

First Cohomology of Some Infinitely Generated Groups

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(ABSTRACT)

The goal of this paper is to explore the first cohomology group of groups G that are not necessarily finitely generated. Our focus is on l^p -cohomology, $1 \leq p \leq \infty$, and what results regarding finitely generated groups change when G is infinitely generated. In particular, for abelian groups and locally finite groups, the l^p -cohomology is non-zero when G is countable, but vanishes when G has sufficient cardinality. We then show that the l^∞ -cohomology remains unchanged for many classes of groups, before looking at several results regarding the injectivity of induced maps from embeddings of G -modules. We present several new results for countable groups, and discuss which results fail to hold in the general uncountable case. Lastly, we present results regarding reduced cohomology, including a useful lemma extending vanishing results for finitely generated groups to the infinitely generated case.

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(GENERAL AUDIENCE ABSTRACT)

The goal of this paper is to use a technique that originated in algebraic topology to study the properties of a structure called a group. Groups are collections of objects that interact with each other through an operation that obeys certain properties. Groups arise when considering many different mathematical questions, and they were first studied when looking at the different symmetries an object can have. Classifying the different properties of a group is an active area of mathematical research. We seek to do this by looking at collections of maps from a particular group to the real or complex numbers, then studying how the group shifts these functions.

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Chapter 1

Introduction

Group cohomology is a technique that arose in the study of algebraic topology that can be used to study some interesting properties of groups. Classifying the cohomology groups for groups with various properties has been useful in several different branches of mathematics including operator theory, geometry, and K-theory, among others. Most of the results about the classification of these cohomology groups, however, have been limited to finitely generated groups. l^2 -cohomology has proven to have some particularly interesting applications to other branches of mathematics [16], and so our initial goal was to see what results about finitely generated groups extend or do not extend to infinitely generated groups when considering coefficients in $l^p(G)$.

Dicks and Dunwoody classify the cohomology groups with coefficients in the group ring AG [4], and particularly interesting is the fact that these cohomology groups are non-zero for countably infinite, locally finite groups, but vanish for uncountable locally finite groups. Holt looks at this specific case in his paper [3], and we found his proof useful in showing that there are indeed several cases when the l^p -cohomology groups are non-zero for countably infinite groups but vanish if the group is uncountable. This is true in particular for uncountable abelian groups and locally finite groups with cardinality greater than the real numbers. It is not clear whether this result remains true for locally finite groups without the extra cardinality condition at this time. We also extend this result to several other G -modules, and show that it will not remain true for cohomology groups with coefficients in $l^\infty(G)$, at least in some cases. Reznikov did some work with coefficients in $l^\infty(G)$ and $c_0(G)$ [13], and we explore some of his results in Chapter 5.

An embedding of G -modules induces a map on cohomology groups, however, this induced map is not injective in general. Bekka and Vallette discuss the induced map from the embedding of the group ring into $l^2(G)$ [1], and Puls does some work with the induced map from the embedding of $l^p(G)$ into $l^q(G)$ where $1 < p < q < \infty$ [22]. Using these ideas, we are able to look at several induced maps, most notably we are able to show that the map induced from $l^p(G) \hookrightarrow l^q(G)$ is an embedding if and only if the group is non-amenable for countable groups G , a result that does not hold for uncountable groups in general.

Finally, we take a look at results in reduced cohomology, and try to extend some of the well-known results to infinitely generated groups. We are able to prove a lemma that extends the vanishing results of reduced cohomology to infinitely generated groups, and we also look at a few results due to Gournay [21]. In particular, he outlines why he believes the reduced cohomology groups with coefficients in $c_0(G)$ vanish, a fact we prove toward the end of this paper. We also look at the vanishing of the reduced cohomology for infinite locally finite groups with coefficients in $c_0(G)$ or $l^p(G)$ for $1 \leq p \leq \infty$.

Chapter 2

Preliminaries

We want to study properties of a general group G by looking at the way that the group acts on an abelian group A , which we will refer to as a G -module.

Definition 2.1. A G -module A will be an abelian group A together with a homomorphism $\phi: G \rightarrow \text{Aut } A$, so that G acts on A on the left by automorphisms.

