N. \tag{B.19}$$

Definition 2.13 *The Meixner polynomials (also known as discrete Laguerre polynomials) $M_k^{(\beta,c)}(t)$, for parameters $\beta>0$ and $0<c<1$, are a set of orthogonal polynomials with weight function $w(t;\beta,c)=\frac{c^t(\beta)_t}{t!}$ on $t\in\mathbb{N}_0$.*

$$M_k^{(\beta,c)}(t)=\tfrac{t!}{(\beta)_t}c^{-t-k}\,\triangle^k\left[\tfrac{c^t(\beta)_t}{(t-k)!}\right],\ \ k\geq 0. \tag{B.20}$$

The recurrence relation of the Meixner polynomials is:

$$\begin{gathered}(k+\beta)cM_{k+1}^{(\beta,c)}(t)=[(c-1)t+(1+c)k+\beta c]M_{k}^{(\beta,c)}(t)-kM_{k-1}^{(\beta,c)}(t),\ \ k\geq 0,\\ M_{0}^{(\beta,c)}(t)=1,\quad M_{1}^{(\beta,c)}(t)=\tfrac{c-1}{\beta c}t+1.\end{gathered}\tag{B.21}$$

Chapter 3

3 Confronting Asset Pricing Theory with Data

3.1 Overview

Due primarily to its simplicity, the CAPM has been subject to sustained criticism for the unrealistic assumptions underlying the model. Undoubtedly, the mathematical justifications of the CAPM are built on assumptions with dubious connections to the real world. Discouraging results pointing out the poor performance of the CAPM are also supported by empirically grounded studies which suggest the inability of the model to explain and predict asset returns. However, almost all the empirical tests of the CAPM, as well as its extensions, are conducted under the assumption that asset returns follow a Normal, Independent, and Identically Distributed (NIID) process. As a matter of fact, very little analysis has been implemented to understand the sensitivity of the statistical inference results to violations of the probabilistic assumptions making up the underlying asset pricing models. Hence, it is of great importance to revisit the empirical adequacy of these models and diagnose the appropriateness and reliability of their inferences.

The primary aim of this chapter is to revisit the empirical adequacy of the statistical CAPM and the Fama-French three-factor model. The chapter begins with section 3.2, by distinguishing between the structural and statistical CAPM, where the latter comprises the probabilistic assumptions imposed on the data. This section brings out clearly the difference between statistical and substantive adequacy.

Section 3.3 briefly discusses the Fama and French (1993) data and the deterministic variables used with a view to capture the statistical regularities exhibited by the data. Section 3.4 thoroughly evaluates the validity of the probabilistic assumptions underlying the statistical CAPM and the Fama-French three-factor model vis-à-vis the data. As the Mis-Specification (M-S) testing results suggest, these models are statistically misspecified.

3.2 The Reliability of CAPM: Data vs. Theory

To motivate the discussion that follows, this section considers the distinctions between the *substantive* (structural) and *statistical* CAPM. Although focus is on CAPM, this discussion is extensive to any empirical testing of asset pricing models, such as the conversion of substantive asset pricing models into (statistical) models which are estimable with a given data set, or pure empirical studies searching for new pricing factors.

A substantive model is a mathematical formulation of a theory that aims to approximate a real-world phenomenon of interest with a view to provide an adequate explanation. Justifiably, such models could be described as simplifications of the real world because they typically rely on strong assumptions. Moreover, by replacing some of the unrealistic assumptions with more realistic ones the theory's fecundity can be enhanced. On the other hand, a statistical model represents the probabilistic assumptions imposed (directly or indirectly) on the data. Its role is to ensure the error-reliability of all statistical inferences based on the estimated model since the inference procedures invoke these assumptions. In practice, though, these assumptions are often not clearly brought out because they are imposed on the data indirectly via the substantive model in conjunction with the error term assumptions and not the observable random variables involved. Hence, the statistical premises are rarely tested thoroughly, and as a result the endeavors to enhance the substantive adequacy of the estimated substantive model are often of questionable value on empirical grounds.

To avoid this conundrum one needs to establish the adequacy of the implicit statistical model before probing for the appropriateness of substantive refinements of the original substantive model; see Spanos (2006b; 2009)

The theory of CAPM describes an equilibrium relationship between expected rates of returns. In practice though, the theoretical concept of expected returns is unobserved. In order to formulate and test the CAPM theory, the unobserved expected returns needs to be replaced with observed random variables. Hence, the theoretical CAPM in (1.1) takes the form of a substantive model:

$$\begin{gathered}\mathcal{M}_{\boldsymbol{\varphi}}(\mathbf{z};\boldsymbol{\varphi})\colon (R_{it}{-}R_{ft}){=}\beta_i(R_{mt}{-}R_{ft})+\varepsilon_{it},\ i{=}1,...,k,\ t{=}1,...,n,\\ \varepsilon_{it}\sim \text{NIID }(0,\sigma^2_{\varepsilon i}),\ \boldsymbol{\varphi}{:=}(\beta_i,\sigma^2_{\varepsilon i}),\end{gathered} \tag{3.1}$$

where R_{it} is the observed return of asset or portfolio i for period t, R_{ft} is the observed risk-free rate, and R_{mt} is the observable counterpart of the market portfolio.

For inference purposes the CAPM is often embedded into a time-series regression which takes the following form (statistical model):

$$\begin{gathered}\mathcal{M}_{\boldsymbol{\theta}}(\mathbf{z};\boldsymbol{\theta})\colon (R_{it}{-}R_{ft}){=}\alpha_i+\beta_i(R_{mt}{-}R_{ft})+\varepsilon_{it},\ i{=}1,...,k,\ t{=}1,...,n,\\ \varepsilon_{it}\sim \text{NIID }(0,\sigma^2_{\varepsilon i}),\ \boldsymbol{\theta}{:=}(\alpha_i,\beta_i,\sigma^2_{\varepsilon i}).\end{gathered} \tag{3.2}$$

Given that the CAPM does not hold exactly when the unobserved expected returns are replaced by the observed random variables, an unobserved random component, ε_{it}, is included as part of the time-series regression in (3.2). This error term is assumed to satisfy the following implicit assumptions:

$$\begin{gathered}\text{(i) } (\varepsilon_{it}|R_{mt}{-}R_{ft})\sim \text{N }(0,\sigma^2_{\varepsilon i}),\ \text{for } \sigma^2_{\varepsilon i}{=}\sigma^2_i{-}\beta^2_i\sigma^2_m,\\ \text{(ii) } Cov(\varepsilon_{it},R_{mt}{-}R_{ft}){=}0,\ \text{(iii) } Cov(\varepsilon_{it},\varepsilon_{jt}){\neq}0,\ \text{for } i{\neq}j,\ i,j{=}1,...,k,\\ \text{(iv) } Cov(\varepsilon_{it},\varepsilon_{is}){=}0,\ \text{for } t{\neq}s,\ t,s{=}1,...,n.\end{gathered} \tag{3.3}$$

Of these assumptions, the zero expected residual assumption, $E(\varepsilon_{it}|R_{mt}{-}R_{ft}){=}0$, is necessary for the interpretation of the statistical results. Under this assumption, the statistical model in (3.2) is directly related to the substantive model in (3.1) if the estimated parameter α_i is zero for every asset or portfolio i; see Jensen (1968). That being the case, a test for the CAPM is constructed by testing the hypothesis of H_0: α_i=0. If this hypothesis is satisfied, the sample data provides evidence in favor of the CAPM theory, otherwise the evidence is against the theory.

Nonetheless, what is often not appreciated by the modeler is that the statistical model in (3.2) invokes a set of explicit testable probabilistic assumptions (table 3.1) which are often not clearly brought out because they are imposed on the data indirectly by the implicit and untestable error term assumptions in (3.3). These error term assumptions are (indirectly) equivalent to the Normal/Homoskedastic LR model assumptions [1]–[4] in table 3.1, yet they do not necessarily require the crucial assumption of [5] t-invariance; see Brenner and Smidt (1977). For a detailed comparison between assumptions (i)-(iv) in (3.3) and [1]–[5] in table 3.1, see Spanos (2010).

Table 3.1 - Normal/Homoskedastic LR Model

$$\boxed{\text{Statistical GM:} \quad y_t{=}\alpha + \beta x_t + u_t,\ t{\in}\mathbb{N}}$$

[1] Normality:	$D(y_t \vert X_t{=}x_t; \boldsymbol{\theta})$ is Normal,
[2] Linearity:	$E(y_t \vert X_t{=}x_t){=}\alpha + \beta x_t$, is linear in x_t,
[3] Homoskedasticity:	$Var(y_t \vert X_t{=}x_t){=}\sigma^2{>}0$, is free of x_t,
[4] Independence:	$\{(y_t \vert X_t{=}x_t), t{\in}\mathbb{N}\}$ independent process,
[5] t-invariance:	$\boldsymbol{\theta}{:=}(\alpha, \beta, \sigma^2)$ are constant over t.

$$\alpha{=}E(y_t){-}E(X_t),\ \beta_i{=}\tfrac{Cov(y_t,X_t)}{Var(X_t)},\ \sigma^2{=}Var(y_t){-}\tfrac{[Cov(y_t,X_t)]^2}{Var(X_t)}.$$

In practice, the modeler needs to make the implicit probabilistic assumptions concerning explicit (table 3.1) in order to specify the statistical model - the inductive premises of interest - and test for its validity; see Spanos (2006a). Indeed, this validity is necessary when estimating the statistical CAPM in (3.2) because the reliability of all the inference results, including the estimated parameters $\boldsymbol{\theta}{:=}(\alpha,\beta,\sigma^2)$, as well as the R^2, relies heavily on the validity of the explicit probabilistic assumptions; see the simulation results in Appendix A. Above all, the reliability of statistical inference is inescapable for testing the validity of the CAPM theory vis-à-vis the data in question because the problem of unreliable inference does not allow the modeler to delineate between two different types of errors (see Spanos, 2006a): (a) *statistical inadequacy* arising when one or more of the probabilistic assumptions (table 3.1) imposed on the data (R_{it}, R_{ft}, R_{mt}) are *invalid*, and (b) *substantive inadequacy* arising when *precise* inference results reflect evidence against the CAPM theory. Indeed the latter presupposes *statistical adequacy*, and it's best to avoid validating or invalidating the CAPM theory, or any theory, based on unreliable statistical inference.

To address the aforementioned problems one needs to separate the substantive and the statistical information ab initio. This will enable the modeler to delineate and probe for different potential errors at different stages of modeling. Indeed, from a purely probabilistic concept a statistical model can be viewed as a parameterization of the observable stochastic process (Spanos, 2006c):

$$\{\mathbf{Z}_{it}{:=}(y_{it},\mathbf{X}_t),\ i{=}1,2,...,k,\ t{\in}\mathbb{N}\}\,, \tag{3.4}$$

giving rise to the observed data. In the case of the CAPM, there are three data series $\mathbf{Z}_{it}{:=}(R_{it}, R_{ft}, R_{mt})$ and one needs to view them as separate entities whose probabilistic structure gives rise to the parameterization of the statistical model. Hence, from this purely probabilistic perspective, the relevant time-series regression (explicit statistical model, viewed as a parameterization of the process underlying the data (see Fama, 1973; Spanos, 2006c)) is:

$$\begin{gathered}\mathcal{M}^*_{\boldsymbol{\theta}}(\mathbf{z};\boldsymbol{\theta}^*){:}\ R_{it}{=}\alpha_i+\beta_{1i}R_{mt}+\beta_{2i}R_{ft}+\varepsilon_{it},\ i{=}1,...,k,\ t{=}1,...,n,\\ \varepsilon_{it}\sim \text{NIID }(0,\sigma^2_{\varepsilon i}),\ \boldsymbol{\theta}^*{:=}(\alpha_i,\beta_{1i},\beta_{2i},\sigma^2_{\varepsilon i}),\end{gathered} \tag{3.5}$$

and the probabilistic assumptions imposed on the underlying observable process take the form of the testable Multivariate Normal/Homoskedastic LR model assumptions

[1]–[5] in table 3.2.

Ergo, the substantive model in (3.1) is a special case of the (explicit) statistical model in (3.5) in the sense that the nesting restrictions relating the substantive $[\boldsymbol{\varphi}{:=}(\beta_i, \sigma_{\varepsilon i}^2)]$ to the statistical $[\boldsymbol{\theta}^*{:=}(\alpha_i, \beta_{1i}, \beta_{2i}, \sigma_{\varepsilon i}^2)]$ parameters take the implicit form:

$$\mathbf{G}(\boldsymbol{\varphi}, \boldsymbol{\theta}^*){=}\mathbf{0}: \quad \alpha_i{=}0,\ \beta_{1i}{+}\beta_{2i}{=}1,\ i{=}1,2,...,k, \tag{3.6}$$

where the substantive parameters $\boldsymbol{\varphi}$ are said to be identified if there exists a unique solution of (3.6) for $\boldsymbol{\varphi}$ in terms of $\boldsymbol{\theta}^*$. In practice, there are more statistical than substantive parameters and probing for substantive adequacy requires one to test the over-identifying restrictions; see Spanos (1990a).

Table 3.2 - Multivariate Normal/Homoskedastic LR Model

Statistical GM:	$y_t{=}\alpha + \boldsymbol{\beta}^\top \mathbf{x}_t + u_t,\ t{\in}\mathbb{N}$
[1] Normality:	$D(y_t \vert \mathbf{X}_t{=}\mathbf{x}_t; \boldsymbol{\theta})$ is Normal,
[2] Linearity:	$E(y_t \vert \mathbf{X}_t{=}\mathbf{x}_t){=}\alpha + \boldsymbol{\beta}^\top \mathbf{x}_t$, is linear in $\mathbf{x}_t$,
[3] Homoskedasticity:	$Var(y_t \vert \mathbf{X}_t{=}\mathbf{x}_t){=}\sigma^2{>}0$, is free of $\mathbf{x}_t$,
[4] Independence:	$\{(y_t \vert \mathbf{X}_t{=}\mathbf{x}_t), t{\in}\mathbb{N}\}$ independent process,
[5] t-invariance:	$\boldsymbol{\theta}{:=}(\alpha, \boldsymbol{\beta}, \sigma^2)$ are constant over t.

$\alpha{=}(\mu_y{-}\boldsymbol{\beta}^\top \boldsymbol{\mu}_{\mathbf{x}}){\in}\mathbb{R},\ \boldsymbol{\beta}{=}(\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\sigma}_{21}){\in}\mathbb{R},\ \sigma^2{=}(\sigma_{11}{-}\boldsymbol{\sigma}_{21}^\top \boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\sigma}_{21}){\in}\mathbb{R}_+,$

$\mu_y{=}E(y_t),\ \boldsymbol{\mu}_{\mathbf{x}}{=}E(\mathbf{X}_t),\ \sigma_{11}{=}Var(y_t),\ \boldsymbol{\sigma}_{21}{=}Cov(y_t, \mathbf{X}_t),\ \boldsymbol{\Sigma}_{22}{=}Cov(\mathbf{X}_t).$

3.3 Data and Relevant Variables

Given that the primary aim of this chapter is to revisit the CAPM and the Fama-French three-factor model with a view to evaluate the validity of the probabilistic assumptions imposed on the Fama and French (1993) data, the sample is the one used in Fama and French (1993). This sample spans a 342 month period from July 1963 to December 1991. In chapter 5, a larger sample size will be used to test for the significance tests of the Fama and French (2015) factors, in the context of a statistically adequate model.

The data are downloaded from Kenneth French's online data library[2] where the reader can find a detailed description for the construction of the data, as well as clarifying details and information. A brief description of the data follows.

The size factor (SMB_t) and value factor (HML_t) are constructed using independent 2×3 sorts on size and book-to-market equity (B/M). The size breakpoint is the NYSE median market equity, whereas the B/M breakpoints are the respective 30^{th} and 70^{th} percentiles for NYSE stocks. The sample used comprise all the NYSE, AMEX, and NASDAQ stocks. The intersections of groups define six value-weighted portfolios formed on size and B/M. The SMB and HML factors are defined as differences between the average returns on the value-weighted portfolios. The market return (R_{mt}) is the return on the market value-weighted portfolio and the risk-free return (R_{ft}) is the one-month Treasury bill rate. Moreover, the 25 *Size-B/M* portfolios (R_{it}) are the average returns of value-weighted portfolios constructed from independent 5×5 sorts of stocks on size and B/M. The size and B/M quintile breakpoints use only NYSE stocks, but the sample used comprise all the NYSE, AMEX, and NASDAQ stocks. In summary, the Fama and French (1993) data consist of the following (a) dependent and (b) explanatory variables.

(a) Monthly returns for 25 stock portfolios formed on size and B/M:

Size\B/M	Low	2	3	4	High
Small	S/L_t	$S/2_t$	$S/3_t$	$S/4_t$	S/H_t
2	$2/L_t$	$2/2_t$	$2/3_t$	$2/4_t$	$2/H_t$
3	$3/L_t$	$3/2_t$	$3/3_t$	$3/4_t$	$3/H_t$
4	$4/L_t$	$4/2_t$	$4/3_t$	$4/4_t$	$4/H_t$
Big	B/L_t	$B/2_t$	$B/3_t$	$B/4_t$	B/H_t

[2]http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/.

(b) Factors in the CAPM and three-factor model:

$R_{mt}-R_{ft}$	Excess market return
R_{mt}	Market return
SMB_t	Size factor (Small Minus Big)
HML_t	Value factor (High Minus Low)
R_{ft}	Risk-free rate

In addition, the following (i)-(iii) additional variables are used with a view to capture the statistical regularities exhibited by the data.

(i) Gram-Schmidt orthonormal trend polynomials:

$$v_k=\frac{\widetilde{v}_k}{\|\widetilde{v}_k\|}, \ \ k=0,1,2,... \tag{3.7}$$

where $\{\widetilde{v}_0,\widetilde{v}_1,\widetilde{v}_2,...,\widetilde{v}_p\}$ is the orthogonal basis of $\{t^0,t,t^2,...,t^p\}$, for $t^0=(1,1,...,1)$, and $t=(1,2,...,n)$.

These polynomials are used to capture any mean or variance heterogeneity in the data because extending ordinary polynomials to orders higher than 4 gives rise to serious near-collinearity problems; see chapter 2 for details.

(ii) Monthly seasonal dummy variables, d_{st}, $s=1,2,...,12$, $t=1,2,...,n$, are used to capture any seasonal mean heterogeneity in the data.

$$S_t\left\{\begin{array}{lll}\delta_1 & \text{if} & t=\text{January}\\ \delta_2 & \text{if} & t=\text{February}\\ \vdots & & \vdots\\ \delta_{12} & \text{if} & t=\text{December}\end{array}\right. ,\quad d_{st}=\left\{\begin{array}{lll}1 & \text{if} & s=i\\ 0 & \text{if} & s\neq i\end{array}\right. ,\forall i=1,2,...,12, \tag{3.8}$$

where, $S_t=\sum_{i=1}^{s}\delta_i d_{it}$ denotes the deterministic seasonality.

(iii) Lags in both dependent and explanatory variables to capture any temporal dependence in the data.

3.4 Testing for Statistical Adequacy

The adequacy of a statistical model is assessed by thoroughly testing its probabilistic assumptions. In the case of the CAPM and its extensions, the implicit statistical model is the Multivariate Normal/ Homoskedastic LR model in table 3.2 which is defined in terms of assumptions [1]–[5]. This statistical model can be viewed as a parameterization of a Normal, Independent and Identically Distributed (NIID) process $\{\mathbf{Z}_t,\ t\in\mathbb{N}\}$ underlying data $\mathbf{Z}_0$.

Formally, the probabilistic reduction takes the form:

$$\begin{aligned} D(\mathbf{Z}_1, \mathbf{Z}_2, ...\mathbf{Z}_n; \boldsymbol{\vartheta}) &\overset{I}{=} \prod_{t=1}^{n} D_t(\mathbf{Z}_t; \boldsymbol{\theta}_{1t}) \\ &\overset{IID}{=} \prod_{t=1}^{n} D(\mathbf{Z}_t; \boldsymbol{\theta}_1) \\ &= \prod_{t=1}^{n} D(y_t|\mathbf{X}_t; \boldsymbol{\theta}_2) D(\mathbf{X}_t; \boldsymbol{\theta}_1^1), \end{aligned} \tag{3.9}$$

where I denotes the imposition of Independence, IID denotes the imposition of both Independence and Identically Distributed assumptions, and $\boldsymbol{\vartheta}$, $\boldsymbol{\theta}_1$, and $\boldsymbol{\theta}_2$ denote the parameters in the joint, marginal, and conditional distributions, respectively.

The imposition of Normality ensures that the marginal distribution $D(\mathbf{X}_t; \boldsymbol{\theta}_1^1)$ in (3.9) can be ignored, and subsequently the distribution underlying the statistical model is the conditional Normal $D(y_t|\mathbf{X}_t;\boldsymbol{\theta}_2)$ with the associated parameterization $\boldsymbol{\theta}_2{:=}(\alpha, \boldsymbol{\beta}, \sigma^2)$; see Engle et al. (1983). Thereby, the statistical model is specified in terms of a complete, internally consistent, and testable list of assumptions [1]–[5]; see Spanos (1986).

3.4.1 Informal Graphical Assessment

The first stage of probing for statistical adequacy is to relate various plots of the original data to model assumptions. In this case one can informally assess the appropriateness of the model assumptions by assessing the validity of the NIID reduction assumptions given in table 3.3.

The graphical displays of the t-plots, histograms, and p-p plots are presented in Appendix D. The t-plots of the data series indicate that the IID assumptions are likely to be invalid because one can detect both trends (heterogeneity) and irregular cycles (temporal dependence). These departures are likely to render the model assumptions [4] and [5] invalid. The histograms and p-p plots depicted indicate certain departures from Normality which are confirmed by the Shapiro-Wilk (W), Anderson-Darling (A^2), and D'Agostino-Pearson (K^2) tests; see Appendix C for a description of these tests and Appendix D for the results. This indicates that one should expect departures from model assumptions [1]–[3] as well.

Table 3.3 - Reduction vs. Model Assumptions

Reduction: $\{\mathbf{Z}_t,\ t\in\mathbb{N}\}$		Model: $\{(y_t\vert\mathbf{X}_t=\mathbf{x}_t),\ t\in\mathbb{N}\}$
N	$\longrightarrow$	[1]–[3]
I	$\longrightarrow$	[4]
ID	$\longrightarrow$	[5]

3.4.2 Joint Mis-Specification (M-S) Testing

Caution is advisable in taking the Normality M-S testing results at face value because the abovementioned tests invoke IID assumptions which are questionable in this case. Hence, in practice it is a better strategy to begin with M-S testing for the other model assumptions since departures from assumptions [2]–[5] call the results of Normality tests into question.

To evaluate the model assumptions [2]–[5] that justify the Multivariate Normal/ Homoskedastic LR model in table 3.2, the estimated residuals of the CAPM and the three-factor model are carefully examined. Joint M-S testing in the form of auxiliary regressions (F) is employed to test for the model assumptions of [2] Linearity, [3] Homoskedasticity, [4] Independence, and [5] t-invariance. The joint M-S testing based on auxiliary regressions has several distinct advantages over other procedures which are based on individual test statistics; see Spanos (2017).

The auxiliary regressions stemming from the first two conditional moments of $(y_t|\mathbf{X}_t{=}\mathbf{x}_t)$ take the following form:

$$\widehat{u}_{it}{=}\gamma_{1i}+\boldsymbol{\gamma}_{2i}^{\top}\mathbf{X}_t+\underbrace{\sum\nolimits_{j=1}^{m}\gamma_{3ji}v_{jt}}_{\overline{[5.1]}}+\underbrace{\sum\nolimits_{j=2}^{s}\gamma_{4ji}d_{jt}}_{\overline{[5.2]}}+\underbrace{\boldsymbol{\gamma}_{5i}^{\top}\boldsymbol{\psi}_t}_{\overline{[2]}}+\underbrace{\sum\nolimits_{j=1}^{p}\boldsymbol{\gamma}_{6ji}^{\top}\mathbf{Z}_{it-j}}_{\overline{[4]}}+\varepsilon_{1it}, \tag{3.10}$$

$$\widehat{u}_{it}^{2}{=}\gamma_{7i}+\underbrace{\sum\nolimits_{j=1}^{m}\gamma_{8ji}v_{jt}}_{\overline{[5.3]}}+\underbrace{\boldsymbol{\gamma}_{9i}^{\top}\mathbf{X}_t+\boldsymbol{\gamma}_{10i}^{\top}\boldsymbol{\psi}_t+\sum\nolimits_{j=1}^{p}\boldsymbol{\gamma}_{11ji}^{\top}\mathbf{Z}_{it-j}^{2}}_{\overline{[3]}+\overline{[4]}}+\varepsilon_{2it}, \tag{3.11}$$

where $\widehat{u}_{it}$ are the estimated residuals of portfolio i for period t; $\mathbf{X}_t{:=}(x_{1t},x_{2t},...,x_{lt})$ are the explanatory variables; v_{jt} denotes the terms of the Gram-Schmidt orthonormal polynomials of order $j{=}1,2,...,m$; $d_{jt}{:=}(d_{2t},d_{3t},...,d_{12t})$ are the monthly dummy variables for the months of February through December; $\boldsymbol{\psi}_t{:=}\{(x_{it}{\cdot}x_{jt}),i{\geq}j,i,j{=}1,2,...,l\}$ are the second-order Kolmogorov-Gabor polynomials; $\mathbf{Z}_{it}{:=}(y_{it},\mathbf{X}_t)$; and $(\varepsilon_{1it},\varepsilon_{2it})$ are the error terms; see Spanos (2010).

The M-S tests employed are F-type and are based on assessing the significance of the additional terms that represent generic departures from assumptions [2]–[5].

Table 3.4 - M-S Testing Hypotheses

[2] Linearity		H_0: $\boldsymbol{\gamma}_5{=}0$ *vs.* H_1: $\boldsymbol{\gamma}_5{\neq}0$,
[3] Homo/city:	static	H_0: $\boldsymbol{\gamma}_9{=}\boldsymbol{\gamma}_{10}{=}0$ *vs.* H_1: $\boldsymbol{\gamma}_9{\neq}\boldsymbol{\gamma}_{10}{\neq}0$,
	dynamic	H_0: $\boldsymbol{\gamma}_{11j}{=}0$ *vs.* H_1: $\boldsymbol{\gamma}_{11j}{\neq}0$, $j{=}1,...,p$,
[4] Independence		H_0: $\boldsymbol{\gamma}_{6j}{=}0$ *vs.* H_1: $\boldsymbol{\gamma}_{6j}{\neq}0$, $j{=}1,...,p$,
[5] t-invariance:		
[5.1] Mean Homogeneity		H_0: $\gamma_{3j}{=}0$ *vs.* H_1: $\gamma_{3j}{\neq}0$, $j{=}1,...,m$,
[5.2] Seasonal Homogeneity		H_0: $\gamma_{4j}{=}0$ *vs.* H_1: $\gamma_{4j}{\neq}0$, $j{=}2,...,s$,
[5.3] Variance Homogeneity		H_0: $\gamma_{8j}{=}0$ *vs.* H_1: $\gamma_{8j}{\neq}0$, $j{=}1,...,m$.

Example 1. *M-S testing for the CAPM model.*

Estimating the statistical model by OLS yields:

$$y_{it} = \underset{(0.632)}{0.714} + \underset{(0.051)}{1.411} x_{1t} - \underset{(1.055)}{2.199} x_{2t} + u_{it},\ \ R^2 = 0.702,\ \ s = 4.253,\ \ n = 342, \tag{3.12}$$

where $y_{it} = S/L_t$ is the return of portfolio i=(smallest size/lowest B/M) for period t, $x_{1t} = R_{mt}$ is the return on the value-weight market portfolio, and $x_{2t} = R_{ft}$ is the return on the risk-free asset.

The t-statistics and their corresponding p-values in square brackets are:

$$\tau_0 = \tfrac{0.714}{0.632} = 1.128[.260],\quad \tau_1 = \tfrac{1.411}{0.051} = 27.935[.000],\quad \tau_2 = -\tfrac{2.199}{1.055} = -2.083[.038]. \tag{3.13}$$

The auxiliary regressions take the following form (p-values in square brackets):

$$\begin{aligned}
\widehat{u}_{it} = {} & \underset{[.074]}{3.506} - \underset{[.563]}{0.027} x_{1t} + \underset{[.497]}{4.057} x_{2t} + \\
& \underbrace{- \underset{[.348]}{4.946} v_1 - \underset{[.072]}{8.311} v_2 + \underset{[.762]}{1.266} v_3 + \underset{[.121]}{6.169} v_4 + \underset{[.000]}{16.837} v_5}_{\overline{[5.1]}:=\textit{mean heterogeneity}} + \\
& \underbrace{- \underset{[.000]}{3.831} d_{2t} - \underset{[.002]}{3.215} d_{3t} - \underset{[.000]}{3.646} d_{4t} - \underset{[.000]}{3.797} d_{5t} - \underset{[.000]}{4.552} d_{6t} - \underset{[.000]}{4.045} d_{7t}}_{\overline{[5.2]}:=\textit{mean heterogeneity (seasonality)}} + \\
& \underbrace{- \underset{[.000]}{4.683} d_{8t} - \underset{[.011]}{2.638} d_{9t} - \underset{[.000]}{5.321} d_{10t} - \underset{[.000]}{4.393} d_{11t} - \underset{[.000]}{4.055} d_{12t}}_{\overline{[5.2]}:=\textit{mean heterogeneity (seasonality)}} + \\
& \underbrace{- \underset{[.093]}{0.009} x_{1t}^2 - \underset{[.630]}{1.699} x_{2t}^2}_{\overline{[2]}:=\textit{linearity}} \ \underbrace{+ \underset{[.036]}{0.115} y_{it-1} + \underset{[.682]}{0.037} x_{1t-1} - \underset{[.380]}{2.504} x_{2t-1}}_{\overline{[4]}:=\textit{temporal dependence}} + \varepsilon_{1it},
\end{aligned} \tag{3.14}$$

$$\widehat{u}_{it}^2 = -\underset{[.839]}{2.432} \ \underbrace{- \underset{[.003]}{105.908} v_1}_{\overline{[5.3]}:=\substack{\textit{variance} \\ \textit{heterogeneity}}} \ \underbrace{+ \underset{[.009]}{0.901} x_{1t} + \underset{[.127]}{56.122} x_{2t} + \underset{[.003]}{0.109} x_{1t}^2 - \underset{[.114]}{39.318} x_{2t}^2}_{\overline{[3]}+\overline{[4]}:=\textit{static and/or dynamic heteroskedasticity}} + \varepsilon_{2it}. \tag{3.15}$$

The M-S testing results for this example are presented in table 3.5; p-values in square brackets below .01 (†) are considered to indicate departures from assumptions [1]–[5].

Example 2. *M-S testing for the Fama-French three-factor model.*

Estimating the statistical model by OLS yields:

$$\begin{gathered} y_{it}=\underset{(0.286)}{0.161}+\underset{(0.026)}{1.032}x_{1t}-\underset{(0.478)}{1.019}x_{2t}+\underset{(0.039)}{1.405}x_{3t}-\underset{(0.044)}{0.285}x_{4t}+u_{it}, \\ R^2=0.940,\ \ s=1.920,\ \ n=342, \end{gathered} \tag{3.16}$$

where $y_{it}=S/L_t$ is the return of portfolio i=(smallest size/lowest B/M) for period t, $x_{1t}=R_{mt}$ is the return on the value-weight market portfolio, $x_{2t}=R_{ft}$ is the return on the risk-free asset, $x_{3t}=SMB_t$ is the size factor, and $x_{4t}=HML_t$ is the value factor.

The t-statistics and their corresponding p-values in square brackets are:

$$\begin{gathered} \tau_0=\tfrac{0.161}{0.286}=0.563[.574],\quad \tau_1=\tfrac{1.032}{0.026}=39.692[.000],\quad \tau_2=-\tfrac{1.019}{0.478}=-2.132[.034], \\ \tau_3=\tfrac{1.405}{0.039}=36.026[.000],\quad \tau_4=-\tfrac{0.285}{0.044}=-6.477[.000]. \end{gathered} \tag{3.17}$$

The auxiliary regressions take the following form (p-values in square brackets):

$$\begin{aligned} \widehat{u}_{it}=&\underset{[.417]}{0.752}-\underset{[.071]}{0.144}x_{1t}+\underset{[.370]}{2.547}x_{2t}+\underset{[.005]}{0.346}x_{3t}+\underset{[.000]}{0.456}x_{4t}+ \\ &\underbrace{-\underset{[.262]}{2.712}v_1-\underset{[.534]}{1.292}v_2+\underset{[.307]}{2.005}v_3+\underset{[.011]}{4.716}v_4+}_{\overline{[5.1]}:=mean\ heterogeneity} \\ &\underbrace{-\underset{[.000]}{1.933}d_{2t}-\underset{[.003]}{1.498}d_{3t}-\underset{[.021]}{1.184}d_{4t}-\underset{[.044]}{1.039}d_{5t}-\underset{[.000]}{1.814}d_{6t}-\underset{[.000]}{1.982}d_{7t}+}_{\overline{[5.2]}:=mean\ heterogeneity\ (seasonality)} \\ &\underbrace{-\underset{[.002]}{1.580}d_{8t}-\underset{[.027]}{1.132}d_{9t}-\underset{[.004]}{1.501}d_{10t}-\underset{[.000]}{2.411}d_{11t}-\underset{[.003]}{1.511}d_{12t}+}_{\overline{[5.2]}:=mean\ heterogeneity\ (seasonality)} \\ &\underbrace{-\underset{[.566]}{0.002}x_{1t}^2+\underset{[.040]}{0.271}x_{1t}x_{2t}-\underset{[.585]}{0.005}x_{1t}x_{3t}-\underset{[.461]}{0.007}x_{1t}x_{4t}-\underset{[.687]}{0.677}x_{2t}^2+}_{\overline{[2]}:=linearity} \\ &\underbrace{-\underset{[.001]}{0.685}x_{2t}x_{3t}-\underset{[.000]}{0.711}x_{2t}x_{4t}+\underset{[.454]}{0.007}x_{3t}^2-\underset{[.006]}{0.034}x_{3t}x_{4t}+\underset{[.141]}{0.016}x_{4t}^2+}_{\overline{[2]}:=linearity} \\ &\underbrace{+\underset{[.348]}{0.051}y_{it-1}-\underset{[.097]}{0.104}x_{1t-1}-\underset{[.483]}{0.953}x_{2t-1}+\underset{[.482]}{0.059}x_{3t-1}-\underset{[.074]}{0.087}x_{4t-1}}_{\overline{[4]}:=temporal\ dependence}+\varepsilon_{2it}, \end{aligned} \tag{3.18}$$

$$
\begin{aligned}
\widehat{u}_{it}^2 = &\underset{[.059]}{5.784} + \underbrace{\underset{[.012]}{24.252}v_1 - \underset{[.815]}{1.876}v_2 - \underset{[.034]}{16.290}v_3 + \underset{[.523]}{4.206}v_4}_{\overline{[5.3]}:=variance\ heterogeneity} + \\
&+\underbrace{\underset{[.001]}{26.427}v_5 + \underset{[.283]}{7.363}v_6 - \underset{[.018]}{18.956}v_7 - \underset{[.031]}{15.626}v_8 + \underset{[.046]}{13.731}v_9 + \underset{[.049]}{13.822}v_{10}}_{\overline{[5.3]}:=variance\ heterogeneity} + \\
&+\underbrace{\underset{[.866]}{0.015}x_{1t} - \underset{[.584]}{4.858}x_{2t} + \underset{[.982]}{0.003}x_{3t} - \underset{[.015]}{0.356}x_{4t} - \underset{[.931]}{0.001}x_{1t}^2 + \underset{[.842]}{1.201}x_{2t}^2}_{\overline{[3]}+\overline{[4]}:=static\ and/or\ dynamic\ heteroskedasticity} + \\
&+\underbrace{\underset{[.436]}{0.021}x_{3t}^2 + \underset{[.000]}{0.127}x_{4t}^2 + \underset{[.023]}{0.015}y_{it-1}^2 - \underset{[.191]}{0.019}x_{1t-1}^2 - \underset{[.122]}{5.274}x_{2t-1}^2}_{\overline{[3]}+\overline{[4]}:=static\ and/or\ dynamic\ heteroskedasticity} + \qquad (3.19) \\
&-\underbrace{\underset{[.244]}{0.043}x_{3t-1}^2 - \underset{[.533]}{0.020}x_{4t-1}^2 + \underset{[.102]}{0.011}y_{it-2}^2 + \underset{[.573]}{0.008}x_{1t-2}^2 + \underset{[.050]}{7.011}x_{2t-2}^2}_{\overline{[3]}+\overline{[4]}:=static\ and/or\ dynamic\ heteroskedasticity} + \\
&-\underbrace{\underset{[.140]}{0.054}x_{3t-2}^2 + \underset{[.041]}{0.066}x_{4t-2}^2 + \underset{[.410]}{0.005}y_{it-3}^2 + \underset{[.514]}{0.009}x_{1t-3}^2}_{\overline{[3]}+\overline{[4]}:=static\ and/or\ dynamic\ heteroskedasticity} + \\
&-\underbrace{\underset{[.007]}{7.630}x_{2t-3}^2 + \underset{[.928]}{0.003}x_{3t-3}^2 - \underset{[.721]}{0.011}x_{4t-3}^2}_{\overline{[3]}+\overline{[4]}:=static\ and/or\ dynamic\ heteroskedasticity} + \varepsilon_{2it}.
\end{aligned}
$$

The M-S testing results for the two examples are presented in table 3.5; p-values in square brackets below .01 (†) are considered to indicate departures from assumptions [1]–[5]. The results indicate that the Multivariate Normal/ Homoskedastic LR model suffers from serious statistical misspecifications. The M-S testing results of the CAPM (example 1) suggest that all the statistical models assumptions except [2] Linearity are invalid for the data in question. On the other hand, the M-S testing results of the Fama-French three-factor model (example 2) indicate similar departures from the model assumptions. There are some differences like the statistically significant improvements of the [5.1] Mean homogeneity and [5.3] Variance homogeneity assumptions, but they are not substantive enough to assure that this model improves the statistical adequacy of the original CAPM. Indeed, the [2] Linearity assumption appear to suggest worsening of the CAPM inadequacy. Also, it is worth mentioning that the contradicting M-S testing results obtained from the three Normality tests brings out the usefulness of combining both parametric and non-parametric tests to enhance the reliability of the diagnosis relying on M-S testing; see Spanos (2010) and Razali and Wah (2011).

Table 3.5 - M-S Testing Results of S/L_t portfolio

Assumption	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W=0.991[.026]	W=0.991[.037]
Anderson-Darling	A^2=1.088[.007]†	A^2=0.413[.336]
D'Agostino-Pearson	K^2=8.999[.011]	K^2=13.536[.001]†
[2] Linearity	F(2,317)=1.497[.225]	F(10,306)=5.017[.000]†
[3] Homoskedasticity	F(4,336)=4.923[.001]†	F(8,328)=3.221[.001]†
[4] Independence	F(3,317)=7.740[.000]†	F(5,306)=3.874[.002]†
[5] t-invariance:		
[5.1] Mean Homogeneity	F(5,317)=4.480[.001]†	F(4,306)=2.432[.048]
[5.2] Seasonal Homogeneity	F(11,317)=3.525[.000]†	F(11,306)=2.946[.001]†
[5.3] Variance Homogeneity	F(1,336)=8.814[.003]†	F(5,328)=1.514[.185]

3.4.3 M-S Testing Results

The M-S testing results for the Multivariate Normal/Homoskedastic LR models are presented in Appendix D. Table 3.6 includes a summary of these results, organized in groups of small, medium, and big size portfolios. The values reported in the table indicate the number of portfolios violating the statistical model assumptions [1]–[5].

The M-S testing results of the CAPM suggest departures from all the model assumptions except the assumption of [2] Linearity. The assumptions of [1] Normality, [3] Homoskedasticity, and [5.3] Variance homogeneity are severely violated for almost all the portfolios. Besides, the departures from [4] Independence, [5.1] Mean homogeneity, and [5.2] Seasonal homogeneity assumptions appear to be more severe for the smaller size portfolios compared to the medium and bigger size portfolios. The heterogeneity distinction related to assumptions [5.1] and [5.2] arises for two reasons. First, the smaller size portfolios include on average a larger number of stocks compared to the medium and bigger size portfolios. Thus, the averaging of these large number of stocks into portfolios generated (more) complicate mean heterogeneous series. Second, the average January returns for the smaller size portfolios are larger relative to the medium and bigger size portfolios; see Keim (1983) and Reinganum (1983). However, it is worth to mention that the not as strong but significant January seasonality for the medium and bigger size portfolios appears to exist primarily in the highest B/M portfolios. Moreover, the dependence distinction associated with assumption [4] is possibly arising because the stocks included in the smaller size portfolios are traded on average less frequently compared to the stocks in the medium and bigger size portfolios.

The M-S testing results of the Fama-French three-factor model indicate that the inclusion of the size and value factors improve significantly the results for assumptions [4]–[5] and marginally the results for assumptions [1] and [3]. Nonetheless, the latter improvements are made on the expense of assumption [2]. In summary, the joint M-S testing using auxiliary regressions indicate clearly that the Multivariate Normal/Homoskedastic LR models suffer from serious statistical misspecifications. The discrepancies from the statistical model assumptions [1]–[5] render the estimated CAPM and Fama-French three-factor model statistically misspecified for the particular data in hand. On the basis of their statistical inadequacy, all the statistical inference results, such as the intercept, the R^2, as well as the significant tests of the added factors, are of questionable reliability; see the simulation results in Appendix A.

Table 3.6 - Summary of M-S Testing Results for Normal/Homoskedastic LR Models		
Assumption	CAPM	three-factor
Smaller Size (10 Portfolios)	**out of 10**	**out of 10**
[1] Normality	10	4
[2] Linearity	0	3
[3] Homoskedasticity	10	5
[4] Independence	8	3
[5] t-invariance:		
[5.1] Mean Homogeneity	7	0
[5.2] Seasonal Homogeneity	9	3
[5.3] Variance Homogeneity	10	4
Medium Size (5 Portfolios)	**out of 5**	**out of 5**
[1] Normality	5	2
[2] Linearity	0	1
[3] Homoskedasticity	4	4
[4] Independence	0	1
[5] t-invariance:		
[5.1] Mean Homogeneity	0	0
[5.2] Seasonal Homogeneity	2	0
[5.3] Variance Homogeneity	4	4
Bigger Size (10 Portfolios)	**out of 10**	**out of 10**
[1] Normality	7	5
[2] Linearity	0	5
[3] Homoskedasticity	10	9
[4] Independence	2	1
[5] t-invariance:		
[5.1] Mean Homogeneity	2	2
[5.2] Seasonal Homogeneity	3	1
[5.3] Variance Homogeneity	8	3
NOTE: the values indicate the number of portfolios violating the model assumptions [1]–[5].		

3.5 Summary and Conclusions

Using the distinction between a substantive and a statistical model (often implicit) the chapter brings out the difference between substantive and statistical assumptions, where the latter refer to the probabilistic assumptions imposed on the data, implicitly or explicitly. The latter distinction enables one to separate the statistical from substantive assumptions, such as 'no relevant variables have been omitted' from the estimated model. This perspective is used to revisit the CAPM and the Fama-French three-factor model with a view to evaluate the validity of the probabilistic assumptions imposed (directly or indirectly) on the Fama and French (1993) data.

As it is shown, the M-S testing results indicate major departures from the statistical model assumptions of Normality, homoskedasticity, independence, and t-invariance, while the only assumption that is clearly satisfied is linearity. Taken together, these results call into question, more generally, the statistical adequacy of asset pricing models which are estimable with a given set of data, as well as the significance tests of empirically motivated pricing factors. As it is argued, before such models can be substantively assessed in an empirically reliable way, one needs to secure their statistical adequacy first, with a view to account for all the statistical systematic information in the data. This is because any form of statistical misspecification is likely to induce large discrepancies between the actual and nominal error probabilities; see simulation results in Appendix A.

In light of the above M-S testing results, chapter 4 will proceed to respecify a statistical model that is rich enough to account for all the statistical systematic information in the data. The respecified statistically adequate model will be used in chapter 5 to test for the significance of the Fama-French factors. As it will be argued, the only way of identifying potentially relevant risk factors in asset pricing tests is, at the outset, to account for all the statistical systematic information in the data.

Appendix C: Normality Tests

This Appendix provides a brief description of the Shapiro-Wilk, Anderson-Darling, and D'Agostino-Pearson tests for Normal distribution.

Shapiro-Wilk. Based on the assumption that $\{z_t,\ t\in\mathbb{N}\}$ is an IID process, the Shapiro-Wilk test statistic for Normality is defined as (Shapiro and Wilk, 1965):

$$W = \left(\sum_{t=1}^{n} \alpha_t z_{(t)}\right)^2 \Big/ \sum_{t=1}^{n} (z_t - \overline{z})^2, \tag{C.1}$$

where $z_{(t)}$ is the t-th order statistic, and $\overline{z}$ denotes the sample mean.

The constants α_t, for $t=1,2,...,n$ are given by:

$$(\alpha_1, \alpha_2, ..., \alpha_n) = \frac{\mathbf{m}^\top \mathbf{V}^{-1}}{\left(\mathbf{m}^\top \mathbf{V}^{-1} \mathbf{V}^{-1} \mathbf{m}\right)^{1/2}}, \tag{C.2}$$

where $\mathbf{m}^\top = (m_1, m_2, ..., m_n)^\top$ denotes the vector of expected values of standard normal order statistics, and $\mathbf{V}$ is the corresponding $(n\times n)$ covariance matrix of these order statistics.

Anderson-Darling. Based on the assumption that $\{z_t,\ t\in\mathbb{N}\}$ is an IID process, the Anderson-Darling test statistic for Normality is defined as (Anderson and Darling, 1954):

$$A^2 = -n - \frac{1}{n}\sum_{t=1}^{n} (2t-1)\left[\ln F(z_{(t)}) + \ln(1 - F(z_{(n-t+1)})\right], \tag{C.3}$$

where $z_{(t)}$ is the t-th order statistic, and $F(.)$ is the cumulative distribution of the Normal distribution.

D'Agostino-Pearson. Based on the assumption that $\{z_t,\ t\in\mathbb{N}\}$ is an IID process, the D'Agostino-Pearson test statistic for Normality is defined as (D'Agostino and Pearson, 1973):

$$K^2 = Z^2(\sqrt{b_1}) + Z^2(b_2) \sim \chi^2(2), \tag{C.4}$$

where $Z(\sqrt{b_1})$ and $Z(b_2)$ are the normal approximations of the estimated third (skewness) and fourth (kurtosis) standardized moments.

The null hypothesis of the tests above is H_0: *Normality* versus the alternative H_1: *non-Normality.* Thus, if the p-value is lower than the chosen significance level, there is evidence that the $\{z_t,\ t\in\mathbb{N}\}$ IID process tested is not normally distributed. It is important to note that for a large sample size, one has to choose much smaller thresholds for significance when using the p-values; see Lehmann and Romano (2006). With a sample size of n=342 a threshold of .01 is more appropriate than the most commonly used threshold of .05.

Appendix D: Graphical Analysis, Regression, and M-S Testing Results

This Appendix presents the graphical analysis of both the (a) dependent variables - monthly returns for 25 stock portfolios formed on size and B/M, and (b) explanatory variables - factors included in the CAPM and Fama-French three-factor model. The graphical representations of the t-plots, histograms, and p-p plots are presented in figures D.1–D.30. The sample is from July 1963 to December 1991, 342 months.

In summary, the t-plots of the data series indicate that the IID assumptions are likely to be invalid because one can detect both trends (heterogeneity) and irregular cycles (temporal dependence). These departures are likely to render the assumptions of [4] Independence and [5] t-invariance invalid. The histograms and p-p plots clearly indicate certain departures from [1] Normality. The latter also suggests the possibility of certain departures from the assumptions of [2] Linearity and [3] Homoskedasticity. This is because Normality is a Linear/Homoskedastic distribution.

In addition, this Appendix presents the regression results of the explicit CAPM and the Fama-French three-factor model estimated by the Normal/ Homoskedastic LR models. The estimated models are (July 1963 to December 1991, 342 months):

$$R_{it}=\alpha_i+\beta_{1i}R_{mt}+\beta_{2i}R_{ft}+u_{it}, \tag{D.1}$$

$$R_{it}=\alpha_i+\beta_{1i}R_{mt}+\beta_{2i}R_{ft}+s_iSMB_t+h_iHML_t+u_{it}, \tag{D.2}$$

where R_{it} is the return of portfolio i for period t; R_{mt} is the return of the value-weighted market portfolio; R_{ft} is the risk-free return; and SMB_t, HML_t, denote the size and value factors, respectively.

The regression results of the 25 *Size-B/M*, for the estimated models in (D.1)-(D.2) are presented in tables D.1.1–D.1.25. These results include coefficient estimates for each corresponding term in (D.1)-(D.2), standard errors and p-values of the coefficients in round and square brackets, respectively, number of observations, coefficient of determination (R^2), standard error, test statistic for the F-test on the regression model and its p-value in square bracket.

Moreover, this Appendix presents the M-S testing results of the explicit CAPM and the Fama-French three-factor model in (D.1)-(D.2). To evaluate the model assumptions [1]–[5] that justify the Normal/Homoskedastic LR model, the estimated residuals of the models in (D.1)-(D.2) are carefully examined. The Shapiro-Wilk (W), Anderson-Darling (A^2), and D'Agostino-Pearson (K^2) tests in Appendix C are employed to test for [1] Normality, and the auxiliary regressions (F) below are employed to test for [2] Linearity, [3] Homoskedasticity, [4] Independence, and [5] t-invariance.

$$\widehat{u}_{it}=\gamma_{1i}+\boldsymbol{\gamma}_{2i}^{\top}\mathbf{X}_t+\underbrace{\sum\nolimits_{j=1}^{m}\gamma_{3ji}v_{jt}}_{\overline{[5.1]}}+\underbrace{\sum\nolimits_{j=2}^{s}\gamma_{4ji}d_{jt}}_{\overline{[5.2]}}+\underbrace{\boldsymbol{\gamma}_{5i}^{\top}\boldsymbol{\psi}_t}_{\overline{[2]}}+\underbrace{\sum\nolimits_{j=1}^{p}\boldsymbol{\gamma}_{6ji}^{\top}\mathbf{Z}_{it-j}}_{\overline{[4]}}+\varepsilon_{1it}, \tag{D.3}$$

$$\widehat{u}_{it}^2=\gamma_{7i}+\underbrace{\sum\nolimits_{j=1}^{m}\gamma_{8ji}v_{jt}}_{\overline{[5.3]}}+\underbrace{\boldsymbol{\gamma}_{9i}^{\top}\mathbf{X}_t+\boldsymbol{\gamma}_{10i}^{\top}\boldsymbol{\psi}_t+\sum\nolimits_{j=1}^{p}\boldsymbol{\gamma}_{11ji}^{\top}\mathbf{Z}_{it-j}^2}_{\overline{[3]}+\overline{[4]}}+\varepsilon_{2it}, \tag{D.4}$$

where $\widehat{u}_{it}$ are the estimated residuals in (D.1)-(D.2) of portfolio i for period t; $\mathbf{X}_t{:=}(x_{1t},...,x_{lt})$ are the explanatory variables in (D.1)-(D.2); v_{jt} denotes the terms of the Gram-Schmidt orthonormal polynomials of order $j{=}1,2,...,m$; $d_{jt}{:=}(d_{2t},...,d_{12t})$ are the monthly dummy variables for the months of February through December; $\boldsymbol{\psi}_t{:=}\{(x_{it}{\cdot}x_{jt}), i{\geq}j, i,j{=}1,2,...,l\}$ are the second-order Kolmogorov-Gabor polynomials; and $\mathbf{Z}_{it}{:=}(y_{it},\mathbf{X}_t)$.

The M-S testing results of the 25 *Size-B/M*, for the Normal/ Homoskedastic LR models are presented in tables D.2.1–D.2.25; p-values in square brackets below .01 are considered to indicate departures from assumptions [1]–[5].