This definition is equivalent to defining a G -module as an R -module over the ring $R = \mathbb{Z}G$ where $\mathbb{Z}G$ is defined as follows:

Definition 2.2. Let R be a ring with identity and G be a group. Define the group ring, denoted RG , to be the set of all finite sums

$$\sum_{g \in S} a_g g$$

where S is a finite subset of G and $a_g \in R$. This is a ring under componentwise addition, and multiplication given by $a_1 g_1 a_2 g_2 = (a_1 a_2)(g_1 g_2)$ extended to all of RG by distributive laws.

We hope to study the G -invariants of the G -module A , which we will denote $A^G := \{a \in A \mid g \cdot a = a \text{ for all } g \in G\}$. In particular, we are interested in the G -invariants of an extension of a G -module A by some other G -module C . Stated another way, the short exact sequence of G -modules

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

induces an exact sequence

$$0 \rightarrow A^G \rightarrow B^G \rightarrow C^G, \quad (2.1)$$

and we want to know whether or not this can be extended to another short exact sequence.

If we view A, B and C as $\mathbb{Z}G$ -modules, then by noting that $A^G = \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$ (where the action of G on \mathbb{Z} is trivial), we find that the group $\text{Ext}_{\mathbb{Z}G}^1(\mathbb{Z}, A)$ is precisely the group that measures the failure of this sequence to extend to a short exact sequence. To get an idea of what this group looks like explicitly, we need to produce a projective resolution of \mathbb{Z} . So consider the projective resolution

$$\dots \xrightarrow{d_2} F_2 \xrightarrow{d_1} F_1 \xrightarrow{d_0} F_0 \xrightarrow{\text{aug}} \mathbb{Z} \rightarrow 0 \quad (2.2)$$

where $F_n = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} \mathbb{Z}G$ (with $n + 1$ factors). When we make F_n a G -module under the G -action $g \cdot g_0 \otimes g_1 \otimes \dots \otimes g_n = gg_0 \otimes g_1 \otimes \dots \otimes g_n$, it can be shown that F_n is a free $\mathbb{Z}G$ -module with basis $1 \otimes g_1 \otimes \dots \otimes g_n$ and $g_i \in G$. Then, using the augmentation map $\text{aug}: \mathbb{Z}G \rightarrow \mathbb{Z}$ defined by $\text{aug}(\sum_{g \in G} \alpha_g g) = \sum_{g \in G} \alpha_g$, and the maps $d_n: F_{n+1} \rightarrow F_n$ for $n \geq 0$ defined by

$$\begin{aligned} d_n(1 \otimes g_1 \otimes \dots \otimes g_{n+1}) &= g_1 \cdot (1 \otimes g_2 \otimes \dots \otimes g_{n+1}) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i (1 \otimes g_1 \otimes \dots \otimes g_{i-1} \otimes g_i g_{i+1} \otimes g_{i+2} \otimes \dots \otimes g_{n+1}) \\ &\quad + (-1)^n (1 \otimes g_1 \otimes \dots \otimes g_n), \end{aligned}$$

(2.1) becomes a free G -module resolution of \mathbb{Z} .

By applying $\mathbb{Z}G$ -module homomorphisms from (2.2) into A and looking at the cohomology groups of the resulting cochain complex, and by noting that homomorphisms from $F_n \rightarrow A$ are determined by where the basis elements are sent, we see that $\text{Ext}_{\mathbb{Z}G}^1(\mathbb{Z}, A)$ is the quotient group of maps $\alpha: G \rightarrow A$ satisfying $0 = \alpha(gh) - \alpha(g) - g \cdot \alpha(h)$ ($\ker d_1^*$) by the group of maps $\beta: G \rightarrow A$ of the form $\beta(g) = g \cdot a - a$ for some $a \in A$ ($\text{Im } d_0^*$). This leads to the following definition of $H^n(G, A)$, which is equivalent to $H^n(G, A) = \text{Ext}_{\mathbb{Z}G}^n(\mathbb{Z}, A)$.

Definition 2.3. Let G be a group and A be a G -module, then we define $C^0(G, A) = A$ and $C^n(G, A)$ to be the set of all maps from G^n to A . We will call these n -cochains of G with values in A . $C^n(G, A)$ is an additive abelian group under pointwise addition, and in the case of $n = 0$ under the group structure of A .

Definition 2.4. For $n \geq 0$, define the n th coboundary homomorphism from $C^n(G, A)$ to $C^{n+1}(G, A)$ by

$$\begin{aligned} d_n(f)(g_1, \dots, g_{n+1}) &= g_1 \cdot f(g_2, \dots, g_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i f(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \dots, g_n). \end{aligned}$$

The fact that $d_n \circ d_{n-1} = 0$ for $n \geq 1$ follows from noting that (2.1) is a projective resolution, but it can be shown directly as well.