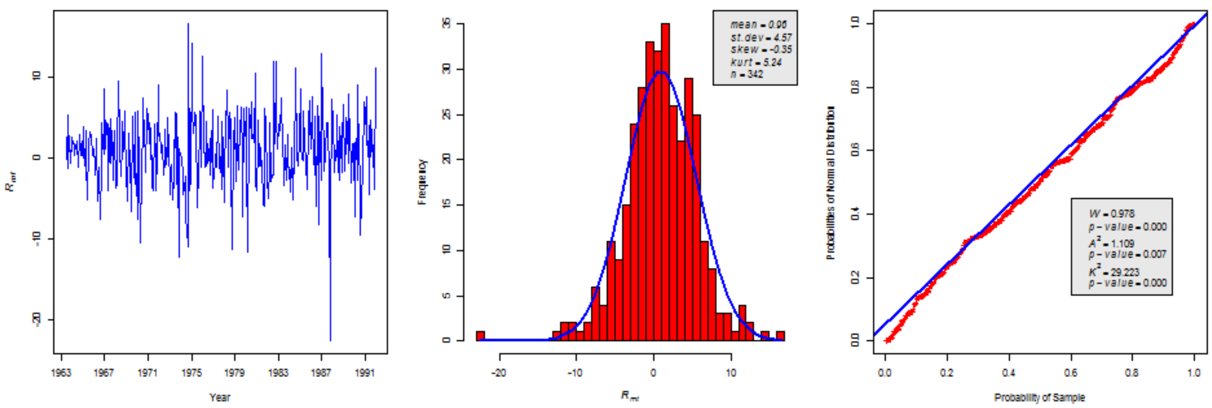

Figure D.1: t-plot, histogram, and p-p plot of market return R_{mt}

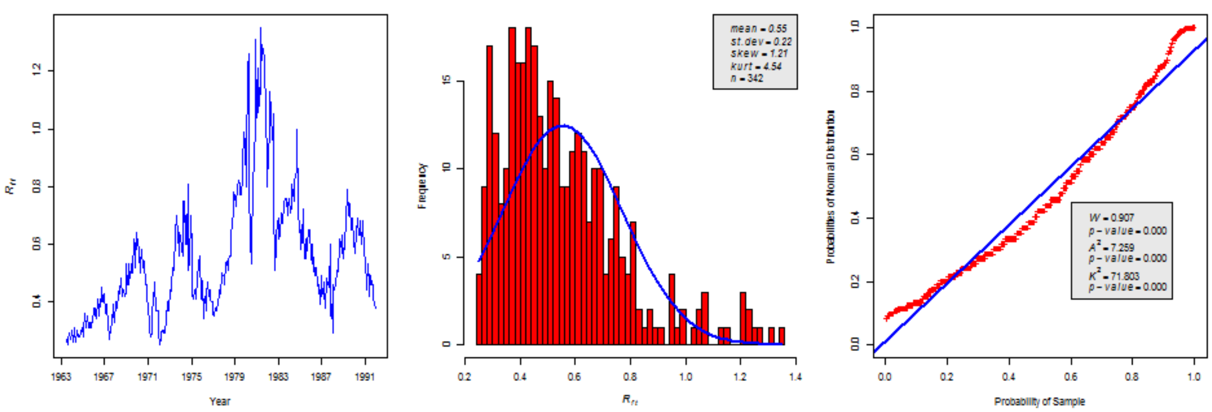

Figure D.2: t-plot, histogram, and p-p plot of risk-free return R_{ft}

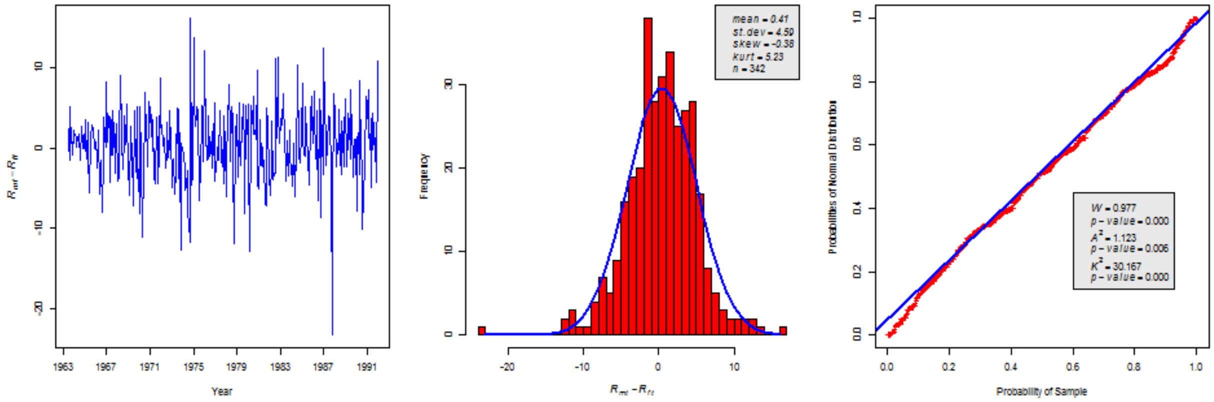

Figure D.3: t-plot, histogram, and p-p plot of market excess return $R_{mt} - R_{ft}$

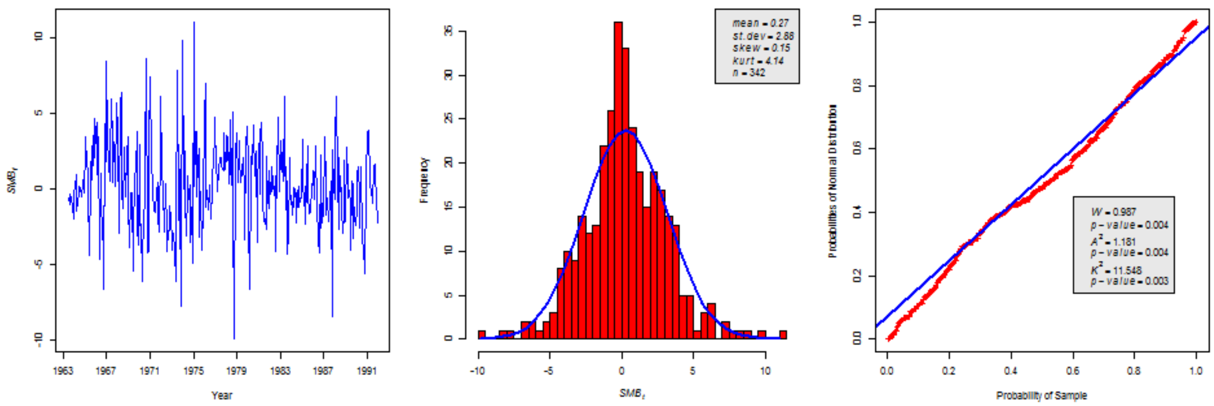

Figure D.4: t-plot, histogram, and p-p plot of size factor SMB_t

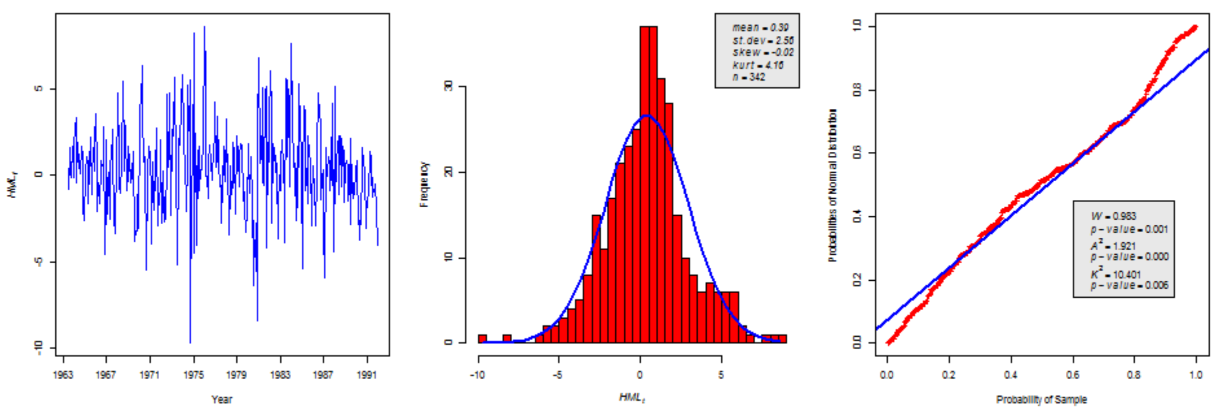

Figure D.5: t-plot, histogram, and p-p plot of value factor HML_t

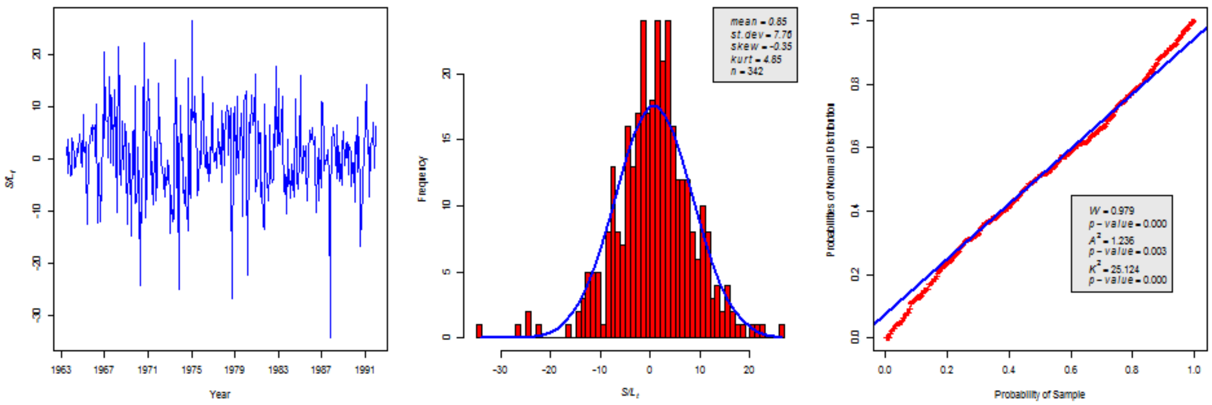

Figure D.6: t-plot, histogram, and p-p plot of S/L_t portfolio

Table D.1.1 - Regression Results (S/L_t)

y_t=S/L_t	CAPM	three-factor
R_{mt}	1.411[.000] (0.051)	1.032[.000] (0.026)
SMB_t		1.405[.000] (0.039)
HML_t		−0.285[.000] (0.044)
R_{ft}	−2.199[.038] (1.055)	−1.019[.034] (0.478)
Constant	0.714[.260] (0.632)	0.161[.574] (0.286)
Observations	342	342
R^2	0.702	0.940
Std. Error	4.253	1.920
F-Statistic	398.770[.000]	1309.455[.000]

Table D.2.1 - M-S Testing Results of S/L_t portfolio

$\widehat{u}_t$=$\widehat{u}_t^{S/L}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W=0.991[.027]	W= 0.991[.037]
Anderson-Darling	A^2=1.087[.007]	A^2= 0.398[.366]
D'Agostino-Pearson	K^2=9.011[.011]	K^2=13.749[.001]
[2] Linearity	F(2,317)=1.497[.225]	F(10,306)= 5.017[.000]
[3] Homoskedasticity	F(4,336)=4.923[.001]	F(23,305)= 3.221[.000]
[4] Independence	F(3,317)=7.740[.000]	F(5,306)= 3.874[.002]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(5,317)=4.480[.001]	F(4,306)= 2.432[.048]
[5.2] Seasonal Homogeneity	F(11,317)=3.525[.000]	F(11,306)= 2.946[.001]
[5.3] Variance Homogeneity	F(1,336)=8.814[.003]	F(10,305)= 2.766[.003]

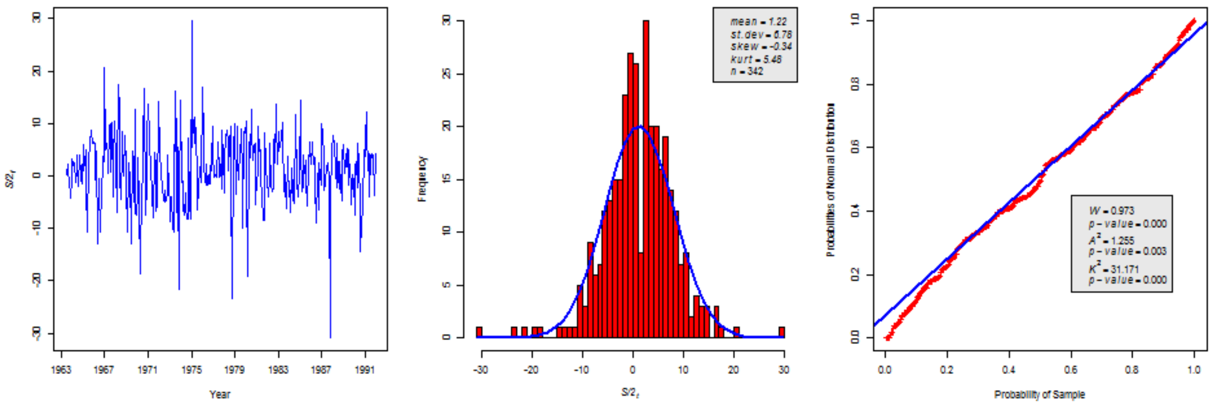

Figure D.7: t-plot, histogram, and p-p plot of $S/2_t$ portfolio

Table D.1.2 - Regression Results ($S/2_t$)

y_t=$S/2_t$	CAPM	three-factor
R_{mt}	1.241[.000] (0.044)	0.964[.000] (0.019)
SMB_t		1.275[.000] (0.028)
HML_t		0.077[.017] (0.032)
R_{ft}	−0.794[.389] (0.920)	0.064[.855] (0.350)
Constant	0.467[.398] (0.551)	−0.113[.589] (0.209)
Observations	342	342
R^2	0.703	0.958
Std. Error	3.708	1.404
F-Statistic	400.895[.000]	1906.822[.000]

Table D.2.2 - M-S Testing Results of $S/2_t$ portfolio

$\widehat{u}_t$=$\widehat{u}_t^{S/2}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.986[.002]	W=0.996[.543]
Anderson-Darling	A^2= 1.275[.003]	A^2=0.381[.401]
D'Agostino-Pearson	K^2=13.900[.001]	K^2=1.216[.544]
[2] Linearity	F(2,317)= 2.505[.083]	F(4,315)=2.287[.060]
[3] Homoskedasticity	F(10,328)= 7.898[.000]	F(4,335)=3.172[.014]
[4] Independence	F(3,317)= 5.962[.001]	F(5,315)=3.653[.003]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(5,317)= 3.201[.008]	F(1,315)=0.879[.349]
[5.2] Seasonal Homogeneity	F(11,317)= 4.821[.000]	F(11,315)=1.670[.079]
[5.3] Variance Homogeneity	F(1,328)=11.081[.001]	F(2,335)=4.506[.012]

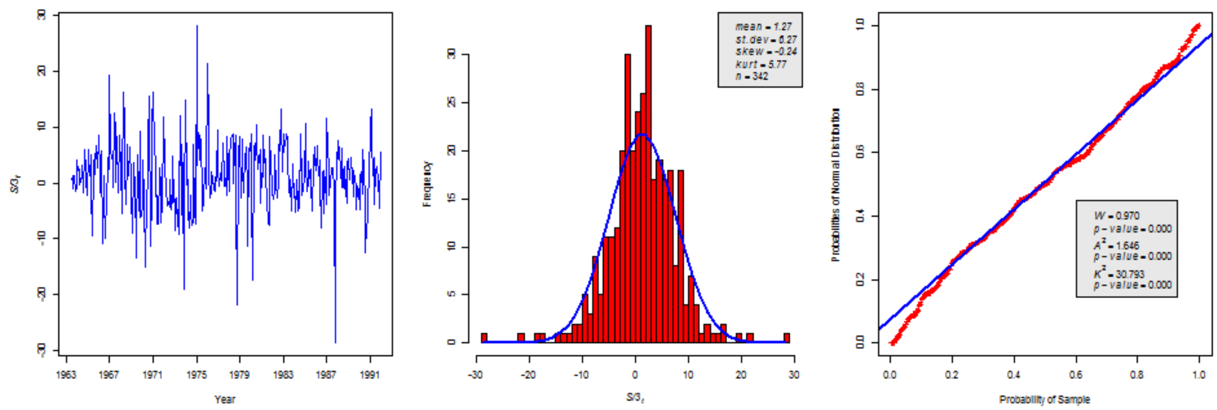

Figure D.8: t-plot, histogram, and p-p plot of $S/3_t$ portfolio

Table D.1.3 - Regression Results ($S/3_t$)

$y_t{=}S/3_t$	CAPM	three-factor
R_{mt}	$\underset{(0.040)}{1.151}$[.000]	$\underset{(0.016)}{0.938}$[.000]
SMB_t		$\underset{(0.024)}{1.159}$[.000]
HML_t		$\underset{(0.027)}{0.262}$[.000]
R_{ft}	$\underset{(0.846)}{-0.522}$[.538]	$\underset{(0.293)}{0.136}$[.643]
Constant	$\underset{(0.507)}{0.454}$[.371]	$\underset{(0.175)}{-0.118}$[.500]
Observations	342	342
R^2	0.706	0.965
Std. Error	3.408	1.175
F-Statistic	407.497[.000]	2343.304[.000]

Table D.2.3 - M-S Testing Results of $S/3_t$ portfolio

$\widehat{u}_t{=}\widehat{u}_t^{S/3}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	$W{=}$ 0.974[.000]	$W{=}$0.994[.177]
Anderson-Darling	$A^2{=}$ 2.304[.000]	$A^2{=}$0.403[.354]
D'Agostino-Pearson	$K^2{=}$25.856[.000]	$K^2{=}$4.132[.127]
[2] Linearity	$F(2,317){=}$ 1.957[.143]	$F(4,315){=}$1.988[.096]
[3] Homoskedasticity	$F(13,324){=}$ 5.987[.000]	$F(15,315){=}$0.806[.671]
[4] Independence	$F(3,317){=}$ 4.946[.002]	$F(5,315){=}$0.879[.495]
[5] t-invariance:		
[5.1] Mean Homogeneity	$F(5,317){=}$ 3.101[.010]	$F(1,315){=}$3.810[.052]
[5.2] Seasonal Homogeneity	$F(11,317){=}$ 6.360[.000]	$F(11,315){=}$2.007[.027]
[5.3] Variance Homogeneity	$F(1,324){=}$11.368[.001]	$F(8,315){=}$2.647[.008]

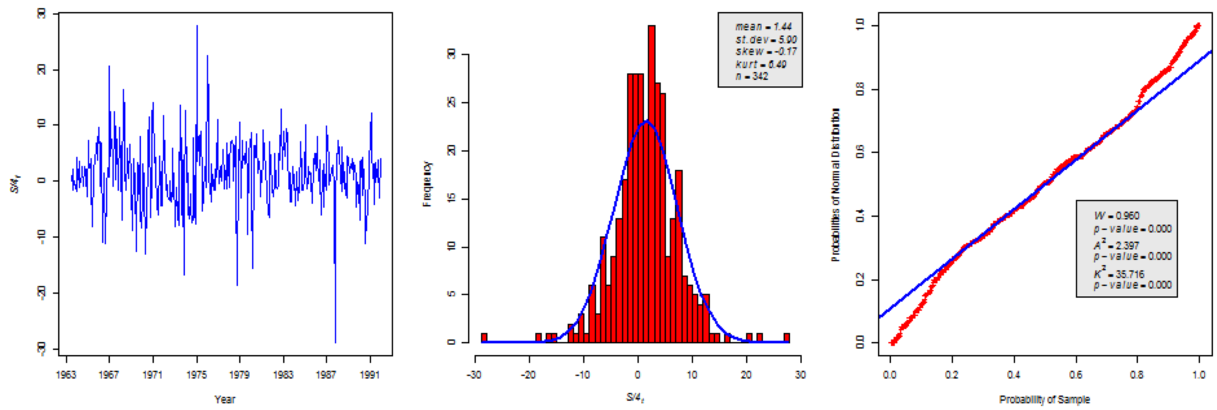

Figure D.9: t-plot, histogram, and p-p plot of $S/4_t$ portfolio

Table D.1.4 - Regression Results ($S/4_t$)		
y_t=$S/4_t$	CAPM	three-factor
R_{mt}	$\underset{(0.040)}{1.065}[.000]$	$\underset{(0.015)}{0.890}[.000]$
SMB_t		$\underset{(0.023)}{1.103}[.000]$
HML_t		$\underset{(0.025)}{0.384}[.000]$
R_{ft}	$\underset{(0.826)}{-0.546}[.509]$	$\underset{(0.278)}{-0.005}[.985]$
Constant	$\underset{(0.495)}{0.720}[.147]$	$\underset{(0.167)}{0.145}[.386]$
Observations	342	342
R^2	0.683	0.965
Std. Error	3.329	1.118
F-Statistic	365.991[.000]	2290.371[.000]

Table D.2.4 - M-S Testing Results of $S/4_t$ **portfolio**		
$\widehat{u}_t$=$\widehat{u}_t^{S/4}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.968[.000]	W=0.992[.073]
Anderson-Darling	A^2= 3.177[.000]	A^2=0.501[.207]
D'Agostino-Pearson	K^2=29.424[.000]	K^2=8.005[.018]
[2] Linearity	F(2,317)= 2.978[.052]	F(4,315)=3.011[.018]
[3] Homoskedasticity	F(10,328)=10.042[.000]	F(14,326)=1.706[.053]
[4] Independence	F(3,317)= 5.891[.001]	F(5,315)=1.482[.195]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(5,317)= 3.263[.007]	F(1,315)=1.880[.171]
[5.2] Seasonal Homogeneity	F(11,317)= 6.794[.000]	F(11,315)=3.001[.001]
[5.3] Variance Homogeneity	F(1,328)= 9.540[.002]	F(1,326)=0.237[.627]

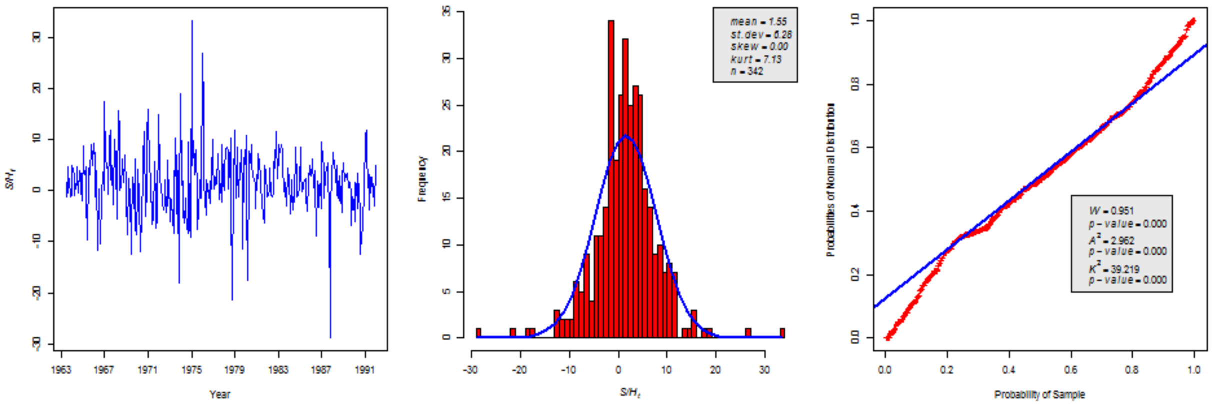

Figure D.10: t-plot, histogram, and p-p plot of S/H_t portfolio

Table D.1.5 - Regression Results (S/H_t)		
$y_t{=}S/H_t$	CAPM	three-factor
R_{mt}	$\underset{(0.045)}{1.095}[.000]$	$\underset{(0.016)}{0.948}[.000]$
SMB_t		$\underset{(0.024)}{1.186}[.000]$
HML_t		$\underset{(0.027)}{0.614}[.000]$
R_{ft}	$\underset{(0.933)}{-1.247}[.182]$	$\underset{(0.297)}{-0.793}[.008]$
Constant	$\underset{(0.559)}{1.189}[.034]$	$\underset{(0.178)}{0.523}[.004]$
Observations	342	342
R^2	0.643	0.964
Std. Error	3.759	1.194
F-Statistic	305.900[.000]	2270.497[.000]

Table D.2.5 - M-S Testing Results of S/H_t **portfolio**		
$\widehat{u}_t{=}\widehat{u}_t^{S/H}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	$W{=}\ 0.950[.000]$	$W{=}0.990[.025]$
Anderson-Darling	$A^2{=}\ 3.365[.000]$	$A^2{=}0.648[.090]$
D'Agostino-Pearson	$K^2{=}66.640[.000]$	$K^2{=}9.202[.010]$
[2] Linearity	$F(2{,}317){=}\ 2.050[.131]$	$F(10{,}309){=}3.591[.000]$
[3] Homoskedasticity	$F(4{,}334){=}10.560[.000]$	$F(4{,}336){=}9.648[.000]$
[4] Independence	$F(3{,}317){=}\ 6.950[.000]$	$F(5{,}309){=}4.037[.001]$
[5] t-invariance:		
[5.1] Mean Homogeneity	$F(5{,}317){=}\ 3.517[.004]$	$F(1{,}309){=}0.037[.848]$
[5.2] Seasonal Homogeneity	$F(11{,}317){=}\ 9.639[.000]$	$F(11{,}309){=}4.339[.000]$
[5.3] Variance Homogeneity	$F(3{,}334){=}\ 4.606[.004]$	$F(1{,}336){=}2.343[.127]$

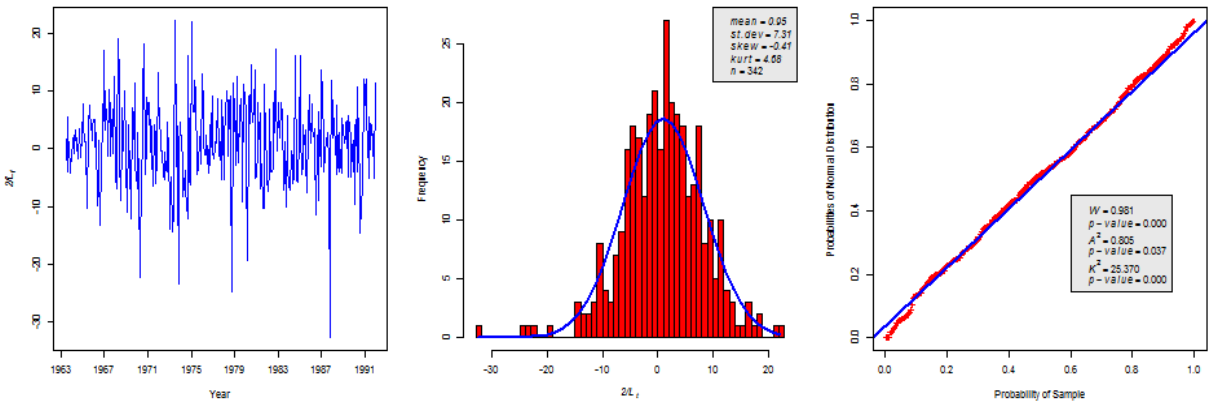

Figure D.11: t-plot, histogram, and p-p plot of $2/L_t$ portfolio

Table D.1.6 - Regression Results $(2/L_t)$		
y_t=$2/L_t$	CAPM	three-factor
R_{mt}	$\underset{(0.039)}{1.428}$[.000]	$\underset{(0.021)}{1.102}$[.000]
SMB_t		$\underset{(0.031)}{1.004}$[.000]
HML_t		$\underset{(0.035)}{-0.479}$[.000]
R_{ft}	$\underset{(0.808)}{-0.848}$[.295]	$\underset{(0.382)}{0.169}$[.658]
Constant	$\underset{(0.484)}{0.046}$[.924]	$\underset{(0.229)}{-0.284}$[.216]
Observations	342	342
R^2	0.803	0.956
Std. Error	3.256	1.535
F-Statistic	689.071[.000]	1847.938[.000]

Table D.2.6 - M-S Testing Results of $2/L_t$ **portfolio**		
$\widehat{u}_t$=$\widehat{u}_t^{2/L}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.988[.006]	W= 0.987[.005]
Anderson-Darling	A^2= 0.566[.141]	A^2= 1.015[.011]
D'Agostino-Pearson	K^2=13.752[.001]	K^2=14.030[.001]
[2] Linearity	F(2,317)= 0.323[.724]	F(10,309)= 1.714[.077]
[3] Homoskedasticity	F(4,336)= 4.226[.002]	F(9,330)= 4.549[.000]
[4] Independence	F(3,317)= 5.233[.002]	F(5,309)= 2.875[.015]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(5,317)= 2.467[.033]	F(1,309)= 0.580[.447]
[5.2] Seasonal Homogeneity	F(11,317)= 1.516[.124]	F(11,309)= 2.223[.013]
[5.3] Variance Homogeneity	F(1,336)=12.220[.001]	F(1,330)= 8.049[.005]

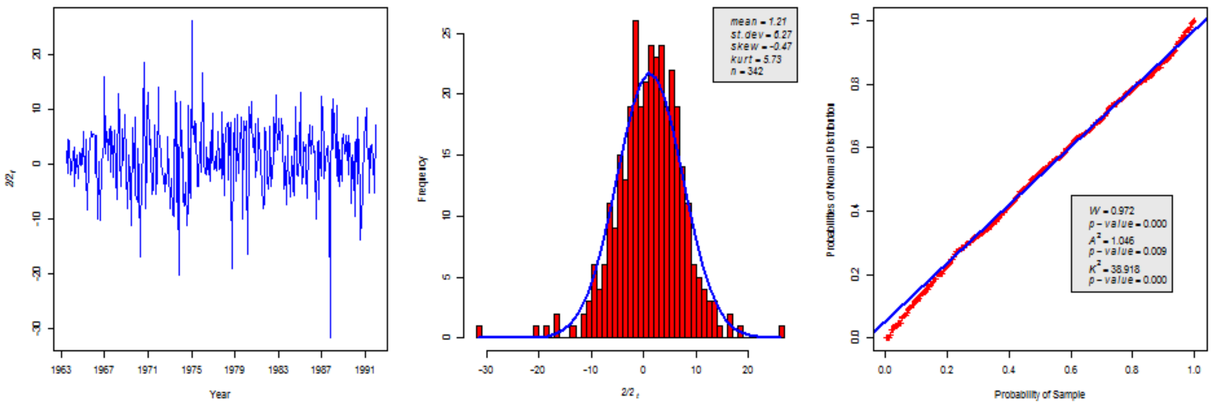

Figure D.12: t-plot, histogram, and p-p plot of $2/2_t$ portfolio

Table D.1.7 - Regression Results ($2/2_t$)

y_t=$2/2_t$	CAPM	three-factor
R_{mt}	1.227[.000] (0.033)	1.017[.000] (0.017)
SMB_t		0.936[.000] (0.026)
HML_t		0.022[.446] (0.029)
R_{ft}	0.036[.959] (0.699)	0.688[.030] (0.316)
Constant	0.007[.987] (0.419)	−0.411[.031] (0.189)
Observations	342	342
R^2	0.800	0.959
Std. Error	2.816	1.271
F-Statistic	675.901[.000]	1990.836[.000]

Table D.2.7 - M-S Testing Results of $2/2_t$ portfolio

$\widehat{u}_t$=$\widehat{u}_t^{2/2}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.974[.000]	W= 0.985[.002]
Anderson-Darling	A^2= 1.800[.000]	A^2= 0.694[.069]
D'Agostino-Pearson	K^2=34.315[.000]	K^2=21.507[.000]
[2] Linearity	F(2,316)= 2.128[.121]	F(4,311)= 1.594[.176]
[3] Homoskedasticity	F(4,330)= 9.342[.000]	F(10,330)= 5.489[.000]
[4] Independence	F(3,316)= 4.107[.007]	F(5,311)= 0.804[.547]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(6,316)= 3.015[.007]	F(5,311)= 1.951[.086]
[5.2] Seasonal Homogeneity	F(11,316)= 3.282[.000]	F(11,311)= 1.029[.420]
[5.3] Variance Homogeneity	F(7,330)= 3.864[.000]	F(1,330)= 1.063[.303]

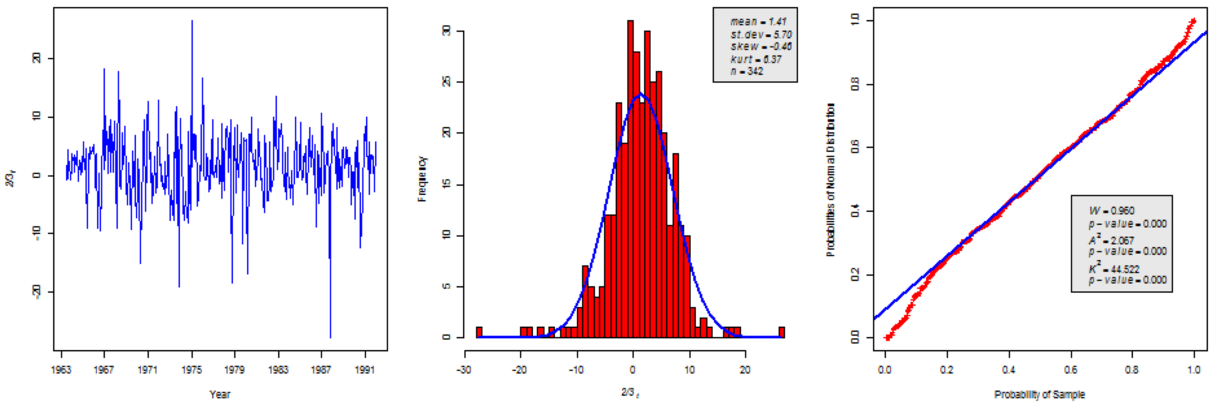

Figure D.13: t-plot, histogram, and p-p plot of $2/3_t$ portfolio

Table D.1.8 - Regression Results $(2/3_t)$		
$y_t = 2/3_t$	CAPM	three-factor
R_{mt}	$1.106[.000]$ (0.031)	$0.962[.000]$ (0.016)
SMB_t		$0.839[.000]$ (0.024)
HML_t		$0.238[.000]$ (0.027)
R_{ft}	$-0.735[.259]$ (0.650)	$-0.290[.320]$ (0.290)
Constant	$0.759[.052]$ (0.390)	$0.333[.056]$ (0.174)
Observations	342	342
R^2	0.790	0.959
Std. Error	2.620	1.166
F-Statistic	638.126[.000]	1953.722[.000]

Table D.2.8 - M-S Testing Results of $2/3_t$ **portfolio**		
$\widehat{u}_t = \widehat{u}_t^{2/3}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	$W= 0.979[.000]$	$W=0.995[.283]$
Anderson-Darling	$A^2= 2.008[.000]$	$A^2=0.333[.509]$
D'Agostino-Pearson	$K^2=17.487[.000]$	$K^2=3.165[.206]$
[2] Linearity	$F(2,317)= 3.527[.031]$	$F(4,295)=3.934[.004]$
[3] Homoskedasticity	$F(10,328)=13.463[.000]$	$F(8,332)=2.955[.003]$
[4] Independence	$F(3,317)= 4.378[.005]$	$F(15,295)=1.637[.064]$
[5] t-invariance:		
[5.1] Mean Homogeneity	$F(5,317)= 1.569[.169]$	$F(9,295)=2.094[.030]$
[5.2] Seasonal Homogeneity	$F(11,317)= 2.914[.001]$	$F(11,295)=1.260[.247]$
[5.3] Variance Homogeneity	$F(1,328)=12.796[.000]$	$F(1,332)=9.569[.002]$

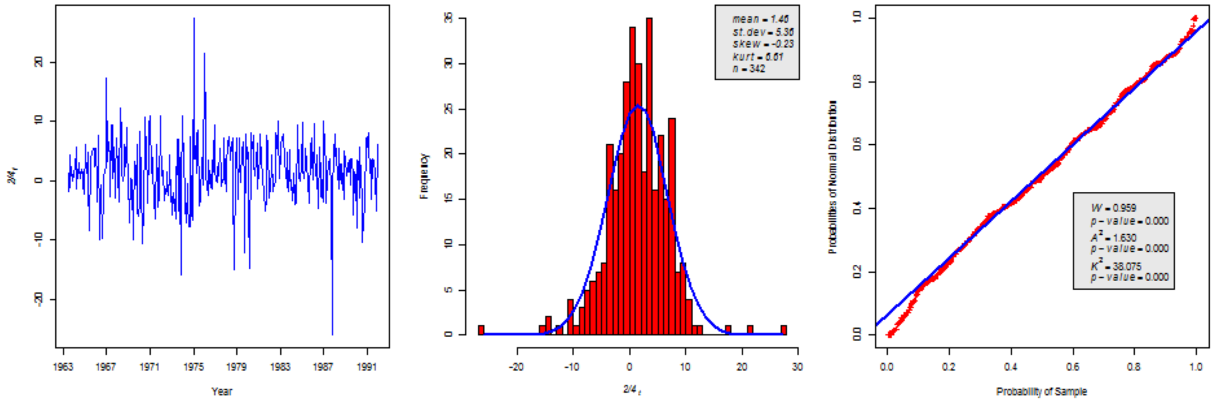

Figure D.14: t-plot, histogram, and p-p plot of $2/4_t$ portfolio

Table D.1.9 - Regression Results $(2/4_t)$		
y_t=$2/4_t$	CAPM	three-factor
R_{mt}	1.037[.000] (0.030)	0.970[.000] (0.015)
SMB_t		0.711[.000] (0.023)
HML_t		0.470[.000] (0.026)
R_{ft}	0.239[.703] (0.626)	0.446[.114] (0.281)
Constant	0.336[.371] (0.375)	−0.087[.604] (0.168)
Observations	342	342
R^2	0.780	0.956
Std. Error	2.522	1.129
F-Statistic	602.014[.000]	1839.222[.000]

Table D.2.9 - M-S Testing Results of $2/4_t$ **portfolio**		
$\widehat{u}_t$=$\widehat{u}_t^{2/4}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.969[.000]	W=0.997[.695]
Anderson-Darling	A^2= 1.617[.000]	A^2=0.337[.504]
D'Agostino-Pearson	K^2=38.915[.000]	K^2=1.928[.381]
[2] Linearity	F(2,321)= 1.985[.139]	F(4,315)=0.425[.791]
[3] Homoskedasticity	F(10,328)=10.528[.000]	F(8,332)=2.276[.022]
[4] Independence	F(3,321)= 2.365[.071]	F(5,315)=0.458[.807]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(1,321)= 2.610[.107]	F(1,315)=0.709[.400]
[5.2] Seasonal Homogeneity	F(11,321)= 4.156[.000]	F(11,315)=1.500[.130]
[5.3] Variance Homogeneity	F(1,328)= 7.291[.007]	F(1,332)=2.982[.085]

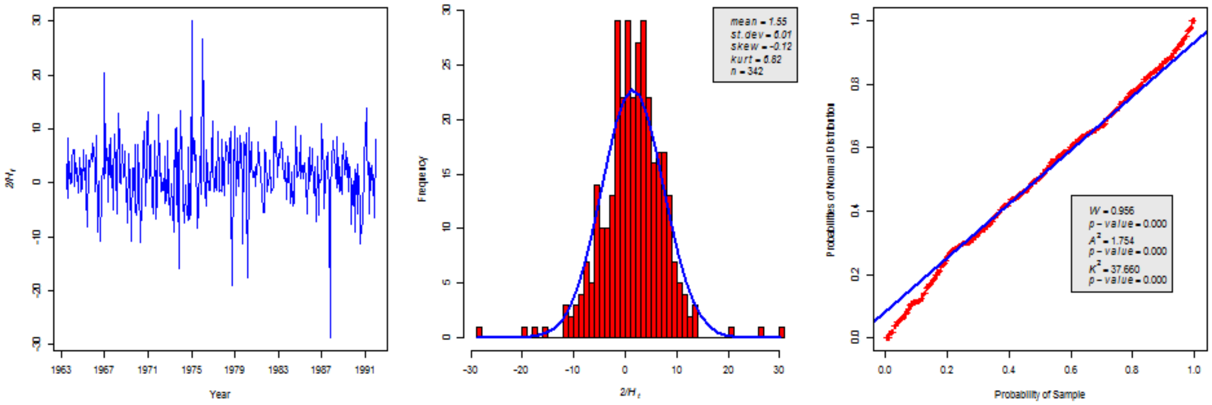

Figure D.15: t-plot, histogram, and p-p plot of $2/H_t$ portfolio

Table D.1.10 - Regression Results $(2/H_t)$		
y_t=$2/H_t$	CAPM	three-factor
R_{mt}	1.120[.000] (0.037)	1.067[.000] (0.017)
SMB_t		0.852[.000] (0.025)
HML_t		0.700[.000] (0.029)
R_{ft}	−0.272[.728] (0.782)	−0.111[.723] (0.312)
Constant	0.629[.181] (0.469)	0.090[.631] (0.187)
Observations	342	342
R^2	0.727	0.957
Std. Error	3.152	1.252
F-Statistic	450.639[.000]	1880.056[.000]

Table D.2.10 - M-S Testing Results of $2/H_t$ **portfolio**		
$\widehat{u}_t$=$\widehat{u}_t^{2/H}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.964[.000]	W=0.992[.064]
Anderson-Darling	A^2= 1.974[.000]	A^2=0.628[.101]
D'Agostino-Pearson	K^2=41.745[.000]	K^2=6.194[.045]
[2] Linearity	F(2,321)= 2.223[.110]	F(10,309)=1.457[.155]
[3] Homoskedasticity	F(10,322)=11.923[.000]	F(14,325)=1.978[.019]
[4] Independence	F(3,321)= 2.208[.087]	F(5,309)=0.931[.461]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(1,321)= 7.630[.006]	F(1,309)=2.242[.135]
[5.2] Seasonal Homogeneity	F(11,321)= 7.034[.000]	F(11,309)=1.143[.327]
[5.3] Variance Homogeneity	F(7,322)= 3.425[.002]	F(2,325)=3.274[.039]

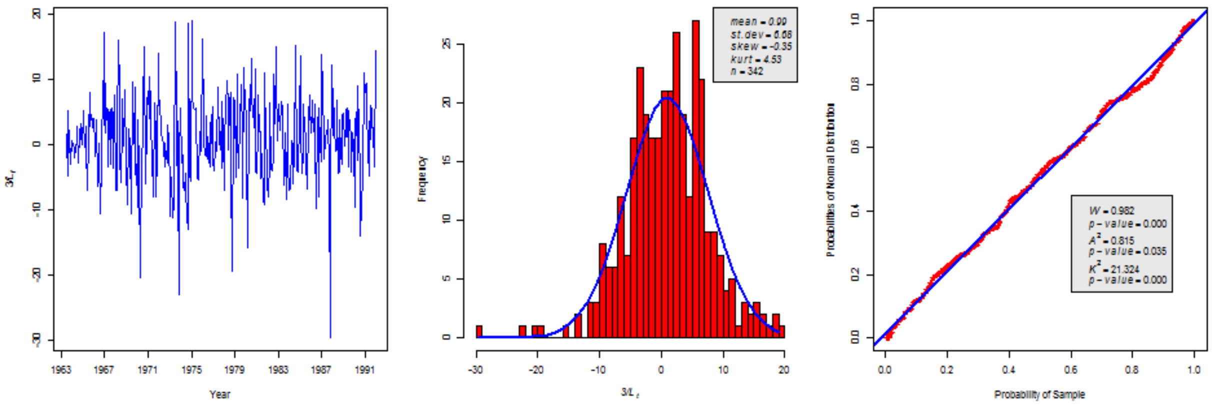

Figure D.16: t-plot, histogram, and p-p plot of $3/L_t$ portfolio

Table D.1.11 - Regression Results ($3/L_t$)		
y_t=$3/L_t$	CAPM	three-factor
R_{mt}	1.355[.000] (0.030)	1.106[.000] (0.018)
SMB_t		0.706[.000] (0.027)
HML_t		−0.433[.000] (0.031)
R_{ft}	−0.336[.589] (0.621)	0.441[.189] (0.335)
Constant	−0.132[.724] (0.372)	−0.341[.089] (0.200)
Observations	342	342
R^2	0.861	0.960
Std. Error	2.502	1.344
F-Statistic	1046.969[.000]	2023.684[.000]

Table D.2.11 - M-S Testing Results of $3/L_t$ **portfolio**		
$\widehat{u}_t$=$\widehat{u}_t^{3/L}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.986[.002]	W=0.994[.202]
Anderson-Darling	A^2= 0.342[.491]	A^2=0.339[.502]
D'Agostino-Pearson	K^2=17.254[.000]	K^2=5.465[.065]
[2] Linearity	F(2,317)= 0.206[.814]	F(4,309)=2.671[.032]
[3] Homoskedasticity	F(4,336)= 2.945[.020]	F(14,320)=3.776[.000]
[4] Independence	F(3,317)= 3.351[.019]	F(10,309)=2.944[.002]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(5,317)= 2.838[.016]	F(1,309)=1.811[.179]
[5.2] Seasonal Homogeneity	F(11,317)= 1.072[.384]	F(11,309)=1.534[.118]
[5.3] Variance Homogeneity	F(1,336)= 6.865[.009]	F(7,320)=3.937[.000]

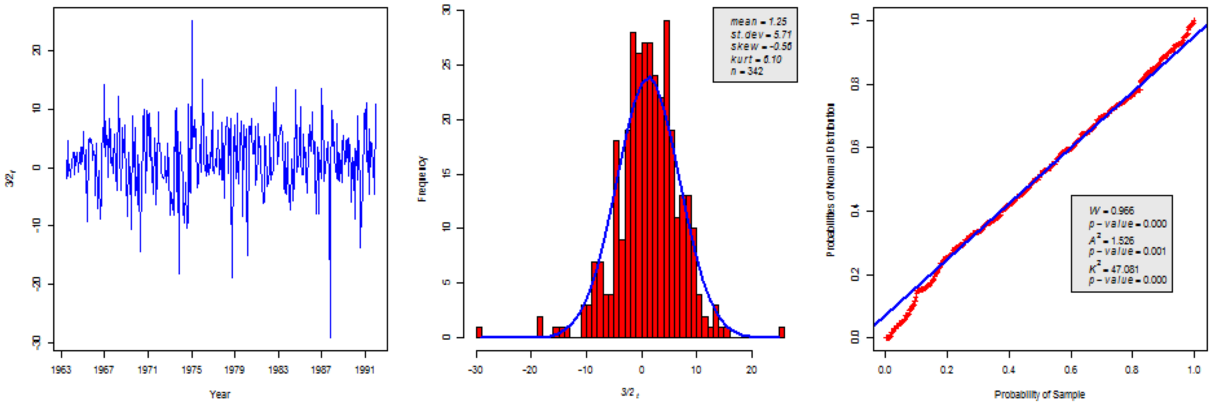

Figure D.17: t-plot, histogram, and p-p plot of $3/2_t$ portfolio

Table D.1.12 - Regression Results $(3/2_t)$

y_t=$3/2_t$	CAPM	three-factor
R_{mt}	1.159[.000] (0.025)	1.024[.000] (0.018)
SMB_t		0.625[.000] (0.027)
HML_t		0.039[.197] (0.030)
R_{ft}	−0.117[.825] (0.530)	0.302[.359] (0.329)
Constant	0.198[.534] (0.318)	−0.087[.660] (0.197)
Observations	342	342
R^2	0.861	0.947
Std. Error	2.137	1.321
F-Statistic	1048.627[.000]	1508.958[.000]

Table D.2.12 - M-S Testing Results of $3/2_t$ **portfolio**

$\widehat{u}_t$=$\widehat{u}_t^{3/2}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.992[.069]	W=0.994[.156]
Anderson-Darling	A^2= 0.473[.241]	A^2=0.671[.079]
D'Agostino-Pearson	K^2= 9.290[.010]	K^2=3.321[.190]
[2] Linearity	F(2,317)= 2.130[.121]	F(4,309)=0.676[.609]
[3] Homoskedasticity	F(11,326)=13.066[.000]	F(8,332)=5.599[.000]
[4] Independence	F(3,317)= 2.496[.060]	F(10,309)=1.802[.060]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(5,317)= 2.227[.051]	F(1,309)=0.020[.888]
[5.2] Seasonal Homogeneity	F(11,317)= 1.336[.203]	F(11,309)=0.977[.468]
[5.3] Variance Homogeneity	F(1,326)= 9.422[.002]	F(1,332)=7.292[.007]

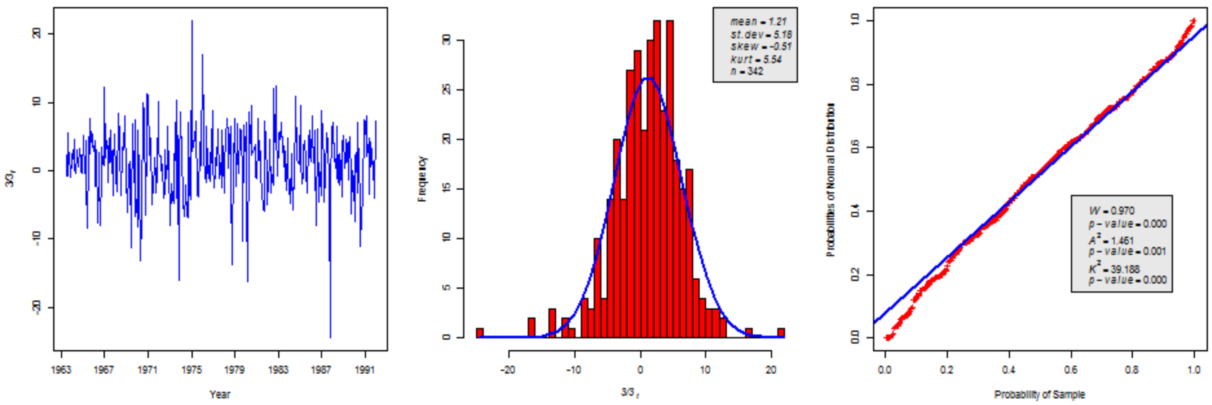

Figure D.18: t-plot, histogram, and p-p plot of $3/3_t$ portfolio

Table D.1.13 - Regression Results ($3/3_t$)

y_t=$3/3_t$	CAPM	three-factor
R_{mt}	1.032[.000] (0.025)	0.970[.000] (0.019)
SMB_t		0.542[.000] (0.028)
HML_t		0.310[.000] (0.031)
R_{ft}	−0.149[.780] (0.532)	0.040[.905] (0.338)
Constant	0.296[.354] (0.319)	−0.015[.940] (0.202)
Observations	342	342
R^2	0.830	0.932
Std. Error	2.145	1.358
F-Statistic	825.405[.000]	1156.295[.000]

Table D.2.13 - M-S Testing Results of $3/3_t$ **portfolio**

$\widehat{u}_t$=$\widehat{u}_t^{3/3}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.973[.000]	W= 0.992[.057]
Anderson-Darling	A^2= 1.929[.000]	A^2= 0.358[.453]
D'Agostino-Pearson	K^2=23.740[.000]	K^2=10.450[.005]
[2] Linearity	F(2,321)= 2.630[.074]	F(4,315)= 1.655[.160]
[3] Homoskedasticity	F(11,326)= 8.450[.000]	F(8,332)= 2.636[.008]
[4] Independence	F(3,321)= 2.278[.079]	F(5,315)= 1.014[.410]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(1,321)= 3.463[.064]	F(1,315)= 0.629[.428]
[5.2] Seasonal Homogeneity	F(11,321)= 1.682[.076]	F(11,315)= 1.994[.028]
[5.3] Variance Homogeneity	F(1,326)= 7.462[.007]	F(1,332)= 7.616[.006]