Definition 2.5. Let $Z^n(G, A) = \ker d_n$ for $n \geq 0$, $B_n(G, A) = \text{Im } d_{n-1}$ for $n \geq 1$, and $B_0(G, A) = 1$. Then, since $d_n \circ d_{n-1} = 0$ for $n \geq 1$, $B^n(G, A)$ is a subgroup of $Z^n(G, A)$, and we can define $H^n(G, A)$ to be the quotient group $Z^n(G, A)/B^n(G, A)$.

We will denote $Z^1(G, A)$ by Z_G , and by definition this will be the subgroup of maps in $C^1(G, A)$ satisfying $\alpha(gh) = \alpha(g) + g \cdot \alpha(h)$ for all $g, h \in G$. These elements will be referred to as 1-cocycles. Then let $B_G = B^1(G, A)$, which by definition is the subgroup of maps in $C^1(G, A)$ satisfying $\alpha(g) = g \cdot a - a$ for some $a \in A$, and we will refer to these maps as 1-coboundaries. This will give us $H^1(G, A) = Z_G/B_G$.

The key point here is that we get a long exact sequence of cohomology groups, showing that $H^1(G, A)$ measures the failure of (2.1) to extend to a short exact sequence. This is stated in the following theorem.

Theorem 2.6. Let

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

be a short exact sequence of G -modules. Then there is a long exact sequence of abelian groups

$$\begin{aligned} 0 \rightarrow A^G \rightarrow B^G \rightarrow C^G \xrightarrow{\delta_0} H^1(G, A) \rightarrow H^1(G, B) \rightarrow \dots \\ \dots \xrightarrow{\delta_{n-1}} H^n(G, A) \longrightarrow H^n(G, B) \longrightarrow \dots \end{aligned}$$

Before defining the specific G -modules that we will be studying in this paper, we make a quick note on some possible confusion with notation. Throughout the paper, we will use the notation \mathbb{C}^G to denote the set of all maps $f : G \rightarrow \mathbb{C}$, not to

be confused with the G -fixed elements of the G -module \mathbb{C} . Since this paper will never consider \mathbb{C} as a G -module, we hope this abuse of notation will not cause any confusion.

Now we define some specific examples of G -modules:

- $\mathbb{C}G := \{f \in \mathbb{C}^G : f(g) = 0 \text{ for all but finitely many } g \in G\}$
- $l^p(G) := \{f \in \mathbb{C}^G : \sum_{g \in G} |f(g)|^p < \infty\}$ ($1 \leq p < \infty$)
- $l^\infty(G) := \{f \in \mathbb{C}^G : \max_{g \in G} \{|f(g)|\} < \infty\}$
- $c_0(G) := \{f \in \mathbb{C}^G : |\{g \in G : |f(g)| > \epsilon\}| < \infty \text{ for any } \epsilon > 0\}$
- $\mathcal{N}(G) := \{f \in l^2(G) : fg \in l^2(G) \text{ for all } g \in l^2(G)\}$ (Von Neumann algebra)
- $C_{red}^* := \{f \in l^2(G) : \sup_{\|g\|_2=1} \{\|fg\|_2\} < \infty\}$
- $l^2(G)l^2(G) := \{\sum_{i=1}^n f_i g_i : f_i, g_i \in l^2(G) \text{ for } 1 \leq i \leq n\}$ (Fourier algebra)

Now, note that all of the above examples are G -submodules of \mathbb{C}^G under the G -action $(g \cdot f)(x) = f(g^{-1}x)$. Since all of the G -modules we will be studying in this paper are G -submodules of \mathbb{C}^G , we would like to take a closer look at the definition of $H^1(G, A)$ when A is of this form. So consider the following proposition.

Proposition 2.7. Let A be a G -submodule of \mathbb{C}^G . Given an element $f \in Z_G$, f is of the form $f(g) = g \cdot \alpha - \alpha$ for some $\alpha \in \mathbb{C}^G$. Conversely, if $\alpha \in \mathbb{C}^G$ is such that $g \cdot \alpha - \alpha \in A$ for all $g \in G$, then $g \cdot \alpha - \alpha \in Z_G$.