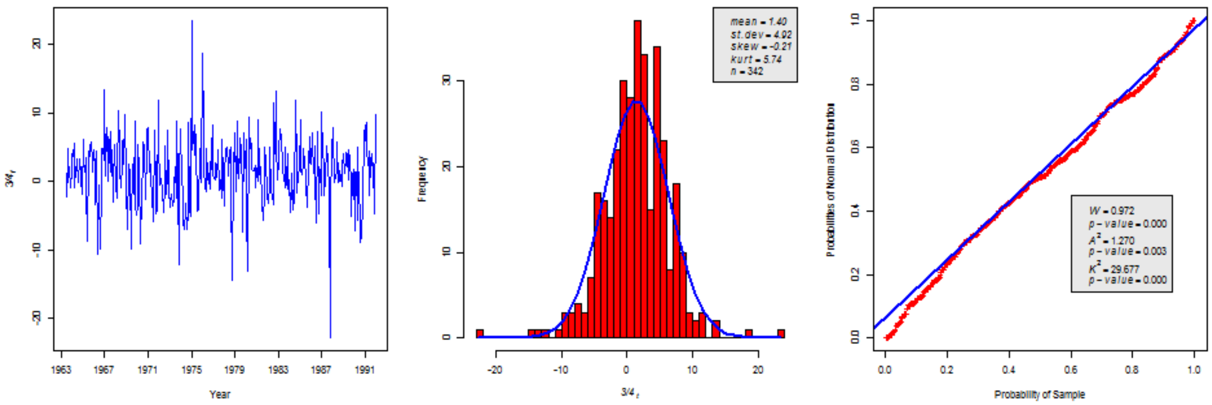

Figure D.19: t-plot, histogram, and p-p plot of $3/4_t$ portfolio

Table D.1.14 - Regression Results $(3/4_t)$

y_t=$3/4_t$	CAPM	three-factor
R_{mt}	0.970[.000] (0.025)	0.969[.000] (0.017)
SMB_t		0.449[.000] (0.025)
HML_t		0.504[.000] (0.028)
R_{ft}	−0.468[.375] (0.527)	−0.468[.124] (0.304)
Constant	0.726[.022] (0.316)	0.410[.025] (0.182)
Observations	342	342
R^2	0.815	0.939
Std. Error	2.125	1.220
F-Statistic	745.431[.000]	1304.138[.000]

Table D.2.14 - M-S Testing Results of $3/4_t$ **portfolio**

$\widehat{u}_t$=$\widehat{u}_t^{3/4}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.978[.000]	W=0.994[.158]
Anderson-Darling	A^2= 1.420[.001]	A^2=0.691[.071]
D'Agostino-Pearson	K^2=20.368[.000]	K^2=2.200[.333]
[2] Linearity	F(2,321)= 1.897[.152]	F(4,315)=3.751[.005]
[3] Homoskedasticity	F(10,328)=10.159[.000]	F(14,326)=3.840[.000]
[4] Independence	F(3,321)= 2.104[.100]	F(5,315)=2.695[.021]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(1,321)= 1.344[.247]	F(1,315)=0.082[.774]
[5.2] Seasonal Homogeneity	F(11,321)= 3.221[.000]	F(11,315)=1.690[.075]
[5.3] Variance Homogeneity	F(1,328)= 8.487[.004]	F(1,326)=5.528[.019]

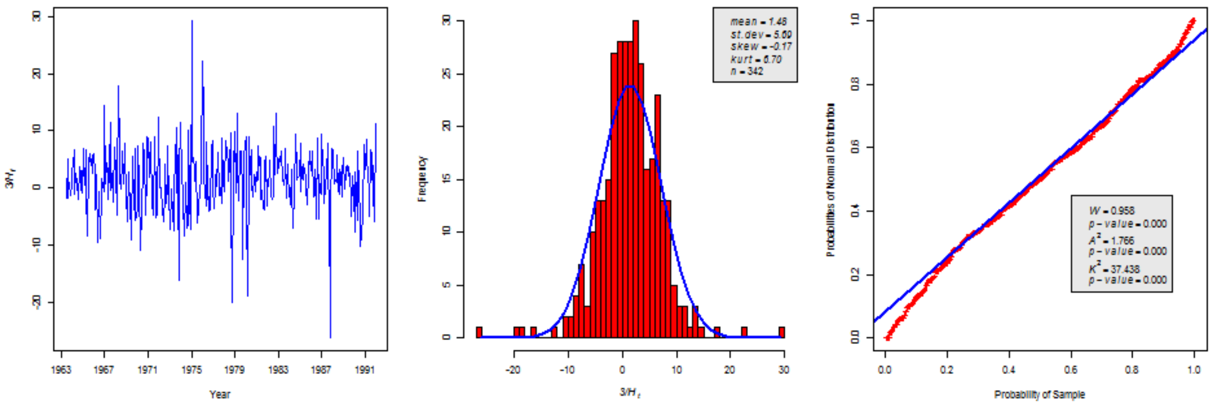

Figure D.20: t-plot, histogram, and p-p plot of $3/H_t$ portfolio

Table D.1.15 - Regression Results $(3/H_t)$		
y_t=$3/H_t$	CAPM	three-factor
R_{mt}	$\underset{(0.035)}{1.073}[.000]$	$\underset{(0.021)}{1.065}[.000]$
SMB_t		$\underset{(0.031)}{0.653}[.000]$
HML_t		$\underset{(0.035)}{0.701}[.000]$
R_{ft}	$\underset{(0.722)}{0.506}[.483]$	$\underset{(0.381)}{0.526}[.168]$
Constant	$\underset{(0.433)}{0.167}[.700]$	$\underset{(0.228)}{-0.285}[.212]$
Observations	342	342
R^2	0.740	0.929
Std. Error	2.909	1.529
F-Statistic	483.147[.000]	1097.779[.000]

Table D.2.15 - M-S Testing Results of $3/H_t$ **portfolio**		
$\widehat{u}_t$=$\widehat{u}_t^{3/H}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.967[.000]	W= 0.986[.002]
Anderson-Darling	A^2= 1.774[.000]	A^2= 1.406[.001]
D'Agostino-Pearson	K^2=42.512[.000]	K^2=10.391[.006]
[2] Linearity	$F(2,321)$= 1.015[.363]	$F(4,315)$= 1.729[.143]
[3] Homoskedasticity	$F(8,330)$= 7.581[.000]	$F(8,329)$= 1.031[.413]
[4] Independence	$F(3,321)$= 2.059[.106]	$F(5,315)$= 0.311[.906]
[5] t-invariance:		
[5.1] Mean Homogeneity	$F(1,321)$= 4.069[.045]	$F(1,315)$= 0.683[.409]
[5.2] Seasonal Homogeneity	$F(11,321)$= 4.381[.000]	$F(11,315)$= 1.424[.161]
[5.3] Variance Homogeneity	$F(1,330)$= 3.261[.072]	$F(4,329)$= 4.161[.003]

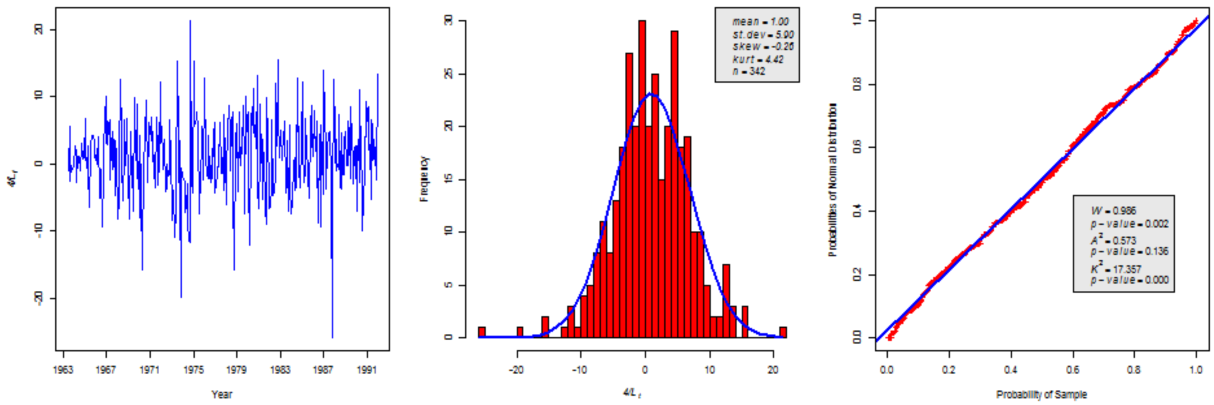

Figure D.21: t-plot, histogram, and p-p plot of $4/L_t$ portfolio

Table D.1.16 - Regression Results $(4/L_t)$

y_t=$4/L_t$	CAPM	three-factor
R_{mt}	1.223[.000] (0.022)	1.063[.000] (0.019)
SMB_t		0.303[.000] (0.028)
HML_t		−0.449[.000] (0.031)
R_{ft}	0.118[.802] (0.468)	0.618[.069] (0.338)
Constant	−0.237[.398] (0.281)	−0.266[.190] (0.202)
Observations	342	342
R^2	0.898	0.948
Std. Error	1.887	1.358
F-Statistic	1,495.306[.000]	1522.409[.000]

Table D.2.16 - M-S Testing Results of $4/L_t$ **portfolio**

$\widehat{u}_t = \widehat{u}_t^{4/L}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.988[.005]	W=0.993[.119]
Anderson-Darling	A^2= 0.646[.091]	A^2=0.509[.198]
D'Agostino-Pearson	K^2=10.996[.004]	K^2=4.333[.115]
[2] Linearity	F(2,317)= 0.999[.370]	F(4,309)=2.043[.088]
[3] Homoskedasticity	F(9,328)= 2.806[.003]	F(20,314)=2.110[.004]
[4] Independence	F(3,317)= 2.220[.086]	F(10,309)=2.177[.019]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(5,317)= 2.007[.077]	F(1,309)=0.135[.714]
[5.2] Seasonal Homogeneity	F(11,317)= 2.784[.002]	F(11,309)=2.194[.015]
[5.3] Variance Homogeneity	F(1,328)= 2.906[.089]	F(5,314)=2.202[.054]

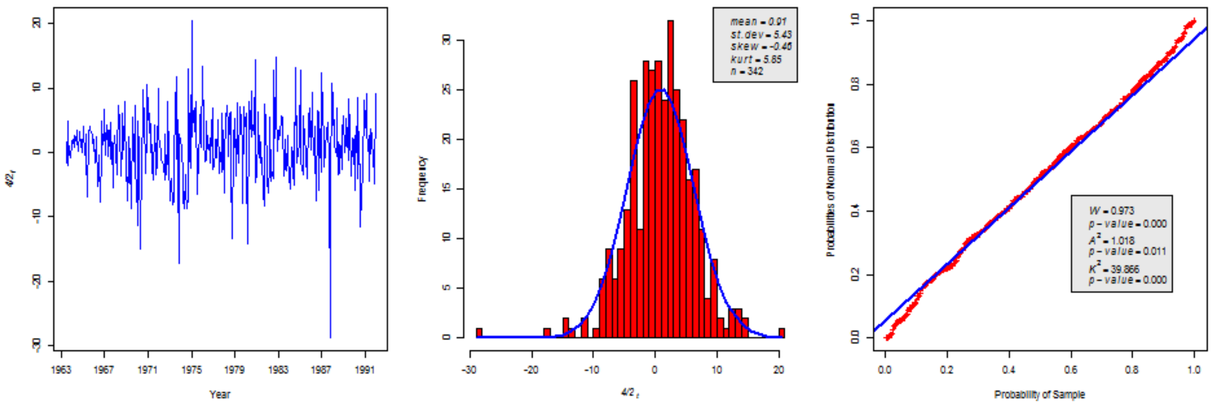

Figure D.22: t-plot, histogram, and p-p plot of $4/2_t$ portfolio

Table D.1.17 - Regression Results $(4/2_t)$

y_t=$4/2_t$	CAPM	three-factor
R_{mt}	1.130[.000] (0.020)	1.073[.000] (0.021)
SMB_t		0.268[.000] (0.031)
HML_t		0.020[.572] (0.035)
R_{ft}	0.081[.845] (0.416)	0.260[.493] (0.378)
Constant	−0.222[.374] (0.249)	−0.345[.128] (0.226)
Observations	342	342
R^2	0.905	0.923
Std. Error	1.676	1.517
F-Statistic	1618.007[.000]	1006.723[.000]

Table D.2.17 - M-S Testing Results of $4/2_t$ **portfolio**

$\widehat{u}_t=\widehat{u}_t^{4/2}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	$W=$ 0.985[.001]	$W=$ 0.989[.013]
Anderson-Darling	$A^2=$ 1.498[.001]	$A^2=$ 1.043[.010]
D'Agostino-Pearson	$K^2=$ 9.857[.007]	$K^2=$ 8.068[.018]
[2] Linearity	$F(2,316)=$ 0.635[.531]	$F(4,315)=$ 0.362[.835]
[3] Homoskedasticity	$F(7,332)=$ 9.988[.000]	$F(8,332)=$ 4.458[.000]
[4] Independence	$F(3,316)=$ 1.629[.183]	$F(5,315)=$ 0.423[.833]
[5] t-invariance:		
[5.1] Mean Homogeneity	$F(6,316)=$ 2.601[.018]	$F(1,315)=$ 0.087[.768]
[5.2] Seasonal Homogeneity	$F(11,316)=$ 1.276[.237]	$F(11,315)=$ 2.157[.017]
[5.3] Variance Homogeneity	$F(1,332)=$10.611[.001]	$F(1,332)=$12.609[.000]

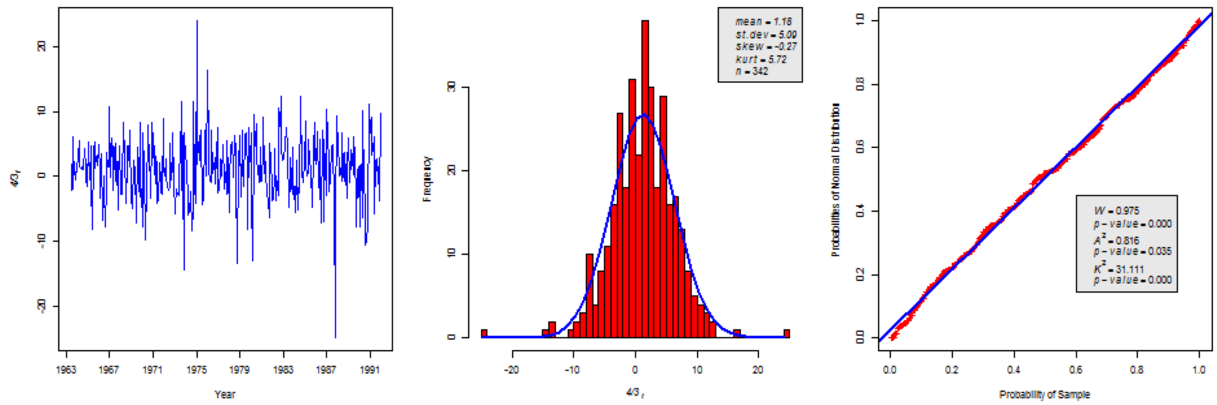

Figure D.23: t-plot, histogram, and p-p plot of $4/3_t$ portfolio

Table D.1.18 - Regression Results $(4/3_t)$		
y_t=$4/3_t$	CAPM	three-factor
R_{mt}	$\underset{(0.022)}{1.040}$[.000]	$\underset{(0.021)}{1.045}$[.000]
SMB_t		$\underset{(0.031)}{0.251}$[.000]
HML_t		$\underset{(0.034)}{0.309}$[.000]
R_{ft}	$\underset{(0.452)}{0.197}$[.662]	$\underset{(0.376)}{0.180}$[.633]
Constant	$\underset{(0.271)}{0.067}$[.804]	$\underset{(0.225)}{-0.116}$[.607]
Observations	342	342
R^2	0.873	0.913
Std. Error	1.821	1.511
F-Statistic	1161.579[.000]	882.085[.000]

Table D.2.18 - M-S Testing Results of $4/3_t$ **portfolio**		
$\widehat{u}_t$=$\widehat{u}_t^{4/3}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.962[.000]	W= 0.985[.001]
Anderson-Darling	A^2= 2.320[.000]	A^2= 1.121[.006]
D'Agostino-Pearson	K^2=34.726[.000]	K^2=14.842[.001]
[2] Linearity	F(2,307)= 0.395[.674]	F(4,315)= 0.378[.824]
[3] Homoskedasticity	F(13,322)= 6.115[.000]	F(10,326)= 3.212[.001]
[4] Independence	F(9,307)= 1.342[.215]	F(5,315)= 1.570[.168]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(7,307)= 2.694[.010]	F(1,315)= 1.549[.214]
[5.2] Seasonal Homogeneity	F(11,307)= 0.892[.549]	F(11,315)= 1.234[.264]
[5.3] Variance Homogeneity	F(3,322)= 4.057[.008]	F(3,326)= 3.854[.010]

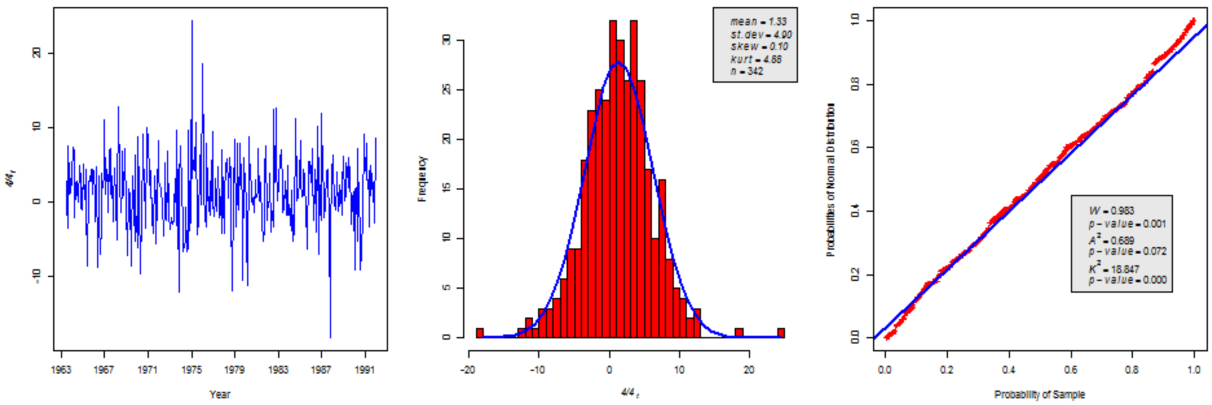

Figure D.24: t-plot, histogram, and p-p plot of $4/4_t$ portfolio

Table D.1.19 - Regression Results $(4/4_t)$

y_t=$4/4_t$	CAPM	three-factor
R_{mt}	$\underset{(0.025)}{0.965}$[.000]	$\underset{(0.020)}{1.029}$[.000]
SMB_t		$\underset{(0.030)}{0.222}$[.000]
HML_t		$\underset{(0.034)}{0.564}$[.000]
R_{ft}	$\underset{(0.521)}{-0.639}$[.220]	$\underset{(0.367)}{-0.839}$[.023]
Constant	$\underset{(0.321)}{0.759}$[.016]	$\underset{(0.220)}{0.529}$[.017]
Observations	342	342
R^2	0.817	0.910
Std. Error	2.098	1.474
F-Statistic	758.763[.000]	855.897[.000]

Table D.2.19 - M-S Testing Results of $4/4_t$ **portfolio**

$\widehat{u}_t$=$\widehat{u}_t^{4/4}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.975[.000]	W= 0.985[.001]
Anderson-Darling	A^2= 1.763[.000]	A^2= 0.727[.058]
D'Agostino-Pearson	K^2=29.750[.000]	K^2=14.424[.001]
[2] Linearity	F(2,321)= 3.059[.048]	F(4,295)= 7.104[.000]
[3] Homoskedasticity	F(13,324)= 7.541[.000]	F(8,332)= 3.121[.002]
[4] Independence	F(3,321)= 0.158[.924]	F(15,295)= 1.912[.022]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(1,321)= 1.261[.262]	F(9,295)= 2.465[.010]
[5.2] Seasonal Homogeneity	F(11,321)= 1.761[.060]	F(11,295)= 2.339[.009]
[5.3] Variance Homogeneity	F(1,324)=14.341[.000]	F(1,332)= 3.969[.047]

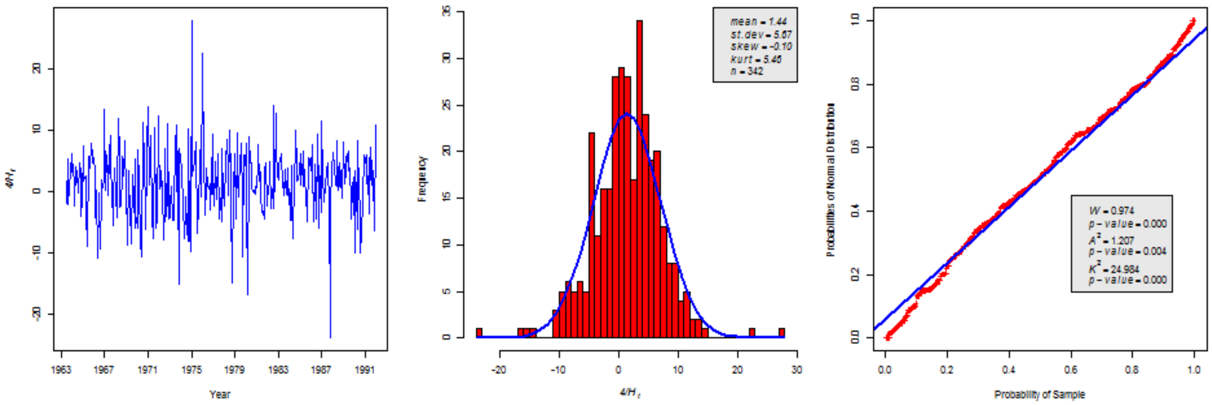

Figure D.25: t-plot, histogram, and p-p plot of $4/H_t$ portfolio

Table D.1.20 - Regression Results ($4/H_t$)

y_t=$4/H_t$	CAPM	three-factor
R_{mt}	1.085[.000] (0.033)	1.153[.000] (0.025)
SMB_t		0.357[.000] (0.037)
HML_t		0.738[.000] (0.042)
R_{ft}	0.083[.903] (0.682)	$-$0.132[.774] (0.459)
Constant	0.353[.389] (0.408)	0.023[.934] (0.274)
Observations	342	342
R^2	0.766	0.896
Std. Error	2.747	1.841
F-Statistic	555.961[.000]	722.912[.000]

Table D.2.20 - M-S Testing Results of $4/H_t$ **portfolio**

$\widehat{u}_t$=$\widehat{u}_t^{4/H}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.986[.002]	W=0.995[.309]
Anderson-Darling	A^2= 0.683[.074]	A^2=0.447[.279]
D'Agostino-Pearson	K^2=16.088[.000]	K^2=0.824[.662]
[2] Linearity	F(2,321)= 0.491[.613]	F(4,311)=1.044[.385]
[3] Homoskedasticity	F(10,328)= 7.232[.000]	F(4,332)=1.762[.136]
[4] Independence	F(3,321)= 0.622[.601]	F(5,311)=2.258[.049]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(1,321)= 0.492[.483]	F(5,311)=2.280[.047]
[5.2] Seasonal Homogeneity	F(11,321)= 2.945[.001]	F(11,311)=1.015[.433]
[5.3] Variance Homogeneity	F(1,328)=10.211[.002]	F(5,332)=2.222[.052]

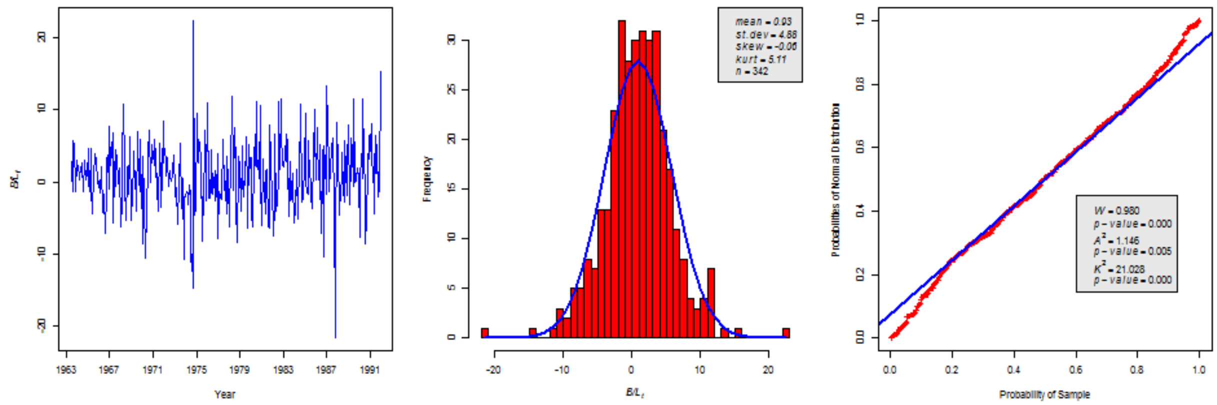

Figure D.26: t-plot, histogram, and p-p plot of B/L_t portfolio

Table D.1.21 - Regression Results (B/L_t)

y_t=B/L_t	CAPM	three-factor
R_{mt}	0.999[.000] (0.020)	0.955[.000] (0.017)
SMB_t		−0.201[.000] (0.025)
HML_t		−0.444[.000] (0.028)
R_{ft}	−0.421[.325] (0.428)	−0.282[.357] (0.306)
Constant	0.200[.435] (0.256)	0.393[.033] (0.183)
Observations	342	342
R^2	0.876	0.937
Std. Error	1.724	1.230
F-Statistic	1200.048[.000]	1261.308[.000]

Table D.2.21 - M-S Testing Results of B/L_t portfolio

$\widehat{u}_t=\widehat{u}_t^{B/L}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	$W=$ 0.992[.077]	W=0.991[.032]
Anderson-Darling	$A^2=$ 0.778[.043]	A^2=0.405[.350]
D'Agostino-Pearson	$K^2=$ 7.006[.030]	K^2=4.696[.096]
[2] Linearity	F(2,320)= 3.887[.021]	F(4,315)=4.684[.001]
[3] Homoskedasticity	F(5,333)=10.534[.000]	F(13,323)=5.873[.000]
[4] Independence	F(3,320)= 0.475[.700]	F(5,315)=1.470[.199]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(2,320)= 5.489[.005]	F(1,315)=1.697[.194]
[5.2] Seasonal Homogeneity	F(11,320)= 2.167[.016]	F(11,315)=1.930[.035]
[5.3] Variance Homogeneity	F(3,333)= 5.178[.002]	F(4,323)=3.087[.016]

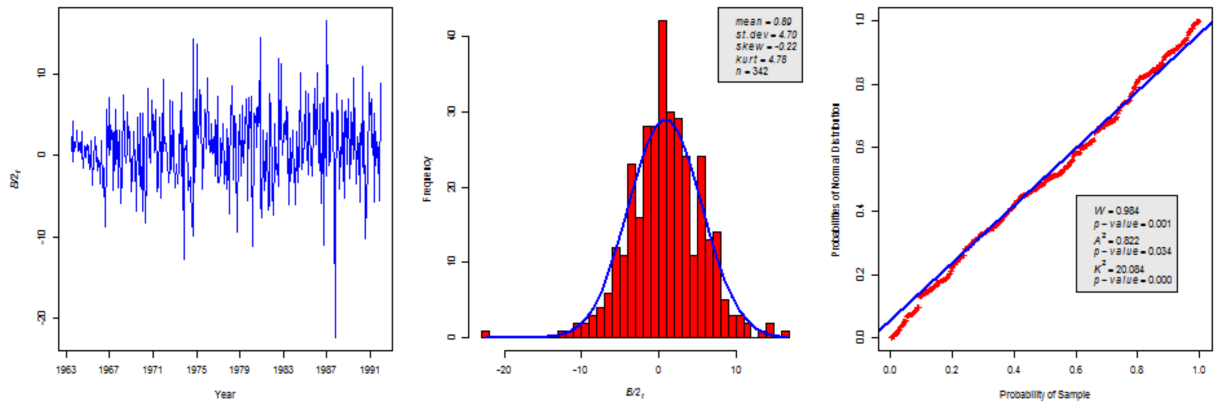

Figure D.27: t-plot, histogram, and p-p plot of $B/2_t$ portfolio

Table D.1.22 - Regression Results $(B/2_t)$

y_t=$B/2_t$	CAPM	three-factor
R_{mt}	0.984[.000] (0.017)	1.023[.000] (0.018)
SMB_t		−0.191[.000] (0.026)
HML_t		−0.026[.376] (0.030)
R_{ft}	0.765[.027] (0.345)	0.646[.046] (0.323)
Constant	−0.480[.021] (0.207)	−0.390[.044] (0.193)
Observations	342	342
R^2	0.913	0.925
Std. Error	1.392	1.296
F-Statistic	1774.917[.000]	1037.904[.000]

Table D.2.22 - M-S Testing Results of $B/2_t$ **portfolio**

$\widehat{u}_t = \widehat{u}_t^{B/2}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W=0.997[.727]	W=0.998[.976]
Anderson-Darling	A^2=0.228[.811]	A^2=0.159[.950]
D'Agostino-Pearson	K^2=1.217[.544]	K^2=0.004[.998]
[2] Linearity	F(2,317)=2.834[.060]	F(10,289)=2.203[.018]
[3] Homoskedasticity	F(3,336)=4.136[.005]	F(8,326)=9.145[.000]
[4] Independence	F(3,317)=1.069[.362]	F(15,289)=1.275[.217]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(5,317)=2.691[.021]	F(9,289)=2.658[.006]
[5.2] Seasonal Homogeneity	F(11,317)=1.002[.444]	F(11,289)=0.496[.905]
[5.3] Variance Homogeneity	F(1,336)=0.169[.681]	F(1,326)=2.340[.024]

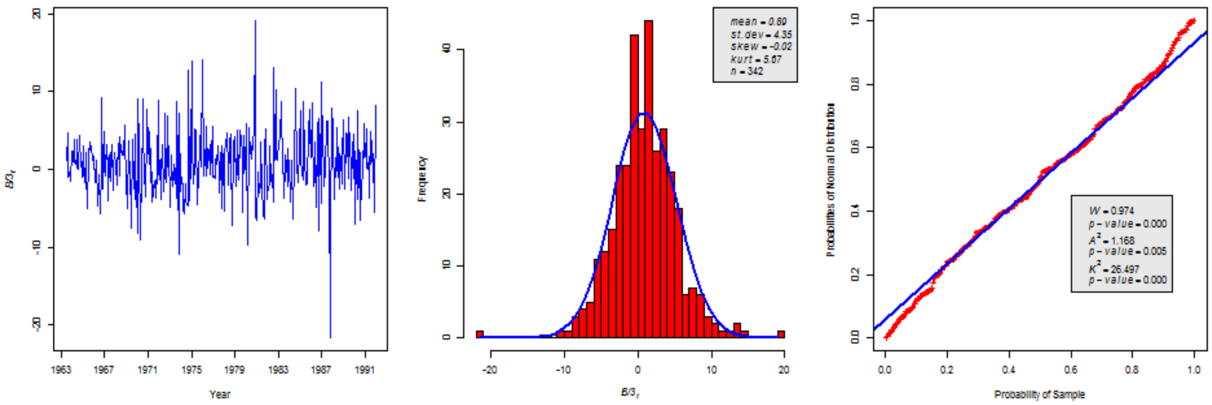

Figure D.28: t-plot, histogram, and p-p plot of $B/3_t$ portfolio

Table D.1.23 - Regression Results ($B/3_t$)

y_t=$B/3_t$	CAPM	three-factor
R_{mt}	0.860[.000] (0.022)	0.961[.000] (0.023)
SMB_t		−0.263[.000] (0.034)
HML_t		0.202[.000] (0.038)
R_{ft}	0.168[.716] (0.462)	−0.147[.723] (0.415)
Constant	−0.026[.927] (0.277)	0.043[.862] (0.248)
Observations	342	342
R^2	0.818	0.855
Std. Error	1.861	1.667
F-Statistic	760.718[.000]	495.105[.000]

Table D.2.23 - M-S Testing Results of $B/3_t$ portfolio

$\widehat{u}_t$=$\widehat{u}_t^{B/3}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.971[.000]	W= 0.966[.000]
Anderson-Darling	A^2= 1.910[.000]	A^2= 1.320[.002]
D'Agostino-Pearson	K^2=37.087[.000]	K^2=50.875[.000]
[2] Linearity	F(2,321)= 3.399[.035]	F(4,310)= 4.598[.001]
[3] Homoskedasticity	F(2,336)= 8.576[.000]	F(14,326)= 8.178[.000]
[4] Independence	F(3,321)= 6.644[.000]	F(5,310)= 2.559[.027]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(1,321)= 0.217[.642]	F(6,310)= 1.944[.073]
[5.2] Seasonal Homogeneity	F(11,321)= 0.885[.555]	F(11,310)= 1.465[.143]
[5.3] Variance Homogeneity	F(1,336)= 4.164[.006]	F(1,326)= 3.786[.053]

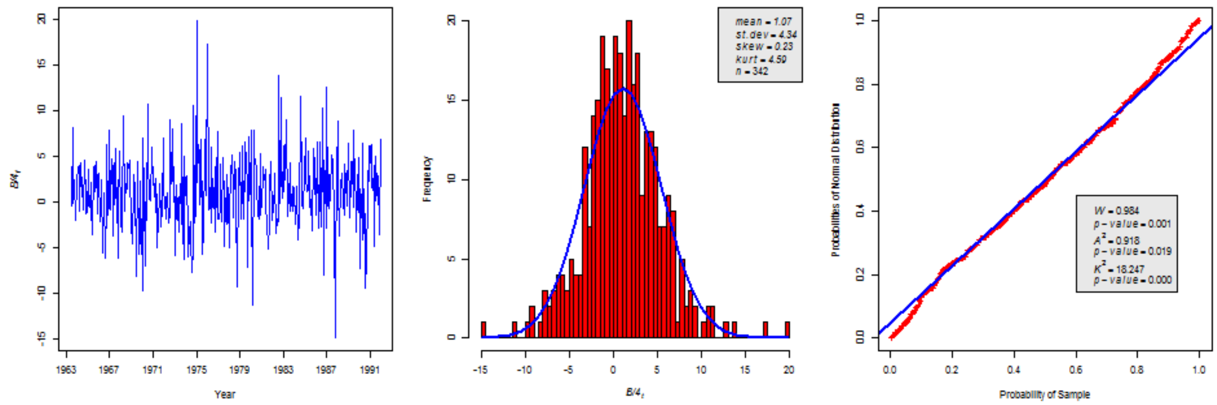

Figure D.29: t-plot, histogram, and p-p plot of $B/4_t$ portfolio

Table D.1.24 - Regression Results ($B/4_t$)

y_t=$B/4_t$	CAPM	three-factor
R_{mt}	0.849[.000] (0.023)	1.007[.000] (0.018)
SMB_t		−0.191[.000] (0.027)
HML_t		0.563[.000] (0.031)
R_{ft}	0.395[.415] (0.485)	−0.098[.771] (0.336)
Constant	0.040[.891] (0.290)	−0.007[.970] (0.201)
Observations	342	342
R^2	0.798	0.904
Std. Error	1.953	1.350
F-Statistic	671.598[.000]	795.332[.000]

Table D.2.24 - M-S Testing Results of $B/4_t$ portfolio

$\widehat{u}_t$=$\widehat{u}_t^{B/4}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W=0.992[.065]	W=0.995[.301]
Anderson-Darling	A^2=0.586[.126]	A^2=0.456[.266]
D'Agostino-Pearson	K^2=8.758[.013]	K^2=3.518[.172]
[2] Linearity	F(2,321)=1.199[.303]	F(10,309)=2.472[.007]
[3] Homoskedasticity	F(13,324)=8.443[.000]	F(4,334)=5.398[.000]
[4] Independence	F(3,321)=4.792[.003]	F(5,309)=3.600[.004]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(1,321)=0.532[.466]	F(1,309)=0.015[.902]
[5.2] Seasonal Homogeneity	F(11,321)=1.328[.207]	F(11,309)=0.707[.732]
[5.3] Variance Homogeneity	F(1,324)=6.812[.009]	F(3,334)=3.138[.026]

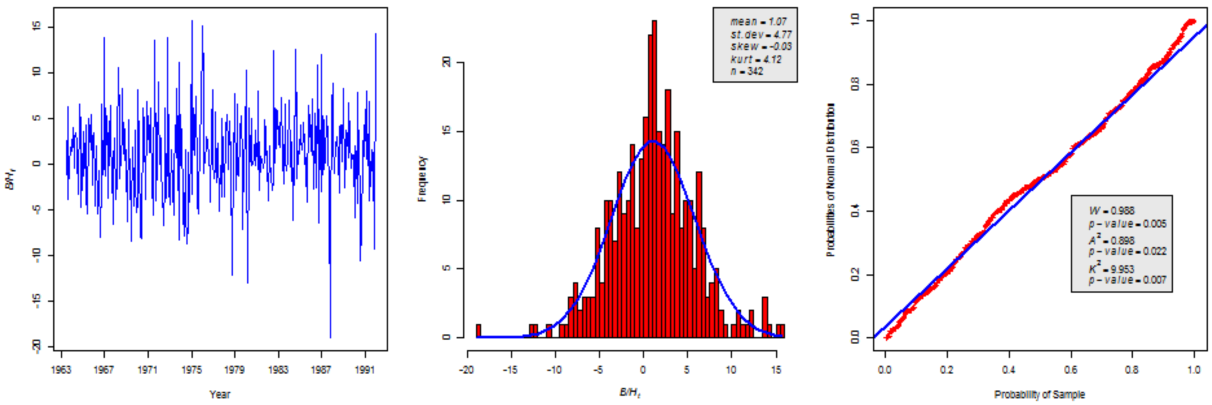

Figure D.30: t-plot, histogram, and p-p plot of B/H_t portfolio

Table D.1.25 - Regression Results (B/H_t)		
y_t=B/H_t	CAPM	three-factor
R_{mt}	0.864[.000] (0.032)	1.027[.000] (0.027)
SMB_t		−0.042[.299] (0.041)
HML_t		0.760[.000] (0.046)
R_{ft}	0.677[.312] (0.669)	0.165[.741] (0.498)
Constant	−0.133[.741] (0.401)	−0.292[.329] (0.298)
Observations	342	342
R^2	0.683	0.826
Std. Error	2.695	2.002
F-Statistic	364.415[.000]	399.736[.000]

Table D.2.25 - M-S Testing Results of B/H_t portfolio		
$\widehat{u}_t$=$\widehat{u}_t^{B/H}$	CAPM	three-factor
[1] Normality:		
Shapiro-Wilk	W= 0.970[.000]	W= 0.983[.000]
Anderson-Darling	A^2= 1.291[.002]	A^2= 1.359[.002]
D'Agostino-Pearson	K^2=25.929[.000]	K^2=14.723[.001]
[2] Linearity	F(3,320)= 3.483[.016]	F(10,309)= 2.376[.010]
[3] Homoskedasticity	F(7,326)= 7.416[.000]	F(8,330)= 8.202[.000]
[4] Independence	F(3,320)= 1.315[.270]	F(5,309)= 2.011[.077]
[5] t-invariance:		
[5.1] Mean Homogeneity	F(1,320)= 0.116[.734]	F(1,309)= 0.398[.529]
[5.2] Seasonal Homogeneity	F(11,320)= 2.008[.027]	F(11,309)= 1.175[.304]
[5.3] Variance Homogeneity	F(7,326)= 4.447[.000]	F(3,330)= 4.401[.005]

Chapter 4

4 Building Blocks of Statistical Adequacy

4.1 Overview

It stands to reason that the reliability of statistical inference results depends crucially on whether the estimated model accounts fully for the statistical regularities in the data. Clearly, the only way to ensure the adequacy of a statistical model is to test the validity of its probabilistic assumptions vis-à-vis the data, and if one or more of these assumptions are found to be invalid, to respecify the original model with a view to account for the information that model did not. The Mis-Specification (M-S) testing results in chapter 3 indicate clearly that the Multivariate Normal/ Homoskedastic LR model suffers from serious statistical misspecifications, and hence the Fama and French (1993) data do not satisfy the probabilistic assumptions imposed on them. Thus, on the basis of their statistical inadequacy, all the inference procedures are of questionable reliability and any empirical evidence is weak to validate or invalidate the CAPM and/or the Fama-French three-factor model against the data in hand.

The purpose of this chapter is to respecify a statistical model, appropriate enough to account for all the statistical systematic information in the data. In light of the M-S testing results in chapter 3, the chapter starts with a first attempt at respecification in section 4.2 where particular emphasis is given on the departures from the independence and t-invariance assumptions. At this stage, the reduction assumptions give rise to a heterogeneous Normal Vector Autoregressive of order 1 model. Yet, the

M-S testing results indicate that the underlying model is inadequate on statistical grounds. Following a brief discussion on the elliptically symmetric family of distributions in section 4.3, and by concluding that the distribution that suggests itself is the Student's t distribution, the second attempt at respecification in section 4.4 employs a heterogeneous Student's t/ Heteroskedastic Vector Autoregressive of order 1 model. As the M-S testing results indicate, the respecified model accounts for all the statistical systematic information in the Fama and French (1993) data.

4.2 First Attempt at Respecification: Normal VAR

At first, an attempt will be made to account for all the departures indicated by the M-S testing results in chapter 3 without changing Normality. There are two reasons for this view. First, there is a possibility that the departures from the [4] Independence and [5] t-invariance assumptions rendered the M-S testing results of the [1]–[3] distributional assumptions unreliable. Second, in light of the fact that the Normality plays a crucial role in the original Markowitz optimal portfolio theory, as well as the CAPM and its extensions, it is important to reappraise this distribution.

Table 4.1 - Normal Vector Autoregressive [VAR(1)] model

$$\boxed{\text{Statistical GM:}\quad \mathbf{Z}_t = \boldsymbol{\alpha} + \boldsymbol{\Gamma}^\top \mathbf{Z}_{t-1} + \mathbf{u}_t,\ t\in\mathbb{N}.}$$

[1] Normality:	$D(\mathbf{Z}_t \vert \boldsymbol{\sigma}(\mathbf{Z}^0_{t-1});\boldsymbol{\theta})$, is Normal, where $\mathbf{Z}_t{:}(l\times 1)$, $\mathbf{Z}^0_{t-1}{:=}(\mathbf{Z}_{t-1},\mathbf{Z}_{t-2},...,\mathbf{Z}_1)$,
[2] Linearity:	$E(\mathbf{Z}_t \vert \boldsymbol{\sigma}(\mathbf{Z}^0_{t-1}))=\boldsymbol{\alpha}+\boldsymbol{\Gamma}^\top\mathbf{Z}_{t-1}$, is linear in $\mathbf{Z}_{t-1}$,
[3] Homoskedasticity:	$Var(\mathbf{Z}_t \vert \boldsymbol{\sigma}(\mathbf{Z}^0_{t-1}))=\boldsymbol{\Omega}>0$, is free of $\mathbf{Z}^0_{t-1}$,
[4] Markov (1):	$\{\mathbf{Z}_t,\ t\in\mathbb{N}\}$ is a Markov process of order 1,
[5] t-invariance:	$\boldsymbol{\theta}{:=}(\boldsymbol{\alpha},\boldsymbol{\Gamma},\boldsymbol{\Omega})$ are constant over t.

$$\boldsymbol{\alpha}=\boldsymbol{\mu}-\boldsymbol{\Gamma}^\top\boldsymbol{\mu},\ \boldsymbol{\Gamma}=\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}_1,\ \boldsymbol{\Omega}=\boldsymbol{\Sigma}_0-\boldsymbol{\Sigma}_1^\top\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\Sigma}_1,$$

$$\boldsymbol{\mu}=E(\mathbf{Z}_t)=E(\mathbf{Z}_{t-1}),\ \boldsymbol{\Sigma}_0=Cov(\mathbf{Z}_t)=Cov(\mathbf{Z}_{t-1}),\ \boldsymbol{\Sigma}_1=Cov(\mathbf{Z}_t,\mathbf{Z}_{t-1}).$$

At this stage, the respecification takes the form of replacing the original reduction assumptions in (3.9) implicitly imposed on the observable stochastic process in (3.4) with more appropriate ones. Departures from assumptions [4] and [5] suggest that the IID assumptions are clearly invalid and need to be replaced with assumptions that allow for temporal dependence and certain forms of heterogeneity. The generic way to do that is to use lags, trends, and seasonal dummy variables. It is important to emphasize that the choice of the number of lags and degree of the trend polynomials needs to be decided on statistical adequacy grounds.

The reduction assumptions of Normality, Markov dependence of order 1 (henceforth Markov (1)), and Stationarity give rise to the Normal VAR of order 1 (henceforth VAR (1)) model in table 4.1. Formally, the probabilistic reduction that gives rise to the Normal VAR (1) model takes the following form:

$$\begin{aligned} D(\mathbf{Z}_1, \mathbf{Z}_2, ..., \mathbf{Z}_n; \boldsymbol{\vartheta}) &\overset{M}{=} D_1(\mathbf{Z}_1; \boldsymbol{\theta}_1) \prod_{t=2}^{n} D_t(\mathbf{Z}_t | \mathbf{Z}_{t-1}; \boldsymbol{\theta}_{2t}) \\ &\overset{MS}{=} D(\mathbf{Z}_1; \boldsymbol{\theta}_1) \prod_{t=2}^{n} D(\mathbf{Z}_t | \mathbf{Z}_{t-1}; \boldsymbol{\theta}_2), \end{aligned} \tag{4.1}$$

where M denotes the imposition of Markov (1), MS denotes the imposition of both Markov (1) and Stationarity, $D(\mathbf{Z}_t|\mathbf{Z}_{t-1}; \boldsymbol{\theta}_2)$ is the conditional Normal distribution, and $\boldsymbol{\vartheta}$, $\boldsymbol{\theta}_1$, and $\boldsymbol{\theta}_2$ denote the parameters in the joint, marginal, and conditional distributions, respectively.

It is important to note that making the further reduction:

$$D(\mathbf{Z}_1, \mathbf{Z}_2, ..., \mathbf{Z}_n; \boldsymbol{\vartheta}) = D(\mathbf{Z}_1; \boldsymbol{\theta}_1) \prod_{t=2}^{n} D(y_t | \mathbf{X}_t, \mathbf{Z}_{t-1}; \boldsymbol{\theta}_2^1) \cdot D(\mathbf{X}_t | \mathbf{Z}_{t-1}; \boldsymbol{\theta}_2^2), \tag{4.2}$$

gives rise to the Normal/Homoskedastic Dynamic LR model in table 4.2, specified in terms of $D(y_t|\mathbf{X}_t, \mathbf{Z}_{t-1}; \boldsymbol{\theta}_2^1)$.

Even though the probabilistic structure of the Normal VAR (1) model (table 4.1) involves t-homogeneity of the parameters, the modeler is not restraint to model time varying volatility. The Gram-Schmidt orthonormal polynomials and seasonal dummy variables can be used with a view to capture the mean and variance heterogeneity exhibited by the data. As a result, the parameters in the probabilistic reductions (4.1)-(4.2) become t-heterogenous.

Table 4.2 - Normal/Homoskedastic Dynamic LR model

$$\boxed{\text{Statistical GM:} \quad y_t{=}\alpha+\boldsymbol{\beta}^\top\mathbf{X}_t+\boldsymbol{\gamma}^\top\mathbf{Z}_{t-1}+u_t,\ t{\in}\mathbb{N}.}$$

[1] Normality: $D(y_t|\boldsymbol{\sigma}(\mathbf{X}_t,\mathbf{Z}^0_{t-1});\boldsymbol{\theta})$, is Normal,
where $\mathbf{Z}^0_{t-1}{:=}(\mathbf{Z}_{t-1},\mathbf{Z}_{t-2},...,\mathbf{Z}_1)$,

[2] Linearity: $E(y_t|\boldsymbol{\sigma}(\mathbf{X}_t,\mathbf{Z}^0_{t-1})){=}\alpha{+}\boldsymbol{\beta}^\top\mathbf{X}_t{+}\boldsymbol{\gamma}^\top\mathbf{Z}_{t-1}$, is linear in $(\mathbf{X}_t,\mathbf{Z}_{t-1})$,
for $\mathbf{Z}_t{:=}(y_t,\mathbf{X}_t)$, where $y_t{:}(1{\times}1)$, $\mathbf{X}_t{:}(l{\times}1)$, $\mathbf{Z}_t{:}((l{+}1)\times 1)$,

[3] Homosked.: $Var(y_t|\boldsymbol{\sigma}(\mathbf{X}_t,\mathbf{Z}^0_{t-1})){=}\sigma^2{>}0$, is free of $(\mathbf{X}_t,\mathbf{Z}^0_{t-1})$,

[4] Markov (1): $\{\mathbf{Z}_t,\ t{\in}\mathbb{N}\}$ is a Markov process of order 1,

[5] t-invariance: $\boldsymbol{\theta}{:=}(\alpha,\boldsymbol{\beta},\boldsymbol{\gamma},\sigma^2)$ are constant over t.

$$\alpha{=}\mu_y{-}\boldsymbol{\beta}^\top\boldsymbol{\mu}_{\mathbf{x}}{-}\boldsymbol{\gamma}^\top\boldsymbol{\mu},\ \boldsymbol{\beta}{=}\mathbf{Q}_1^{-1}\boldsymbol{\sigma}_{21,1},\ \boldsymbol{\gamma}{=}\mathbf{Q}_2^{-1}\boldsymbol{\sigma}_{21,2},\ \sigma^2{=}\sigma_{11}{-}\boldsymbol{\sigma}_{21}^\top\mathbf{Q}^{-1}\boldsymbol{\sigma}_{21},$$

$$\mu_y{=}E(y_t),\ \boldsymbol{\mu}_{\mathbf{x}}{=}E(\mathbf{X}_t),\ \boldsymbol{\mu}{=}E(\mathbf{Z}_{t-1}),$$

$$\mathbf{W}_t{:=}(\mathbf{X}_t,\mathbf{Z}_{t-1}),\ \mathbf{Q}{=}Cov(\mathbf{W}_t),\ \mathbf{Q}_1{=}Cov(\mathbf{X}_t),\ \mathbf{Q}_2{=}Cov(\mathbf{Z}_{t-1}),$$

$$\sigma_{11}{=}Var(y_t),\ \boldsymbol{\sigma}_{21}{=}Cov(y_t,\mathbf{W}_t),\ \boldsymbol{\sigma}_{21,1}{=}Cov(y_t,\mathbf{X}_t),\ \boldsymbol{\sigma}_{21,2}{=}Cov(y_t,\mathbf{Z}_{t-1}).$$

The estimation of the heterogeneous Normal VAR (1) model is equivalent to the following system of three equations:

$$y_{it}{=}\alpha_{1i}{+}\sum_{j=1}^{m^*}\delta_{11ji}v_{jt}{+}\sum_{j=2}^{s}\delta_{12ji}d_{jt}{+}\gamma_{11i}y_{it-1}{+}\gamma_{12i}R_{mt-1}{+}\gamma_{13i}R_{ft-1}{+}u_{1it}, \tag{4.3}$$

$$R_{mt}{=}\alpha_{2i}{+}\sum_{j=1}^{m^*}\delta_{21ji}v_{jt}{+}\sum_{j=2}^{s}\delta_{22ji}d_{jt}{+}\gamma_{21i}y_{it-1}{+}\gamma_{22i}R_{mt-1}{+}\gamma_{23i}R_{ft-1}{+}u_{2it}, \tag{4.4}$$

$$R_{ft}{=}\alpha_{3i}{+}\sum_{j=1}^{m^*}\delta_{31ji}v_{jt}{+}\sum_{j=2}^{s}\delta_{32ji}d_{jt}{+}\gamma_{31i}y_{it-1}{+}\gamma_{32i}R_{mt-1}{+}\gamma_{33i}R_{ft-1}{+}u_{3it}, \tag{4.5}$$

for

$$(u_{jit}|\mathbf{Z}_{it-1}) \backsim \mathrm{N}(0,\sigma_i^2(t)),\ \mathbf{Z}_{it}{:=}(y_{it},R_{mt},R_{ft}),$$

$$j{=}\{1,2,3\},\ i{=}1,2,...,k,\ t{\in}\mathbb{N},$$

where y_{it} are the monthly returns of portfolio i for period t; v_{jt} denotes the terms of the Gram-Schmidt orthonormal polynomials of order $j{=}1, 2, ..., m^{*}$; $d_{jt}{:=}(d_{2t}, d_{3t}, ..., d_{12t})$ are the monthly dummy variables for the months of February through December; and $\sigma_i^2(t)$ is homoskedastic and heterogeneous, stemming from the Multivariate Normal distribution.