Proof. To see the first part of the proposition, let $\alpha \in H^1(G, A)$, and define $f(g) = -\alpha(g)(g)$ for all $g \in G$. Then given any $g, h \in G$, we have that

$$\begin{aligned}
 (g \cdot f - f)(h) &= (g \cdot f)(h) - f(h) = f(g^{-1}h) - f(h) \\
 &= -\alpha(g^{-1}h)(g^{-1}h) + \alpha(h)(h) \\
 &= -\alpha(g^{-1}h)(g^{-1}h) + \alpha(gg^{-1}h)(h) \\
 &= -\alpha(g^{-1}h)(g^{-1}h) + (\alpha(g) + g \cdot \alpha(g^{-1}h))(h) \\
 &= -\alpha(g^{-1}h)(g^{-1}h) + \alpha(g)(h) + \alpha(g^{-1}h)(g^{-1}h) = \alpha(g)(h),
 \end{aligned}$$

which shows that $\alpha(g) = g \cdot f - f$ for all g .

For the converse, note that $\alpha(g) = g \cdot f - f$ is a map from $G \rightarrow A$ by assumption, so all that we need to show is that $\alpha(g) - \alpha(gh) + g \cdot \alpha(h) = 0$ for any $g, h \in G$. So given $g, h \in G$,

$$\alpha(g) - \alpha(gh) + g \cdot \alpha(h) = g \cdot f - f - (gh \cdot f - f) + g \cdot (h \cdot f - f) = 0.$$

Therefore, $\alpha \in Z_G$, so it represents an element in the quotient $H^1(G, A)$. \blacksquare

This proposition gives us the ability to give an equivalent definition of $H^1(G, A)$ when A is a G -submodule of \mathbb{C}^G .

Definition 2.8. Let A be a G -submodule of \mathbb{C}^G . Then Z_G is the group of all maps $\alpha : G \rightarrow A$ of the form $\alpha(g) = g \cdot f - f$ for some $f \in \mathbb{C}^G$ such that $g \cdot f - f \in A$ for all $g \in G$. B_G is the group of all maps $\beta : G \rightarrow A$ of the form $\beta(g) = g \cdot f - f$ where $f \in A$, noting that if $f \in A$, then $g \cdot f - f \in A$ for all $g \in G$ simply because A is a G -module. Then $H^1(G, A) = Z_G/B_G$.

So, given any $f \in \mathbb{C}^G$, $g \cdot f - f$ represents a coset in $H^1(G, A)$ if and only if $g \cdot f - f \in A$ for all $g \in G$. If $f \in A$, then $g \cdot f - f$ is a representative of the identity coset (B_G), and if $f \notin A$, $g \cdot f - f$ represents a non-zero coset of $H^1(G, A)$. So, if we hope to show that $H^1(G, A) = 0$, we need to show that for any $f \in \mathbb{C}^G$ such that $g \cdot f - f \in A$ for all $g \in G$, $f \in A$. If we hope to show that $H^1(G, A) \neq 0$, then we only need to find one $f \in \mathbb{C}^G$ such that $g \cdot f - f \in A$ for all $g \in G$, but $f + C \notin A$ for any constant $C \in \mathbb{C}^G$. The need for dealing with the extra constant C arises from the fact that the maps $g \cdot f - f$ and $g \cdot (f + C) - (f + C)$ are equivalent as maps from G to A .

Now that we have a satisfactory definition of the first cohomology groups, we note that many of the G -modules in our examples are actually normed spaces. This means that we can give $H^1(G, A)$ the topology of pointwise convergence. When G is countable, \mathbb{C}^G is a Fréchet space, so by the open mapping theorem we get that $H^1(G, A)$ is Hausdorff under the topology of pointwise convergence if and only if B_G is complete. As it turns out, the first cohomology group is often not Hausdorff, and we will look at this in great detail, however, this gives rise to the question, what does the space Z_G/B_G look like? This leads to the following definition of the reduced cohomology group.

Definition 2.9. Let A be a G -module with a norm and Z_G and B_G be defined

as before. Then the reduced first cohomology group of G with coefficients in A is given by $\overline{H^1}(G, A) = Z_G/\overline{B_G}$, where the closure of B_G is taken using the topology of pointwise convergence arising from the norm on A .

The techniques used to study these reduced cohomology groups have applications in many other fields of mathematics including group theory, operator theory, geometry and algebraic K -theory [16]. Determining whether or not the reduced first cohomology group vanishes asks whether or not we can find a sequence of elements $g \cdot f_n - f_n \in B_G$ converging pointwise to $g \cdot f - f \in Z_G$. We discuss some previously known results on the reduced cohomology groups in Chapter 5.

Next, we aim to use homomorphisms between groups and corresponding G -modules to construct maps between cohomology groups. These induced maps will be useful in using known cohomology groups to determine properties of unknown cohomology groups. However, not all homomorphisms between groups and G -modules induce a well-defined map between cohomology groups. The definition below gives us the necessary and sufficient condition to get an induced homomorphism.

Definition 2.10. Let A be a G -module and A' be a G' -module, then the group homomorphisms $\phi: G' \rightarrow G$ and $\psi: A \rightarrow A'$ are said to be compatible if $\psi(\phi(g')a) = g'\psi(a)$ for all $g' \in G'$ and $a \in A$.

These compatible homomorphisms induce group homomorphisms on the corresponding cohomology groups. Note that we can make A into a G' -module via ϕ , and the condition for ψ and ϕ to be compatible is precisely the condition that ψ is a G' -module homomorphism when A is made into a G' -module in this way.

To see that compatible maps induce group homomorphisms on the corresponding cohomology groups, first induce a homomorphism $\phi^n: (G')^n \rightarrow G^n$. This gives us a map from $C^n(G, A)$ to $C^n(G', A)$, where f maps to $f \circ \phi^n$. Then ψ induces a homomorphism from $C^n(G', A)$ to $C^n(G', A')$ where f is mapped to $\psi \circ f$. Let λ_n denote the resulting map from $C^n(G, A)$ to $C^n(G', A')$ that maps f to $\psi \circ f \circ \phi^n$.

If ψ and ϕ are compatible, then we can check that coboundaries are sent to coboundaries and cocycles to cocycles, and thus we get an induced group homomorphism on the cohomology $\lambda_n: H^n(G, A) \rightarrow H^n(G', A')$.

In particular, we will often look at the homomorphism between cohomology groups induced by the inclusion map when A is a G -submodule of A' . This is easily checked to be a compatible homomorphism with the identity on G , and the induced map is easily understood since if $g \cdot f - f \in A$ for all $g \in G$, then certainly $g \cdot f - f \in A'$ for all $g \in G$, so $g \cdot f - f$ is certainly an element of $H^1(G, A')$. The map is well-defined since if $f \in A$, then $f \in A'$. The question we will often ask in this paper, however, is whether or not this induced map is also an injection.

Finally, although seemingly somewhat off topic, we make the following equivalent definitions of amenability, which will be a key property of groups discussed throughout this paper.

Definition 2.11. A group G is said to be amenable if given $\epsilon > 0$ and finite $F \subset G$, there exists a finite subset U of G such that $|U \Delta g \cdot U|/|U| < \epsilon$ for all $g \in F$. If we can always find such a U we say G satisfies the Følner condition. (Here Δ denotes the symmetric difference of sets).

Definition 2.12. A group G is said to be amenable if given any finite subset F of G and $\epsilon > 0$, we can find some $f \in l^1(G)$ with $f \geq 0$, $\|f\|_1 = 1$, and $\|g \cdot f - f\|_1 < \epsilon$ for all $g \in F$. This is referred to as Reiter's Property.

Of particular importance is the fact that finite groups, abelian groups and solvable groups all satisfy the conditions of amenability. The key non-amenable group is $F(x_1, x_2)$, the free group on two generators, or any group containing a subgroup isomorphic to $F(x_1, x_2)$.

Some Properties of Amenability.

1. If G is amenable, then every subgroup H of G is amenable.
2. Given normal subgroup N of G , G is amenable if and only if N and G/N are amenable.
3. G is amenable if and only if every finitely generated subgroup of G is amenable.

4. If G is abelian then G is amenable.

The proofs of these properties is omitted, but [8] looks at these and other properties in more detail. We will always be viewing our group G with the discrete topology, so G will certainly be locally compact.