To appraise the statistical adequacy of the heterogeneous Normal VAR (1) model, analogous M-S testing as in chapter 3 is applied.

4.2.1 M-S Testing

To evaluate the model assumptions [1]–[5] that justify the heterogeneous Normal/ Homoskedastic VAR (1), the estimated residuals of the three equations in (4.3)-(4.5) are carefully examined. The Shapiro-Wilk (W), Anderson-Darling (A^2), and D'Agostino-Pearson (K^2) tests (see Appendix C) are employed to test for [1] Normality, and the auxiliary regressions (F) in (3.10)-(3.11) are extended to test for the model assumptions of [2] Linearity, [3] Homoskedasticity, [4] Markov (1), and [5] t-invariance. The extended auxiliary regressions take the following form:

$$\widehat{u}_{it}{=}\gamma_{1i} + \boldsymbol{\gamma}_{2i}^{\top}\boldsymbol{\Xi}_{it}^{*} + \underbrace{\sum\nolimits_{j=m^{*}+1}^{m} \gamma_{3ji}v_{jt}}_{\overline{[5.1]}} + \underbrace{\boldsymbol{\gamma}_{4i}^{\top}\boldsymbol{\psi}_{t}}_{\overline{[2]}} + \underbrace{\sum\nolimits_{j=2}^{p} \boldsymbol{\gamma}_{5ji}^{\top}\mathbf{Z}_{it-j}}_{\overline{[4]}} + \varepsilon_{1it}, \tag{4.6}$$

$$\widehat{u}_{it}^{2}{=}\gamma_{6i} + \underbrace{\sum\nolimits_{j=1}^{m} \gamma_{7ji}v_{jt}}_{\overline{[5.3]}} + \underbrace{\boldsymbol{\gamma}_{8i}^{\top}\mathbf{Z}_{it-1} + \boldsymbol{\gamma}_{9i}^{\top}\boldsymbol{\psi}_{t} + \sum\nolimits_{j=2}^{p} \boldsymbol{\gamma}_{10ji}^{\top}\mathbf{Z}_{it-j}^{2}}_{\overline{[3]}+\overline{[4]}} + \varepsilon_{2it}, \tag{4.7}$$

where $\widehat{u}_{it}$ are the estimated residuals in (4.3)-(4.5) of portfolio i for period t; and let $\boldsymbol{\Xi}_{it}^{*}{:=}(v_{1t}, v_{2t}, ..., v_{mt}^{*}, d_{2t}, d_{3t}, ..., d_{12t}, \mathbf{Z}_{it-1})$ to denote the collection of all the deterministic and lagged variables included in the models.

The M-S testing results for the heterogeneous Normal/ Homoskedastic VAR (1) model are presented in Appendix F. Table 4.3 includes a representative summary of these results.

The heterogeneous Normal/ Homoskedastic VAR (1) model appears to account for the temporal dependence and heterogeneity in the data. The M-S testing results indicate that the [2] Linearity, [4] Markov (1), and [5] t-invariance assumptions are valid for the data in question. Nevertheless, the assumptions of [1] Normality and [3] Homoskedasticity are seriously violated. Notably, these assumptions are invalid for all the 25 *Size-B/M* portfolios.

Table 4.3 - Representative M-S Testing Results for heterogeneous Normal VAR (1) Model

Assumption	y_{it}	R_{mt}	R_{ft}
[1] Normality	$\times$	$\times$	$\times$
[2] Linearity	✓	✓	✓
[3] Homoskedasticity	$\times$	$\times$	$\times$
[4] Markov (1)	✓	✓	✓
[5] t-invariance:			
[5.1] Mean Heterogeneity	✓	✓	✓
[5.3] Variance Heterogeneity	✓	✓	✓

Although on statistical adequacy grounds the heterogeneous Normal VAR (1) model is an improvement over the original Multivariate Normal LR model, it is not statistically adequate. However, the M-S testing results provide us with helpful information with regard to the next attempt at respecification. Specifically, the clear departures from the Normality and homoskedasticity assumptions, suggest that the best way forward is to consider another member of the elliptically symmetric family of distributions.

4.3 Elliptically Symmetric Family of Distributions

The distributions within the Elliptically Symmetric (ES) family are extensions/ generalizations of the Multivariate Normal distribution and play a central role in the joint modeling of asset returns. There is a large literature proposing several ES distributions due to the heavy-tailed and highly peaked nature of the statistical distribution of asset returns; see section 1.4 for some of the best-known alternatives to the Normal distribution. This family of distributions was formally introduced by Kelker (1970), yet its origins can be traced back to Maxwell (1860), Bartlett (1934), and Hartman and Wintner (1940); for a review and bibliography of this family of distributions, see Chmielewski (1981). For the discussion below, as well as a more detailed discussion on the ES family of distributions, see Fang et al. (1990).

A random vector $\mathbf{Z}{:=}(Z_1, Z_2, ..., Z_n)^\top$ is said to have an *elliptically symmetric* distribution, denoted by:

$$\mathbf{Z} \backsim \mathsf{ES}_n\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi\right), \tag{4.8}$$

if $\mathbf{Z}$ has a stochastic representation and if its *characteristic function* takes the following form:

$$\psi_{\mathbf{z}}(\mathbf{t}){=}E(e^{i\mathbf{t}^\top\mathbf{z}}){=}e^{i\mathbf{t}^\top\boldsymbol{\mu}}\phi(\mathbf{t}^\top\boldsymbol{\Sigma}\mathbf{t}), \tag{4.9}$$

where $\boldsymbol{\mu}$ is an $(n\times 1)$ mean vector, $\boldsymbol{\Sigma}$ is an $(n\times n)$ positive definite covariance matrix, $i{=}\sqrt{-1}$, and ϕ is some scalar function called the *characteristic generator.*

The first four *moments* of $\mathbf{Z}$ (if exist) are as follows:

$$E(\mathbf{Z}){=}\boldsymbol{\mu}, \tag{4.10}$$

$$Cov(\mathbf{Z}){=}\lambda\boldsymbol{\Sigma},\ \text{for}\ \lambda{=}-2\phi'(0),\ \phi'(0){=}\left.\frac{\partial\phi(\mathbf{t}^\top\boldsymbol{\Sigma}\mathbf{t})}{\partial\mathbf{t}}\right|_{\mathbf{t}=\mathbf{0}}, \tag{4.11}$$

$$Skew(\mathbf{Z}){=}0, \tag{4.12}$$

$$Kurt(\mathbf{Z}){=}\frac{\phi''(0)}{(\phi'(0))^2}-1. \tag{4.13}$$

It is important to notice that the ES family includes both leptokurtic (>3) and platykurtic (<3) distributions; in addition to the Normal, a mesokurtic (=3) distribution.

The joint *probability density function* (pdf) of $\mathbf{Z}$ (if exists) takes the generic form:

$$f_{\mathbf{z}}(\mathbf{z})=\frac{C_n}{\sqrt{|\boldsymbol{\Sigma}|}}h_n\left((\mathbf{z}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right), \tag{4.14}$$

where $|\cdot|$ denotes the determinant, $h_n(.)$ is a non-negative function called the *density generator*, and $C_n>0$ is a normalizing constant expressed as:

$$C_n=\frac{\Gamma(n/2)}{(2\pi)^{n/2}}\left[\int_0^\infty u^{n-1}h_n(u^2)du\right]^{-1}, \text{ for } u>0. \tag{4.15}$$

Let $\mathbf{Z}\backsim \mathsf{ES}_n\left(\boldsymbol{\mu},\boldsymbol{\Sigma},\,\phi\right)$ with $\boldsymbol{\Sigma}>\mathbf{0}$, partitioned as follows:

$$\begin{pmatrix}\mathbf{Z}_1\\ \mathbf{Z}_2\end{pmatrix}\backsim \mathsf{ES}_n\left[\begin{pmatrix}\boldsymbol{\mu}_1\\ \boldsymbol{\mu}_2\end{pmatrix},\begin{pmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12}\\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{pmatrix},\phi\right], \tag{4.16}$$

where $\mathbf{Z}^\top:=\left(\mathbf{Z}_1^\top,\mathbf{Z}_2^\top\right)$ have dimensions $(d_1\times 1)$ and $(d_2\times 1)$, respectively, for $d_1+d_2=n$.

Then the *marginal distribution*, which is also elliptical is given by (w.l.o.g):

$$f_{\mathbf{Z}_1}(\mathbf{Z}_1)\backsim \mathsf{ES}_{d1}(\boldsymbol{\mu}_1,\boldsymbol{\Sigma}_{11},\ \phi), \tag{4.17}$$

and the *conditional distribution* is given by (w.l.o.g.):

$$f_{\mathbf{Z}_1|\mathbf{Z}_2}(\mathbf{Z}_1|\mathbf{Z}_2)\backsim \mathsf{ES}_{d1}(\boldsymbol{\mu}_{\mathbf{Z}_1|\mathbf{Z}_2}(\mathbf{Z}_1|\mathbf{Z}_2),\Sigma_{\mathbf{Z}_1|\mathbf{Z}_2}(\mathbf{Z}_1|\mathbf{Z}_2),\ \phi_{\mathbf{Z}_1|\mathbf{Z}_2}(\mathbf{Z}_1|\mathbf{Z}_2)), \tag{4.18}$$

for

$$\phi_{\mathbf{Z}_1|\mathbf{Z}_2}(\mathbf{Z}_1|\mathbf{Z}_2)=\phi_{d2}, \tag{4.19}$$

$$\boldsymbol{\mu}_{\mathbf{Z}_1|\mathbf{Z}_2}(\mathbf{Z}_1|\mathbf{Z}_2)=E(\mathbf{Z}_1|\mathbf{Z}_2)=\boldsymbol{\mu}_1+\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{Z}_2-\boldsymbol{\mu}_2), \tag{4.20}$$

$$\Sigma_{\mathbf{Z}_1|\mathbf{Z}_2}(\mathbf{Z}_1|\mathbf{Z}_2)=Var(\mathbf{Z}_1|\mathbf{Z}_2)=g_{d1}(\mathbf{Z}_2)\left(\boldsymbol{\Sigma}_{11}-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right), \tag{4.21}$$

where $g_{d1}(.)$ is a non-negative function determined by $h_n(.)$.

Theorem 4.1 ***(Kelker, 1970)*** *Suppose* $\mathbf{Z}\backsim \mathit{ES}_n\left(\boldsymbol{\mu},\boldsymbol{\Sigma},\,\phi\right)$, *with* $\mathbf{Z}^\top:=\left(\mathbf{Z}_1^\top,\mathbf{Z}_2^\top\right)$ *and with* $\mathbf{Z}_2$ *a* d_2*-component vector. The conditional covariance matrix of* $\mathbf{Z}_1$ *given* $\mathbf{Z}_2=\mathbf{z}_2$ *is independent of* $\mathbf{Z}_2$ *only if* $\mathbf{Z}$ *has a Normal distribution.*

The theorem implies that Normal is the only member of the ES family whose conditional variance in (4.21) is characterized by homoskedasticity, i.e., w.l.o.g., $g_{d1}(\mathbf{Z}_2)=1$. This is in contrast to the rest of the ES distributions which have heteroskedastic conditional variances; see Kelker (1970) for the formal proof.

A brief summary of the most widely known ES distributions is given below; for additional information, see Fang et al. (1990).

4.3.1 Multivariate Kotz Family

A random vector $\mathbf{Z}:=(Z_1, Z_2, ..., Z_n)^\top$ belongs to the family of Multivariate Kotz distributions, denoted by $\mathbf{Z} \backsim \mathsf{K}_n\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}; r, s, N\right)$, if its joint probability density function (pdf) takes the form:

$$f_{\mathbf{Z}}(\mathbf{z})=\tfrac{C_n}{\sqrt{|\boldsymbol{\Sigma}|}}\left[(\mathbf{z}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right]^{N-1}\exp\left\{-r\left[(\mathbf{z}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right]^{s}\right\}, \quad (4.22)$$

where $r, s>0,\ N>(2-n)/2$, and

$$C_n=\tfrac{s\Gamma(n/2)}{\pi^{n/2}\Gamma((2N+n-2)/2s)}r^{(2N+n-2)/2s}, \quad (4.23)$$

is the normalizing constant obtained using (4.15).

The joint pdf of the Multivariate Kotz family in (4.22) is general enough that it includes some families of distributions as special cases. For instance, when $s=1$, the joint pdf in (4.22) reduces to the original Kotz distribution; see Kotz (1975). The special cases of the Multivariate Normal and Multivariate Exponential Power family are briefly described below.

4.3.2 Multivariate Normal

When $N=s=1$ and $r=1/2$, the joint pdf in (4.22) reduces to the Multivariate Normal distribution. A random vector $\mathbf{Z}:=(Z_1, Z_2, ..., Z_n)^\top$ is said to have a Multivariate

Normal distribution, denoted by $\mathbf{Z} \backsim \mathsf{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, if its joint pdf takes the form:

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{C_n}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\left\{-\tfrac{1}{2}(\mathbf{z}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right\}, \tag{4.24}$$

where

$$C_n = (2\pi)^{-n/2}, \tag{4.25}$$

is the normalizing constant obtained using (4.15).

4.3.3 Multivariate Exponential Power Family

When $N=1$, the joint pdf in (4.22) reduces to the family of Multivariate Exponential Power distributions. A random vector $\mathbf{Z} := (Z_1, Z_2, ..., Z_n)^\top$ belongs to the family of Multivariate Exponential Power distributions, denoted by $\mathbf{Z} \backsim \mathsf{EP}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}; r, s)$, if its joint pdf takes the form:

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{C_n}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\left\{-r\left[(\mathbf{z}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right]^s\right\}, \tag{4.26}$$

where $r, s > 0$, and

$$C_n = \frac{s\Gamma(n/2)}{\pi^{n/2}\Gamma(n/2s)} r^{n/2s}, \tag{4.27}$$

is the normalizing constant obtained using (4.15).

4.3.4 Multivariate Pearson Type II Family

A random vector $\mathbf{Z} := (Z_1, Z_2, ..., Z_n)^\top$ belongs to the family of Multivariate Pearson Type II distributions, denoted by $\mathbf{Z} \backsim \mathsf{PII}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}; m)$, if its joint pdf takes the form:

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{C_n}{\sqrt{|\boldsymbol{\Sigma}|}} \left[1 - (\mathbf{z}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right]^m, \tag{4.28}$$

where $0 \leq \left[(\mathbf{z}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right] \leq 1$, $m > -1$, and

$$C_n = \frac{\Gamma(n/2+m+1)}{\pi^{n/2}\Gamma(m+1)}, \tag{4.29}$$

is the normalizing constant obtained using (4.15).

4.3.5 Multivariate Pearson Type VII Family

A random vector $\mathbf{Z}{:=}(Z_1, Z_2, ..., Z_n)^\top$ belongs to the family of Multivariate Pearson Type VII distributions, denoted by $\mathbf{Z} \backsim \mathsf{PVII}_n\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}; p, v\right)$, if its joint pdf takes the form:

$$f_{\mathbf{Z}}(\mathbf{z}){=}\tfrac{C_n}{\sqrt{|\boldsymbol{\Sigma}|}}\left[1+\tfrac{1}{\nu}(\mathbf{z}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right]^{-p}, \tag{4.30}$$

where $p{>}n/2$, $\nu{>}0$ denotes the degrees of freedom (dfs), and

$$C_n{=}\tfrac{\Gamma(p)}{(\pi v)^{n/2}\Gamma(p-n/2)}, \tag{4.31}$$

is the normalizing constant obtained using (4.15).

The joint pdf of the Multivariate Pearson Type VII family in (4.30) is general enough that it includes some families of distributions as special cases. The special cases of the Multivariate Student's t family and Multivariate Cauchy are briefly described below.

4.3.6 Multivariate Student's t Family

When $p{=}(n+\nu)/2$, the joint pdf in (4.30) reduces to the family of Multivariate Student's t distributions. A random vector $\mathbf{Z}{:=}(Z_1, Z_2, ..., Z_n)^\top$ belongs to the family of Multivariate Student's t distributions, denoted by $\mathbf{Z} \backsim \mathsf{St}_n\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}; \nu\right)$, if its joint pdf takes the form:

$$f_{\mathbf{Z}}(\mathbf{z}){=}\tfrac{C_n}{\sqrt{|\boldsymbol{\Sigma}|}}\left[1+\tfrac{1}{\nu}(\mathbf{z}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right]^{-(n+\nu)/2}, \tag{4.32}$$

where $\nu{>}0$ denotes the dfs, and

$$C_n{=}\tfrac{\Gamma((n+\nu)/2)}{(\pi\nu)^{n/2}\Gamma(\nu/2)}, \tag{4.33}$$

is the normalizing constant obtained using (4.15).

4.3.7 Multivariate Cauchy

When $p{=}(n+1)/2$, the joint pdf in (4.30) reduces to the Multivariate Cauchy distribution. However, the moments of this distribution do not exist. A random vector $\mathbf{Z}{:=}(Z_1, Z_2, ..., Z_n)^\top$ is said to have a Multivariate Cauchy distribution, denoted by $\mathbf{Z} \backsim \mathsf{C}_n\left(\mathbf{0}, \mathbf{I}\right)$, if its joint pdf takes the form:

$$f_{\mathbf{z}}(\mathbf{z}){=}C_n\left[1+\mathbf{z}^\top\mathbf{z}\right]^{-(n+1)/2}, \tag{4.34}$$

where

$$C_n{=}\tfrac{\Gamma((n+1)/2)}{n^{(n+1)/2}}, \tag{4.35}$$

is the normalizing constant obtained using (4.15).

Alternatively, the Multivariate Cauchy distribution can be viewed as a special case of the Multivariate Student's t family with df of $\nu{=}1$.

4.3.8 Multivariate Bessel Family

A random vector $\mathbf{Z}{:=}(Z_1, Z_2, ..., Z_n)^\top$ belongs to the family of Multivariate Bessel distributions, denoted by $\mathbf{Z} \backsim \mathsf{B}_n\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}; \alpha, \beta\right)$, if its joint pdf takes the form:

$$f_{\mathbf{z}}(\mathbf{z}){=}\tfrac{C_n}{\sqrt{|\boldsymbol{\Sigma}|}}\left(\tfrac{1}{\beta}\left[(\mathbf{z}{-}\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{z}{-}\boldsymbol{\mu})\right]^{1/2}\right)^{\alpha}K_a\left(\tfrac{1}{\beta}\left[(\mathbf{z}{-}\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{z}{-}\boldsymbol{\mu})\right]^{1/2}\right), \tag{4.36}$$

where $\alpha{>}-n/2$, $\beta{>}0$, $K_a(\cdot)$ denotes the modified Bessel function of the third kind:

$$\begin{aligned} &K_a(y){=}\tfrac{\pi}{2}\tfrac{I_{-\alpha}(y)-I_{\alpha}(y)}{\sin(\alpha\pi)}, \quad |\arg(y)|{<}\pi, \quad \alpha{\in}\mathbb{Z}, \\ &\text{for } \; I_{\alpha}(y)=\sum_{k=0}^{\infty}\tfrac{1}{k!\Gamma(k+\alpha+1)}\left(\tfrac{y}{2}\right)^{\alpha+2k}, \quad |y|{<}\infty, \quad |\arg(y)|{<}\pi, \end{aligned} \tag{4.37}$$

and

$$C_n{=}\left[2^{\alpha+n-1}\pi^{n/2}\beta^n\Gamma(\alpha+n/2)\right]^{-1}, \tag{4.38}$$

is the normalizing constant obtained using (4.15).

The joint pdf of the Multivariate Bessel family in (4.36) is general enough that it includes some families of distributions as special cases. The special case of the Multivariate Laplace family is briefly described below.

4.3.9 Multivariate Laplace Family

When α=0 and β=$\sigma/\sqrt{2}$ the joint pdf in (4.36) reduces to the family of Multivariate Laplace distributions. A random vector $\mathbf{Z}$:=$(Z_1, Z_2, ..., Z_n)^\top$ belongs to the family of Multivariate Laplace distributions, denoted by $\mathbf{Z} \backsim \mathsf{Lap}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}; \sigma)$, if its joint pdf takes the form:

$$f_{\mathbf{z}}(\mathbf{z}) = \tfrac{C_n}{\sqrt{|\boldsymbol{\Sigma}|}} K_0\left(\tfrac{1}{\sigma}\left[2(\mathbf{z}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right]^{1/2}\right), \tag{4.39}$$

where σ>0, $K_0(\cdot)$ denotes the modified Bessel function in (4.37), for α=0, and

$$C_n = \left[2^{n/2-1}\pi^{n/2}\sigma^n\Gamma(n/2)\right]^{-1}, \tag{4.40}$$

is the normalizing constant obtained using (4.15).

4.3.10 Multivariate Logistic

A random vector $\mathbf{Z}$:=$(Z_1, Z_2, ..., Z_n)^\top$ is said to have a Multivariate Logistic distribution, denoted by $\mathbf{Z} \backsim \mathsf{Log}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, if its joint pdf takes the form:

$$f_{\mathbf{z}}(\mathbf{z}) = \tfrac{C_n}{\sqrt{|\boldsymbol{\Sigma}|}} \frac{\exp\left\{-(\mathbf{z}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right\}}{\left[1+\exp\left\{-(\mathbf{z}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right\}\right]^2}, \tag{4.41}$$

where $\left[(\mathbf{z}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right] \geq 0$, and

$$C_n = \tfrac{\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty y^{n/2-1} \tfrac{e^{-y}}{(1+e^{-y})^2} dy, \tag{4.42}$$

is the normalizing constant obtained using (4.15).

4.3.11 Which One is Most Appropriate?

In light of the clear departures from the Normality and homoskedasticity assumptions reported in section 4.2, the primary aim at this stage is to replace the Normal distribution with another member of the ES family of distributions. Given that all the members of the ES family are symmetric and have heteroskedastic conditional variance (except the Normal), special emphasis is placed upon the shape characteristics, and particularly the peakedness in relation to the tail heaviness of these distributions. As to observe the shape characteristics of the most common ES distributions, sophisticated graphical techniques are used.

First, data is simulated from these distributions and their smoothed density functions are compared to the theoretical Normal pdf; see figures 4.1–4.6. Thereafter, the smoothed histograms of the estimated residuals in (4.3)-(4.5) are compared to the smoothed density functions of the candidate distributions. In addition, standardized p-p plots are used to compare the empirical cumulative distribution functions (cdf) of the estimated residuals in (4.3)-(4.5) to the theoretical cdf of the candidate distributions. The density function and cdf of the leptokurtic Student's t are the ones closest to the smoothed histograms and empirical cdf of the estimated residuals. Thus, the distribution that suggests itself is the Student's t distribution.

Of particular interest is the conditional Student's t distribution (w.l.o.g.):

$$f_{\mathbf{Z}_1|\mathbf{Z}_2}(\mathbf{Z}_1|\mathbf{Z}_2)\backsim \mathsf{St}_{d1}(\mathbf{Z}_1; E(\mathbf{Z}_1|\mathbf{Z}_2), Var(\mathbf{Z}_1|\mathbf{Z}_2), \nu(\mathbf{Z}_1|\mathbf{Z}_2)), \tag{4.43}$$

where the dfs and the first two moments of the conditional distribution take the form:

$$\nu(\mathbf{Z}_1|\mathbf{Z}_2)=\nu + d_2, \tag{4.44}$$

$$E(\mathbf{Z}_1|\mathbf{Z}_2)=\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{Z}_2 - \boldsymbol{\mu}_2), \tag{4.45}$$

$$Var(\mathbf{Z}_1|\mathbf{Z}_2)=\tfrac{\nu+(\mathbf{Z}_2-\boldsymbol{\mu}_2)^{\top}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{Z}_2-\boldsymbol{\mu}_2)}{\nu+d_2}\left(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right). \tag{4.46}$$

The first conditional moment of Student's t in (4.45) is of the same form as of the Normal, but the second conditional moment in (4.46) is heteroskedastic in contrast to the homoskedastic of the Normal. Besides, it is worth to mention that when the dfs in (4.44) approach infinity, the Student's t reduces to the Normal distribution, and the conditional variance in (4.46) becomes homoskedastic.

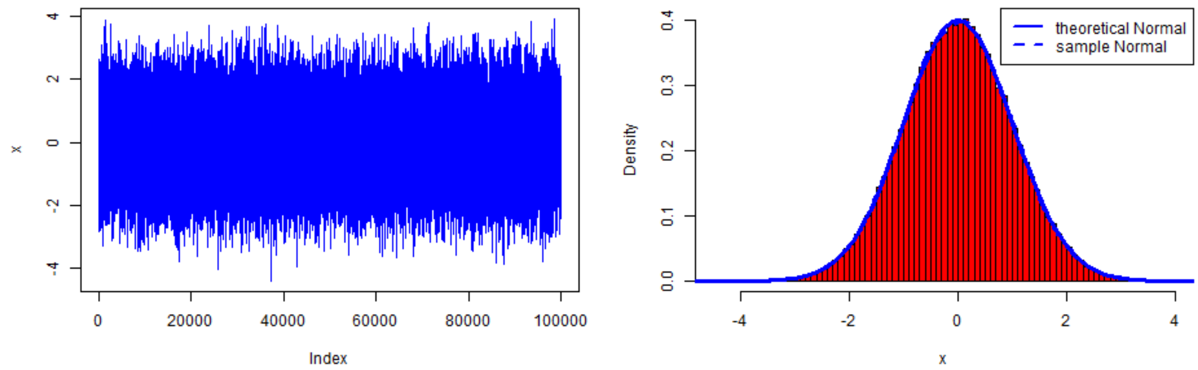

Figure 4.1: t-plot and smoothed histogram of simulated Normal IID data

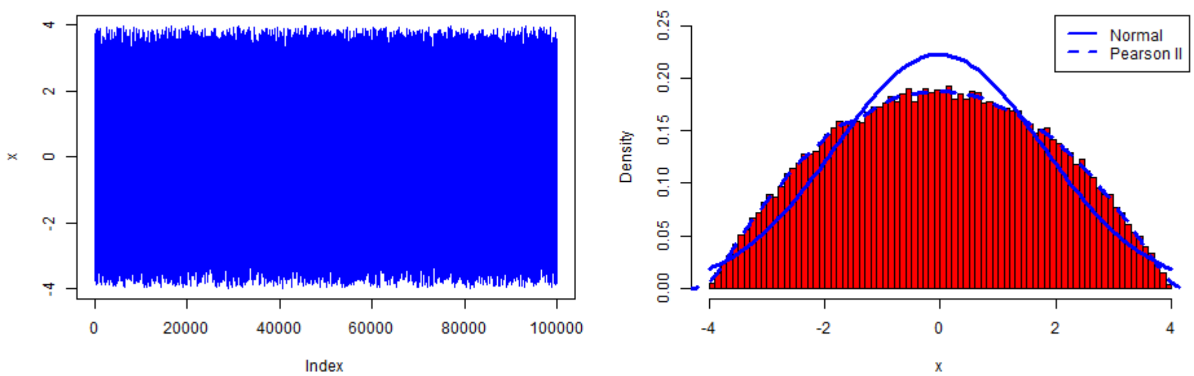

Figure 4.2: t-plot and smoothed histogram of simulated Pearson Type II IID data

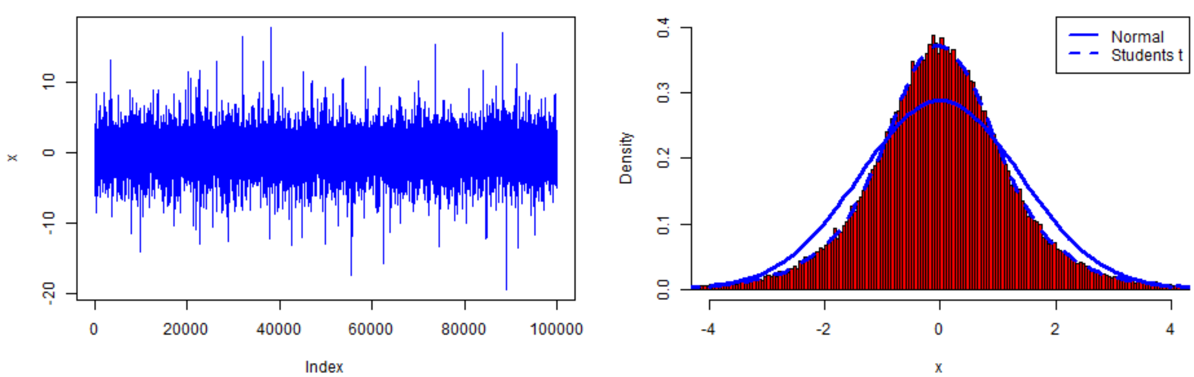

Figure 4.3: t-plot and smoothed histogram of simulated Student's t IID data

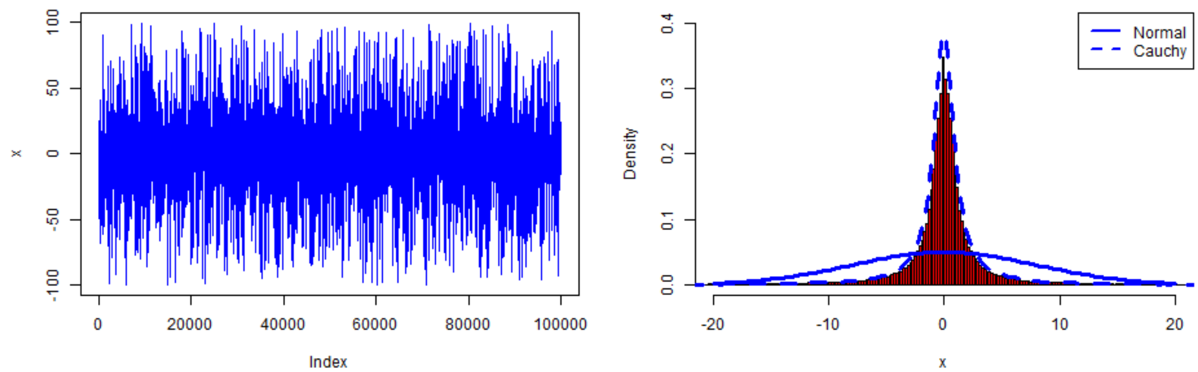

Figure 4.4: t-plot and smoothed histogram of simulated Cauchy IID data

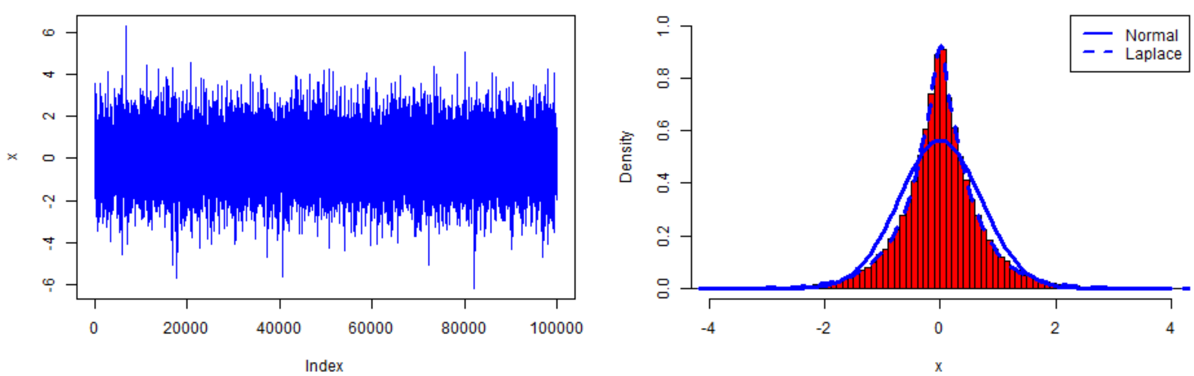

Figure 4.5: t-plot and smoothed histogram of simulated Laplace IID data

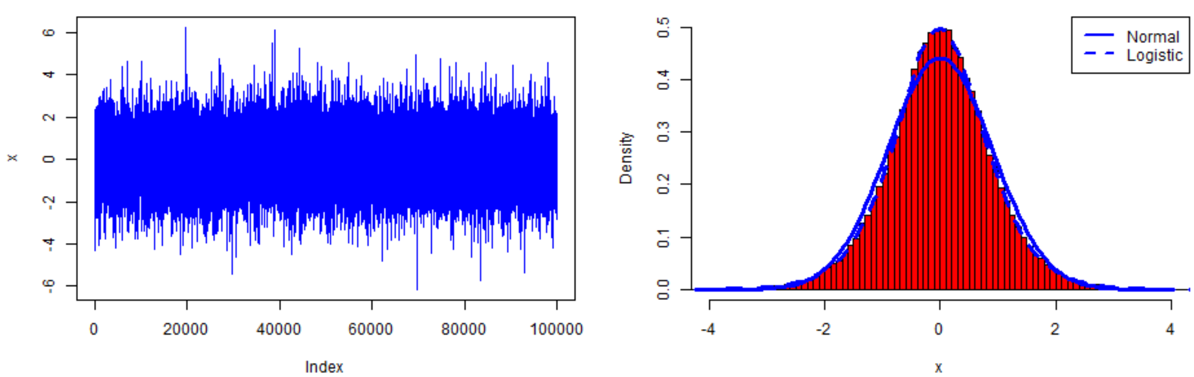

Figure 4.6: t-plot and smoothed histogram of simulated Logistic IID data

4.4 Second Attempt at Respecification: Student's t VAR

The M-S testing results in section 4.2 guided light for a second attempt at respecification. At this stage, the aim is to respecify a more appropriate probabilistic structure compared to the Normal VAR (1) model, with a view to account for all the departures indicated by the M-S testing. Following the discussion of the ES family of distributions, the reduction assumption of Student's t supplemental to the reduction assumptions of Markov (1) and Stationarity, gives rise to the Student's t Vector Autoregressive of order 1 with ν dfs (henceforth VAR $(1;\nu)$) model in table 4.4.

Table 4.4 - Student's t Vector Autoregressive [VAR $(1;\nu)$] model

$$\boxed{\text{Statistical GM:} \quad \mathbf{Z}_t{=}\boldsymbol{\alpha}+\boldsymbol{\Gamma}^{\top}\mathbf{Z}_{t-1}+\mathbf{u}_t,\ t{\in}\mathbb{N}.}$$

[1] Student's t:	$D(\mathbf{Z}_t \vert \boldsymbol{\sigma}(\mathbf{Z}^0_{t-1});\nu,\boldsymbol{\theta})$, is Student's t with ν dfs, where $\mathbf{Z}_t{:}(l{\times}1)$, $\mathbf{Z}^0_{t-1}{:=}(\mathbf{Z}_{t-1},\mathbf{Z}_{t-2},...,\mathbf{Z}_1)$,
[2] Linearity:	$E(\mathbf{Z}_t \vert \boldsymbol{\sigma}(\mathbf{Z}^0_{t-1})){=}\boldsymbol{\alpha}+\boldsymbol{\Gamma}^{\top}\mathbf{Z}_{t-1}$, is linear in $\mathbf{Z}_{t-1}$,
[3] Heteroskedasticity:	$Var(\mathbf{Z}_t \vert \boldsymbol{\sigma}(\mathbf{Z}^0_{t-1})){=}\left(\frac{\nu}{\nu+l-2}\right)\boldsymbol{\Omega}\left(1+\frac{2\varrho^2}{\nu}\right)$, $\varrho^2{=}\frac{1}{2}(\mathbf{Z}_{t-1}{-}\boldsymbol{\mu})^{\top}\boldsymbol{\Sigma}_0^{-1}(\mathbf{Z}_{t-1}{-}\boldsymbol{\mu})$, is not free of $\mathbf{Z}^0_{t-1}$,
[4] Markov (1):	$\{\mathbf{Z}_t,\ t{\in}\mathbb{N}\}$ is a Markov process of order 1,
[5] t-invariance:	$\boldsymbol{\theta}{:=}(\boldsymbol{\alpha},\boldsymbol{\Gamma},\boldsymbol{\mu},\boldsymbol{\Omega},\boldsymbol{\Sigma}_0)$ are constant over t.

The probabilistic reduction that gives rise to the Student's t VAR $(1;\nu)$ model takes the same form as in (4.1), where $D(\mathbf{Z}_t|\mathbf{Z}_{t-1};\boldsymbol{\theta}_2)$ denotes the conditional Student's t distribution. Moreover, the reduction in (4.2) gives rise to the Student's t/ Heteroskedastic Dynamic LR model in table 4.5, specified in terms of $D(y_t|\mathbf{X}_t,\mathbf{Z}_{t-1};\boldsymbol{\theta}^1_2)$.

The heterogeneous form of the Student's t/ Heteroskedastic VAR $(1;\nu)$ model is equivalent to the system of the three Student's t equations in (4.3)-(4.5), for:

$$(u_{jit}|\mathbf{Z}_{it-1}) \backsim \text{St}(0,\sigma_i^2(t)),\ \mathbf{Z}_{it}{:=}(y_{it},R_{mt},R_{ft}),$$
$$j{=}\{1,2,3\},\ i{=}1,2,...,k,\ t{\in}\mathbb{N},$$

where $\sigma_i^2(t)$ is both heteroskedastic and heterogeneous, stemming from the Multivariate Student's t distribution.

Table 4.5 - Student's t/Heteroskedastic Dynamic LR (1;ν) model

Statistical GM:	$y_t{=}\alpha + \boldsymbol{\beta}^\top \mathbf{X}_t + \boldsymbol{\gamma}^\top \mathbf{Z}_{t-1} + u_t,\ t{\in}\mathbb{N}.$
[1] Student's t:	$D(y_t \vert \boldsymbol{\sigma}(\mathbf{X}_t, \mathbf{Z}_{t-1}^0); \nu, \boldsymbol{\theta})$, is Student's t with ν dfs, where $\mathbf{Z}_{t-1}^0{:=}(\mathbf{Z}_{t-1}, \mathbf{Z}_{t-2}, ..., \mathbf{Z}_1)$,
[2] Linearity:	$E(y_t \vert \boldsymbol{\sigma}(\mathbf{X}_t, \mathbf{Z}_{t-1}^0)){=}\alpha{+}\boldsymbol{\beta}^\top \mathbf{X}_t{+}\boldsymbol{\gamma}^\top \mathbf{Z}_{t-1}$, is linear in $(\mathbf{X}_t, \mathbf{Z}_{t-1})$, for $\mathbf{Z}_t{:=}(y_t, \mathbf{X}_t)$, where $y_t{:}(1{\times}1),\ \mathbf{X}_t{:}(l{\times}1),\ \mathbf{Z}_t{:}((l{+}1)\times 1)$,
[3] Heterosked.:	$Var(y_t \vert \boldsymbol{\sigma}(\mathbf{X}_t, \mathbf{Z}_{t-1}^0)){=}\left(\frac{\nu\sigma^2}{\nu+l^*-2}\right)\left(1+\frac{2\varrho^2}{\nu}\right),\ l^*{=}2l{+}1,$ $\varrho^2{=}\frac{1}{2}\left[(\mathbf{X}_t{-}\boldsymbol{\mu}_{\mathbf{x}})^\top \mathbf{Q}_1^{-1}(\mathbf{X}_t{-}\boldsymbol{\mu}_{\mathbf{x}}) + (\mathbf{Z}_{t-1}{-}\boldsymbol{\mu})^\top \mathbf{Q}_2^{-1}(\mathbf{Z}_{t-1}{-}\boldsymbol{\mu})\right],$
[4] Markov (1):	$\{\mathbf{Z}_t,\ t{\in}\mathbb{N}\}$ is a Markov process of order 1,
[5] t-invariance:	$\boldsymbol{\theta}{:=}(\alpha, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\mu}, \sigma^2, \mathbf{Q}_1, \mathbf{Q}_2)$ are constant over t.

The probabilistic structures of the Normal VAR (1) and Student's t VAR $(1;\nu)$ models are identical in terms of assumptions [2] and [4]–[5]. The two models have linear autoregressive functions with the same parameterization and share the same Markov (1) and stationarity assumptions. Nonetheless, the probabilistic structure of the Student's t VAR $(1;\nu)$ differs from that of the Normal VAR (1) model with respect to assumptions [1] and [3]. In contrast to the mesokurtic Normal distribution, the Student's t is a leptokurtic distribution that accounts for higher concentration around the mean and allows for fatter tails; see figure 4.3. In addition, the heteroskedastic autoskedastic function of the Student's t VAR $(1;\nu)$ is a quadratic and dynamic function of the conditioning variables, in contrast to the homoskedastic/static autoskedastic function of the Normal VAR (1) model.

The autoskedastic function of the Student's t VAR $(1;\nu)$ is expected to capture the second order temporal dependence (dynamic heteroskedasticity) in the data, not captured by the autoskedastic function of the Normal VAR (1) model. The second order temporal dependence is a common stylized fact of financial returns, observed as extended periods of high volatility followed by extended periods of low volatility, and vice versa. Besides, the heterogeneous form of the autoskedastic function of the heterogeneous Student's t VAR $(1;\nu)$ model is also expected to capture the variance heterogeneity in the data, even though it is captured to a high degree by the heterogeneous Normal VAR (1) model. Equally, variance heterogeneity is another common stylized fact of financial returns, observed in the form of asymmetrical second order temporal dependence, i.e., recessions differ in magnitude over time.

As might be expected, the heterogeneous Student's t VAR $(1;\nu)$ model will account for all the statistical systematic information in the data. Yet, it is important to keep in mind that M-S testing is highly vulnerable to the fallacy of rejection. For this reason, it is of great importance to further evaluate the probabilistic assumptions underlying the respecified model.

4.4.1 M-S Testing

To evaluate the model assumptions [1]–[5] that justify the heterogeneous Student's t VAR $(1;\nu)$, the estimated standardized residuals of the three Student's t equations in (4.3)-(4.5) are carefully examined. The Kolmogorov-Smirnov (D), and Anderson-Darling (A^2) tests (see Appendix E) are employed to test for [1] Student's t, and similar auxiliary regressions (F) as in (4.6)-(4.7) are used to test for the model assumptions of [2] Linearity, [3] Heteroskedasticity, [4] Markov (1), and [5] t-invariance. Formally, the two auxiliary regressions take the form:

$$(\widehat{u}_{it})^{st} = \gamma_{1i} + \boldsymbol{\gamma}_{2i}^{\top}\boldsymbol{\Xi}_{it}^{*} + \underbrace{\sum\nolimits_{j=m^*+1}^{m} \gamma_{3ji} v_{jt}}_{[5.1]} + \underbrace{\boldsymbol{\gamma}_{4i}^{\top}\boldsymbol{\psi}_t}_{[2]} + \underbrace{\sum\nolimits_{j=2}^{p} \boldsymbol{\gamma}_{5ji}^{\top}\mathbf{Z}_{it-j}}_{[4]} + \varepsilon_{1it}, \tag{4.47}$$

$$(\widehat{u}_{it}^2)^{st} = \gamma_{6i} + \underbrace{\sum\nolimits_{j=1}^{m} \gamma_{7ji} v_{jt}}_{[5.3]} + \underbrace{\boldsymbol{\gamma}_{8i}^{\top}\mathbf{Z}_{it-1} + \boldsymbol{\gamma}_{9i}^{\top}\boldsymbol{\psi}_t + \sum\nolimits_{j=2}^{p} \boldsymbol{\gamma}_{10ji}^{\top}\mathbf{Z}_{it-j}^2}_{[3]+[4]} + \varepsilon_{2it}, \tag{4.48}$$

where $(\widehat{u}_{it})^{st}$ denote the estimated standardized residuals of portfolio i for period t.

To construct the standardized residuals:

$$(\widehat{\mathbf{u}}_{it})^{st} = \mathbf{L}_{it}^{-1}(\mathbf{Z}_{it} - \widehat{\mathbf{Z}}_{it}),\ \mathbf{L}_{it}\mathbf{L}_{it}^{\top} = \widehat{Var}(\mathbf{Z}_{it}|\boldsymbol{\sigma}(\mathbf{Z}_{it-1}^{0})), \tag{4.49}$$

the estimated (unscaled) residuals $\widehat{u}_{it}$ of the Student's t equations in (4.3)-(4.5) are scaled by a t-varying conditional variance in order to account for the heteroskedastic conditional variance.

Alternatively, the auxiliary regression for M-S testing that concerns the skedastic function in (4.48) can take the following simplified form:

$$(\widehat{u}_{it}^{2})^{st} = \delta_{1i} + \underbrace{\sum\nolimits_{j=1}^{m} \delta_{2ji} v_{jt}}_{\overline{[5.3]}} + \delta_{3i}\widehat{\widehat{u}}_{it-1}^{2} + \underbrace{\sum\nolimits_{j=2}^{p} \delta_{4ji}\widehat{\widehat{u}}_{it-j}^{2}}_{\overline{[3]}+\overline{[4]}} + \epsilon_{2it}. \tag{4.50}$$

The derivation of this alternative auxiliary regression is shown below.

Step 1. Estimate the following auxiliary regressions:

$$y_{it-1} = \delta_{10i} + \sum\nolimits_{j=1}^{m^*} \delta_{11ji} v_{jt} + \sum\nolimits_{j=2}^{12} \delta_{12ji} d_{jt} + \eta_{1it}, \tag{4.51}$$

$$R_{mt-1} = \delta_{20i} + \sum\nolimits_{j=1}^{m^*} \delta_{21ji} v_{jt} + \sum\nolimits_{j=2}^{12} \delta_{22ji} d_{jt} + \eta_{2it}, \tag{4.52}$$

$$R_{ft-1} = \delta_{30i} + \sum\nolimits_{j=1}^{m^*} \delta_{31ji} v_{jt} + \sum\nolimits_{j=2}^{12} \delta_{32ji} d_{jt} + \eta_{3it}. \tag{4.53}$$

Step 2. Obtain the estimated residuals from the three equations in (4.51)-(4.53) which represent the mean deviations that come into the conditional variance, and use these residuals to estimate the following quadratic equation:

$$\widehat{u}_{it-1}^{2} = \gamma_{0i} + \gamma_{1i}\widehat{\eta}_{1it}^{2} + \gamma_{2i}\widehat{\eta}_{2it}^{2} + \gamma_{3i}\widehat{\eta}_{3it}^{2} + \gamma_{4i}\widehat{\eta}_{1it}\widehat{\eta}_{2it} + \gamma_{5i}\widehat{\eta}_{1it}\widehat{\eta}_{3it} + \gamma_{6i}\widehat{\eta}_{2it}\widehat{\eta}_{3it} + \zeta_{it}, \tag{4.54}$$

where $\widehat{u}_{it}^{2}$ denotes the estimated (unscaled) residuals from the three Student's t equations in (4.3)-(4.5).

Step 3. The simplified auxiliary regression for M-S testing that concerns the skedastic function in (4.48) takes the form in (4.50), where $\widehat{\widehat{u}}_{it-1}^{2}$ denotes the fitted values from (4.54).

The M-S testing results for the heterogeneous Student's t VAR $(1;\nu)$ model are presented in Appendix G. Table 4.6 includes a representative summary of these results.

Table 4.6 - Representative M-S Testing Results for heterogeneous Students't VAR $(1;\nu)$ Model			
Assumption	y_{it}	R_{mt}	R_{ft}
[1] Student's t	✓	✓	✓
[2] Linearity	✓	✓	✓
[3] Heteroskedasticity	✓	✓	✓
[4] Markov (1)	✓	✓	✓
[5] t-invariance:			
[5.1] Mean Heterogeneity	✓	✓	✓
[5.3] Variance Heterogeneity	✓	✓	✓

These results indicate that the heterogeneous Student's t VAR $(1;\nu)$ model is statistically adequate; that is, it accounts for all the statistical systematic information in the Fama-French (1993) data.

4.5 Summary and Conclusions

The heterogeneous Student's t/ Heteroskedastic VAR $(1;\nu)$ model can be viewed as the appropriate (adequate) statistical model underlying the CAPM and the Fama-French three-factor model. The M-S testing results of the latter model show that it accounts for all the statistical systematic information in the data. This model provides a sound basis for revisiting the empirical (statistical and substantive) adequacy of the CAPM and the Fama-French three-factor model, as well as that of other empirical asset pricing models.

Viewing the above empirical results from a theoretical perspective, it is clear that some of the key assumptions of the CAPM, as well as of other empirical asset pricing models, are rejected by the data. The Normality assumption, which provides the cornerstone of these models as well as of the Markowitz portfolio theory, is clearly rejected by the data in favor of the Student's t distribution. This constitutes a major break from these theories primarily because of the heteroskedastic conditional variance of the latter distribution. In addition, the presence of heterogeneity as well as temporal dependence also call into question the assumption of equilibrium among agents using rational expectations.

In the next chapter, a factor selection procedure will be proposed. In this procedure, both the modeling of heterogeneity in portfolio data (chapter 2) and the respecification that yields the statistically adequate model (chapter 4) will play a crucial role. That being so, the next chapter will provide a coherent link between the different stages discussed so far.

Appendix E: Student's t Tests

This Appendix provides a brief description of the Kolmogorov-Smirnov and Anderson-Darling tests for Student's t distribution.

Kolmogorov-Smirnov. Based on the assumption that $\{z_t,\ t \in \mathbb{N}\}$ is an IID process, the Kolmogorov-Smirnov test statistic for Student's t is defined as (Kolmogorov, 1933; Smirnov, 1939):

$$D = \sup_{z \in \mathbb{R}} |F^*(z) - F(z)|, \tag{E.1}$$

where $F^*(z)$ denotes the (known) Student's t cumulative distribution function (cdf) and $F(z)$ is the empirical cdf defined as:

$$F(z) = \tfrac{1}{n} \sum_{t=1}^{n} I_{(-\infty, z]}(z_t), \tag{E.2}$$

where I denotes an indicator function: $I=1$ if $z_t \leq z$ and $I=0$ otherwise.

Anderson-Darling. Based on the assumption that $\{z_t,\ t \in \mathbb{N}\}$ is an IID process, the Anderson-Darling test statistic for Student's t is defined as (Anderson and Darling, 1954):

$$A^2 = -n - \tfrac{1}{n} \sum_{t=1}^{n} (2t-1) \left[\ln F(z_{(t)}) + \ln(1 - F(z_{(n-t+1)})) \right], \tag{E.3}$$

where $z_{(t)}$ is the t-th order statistic, and $F(.)$ is the cumulative distribution of the Student's t distribution.

The null hypothesis of the tests above is H_0: *Student's t* versus the alternative H_1: *non-Student's t.* Thus, if the p-value is lower than the chosen significance level, there is evidence that the $\{z_t,\ t \in \mathbb{N}\}$ IID process tested is not Student's t distributed. It is important to note that for a large sample size, one has to choose much smaller thresholds for significance when using the p-values; see Lehmann and Romano (2006). With a sample size of $n=342$ a threshold of .01 is more appropriate than the most commonly used threshold of .05.

Appendix F: M-S Testing Results for Normal VAR

This Appendix presents the M-S testing results of the CAPM estimated by the heterogeneous Normal Vector Autoregressive of order 1 [heterogeneous Normal VAR (1)] model. The estimation of the model is equivalent to the following system of three equations (July 1963 to December 1991, 342 months):

$$y_{it}{=}\alpha_{1i}{+}\sum_{j=1}^{m^*}\delta_{11ji}v_{jt}{+}\sum_{j=2}^{s}\delta_{12ji}d_{jt}{+}\gamma_{11i}y_{it-1}{+}\gamma_{12i}R_{mt-1}{+}\gamma_{13i}R_{ft-1}{+}u_{1it}, \tag{F.1}$$

$$R_{mt}{=}\alpha_{2i}{+}\sum_{j=1}^{m^*}\delta_{21ji}v_{jt}{+}\sum_{j=2}^{s}\delta_{22ji}d_{jt}{+}\gamma_{21i}y_{it-1}{+}\gamma_{22i}R_{mt-1}{+}\gamma_{23i}R_{ft-1}{+}u_{2it}, \tag{F.2}$$

$$R_{ft}{=}\alpha_{3i}{+}\sum_{j=1}^{m^*}\delta_{31ji}v_{jt}{+}\sum_{j=2}^{s}\delta_{32ji}d_{jt}{+}\gamma_{31i}y_{it-1}{+}\gamma_{32i}R_{mt-1}{+}\gamma_{33i}R_{ft-1}{+}u_{3it}, \tag{F.3}$$

$$\text{for } (u_{jit}|\mathbf{Z}_{it-1}) \backsim \text{N}(0,\sigma_i^2(t)),\ j{=}\{1,2,3\},\ i{=}1,2,...,k,\ t{\in}\mathbb{N},$$

where $y_{it}{=}R_{it}$ is the return of portfolio i for period t; v_{jt} denotes the terms of the Gram-Schmidt orthonormal polynomials of order $j{=}1,2,...,m^*$; $d_{jt}{:=}(d_{2t},...,d_{12t})$ are the monthly dummy variables for the months of February through December; R_{mt} is the return of the value-weighted market portfolio; R_{ft} is the risk-free return; and $\sigma_i^2(t)$ is homoskedastic and heterogeneous stemming from the Multivariate Normal distribution.