Chapter 3

$H^1(G, \mathbb{C}G)$

Most of our results in this paper deal with G -modules that are Banach Spaces, however, we first look at some useful results for the G -module $\mathbb{C}G$. While much work has been done on cohomology groups with coefficients in $l^p(G)$ for finitely generated groups, many of the techniques do not work for G not necessarily finitely generated. So, in order to find some useful techniques for the more general case, we look at coefficients in $\mathbb{C}G$, where much more is known about the vanishing and non-vanishing of the first cohomology groups for groups not necessarily finitely generated. We start, however, by looking at the case when G is finite. The idea behind Proposition 3.1 is that if G is finite, then $\mathbb{C}G = \mathbb{C}^G$.

Proposition 3.1. Let G be a finite group. Then $H^1(G, \mathbb{C}G) = 0$.

Proof. We first note that $\mathbb{C}G$ can be written as $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}G$, and when G is finite, we have $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}G \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}G, \mathbb{C})$. Therefore, by Shapiro's Lemma we get the following equivalences,

$$H^1(G, \mathbb{C}G) \cong H^1(G, \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}G) \cong H^1(G, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}G, \mathbb{C})) \cong H^1(1, \mathbb{C}) = 0.$$

■

To characterize the cohomology groups when G is not finite, we first take a look at defining the ends of the group, since these ideas will turn out to be closely related. To look at the ends of a group, we want to consider almost invariant subsets of the group, so we make the following definitions.

Definition 3.2. We define the equivalence relation of almost equality on the group G by saying that if $A, B \subset G$, then $A \sim_a B \iff |A \Delta B| < \infty$. This can easily be checked to be an equivalence relation.

Definition 3.3. We say that a subset A of G is almost (left) invariant if $gA \sim_a A$ for all $g \in G$. Note here that any subset of G that is almost equal to an almost invariant subset is also almost invariant.

The definition of the ends of G is well understood by considering the cohomology groups of G with coefficients in \mathbb{Z}_2G (where \mathbb{Z}_2 is the field with 2 elements), but we would like to relate this definition of ends to the cohomology groups with coefficients in $\mathbb{C}G$ as best we can. For now, we focus on \mathbb{Z}_2 and follow the process of Cohen in [5].

Definition 3.4. Let F be a field and G be a group. Then define $\overline{FG} := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}G, F)$.

Now note that $\overline{\mathbb{Z}_2G}$ can be viewed as the set of all subsets of G (or the set of all maps from G to \mathbb{Z}_2) by looking at where the elements of G are sent under a particular homomorphism. Then \mathbb{Z}_2G is a G -submodule of $\overline{\mathbb{Z}_2G}$ and is identified with all finite subsets of G . If we let $\phi G = \overline{\mathbb{Z}_2G}/\mathbb{Z}_2G$, we see that a subset of G is almost invariant if and only if its image in ϕG is invariant. So, we would like to look at the invariant subsets of ϕG , or $H^0(G, \phi G)$.

Definition 3.5. The number of ends of a group G , denoted $e(G)$, is defined to be the $\dim H^0(G, \phi G)$ (as a \mathbb{Z}_2 vector space).

To study the dimension of this \mathbb{Z}_2 -vector space, we consider the short exact sequence

$$0 \rightarrow \mathbb{Z}_2G \rightarrow \overline{\mathbb{Z}_2G} \rightarrow \phi G \rightarrow 0$$

which gives rise to the exact sequence of \mathbb{Z}_2 -vector spaces

$$0 \rightarrow H^0(G, \mathbb{Z}_2G) \rightarrow H^0(G, \overline{\mathbb{Z}_2G}) \rightarrow H^0(G, \phi G) \rightarrow H^1(G, \mathbb{Z}_2G) \rightarrow H^1(G, \overline{\mathbb{Z}_2G}).$$

Now noting that by Shapiro's Lemma, $H^1(G, \overline{\mathbb{Z}_2G}) = H^1(\{e\}, \mathbb{Z}_2) = 0$, and that $H^0(G, \mathbb{Z}_2G) = 0$ and $H^0(G, \overline{\mathbb{Z}_2G}) = \mathbb{Z}_2$, we find that $\dim H^0(G, \phi G) =$

$1 + \dim H^1(G, \mathbb{Z}_2G)$, i.e. the number of ends of a group G is $1 + \dim H^1(G, \mathbb{Z}_2G)$.

So, we can equivalently define the number of ends of a group as follows.

Definition 3.6. The number of ends of a group G is $1 + \dim H^1(G, \mathbb{Z}_2G)$.

To relate this to the cohomology groups with coefficients in $\mathbb{C}G$, note that $\overline{\mathbb{C}G}$ can be viewed as the set of maps $\{f: G \rightarrow \mathbb{C}\}$. Again, we can view $\mathbb{C}G$ as a G -submodule of $\overline{\mathbb{C}G}$, and it will correspond to the functions with compact (finite) support. Now, considering $\psi G = \overline{\mathbb{C}G}/\mathbb{C}G$, we want to use a basis for $H^0(G, \phi G)$ to define a linearly independent set of functions in $H^0(G, \psi G)$ (i.e. we take a basis of almost invariant sets, and make a linearly independent set of almost invariant functions.)

So let $\{E_i\}_{i \in I}$ be a basis for $H^0(G, \phi G)$. Let $\{\overline{E}_i\}_{i \in I}$ denote the corresponding elements of $l^\infty(G)$, defined to be 1 on the set E_i and 0 outside that set. To see the linear independence of the \overline{E}_i , assume that $\sum_{j \in J} a_j \overline{E}_j = 0$ in $H^0(G, \psi G)$, where $a_j \in \mathbb{C}$. Then we have that $g \cdot \sum_{j \in J} a_j \overline{E}_j = 0$ for all $g \in G$, and since $g \cdot \overline{E}_j = \overline{E}_j$ in $H^0(G, \psi G)$, we must have that $\bigcup_{j \in J} E_j = \emptyset$ in $H^0(G, \phi G)$, i.e. $\sum_{j \in J} b_j E_j = 0$ where $b_j = 0, 1$. Since the E_i are a basis in $H^0(G, \phi G)$, we see that all of the $b_j = 0$, and hence all of the $a_j = 0$.

In general, however, these functions will not span $\psi(G)$, at least in the case when G is locally finite. We give an example of a function which cannot be written as a sum of such elements in the results section of this paper. While we do believe $\dim_{\mathbb{Z}_2} H^1(G, \mathbb{Z}_2G) = \dim_{\mathbb{C}} H^1(G, \mathbb{C}G)$, it is not in the natural way. However, since these elements of $\psi(G)$ are linearly independent, we do see clearly that $\dim_{\mathbb{Z}_2} H^1(G, \mathbb{Z}_2G) \leq \dim_{\mathbb{C}} H^1(G, \mathbb{C}G)$, and in particular if $H^1(G, \mathbb{C}G) = 0$ then G has one end. Also, if $e(G) > 1$ then $H^1(G, \mathbb{C}G) \neq 0$. For the scope of this paper this alone will be useful.

The most general result on the cohomology groups of G with coefficients in $\mathbb{C}G$ can be found in [4], and it is proven using the theory of groups acting on graphs. While we will focus specifically on the case when G is locally finite, the following theorem gives a complete characterization of when the first cohomology group does or does not vanish.

Theorem 3.7.([4]) For any nonzero abelian group A , the following are equivalent:

- (a) $e(G) > 1$.
- (b) $H^1(G, AG) \neq 0$.
- (c) One of the following holds:
 - (i) $G = B *_C D$, where $B \neq C \neq D$ and C is finite;
 - (ii) $G = B *_C x$, where C is finite;
 - (iii) G is countably-infinite locally-finite.
- (d) $e(G)$ is 2 or ∞ .

In this theorem, $B *_C D$ refers to the free product of B and D amalgamating C , and $B *_C x$ denotes the HNN extension of A by $x: C \rightarrow A$. AG denotes the tensor product $A \otimes_{\mathbb{Z}} \mathbb{Z}G$. The theory of groups acting on graphs actually goes further to produce the following result on the number of ends of G .

Theorem 3.8. $e(G) = 2$ if and only if G has an infinite cyclic subgroup of finite index.

The case when G is locally-finite is of particular interest in this paper, and we show a direct proof of G countably-infinite locally finite implies $H^1(G, \mathbb{C}G) \neq 0$ in the last section of this paper. For now, though, we want to look carefully at the case when G is uncountable locally-finite, since in this case $H^1(G, \mathbb{C}G)$ vanishes. We look at why this happens as discussed by Holt in [3]. We start with a lemma regarding functions in \mathbb{C}^G .

Lemma 3.9. Let H and K be proper subgroups of a periodic group G , with $\langle H, K \rangle = G$. Let $f: G \rightarrow \mathbb{C}$ be constant on each coset $Hg \neq H$ and $Kg \neq K$. Then f is constant on $G - (H \cap K)$. Further, if f is also constant on K , f is constant on all of G .