To evaluate the model assumptions [1]–[5] that justify the heterogeneous Normal VAR (1) model, the estimated residuals of the three equations in (F.1)-(F.3) are carefully examined. The Shapiro-Wilk (W), Anderson-Darling (A^2), and D'Agostino-Pearson (K^2) tests in Appendix C are employed to test for [1] Normality, and the auxiliary regressions (F) below are employed to test for [2] Linearity, [3] Homoskedasticity, [4] Markov Dependence of order 1, and [5] t-invariance.

$$\widehat{u}_{it}{=}\gamma_{1i}+\boldsymbol{\gamma}_{2i}^{\top}\boldsymbol{\Xi}_{it}^{*}+\underbrace{\sum\nolimits_{j=m^*+1}^{m}\gamma_{3ji}v_{jt}}_{\overline{[5.1]}}+\underbrace{\boldsymbol{\gamma}_{4i}^{\top}\boldsymbol{\psi}_t}_{\overline{[2]}}+\underbrace{\sum\nolimits_{j=2}^{p}\boldsymbol{\gamma}_{5ji}^{\top}\mathbf{Z}_{it-j}}_{\overline{[4]}}+\varepsilon_{1it}, \tag{F.4}$$

$$\widehat{u}_{it}^2=\gamma_{6i}+\underbrace{\sum_{j=1}^{m}\gamma_{7ji}v_{jt}}_{\overline{[5.3]}}+\underbrace{\boldsymbol{\gamma}_{8i}^{\top}\mathbf{Z}_{it-1}+\boldsymbol{\gamma}_{9i}^{\top}\boldsymbol{\psi}_t+\sum_{j=2}^{p}\boldsymbol{\gamma}_{10ji}^{\top}\mathbf{Z}_{it-j}^2}_{\overline{[3]}+\overline{[4]}}+\varepsilon_{2it}, \tag{F.5}$$

where $\widehat{u}_{it}$ are the estimated residuals in (F.1)-(F.3) of portfolio i for period t; $\mathbf{Z}_{it}$:=(R_{it},R_{mt},R_{ft}); $\boldsymbol{\Xi}_{it}^*$:=$(v_{1t},v_{2t},...,v_{mt}^*,d_{2t},d_{3t},...,d_{12t},\mathbf{Z}_{it-1})$ denotes the collection of all the deterministic and lagged variables included in the models; and $\boldsymbol{\psi}_t$:=$\{(z_{it}\cdot z_{jt}),i\geq j,i,j=1,2,...,l\}$ are the second-order Kolmogorov-Gabor polynomials.

The M-S testing results of the 25 *Size-B/M* portfolios for the heterogeneous Normal VAR (1) model are presented in tables F.1–F.25; p-values in square brackets below .01 are considered to indicate departures from assumptions [1]–[5].

Table F.1 - M-S Testing Results of S/L_t portfolio

Assumption	y_t=S/L_t	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W= 0.965[.000]	W=2.196[.000]	W=25.478[.000]
Anderson-Darling	A^2= 0.977[.000]	A^2=0.989[.013]	A^2=29.640[.000]
D'Agostino-Pearson	K^2= 0.922[.000]	K^2=5.654[.000]	K^2=44.699[.000]
[2] Linearity	$F(3,311)$= 0.594[.619]	$F(3,311)$=0.072[.975]	$F(3,311)$= 2.605[.052]
[3] Homoskedasticity	$F(12,325)$= 2.799[.001]	$F(12,325)$=2.879[.001]	$F(3,336)$=33.679[.000]
[4] Markov Dependence (1)	$F(3,311)$= 2.172[.091]	$F(3,311)$=1.607[.188]	$F(3,311)$= 0.264[.851]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$= 0.517[.473]	$F(1,311)$=0.008[.928]	$F(1,311)$= 2.222[.137]
[5.3] Variance Heterogeneity	$F(1,325)$=13.595[.000]	$F(1,325)$=0.752[.387]	$F(1,336)$= 5.321[.022]

Table F.2 - M-S Testing Results of $S/2_t$ portfolio

Assumption	y_t=$S/2_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W= 0.969[.000]	W=1.664[.000]	W=29.102[.000]
Anderson-Darling	A^2= 0.977[.000]	A^2=0.937[.017]	A^2=29.888[.000]
D'Agostino-Pearson	K^2= 0.918[.000]	K^2=6.094[.000]	K^2=45.087[.000]
[2] Linearity	$F(3,311)$= 0.485[.693]	$F(3,311)$=0.058[.982]	$F(3,311)$= 2.826[.039]
[3] Homoskedasticity	$F(12,325)$= 3.851[.000]	$F(12,325)$=3.316[.000]	$F(3,336)$=35.098[.000]
[4] Markov Dependence (1)	$F(3,311)$= 1.825[.143]	$F(3,311)$=2.002[.114]	$F(3,311)$= 0.590[.622]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$= 0.172[.678]	$F(1,311)$=0.002[.965]	$F(1,311)$= 2.083[.150]
[5.3] Variance Heterogeneity	$F(1,325)$=12.320[.001]	$F(1,325)$=0.681[.410]	$F(1,336)$= 5.774[.017]

Table F.3 - M-S Testing Results of $S/3_t$ portfolio			
Assumption	y_t=$S/3_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.970[.000]	W=1.693[.000]	W=27.980[.000]
Anderson-Darling	A^2=0.978[.000]	A^2=0.883[.024]	A^2=29.922[.000]
D'Agostino-Pearson	K^2=0.920[.000]	K^2=5.906[.000]	K^2=44.447[.000]
[2] Linearity	$F(3,311)$=0.779[.506]	$F(3,311)$=0.293[.831]	$F(3,311)$= 2.990[.031]
[3] Homoskedasticity	$F(12,325)$=3.749[.000]	$F(12,325)$=3.229[.000]	$F(3,336)$=33.364[.000]
[4] Markov Dependence (1)	$F(3,311)$=1.943[.123]	$F(3,311)$=1.776[.152]	$F(3,311)$= 0.457[.713]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$= 0.070[.792]	$F(1,311)$=0.001[.999]	$F(1,311)$= 1.762[.185]
[5.3] Variance Heterogeneity	$F(1,325)$=11.037[.001]	$F(1,325)$=0.436[.510]	$F(1,336)$= 5.286[.022]

Table F.4 - M-S Testing Results of $S/4_t$ portfolio			
Assumption	y_t=$S/4_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.963[.000]	W=2.397[.000]	W=34.105[.000]
Anderson-Darling	A^2=0.977[.000]	A^2=0.970[.014]	A^2=29.875[.000]
D'Agostino-Pearson	K^2=0.919[.000]	K^2=5.986[.000]	K^2=44.715[.000]
[2] Linearity	$F(3,311)$=0.540[.655]	$F(3,311)$=0.075[.973]	$F(3,311)$= 3.151[.025]
[3] Homoskedasticity	$F(9,326)$=3.666[.000]	$F(12,325)$=3.188[.000]	$F(3,336)$=33.985[.000]
[4] Markov Dependence (1)	$F(3,311)$=1.720[.163]	$F(3,311)$=1.833[.141]	$F(3,311)$= 0.510[.676]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.046[.831]	$F(1,311)$=0.001[.998]	$F(1,311)$= 1.713[.192]
[5.3] Variance Heterogeneity	$F(3,326)$=5.180[.002]	$F(1,325)$=0.425[.515]	$F(1,336)$= 5.342[.021]

Table F.5 - M-S Testing Results of S/H_t portfolio			
Assumption	y_t=S/H_t	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.957[.000]	W=2.520[.000]	W=36.038[.000]
Anderson-Darling	A^2=0.977[.000]	A^2=0.952[.016]	A^2=29.751[.000]
D'Agostino-Pearson	K^2=0.921[.000]	K^2=5.868[.000]	K^2=44.144[.000]
[2] Linearity	$F(3,311)$=0.932[.425]	$F(3,311)$=0.211[.889]	$F(3,311)$= 3.018[.030]
[3] Homoskedasticity	$F(9,328)$=3.639[.000]	$F(12,325)$=2.830[.001]	$F(3,336)$=32.842[.000]
[4] Markov Dependence (1)	$F(3,311)$=2.458[.063]	$F(3,311)$=1.693[.169]	$F(3,311)$= 0.551[.648]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.107[.744]	$F(1,311)$=0.001[.972]	$F(1,311)$= 1.623[.204]
[5.3] Variance Heterogeneity	$F(1,328)$=4.073[.044]	$F(1,325)$=0.122[.727]	$F(1,336)$= 4.576[.033]

Table F.6 - M-S Testing Results of $2/L_t$ portfolio			
Assumption	y_t=$2/L_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.979[.000]	W=1.018[.011]	W=19.372[.000]
Anderson-Darling	A^2=0.977[.000]	A^2=0.930[.018]	A^2=29.579[.000]
D'Agostino-Pearson	K^2=0.919[.000]	K^2=5.949[.000]	K^2=45.332[.000]
[2] Linearity	$F(3,311)$=0.462[.709]	$F(3,311)$=0.226[.878]	$F(3,311)$= 3.142[.026]
[3] Homoskedasticity	$F(6,333)$=3.639[.002]	$F(12,325)$=3.292[.000]	$F(3,336)$=34.547[.000]
[4] Markov Dependence (1)	$F(3,311)$=1.034[.378]	$F(3,311)$=1.252[.291]	$F(3,311)$= 0.712[.546]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.002[.961]	$F(1,311)$=0.007[.936]	$F(1,311)$= 2.011[.157]
[5.3] Variance Heterogeneity	$F(1,333)$=3.340[.069]	$F(1,325)$=0.734[.392]	$F(1,336)$= 4.933[.027]

Table F.7 - M-S Testing Results of $2/2_t$ portfolio			
Assumption	y_t=$2/2_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.974[.000]	W=1.257[.003]	W=29.793[.000]
Anderson-Darling	A^2=0.978[.000]	A^2=0.861[.027]	A^2=28.390[.000]
D'Agostino-Pearson	K^2=0.919[.000]	K^2=6.144[.000]	K^2=45.000[.000]
[2] Linearity	$F(3,311)$=0.337[.799]	$F(3,311)$=0.223[.881]	$F(3,311)$= 3.294[.021]
[3] Homoskedasticity	$F(9,326)$=3.388[.001]	$F(12,325)$=3.511[.000]	$F(3,336)$=33.080[.000]
[4] Markov Dependence (1)	$F(3,311)$=1.553[.201]	$F(3,311)$=1.538[.205]	$F(3,311)$= 0.520[.669]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.028[.868]	$F(1,311)$=0.006[.938]	$F(1,311)$= 1.691[.194]
[5.3] Variance Heterogeneity	$F(3,326)$=2.659[.048]	$F(1,325)$=0.206[.650]	$F(1,336)$= 4.479[.035]

Table F.8 - M-S Testing Results of $2/3_t$ portfolio			
Assumption	y_t=$2/3_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.962[.000]	W=1.852[.000]	W=37.408[.000]
Anderson-Darling	A^2=0.977[.000]	A^2=0.891[.023]	A^2=29.842[.000]
D'Agostino-Pearson	K^2=0.922[.000]	K^2=5.839[.000]	K^2=43.602[.000]
[2] Linearity	$F(3,311)$=0.204[.894]	$F(3,311)$=0.028[.994]	$F(3,311)$= 3.071[.028]
[3] Homoskedasticity	$F(9,328)$=4.991[.000]	$F(12,325)$=3.531[.000]	$F(3,336)$=33.476[.000]
[4] Markov Dependence (1)	$F(3,311)$=1.861[.136]	$F(3,311)$=1.475[.221]	$F(3,311)$= 0.508[.677]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.040[.841]	$F(1,311)$=0.017[.898]	$F(1,311)$= 1.410[.236]
[5.3] Variance Heterogeneity	$F(1,328)$=1.840[.176]	$F(1,325)$=0.236[.628]	$F(1,336)$= 4.858[.028]

Table F.9 - M-S Testing Results of $2/4_t$ portfolio			
Assumption	y_t=$2/4_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.966[.000]	W=1.433[.001]	W=33.523[.000]
Anderson-Darling	A^2=0.976[.000]	A^2=0.994[.013]	A^2=30.373[.000]
D'Agostino-Pearson	K^2=0.927[.000]	K^2=5.110[.000]	K^2=43.396[.000]
[2] Linearity	$F(3,311)$=0.247[.864]	$F(3,311)$=0.095[.963]	$F(3,311)$= 3.295[.021]
[3] Homoskedasticity	$F(9,328)$=5.226[.000]	$F(12,325)$=3.091[.000]	$F(3,336)$=30.942[.000]
[4] Markov Dependence (1)	$F(3,311)$=2.085[.106]	$F(3,311)$=1.755[.156]	$F(3,311)$= 1.223[.302]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.049[.825]	$F(1,311)$=0.009[.926]	$F(1,311)$= 1.018[.314]
[5.3] Variance Heterogeneity	$F(1,328)$=2.805[.095]	$F(1,325)$=0.110[.741]	$F(1,336)$= 4.396[.037]

Table F.10 - M-S Testing Results of $2/H_t$ portfolio			
Assumption	y_t=$2/H_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.969[.000]	W=1.318[.002]	W=31.191[.000]
Anderson-Darling	A^2=0.977[.000]	A^2=1.000[.012]	A^2=29.456[.000]
D'Agostino-Pearson	K^2=0.919[.000]	K^2=5.999[.000]	K^2=45.135[.000]
[2] Linearity	$F(3,311)$=0.145[.933]	$F(3,311)$=0.160[.924]	$F(3,311)$= 3.430[.017]
[3] Homoskedasticity	$F(9,326)$=3.739[.000]	$F(12,325)$=2.993[.001]	$F(3,336)$=32.077[.000]
[4] Markov Dependence (1)	$F(3,311)$=2.288[.079]	$F(3,311)$=1.785[.150]	$F(3,311)$= 0.419[.740]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.160[.690]	$F(1,311)$=0.001[.974]	$F(1,311)$= 1.514[.220]
[5.3] Variance Heterogeneity	$F(3,326)$=3.238[.022]	$F(1,325)$=0.028[.868]	$F(1,336)$= 4.192[.041]

Table F.11 - M-S Testing Results of $3/L_t$ portfolio			
Assumption	y_t=$3/L_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.980[.000]	W=0.956[.016]	W=18.148[.000]
Anderson-Darling	A^2=0.978[.000]	A^2=0.860[.027]	A^2=28.534[.000]
D'Agostino-Pearson	K^2=0.918[.000]	K^2=6.136[.000]	K^2=45.209[.000]
[2] Linearity	$F(3,311)$=1.290[.278]	$F(3,311)$=0.632[.595]	$F(3,311)$= 3.109[.027]
[3] Homoskedasticity	$F(12,323)$=3.211[.000]	$F(12,325)$=3.187[.000]	$F(3,336)$=34.010[.000]
[4] Markov Dependence (1)	$F(3,311)$=1.217[.304]	$F(3,311)$=1.347[.259]	$F(3,311)$= 0.511[.675]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.091[.764]	$F(1,311)$=0.029[.865]	$F(1,311)$= 1.767[.185]
[5.3] Variance Heterogeneity	$F(3,323)$=3.749[.011]	$F(1,325)$=0.895[.345]	$F(1,336)$= 4.873[.028]

Table F.12 - M-S Testing Results of $3/2_t$ portfolio			
Assumption	y_t=$3/2_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.969[.000]	W=1.400[.001]	W=38.018[.000]
Anderson-Darling	A^2=0.977[.000]	A^2=0.881[.024]	A^2=30.356[.000]
D'Agostino-Pearson	K^2=0.923[.000]	K^2=5.625[.000]	K^2=43.210[.000]
[2] Linearity	$F(3,311)$=0.171[.916]	$F(3,311)$=0.023[.995]	$F(3,311)$= 3.272[.021]
[3] Homoskedasticity	$F(12,325)$=5.119[.000]	$F(12,325)$=4.037[.000]	$F(3,336)$=33.836[.000]
[4] Markov Dependence (1)	$F(3,311)$=1.247[.293]	$F(3,311)$=1.531[.207]	$F(3,311)$= 0.399[.754]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.002[.969]	$F(1,311)$=0.008[.930]	$F(1,311)$= 2.080[.150]
[5.3] Variance Heterogeneity	$F(1,325)$=2.179[.141]	$F(1,325)$=0.145[.703]	$F(1,336)$= 4.069[.044]

Table F.13 - M-S Testing Results of $3/3_t$ portfolio			
Assumption	y_t=$3/3_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.976[.000]	W=1.317[.002]	W=28.426[.000]
Anderson-Darling	A^2=0.978[.000]	A^2=0.853[.028]	A^2=28.604[.000]
D'Agostino-Pearson	K^2=0.930[.000]	K^2=5.204[.000]	K^2=40.291[.000]
[2] Linearity	$F(3,311)$=0.127[.944]	$F(3,311)$=0.028[.994]	$F(3,311)$= 3.645[.013]
[3] Homoskedasticity	$F(12,325)$=5.035[.000]	$F(12,325)$=3.534[.000]	$F(3,336)$=30.585[.000]
[4] Markov Dependence (1)	$F(3,311)$=1.886[.132]	$F(3,311)$=1.406[.241]	$F(3,311)$= 0.567[.638]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.039[.843]	$F(1,311)$=0.003[.956]	$F(1,311)$= 1.891[.170]
[5.3] Variance Heterogeneity	$F(1,325)$=5.258[.022]	$F(1,325)$=0.157[.692]	$F(1,336)$= 4.215[.041]

Table F.14 - M-S Testing Results of $3/4_t$ portfolio			
Assumption	y_t=$3/4_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.973[.000]	W=1.498[.001]	W=27.385[.000]
Anderson-Darling	A^2=0.977[.000]	A^2=1.089[.007]	A^2=28.540[.000]
D'Agostino-Pearson	K^2=0.938[.000]	K^2=4.489[.000]	K^2=36.785[.000]
[2] Linearity	$F(3,311)$=0.143[.934]	$F(3,311)$=0.093[.964]	$F(3,311)$= 3.598[.139]
[3] Homoskedasticity	$F(9,328)$=4.616[.000]	$F(9,328)$=2.715[.005]	$F(3,336)$=28.987[.000]
[4] Markov Dependence (1)	$F(3,311)$=1.542[.204]	$F(3,311)$=1.790[.149]	$F(3,311)$= 0.752[.522]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.344[.558]	$F(1,311)$=0.153[.696]	$F(1,311)$= 0.768[.382]
[5.3] Variance Heterogeneity	$F(1,328)$=1.077[.300]	$F(1,328)$=0.138[.711]	$F(1,336)$= 3.162[.076]

Table F.15 - M-S Testing Results of $3/H_t$ portfolio			
Assumption	y_t=$3/H_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.963[.000]	W=1.676[.000]	W=33.193[.000]
Anderson-Darling	A^2=0.977[.000]	A^2=1.011[.011]	A^2=29.646[.000]
D'Agostino-Pearson	K^2=0.923[.000]	K^2=5.480[.000]	K^2=43.571[.000]
[2] Linearity	$F(3,311)$=0.494[.686]	$F(3,311)$=0.419[.740]	$F(3,311)$= 3.113[.027]
[3] Homoskedasticity	$F(9,326)$=3.286[.001]	$F(12,325)$=2.771[.001]	$F(3,336)$=33.169[.000]
[4] Markov Dependence (1)	$F(3,311)$=2.661[.048]	$F(3,311)$=1.397[.244]	$F(3,311)$= 0.449[.718]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.398[.529]	$F(1,311)$=0.001[.996]	$F(1,311)$= 1.450[.230]
[5.3] Variance Heterogeneity	$F(3,326)$=3.093[.027]	$F(1,325)$=0.062[.803]	$F(1,336)$= 4.579[.033]

Table F.16 - M-S Testing Results of $4/L_t$ portfolio			
Assumption	y_t=$4/L_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.983[.001]	W=0.664[.082]	W=17.322[.000]
Anderson-Darling	A^2=0.978[.000]	A^2=0.885[.023]	A^2=28.977[.000]
D'Agostino-Pearson	K^2=0.918[.000]	K^2=6.121[.000]	K^2=45.369[.000]
[2] Linearity	$F(3,311)$=2.660[.048]	$F(3,311)$=2.025[.110]	$F(3,311)$= 5.224[.002]
[3] Homoskedasticity	$F(12,325)$=3.202[.000]	$F(12,325)$=2.367[.006]	$F(3,336)$=31.646[.000]
[4] Markov Dependence (1)	$F(3,311)$=0.923[.430]	$F(3,311)$=1.266[.286]	$F(3,311)$= 0.465[.707]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.130[.719]	$F(1,311)$=0.018[.893]	$F(1,311)$= 1.504[.221]
[5.3] Variance Heterogeneity	$F(1,325)$=0.956[.329]	$F(1,325)$=0.067[.796]	$F(1,336)$= 3.795[.052]

Table F.17 - M-S Testing Results of $4/2_t$ portfolio			
Assumption	y_t=$4/2_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.968[.000]	W=1.576[.000]	W=37.664[.000]
Anderson-Darling	A^2=0.977[.000]	A^2=1.067[.008]	A^2=31.165[.000]
D'Agostino-Pearson	K^2=0.916[.000]	K^2=6.225[.000]	K^2=46.057[.000]
[2] Linearity	$F(3,311)$=1.641[.180]	$F(3,311)$=0.672[.570]	$F(3,311)$= 2.748[.043]
[3] Homoskedasticity	$F(12,325)$=3.084[.000]	$F(12,325)$=3.187[.000]	$F(3,336)$=30.357[.000]
[4] Markov Dependence (1)	$F(3,311)$=1.110[.345]	$F(3,311)$=1.288[.279]	$F(3,311)$= 0.399[.754]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.068[.795]	$F(1,311)$=0.023[.879]	$F(1,311)$= 1.915[.167]
[5.3] Variance Heterogeneity	$F(1,325)$=0.390[.533]	$F(1,325)$=0.002[.966]	$F(1,336)$= 2.880[.091]

Table F.18 - M-S Testing Results of $4/3_t$ portfolio			
Assumption	y_t=$4/3_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.978[.000]	W=0.823[.033]	W=25.970[.000]
Anderson-Darling	A^2=0.977[.000]	A^2=1.001[.012]	A^2=29.377[.000]
D'Agostino-Pearson	K^2=0.934[.000]	K^2=4.612[.000]	K^2=40.563[.000]
[2] Linearity	$F(3,311)$=0.013[.998]	$F(3,311)$=0.182[.908]	$F(3,311)$= 3.186[.024]
[3] Homoskedasticity	$F(9,328)$=3.367[.001]	$F(6,333)$=5.665[.000]	$F(3,336)$=32.276[.000]
[4] Markov Dependence (1)	$F(3,311)$=0.860[.462]	$F(3,311)$=1.112[.344]	$F(3,311)$= 0.636[.592]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.101[.751]	$F(1,311)$=0.003[.959]	$F(1,311)$= 1.596[.207]
[5.3] Variance Heterogeneity	$F(1,328)$=0.461[.498]	$F(1,333)$=3.072[.081]	$F(1,336)$= 2.819[.094]

Table F.19 - M-S Testing Results of $4/4_t$ portfolio			
Assumption	y_t=$4/4_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.987[.005]	W=0.606[.114]	W=14.203[.001]
Anderson-Darling	A^2=0.977[.000]	A^2=1.048[.009]	A^2=29.769[.000]
D'Agostino-Pearson	K^2=0.947[.000]	K^2=3.927[.000]	K^2=34.393[.000]
[2] Linearity	$F(3,311)$=0.294[.830]	$F(3,311)$=0.322[.809]	$F(3,311)$= 1.818[.144]
[3] Homoskedasticity	$F(9,328)$=3.624[.000]	$F(12,325)$=2.625[.002]	$F(3,336)$=25.576[.000]
[4] Markov Dependence (1)	$F(3,311)$=1.275[.283]	$F(3,311)$=1.374[.251]	$F(3,311)$= 1.224[.301]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.147[.702]	$F(1,311)$=0.040[.842]	$F(1,311)$= 0.631[.428]
[5.3] Variance Heterogeneity	$F(1,328)$=1.144[.286]	$F(1,325)$=0.004[.949]	$F(1,336)$= 3.396[.066]

Table F.20 - M-S Testing Results of $4/H_t$ portfolio			
Assumption	y_t=$4/H_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.982[.000]	W=0.912[.020]	W=20.295[.000]
Anderson-Darling	A^2=0.977[.000]	A^2=1.013[.011]	A^2=29.275[.000]
D'Agostino-Pearson	K^2=0.928[.000]	K^2=5.155[.000]	K^2=41.911[.000]
[2] Linearity	$F(3,311)$=1.203[.309]	$F(3,311)$=1.838[.140]	$F(3,311)$= 3.655[.013]
[3] Homoskedasticity	$F(9,328)$=2.869[.003]	$F(12,325)$=2.517[.004]	$F(3,336)$=30.929[.000]
[4] Markov Dependence (1)	$F(3,311)$=0.600[.615]	$F(3,311)$=1.150[.329]	$F(3,311)$= 0.316[.814]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.251[.616]	$F(1,311)$=0.009[.923]	$F(1,311)$= 1.850[.175]
[5.3] Variance Heterogeneity	$F(1,328)$=1.829[.177]	$F(1,325)$=0.082[.775]	$F(1,336)$= 3.933[.048]

Table F.21 - M-S Testing Results of B/L_t portfolio			
Assumption	y_t=B/L_t	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.974[.000]	W=1.165[.005]	W=26.784[.000]
Anderson-Darling	A^2=0.977[.000]	A^2=1.020[.011]	A^2=29.571[.000]
D'Agostino-Pearson	K^2=0.918[.000]	K^2=6.082[.000]	K^2=45.547[.000]
[2] Linearity	$F(3,311)$=0.232[.874]	$F(3,311)$=0.227[.878]	$F(3,311)$= 3.497[.016]
[3] Homoskedasticity	$F(9,328)$=3.104[.001]	$F(9,328)$=3.565[.000]	$F(6,333)$=16.536[.000]
[4] Markov Dependence (1)	$F(3,311)$=1.004[.391]	$F(3,311)$=1.226[.300]	$F(3,311)$= 0.849[.468]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.075[.784]	$F(1,311)$=0.018[.894]	$F(1,311)$= 2.103[.148]
[5.3] Variance Heterogeneity	$F(1,328)$=0.611[.435]	$F(1,328)$=0.236[.628]	$F(1,333)$= 3.644[.057]

Table F.22 - M-S Testing Results of $B/2_t$ portfolio			
Assumption	y_t=$B/2_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.981[.000]	W=1.049[.009]	W=23.645[.000]
Anderson-Darling	A^2=0.978[.000]	A^2=1.035[.010]	A^2=28.945[.000]
D'Agostino-Pearson	K^2=0.921[.000]	K^2=5.977[.000]	K^2=43.759[.000]
[2] Linearity	$F(3,311)$=0.748[.524]	$F(3,311)$=0.248[.863]	$F(3,311)$= 4.461[.004]
[3] Homoskedasticity	$F(12,325)$=1.832[.042]	$F(12,325)$=2.615[.002]	$F(3,336)$=31.705[.000]
[4] Markov Dependence (1)	$F(3,311)$=1.477[.221]	$F(3,311)$=1.156[.327]	$F(3,311)$= 0.450[.718]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.035[.852]	$F(1,311)$=0.001[.979]	$F(1,311)$= 2.230[.136]
[5.3] Variance Heterogeneity	$F(1,325)$=2.421[.121]	$F(1,325)$=0.256[.613]	$F(1,336)$= 4.297[.039]

Table F.23 - M-S Testing Results of $B/3_t$ portfolio			
Assumption	y_t=$B/3_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.970[.000]	W=1.409[.001]	W=30.747[.000]
Anderson-Darling	A^2=0.977[.000]	A^2=0.889[.023]	A^2=27.500[.000]
D'Agostino-Pearson	K^2=0.922[.000]	K^2=5.880[.000]	K^2=43.792[.000]
[2] Linearity	$F(3,311)$=0.914[.435]	$F(3,311)$=0.085[.968]	$F(3,311)$= 3.761[.011]
[3] Homoskedasticity	$F(6,333)$=4.434[.000]	$F(6,333)$=9.179[.000]	$F(3,336)$=43.012[.000]
[4] Markov Dependence (1)	$F(3,311)$=2.120[.098]	$F(3,311)$=1.710[.165]	$F(3,311)$= 0.342[.795]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.022[.882]	$F(1,311)$=0.001[.989]	$F(1,311)$= 2.116[.147]
[5.3] Variance Heterogeneity	$F(1,333)$=0.376[.540]	$F(1,333)$=2.675[.103]	$F(1,336)$= 3.908[.049]

Table F.24 - M-S Testing Results of $B/4_t$ portfolio			
Assumption	y_t=$B/4_t$	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.987[.003]	W=0.846[.029]	W=12.530[.002]
Anderson-Darling	A^2=0.977[.000]	A^2=0.958[.015]	A^2=29.785[.000]
D'Agostino-Pearson	K^2=0.924[.000]	K^2=5.531[.000]	K^2=43.585[.000]
[2] Linearity	$F(3,311)$=0.672[.570]	$F(3,311)$=1.265[.286]	$F(3,311)$= 2.675[.047]
[3] Homoskedasticity	$F(12,325)$=4.465[.000]	$F(12,325)$=2.422[.005]	$F(3,336)$=29.499[.000]
[4] Markov Dependence (1)	$F(3,311)$=0.775[.509]	$F(3,311)$=1.469[.223]	$F(3,311)$= 0.238[.870]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.079[.779]	$F(1,311)$=0.005[.944]	$F(1,311)$= 1.917[.167]
[5.3] Variance Heterogeneity	$F(1,325)$=0.594[.441]	$F(1,325)$=0.001[.992]	$F(1,336)$= 3.042[.082]

Table F.25 - M-S Testing Results of B/H_t portfolio			
Assumption	y_t=B/H_t	R_{mt}	R_{ft}
[1] Normality:			
Shapiro-Wilk	W=0.991[.044]	W=0.593[.121]	W= 8.094[.017]
Anderson-Darling	A^2=0.977[.000]	A^2=1.028[.010]	A^2=29.638[.000]
D'Agostino-Pearson	K^2=0.921[.000]	K^2=5.750[.000]	K^2=45.126[.000]
[2] Linearity	$F(3,311)$=0.489[.691]	$F(3,311)$=0.127[.944]	$F(3,311)$= 3.274[.021]
[3] Homoskedasticity	$F(12,323)$=1.803[.047]	$F(12,325)$=3.078[.000]	$F(3,336)$=29.947[.000]
[4] Markov Dependence (1)	$F(3,311)$=0.653[.582]	$F(3,311)$=1.041[.375]	$F(3,311)$= 0.488[.691]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.101[.750]	$F(1,311)$=0.001[.997]	$F(1,311)$= 2.018[.156]
[5.3] Variance Heterogeneity	$F(3,323)$=2.472[.062]	$F(1,325)$=0.001[.973]	$F(1,336)$= 3.102[.079]

Appendix G: M-S Testing Results for Student's t VAR

This Appendix presents the M-S testing results of the CAPM estimated by the heterogeneous Student's t Vector Autoregressive of order 1 with ν degrees of freedom [heterogeneous Student's t VAR $(1;\nu)$] model. The estimation of the model is equivalent to the following system of three equations (July 1963 to December 1991, 342 months):

$$y_{it}=\alpha_{1i}+\sum_{j=1}^{m^*}\delta_{11ji}v_{jt}+\sum_{j=2}^{s}\delta_{12ji}d_{jt}+\gamma_{11i}y_{it-1}+\gamma_{12i}R_{mt-1}+\gamma_{13i}R_{ft-1}+u_{1it}, \tag{G.1}$$

$$R_{mt}=\alpha_{2i}+\sum_{j=1}^{m^*}\delta_{21ji}v_{jt}+\sum_{j=2}^{s}\delta_{22ji}d_{jt}+\gamma_{21i}y_{it-1}+\gamma_{22i}R_{mt-1}+\gamma_{23i}R_{ft-1}+u_{2it}, \tag{G.2}$$

$$R_{ft}=\alpha_{3i}+\sum_{j=1}^{m^*}\delta_{31ji}v_{jt}+\sum_{j=2}^{s}\delta_{32ji}d_{jt}+\gamma_{31i}y_{it-1}+\gamma_{32i}R_{mt-1}+\gamma_{33i}R_{ft-1}+u_{3it}, \tag{G.3}$$

$$\text{for } (u_{jit}|\mathbf{Z}_{it-1}) \backsim \text{St}(0,\sigma_i^2(t)),\ j=\{1,2,3\},\ i=1,2,...,k,\ t\in\mathbb{N},$$

where $y_{it}=R_{it}$ is the return of portfolio i for period t; v_{jt} denotes the terms of the Gram-Schmidt orthonormal polynomials of order $j=1,2,...,m^*$; $d_{jt}:=(d_{2t},...,d_{12t})$ are the monthly dummy variables for the months of February through December; R_{mt} is the return of the value-weighted market portfolio; R_{ft} is the risk-free return; and $\sigma_i^2(t)$ is heteroskedastic and heterogeneous stemming from the Multivariate Student's t distribution.

To evaluate the model assumptions [1]–[5] that justify the heterogeneous Student's t VAR $(1;\nu)$ model, the estimated standardized residuals of the three equations in (G.1)-(G.3) are carefully examined. The Kolmogorov-Smirnov (D), and Anderson-Darling (A^2) tests in Appendix E are employed to test for [1] Student's t, and the auxiliary regressions (F) below are employed to test for the model assumptions of [2] Linearity, [3] Heteroskedasticity, [4] Markov Dependence of order 1, [5] t-invariance.

$$(\widehat{u}_{it})^{st}=\gamma_{1i}+\boldsymbol{\gamma}_{2i}^{\top}\boldsymbol{\Xi}_{it}^{*}+\underbrace{\sum\nolimits_{j=m^*+1}^{m}\gamma_{3ji}v_{jt}}_{[5.1]}+\underbrace{\boldsymbol{\gamma}_{4i}^{\top}\boldsymbol{\psi}_t}_{[2]}+\underbrace{\sum\nolimits_{j=2}^{p}\boldsymbol{\gamma}_{5ji}^{\top}\mathbf{Z}_{it-j}}_{[4]}+\varepsilon_{1it}, \tag{G.4}$$

$$\left(\widehat{u}_{it}^{2}\right)^{st}=\gamma_{6i}+\underbrace{\sum\nolimits_{j=1}^{m}\gamma_{7ji}v_{jt}}_{\overline{[5.3]}}+\underbrace{\boldsymbol{\gamma}_{8i}^{\top}\mathbf{Z}_{it-1}+\boldsymbol{\gamma}_{9i}^{\top}\boldsymbol{\psi}_{t}+\sum\nolimits_{j=2}^{p}\boldsymbol{\gamma}_{10ji}^{\top}\mathbf{Z}_{it-j}^{2}}_{\overline{[3]}+\overline{[4]}}+\varepsilon_{2it}, \tag{G.5}$$

where $\left(\widehat{u}_{it}\right)^{st}$ are the estimated standardized residuals in (G.1)-(G.3) of portfolio i for period t; $\mathbf{Z}_{it}:=(R_{it}, R_{mt}, R_{ft})$; $\boldsymbol{\Xi}_{it}^{*}:=(v_{1t}, v_{2t}, ..., v_{mt}^{*}, d_{2t}, ..., d_{12t}, \mathbf{Z}_{it-1})$ denotes the collection of all the deterministic and lagged variables included in the models; and $\boldsymbol{\psi}_{t}:=\{(z_{it}{\cdot}z_{jt}), i{\geq}j, i, j{=}1, 2, ..., l\}$ are the second-order Kolmogorov-Gabor polynomials.

The M-S testing results of the 25 *Size-B/M* portfolios for the heterogeneous Student's t VAR $(1;\nu)$ model are presented in tables H.1–H.25; p-values in square brackets below .01 are considered to indicate departures from assumptions [1]–[5].

Table G.1 - M-S Testing Results of S/L_t portfolio

Assumption	$y_t{=}S/L_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	$D{=}0.062[.075]$	$D{=}0.048[.209]$	$D{=}0.478[.000]$
Anderson-Darling	$A^2{=}3.994[.009]$	$A^2{=}1.544[.166]$	$A^2{=}127.613[.000]$
[2] Linearity	$F(3,310){=}0.346[.792]$	$F(3,310){=}0.016[.997]$	$F(3,310){=}1.120[.341]$
[3] Heteroskedasticity	$F(1,336){=}1.554[.213]$	$F(1,336){=}0.266[.607]$	$F(1,336){=}0.782[.377]$
[4] Markov Dependence (1)	$F(3,310){=}1.351[.258]$	$F(3,310){=}1.517[.326]$	$F(3,310){=}1.954[.121]$
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,310){=}0.003[.960]$	$F(1,310){=}0.074[.787]$	$F(1,310){=}0.148[.285]$
[5.3] Variance Heterogeneity	$F(1,336){=}8.689[.003]$	$F(1,336){=}0.988[.321]$	$F(1,336){=}0.097[.755]$

Table G.2 - M-S Testing Results of $S/2_t$ portfolio

Assumption	$y_t{=}S/2_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	$D{=}0.034[.455]$	$D{=}0.046[.243]$	$D{=}0.477[.000]$
Anderson-Darling	$A^2{=}1.013[.350]$	$A^2{=}1.847[.112]$	$A^2{=}127.704[.000]$
[2] Linearity	$F(3,310){=}0.285[.837]$	$F(3,310){=}0.001[.999]$	$F(3,310){=}1.315[.269]$
[3] Heteroskedasticity	$F(1,336){=}1.061[.304]$	$F(1,336){=}0.265[.607]$	$F(1,336){=}0.852[.357]$
[4] Markov Dependence (1)	$F(3,310){=}1.227[.300]$	$F(3,310){=}1.289[.278]$	$F(3,310){=}1.094[.352]$
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,310){=}0.081[.895]$	$F(1,310){=}0.305[.581]$	$F(1,310){=}1.027[.312]$
[5.3] Variance Heterogeneity	$F(1,336){=}12.722[.000]$	$F(1,336){=}2.081[.150]$	$F(1,336){=}0.696[.405]$

Table G.3 - M-S Testing Results of $S/3_t$ portfolio			
Assumption	y_t=$S/3_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.027[.613]	D=0.046[.240]	D=0.480[.000]
Anderson-Darling	A^2=0.567[.679]	A^2=2.690[.039]	A^2=127.691[.000]
[2] Linearity	$F(3,312)$=0.336[.800]	$F(3,312)$=0.170[.916]	$F(3,312)$=1.202[.309]
[3] Heteroskedasticity	$F(1,336)$=0.066[.797]	$F(1,336)$=0.605[.437]	$F(1,336)$=0.092[.762]
[4] Markov Dependence (1)	$F(3,312)$=1.320[.268]	$F(3,312)$=1.247[.293]	$F(3,312)$=1.525[.208]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=0.898[.344]	$F(1,312)$=0.240[.625]	$F(1,312)$=7.957[.005]
[5.3] Variance Heterogeneity	$F(1,336)$=7.686[.006]	$F(1,336)$=0.108[.742]	$F(1,336)$=0.031[.860]

Table G.4 - M-S Testing Results of $S/4_t$ portfolio			
Assumption	y_t=$S/4_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.029[.566]	D=0.043[.285]	D=0.480[.000]
Anderson-Darling	A^2=0.846[.449]	A^2=2.305[.063]	A^2=127.684[.000]
[2] Linearity	$F(3,312)$=0.170[.917]	$F(3,312)$=0.065[.978]	$F(3,312)$=1.313[.270]
[3] Heteroskedasticity	$F(1,336)$=0.001[.990]	$F(1,336)$=0.479[.489]	$F(1,336)$=0.549[.459]
[4] Markov Dependence (1)	$F(3,312)$=1.298[.275]	$F(3,312)$=1.454[.227]	$F(3,312)$=1.515[.211]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=0.357[.551]	$F(1,312)$=0.351[.554]	$F(1,312)$=7.416[.007]
[5.3] Variance Heterogeneity	$F(1,336)$=6.370[.012]	$F(1,336)$=0.098[.755]	$F(1,336)$=0.005[.945]

Table G.5 - M-S Testing Results of S/H_t portfolio			
Assumption	y_t=S/H_t	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.021[.739]	D=0.026[.623]	D=0.480[.000]
Anderson-Darling	A^2=0.742[.524]	A^2=0.729[.535]	A^2=127.298[.000]
[2] Linearity	$F(3,312)$=0.259[.855]	$F(3,312)$=0.014[.998]	$F(3,312)$=1.115[.343]
[3] Heteroskedasticity	$F(1,336)$=0.025[.875]	$F(1,336)$=0.211[.646]	$F(1,336)$=0.175[.676]
[4] Markov Dependence (1)	$F(3,312)$=1.256[.290]	$F(3,312)$=0.965[.410]	$F(3,312)$=1.986[.116]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=0.370[.543]	$F(1,312)$=0.519[.472]	$F(1,312)$=6.604[.011]
[5.3] Variance Heterogeneity	$F(1,336)$=8.265[.004]	$F(1,336)$=0.061[.805]	$F(1,336)$=0.244[.622]

Table G.6 - M-S Testing Results of $2/L_t$ portfolio			
Assumption	y_t=$2/L_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.080[.013]	D=0.069[.040]	D=0.481[.000]
Anderson-Darling	A^2=3.656[.013]	A^2=3.976[.009]	A^2=127.899[.000]
[2] Linearity	$F(3,312)$=0.415[.743]	$F(3,312)$=0.081[.970]	$F(3,312)$=1.448[.229]
[3] Heteroskedasticity	$F(1,336)$=0.709[.400]	$F(1,336)$=0.599[.439]	$F(1,336)$=1.171[.280]
[4] Markov Dependence (1)	$F(3,312)$=0.728[.536]	$F(3,312)$=0.898[.443]	$F(3,312)$=1.343[.260]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=0.956[.329]	$F(1,312)$=0.512[.475]	$F(1,312)$=5.597[.019]
[5.3] Variance Heterogeneity	$F(1,336)$=3.171[.076]	$F(1,336)$=0.015[.903]	$F(1,336)$=0.890[.346]

Table G.7 - M-S Testing Results of $2/2_t$ portfolio			
Assumption	y_t=$2/2_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.026[.635]	D=0.062[.070]	D=0.481[.000]
Anderson-Darling	A^2=0.320[.922]	A^2=3.564[.014]	A^2=127.909[.000]
[2] Linearity	$F(3,312)$=0.193[.901]	$F(3,312)$=0.062[.980]	$F(3,312)$=1.464[.224]
[3] Heteroskedasticity	$F(1,336)$=0.789[.375]	$F(1,336)$=0.570[.451]	$F(1,336)$=0.296[.587]
[4] Markov Dependence (1)	$F(3,312)$=0.876[.454]	$F(3,312)$=0.909[.437]	$F(3,312)$=1.273[.284]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=0.773[.380]	$F(1,312)$=0.497[.481]	$F(1,312)$=8.470[.004]
[5.3] Variance Heterogeneity	$F(1,336)$=1.183[.278]	$F(1,336)$=0.046[.831]	$F(1,336)$=0.361[.548]

Table G.8 - M-S Testing Results of $2/3_t$ portfolio			
Assumption	y_t=$2/3_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.047[.223]	D=0.037[.386]	D=0.478[.000]
Anderson-Darling	A^2=1.055[.330]	A^2=0.484[.763]	A^2=127.122[.000]
[2] Linearity	$F(3,312)$=0.069[.976]	$F(3,312)$=0.044[.988]	$F(3,312)$=1.375[.250]
[3] Heteroskedasticity	$F(1,336)$=0.279[.598]	$F(1,336)$=1.164[.281]	$F(1,336)$=0.121[.729]
[4] Markov Dependence (1)	$F(3,312)$=0.979[.403]	$F(3,312)$=0.884[.450]	$F(3,312)$=2.103[.100]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=1.135[.288]	$F(1,312)$=0.422[.516]	$F(1,312)$=6.078[.014]
[5.3] Variance Heterogeneity	$F(1,336)$=6.003[.015]	$F(1,336)$=0.323[.570]	$F(1,336)$=0.030[.862]

Table G.9 - M-S Testing Results of $2/4_t$ portfolio			
Assumption	y_t=$2/4_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.033[.476]	D=0.035[.430]	D=0.479[.000]
Anderson-Darling	A^2=0.668[.586]	A^2=0.512[.735]	A^2=127.117[.000]
[2] Linearity	$F(3,312)$=0.096[.962]	$F(3,312)$=0.031[.993]	$F(3,312)$=1.159[.326]
[3] Heteroskedasticity	$F(1,336)$=0.382[.537]	$F(1,336)$=0.660[.417]	$F(1,336)$=0.056[.813]
[4] Markov Dependence (1)	$F(3,312)$=0.779[.506]	$F(3,312)$=0.969[.408]	$F(3,312)$=3.409[.018]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=1.082[.299]	$F(1,312)$=0.623[.431]	$F(1,312)$=7.446[.007]
[5.3] Variance Heterogeneity	$F(1,336)$=6.006[.015]	$F(1,336)$=0.023[.879]	$F(1,336)$=0.013[.908]

Table G.10 - M-S Testing Results of $2/H_t$ portfolio			
Assumption	y_t=$2/H_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.056[.120]	D=0.026[.642]	D=0.479[.000]
Anderson-Darling	A^2=3.106[.024]	A^2=0.349[.898]	A^2=127.106[.000]
[2] Linearity	$F(3,313)$=0.036[.991]	$F(3,313)$=0.001[.999]	$F(3,313)$=1.399[.243]
[3] Heteroskedasticity	$F(1,336)$=0.657[.418]	$F(1,336)$=0.241[.624]	$F(1,336)$=0.082[.775]
[4] Markov Dependence (1)	$F(3,313)$=1.293[.277]	$F(3,313)$=0.989[.398]	$F(3,313)$=2.491[.141]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,313)$=0.179[.673]	$F(1,313)$=0.495[.482]	$F(1,313)$=1.563[.212]
[5.3] Variance Heterogeneity	$F(1,336)$=5.684[.018]	$F(1,336)$=0.001[.971]	$F(1,336)$=0.061[.805]

Table G.11 - M-S Testing Results of $3/L_t$ portfolio			
Assumption	y_t=$3/L_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.042[.296]	D=0.061[.081]	D=0.481[.000]
Anderson-Darling	A^2=0.781[.495]	A^2=4.029[.008]	A^2=128.018[.000]
[2] Linearity	$F(3,312)$=0.918[.432]	$F(3,312)$=0.417[.741]	$F(3,312)$=1.764[.154]
[3] Heteroskedasticity	$F(1,336)$=2.275[.132]	$F(1,336)$=0.642[.424]	$F(1,336)$=2.238[.136]
[4] Markov Dependence (1)	$F(3,312)$=0.569[.636]	$F(3,312)$=0.595[.619]	$F(3,312)$=1.153[.328]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=0.579[.447]	$F(1,312)$=0.453[.501]	$F(1,312)$=6.103[.014]
[5.3] Variance Heterogeneity	$F(1,336)$=1.184[.277]	$F(1,336)$=0.119[.730]	$F(1,336)$=1.196[.275]

Table G.12 - M-S Testing Results of $3/2_t$ portfolio			
Assumption	y_t=$3/2_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.045[.248]	D=0.054[.137]	D=0.479[.000]
Anderson-Darling	A^2=0.752[.517]	A^2=1.603[.154]	A^2=127.391[.000]
[2] Linearity	$F(3,312)$=0.106[.957]	$F(3,312)$=0.011[.999]	$F(1,312)$=1.253[.291]
[3] Heteroskedasticity	$F(1,336)$=0.073[.787]	$F(1,336)$=0.433[.511]	$F(1,336)$=0.162[.688]
[4] Markov Dependence (1)	$F(3,312)$=0.821[.483]	$F(3,312)$=0.993[.396]	$F(3,312)$=1.391[.245]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=0.192[.662]	$F(1,312)$=0.229[.632]	$F(1,312)$=6.678[.010]
[5.3] Variance Heterogeneity	$F(1,336)$=0.615[.434]	$F(1,336)$=0.008[.927]	$F(1,336)$=0.238[.626]

Table G.13 - M-S Testing Results of $3/3_t$ portfolio			
Assumption	y_t=$3/3_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.041[.315]	D=0.048[.208]	D=0.477[.000]
Anderson-Darling	A^2=0.710[.551]	A^2=0.716[.546]	A^2=127.048[.000]
[2] Linearity	$F(3,312)$=0.179[.911]	$F(3,312)$=0.128[.943]	$F(3,312)$=1.455[.227]
[3] Heteroskedasticity	$F(1,336)$=0.031[.861]	$F(1,336)$=0.579[.447]	$F(1,336)$=0.099[.753]
[4] Markov Dependence (1)	$F(3,312)$=1.121[.341]	$F(3,312)$=0.648[.585]	$F(3,312)$=2.237[.084]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=0.304[.582]	$F(1,312)$=0.682[.410]	$F(1,312)$=4.964[.027]
[5.3] Variance Heterogeneity	$F(1,336)$=4.090[.044]	$F(1,336)$=0.025[.875]	$F(1,336)$=0.082[.774]

Table G.14 - M-S Testing Results of $3/4_t$ portfolio			
Assumption	y_t=$3/4_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.041[.315]	D=0.034[.460]	D=0.480[.000]
Anderson-Darling	A^2=0.441[.808]	A^2=0.567[.680]	A^2=127.167[.000]
[2] Linearity	$F(3,312)$=0.079[.972]	$F(3,312)$=0.021[.996]	$F(3,312)$=1.584[.193]
[3] Heteroskedasticity	$F(1,336)$=0.248[.619]	$F(1,336)$=0.511[.475]	$F(1,336)$=0.555[.457]
[4] Markov Dependence (1)	$F(3,312)$=0.761[.516]	$F(3,312)$=0.904[.439]	$F(3,312)$=1.676[.172]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=0.216[.642]	$F(1,312)$=0.679[.411]	$F(1,312)$=5.526[.019]
[5.3] Variance Heterogeneity	$F(1,336)$=3.530[.061]	$F(1,336)$=0.011[.918]	$F(1,336)$=0.253[.616]

Table G.15 - M-S Testing Results of $3/H_t$ portfolio			
Assumption	y_t=$3/H_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.050[.181]	D=0.035[.433]	D=0.478[.000]
Anderson-Darling	A^2=2.047[.087]	A^2=0.428[.820]	A^2=126.948[.000]
[2] Linearity	$F(3,312)$=0.158[.925]	$F(3,312)$=0.042[.989]	$F(3,312)$=1.203[.309]
[3] Heteroskedasticity	$F(1,336)$=0.005[.945]	$F(1,336)$=0.154[.695]	$F(1,336)$=0.019[.890]
[4] Markov Dependence (1)	$F(3,312)$=1.341[.261]	$F(3,312)$=0.575[.632]	$F(3,312)$=2.088[.102]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=0.185[.668]	$F(1,312)$=0.759[.384]	$F(1,312)$=5.132[.024]
[5.3] Variance Heterogeneity	$F(1,336)$=5.863[.016]	$F(1,336)$=0.211[.646]	$F(1,336)$=0.483[.488]

Table G.16 - M-S Testing Results of $4/L_t$ portfolio			
Assumption	y_t=$4/L_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.065[.054]	D=0.066[.049]	D=0.480[.000]
Anderson-Darling	A^2=1.523[.171]	A^2=3.214[.021]	A^2=129.833[.000]
[2] Linearity	$F(3,312)$=1.495[.216]	$F(3,312)$=0.850[.467]	$F(3,312)$=2.404[.068]
[3] Heteroskedasticity	$F(1,336)$=1.381[.241]	$F(1,336)$=1.148[.285]	$F(1,336)$=1.500[.222]
[4] Markov Dependence (1)	$F(3,312)$=0.433[.730]	$F(3,312)$=0.662[.576]	$F(3,312)$=1.139[.334]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=0.139[.709]	$F(1,312)$=0.146[.703]	$F(1,312)$=8.468[.004]
[5.3] Variance Heterogeneity	$F(1,336)$=0.669[.414]	$F(1,336)$=0.001[.972]	$F(1,336)$=0.554[.457]

Table G.17 - M-S Testing Results of $4/2_t$ portfolio			
Assumption	y_t=$4/2_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.054[.140]	D=0.036[.408]	D=0.476[.000]
Anderson-Darling	A^2=0.922[.401]	A^2=0.398[.851]	A^2=127.101[.000]
[2] Linearity	$F(3,312)$=0.748[.525]	$F(3,312)$=0.252[.860]	$F(3,312)$=1.037[.376]
[3] Heteroskedasticity	$F(1,336)$=0.775[.379]	$F(1,336)$=0.695[.405]	$F(1,336)$=1.861[.174]
[4] Markov Dependence (1)	$F(3,312)$=0.471[.703]	$F(3,312)$=0.608[.610]	$F(3,312)$=1.202[.309]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=0.569[.451]	$F(1,312)$=0.935[.335]	$F(1,312)$=6.868[.359]
[5.3] Variance Heterogeneity	$F(1,336)$=0.208[.649]	$F(1,336)$=0.530[.467]	$F(1,336)$=0.150[.699]

Table G.18 - M-S Testing Results of $4/3_t$ portfolio			
Assumption	y_t=$4/3_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.073[.027]	D=0.053[.148]	D=0.467[.000]
Anderson-Darling	A^2=5.147[.002]	A^2=2.316[.062]	A^2=126.017[.000]
[2] Linearity	$F(3,311)$=0.054[.983]	$F(3,311)$=0.010[.999]	$F(10,310)$=0.786[.502]
[3] Heteroskedasticity	$F(1,336)$=1.447[.230]	$F(1,336)$=0.386[.535]	$F(12,325)$=0.034[.854]
[4] Markov Dependence (1)	$F(3,311)$=0.350[.789]	$F(3,311)$=0.565[.638]	$F(3,310)$=1.134[.335]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.162[.688]	$F(1,311)$=0.006[.937]	$F(1,310)$=1.647[.200]
[5.3] Variance Heterogeneity	$F(1,336)$=0.128[.720]	$F(1,336)$=0.007[.410]	$F(1,325)$=0.362[.548]

Table G.19 - M-S Testing Results of $4/4_t$ portfolio			
Assumption	y_t=$4/4_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.075[.021]	D=0.043[.278]	D=0.471[.000]
Anderson-Darling	A^2=4.893[.003]	A^2=2.343[.060]	A^2=126.165[.000]
[2] Linearity	$F(3,314)$=0.013[.998]	$F(3,314)$=0.013[.998]	$F(3,314)$=0.484[.693]
[3] Heteroskedasticity	$F(1,336)$=0.008[.931]	$F(1,336)$=0.427[.514]	$F(1,336)$=0.884[.348]
[4] Markov Dependence (1)	$F(3,314)$=0.422[.738]	$F(3,314)$=0.322[.810]	$F(3,314)$=4.282[.006]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,314)$=1.168[.281]	$F(1,314)$=1.426[.233]	$F(1,314)$=0.477[.490]
[5.3] Variance Heterogeneity	$F(1,336)$=6.518[.011]	$F(1,336)$=0.001[.990]	$F(1,336)$=0.502[.479]

Table G.20 - M-S Testing Results of $4/H_t$ portfolio			
Assumption	y_t=$4/H_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.100[.001]	D=0.056[.115]	D=0.469[.000]
Anderson-Darling	A^2=13.936[.000]	A^2=4.513[.005]	A^2=125.812[.000]
[2] Linearity	$F(3,311)$=0.129[.943]	$F(3,311)$=0.181[.909]	$F(3,311)$=1.019[.385]
[3] Heteroskedasticity	$F(1,336)$=1.337[.248]	$F(1,336)$=0.953[.330]	$F(1,336)$=4.438[.036]
[4] Markov Dependence (1)	$F(3,311)$=0.203[.894]	$F(3,311)$=0.363[.780]	$F(3,311)$=1.424[.236]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,311)$=0.067[.796]	$F(1,311)$=0.044[.834]	$F(1,311)$=1.517[.219]
[5.3] Variance Heterogeneity	$F(1,336)$=5.116[.024]	$F(1,336)$=0.013[.908]	$F(1,336)$=0.259[.611]

Table G.21 - M-S Testing Results of B/L_t portfolio			
Assumption	y_t=B/L_t	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.036[.409]	D=0.038[.369]	D=0.479[.000]
Anderson-Darling	A^2=0.545[.701]	A^2=0.627[.623]	A^2=127.126[.000]
[2] Linearity	$F(3,312)$=0.026[.994]	$F(3,312)$=0.009[.999]	$F(3,312)$=1.154[.328]
[3] Heteroskedasticity	$F(1,336)$=0.241[.624]	$F(1,336)$=0.240[.625]	$F(1,336)$=0.101[.751]
[4] Markov Dependence (1)	$F(3,312)$=0.417[.741]	$F(3,312)$=0.549[.649]	$F(3,312)$=1.032[.379]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,312)$=0.730[.394]	$F(1,312)$=0.642[.424]	$F(1,312)$=5.582[.019]
[5.3] Variance Heterogeneity	$F(1,336)$=0.020[.887]	$F(1,336)$=0.081[.776]	$F(1,336)$=1.217[.271]

Table G.22 - M-S Testing Results of $B/2_t$ portfolio			
Assumption	y_t=$B/2_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.049[.197]	D=0.032[.500]	D=0.480[.000]
Anderson-Darling	A^2=0.525[.722]	A^2=0.380[.868]	A^2=127.105[.000]
[2] Linearity	$F(3,313)$=0.184[.907]	$F(3,313)$=0.146[.932]	$F(3,313)$=1.822[.143]
[3] Heteroskedasticity	$F(1,336)$=0.730[.394]	$F(1,336)$=0.915[.339]	$F(1,336)$=0.072[.788]
[4] Markov Dependence (1)	$F(3,313)$=0.770[.512]	$F(3,313)$=0.438[.726]	$F(3,313)$=1.926[.125]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,313)$=0.972[.325]	$F(1,313)$=1.022[.313]	$F(1,313)$=1.060[.304]
[5.3] Variance Heterogeneity	$F(1,336)$=0.086[.770]	$F(1,336)$=0.808[.369]	$F(1,336)$=0.995[.319]

Table G.23 - M-S Testing Results of $B/3_t$ portfolio			
Assumption	y_t=$B/3_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.094[.002]	D=0.108[.000]	D=0.462[.000]
Anderson-Darling	A^2=13.299[.000]	A^2=19.974[.000]	A^2=123.618[.000]
[2] Linearity	$F(3,318)$=0.181[.910]	$F(3,318)$=0.233[.873]	$F(3,318)$=0.949[.417]
[3] Heteroskedasticity	$F(1,336)$=0.547[.460]	$F(1,336)$=1.558[.213]	$F(1,336)$=0.358[.550]
[4] Markov Dependence (1)	$F(3,318)$=0.667[.573]	$F(3,318)$=0.546[.651]	$F(3,318)$=1.915[.127]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,318)$=1.600[.207]	$F(1,318)$=1.816[.179]	$F(1,318)$=0.246[.621]
[5.3] Variance Heterogeneity	$F(1,336)$=0.761[.384]	$F(1,336)$=0.226[.635]	$F(1,336)$=3.839[.051]

Table G.24 - M-S Testing Results of $B/4_t$ portfolio			
Assumption	y_t=$B/4_t$	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.111[.000]	D=0.124[.000]	D=0.462[.000]
Anderson-Darling	A^2=17.035[.000]	A^2=21.043[.000]	A^2=123.411[.000]
[2] Linearity	$F(3,318)$=0.236[.871]	$F(3,318)$=0.322[.810]	$F(3,318)$=0.860[.462]
[3] Heteroskedasticity	$F(1,336)$=0.502[.479]	$F(1,336)$=0.826[.364]	$F(1,336)$=0.833[.362]
[4] Markov Dependence (1)	$F(3,318)$=0.848[.469]	$F(3,318)$=0.979[.403]	$F(3,318)$=1.373[.251]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,318)$=1.729[.190]	$F(1,318)$=1.894[.170]	$F(1,318)$=0.216[.643]
[5.3] Variance Heterogeneity	$F(1,336)$=0.319[.573]	$F(1,336)$=1.714[.191]	$F(1,336)$=4.276[.039]

Table G.25 - M-S Testing Results of B/H_t portfolio			
Assumption	y_t=B/H_t	R_{mt}	R_{ft}
[1] Student's t:			
Kolmogorov-Smirnov	D=0.109[.000]	D=0.121[.000]	D=0.464[.000]
Anderson-Darling	A^2=24.144[.000]	A^2=22.398[.000]	A^2=123.405[.000]
[2] Linearity	$F(3,318)$=0.156[.926]	$F(3,318)$=0.130[.942]	$F(3,318)$=1.278[.282]
[3] Heteroskedasticity	$F(1,336)$=0.693[.406]	$F(1,336)$=0.346[.557]	$F(1,336)$=1.543[.215]
[4] Markov Dependence (1)	$F(3,318)$=0.497[.684]	$F(3,318)$=0.380[.767]	$F(3,318)$=1.092[.353]
[5] t-invariance:			
[5.1] Mean Heterogeneity	$F(1,318)$=0.462[.497]	$F(1,318)$=1.400[.238]	$F(1,318)$=0.374[.541]
[5.3] Variance Heterogeneity	$F(1,336)$=0.083[.774]	$F(1,336)$=2.330[.128]	$F(1,336)$=3.792[.052]

Chapter 5

5 On Selecting Factors in Multi-Factor Models

5.1 Overview

The results in chapter 3 indicate clearly that the Fama and French (1993) data do not satisfy the probabilistic assumptions imposed on them for both the CAPM and the Fama-French three factor model. In light of these results, chapter 4 respecified a statistically adequate model which is shown to account for all the statistical regularities in the data. The latter is particularly important because the reliability of the significance tests of the Fama-French factors depends crucially on whether the estimation of the model prior the factor inclusion is statistically adequate. As argued, when that is not the case, the tests of significance are likely to be unreliable because of sizeable discrepancies between their actual and nominal error probabilities.

This chapter has two main objectives. First, to propose a factor selection procedure. This procedure has the advantage of providing a coherent basis for selecting the relevant factors from a set of possible ones. Second, to employ the factor selection procedure to test for the significance of the Fama-French factors using the Fama and French (2015) data. A brief outline of the chapter follows. Section 5.2 provides a brief discussion of the factor selection problem. Section 5.3 briefly discusses the Fama and French (2015) data. Section 5.4 discusses the building blocks of the proposed factor selection procedure. The general notion of this procedure it to account for all the regularities in the data using deterministic variables, and then, on the basis of a statistically adequate model, attempt to replace these variables with observed factors. By testing for the significance of the Fama-French factors, the empirical results indicate that the value factor HML is not redundant, as conjectured by Fama and French (2015), whereas the redundant factor for describing average returns is the investment factor CMA. In addition, the section argues that although the Fama-French multi-factor models extend the CAPM in the right direction, the lingering heterogeneity in the data indicate that there are still missing relevant factors.

5.2 The Problem of Factor Selection

The Intertemporal CAPM (ICAPM) of Merton (1973) and Arbitrage Pricing Theory (APT) developed by Ross (1976) can be interpreted as providing the justifications for the study of multi-factor models. This interpretation is stemming from the existence of a limited number of factors, a general assumption underlying these models. Explicitly, the ICAPM allows for a number of 'state variables' arising from investors demand to hedge uncertainty about future investment opportunities, and the APT provides an approximation relation of expected asset returns with a number of factors.

In both the ICAPM and APT, the appropriate number of factors is unknown and the factors themselves are not identified. Although one may argue that the latter can be viewed as an advantage from a purely theoretical concept, the lack of consensus in identifying the appropriate set of factors is an unfavorable condition once the factor selection becomes an empirical issue. Thus, the selection of factors is often ad hoc and studies of multi-factor models are often criticized as being motivated by purely empirical conceptions; for a detailed discussion of the factor selection problem, as well as the ICAPM and the APT, see Campbell et al. (1997, pp. 219-251).

Fama (1991) was among the first to raise awareness of this fundamental problem:

> "Since multifactor models offer at best vague predictions about the variables that are important in returns and expected returns, there is the danger that measured relations between returns and economic factors are spurious [empha-

> sis added], the result of special features of a particular sample (factor dredging)." Fama (1991, p. 1595).
>
> "One can argue that an open competition among the SLB [meaning the CAPM], multifactor, and consumption models is biased in favor of the multifactor model. The expected-return variables of the SLB and consumption models (market and consumption β's) are clearly specified. In contrast, the multifactor models are licenses [emphasis added] to search the data for variables that, ex post, describe the cross-section of average returns. It is perhaps no surprise, then, that these variables do well in competitions on the data used to identify them." Fama (1991, p. 1598).

Although this ad hoc selection problem has been well documented in the literature (e.g., Ferson, 1996; Lewellen et al., 2010), several decades after the first empirical tests, we still lack an empirical base sufficient for selecting factors. Worse, we still try to understand what are the state variables that drive the different factors, and why do they lead to variation in expected returns missed by the market beta. As Fama (2011) stated, any evidence proposed in the literature so far is unconvincing.

A recently released study by Harvey et al. (2015) attempts to address one part of the problem, and specifically the factor spuriousness arising due to extensive data mining. As the authors argue, it is a serious mistake to use the usual criteria for establishing factor significance. Instead, the significance criteria should vary through time as more factors are data mined. Although the current study agrees with the latter argument, it identifies two additional problems which are worth attention and should be addressed separately.

The first problem is the significance 'artifact' of the large sample size; see Good (1988), Lehmann and Romano (2006). In empirical studies of asset pricing searching for factors, the employed sample sizes are usually large enough to violate the most commonly used threshold of t-statistic of 2.0 or p-value of .05. In addition, the median sample size employed in empirical asset pricing studies is likely to increase through time. For instance, the studies of the three-factor and five-factor models (see Fama and French, 1993; 2015) employ sample sizes of n=342 and n=606, respectively. Hence, in practise one has to choose much smaller thresholds for significance when using p-values (or much larger thresholds when using t-statistics).[3]

The second problem is the problem of statistical misspecification that has been discussed already in great length. This problem is indeed not much different than the problem of data mining. There are potentially thousands of plausible statistical models and techniques which offer the modeler the flexibility of choosing a model that produces attractive results. It would not be outrageous to argue that the statistical assumptions underlying the models employed are almost never stated explicitly by the authors. As mentioned earlier, if one or more of these assumptions are invalid, the significance tests of the added factors are likely to be unreliable because of sizeable discrepancies between their actual and nominal error probabilities.

The latter can be characterized using the simulation results in Appendix A. Intuitively, one can pose the question of omitted variables (i.e., factors) in the context of both the statistically adequate and misspecified models, by treating x_{2t} as an omitted variable. When this variable is included in a statistically adequate model, its empirical Type I error probability associated with the t-test is expected to be very close to the nominal error probability of α=0.05. On the other hand, if the same variable is included in a statistically misspecified model, the corresponding empirical Type I error probability is expected to be very different from the nominal one. For instance, the simulation results in tables A1–A2 suggest that even when one of the probabilistic assumptions is invalid, the inclusion of omitted variables in statistically misspecified models may result in rejecting the true null hypothesis 100% of the time. Thus, 'inference mining'.

In what follows, the chapter will propose a factor selection procedure. This procedure is discussed below and later used to test for the significance of the Fama-French

[3] For the current study a t-statistic of 2.58 (p-value of 0.01) is employed for a sample size of 342, and a t-statistic of 3.29 (p-value of 0.001) for a sample size of 606.

factors, as well as to identify if there are still missing relevant factors for the data in question.

5.3 Data and Relevant Variables

Given that the primary aim of this chapter is to test for the significance of the Fama and French (2015) factors on the basis of a statistically adequate model, the sample is the one used in Fama and French (2015). This sample spans a 606 month period from July 1963 to December 2013. The data are downloaded from Kenneth French's online data library[4] where the reader can find a detailed description for the construction of the data, as well as clarifying details and information. A brief description of the data follows.

To size (SMB_t), value (HML_t), profitability (RMW_t), and investment (CMA_t) factors are constructed using independent 2×3 sorts on size and each of book-to-market equity (B/M), operating profitability (OP), and investment (Inv). The size breakpoint is the NYSE median market equity, whereas the B/M, OP, and Inv breakpoints are the respective 30^{th} and 70^{th} percentiles for NYSE stocks. The sample used comprise all the NYSE, AMEX, and NASDAQ stocks. The intersections of groups define, six value-weighted portfolios formed on size and B/M, six value-weighted portfolios formed on size and OP, and six value-weighted portfolios formed on size and Inv. The four factors are defined as differences between the average returns on the value-weighted portfolios. The market return (R_{mt}) is the return on the market value-weighted portfolio and the risk-free return (R_{ft}) is the one-month Treasure bill rate. Moreover, the 25 *Size-B/M*, 25 *Size-OP*, and 25 *Size-Inv* portfolios (R_{it}) are the average returns of value-weighted portfolios constructed on independent 5×5 sorts of stocks on size and each of B/M, OP, and Inv. The size, B/M, OP, and Inv quintile breakpoints use only NYSE stocks, but the sample used comprise all the NYSE, AMEX, and NASDAQ stocks.

[4] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/.

In addition, the following (i)-(iii) additional variables are used with a view to capture the statistical regularities exhibited by the data: (i) Gram-Schmidt orthonormal trend polynomials, (ii) monthly seasonal dummy variables, and (iii) lags in both dependent and explanatory variables; see section 3.3 for additional details on the construction of these variables.

5.4 The Factor Selection Procedure

The proposed factor selection procedure is a three-step procedure. The first step is the establishment of a statistically adequate model that ensures for the reliability of statistical inference results; that is, a model that accounts for all the statistical systematic information in the data and is likely to include deterministic variables (i.e., trends, seasonal dummy variables) and assume an appropriate distribution that fits well the data in hand. The second step is the inclusion of the observed factors to the statistically adequate model in the process of testing for statistical significance. The importance of these first two steps is stemming from the fact that the reliability of the significance tests of the added factors depends crucially on whether the model prior the inclusion is statistically adequate.

The third step is the observation of how much of the variation in returns remains unexplained by examining the change of the statistical significance of the deterministic variables already included in the model. The inclusion of these deterministic variables in an asset pricing model can provide a generic way to account for the regularities in the data, but these variables do not explain what gave rise to the regularities in the data. On that account, the presence of deterministic variables indicates substantive ignorance by suggesting that certain random variables (i.e., factors, seasonal explanations, etc.) are clearly missing.

The factor selection procedure is used below to test for the significance of the Fama-French factors, as well as to identify if there are still missing relevant factors for the data in question.

5.4.1 Step I: Testing for Statistical Adequacy

As discussed in chapter 3, a statistical model can be viewed as a reparameterization of an observable stochastic process. Along these lines, the models resulted from the second attempt at respecification in section 4.4 are not guaranteed to be statistically adequate for any observable stochastic process involved. The Fama and French (2015) data differ from the Fama and French (1993) data considered earlier, in terms of the sample size and the number of variables. Consequently, it is of paramount importance to confirm that the previously respecified models are entirely satisfactory to probe questions of interests for the new data in question.

A similar procedure as in chapters 3 and 4 is followed and briefly discussed below. Yet, at this point, it would be more interesting for comparison purposes, to evaluate the statistical model assumptions [1]–[5] for the models employed in the literature. Hence, the estimated residuals of the CAPM, three-factor, and five-factor models in (1.3) are carefully examined. The statistical assumptions for the CAPM take the form of the model assumptions [1]–[5] that justify the Bivariate Normal/ Homoskedastic LR model in table 3.1, and for the Fama-French multi-factor models take the form of [1]–[5] that justify the Multivariate Normal/Homoskedastic LR model in table 3.2. The Shapiro-Francia (W') test (see Appendix H) is employed to test for [1] Normality, and the two auxiliary regressions (F) in (3.10)-(3.11) are employed to test for the model assumptions of [2] Linearity, [3] Homoskedasticity, [4] Independence, and [5] t-invariance.

The M-S testing results of the 25 *Size-B/M*, 25 *Size-OP*, and 25 *Size-Inv* portfolios, for the CAPM, three-factor, and five-factor models are presented in Appendix I. Table 5.1 includes a representative summary of these M-S results, divided into smaller and bigger size portfolios.

The M-S testing results of the CAPM are very similar to the results reported in section 3.4.3. In short, the results suggest departures from the all the assumptions except [2] Linearity; the assumptions of [1] Normality, [3] Homoskedasticity, and [5.3]

Variance homogeneity are severely violated; and the departures from [4] Independence, [5.1] Mean homogeneity, and [5.2] Seasonal homogeneity are more severe for the smaller size portfolios compared to the bigger size portfolios. The only other thing worth to mention for these results, is that not strong but significant January seasonality appears to exist in the highest B/M, lowest OP, and lowest Inv portfolios.

Table 5.1 - Representative M-S Results for Normal LR Models

Assumption	CAPM	three-factor	five-factor
Smaller Size Portfolios			
[1] Normality	×	×	×
[2] Linearity	✓	×	×
[3] Homoskedasticity	×	×	×
[4] Independence	×	×	×
[5] t-invariance:			
[5.1] Mean Homogeneity	×	✓	✓
[5.2] Seasonal Homogeneity	×	✓	✓
[5.3] Variance Homogeneity	×	✓	✓
Bigger Size Portfolios			
[1] Normality	×	×	×
[2] Linearity	✓	×	×
[3] Homoskedasticity	×	×	×
[4] Independence	✓	✓	✓
[5] t-invariance:			
[5.1] Mean Homogeneity	✓	✓	✓
[5.2] Seasonal Homogeneity	✓	✓	✓
[5.3] Variance Homogeneity	×	✓	✓

The M-S testing results of the Fama-French multi-factor models are similar to the results reported in section 3.4.3 with respect to assumptions [2] and [4]–[5]. That is, the inclusion of the additional factors to the CAPM, significantly improved the results for assumptions [4]–[5] (with the only exception of [4] for the smaller size portfolios) on the expense of assumption [2]. Nonetheless, the marginal improvements (see section 3.4.3) concerning assumptions [1] and [3] are diminished. Remarkably, these assumptions are severely violated for almost all the portfolios.

The above M-S testing results render the estimated CAPM and Fama-French multi-factor models statistically misspecified for the data in question. In order to account for the departures indicated by the above M-S testing results, the Normal/Homoskedastic LR model is replaced with the Student's t/Heteroskedastic Dynamic LR model defined in table 4.5. This model includes the Gram-Schmidt orthonormal polynomials and monthly seasonal dummy variables with a view to capture the heterogeneity exhibited by the data, as well as lags with a view to capture the temporal dependence. Since the inclusion of trend, seasonality, and lagged terms give no meaningful interpretation of the zero intercept term, the estimations of the heterogeneous Student's t/Heteroskedastic Dynamic LR model take the form of the explicit statistical CAPM in (3.5) which involves the additional nesting restriction of $\beta_1+\beta_2=1$ (Fama, 1973; Spanos, 2006c):

$$\begin{aligned}\underbrace{y_{it}=\alpha_i+\beta_{1i}x_{1t}+\beta_{2i}x_{2t}}_{\text{explicit statistical CAPM}}+\sum_{j=1}^{m^*}\delta_{1ji}v_{jt}+\sum_{j=2}^{s}\delta_{2ji}d_{jt}+\boldsymbol{\gamma}_i^\top\mathbf{Z}_{it-1}+u_{it},\\ i=1,2,...,k,\ \ t\in\mathbb{N},\end{aligned}\tag{5.1}$$

where $y_{it}=R_{it}$ is the return of portfolio i for period t, $x_{1t}=R_{mt}$ is the return of the value-weighted market portfolio, $x_{2t}=R_{ft}$ is the risk-free return, v_{jt} denotes the terms of the Gram-Schmidt orthonormal polynomials of order $j=1,2,...,m^*$, $d_{jt}:=(d_{2t},d_{3t},...,d_{12t})$ are the monthly dummy variables for the months of February through December, $\mathbf{Z}_{it}:=(y_{it},\mathbf{X}_t)$, for $\mathbf{X}_t:=(x_{1t},x_{2t})$.

To evaluate the model assumptions [1]–[5] that justify the heterogeneous Student's t/ Heteroskedastic Dynamic LR model, the estimated standardized residuals (see section 4.4.1) of the model in (5.1) are carefully examined. The Kolmogorov-Smirnov

(D) test (see Appendix E) is employed to test for [1] Student's t, and the auxiliary regressions (F) in (4.47)-(4.48) are extended to test for the model assumptions of [2] Linearity, [3] Heteroskedasticity, [4] Markov (1), and [5] t-invariance.

Formally, the two auxiliary regressions take the form:

$$(\widehat{u}_{it})^{st} = \gamma_{1i} + \boldsymbol{\gamma}_{2i}^{\top}\boldsymbol{\Xi}_{it}^{*} + \underbrace{\sum\nolimits_{j=m^{*}+1}^{m}\gamma_{3ji}v_{jt}}_{\boxed{5.1}} + \underbrace{\boldsymbol{\gamma}_{4i}^{\top}\boldsymbol{\psi}_{t}}_{\boxed{2}} + \underbrace{\sum\nolimits_{j=2}^{p}\boldsymbol{\gamma}_{5ji}^{\top}\mathbf{Z}_{it-j}}_{\boxed{4}} + \varepsilon_{1it}, \tag{5.2}$$

$$(\widehat{u}_{it}^{2})^{st} = \gamma_{6i} + \underbrace{\sum\nolimits_{j=1}^{m}\gamma_{7ji}v_{jt}}_{\boxed{5.3}} + \underbrace{\boldsymbol{\gamma}_{8i}^{\top}\mathbf{W}_{it} + \boldsymbol{\gamma}_{9i}^{\top}\boldsymbol{\psi}_{t} + \sum\nolimits_{j=2}^{p}\boldsymbol{\gamma}_{10ji}^{\top}\mathbf{Z}_{it-j}^{2}}_{\boxed{3}+\boxed{4}} + \varepsilon_{2it}, \tag{5.3}$$

where $(\widehat{u}_{it})^{st}$ are the estimated standardized residuals in (5.1) of portfolio i for period t; $\mathbf{W}_{it}:=(\mathbf{X}_{t}, \mathbf{Z}_{it-1})=(w_{1t}, ..., w_{lt})$; $\boldsymbol{\psi}_{t}:=\{(w_{it}\cdot w_{jt}), i\geq j, i, j=1, 2, ..., l\}$ are the second-order Kolmogorov-Gabor polynomials; $\boldsymbol{\Xi}_{it}^{*}:=(v_{1t}, v_{2t}, ..., v_{mt}^{*}, d_{2t}, d_{3t}, ..., d_{12t}, \mathbf{W}_{it})$.

The M-S testing results of the 25 *Size-B/M*, 25 *Size-OP*, and 25 *Size-Inv* portfolios, for the explicit CAPM are presented in Appendix J. These results indicate that the underlying model is a statistically adequate model since all the model assumptions [1]–[5] are satisfied with minor exceptions for a small number of portfolios.

Testing the Over-identifying Restrictions. Since the heterogeneous Student's t/Heteroskedastic Dynamic LR $(1; \nu)$ model in (5.1) ensures for the reliability of statistical inference results, it can be used as a sound basis for testing the over-identifying restrictions. The over-identifying restrictions take the form of:

$$H_0\text{: } \mathbf{G}(\boldsymbol{\varphi}, \boldsymbol{\theta})=\mathbf{0}, \text{ vs. } H_1\text{: } \mathbf{G}(\boldsymbol{\varphi}, \boldsymbol{\theta})\neq\mathbf{0}, \text{ for } \boldsymbol{\varphi}\in\Phi,\ \boldsymbol{\theta}\in\Theta. \tag{5.4}$$

The relevant test is based on the likelihood ratio statistic:

$$\lambda_n(\mathbf{Z}) = \frac{\max_{\boldsymbol{\varphi}\in\Phi} L(\boldsymbol{\varphi};\mathbf{Z})}{\max_{\boldsymbol{\theta}\in\Theta} L(\boldsymbol{\theta};\mathbf{Z})} = \frac{L(\widetilde{\boldsymbol{\varphi}};\mathbf{Z})}{L(\widehat{\boldsymbol{\theta}};\mathbf{Z})} \Rightarrow -2\ln\lambda_n(\mathbf{Z}) \overset{H_0}{\underset{a}{\backsim}} \chi^2(r). \tag{5.5}$$

where $L(\widetilde{\boldsymbol{\varphi}}; \mathbf{Z})$ is the likelihood function of the substantive model in (3.1), and $L(\widehat{\boldsymbol{\theta}}; \mathbf{Z})$ is the likelihood function of the statistical model in (5.1).

For r=27 and α=.001, the observed test statistic of the (smallest size/lowest B/M) portfolio, yields:

$$-2\ln\lambda_n(\mathbf{Z}_0)=115.63[6.128\times10^{-13}], \tag{5.6}$$

where the number in square brackets denotes the p-value. This testing result is typical of the results pertaining to the validity of the over-identifying restrictions imposed by the CAPM and its extensions. The tiny p-value in (5.6) suggests that the Fama and French (2015) data provide strong evidence against the models estimated by the Normal/Homoskedastic LR models. This calls for a reconsideration of empirical asset pricing models more generally, with a view to explain the regularities found in stocks returns. In particular, the strong leptokurticity and the lingering heterogeneity need to be accounted first, before probing any substantive questions of interest like testing for the significance of factors.

The statistical adequacy of the heterogeneous Student's t/ Heteroskedastic Dynamic LR model in (5.1) allows one to include the Fama-French factors in the model, in an attempt to identify which of these factors capture variation in returns that is left unexplained by the original CAPM.

5.4.2 Step II: Testing for Candidate Factors

In the second step, the objective is to test for the significance of the Fama-French factors, by posing the general question of omitted relevant variables on the basis of a statistically adequate model. The statistically adequate model in (5.1) is estimated by including one of the four factors, combinations of two, three, and all the four factors. At this stage, the HML is not treated as a redundant factor in describing average returns since the regressions of each of the five factors on the other four (see Fama and French, 2015) are statistically misspecified.[5]

[5] The estimated residuals of the regression to explain HML on the other four factors have been examined. The Shapiro Francia (W') test is employed to test for [1] Normality, and the auxilliary regressions (F) in (3.10)-(3.11) are employed to test for model assumptions of [2] Linearity, [3] Homoskedasticity, [4] Independence, and [5] t-invariance. The M-S testing results are the following: [1] 0.969[.000]; [2] 5.082[.000]; [3] 94.985[.000]; [4] 2.769[.018]; [5.1] 0.774 [.379], [5.2] 2.492[.005], [5.3]

When the regressions for the 25 *Size-B/M*, 25 *Size-OP*, and 25 *Size-Inv* portfolios include only one of the SMB, HML, RMW, or CMA, the added factor is always statistically significant with no exceptions. In addition, the statistical significance of the added factors remains unchanged in regressions that include combinations of the SMB and one of the HML, RMW, or CMA. These results suggest two things. First, all the four factors capture variation in returns that is left unexplained by the original CAPM, and second the SMB does not absorb any average returns of the other three factors. By that means, the attention is put on whether the average returns of HML, RMW, or CMA, are captured by their exposures with respect to each other.

Table 5.2 shows the estimated coefficients and p-values of HML, RMW, and CMA, for the 25 *Size-B/M* portfolios. The HML slopes are statistically significant, and increasing monotonically from strongly negative for low B/M portfolios to strongly positive for high B/M portfolios - value effect. This is not surprising given that these portfolios are formed on B/M ratios. Besides, the slopes of RMW and CMA are statistically significant for nearly half of the portfolios. The latter makes it interesting to identify if the average returns of the two factors are captured by exposures to HML, or with respect to each other. Concerning the RMW, there is no clear evidence indicating that the average returns of the factor are captured by exposures to any of the other two factors. For instance, the RMW is highly statistically significant in regressions that include RMW and one of HML or CMA, as well as in regressions that include all RMW, HML, and CMA. On the contrary, the evidence for the CMA is clearer. In regressions that include only CMA and RMW, the former is highly statistically significant, whereas in regressions that include only CMA and HML, the factor is statistically insignificant for the majority of the portfolios. The latter indicates that the average returns of CMA are captured by their exposure to the HML factor.

4.029 [.001]. The p-values in square brackets indicate departures from the model assumptions [1]–[5] which render all inferences untrustworthy, including the close to zero intercept. The misspecification arising is similar to the simulation results in the Appendix table A.2, where the sampling distribution of the intercept, $\widehat{\alpha}_0$, is shifted to the left ($\widehat{\alpha}_0$=.016) of the true value (α_0^*=1.527).

Table 5.2 - Regressions for 25 Size-B/M Portfolios										
The estimated model is (July 1963 to December 2013, 606 months): $R_{it}=\alpha_i+\beta_{1i}R_{mt}+\beta_{2i}R_{ft}+\sum_{j=1}^{m^*}\delta_{1ji}v_{jt}+\sum_{j=2}^{s}\delta_{2ji}d_{jt}+\boldsymbol{\gamma}_i^{\top}\mathbf{Z}_{it-1}$ $+s_iSMB_t+h_iHML_t+r_iRMW_t+c_iCMA_t+u_{it},\ i=1,2,...,k,\ t\in\mathbb{N}$, where R_{it} are the monthly returns on the 25 *Size-B/M* portfolios.										
Size\B/M	Low	2	3	4	High	Low	2	3	4	High
			h_i					$p(h_i)$		
Small	-0.414	-0.085	0.106	0.277	0.444	.000	.001	.000	.000	.000
2	-0.467	-0.043	0.273	0.403	0.659	.000	.038	.000	.000	.000
3	-0.421	0.018	0.308	0.490	0.630	.000	.484	.000	.000	.000
4	-0.374	-0.001	0.325	0.492	0.743	.000	.970	.000	.000	.000
Big	-0.345	0.032	0.210	0.530	0.928	.000	.162	.000	.000	.000
			r_i					$p(r_i)$		
Small	-0.384	-0.228	-0.036	0.017	0.048	.000	.000	.185	.508	.079
2	-0.160	0.044	0.118	0.081	0.089	.000	.089	.000	.003	.002
3	-0.074	0.084	0.145	0.050	0.062	.015	.005	.000	.090	.083
4	-0.015	0.036	0.025	-0.030	-0.029	.617	.268	.445	.364	.487
Big	0.220	0.129	-0.052	0.003	-0.232	.000	.000	.114	.926	.000
			c_i					$p(c_i)$		
Small	-0.042	-0.026	0.070	0.077	0.174	.369	.476	.028	.011	.000
2	-0.110	0.040	-0.021	0.084	0.071	.002	.177	.493	.007	.038
3	-0.211	0.014	0.016	0.037	0.082	.000	.692	.643	.278	.048
4	-0.079	0.163	0.016	0.016	-0.076	.023	.000	.672	.681	.111
Big	-0.043	0.170	0.122	0.020	-0.270	.093	.000	.001	.565	.000

In regard to the estimated coefficients, the negative signs of the RMW and CMA slopes of the smaller size - lowest B/M portfolios are consistent with the reasoning of Fama and French (2015) that these portfolios may contain small stocks of firms that invest a lot despite low profitability. However the magnitude of these slopes is significantly increased. For instance, the strongest negative CMA slope (-0.21) among these portfolios is significantly larger compared to the respective estimated coefficient of -0.67 reported by Fama and French. Also, the CMA slope of the (smallest size/lowest B/M) portfolio is close to zero (-0.04) and statistically insignificant, in contrast to the highly significant and strongly negative estimated coefficient (-0.57) of Fama and French. Similarly, the RMW slope of the (smallest size/lowest B/M) portfolio (-0.38) is about 20 basis points greater than the corresponding estimated coefficient of -0.58; see Fama and French (2015, p. 13) for the 25 *Size-B/M* regression results.

Table 5.3 - Regressions for 25 Size-OP Portfolios

The estimated model is (July 1963 to December 2013, 606 months):

$$R_{it}=\alpha_i+\beta_{1i}R_{mt}+\beta_{2i}R_{ft}+\sum_{j=1}^{m^*}\delta_{1ji}v_{jt}+\sum_{j=2}^{s}\delta_{2ji}d_{jt}+\boldsymbol{\gamma}_i^{\top}\mathbf{Z}_{it-1}$$

$$+s_iSMB_t+h_iHML_t+r_iRMW_t+c_iCMA_t+u_{it},\ i=1,2,...,k,\ t\in\mathbb{N},$$

where R_{it} are the monthly returns on the 25 *Size-OP* portfolios.

Size\OP	Low	2	3	4	High	Low	2	3	4	High
			h_i					$p(h_i)$		
Small	-0.122	0.148	0.226	0.193	0.106	.000	.000	.000	.000	.000
2	-0.150	0.109	0.179	0.163	0.061	.000	.000	.000	.000	.016
3	-0.071	0.146	0.182	0.115	0.055	.020	.000	.000	.000	.046
4	0.065	0.208	0.128	0.071	0.044	.051	.000	.000	.003	.087
Big	0.151	0.119	0.057	0.011	-0.114	.000	.000	.016	.513	.000
			r_i					$p(r_i)$		
Small	-0.527	0.039	0.107	0.259	0.212	.000	.174	.000	.000	.000
2	-0.647	0.011	0.092	0.348	0.434	.000	.710	.000	.000	.000
3	-0.786	-0.052	0.114	0.279	0.434	.000	.087	.000	.000	.000
4	-0.739	-0.283	0.074	0.248	0.332	.000	.000	.012	.000	.000
Big	-0.855	-0.304	-0.108	0.146	0.397	.000	.000	.000	.000	.000
			c_i					$p(c_i)$		
Small	0.103	0.065	0.004	0.055	-0.031	.006	.054	.914	.114	.402
2	0.060	0.076	-0.046	-0.011	-0.035	.090	.023	.127	.754	.313
3	-0.079	0.013	-0.032	0.034	-0.095	.069	.712	.337	.307	.014
4	-0.059	-0.000	0.062	0.077	-0.060	.199	.991	.071	.020	.100
Big	-0.178	0.082	0.105	0.018	-0.027	.000	.012	.001	.457	.286

Table 5.3 shows the estimated coefficients and p-values of HML, RMW, and CMA, for the 25 *Size-OP* portfolios. Given that these portfolios are formed on profitabil-

ity, the RMW slopes are statistically significant, and increasing monotonically from strongly negative for low OP portfolios to strongly positive for high OP portfolios - profitability effect. Likewise, the HML slopes are highly statistically significant. Nonetheless, as in the case of the 25 *Size-B/M* portfolios, the CMA slopes are close to zero and are almost never statistically significant. This insignificance puts forth for discussion, whether CMA is redundant for describing average returns. In light of the regressions that include CMA and one of HML or RMW, the factor is statistically significant for a very small number of portfolios, which suggests that the average CMA returns are captured by their exposure to the HML and RMW factors; see Fama and French (2015, p. 16) for the 25 *Size-OP* regression results.

Table 5.4 shows the estimated coefficients and p-values of HML, RMW, and CMA, for the 25 *Size-Inv* portfolios. In this instance, it comes as no surprise that the CMA slopes are statistically significant because these portfolios are formed on investment. The CMA slopes are decreasing monotonically from strongly positive for low Inv to strongly negative for high Inv portfolios - investment effect. The negative exposures of RMW and CMA slopes for the highest Inv portfolios cite with the results of Fama and French, with the magnitude of the CMA slopes being greater by about 10 basis points on average; see Fama and French (2015, p. 17) for the 25 *Size-Inv* regression results.

The above results suggest that the CMA factor is redundant for describing average returns. As it is shown, when the CMA factor is included in regressions for the 25 *Size-B/M* and 25 *Size-OP* portfolios, the CMA slopes are close to zero and statistically insignificant. The average CMA returns are mostly absorbed by the HML factor and to a smaller degree by the RMW factor.[6] The CMA slopes appear to be statistically significant only for the 25 *Size-Inv* regressions, for which the portfolios are formed on investment and thus, including the CMA factor is crucial. Besides, the HML factor is statistically significant in regressions for all the 25 *Size-B/M*, 25 *Size-OP*,

[6] It is worth to mention that Fama and French (2017) report redundancy of the CMA factor for Europe and Japan; sample period is July 1990 to December 2015.

and 25 *Size-Inv* portfolios. The latter suggests that this factor is not redundant, as conjectured by Fama and French (2015), at least for the data in question.

Table 5.4 - Regressions for 25 Size-Inv Portfolios

The estimated model is (July 1963 to December 2013, 606 months):

$$R_{it}{=}\alpha_i{+}\beta_{1i}R_{mt}{+}\beta_{2i}R_{ft}{+}\sum\nolimits_{j=1}^{m^*}\delta_{1ji}v_{jt}{+}\sum\nolimits_{j=2}^{s}\delta_{2ji}d_{jt}{+}\boldsymbol{\gamma}_i^\top\mathbf{Z}_{it-1}$$

$$+s_iSMB_t{+}h_iHML_t{+}r_iRMW_t{+}c_iCMA_t{+}u_{it},\ \ i{=}1,2,...,k,\ \ t{\in}\mathbb{N},$$

where R_{it} are the monthly returns on the 25 *Size-Inv* portfolios.

Size\Inv	Low	2	3	4	High	Low	2	3	4	High
			h_i					$p(h_i)$		
Small	-0.053	0.187	0.179	0.120	-0.055	.047	.000	.000	.000	.026
2	0.028	0.225	0.189	0.201	-0.131	.241	.000	.000	.000	.000
3	0.066	0.207	0.193	0.208	-0.082	.034	.000	.000	.000	.001
4	0.106	0.223	0.224	0.128	-0.087	.000	.000	.000	.000	.002
Big	-0.147	-0.025	0.107	0.029	-0.065	.000	.198	.000	.124	.007
			r_i					$p(r_i)$		
Small	-0.350	-0.010	0.102	0.009	-0.237	.000	.716	.000	.759	.000
2	-0.195	0.131	0.102	0.165	-0.175	.000	.000	.000	.000	.000
3	-0.049	0.048	0.173	0.136	-0.119	.187	.104	.000	.000	.000
4	-0.096	0.018	0.049	0.081	-0.134	.007	.544	.068	.006	.000
Big	0.022	0.038	0.089	0.134	0.036	.538	.104	.000	.000	.215
			c_i					$p(c_i)$		
Small	0.338	0.175	0.119	0.023	-0.248	.000	.000	.000	.506	.000
2	0.443	0.192	0.153	-0.090	-0.361	.000	.000	.000	.003	.000
3	0.402	0.274	0.102	-0.185	-0.489	.000	.000	.001	.000	.000
4	0485	0.250	0.065	-0.092	-0.445	.000	.000	.042	.007	.000
Big	0.811	0.524	0.078	-0.191	-0.614	.000	.000	.002	.000	.000

5.4.3 Step III: Missing Additional Factors?

If one or more of the Fama-French factors capture variation in returns (i.e., one or more of the factors are statistically significant), it is expected to account for some or all the lingering heterogeneity in the data; that is, statistically significant generic trends becoming statistically insignificant after the inclusion of the factors into the asset pricing models. Nonetheless, if a number of trends remain statistically significant, there is an indication of additional missing relevant factors.

From the probabilistic perspective, the inclusion of deterministic trends in the asset pricing models provide a generic way to capture the heterogeneity in the data. From the theory perspective, however, the presence of trends indicates substantive ignorance by suggesting that certain explanatory variables are missing. It is therefore interesting to view whether the Fama-French factors account for the lingering heterogeneity in the data, or at least part of it. Table 5.5 shows the highest statistically significant order (m^*) of the Gram-Schmidt orthonormal polynomials included in the models prior and posterior the inclusion of the Fama-French factors. It is important to note that the aggregation induced heterogeneity appears to be more complicated for the regressions of the smaller size portfolios since these portfolios contain on average a larger number of stocks compared to other portfolios.

The results in table 5.5 suggest that the Fama-French factors on average account for nearly 50% of the lingering heterogeneity in the data. For instance, the regressions of the smaller size portfolios for the 25 *Size-B/M* portfolios include 11 or 12 trends in the CAPM model, whereas on average 6 of these trends are becoming statistically insignificant after the inclusion of the factors into the regressions. For some of the regressions, like for the bigger size - highest OP portfolios, the Fama-French factors account for all the trends. In other instances, the factors account for no or a few number of trends, like in the regressions of the bigger size - medium Inv portfolios.

The lingering heterogeneity in the data indicates that a number of relevant factors are still missing and further work is needed to enhance the substantive adequacy of the Fama-French multi-factor models in order to replace the generic trends with proper explanatory variables. The current study does not pose any further questions of omitted factors, other than the one posed on the Fama-French factors. These questions are left for future research.

Table 5.5 - Highest Order of Trends in Regressions

The estimated models are (July 1963 to December 2013, 606 months):

$$R_{it}=\alpha_i+\beta_{1i}R_{mt}+\beta_{2i}R_{ft}+\sum\nolimits_{j=1}^{m^*}\delta_{1ji}v_{jt}+\sum\nolimits_{j=2}^{s}\delta_{2ji}d_{jt}+\boldsymbol{\gamma}_i^\top\mathbf{Z}_{it-1},$$

and the extended model posterior the inclusion of the Fama-French factors:

$$...+s_iSMB_t+h_iHML_t+r_iRMW_t+c_iCMA_t+u_{it},\ i=1,2,...,k,\ t\in\mathbb{N},$$

where R_{it} are the monthly returns on the 25 *Size-B/M*, 25 *Size-OP*, 25 *Size-Inv* portfolios. The table presents the highest order (m^*) of the trends included in the regressions.

	CAPM					five-factor model				
	Low	2	3	4	High	Low	2	3	4	High
Size\B/M			m^*					m^*		
Small	12	12	12	12	11	6	5	7	7	3
2	11	11	11	11	11	–	6	8	–	–
3	11	11	9	11	7	6	5	9	6	–
4	11	9	–	1	10	6	9	–	1	10
Big	11	1	–	–	–	11	1	–	–	–
Size\OP			m^*					m^*		
Small	13	11	11	11	11	6	6	11	7	9
2	11	11	11	11	11	8	10	9	9	–
3	11	11	11	11	11	–	9	6	10	–
4	–	5	11	10	8	–	5	2	10	–
Big	–	1	3	11	11	–	1	3	11	–
Size\Inv			m^*					m^*		
Small	11	11	12	12	12	6	6	7	8	12
2	11	11	11	11	11	–	6	–	–	8
3	9	9	7	9	11	9	9	–	6	9
4	–	–	8	10	11	–	–	8	10	10
Big	–	–	11	9	9	–	–	11	9	3

5.4.4 Testing for the Nesting Restrictions

In light of the fact that the estimated statistical model in (5.1) ensures the reliability of the inference, it can provide a sound basis to test the CAPM nesting restriction of $\beta_1+\beta_2=1$. The testing will provide evidence whether these restrictions are data-acceptable, and constitutes part of probing the substantive adequacy of the CAPM and the Fama-French multi-factor models. Table 5.6 shows the estimated coefficients of the restriction (β_1+β_2) and p-values for the regressions of the 25 *Size-B/M*, 25 *Size-OP*, and 25 *Size-Inv* portfolios. The large p-values do not indicate rejection of these restrictions for any of the estimated regressions, and thus one can use the restricted model to probe further the substantive adequacy of the Fama-French multi-factor models; see Appendix K for details.

Table 5.6 - Nesting Restrictions

The estimated model is (July 1963 to December 2013, 606 months):

$$R_{it}{=}\alpha_i{+}\beta_{1i}R_{mt}{+}\beta_{2i}R_{ft}{+}\sum\nolimits_{j=1}^{m^*}\delta_{1ji}v_{jt}{+}\sum\nolimits_{j=2}^{s}\delta_{2ji}d_{jt}{+}\boldsymbol{\gamma}_i^\top\mathbf{Z}_{it-1}$$

$$+s_iSMB_t{+}h_iHML_t{+}r_iRMW_t{+}c_iCMA_t{+}u_{it},\ i{=}1,2,...,k,\ t{\in}\mathbb{N},$$

where R_{it} are the monthly returns on the 25 *Size-B/M*, 25 *Size-OP*, 25 *Size-Inv* portfolios.

	Low	2	3	4	High	Low	2	3	4	High
Size\B/M	$\beta_{1i}+\beta_{2i}$					$p(\beta_{1i}+\beta_{2i})$				
Small	-0.927	0.308	0.029	0.374	-0.893	.172	.524	.317	.489	.045
2	2.297	2.232	1.650	3.034	0.846	.229	.179	.476	.036	.884
3	2.721	1.005	2.154	0.162	2.180	.112	.996	.283	.413	.361
4	1.273	1.809	1.041	0.976	1.610	.795	.480	.971	.984	.676
Big	1.060	2.212	2.979	1.489	-0.620	.939	.230	.097	.654	.314
Size\OP	$\beta_{1i}+\beta_{2i}$					$p(\beta_{1i}+\beta_{2i})$				
Small	-0.450	0.371	-0.864	0.943	0.077	.189	.532	.058	.959	.411
2	1.882	3.219	1.828	0.946	4.023	.437	.029	.376	.960	.004
3	3.089	0.431	1.045	2.103	2.799	.112	.598	.962	.260	.120
4	-0.357	-0.092	0.594	1.390	1.964	.366	.324	.696	.695	.387
Big	1.562	2.679	0.548	1.423	1.086	.643	.100	.665	.561	.909
Size\Inv	$\beta_{1i}+\beta_{2i}$					$p(\beta_{1i}+\beta_{2i})$				
Small	0.064	-0.877	0.987	-0.823	-1.009	.389	.049	.991	.081	.053
2	1.721	2.224	2.198	1.446	2.926	.487	.234	.237	.629	.032
3	3.386	2.075	1.451	1.022	1.520	.073	.311	.651	.981	.624
4	0.675	1.745	0.854	1.604	1.340	.805	.484	.879	.555	.776
Big	2.394	0.452	0.778	1.321	1.595	.283	.513	.771	.683	.573

5.5 Summary and Conclusions

After discussing in brief the problem of factor selection, the chapter proposed a simplified procedure for factor selection. This procedure consists of three steps. The first step involves the specification of a statistically adequate model that is likely to include deterministic variables. The second step involves the testing of the candidate factors in terms of their statistical significance. The third step involves the assessment of the deterministic trends included prior the factor testing. The main advantage of this three-step procedure is that it provides a coherent basis for selecting the relevant factors from a set of possible ones.

The proposed procedure is used to test for the significance of the Fama-French factors. These factors are included to the heterogeneous Student's t/Heteroskedastic Dynamic LR model which accounts for all the statistical systematic information exhibited by the data in question. As it is shown, the value factor HML is not redundant, as conjectured by Fama and French (2015), whereas the redundant factor for describing average returns is the investment factor CMA. In addition, the statistical significance of the preincluded deterministic trends is assessed. As argued the data indicate that there are still missing relevant factors and further work is needed to enhance the substantive adequacy of the Fama-French models in order to replace the generic trends with proper explanatory variables.

In conclusion, it is important to emphasize that one can use the same procedure to establish the relevance of any seasonal effects. For instance, one can construct mimicking variables for the tax-loss selling hypothesis, window dressing, etc., and test for the significance on these variables on the basis of a statistically adequate model. The assessment associated to the statistical significance of the seasonal dummy variables prior and posterior the inclusion of the constructed mimicking variables allows one to observe the extend that the seasonal effect is explained by the constructed mimicking variables.

Appendix H: Normality and Student's t Tests

This Appendix provides a brief description of the Shapiro-Francia test for Normal distribution and Kolmogorov-Smirnov test for Student's t distribution.

Shapiro-Francia. Based on the assumption that $\{z_t,\ t\in\mathbb{N}\}$ is an IID process, the Shapiro-Francia test statistic for Normality is defined as (Shapiro and Francia, 1972):

$$W'=\left(\sum_{t=1}^{n} b_t z_{(t)}\right)^2 \Big/ \sum_{t=1}^{n} (z_t-\overline{z})^2, \tag{H.1}$$

where $z_{(t)}$ is the t-th order statistic, and $\overline{z}$ denotes the sample mean.

The constants b_t, for $t{=}1,2,...,n$ are given by:

$$(b_1, b_2, ..., b_n)=\frac{\mathbf{m}^\top}{\left(\mathbf{m}^\top\mathbf{m}\right)^{1/2}}, \tag{H.2}$$

where $\mathbf{m}^\top{=}(m_1, m_2, ..., m_n)^\top$ denotes the vector of expected values of standard normal order statistics.

This test is a simplification of the Shapiro-Wilk test where the covariance matrix of the order statistics $\mathbf{V}$ is replaced with the identity matrix $\mathbf{I}{:=}diag(1,1,...,1)$; see Appendix C for a description of the Shapiro-Wilk test.

Kolmogorov-Smirnov. See Appendix E for a description of this test.

The null hypothesis of the tests above is H_0: *Normal* (*Student's t*) versus the alternative H_1: *non-Normal* (*non-Student's t*)*.* Thus, if the p-value is lower than the chosen significance level, there is evidence that the $\{z_t,\ t\in\mathbb{N}\}$ IID process tested is not Normal/Student's t distributed. It is important to note that for a large sample size, one has to choose much smaller thresholds for significance when using the p-values; see Lehmann and Romano (2006). With a sample size of $n{=}606$ a threshold of 0.001 is more appropriate than the most commonly used threshold of 0.05.

Appendix I: M-S Testing Results for Normal LR

This Appendix presents the M-S testing results of the CAPM and the Fama-French multi-factor models estimated by the Normal/ Homoskedastic LR model. The estimated models are (July 1963 to December 2013, 606 months):

$$(R_{it}-R_{ft})=\alpha_i+\beta_i(R_{mt}-R_{ft})+u_{it}, \tag{I.1}$$

$$(R_{it}-R_{ft})=\alpha_i+\beta_i(R_{mt}-R_{ft})+s_iSMB_t+h_iHML_t+u_{it}, \tag{I.2}$$

$$(R_{it}-R_{ft})=\alpha_i+\beta_i(R_{mt}-R_{ft})+s_iSMB_t+h_iHML_t+r_iRMW_t+c_iCMA_t+u_{it}, \tag{I.3}$$

where R_{it} is the return of portfolio i for period t; R_{ft} is the risk-free return; R_{mt} is the return of the value-weighted market portfolio; and $SMB_t, HML_t, RMW_t, CMA_t$ denote the size, value, profitability, and investment factors, respectively.

To evaluate the model assumptions [1]–[5] that justify the Normal/Homoskedastic LR model, the estimated residuals of the models in (I.1)-(I.3) are carefully examined. The Shapiro-Francia (W') test in Appendix I is employed to test for [1] Normality, and the auxiliary regressions (F) below are employed to test for [2] Linearity, [3] Homoskedasticity, [4] Independence, and [5] t-invariance.

$$\widehat{u}_{it}=\gamma_{1i}+\boldsymbol{\gamma}_{2i}^{\top}\mathbf{X}_t+\underbrace{\sum\nolimits_{j=1}^{m}\gamma_{3ji}v_{jt}}_{\overline{[5.1]}}+\underbrace{\sum\nolimits_{j=2}^{s}\gamma_{4ji}d_{jt}}_{\overline{[5.2]}}+\underbrace{\boldsymbol{\gamma}_{5i}^{\top}\boldsymbol{\psi}_t}_{\overline{[2]}}+\underbrace{\sum\nolimits_{j=1}^{p}\boldsymbol{\gamma}_{6ji}^{\top}\mathbf{Z}_{it-j}}_{\overline{[4]}}+\varepsilon_{1it}, \tag{I.4}$$

$$\widehat{u}_{it}^2=\gamma_{7i}+\underbrace{\sum\nolimits_{j=1}^{m}\gamma_{8ji}v_{jt}}_{\overline{[5.3]}}+\underbrace{\boldsymbol{\gamma}_{9i}^{\top}\mathbf{X}_t+\boldsymbol{\gamma}_{10i}^{\top}\boldsymbol{\psi}_t+\sum\nolimits_{j=1}^{p}\boldsymbol{\gamma}_{11ji}^{\top}\mathbf{Z}_{it-j}^2}_{\overline{[3]}+\overline{[4]}}+\varepsilon_{2it}, \tag{I.5}$$

where $\widehat{u}_{it}$ are the estimated residuals in (I.1)-(I.3) of portfolio i for period t; $\mathbf{X}_t{:=}(x_{1t}, x_{2t}, ..., x_{lt})$ are the explanatory variables in (I.1)-(I.3); v_{jt} denotes the terms of the Gram-Schmidt orthonormal polynomials of order $j{=}1, 2, ..., m$; $d_{jt}{:=}(d_{2t}, ..., d_{12t})$ are the monthly dummy variables for the months of February through December; $\boldsymbol{\psi}_t{:=}\{(x_{it}{\cdot}x_{jt}), i{\geq}j, i, j{=}1, 2, ..., l\}$ are the second-order Kolmogorov-Gabor polynomials; and $\mathbf{Z}_{it}{:=}(y_{it}, \mathbf{X}_t)$.

The M-S testing results of the 25 *Size-B/M*, 25 *Size-OP*, and 25 *Size-Inv* portfolios, for the CAPM, three- and five-factor models are presented in tables J.1–J.9; values of $p(.)$ below .001 are considered to indicate departures from assumptions [1]–[5].

Table I.1 - M-S Testing Results for 25 Size-B/M portfolios: CAPM

Size\B/M	Low	2	3	4	High	Low	2	3	4	High
	[1] Normality (S-F)					$p([1])$				
Small	0.922	0.897	0.949	0.960	0.963	.000	.000	.000	.000	.000
2	0.956	0.972	0.983	0.982	0.964	.000	.000	.000	.000	.000
3	0.927	0.979	0.970	0.966	0.969	.000	.000	.000	.000	.000
4	0.898	0.971	0.928	0.967	0.972	.000	.000	.000	.000	.000
Big	0.995	0.951	0.957	0.918	0.959	.036	.000	.000	.000	.000
	[2] Linearity					$p([2])$				
Small	1.459	2.373	1.623	2.208	1.909	.228	.124	.203	.138	.168
2	0.387	2.160	2.069	2.630	0.902	.534	.142	.151	.105	.343
3	0.530	1.019	1.774	0.348	1.292	.467	.313	.184	.556	.256
4	0.154	2.285	0.755	2.284	0.328	.695	.131	.385	.131	.567
Big	4.634	0.264	0.154	3.609	0.090	.032	.608	.695	.058	.764
	[3] Homoskedasticity					$p([3])$				
Small	7.664	9.950	3.589	6.493	8.150	.000	.000	.001	.000	.000
2	5.009	3.172	15.454	13.330	10.382	.000	.003	.000	.000	.000
3	10.096	10.959	11.198	4.639	10.868	.000	.000	.000	.000	.000
4	10.236	9.885	6.929	8.490	7.900	.000	.000	.000	.000	.000
Big	7.567	1.959	0.857	5.895	3.778	.000	.059	.540	.000	.001
	[4] Independence					$p([4])$				
Small	6.032	4.966	3.965	5.403	10.566	.000	.000	.001	.000	.000
2	2.629	2.698	4.140	2.172	3.157	.016	.014	.000	.044	.005
3	2.385	3.002	2.353	2.720	3.855	.028	.007	.030	.013	.001
4	2.395	6.364	1.847	0.931	1.267	.027	.000	.088	.472	.271
Big	0.257	0.409	1.813	1.307	0.510	.956	.873	.094	.252	.801
	[5.1] Mean Homogeneity					$p([5.1])$				
Small	3.844	3.118	3.207	3.034	2.186	.000	.000	.000	.000	.011
2	2.602	2.300	1.654	1.513	1.175	.002	.007	.073	.115	.298
3	1.981	1.415	1.221	1.035	0.676	.024	.154	.264	.415	.775
4	0.801	1.074	1.388	1.412	0.979	.650	.380	.167	.156	.468
Big	1.401	1.048	0.624	0.365	0.485	.161	.403	.823	.975	.924
	[5.2] Seasonal Homogeneity					$p([5.2])$				
Small	3.418	3.251	4.690	5.151	8.309	.000	.000	.000	.000	.000
2	2.248	2.986	2.551	2.683	4.284	.011	.001	.004	.002	.000
3	1.324	1.813	1.369	1.918	2.960	.207	.049	.184	.035	.001
4	1.828	1.217	1.222	1.526	1.929	.046	.272	.269	.118	.033
Big	1.911	0.918	0.384	1.158	1.730	.035	.522	.962	.314	.063
	[5.3] Variance Homogeneity					$p([5.3])$				
Small	2.135	1.738	3.322	3.765	2.332	.014	.056	.000	.000	.006
2	2.375	5.298	5.113	6.417	3.176	.005	.000	.000	.000	.000
3	2.184	4.293	4.117	7.496	4.195	.011	.000	.000	.000	.000
4	0.963	8.993	5.474	5.662	5.140	.484	.000	.000	.000	.000
Big	2.543	5.476	4.086	6.997	3.355	.003	.000	.000	.000	.000

Table I.2 - M-S Testing Results for 25 Size-OP Portfolios: CAPM										
Size\OP	Low	2	3	4	High	Low	2	3	4	High
	[1] Normality (S-F)					$p([1])$				
Small	0.893	0.967	0.978	0.978	0.983	.000	.000	.000	.000	.000
2	0.937	0.983	0.978	0.978	0.981	.000	.000	.000	.000	.000
3	0.959	0.948	0.977	0.981	0.986	.000	.000	.000	.000	.000
4	0.960	0.953	0.971	0.943	0.970	.000	.000	.000	.000	.000
Big	0.982	0.970	0.978	0.973	0.991	.000	.000	.000	.000	.001
	[2] Linearity					$p([2])$				
Small	1.537	1.520	1.984	1.172	2.344	.216	.218	.160	.280	.126
2	1.105	5.103	1.759	0.325	1.389	.294	.024	.185	.569	.239
3	2.457	1.518	0.273	0.855	0.080	.118	.218	.602	.356	.778
4	0.003	0.001	0.012	0.003	0.001	.960	.999	.912	.958	.984
Big	0.007	0.101	1.377	4.614	0.244	.935	.751	.241	.032	.622
	[3] Homoskedasticity					$p([3])$				
Small	6.889	6.102	7.616	11.375	5.239	.000	.000	.000	.000	.000
2	6.095	5.619	12.676	12.937	7.138	.000	.000	.000	.000	.000
3	3.973	3.749	9.093	9.139	7.145	.000	.001	.000	.000	.000
4	6.712	1.746	7.773	3.877	7.543	.000	.096	.000	.000	.000
Big	8.941	0.941	3.908	8.459	1.397	.000	.474	.000	.000	.204
	[4] Independence					$p([4])$				
Small	8.120	4.953	5.196	5.523	7.121	.000	.000	.000	.000	.000
2	2.938	1.868	2.838	3.907	3.448	.008	.084	.010	.001	.002
3	2.368	2.886	2.584	2.708	3.743	.029	.009	.018	.013	.001
4	1.365	0.415	2.768	4.180	2.229	.227	.870	.012	.000	.039
Big	3.321	1.418	2.577	1.331	0.682	.003	.205	.018	.241	.665
	[5.1] Mean Homogeneity					$p([5.1])$				
Small	4.184	2.843	2.455	2.281	2.781	.000	.001	.004	.001	.001
2	2.970	2.621	2.136	1.775	1.394	.000	.002	.013	.049	.164
3	2.485	1.765	1.104	1.917	1.175	.004	.051	.354	.030	.297
4	0.807	1.027	0.990	0.994	1.557	.643	.422	.457	.453	.100
Big	0.575	1.002	0.765	2.315	1.178	.863	.446	.687	.007	.295
	[5.2] Seasonal Homogeneity					$p([5.2])$				
Small	6.396	4.716	3.742	3.681	3.075	.000	.000	.000	.000	.001
2	3.544	2.618	2.243	1.640	3.070	.000	.003	.011	.084	.001
3	2.985	1.294	1.763	1.222	1.758	.001	.224	.057	.269	.058
4	1.637	1.078	1.702	1.099	2.499	.085	.376	.069	.360	.005
Big	0.340	0.994	1.393	1.640	1.336	.977	.450	.172	.084	.201
	[5.3] Variance Homogeneity					$p([5.3])$				
Small	1.663	5.120	3.989	3.962	3.657	.071	.000	.000	.000	.000
2	2.146	5.211	6.037	6.165	2.563	.013	.000	.000	.000	.003
3	3.676	5.667	4.009	5.899	4.801	.000	.000	.000	.000	.000
4	2.142	4.879	10.310	8.868	5.094	.013	.000	.000	.000	.000
Big	1.116	5.126	3.139	3.133	3.863	.344	.000	.000	.000	.000

Table I.3 - M-S Testing Results for 25 Size-Inv Portfolios: CAPM

Size\Inv	Low	2	3	4	High	Low	2	3	4	High
	[1] Normality (S-F)					p([1])				
Small	0.890	0.954	0.952	0.976	0.969	.000	.000	.000	.000	.000
2	0.961	0.984	0.957	0.973	0.973	.000	.000	.000	.000	.000
3	0.979	0.972	0.974	0.962	0.977	.000	.000	.000	.000	.000
4	0.993	0.953	0.947	0.925	0.924	.006	.000	.000	.000	.000
Big	0.990	0.987	0.980	0.969	0.971	.001	.000	.000	.000	.000
	[2] Linearity					p([2])				
Small	2.032	2.371	2.762	4.493	1.236	.155	.124	.097	.034	.267
2	1.855	1.439	1.615	1.039	1.814	.174	.231	.204	.308	.179
3	0.746	1.139	1.780	1.511	1.101	.388	.286	.183	.220	.295
4	0.195	1.357	0.280	0.195	1.958	.659	.245	.597	.659	.162
Big	3.349	0.224	0.151	1.421	8.513	.068	.636	.698	.234	.004
	[3] Homoskedasticity					p([3])				
Small	7.014	5.974	5.742	5.165	4.831	.000	.000	.000	.000	.000
2	1.910	5.078	4.470	13.550	2.447	.066	.000	.000	.000	.018
3	8.229	9.321	10.032	3.634	2.295	.000	.000	.000	.001	.026
4	3.201	5.081	5.989	4.289	9.213	.002	.000	.000	.000	.000
Big	4.609	2.536	2.702	1.979	4.857	.000	.014	.009	.056	.000
	[4] Independence					p([4])				
Small	7.001	6.360	3.904	4.978	8.645	.000	.000	.001	.000	.000
2	3.591	4.023	2.047	3.649	4.068	.002	.001	.058	.001	.001
3	1.807	1.006	1.486	2.211	2.039	.096	.420	.181	.041	.059
4	0.937	1.488	2.781	2.801	2.815	.468	.180	.011	.011	.010
Big	1.402	1.579	1.726	1.482	0.486	.212	.151	.113	.182	.819
	[5.1] Mean Homogeneity					p([5.1])				
Small	3.081	2.451	2.919	3.775	5.258	.000	.004	.001	.000	.000
2	1.951	1.190	1.924	1.951	3.214	.027	.286	.029	.027	.000
3	1.326	1.485	1.486	2.201	2.771	.199	.125	.125	.011	.001
4	1.347	0.793	1.093	1.521	1.406	.188	.658	.363	.112	.158
Big	0.401	1.568	1.553	3.220	1.721	.963	.097	.102	.000	.059
	[5.2] Seasonal Homogeneity					p([5.2])				
Small	5.942	4.488	4.788	3.745	4.642	.000	.000	.000	.000	.000
2	3.253	3.898	2.397	2.519	2.729	.000	.000	.007	.004	.002
3	3.226	1.686	1.590	1.199	1.665	.000	.073	.098	.284	.078
4	1.907	0.981	1.759	1.822	1.704	.036	.463	.058	.047	.069
Big	1.250	1.443	1.112	1.209	1.282	.250	.150	.349	.278	.231
	[5.3] Variance Homogeneity					p([5.3])				
Small	1.627	2.905	3.787	5.040	3.923	.080	.001	.000	.000	.000
2	2.983	3.592	4.417	4.762	5.550	.000	.000	.000	.000	.000
3	1.871	5.220	4.498	4.583	5.315	.035	.000	.000	.000	.000
4	3.482	6.871	5.194	5.206	1.454	.000	.000	.000	.000	.137
Big	4.077	1.685	5.295	3.734	3.617	.000	.066	.000	.000	.000

Table I.4 - M-S Testing Results for 25 Size-B/M Portfolios: three-factor										
Size\B/M	Low	2	3	4	High	Low	2	3	4	High
	[1] Normality (S-F)					$p([1])$				
Small	0.957	0.959	0.988	0.963	0.979	.000	.000	.000	.000	.000
2	0.987	0.971	0.978	0.986	0.984	.000	.000	.000	.000	.000
3	0.993	0.952	0.971	0.951	0.956	.007	.000	.000	.000	.000
4	0.967	0.976	0.952	0.986	0.993	.000	.000	.000	.000	.008
Big	0.992	0.977	0.970	0.975	0.976	.002	.000	.000	.000	.000
	[2] Linearity					$p([2])$				
Small	4.165	6.428	4.823	3.358	4.467	.000	.000	.000	.003	.000
2	2.097	1.559	3.076	0.608	1.171	.052	.157	.006	.724	.320
3	0.667	2.990	3.279	5.122	0.896	.677	.007	.004	.000	.497
4	4.092	2.747	1.154	3.522	0.940	.000	.012	.330	.002	.466
Big	3.679	3.195	4.031	1.473	1.935	.001	.004	.001	.185	.073
	[3] Homoskedasticity					$p([3])$				
Small	6.886	23.167	5.354	9.782	4.821	.000	.000	.000	.000	.000
2	3.588	4.171	8.405	3.983	1.850	.000	.000	.000	.000	.018
3	9.444	11.997	17.145	8.941	7.234	.000	.000	.000	.000	.000
4	23.507	14.678	10.243	3.990	2.239	.000	.000	.000	.000	.002
Big	4.273	13.609	5.598	4.614	8.110	.000	.000	.000	.000	.000
	[4] Independence					$p([4])$				
Small	3.904	2.658	2.109	3.860	8.153	.000	.002	.015	.000	.000
2	1.146	0.960	1.935	1.532	1.933	.320	.486	.028	.108	.028
3	5.167	2.377	1.129	1.847	2.519	.000	.005	.334	.038	.003
4	2.410	2.245	1.220	1.065	0.736	.005	.009	.266	.387	.716
Big	1.321	0.747	1.751	1.848	3.031	.202	.706	.053	.038	.000
	[5.1] Mean Homogeneity					$p([5.1])$				
Small	1.452	0.573	1.362	2.591	0.682	.138	.865	.180	.002	.770
2	0.963	0.888	1.288	0.634	0.590	.483	.559	.221	.814	.851
3	0.945	0.523	0.599	1.244	0.423	.501	.900	.844	.249	.954
4	0.747	0.617	1.054	2.116	0.913	.706	.829	.398	.015	.533
Big	1.057	1.044	1.250	0.975	0.756	.394	.407	.245	.471	.696
	[5.2] Seasonal Homogeneity					$p([5.2])$				
Small	4.360	1.869	2.460	2.991	6.150	.000	.041	.005	.001	.000
2	2.037	2.299	2.346	1.324	1.611	.023	.009	.008	.207	.092
3	1.796	2.700	2.666	1.703	1.714	.052	.002	.002	.069	.067
4	2.280	2.201	1.871	1.464	1.666	.010	.013	.041	.141	.078
Big	0.952	0.984	0.662	1.551	0.945	.489	.459	.775	.110	.497
	[5.3] Variance Homogeneity					$p([5.3])$				
Small	2.925	1.917	2.926	1.917	1.855	.001	.030	.001	.030	.037
2	3.099	2.233	2.048	1.232	2.406	.000	.009	.019	.257	.005
3	1.952	1.090	1.678	1.332	1.410	.026	.366	.068	.196	.157
4	1.883	1.505	1.285	1.879	2.397	.034	.118	.223	.034	.005
Big	3.439	0.931	0.879	1.253	1.079	.000	.515	.569	.243	.375

Table I.5 - M-S Testing Results for 25 Size-OP Portfolios: three-factor										
Size\OP	Low	2	3	4	High	Low	2	3	4	High
	[1] Normality (S-F)					p([1])				
Small	0.931	0.958	0.977	0.947	0.913	.000	.000	.000	.000	.000
2	0.986	0.976	0.987	0.987	0.938	.000	.000	.000	.000	.000
3	0.984	0.965	0.977	0.982	0.948	.000	.000	.000	.000	.000
4	0.980	0.961	0.983	0.960	0.967	.000	.000	.000	.000	.000
Big	0.978	0.954	0.983	0.976	0.984	.000	.000	.000	.000	.000
	[2] Linearity					p([2])				
Small	6.662	2.631	2.893	7.270	11.079	.000	.016	.009	.000	.000
2	2.576	3.607	1.476	3.441	4.060	.018	.002	.184	.002	.001
3	2.093	3.668	2.551	2.128	7.590	.052	.001	.019	.049	.000
4	2.202	4.475	2.039	2.070	2.823	.041	.000	.059	.055	.010
Big	3.516	2.642	2.719	1.006	4.697	.002	.016	.013	.420	.000
	[3] Homoskedasticity					p([3])				
Small	14.140	18.552	20.198	47.613	66.642	.000	.000	.000	.000	.000
2	4.406	15.973	5.280	11.497	94.698	.000	.000	.000	.000	.000
3	6.205	21.988	7.481	9.055	37.744	.000	.000	.000	.000	.000
4	9.004	25.934	12.206	11.361	10.517	.000	.000	.000	.000	.000
Big	10.035	3.125	6.427	3.662	9.418	.000	.000	.000	.000	.000
	[4] Independence					p([4])				
Small	3.893	3.326	2.409	3.630	6.127	.000	.000	.005	.000	.000
2	1.300	1.429	0.584	1.071	1.065	.214	.148	.856	.382	.388
3	4.862	3.560	1.396	0.914	2.296	.000	.000	.163	.533	.007
4	1.534	1.915	0.901	1.547	1.218	.108	.030	.546	.103	.266
Big	1.941	1.627	0.975	1.389	1.762	.028	.080	.472	.167	.051
	[5.1] Mean Homogeneity					p([5.1])				
Small	1.535	1.392	1.279	1.421	1.273	.107	.165	.227	.152	.231
2	1.992	0.678	1.108	2.122	0.850	.023	.773	.350	.014	.598
3	0.845	1.769	0.446	1.177	0.637	.604	.050	.945	.296	.811
4	0.591	0.922	0.832	0.698	1.038	.851	.524	.617	.754	.412
Big	0.473	0.743	0.876	0.942	0.841	.931	.710	.572	.504	.608
	[5.2] Seasonal Homogeneity					p([5.2])				
Small	8.291	1.715	1.197	1.116	0.693	.000	.067	.286	.346	.746
2	1.705	1.387	2.424	2.257	1.650	.069	.175	.006	.011	.081
3	1.967	2.802	3.347	3.794	2.115	.029	.001	.000	.000	.018
4	0.623	2.755	2.493	2.673	3.239	.810	.002	.005	.002	.000
Big	0.357	0.939	0.747	0.745	0.211	.972	.502	.693	.696	.997
	[5.3] Variance Homogeneity					p([5.3])				
Small	2.544	4.703	2.721	2.866	1.130	.003	.000	.001	.001	.332
2	4.233	1.731	3.444	4.611	2.127	.000	.057	.000	.000	.014
3	4.404	1.163	1.970	2.503	1.445	.000	.306	.025	.003	.141
4	2.304	1.464	2.486	2.055	2.320	.007	.134	.004	.018	.007
Big	0.361	4.420	1.356	1.290	3.513	.976	.000	.183	.220	.000

Table I.6 - M-S Testing Results for 25 Size-Inv Portfolios: three-factor										
Size\Inv	Low	2	3	4	High	Low	2	3	4	High
	[1] Normality (S-F)					$p([1])$				
Small	0.915	0.991	0.967	0.987	0.976	.000	.002	.000	.000	.000
2	0.992	0.971	0.994	0.976	0.984	.004	.000	.014	.000	.000
3	0.992	0.986	0.983	0.964	0.979	.003	.000	.000	.000	.000
4	0.987	0.970	0.972	0.952	0.983	.000	.000	.000	.000	.000
Big	0.986	0.993	0.988	0.956	0.977	.000	.006	.000	.000	.000
	[2] Linearity					$p([2])$				
Small	6.961	1.634	2.241	0.815	1.419	.000	.136	.038	.558	.205
2	2.169	5.587	2.263	3.977	0.705	.045	.000	.036	.001	.646
3	6.104	2.425	2.431	2.339	1.130	.000	.025	.025	.031	.343
4	2.904	1.507	3.489	8.133	3.307	.009	.174	.002	.000	.003
Big	4.027	5.550	5.323	5.391	3.819	.001	.000	.000	.000	.001
	[3] Homoskedasticity					$p([3])$				
Small	29.576	5.404	3.810	6.460	8.460	.000	.000	.000	.000	.000
2	4.290	17.960	4.442	8.770	5.526	.000	.000	.000	.000	.000
3	7.324	7.170	7.336	7.767	2.895	.000	.000	.000	.000	.000
4	8.972	9.503	6.410	16.717	4.941	.000	.000	.000	.000	.000
Big	7.820	6.173	7.635	6.116	5.591	.000	.000	.000	.000	.000
	[4] Independence					$p([4])$				
Small	3.228	4.240	1.931	4.007	7.041	.000	.000	.029	.000	.000
2	2.148	2.116	1.964	3.205	1.920	.013	.015	.025	.000	.030
3	2.748	2.054	1.201	2.638	4.049	.001	.018	.279	.002	.000
4	0.434	0.924	1.558	1.438	1.776	.950	.523	.100	.144	.049
Big	0.973	1.179	0.703	2.045	0.840	.474	.295	.749	.019	.609
	[5.1] Mean Homogeneity					$p([5.1])$				
Small	0.895	1.221	1.704	1.835	3.271	.552	.264	.062	.040	.000
2	1.057	0.910	1.255	1.399	2.062	.395	.537	.242	.162	.018
3	0.757	1.074	1.229	0.889	1.795	.695	.380	.259	.558	.046
4	0.970	0.933	0.627	0.898	1.229	.477	.513	.820	.549	.259
Big	0.383	0.553	1.948	2.493	2.123	.970	.879	.027	.003	.014
	[5.2] Seasonal Homogeneity					$p([5.2])$				
Small	6.293	1.689	2.571	1.348	3.764	.000	.072	.003	.194	.000
2	0.507	0.958	3.651	1.817	2.235	.899	.484	.000	.048	.012
3	1.625	3.054	1.907	3.364	1.506	.088	.001	.036	.000	.125
4	1.105	2.322	2.893	3.386	1.665	.355	.009	.001	.000	.078
Big	1.221	1.372	2.323	0.971	1.296	.270	.182	.009	.472	.223
	[5.3] Variance Homogeneity					$p([5.3])$				
Small	3.103	2.948	2.950	2.305	0.947	.000	.001	.001	.007	.499
2	2.231	1.841	3.524	1.625	3.900	.009	.039	.000	.081	.000
3	1.574	0.788	2.117	1.100	4.523	.095	.663	.014	.358	.000
4	2.140	1.149	1.913	1.382	2.288	.013	.317	.030	.170	.008
Big	1.631	1.086	2.029	0.735	3.941	.079	.369	.020	.718	.000

Table I.7 - M-S Testing Results for 25 Size-B/M Portfolios: five-factor										
Size\B/M	Low	2	3	4	High	Low	2	3	4	High
	[1] Normality (S-F)					$p([1])$				
Small	0.962	0.977	0.987	0.960	0.977	.000	.000	.000	.000	.000
2	0.985	0.977	0.984	0.991	0.983	.000	.000	.000	.001	.000
3	0.989	0.963	0.985	0.960	0.963	.000	.000	.000	.000	.000
4	0.973	0.989	0.956	0.987	0.992	.000	.000	.000	.000	.003
Big	0.988	0.986	0.969	0.975	0.977	.000	.000	.000	.000	.000
	[2] Linearity					$p([2])$				
Small	2.955	3.899	1.910	2.735	2.982	.000	.000	.020	.000	.000
2	2.000	1.059	1.669	0.821	1.124	.014	.393	.053	.654	.331
3	1.240	2.806	2.550	2.706	0.773	.237	.000	.001	.001	.709
4	2.097	2.351	3.009	3.719	0.834	.009	.003	.000	.000	.639
Big	1.890	1.619	3.075	1.317	1.984	.022	.064	.000	.186	.015
	[3] Homoskedasticity					$p([3])$				
Small	13.343	5.860	5.459	7.592	7.296	.000	.000	.000	.000	.000
2	4.355	3.058	3.487	3.315	2.250	.000	.000	.000	.000	.000
3	8.628	10.914	9.231	9.611	6.270	.000	.000	.000	.000	.000
4	15.422	8.157	7.979	4.311	2.076	.000	.000	.000	.000	.001
Big	2.563	7.987	5.747	3.607	8.007	.000	.000	.000	.000	.000
	[4] Independence					$p([4])$				
Small	6.244	3.404	1.834	3.019	6.480	.000	.000	.019	.000	.000
2	2.051	1.330	1.595	1.973	1.545	.007	.163	.057	.010	.070
3	4.087	2.560	1.180	2.698	1.900	.000	.000	.273	.000	.014
4	2.244	1.853	1.341	1.791	0.606	.002	.017	.157	.023	.896
Big	0.686	0.450	2.019	2.012	2.398	.826	.976	.008	.008	.001
	[5.1] Mean Homogeneity					$p([5.1])$				
Small	1.186	0.832	1.101	2.203	0.792	.289	.618	.356	.011	.659
2	0.718	1.225	0.892	0.575	0.586	.734	.262	.555	.863	.855
3	0.926	0.781	1.032	1.401	0.343	.521	.671	.418	.161	.981
4	0.926	1.094	0.980	1.932	1.013	.520	.362	.467	.028	.435
Big	1.107	0.808	1.250	0.978	0.718	.351	.642	.245	.469	.735
	[5.2] Seasonal Homogeneity					$p([5.2])$				
Small	3.422	1.522	2.419	2.874	5.942	.000	.120	.006	.001	.000
2	2.464	1.680	1.622	0.916	1.653	.005	.074	.089	.524	.081
3	2.285	2.529	2.166	1.485	1.694	.010	.004	.015	.133	.071
4	2.551	1.924	1.488	1.312	1.620	.004	.034	.132	.214	.090
Big	1.265	0.670	0.714	1.332	0.807	.241	.767	.725	.203	.634
	[5.3] Variance Homogeneity					$p([5.3])$				
Small	0.679	2.418	1.203	1.695	1.579	.772	.005	.277	.064	.093
2	2.842	1.911	1.072	1.467	2.061	.001	.031	.382	.132	.018
3	2.087	1.323	1.826	1.421	1.304	.016	.201	.041	.152	.212
4	1.850	1.811	2.129	2.168	2.149	.038	.043	.014	.012	.013
Big	3.128	1.394	1.083	1.768	1.295	.000	.164	.372	.050	.217

Table I.8 - M-S Testing Results for 25 Size-OP Portfolios: five-factor										
Size\OP	Low	2	3	4	High	Low	2	3	4	High
	[1] Normality (S-F)					$p([1])$				
Small	0.953	0.968	0.986	0.975	0.963	.000	.000	.000	.000	.000
2	0.994	0.986	0.991	0.989	0.981	.012	.000	.002	.000	.000
3	0.987	0.971	0.980	0.989	0.984	.000	.000	.000	.000	.000
4	0.987	0.975	0.990	0.985	0.979	.000	.000	.001	.000	.000
Big	0.990	0.958	0.982	0.972	0.991	.001	.000	.000	.000	.001
	[2] Linearity					$p([2])$				
Small	3.783	3.693	2.821	3.646	4.776	.000	.000	.000	.000	.000
2	1.045	2.429	2.029	1.075	2.942	.406	.002	.012	.377	.000
3	2.315	4.072	2.401	1.954	2.395	.003	.000	.002	.017	.002
4	1.326	4.059	2.428	2.274	2.558	.181	.000	.002	.004	.001
Big	1.768	2.231	2.378	2.091	2.605	.036	.005	.002	.009	.001
	[3] Homoskedasticity					$p([3])$				
Small	11.183	7.373	6.894	10.152	18.194	.000	.000	.000	.000	.000
2	2.949	5.601	1.969	2.134	6.923	.000	.000	.001	.000	.000
3	4.631	19.947	8.440	1.848	7.952	.000	.000	.000	.003	.000
4	8.948	11.617	5.202	7.949	7.475	.000	.000	.000	.000	.000
Big	10.193	1.924	4.730	4.838	4.022	.000	.002	.000	.000	.000
	[4] Independence					$p([4])$				
Small	7.646	2.810	2.174	2.160	4.487	.000	.000	.003	.004	.000
2	0.812	1.351	1.030	1.691	1.568	.688	.151	.423	.037	.063
3	3.085	3.217	1.723	1.728	2.253	.000	.000	.032	.031	.002
4	1.225	2.114	1.444	1.148	1.221	.235	.005	.106	.302	.239
Big	0.512	1.333	1.648	1.127	0.907	.953	.161	.045	.322	.570
	[5.1] Mean Homogeneity					$p([5.1])$				
Small	2.140	1.695	1.530	1.614	1.352	.013	.064	.109	.084	.185
2	1.780	1.325	0.818	2.689	1.278	.048	.200	.632	.002	.228
3	0.859	2.339	0.642	1.500	0.802	.590	.006	.806	.120	.649
4	1.146	0.975	1.177	0.853	1.354	.320	.472	.297	.596	.184
Big	0.457	1.109	0.686	0.862	1.318	.939	.350	.765	.586	.204
	[5.2] Seasonal Homogeneity					$p([5.2])$				
Small	7.642	1.910	1.603	1.450	1.106	.000	.036	.094	.147	.354
2	1.246	1.301	2.012	1.222	1.040	.253	.220	.025	.269	.410
3	1.859	3.249	2.766	3.013	1.759	.042	.000	.002	.001	.058
4	1.173	3.367	1.985	2.014	3.101	.303	.000	.028	.025	.000
Big	0.862	0.804	0.889	0.760	0.812	.578	.636	.551	.680	.628
	[5.3] Variance Homogeneity					$p([5.3])$				
Small	0.737	3.334	1.665	2.257	1.215	.716	.000	.071	.009	.269
2	2.041	1.650	1.426	5.980	2.239	.019	.074	.149	.000	.009
3	1.082	2.749	1.131	3.005	3.179	.373	.001	.332	.000	.000
4	0.853	1.159	1.817	1.349	2.265	.595	.310	.042	.187	.008
Big	0.605	5.086	1.804	0.829	2.722	.839	.000	.044	.621	.001

Table I.9 - M-S Testing Results for 25 Size-Inv Portfolios: five-factor										
Size\Inv	Low	2	3	4	High	Low	2	3	4	High
	[1] Normality (S-F)					p([1])				
Small	0.943	0.988	0.964	0.987	0.983	.000	.000	.000	.000	.000
2	0.992	0.984	0.994	0.985	0.990	.003	.000	.022	.000	.000
3	0.988	0.992	0.990	0.951	0.975	.000	.002	.001	.000	.000
4	0.977	0.976	0.979	0.953	0.980	.000	.000	.000	.000	.000
Big	0.992	0.991	0.995	0.962	0.992	.003	.002	.037	.000	.004
	[2] Linearity					p([2])				
Small	3.643	0.943	2.674	0.943	2.925	.000	.516	.001	.515	.000
2	0.800	3.065	1.530	2.056	1.732	.679	.000	.090	.011	.042
3	3.931	1.835	1.055	4.334	2.220	.000	.027	.397	.000	.005
4	1.832	1.565	2.329	6.482	1.967	.028	.079	.003	.000	.016
Big	1.857	2.291	2.727	3.460	2.949	.025	.004	.001	.000	.000
	[3] Homoskedasticity					p([3])				
Small	8.966	4.441	6.856	3.962	10.017	.000	.000	.000	.000	.000
2	1.217	7.473	3.928	2.862	5.421	.192	.000	.000	.000	.000
3	7.512	5.189	5.433	12.049	2.449	.000	.000	.000	.000	.000
4	23.079	10.411	6.921	23.544	3.564	.000	.000	.000	.000	.000
Big	4.248	2.719	2.500	4.331	2.512	.000	.000	.000	.000	.000
	[4] Independence					p([4])				
Small	5.281	3.302	1.972	3.398	7.519	.000	.000	.010	.000	.000
2	1.843	1.696	2.161	2.992	2.074	.018	.036	.004	.000	.006
3	2.972	1.980	1.881	3.225	3.379	.000	.009	.015	.000	.000
4	1.245	0.925	1.219	2.190	1.410	.220	.548	.240	.003	.121
Big	1.515	1.604	1.349	2.656	1.384	.079	.054	.152	.000	.133
	[5.1] Mean Homogeneity					p([5.1])				
Small	1.587	0.819	1.390	1.741	2.865	.091	.631	.166	.055	.001
2	1.132	0.457	0.935	1.654	1.452	.331	.939	.511	.074	.139
3	1.054	1.827	0.748	1.174	1.660	.397	.041	.704	.299	.072
4	1.201	0.724	0.671	0.861	1.160	.279	.729	.780	.587	.309
Big	1.191	0.998	1.776	3.069	1.461	.286	.449	.049	.000	.135
	[5.2] Seasonal Homogeneity					p([5.2])				
Small	5.963	1.522	2.734	1.396	2.994	.000	.120	.002	.171	.001
2	0.428	0.730	2.972	1.244	2.599	.944	.710	.001	.254	.003
3	1.745	2.445	1.593	3.235	2.297	.061	.006	.097	.000	.009
4	1.049	1.873	2.476	2.907	2.492	.402	.040	.005	.001	.005
Big	1.521	1.295	2.031	1.131	1.618	.120	.224	.024	.334	.090
	[5.3] Variance Homogeneity					p([5.3])				
Small	2.345	1.725	2.384	1.320	0.633	.006	.058	.005	.203	.814
2	2.873	1.844	3.397	1.688	1.037	.001	.039	.000	.066	.413
3	1.949	0.980	1.542	0.964	3.278	.027	.466	.105	.482	.000
4	3.159	1.126	1.897	1.520	1.663	.000	.336	.032	.112	.071
Big	1.731	0.979	1.284	0.594	4.069	.057	.468	.224	.848	.000

Appendix J: M-S Testing Results for Student's t Dynamic LR

This Appendix presents the M-S testing results of the explicit CAPM estimated by the heterogeneous Student's t/ Heteroskedastic Dynamic LR $(1;\nu)$ model. The estimated model is (July 1963 to December 2013, 606 months):

$$y_{it}{=}\alpha_i{+}\beta_{1i}x_{1t}{+}\beta_{2i}x_{2t}{+}\sum\nolimits_{j=1}^{m^*}\delta_{1ji}v_{jt}{+}\sum\nolimits_{j=2}^{s}\delta_{2ji}d_{jt}{+}\boldsymbol{\gamma}_i^{\top}\mathbf{Z}_{it-1}{+}u_{it}, \quad i{=}1,2,...,k,\ t{\in}\mathbb{N}, \tag{J.1}$$

where $y_{it}{=}R_{it}$ is the return of portfolio i for period t; $x_{1t}{=}R_{mt}$ is the return of the value-weighted market portfolio; $x_{2t}{=}R_{ft}$ is the risk-free return; v_{jt} denotes the terms of the Gram-Schmidt orthonormal polynomials of order $j{=}1,2,...,m^*$; $d_{jt}{:=}(d_{2t},d_{3t},...,d_{12t})$ are the monthly dummy variables for the months of February through December; and $\mathbf{Z}_{it}{:=}(y_{it},\mathbf{X}_t)$, for $\mathbf{X}_t{:=}(x_{1t},x_{2t})$.

To evaluate the model assumptions [1]–[5] that justify the heterogeneous Student's t/ Heteroskedastic Dynamic LR $(1;\nu)$ model, the estimated standardized residuals of the model in (J.1) are carefully examined. The Kolmogorov-Smirnov (D) test in Appendix F is employed to test for [1] Student's t, and the auxiliary regressions (F) below are employed to test for [2] Linearity, [3] Heteroskedasticity, [4] Markov Dependence of order 1, [5] t-invariance.

$$(\widehat{u}_{it})^{st}=\gamma_{1i}+\boldsymbol{\gamma}_{2i}^{\top}\boldsymbol{\Xi}_{it}^{*}+\underbrace{\sum\nolimits_{j=m^*+1}^{m}\gamma_{3ji}v_{jt}}_{\overline{[5.1]}}+\underbrace{\boldsymbol{\gamma}_{4i}^{\top}\boldsymbol{\psi}_t}_{\overline{[2]}}+\underbrace{\sum\nolimits_{j=2}^{p}\boldsymbol{\gamma}_{5ji}^{\top}\mathbf{Z}_{it-j}}_{\overline{[4]}}+\varepsilon_{1it}, \tag{J.2}$$

$$(\widehat{u}_{it}^{2})^{st}=\gamma_{6i}+\underbrace{\sum\nolimits_{j=1}^{m}\gamma_{7ji}v_{jt}}_{\overline{[5.3]}}+\underbrace{\boldsymbol{\gamma}_{8i}^{\top}\mathbf{W}_{it}+\boldsymbol{\gamma}_{9i}^{\top}\boldsymbol{\psi}_t+\sum\nolimits_{j=2}^{p}\boldsymbol{\gamma}_{10ji}^{\top}\mathbf{Z}_{it-j}^{2}}_{\overline{[3]}+\overline{[4]}}+\varepsilon_{2it}, \tag{J.3}$$

where $(\widehat{u}_{it})^{st}$ are the estimated standardized residuals in (J.1) of portfolio i for period t; $\mathbf{W}_{it}{:=}(\mathbf{X}_t,\mathbf{Z}_{it-1}){=}(w_{1t},...,w_{lt})$; $\boldsymbol{\psi}_t{:=}\{(w_{it}{\cdot}w_{jt}),i{\geq}j,i,j{=}1,2,...,l\}$ are the second-order Kolmogorov-Gabor polynomials; $\boldsymbol{\Xi}_{it}^{*}{:=}(v_{1t},v_{2t},...,v_{mt}^{*},d_{2t},d_{3t},...,d_{12t},\mathbf{W}_{it})$.

The M-S testing results of the 25 *Size-B/M*, 25 *Size-OP*, and 25 *Size-Inv* portfolios, are presented in tables J.1–J.3; values of $p(.)$ below .001 are considered to indicate departures from assumptions [1]–[5].

Table J.1 - M-S Testing Results for 25 Size-B/M Portfolios: CAPM

Size\B/M	Low	2	3	4	High	Low	2	3	4	High
	[1] Student's t (K-S)					$p([1])$				
Small	0.040	0.017	0.017	0.024	0.035	.146	.708	.698	.508	.234
2	0.020	0.024	0.023	0.033	0.023	.612	.494	.532	.267	.528
3	0.016	0.037	0.025	0.041	0.016	.741	.192	.463	.128	.744
4	0.038	0.049	0.054	0.021	0.039	.175	.057	.028	.577	.162
Big	0.035	0.051	0.042	0.020	0.031	.233	.040	.123	.621	.322
	[2] Linearity					$p([2])$				
Small	4.039	2.840	1.153	1.246	1.597	.001	.015	.331	.286	.159
2	2.736	0.750	0.353	0.398	0.678	.019	.586	.880	.851	.640
3	2.275	0.807	0.485	0.553	0.604	.046	.545	.788	.736	.697
4	0.702	1.586	1.041	0.219	0.601	.622	.162	.393	.954	.699
Big	0.912	0.743	0.467	0.381	0.231	.473	.592	.801	.862	.949
	[3] Heteroskedasticity					$p([3])$				
Small	0.025	2.933	3.763	1.689	3.538	.875	.087	.053	.194	.060
2	0.013	0.651	0.009	2.758	0.320	.909	.420	.923	.097	.572
3	1.027	1.918	3.145	19.484	3.353	.311	.167	.077	.000	.068
4	11.669	11.269	16.917	7.873	2.224	.001	.001	.000	.005	.136
Big	2.666	19.612	12.235	54.312	0.764	.103	.000	.001	.000	.382
	[4] Markov Dependence (1)					$p([4])$				
Small	0.655	0.607	0.495	0.678	1.045	.580	.611	.686	.566	.372
2	0.081	0.105	1.375	0.234	0.292	.970	.957	.249	.873	.831
3	0.290	1.338	1.452	0.861	3.673	.833	.261	.227	.461	.012
4	4.468	4.927	1.032	0.913	0.408	.004	.002	.378	.434	.747
Big	0.631	0.367	0.474	1.014	0.182	.595	.777	.701	.386	.909
	[5.1] Mean Heterogeneity					$p([5.1])$				
Small	2.259	2.650	2.580	1.729	0.909	.133	.104	.109	.189	.341
2	3.077	3.143	3.772	1.487	1.376	.080	.077	.053	.223	.241
3	1.658	0.760	0.001	5.050	1.399	.198	.384	.978	.025	.237
4	1.027	0.952	4.562	0.731	1.526	.311	.330	.033	.393	.217
Big	0.010	0.490	0.320	0.089	0.006	.922	.484	.572	.766	.940
	[5.3] Variance Heterogeneity					$p([5.3])$				
Small	0.762	0.947	0.275	1.420	0.024	.383	.331	.600	.234	.876
2	0.924	0.704	2.210	2.369	0.976	.337	.402	.138	.124	.324
3	1.812	1.943	6.541	5.199	5.288	.179	.164	.011	.023	.022
4	0.512	3.278	3.533	0.028	2.557	.475	.071	.061	.867	.110
Big	1.511	0.064	0.061	0.149	0.256	.219	.800	.806	.699	.613

Table J.2 - M-S Testing Results for 25 Size-OP Portfolios: CAPM

Size\OP	Low	2	3	4	High	Low	2	3	4	High
	[1] Student's t (K-S)					$p([1])$				
Small	0.052	0.021	0.037	0.026	0.024	.039	.579	.186	.456	.510
2	0.043	0.032	0.040	0.039	0.012	.106	.281	.147	.158	.833
3	0.026	0.038	0.030	0.045	0.024	.438	.172	.329	.088	.496
4	0.041	0.041	0.040	0.044	0.042	.126	.132	.146	.093	.115
Big	0.030	0.043	0.038	0.055	0.041	.337	.112	.172	.026	.126
	[2] Linearity					$p([2])$				
Small	2.916	1.076	1.379	1.080	1.759	.013	.372	.230	.370	.119
2	1.681	0.771	0.338	0.633	0.718	.137	.571	.890	.675	.610
3	1.603	0.832	0.757	0.532	0.393	.157	.527	.581	.752	.854
4	0.806	1.112	0.491	0.267	0.623	.545	.353	.783	.931	.682
Big	0.614	0.228	0.816	3.088	1.849	.690	.951	.539	.009	.102
	[3] Heteroskedasticity					$p([3])$				
Small	0.711	6.215	4.045	6.095	5.099	.399	.013	.045	.014	.024
2	0.571	1.188	0.012	0.676	0.084	.450	.276	.911	.411	.772
3	0.837	1.156	0.049	0.189	0.211	.361	.283	.824	.664	.646
4	26.535	0.493	5.263	17.682	0.114	.000	.483	.022	.000	.736
Big	0.259	0.283	9.573	3.136	2.082	.611	.206	.766	.301	.739
	[4] Markov Dependence (1)					$p([4])$				
Small	0.736	0.753	0.661	0.546	1.268	.531	.521	.576	.651	.285
2	0.346	0.034	0.342	0.977	0.515	.792	.992	.795	.403	.673
3	0.906	2.183	2.655	0.723	0.994	.438	.089	.048	.539	.395
4	1.313	0.281	0.930	2.458	1.400	.269	.840	.426	.062	.242
Big	0.783	1.528	0.382	1.221	0.420	.504	.206	.766	.301	.739
	[5.1] Mean Heterogeneity					$p([5.1])$				
Small	0.004	2.519	2.750	2.339	2.583	.953	.113	.098	.127	.109
2	2.549	4.044	6.127	0.881	1.089	.111	.045	.014	.348	.297
3	2.348	0.801	1.284	2.674	0.461	.126	.371	.258	.103	.497
4	3.970	0.060	0.206	0.051	0.062	.047	.807	.650	.821	.804
Big	0.121	0.011	1.871	3.755	0.090	.728	.916	.172	.053	.765
	[5.3] Variance Heterogeneity					$p([5.3])$				
Small	0.292	2.033	0.074	0.068	0.001	.589	.155	.786	.794	.971
2	1.069	0.904	2.195	0.722	0.566	.302	.342	.139	.396	.452
3	2.216	1.197	5.597	2.985	0.277	.137	.274	.018	.085	.599
4	0.955	0.959	3.198	2.938	0.224	.329	.328	.074	.087	.637
Big	1.003	0.173	3.209	0.017	0.518	.317	.678	.074	.896	.472

Table J.3 - M-S Testing Results for 25 Size-Inv Portfolios: CAPM

Size\Inv	Low	2	3	4	High	Low	2	3	4	High
	[1] Student's t (K-S)					$p([1])$				
Small	0.046	0.015	0.040	0.008	0.020	.081	.773	.142	.926	.609
2	0.032	0.026	0.026	0.034	0.020	.300	.442	.449	.256	.624
3	0.029	0.026	0.028	0.037	0.021	.361	.437	.388	.199	.577
4	0.020	0.043	0.052	0.040	0.036	.628	.104	.039	.149	.206
Big	0.022	0.037	0.043	0.042	0.043	.554	.199	.107	.114	.106
	[2] Linearity					$p([2])$				
Small	2.979	1.298	1.153	1.260	2.507	.012	.263	.331	.280	.029
2	1.585	0.556	0.287	0.414	0.953	.162	.734	.920	.839	.447
3	0.906	0.775	1.091	0.667	0.514	.477	.568	.364	.649	.766
4	0.475	1.052	1.193	2.610	0.525	.795	.386	.311	.024	.757
Big	0.488	0.232	2.011	0.628	2.192	.785	.949	.076	.679	.054
	[3] Heteroskedasticity					$p([3])$				
Small	0.083	2.293	3.316	3.830	3.788	.774	.131	.069	.051	.052
2	0.891	0.315	1.063	0.250	0.274	.346	.575	.303	.617	.601
3	0.090	1.113	0.962	0.378	2.098	.764	.292	.327	.539	.148
4	0.398	0.116	14.516	0.451	13.252	.528	.733	.000	.502	.000
Big	3.300	0.589	40.047	1.009	0.196	.070	.443	.000	.316	.658
	[4] Markov Dependence (1)					$p([4])$				
Small	0.814	1.347	0.604	0.280	1.136	.487	.258	.612	.840	.334
2	0.953	0.692	0.228	0.400	0.179	.415	.557	.877	.753	.911
3	0.162	0.218	1.187	0.201	0.477	.922	.884	.314	.896	.699
4	0.658	1.221	4.667	1.812	2.198	.578	.301	.003	.142	.087
Big	0.491	0.602	0.428	0.168	0.385	.689	.614	.733	.918	.764
	[5.1] Mean Heterogeneity					$p([5.1])$				
Small	4.575	4.568	3.739	1.443	1.901	.033	.033	.054	.230	.169
2	3.664	3.026	2.100	2.057	1.661	.056	.082	.148	.152	.198
3	0.644	0.129	1.642	0.728	0.352	.423	.720	.201	.394	.553
4	2.019	0.557	0.835	0.253	0.382	.156	.456	.361	.616	.537
Big	1.153	0.401	1.362	0.714	1.071	.283	.527	.712	.399	.301
	[5.3] Variance Heterogeneity					$p([5.3])$				
Small	0.131	0.599	0.841	0.089	0.407	.718	.439	.360	.766	.524
2	0.850	0.109	1.836	0.265	3.287	.357	.742	.176	.607	.070
3	0.059	5.009	6.695	4.075	1.127	.809	.026	.010	.044	.289
4	0.041	4.815	2.245	0.728	0.322	.840	.029	.135	.394	.571
Big	0.455	4.786	2.270	0.607	0.101	.500	.029	.132	.436	.751

Appendix K: Testing for the Nesting Restrictions

The current form of the estimated statistical model in (5.1) cannot be used to test for the CAPM nesting restriction of $\beta_{1i}+\beta_{2i}=1$. For this reason, the model needs to be reparameterized. A sample proof is shown below.
The unrestricted statistical CAPM model takes the following form:

$$\begin{aligned} &(R_{it}-R_{ft})=\beta_{0i}+\beta_{1i}(R_{mt}-R_{ft})+u_{it}, \\ &\rightarrow R_{it}=\beta_{0i}+\beta_{1i}R_{mt}+\beta_{2i}R_{ft}+u_{it},\ i=1,2,...,k,\ t\in N. \end{aligned} \tag{K.1}$$

The restricted statistical CAPM model takes the following form:

$$\begin{aligned} &(R_{it}-R_{ft})=\alpha_{0i}+\alpha_{1i}(R_{mt}-R_{ft})+\eta_{it}, \\ &\rightarrow R_{it}=\alpha_{0i}+\alpha_{1i}R_{mt}+(1-\alpha_{1i})R_{ft}+\eta_{it},\ i=1,2,...,k,\ t\in N. \end{aligned} \tag{K.2}$$

The reparameterized unrestricted CAPM model takes the following form:

$$\begin{aligned} &(R_{it}-R_{ft})=\gamma_{0i}+\gamma_{1i}(R_{mt}-R_{ft})+\gamma_{2i}R_{ft}+\upsilon_{it}, \\ &\rightarrow R_{it}=\gamma_{0i}+R_{ft}+\gamma_{1i}R_{mt}-\gamma_{1i}R_{ft}+\gamma_{2i}R_{ft}+\upsilon_{it}, \\ &\rightarrow R_{it}=\gamma_{0i}+\gamma_{1i}R_{mt}+(1-\gamma_{1i}+\gamma_{2i})R_{ft}+\upsilon_{it},\ i=1,2,...,k,\ t\in N. \end{aligned} \tag{K.3}$$

(K.2) relates to (K.1) as follows:

$$\begin{aligned} &\alpha_{0i}=\beta_{0i},\ \alpha_{1i}=\beta_{1i},\ (1-\alpha_{1i})=\beta_{2i},\ \eta_{it}=u_{it}, \\ &\rightarrow \beta_{1i}+\beta_{2i}=1. \end{aligned}$$

(K.3) relates to (K.1) as follows:

$$\gamma_{0i}=\beta_{0i},\ \gamma_{1i}=\beta_{1i},\ (1-\gamma_{1i}+\gamma_{2i})=\beta_{2i},\ \upsilon_{it}=u_{it}.$$

(K.3) relates to (K.2) as follows:

$$(\gamma_{2i}=0)\rightarrow \gamma_{0i}=\alpha_{0i},\ \gamma_{1i}=\alpha_{1i},\ \upsilon_{it}=\eta_{it}.$$

Hence, testing H_0:$\gamma_{2i}=0$ in (K.3) is equivalent to testing $\beta_{1i}+\beta_{2i}=1$ in (K.1).

Chapter 6

6 On Forecasting Market Risk

6.1 Overview

The Basel Committee on Banking Supervision has recently proposed to replace the most commonly used market risk measure of Value-at-Risk (VaR) with Expected Shortfall (ES). This transition stems from a number of problems associated with VaR, among them, its inability to capture the "tail risk" and lack of coherence. The primary aim of this chapter is to revisit the risk measures of VaR and ES, by paying particular attention to the validity of the internal models used by banks to forecast market risk. The latter validity is particularly important because the reliability of both the VaR and ES relies heavily on the probabilistic assumptions underlying these internal models.

A brief outline of the chapter follows. Section 6.2 provides a brief history of the Basel Accords. Section 6.3 discusses the differences between VaR and ES, by delineating between conceptual and probabilistic weaknesses. As it is argued, the "tail risk" and lack of coherence weaknesses refer only to conceptual weaknesses, whereas seeing things from a purely probabilistic perspective, the two risk measures can be weak if the probabilistic assumptions underlying the internal risk forecast models are invalid. Section 6.4 attempts to assess the performance of the most commonly used risk forecast models by employing a simplistic but suggestive empirical example. In this example, the violation ratios of the most commonly used internal models are compared to the ones of the proposed heterogeneous Student's t/ Heteroskedastic Autoregressive model, which is shown to have good forecasting ability. In view of these results, the chapter argues that the process of measuring market risk capital requirements is deemed inadequate, primarily because of the statistical inadequacy of the risk forecast models used, and not necessarily to the mathematical methods for computing market risk. Finally, the chapter suggests that the Basel Committee should severely restrict the scope of internal modeling, to models which are likely to pass some kind of statistical adequacy test.

6.2 A Brief History of the Basel Accords

At the end of 1974, the central bank governors of the G10 countries established a Committee on Banking Regulations and Supervisory Practises, in response to a series of adverse events[7] and disruptions in the international financial markets. Later renamed the Basel Committee on Banking Supervision (BCBS), the Committee was designed as a forum for regular cooperation between its member countries on banking supervisory matters:

> "Its aim was and is to enhance financial stability by improving supervisory knowhow and the quality of banking supervision worldwide. The Committee seeks to achieve its aims by setting minimum standards for the regulation and supervision of banks; by sharing supervision issues, approaches and techniques to promote common understanding and improve cross-border cooperation; and by exchanging information on developments in the banking sector and financial markets to identify current and emerging risks for the global financial system. Also, to engage with the challenges presented by diversified financial conglomerates, the Committee also works with other standard-setting bodies." (BCBS, 2014).

[7] Viz., the breakdown of Bretton Woods system of managed exchange rates in 1973, the closure and liquidation of Bankhaus Herstatt's in 1974, and the collapse of Franklin National Bank of New York in 1974.

Prior the first attempt of the Committee to set international risk-based standards for capital adequacy, the bank regulators of each country tended to regulate bank capital by setting minimum levels for the ratio of capital to total assets. In 1988, the Committee made the first attempt to assess capital mainly in relation with credit risk. The Basel Capital Accord, or Basel I as it has been known more recently, introduced a minimum capital ratio of capital to risk-weighted assets of 8%. After several revisions, the Committee refined the original framework to address risks other than credit risk. In 1996, the Committee issued the Market Risk Amendment to the Capital Accord, designed to incorporate a capital requirement for the market risks arising from banks' exposures to foreign exchange, traded debt securities, equities, commodities, and options. An important aspect of 1996 Market Risk Amendment was that banks with well-established risk management functions were, for the first time, allowed to use internal models as a basis for measuring their market risk capital requirements. This involved calculating the Value-at-Risk (VaR) measure based on the output of the internal risk forecast models, and converting it into capital requirement using a prespecified formula.

Over the years, the reforms in the successive accords of Basel II and Basel II.5 improved the effectiveness of supervision and strengthened the regulatory capital framework. Likewise, banks themselves refined the practise of risk management by the development of new techniques respecting the internal risk measurement. Nevertheless, the recent financial crisis revealed structural weaknesses in the financial system, regulation, and supervision. In response, to these weaknesses, the Committee proposed a stronger regulatory framework, known as Basel III, concerning better quality of capital, increased coverage of risk for capital market activities, and better liquidity standards, among other benefits. As part of the revised regulatory approach, the Committee has also proposed a transition from VaR to Expected Shortfall (ES):

> "A number of weaknesses have been identified with using VaR in determining regulatory capital requirements, including its inability to capture "tail risk".

For this reason, the Committee proposed in May 2012 to replace VaR with ES. ES measures the riskiness of a position by considering both the size and the likelihood losses above a certain confidence level. The Committee has agreed to use a 97.5% ES for the internal models-based approach and has also used that approach to calibrate capital requirements under the revised market risk standardized approach." (BCBS, 2013).

A more careful and judicious discussion on the VaR and ES risk measures is provided in the next section.

6.3 Value-at-Risk vs. Expected Shortfall

The Value-at-Risk (VaR) analysis was formally introduced as a market risk management tool in the technical document Riskmetrics, published by J. P. Morgan in 1993.[8] During that time VaR was adopted by a number of financial institutions as a tool to measure and quantify the level of financial risk. Yet, it was the endorsement of the use of VaR for regulatory purposes by the Basel Committee in 1996, that established VaR as the most commonly used risk measure.

Definition 6.1 ***Value-at-Risk (VaR):*** *Given some confidence level* $\alpha \in (0,1)$, *VaR is the loss level,* l, *such that the probability of loss* L *exceeding* l *is no larger than* $(1-\alpha)$*:*

$$\begin{aligned} VaR(\alpha) &= \inf\{l \in \mathbb{R} : \Pr(L > l) \leq (1-\alpha)\} \\ &= \inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\}, \end{aligned} \tag{6.1}$$

where F_L *denotes the loss distribution function.*

Despite its simple appearance, the VaR risk measure has been subject of sustained criticism, with Artzner et al. (1997; 1999) being the most noticeable. In their seminal papers, Artzner et al. (1997; 1999) have cited two major weaknesses of VaR. A first weakness is associated with the "tail risk", stated by BCBS (2013). Indeed, the VaR

[8] Although formally introduced by J. P. Morgan, VaR can be traced back to Leavens (1945). In his paper, Leavens did not explicitly identify a VaR metric, but he illustrated a quantitative example which may be the first VaR measure ever published.

does not provide any information about the downside risk, other than what is the most optimistic of the worst case scenarios. By taking a closer look at the definition of VaR, it is clear that the actual loss, L, is expected to exceed the potential loss level, l, with a probability of less than α, which entails that VaR is inadequate in capturing the risk of extreme movements. A second weakness of VaR is that it does not constitute a coherent risk measure; see Artzner et al. (1999).

Definition 6.2 ***Coherent Risk Measure:*** *Let X and Y be random variables denoting the losses of two individual portfolios, and let c denote a real number. Then, the function $\wp(.)$:$X, Y \rightarrow \mathbb{R}$, is characterized coherent risk measure if it satisfies the following four axioms:*

Axiom 6.1 *Translation Invariance:* $\wp(X + c)=\wp(X) - c, \ \forall \ X, c.$

Axiom 6.2 *Subadditivity:* $\wp(X + Y) \leq \wp(X) + \wp(Y), \ \forall \ X, Y.$

Axiom 6.3 *Monotonicity:* $\wp(X) \geq \wp(Y), \ \forall \ X, Y, \ X \leq Y.$

Axiom 6.4 *Positive Homogeneity:* $\wp(cX)=c\wp(X), \ \forall \ X, c.$

VaR can not be characterized as a coherent risk measure because it does not always satisfy the axiom of subadditivity. The violation of this axiom is of particular concern since it contradicts with the principle of diversification. Having said that, if the regulator uses a non-subadditive risk measure in determining the regulatory capital for banks, these banks have an incentive to legally break up into various subsidiaries in order to reduce their regulatory capital requirements; see McNeil et al. (2005). Undoubtedly, both the "tail risk" weakness and the lack of coherence for the VaR risk measure influenced the decision of Basel Committee to propose a transition from VaR to ES:

> "... it [meaning the VaR] has been criticized in the literature for lacking subadditivity."; "The most prominent alternative to VaR is ES, which is subadditive."; "...ES does account for the severity of losses beyond the confidence threshold.", and "...it is always subadditive and coherent."; "... ES avoids the major flaws of VaR bus its fundamental difference from VaR - that it accounts

for the magnitude of losses beyond a threshold - is an equally important advantage." (BCBS, 2011).

"A number of weaknesses have been identified with using VaR for determining regulatory capital requirements, including its inability to capture "tail risk". For this reason the Committee has considered alternative risk metrics, in particular ES."; "Accordingly, the Committee is proposing the use of ES for the internal models-based approach...." (BCBS, 2012).

"... the Committee has identified a number of weaknesses with risk measurement under the models-based approach under the 1996 market risk amendment. Specifically, the 10-day VaR calculation did not adequately capture credit risk or market liquidity risks; incentivised banks to take on tail risk..."; "To strengthen model standards, the Committee has agreed to, ... move to an ES metric...." (BCBS, 2013).

The risk measure of ES has been proposed by Artzner et al. (1997) to overcome the "tail risk" and lack of coherence weaknesses inherent in VaR.

Definition 6.3 ***Expected Shortfall (ES):*** *Given some confidence level* $\alpha \in (0,1)$*, ES is the expected loss,* L*, conditional on being lower than negative* $VaR(\alpha)$*:*

$$ES \;\; = - \, E\left[L | L \leq - \, VaR(\alpha)\right], \tag{6.2}$$

where $VaR(\alpha)$ *is the potential loss level,* l*, from the VaR risk measure.*

Whereas VaR does not account for the tail of the distribution, ES involves the expectation of the actual loss, L, and thus, captures the "tail risk". Hence, the ES risk measure is expected to give a lower loss level than the VaR. For instance, if the VaR(α) yields a potential loss level of l, the ES is expected to yield a potential loss, L, in the interval $[l, \infty]$. Moreover, ES can be characterized as a coherent risk measure since it satisfies the axiom of subadditivity; see Artnzer et al. (1999).

In view of the two major weaknesses of VaR, one may argue that both constitute pure conceptual weaknesses. From a purely probabilistic perspective, even though the definition of VaR involves a loss distribution function (F_L), the inability of VaR to capture the "tail risk" will remain for any underlying empirical distribution. On the other hand, in contempt of the fact that the axiom of subadditivity is violated only for fat-tailed distributions (see Artzner et al., 1999; Danielsson et al., 2013), employing the normal, or another thin-tailed distribution whose tail index exceeds 2 in an attempt to satisfy subadditivity, will falsely assume less risk than actually is present. This is because the tails of financial asset return distributions are much thicker than for the normal distribution; see Mandelbrot (1963), Fama (1965).

In practise the computation of VaR:[9]

$$VaR(\alpha) = -\,\sigma \times F_{\alpha}^{-1} \times W_{t-1}, \tag{6.3}$$

where F_α denotes the cumulative distribution function of the underlying distribution, and W_{t-1} is the value of the portfolio, involves the estimated standard deviation, σ, which depends crucially on the validity of the (implicit) statistical assumptions of the internal risk forecast models employed. Similarly, the computation of the conceptually improved ES involves the same estimated standard deviation:[10]

$$ES = -\,\sigma \times \frac{f(F_{\alpha}^{-1})}{\alpha} \times W_{t-1}, \tag{6.4}$$

where $f(.)$ denotes the probability density function.

[9] The definition of VaR can be rewritten as: $\Pr(R_t \leq -\,VaR(\alpha)) = \alpha$, where $R_t = \frac{W_t - W_{t-1}}{W_{t-1}}$ is the change in the value of the portfolio. Thus,

$$\begin{aligned} \alpha &= \Pr(W_{t-1}R_t \leq -VaR(\alpha)), \\ \alpha &= \Pr(\tfrac{R_t}{\sigma} \leq -\tfrac{VaR(\alpha)}{W_{t-1}\sigma}), \\ F_{\alpha}^{-1} &= -\tfrac{VaR(\alpha)}{W_{t-1}\sigma}, \\ VaR(\alpha) &= -\sigma \times F_{\alpha}^{-1} \times W_{t-1}. \end{aligned}$$

[10] The computation formula of ES is derived as follows:

$$\begin{aligned} ES &= -E\left[L | L \leq -\mathrm{VaR}(\alpha)\right] \\ &= -\tfrac{f(-\mathrm{VaR}(\alpha))}{F(-\mathrm{VaR}(\alpha))} \\ &= -\tfrac{f(F^{-1}(\alpha))}{\alpha} \\ ES &= -\sigma \times \tfrac{f(F^{-1}(\alpha))}{\alpha} \times W_{t-1} \end{aligned}$$

Although, from a conceptual perspective ES is considered superior to VaR, it can be contended that from a purely probabilistic perspective, both the two risk measures can be weak if the probabilistic assumptions underlying the specifications of the internal risk forecast models are invalid. As a matter of fact, ES is more uncertain than VaR since the estimation of the former needs to determine the latter before obtaining the expectation of the "tail risk". By that means, invalid probabilistic structure comprise an important (probabilistic) weakness, which is far more crucial than any other (conceptual) weakness.

6.4 Empirical Example

To assess the performance of the most commonly used risk forecast models, as well as of the heterogeneous Student's t/ Heteroskedastic Autoregressive (StAR) model (see Spanos, 1995) the simple procedure of backtesting will be used. This procedure can be used to compare the various risk models by comparing the ex ante VaR forecasts from a particular model to the ex post realized returns. A brief description of the backtesting procedure is given below; for additional information, see Danielsson (2011).

If the realized loss exceeds the VaR forecast on a particular day, then a VaR violation occurs. Formally a VaR violation is defined as:

$$\eta_t := \begin{cases} 1 \text{ if } y_t \leq -VaR_t, \\ 0 \text{ if } y_t > -VaR_t. \end{cases} \tag{6.5}$$

where y_t is the realized return on a particular day t, and VaR_t denotes the VaR forecasted for the same day.

The observed number of violations ($\sum \eta_t$) for a given data sample (called the testing window (W_T)) is used to assess the performance of a particular model. The latter assessment is simply a comparison between the observed number of violations ($\sum \eta_t$) and the expected number of violations ($p \times W_T$). This tool is called the violation ratio:

$$VR = \frac{\sum \eta_t}{p \times W_T}. \tag{6.6}$$

A closer look at the violation ratio in (6.6) reveals that the ratio is simply an ex post comparison of the actual and nominal error probabilities of a particular forecast model. Given that the nominal error probability is set to 1% by the Basel Committee, a model is expected to have good performance, when the actual error probability is close to 1%. Hence, one can think of this ratio as an ex post M-S test which is expected to give a value close to 1 if and only if the ex ante actual error probabilities approximate the nominal ones.

6.4.1 Internal Risk Forecast Models

Financial institutions with well-established risk management functions are allowed to use internal model-based approach for setting market risk capital. The most commonly used internal risk forecast models are: (a) Historical Simulation, (b) Moving Average, (c) Exponentially Weighted Moving Average, (d) Normal GARCH, and (e) Student's t GARCH. These models are briefly discussed below; for a detailed description of these models, see Danielsson (2011).

Historical Simulation (HS). A non-parametric method where the VaR at confidence level α is defined as the negative $(T \times \alpha)^{th}$ value in the sorted return vector, multiplied be the value of the portfolio.

Moving Average (MA).

$$\widehat{\sigma}_t^2 = \frac{1}{W_E} \sum_{i=1}^{W_E} y_{t-i}^2, \tag{6.7}$$

where $\widehat{\sigma}_t$ is the estimated conditional volatility of returns, W_E denotes the length of the estimation window, and y_t is the observed return for period t.

Exponentially Weighted Moving Average (EWMA).

$$\widehat{\sigma}_t^2 = (1-\lambda) y_{t-i}^2 + \lambda \widehat{\sigma}_{t-1}^2, \tag{6.8}$$

where λ denotes a decay factor, for $0 < \lambda < 1$.

Normal GARCH (NGARCH).

$$\widehat{\sigma}_t^2 = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \widehat{\sigma}_{t-j}^2, \tag{6.9}$$

where $\omega,\ \alpha_i,\ \beta_j,\ i=1,2,...,p,\ j=1,2,...,q$ refer to estimated parameters.

The most common version only employs one lag in the GARCH(1,1) model.

Student's *t* GARCH (StGARCH). The residuals of the GARCH model are Student's t distributed with ν degrees of freedom.

6.4.2 Empirical Results

Table 6.1 reports the violation ratios of the most commonly used risk forecast models and StAR. The sample used is the daily S&P 500 returns from May 29, 2002 to December 30, 2016; 3676 days. The estimation window is W_E=300 [i.e., the first W_E is from May 29, 2002 to August 5, 2003] and the testing window is W_T=3376 [August 6, 2003 to December 30, 2016]. The estimation windows for the 3 sub-sample periods are: W_{T_1}=895 [August 6, 2003 to February 26, 2007]; W_{T_2}=1270 [February 27, 2007 to March 9, 2012]; and W_{T_3}=1211 [March 10, 2012 to December 30, 2016].

Table 6.1 - S&P500: 99% VaR Violation Ratios for W_E=300

Model\Period	Sample 1	Sample 2	Sample 3	Full Sample
MA	0.670 [.006]	4.173 [.015]	1.982 [.005]	2.459 [.012]
EWMA	1.453 [.003]	2.992 [.020]	2.230 [.005]	2.310 [.015]
HS	0.670 [.006]	1.890 [.025]	0.826 [.008]	1.185 [.020]
NGARCH	1.117 [.002]	3.228 [.020]	1.899 [.005]	2.192 [.015]
StGARCH	0.000 [.004]	0.315 [.033]	0.165 [.009]	0.178 [.024]
StAR	0.782 [.015]	0.945 [.066]	0.991 [.017]	0.889 [.032]

As the violation ratios indicate, the most commonly used risk forecast models do not have good forecasting ability. Specifically, the MA, EWMA, HS, and NGARCH models consistently underestimate risk, whereas the StGARCH model consistently overestimates risk. The only violation ratios which are considered acceptable are for the low volatile period (sample 1) where the MA, EWMA, HS, and NGARCH are forecasting risk relatively well. Comparing these violation ratios with the results of the StAR, it is clear that the performance of the latter model is much better. The violation ratios are very close to 1, especially during periods of high volatility (samples 2 and 3). In addition, the model forecasts risk relatively well for periods of low volatility as well (sample 1). The good performance of the StAR model is stemming from the statistical adequacy of the model. This model accounts for all the statistical systematic information in the data and because of that its actual error probabilities approximate the nominal ones.

6.5 Summary and Conclusions

By delineating between conceptual and probabilistic weaknesses, the chapter argues that the "tail risk" and lack of coherence refer only to conceptual weaknesses, whereas both the risk measures of VaR and ES can be weak if the probabilistic assumptions underlying the internal risk forecast models used by banks are invalid. The latter argument led to the assessment of the most commonly used risk forecast models using an empirical example. As it is shown, the proposed heterogeneous Student's t/ Heteroskedastic Autoregressive model has good forecasting ability as it accounts for all the statistical systematic information in the data. The adequate forecasting ability of this model shows that the process of measuring market risk capital requirements relies heavily on the validity of the probabilistic assumptions underlying the internal risk forecast models. As the chapter argues, in order for the Basel Committee to improve the process, the scope of internal modeling should be severely restricted to models which are likely to pass some kind of statistical adequacy test.

Chapter 7

7 Summary and Conclusions

The primary concern of this dissertation is to revisit the CAPM and the Fama-French multi-factor models with a view to evaluate the validity of the probabilistic assumptions imposed (directly or indirectly) on the Fama and French (1993; 2015) data, as well as the selection of the additional factors for explaining asset returns. The concern with model validation stems from the primary aim of empirical modeling, which is to learn from the data about phenomena of interest. Thus, this dissertation pays particular attention to this 'learning from the data' which depends crucially on the task of specifying a probabilistic representation (statistical model) of a real world financial phenomenon that accounts for the chance regularities in the data using probabilistic assumptions from three broad categories: *Distribution*, *Dependence*, and *Heterogeneity*.

As has been discussed in chapter 2, the heterogeneity category has been the least developed in the sense of devising concepts to account for different forms of heterogeneity in financial data. For this reason, this dissertation starts by addressing the serious issues raised when modeling complicated forms of heterogeneity exhibited by aggregated data (i.e., portfolio data) in the context of Linear Regression (LR) models, using trend polynomials. In this chapter a number of different ways one could deal with the near-collinearity problem arising from the high degree of the polynomials are evaluated. These include the (scaled) ordinary, continuous, discrete, and

Gram-Schmidt (G-S) polynomials.

The evaluations provide a number of recommendations to the practitioner. The most noteworthy recommendation is that the problem of near-collinearity can be addressed by combining (a) continuous orthogonal polynomials whose interval of orthogonality is [-1,1], and (b) scaling the time ordering so that the range of values lies within the same interval. These results suggest that the traditional textbook recommendations associated to the orthogonality property and the scaling of time ordering are not sufficient enough to effectively overcome the instability in the computations. Whereas the two recommendations are necessary, the interval of orthogonality also matters. For instance, orthogonal polynomials whose interval of orthogonality is outside the interval [-1,1], like the Hermite, do not help to overcome the problem. In addition to the aforementioned recommendation, an alternative way that effectively addresses the problem is the usage of the orthonormal Gram-Schmidt polynomials (not orthogonal). The results of this chapter are considered the basis of this dissertation since the G-S orthonormal polynomials play a very crucial role in the Mis-Specification (M-S) and respecification testing in chapters 3-4, as well as the factor selection procedure in chapter 5 and market risk forecasting in chapter 6.

In chapter 3, the CAPM and the Fama-French three-factor model are revisited with a view to evaluate their statistical adequacy vis-à-vis the Fama and French (1993) data. In this chapter, thorough M-S testing is used to evaluate the probabilistic assumptions imposed on the data in question. As it is shown, the model assumptions of Normality, Linearity, Homoskedasticity, Independence, and t-invariance, implicitly imposed on these empirical asset pricing models are severely violated. In light of the M-S testing results, chapter 4 proceeds to respecify the statistical Normal/ Homoskedastic LR model underlying the CAPM and the Fama-French three-factor model with a probabilistic structure that is rich enough to account for all the statistical regularities in the data. This takes the form of the heterogeneous Student's t/ Heteroskedastic Vector Autoregressive (VAR) model and the reparameterized heterogeneous Student's t/ Heteroskedastic Dynamic LR model.

On the basis of their statistical adequacy, the respecified models shed light on the existing models in the literature. Whereas the finance literature has focused mostly on the departures from homoskedasticity (e.g., GARCH-type models) and weakening distributional assumptions (e.g., GMM), largely ignoring the leptokurticity, as well as the different forms of dependence and heterogeneity in the data, the finite-dimensional models respecified in the current study are shown to account for all the systematic statistical information in the data, rather than solely a subset of these information.

Even though the respecified models are used exclusively to elucidate the CAPM and the Fama-French multi-factor models, they can equally provide a sound basis to revisit any empirical testing of asset pricing models, such as the conversion of asset pricing theories into models which are estimable with a given set of data, or empirical studies searching for potential sources of risk. For instance, future research may evaluate the mutual fund performance evaluation of the Carhart four-factor model. Besides, the respecified models can also be used to revisit alternative frameworks, delimiting what there is to explain by capturing the statistical regularities in the data. One such specimen is presented in chapter 6, where the heterogeneous Student's t Autoregressive (AR) model is shown to have better predictive ability compared to other commonly used market risk forecast models, such as the Historical Simulation (HS), Normal GARCH (NGARCH), and Student's t GARCH (StGARCH).

Moreover, the respecification that yields the statistically adequate models is not confined to the reliability of the inference, but also provides a sound basis for probing substantive questions of interest. The factor selection procedure proposed in chapter 5 indicates that the specification of a statistically adequate model can serve as a basis for selecting a set of factors when studying multi-factor models. The proposed procedure has the advantage of providing a coherent basis for selecting the relevant factors from the set of possible ones. By employing the factor selection procedure to test for the significance of the Fama-French factors, the empirical results suggest that the value factor HML is not redundant as conjectured by Fama and French (2015), whereas the redundant factor is the investment factor CMA. In addition, the lingering heterogeneity in the data indicates that a number of relevant factors are still missing and further work is needed to enhance the substantive adequacy of the Fama-French multi-factor models in order to replace the generic trends with proper explanatory variables. In an analogous way, one can use the same procedure to establish the relevance of any seasonal effects.

References

[1] Affleck-Graves, J., and McDonald, B. (1989), "Nonnormalities and Tests of Asset Pricing Theories," *Journal of Finance*, 44(4), 889-908.

[2] Anderson, T. W., and Darling, D. A. (1954), "A Test of Goodness of Fit," *Journal of the American Statistical Association*, 49(268), 765-769.

[3] Andreou, E., and Spanos A. (2003), "Statistical Adequacy and the Testing of Trend Versus Difference Stationarity," *Econometric Reviews*, 22(3), 217-237.

[4] Ariel, R. A. (1987), "A Monthly Effect in Stock Returns," *Journal of Financial Economics*, 18(1), 161-174.

[5] Artzner, P., Delbaen, F., Eber, J. M., and Heath, D. (1997), "Thinking Coherently," *Risk*, 10(11), 68-71.

[6] Artzner, P., Delbaen, F., Eber, J. M., and Heath, D. (1999), "Coherent Measures of Risk," *Mathematical Finance*, 9(3), 203-228.

[7] Bachelier, L. (1900), "Théorie de la spéculation," *Annales Scientifiques de l'École Normale Supérieure*, 3(17), 21–86. Reprinted in English in 1964, in *The Random Character of Stock Market Prices*, edited by Cootner, P. H., pp. 17-78, M.I.T. Press, Cambridge.

[8] Bailey, R. E. (2005), *The Economics of Financial Markets*, Cambridge University Press, New York.

[9] Banz, R. W. (1981), "The Relationship Between Return and Market Value of Common Stocks," *Journal of Financial Economics*, 9(1), 3-18.

[10] Barone-Adesi, G., and Talwar, P. P. (1983), "Market Models and Heteroscedasticity of Residual Security Returns," *Journal of Business & Economic Statistics*, 1(2), 163-168.

[11] Bartlett, M. S. (1934), "The Vector Representation of a Sample," *Mathematical Proceedings of the Cambridge Philosophical Society*, 30(3), 327-340.

[12] Basel Committee on Banking Supervision. (2011), *Messages from the Academic Literature on Risk Management for the Trading Book* (working paper no.19). Retrieved from Bank for International Settlements website: http://bis.org/publ/bcbs_wp19.pdf.

[13] Basel Committee on Banking Supervision. (2012), *Consulative Document: Fundamental Review of the Trading Book*. Retrieved from Bank for International Settlements website: http://bis.org/publ/bcbs219.pdf.

[14] Basel Committee on Banking Supervision. (2013), *Consulative Document: Fundamental Review of the Trading Book: A Revised Market Risk Framework 5*. Retrieved from Bank for International Settlements website: http://bis.org/publ/bcbs265.pdf.

[15] Basel Committee on Banking Supervision. (2014), *A Brief History of the Basel Committee*. Retrieved from Bank for International Settlements website: http://bis.org/bcbs/history.pdf.

[16] Basu, S. (1983), "The Relationship Between Earnings' Yield, Market Value and Return for NYSE Common Stocks: Further Evidence," *Journal of Financial Economics*, 12(1), 129-156.

[17] Beaver, W., Kettler, P., and Scholes, M. (1970), "The Association Between Market Determined and Accounting Determined Risk Measures," *Accounting Review*, 45(4), 654-682.

[18] Belkaoui, A. (1977), "Canadian Evidence of Heteroscedasticity in the Market Model," *Journal of Finance*, 32(4), 1320-1324.

[19] Bey, R. P., and Pinches, G. E. (1980), "Additional Evidence of Heteroscedasticity in the Market Model," *Journal of Financial and Quantitative Analysis*, 15(2), 299-322.

[20] Bhandari, L. C. (1988), "Debt/Equity Ratio and Expected Common Stock Returns: Empirical Evidence," *Journal of Finance*, 43(2), 507-528.

[21] Blattberg, R. C., and Gonedes, N. J. (1974), "A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices," *Journal of Business*, 47(2), 244-280.

[22] Black, F. (1972), "Capital Market Equilibrium with Restricted Borrowing," *Journal of Business*, 45(3), 444-455.

[23] Black, F. (1993), "Beta and Return," *Journal of Portfolio Management*, 20(1), 8-18.

[24] Black, F., Jensen, M. C., and Scholes, M. S. (1972), "The Capital Asset Pricing Model: Some Empirical Tests," in *Studies in the Theory of Capital Markets*, edited by Jensen, M. C., pp. 79-121, Praeger, New York.

[25] Blume, M. E. (1968), *The Assessment of Portfolio Performance: An Application of Portfolio Theory*, Doctoral dissertation, University of Chicago.

[26] Blume, M. E. (1970), "Portfolio Theory: A Step Toward Its Practical Application," *Journal of Business*, 43(2), 152-173.

[27] Blume, M. E. (1971), "On the Assessment of Risk," *Journal of Finance*, 26(1), 1-10.

[28] Blume, M. E. (1975), "Betas and Their Regression Tendencies," *Journal of Finance*, 30(3), 785-795.

[29] Bollerslev, T. (1986), "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, 31(3), 307-327.

[30] Bookstaber, R. M., and McDonald, J. B. (1987), "A General Distribution for Describing Security Price Returns," *Journal of Business*, 60(3), 401-424.

[31] Bouman, S., and Jacobsen, B. (2002), "The Halloween Indicator," Sell in May and Go Away": Another Puzzle," *American Economic Review*, 92(5), 1618-1635.

[32] Box, G. E. P., and Tiao, G. C. (1973), *Bayesian Inference in Statistical Analysis*, Addison-Wesley, Reading, MA.

[33] Brenner, M., and Smidt, S. (1977), "A Simple Model of Non-stationarity of Systematic Risk," *Journal of Finance*, 32(4), 1081-1092.

[34] Brown, S. J. (1977), "Heteroscedasticity in the Market Model: A Comment," *Journal of Business*, 50(1), 80-83.

[35] Campbell, J. Y., Lo, A. W., and MacKinlay, A. C. (1997), *The Econometrics of Financial Markets*, Princeton University Press, Princeton, NJ.

[36] Carhart, M. M. (1997), "On Persistence in Mutual Fund Performance," *Journal of Finance*, 52(1), 57-82.

[37] Chen, N. F., Roll, R., and Ross, S. A. (1986), "Economic Forces and the Stock Market," *Journal of Business*, 59(3), 383-403.

[38] Chmielewski, M. A. (1981), "Elliptically Symmetric Distributions: A Review and Bibliography," *International Statistical Review*, 49(1), 67-74.

[39] Choi, I. (2015), *Almost All About Unit Roots: Foundations, Developments, and Applications*, Cambridge University Press, Cambridge.

[40] Clark, P. K. (1973), "A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices," *Econometrica*, 41(1), 135-155.

[41] Danielsson, J. (2011), *Financial Risk Forecasting: The Theory and Practise of Forecasting Market Risk with Implementation in R and Matlab*, John Wiley and Sons Ltd, Chicester.

[42] Danielsson, J., Jorgensen, B. N., Samorodnitsky, G., Sarma, M., and de Vries, C. G. (2013), "Fat tails, VaR and subadditivity," *Journal of Econometrics*, 172(2), 283-291.

[43] D'Agostino, R., and Pearson, E. S. (1973), "Tests for Departure from Normality. Empirical Results for the Distributions of b_2 and $\sqrt{b_1}$," *Biometrika*, 60(3), 613-622.

[44] Dunkl, C. F., and Xu, Y. (2014), *Orthogonal Polynomials of Several Variables*, 2nd ed., Encyclopedia of Mathematics and Its Applications, vol. 155, Cambridge University Press, Cambridge.

[45] Engle, R. F. (1982), "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation," *Econometrica*, 50(4), 987-1007.

[46] Engle, R. F., Hendry, D. F., and Richard, J-F. (1983), "Exogeneity," *Econometrica*, 51(2), 277-304.

[47] Fama, E. F. (1976), *Foundations of Finance: Portfolio Decisions and Securities Prices*, Basic Books, New York.

[48] Fama, E. F. (1965), "The Behavior of Stock-Market Prices," *Journal of Business*, 38(1), 34-105.

[49] Fama, E. F. (1973), "A Note on the Market Model and the Two-Parameter Model," *Journal of Finance*, 28(5), 1181-1185.

[50] Fama, E. F. (1991), "Efficient Capital Markets: II," *Journal of Finance*, 46(5), 1575-1617.

[51] Fama, E. F. (1996), "Multifactor Portfolio Efficiency and Multifactor Asset Pricing," *Journal of Financial and Quantitative Analysis*, 31(4), 441-465.

[52] Fama, E. F. (2011), "My Life in Finance," *Annual Review of Financial Economics*, 3(1), 1-15.

[53] Fama, E. F., Fisher, L., Jensen, M. C., and Roll, R. (1969), "The Adjustment of Stock Prices to New Information," *International Economic Review*, 10(1), 1-21.

[54] Fama, E. F., and French, K. R. (1992), "The Cross-Section of Expected Stock Returns," *Journal of Finance*, 47(2), 427-465.

[55] Fama, E. F., and French, K. R. (1993), "Common Risk Factors in the Returns on Stocks and Bonds," *Journal of Financial Economics*, 33(1), 3-56.

[56] Fama, E. F., and French, K. R. (2015), "A Five-Factor Asset Pricing Model," *Journal of Financial Economics*, 116(1), 1-22.

[57] Fama, E. F., and French, K. R. (2017), "International Tests of a Five-Factor Asset Pricing Model," *Journal of Financial Economics*, 123(3), 441-463.

[58] Fama, E. F., and MacBeth, J. D. (1973), "Risk, Return, and Equilibrium: Empirical Tests," *Journal of Political Economy*, 81(3), 607-636.

[59] Fang, K.-T., Kotz, S., and Ng, K. W. (1990), *Symmetric Multivariate and Related Distributions*, Chapman and Hall, London.

[60] Ferson, W. (1996), "Warning: Attribute-sorted Portfolios can be Hazardous to your Research," in *Modern Finance Theory and its Applications*, edited by Saitou, S., Sawaki, K., and Kubota, K., pp. 21-32, Center of Academic Societies, Osaka.

[61] French, K. R. (1980), "Stock Returns and the Weekend Effect," *Journal of Financial Economics*, 8(1), 55-69.

[62] Friend, I., and Blume, M. (1970), "Measurement of Portfolio Performance Under Uncertainty," *American Economic Review*, 60(4), 561-575.

[63] Gautschi, W. (2004), *Orthogonal Polynomials: Computation and Approximation*, Oxford University Press, New York.

[64] Gautschi, W. (1990), "How (Un)stable Are Vandermonde Systems," in *Asymptotic and Computational Analysis: Conference in Honor of Frank W. J. Olver's 65th Birthday*, vol. 124, edited by Wong, R., pp. 193-210, Marcel Dekker, New York.

[65] Gautschi, W., and Inglese, G. (1988), "Lower Bounds for the Condition Number of Vandermonde Matrices," *Numerische Mathematik*, 52(3), 241-250.

[66] Granger, C. W. J., and Morgenstern, O. (1963), "Spectral Analysis of New York Stock Market Prices," *Kyklos*, 16(1), 1-27.

[67] Granger, C. W. J., and Newbold, P. (1974), "Spurious Regressions in Econometrics," *Journal of Econometrics*, 2(2), 111–120.

[68] Greene, W. H. (2012), *Econometric Analysis*, 7th ed., Prentice Hall, New Jersey.

[69] Greene, M. T., and Fielitz, B. D. (1977), "Long-term Dependence in Common Stock Returns," *Journal of Financial Economics*, 4(3), 339-349.

[70] Golub, G. H., and Van Loan, C. F. (2013), *Matrix Computations*, 4th ed., Johns Hopkins University Press, Baltimore.

[71] Gonedes, N. J. (1973), "Evidence on the Information Content of Accounting Numbers: Accounting-based and Market-based Estimates of Systematic Risk," *Journal of Financial and Quantitative Analysis*, 8(3), 407-443.

[72] Good, I. J. (1988), "The Interface between Statistics and Philosophy of Science," *Statistical Science*, 3(4), 386-397.

[73] Hagerman, R. L. (1978), "More Evidence on the Distribution of Security Returns," *Journal of Finance*, 33(4), 1213-1221.

[74] Hansen, L. P. (1982), "Large Sample Properties of Generalized Method of Moments Estimators," *Econometrica*, 50(4), 1029-1054.

[75] Hartman, P., and Wintner, A. (1940), "On the Spherical Approach to the Normal Distribution Law," *American Journal of Mathematics*, 62(1), 759-779.

[76] Harvey, C. R., Liu, Y., and Zhu, H. (2015), "... and the Cross-Section of Expected Returns," *Review of Financial Studies*, 29(1), 5-68.

[77] Hooker, R. H. (1901), "Correlation of the Marriage-Rate with Trade," *Journal of the Royal Statistical Society*, 64(3), 485-492.

[78] Hooker, R. H. (1905), "On the Correlation of Successive Observations," *Journal of the Royal Statistical Society*, 68(4), 696-703.

[79] Jensen, M. C. (1968), "The Performance of Mutual Funds in the Period 1945-1964," *Journal of Finance*, 23(2), 389-416.

[80] Jensen, M. C. (1969), "Risk, The Pricing of Capital Assets, and The Evaluation of Investment Portfolios," *Journal of Business*, 42(2), 167-247.

[81] Keim, D. B. (1983), "Size-Related Anomalies and Stock Return Seasonality: Further Empirical Evidence," *Journal of Financial Economics*, 12(1), 13-32.

[82] Kelker, D. (1970), "Distribution Theory of Spherical Distributions and a Location-Scale Parameter Generalization," *Sankhyā: The Indian Journal of Statistics, Series A*, 32(4), 419-430.

[83] King, B. F. (1966), "Market and Industry Factors in Stock Price Behavior," *Journal of Business*, 39(1), 139-190.

[84] Kolmogorov, A. N. (1933), "Sulla Determinazione Empirica di una Legge di Distribuzione," (in Italian) *Giornale dell' Instituto Italiano degli Attuari*, 4(1), 83-91.

[85] Kothari, S. P., Shanken, J., and Sloan, R. G. (1995), "Another Look at the Cross-Section of Expected Stock Returns," *Journal of Finance*, 50(1), 185-224.

[86] Kotz, S. (1975), "Multivariate Distributions at a Cross Road," in *A Modern Course on Statistical Distributions in Scientific Work*, vol. 1, edited by Patil, G. P., Kotz, S., and Ord, J. K., pp. 247-270, D. Reidel Publishing Company, Dordrecht, Holland.

[87] Leavens, D. H. (1945), "Diversification of Investments," *Trusts and Estates*, 80(5), 469-473.

[88] Lehmann, E. L., and Romano, J. P. (2006), *Testing Statistical Hypotheses*, 3rd ed., Springer, New York, NY.

[89] Levy, H. (2012), *The Capital Asset Pricing Model in the 21st Century: Analytical, Empirical, and Behavioral Perspectives*, Cambridge University Press, New York.

[90] Lewellen, J., Nagel, S., and Shanken, J. (2010), "A Skeptical Appraisal of Asset Pricing Tests," *Journal of Financial Economics*, 96(2), 175-194.

[91] Lintner, J. (1965a), "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics*, 47(1), 13-37.

[92] Lintner, J. (1965b), "Security Prices, Risk, and Maximal Gains from Diversification," *Journal of Finance*, 20(4), 587-615.

[93] Mandelbrot, B. (1963), "The Variation of Certain Speculative Prices," *Journal of Business*, 36(4), 394-419.

[94] Markowitz, H. (1952), "Portfolio Selection," *Journal of Finance*, 7(1), 77-91.

[95] Martin, J. D., and Klemkosky, R. C. (1975), "Evidence of Heteroscedasticity in the Market Model," *Journal of Business*, 48(1), 81-86.

[96] Maxwell, J. C. (1860), "V. Illustrations of the Dynamical Theory of Gases. - Part I. On the Motions and Collisions of Perfectly Elastic Spheres," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 19(124), 19-32.

[97] McNeil, A. J., Frey, R., and Embrechts, P. (2005), *Quantitative Risk Management*, Princeton University Press, Princeton, NJ.

[98] Merton, R. C. (1973), "An Intertemporal Capital Asset Pricing Model," *Econometrica*, 41(5), 867-887.

[99] Meyers, S. L. (1973), "The Stationarity Problem in the Use of the Market Model of Security Price Behavior," *Accounting Review*, 48(2), 318-322.

[100] Miller, M. H., and Scholes, M. S. (1972), "Rates of Return in Relation to Risk: A Reexamination of Some Recent Findings," in *Studies in the Theory of Capital Markets*, edited by Jensen, M. C., pp. 47-78, Praeger, New York.

[101] Modigliani, F., and Miller, H. M. (1958), "The Cost of Capital, Corporation Finance and the Theory of Investment," *American Economic Review*, 48(3), 261-297.

[102] Moore, H. L. (1914), *Economic Cycles: Their Law and Cause*, Macmillan, London.

[103] Morgan, I. G. (1976), "Stock Prices and Heteroscedasticity," *Journal of Business*, 49(4), 496-508.

[104] Morgan, M. S. (1990), *The History of Econometric Ideas*, Cambridge University Press, Cambridge.

[105] Mossin, J. (1966), "Equilibrium in a Capital Asset Market," *Econometrica*, 34(4), 768-783.

[106] Officer, R. R. (1972), "The Distribution of Stock Returns," *Journal of the American Statistical Association*, 67(340), 807-812.

[107] Osborne, M. F. M. (1959), "Brownian Motion in the Stock Market," *Operations Research*, 7(2), 145-173.

[108] Pástor, Ľ., and Stambaugh, R. F. (2003), "Liquidity Risk and Expected Stock Returns," *Journal of Political Economy*, 111(3), 642-685.

[109] Perry, P. R. (1982), "The Time-Variance Relationship of Security Returns: Implications for the Return-Generating Stochastic Process," *Journal of Finance*, 37(3), 857-870.

[110] Pesaran, M. H. (2015), *Time Series and Panel Data Econometrics*, Oxford University Press, New York.

[111] Phillips, P. C. B. (1986), "Understanding Spurious Regressions in Econometrics," *Journal of Econometrics*, 33(3), 311-340.

[112] Phillips, P. C. B. (2005), "Challenges of Trending Time Series Econometrics," *Mathematics and Computers in Simulation*, 68(5-6), 401–416.

[113] Praetz, P. D. (1969), "Australian Share Prices and the Random Walk Hypothesis," *Australian & New Zealand Journal of Statistics*, 11(3), 123-139.

[114] Praetz, P. D. (1972), "The Distribution of Share Price Changes," *Journal of Business*, 45(1), 49-55.

[115] Press, S. J. (1967), "A Compound Events Model for Security Prices," *Journal of Business*, 40(3), 317-335.

[116] Razali, N. M., and Wah, Y. B. (2011), "Power Comparisons of Shapiro-Wilk, Kolmogorov-Smirnov, Lilliefors and Anderson-Darling Tests," *Journal of Statistical Modeling and Analytics*, 2(1), 21-33.

[117] Reinganum, M. R. (1983), "The Anomalous Stock Market Behavior of Small Firms in January: Empirical Tests for Tax-Loss Selling Effects," *Journal of Financial Economics*, 12(1), 89-104.

[118] Rosenberg, B., Reid, K., and Lanstein, R. (1985), "Persuasive Evidence of Market Inefficiency," *Journal of Portfolio Management*, 11(3), 9-16.

[119] Ross, S. A. (1976), "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory*, 13(3), 341-360.

[120] Rozeff, M. S., and Kinney, W. R. (1976), "Capital Market Seasonality: The Case of Stock Returns," *Journal of Financial Economics*, 3(4), 379-402.

[121] Ruppert, D., Wand, M. P., and Carroll, R. J. (2003), *Semiparametric Regression*, Cambridge University Press, Cambridge.

[122] Schwert, G. W., and Seguin, P. J. (1990), "Heteroskedasticity in Stock Returns," *Journal of Finance*, 45(4), 1129-1155.

[123] Seber, G. A. F., and Lee, A. J. (2003), *Linear Regression Analysis*, 2nd ed., Wiley, Hoboken, NJ.

[124] Shapiro, S. S., and Francia, R. S. (1972), "An Approximate Analysis of Variance Test for Normality," *Journal of the American Statistical Association*, 67(334), 215-216.

[125] Shapiro, S. S., and Wilk, M. B. (1965), "An Analysis of Variance Test for Normality (Complete Samples)," *Biometrika*, 52(3-4), 591-611.

[126] Sharpe, W. F. (1964), "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," *Journal of Finance*, 19(3), 425-442.

[127] Smirnov, N. V. (1939), "On the Estimation of the Discrepancy Between Empirical Curves of Distribution for Two Independent Samples," (in Russian) *Bulletin Mathematique de l'Universite de Moscou*, 2(2), 3-16.

[128] Smith, J. (1981), *The Probability Distribution of Market Returns: A Logistic Hypothesis*, Doctoral dissertation, University of Utah.

[129] Spanos, A., (1986), *Statistical Foundations of Econometric Modelling*, Cambridge University Press, Cambridge.

[130] Spanos, A. (1999), *Probability Theory and Statistical Inference: Econometric Modeling with Observational Data*, Cambridge University Press, Cambridge.

[131] Spanos, A. (1990a), "The Simultaneous-Equations Model Revisited: Statistical Adequacy and Identification," *Journal of Econometrics*, 44(1-2), 87-105.

[132] Spanos, A. (1990b), "Unit Roots and their Dependence on the Implicit Conditioning Information Set," *Advances in Econometrics*, 8, 271-292.

[133] Spanos, A. (1994), "On Modeling Heteroskedasticity: The Student's t and Elliptical Linear Regression Models," *Econometric Theory*, 10(2), 286-315.

[134] Spanos, A. (1995), "On Theory Testing in Econometrics: Modeling with Non-experimental data," Journal of Econometrics, 67(1), 189-226.

[135] Spanos, A. (2006a), "Econometrics in Retrospect and Prospect," in *New Palgrave Handbook of Econometrics*, vol. 1, edited by Mills, T. C., and Patterson, K., pp. 3-58, MacMillan, London.

[136] Spanos, A. (2006b), "Revisiting the Omitted Variables Argument: Substantive vs. Statistical Adequacy," *Journal of Economic Methodology*, 13(2), 179–218.

[137] Spanos, A. (2006c), "Where Do Statistical Models Come From? Revisiting the Problem of Specification," in *Optimality: The Second Erich L. Lehmann Symposium*, vol. 49, Lecture Notes-Monograph Series, edited by Rojo, J., pp. 98-119, Institute of Mathematical Statistics, Beachwood, OH.

[138] Spanos, A. (2009), "Statistical Misspecification and the Reliability of Inference: The Simple t-test in the Presence of Markov Dependence," *Korean Economic Review*, 25(2), 165-213.

[139] Spanos, A. (2010), "Statistical Adequacy and the Trustworthiness of Empirical Evidence: Statistical vs. Substantive Information," *Economic Modelling*, 27(6), 1436–1452.

[140] Spanos, A. (2017), "Mis-Specification Testing in Retrospect," *Journal of Economic Surveys*, forthcoming.

[141] Spanos, A., and McGuirk, A. (2002), "The Problem of Near-multicollinearity Revisited: Erratic vs. Systematic Volatility," *Journal of Econometrics*, 108(2), 365-393.

[142] Stigler, S.M. (1986), *The History of Statistics: The measurement of uncertainty before 1900*, Harvard University Press.

[143] Teichmoeller, J. (1971), "A Note on the Distribution of Stock Price Changes," *Journal of the American Statistical Association*, 66(334), 282-284.

[144] Tobin, J. (1958), "Liquidity Preference as Behavior Towards Risk," *Review of Economic Studies*, 25(2), 65-86.

[145] Trefethen, L. N., and Bau III, D. (1997), *Numerical Linear Algebra*, Society for Industrial and Applied Mathematics, Philadelphia, PA.

[146] Treynor, J. L. (1962), "Toward a Theory of Market Value of Risky Assets," unpublished manuscript. A final version was published in 1999, in *Asset Pricing and Portfolio Performance: Models, Strategy, and Performance Metrics,* edited by Korajczyk, R. A., pp. 15-22, Risk Books, London.

[147] Tyrtyshnikov, E. E. (1994), "How Bad are Hankel Matrices?," *Numerische Mathematik*, 67(2), 261-269.

[148] Westerfield, R. (1977), "The Distribution of Common Stock Price Changes: An Application of Transactions Time and Subordinated Stochastic Models," *Journal of Financial and Quantitative Analysis*, 12(5), 743-765.

[149] Wooldridge, J. M. (2012), *Introductory Econometrics: A Modern Approach*, 5th ed., South-Western, Mason, OH.

[150] Young, W. E. (1971), "Random Walk of Stock Prices: A Test of the Variance-Time Function," *Econometrica*, 39(5), 797-812.

[151] Yule, G. U. (1903), "Notes on the Theory of Association of Attributes in Statistics," *Biometrika*, 2(2), 121-134.

[152] Zhou, G. (1993), "Asset-pricing Tests under Alternative Distributions," *Journal of Finance*, 48(5), 1927-1942.