Proof. Since G is locally finite, we know that given $h \in H - K$ and $k \in K - H$, $(kh^{-1})^n = 1$ and $(hk^{-1})^m = 1$ for some n and m . Therefore, we have a word of the form $hkh^{-1}hk^{-1} \dots hkh^{-1}h \in K$ and a word of the form $kh^{-1}kh^{-1} \dots kh^{-1}k \in H$. Without loss of generality assume $w_1 := hkh^{-1}hk^{-1} \dots hkh^{-1}h$ is the shortest word with one of these properties. Then $hkh^{-1}hk^{-1} \dots hkh^{-1}h \in K$, and so $hkh^{-1}hk^{-1} \dots hkh^{-1}h = k_1$

for some $k_1 \in K$. By rearranging terms, we see that

$$\begin{aligned} hk^{-1}hk^{-1}\dots hk^{-1}h &= k_1 \\ \Rightarrow hk_1^{-1}hk^{-1}\dots hk^{-1}h &= k. \end{aligned}$$

Since $h \notin K$ and f is constant on $Kg \neq K$, we see that $f(h) = f(k^{-1}h)$. Now, $k^{-1} \notin H$, so $k^{-1}h \notin H$, and so f constant on $Hg \neq H$ implies that $f(k^{-1}h) = f(hk^{-1}h)$. Note here that no terminal segment of w_1 can be in $H \cup K$, since this would contradict the fact that w_1 is the shortest word having one of the previously discussed properties. So, we use that f is constant on the cosets $Hg \neq H$ and $Kg \neq K$ to get that

$$\begin{aligned} f(h) &= f(k^{-1}h) = f(hk^{-1}h) = \dots \\ &= f(hk_1^{-1}hk^{-1}\dots hk^{-1}hk^{-1}h) = f(k). \end{aligned}$$

Therefore, f is constant on $(H \cup K) - (H \cap K)$.

Now let $g \in G - (H \cap K)$. Since $G = \langle H, K \rangle$, we can write $g = g_1 \dots g_r$ where $g_i \in H \cup K$ for all i . We can also choose r to be minimal, so that $g_r \in H \cup K$, but $g_i \dots g_r \notin H \cup K$ for any $1 < i < r$. (We can do this by replacing g_i with $g_i \dots g_r$ if $g_i \dots g_r \in H \cup K$ for some i). Note here that by assuming r is minimal, we also have that $g_r \notin H \cap K$, since this would imply that $g_1 g_2 \in H \cup K$.

So, if $g_r \in H$, $g_{r-1} \notin H$, so $f(g_r) = f(g_{r-1}g_r)$. Then, $g_{r-1}g_r \notin H \cup K$, so $f(g_{r-1}g_r) = f(g_{r-2}g_{r-1}g_r)$. Continuing in this way, we see that $f(g_r) = f(g_{r-1}g_r) = \dots = f(g_1 g_2 \dots g_r) = f(g)$, and since $g_r \in (H \cup K) - (H \cap K)$, we have that f is constant on $G - (H \cap K)$.

To see the last part of the lemma, assume further that f is constant on K . Then, we know that f is constant on $G - (H \cap K)$, and since $K - H$ is non-empty, let $g_1 \in K - H$. For any $g_2 \in H \cap K$, $f(g_1) = f(g_2)$, and thus f is constant on all of G . ■

We now move to proving the following theorem:

Theorem 3.10. Let G be a locally finite group and $f \in H^1(G, \mathbb{C}G)$. If there exists an infinite, proper subgroup H of G such that $f(H) \subseteq \mathbb{C}H$, then f is of the form $f(g) = g \cdot \alpha - \alpha$ for some $\alpha \in \mathbb{C}G$. (i.e. $f \in B_G$)

Proof. We know that if we define $\alpha \in \mathbb{C}^G$ by $\alpha(g) = -f(g)(g)$, then we have $f(g) = g \cdot \alpha - \alpha$ for all $g \in G$ by the previous proposition. Now note that for any constant $\beta \in \mathbb{C}^G$, we have $f(g) = g \cdot (\alpha + \beta) - (\alpha + \beta)$ for all $g \in G$. Therefore, if we can show that α is constant outside some finite subset of G , then we can write $f(g) = g \cdot (\alpha - \beta) - (\alpha - \beta)$ where $\alpha - \beta \in \mathbb{C}G$, which is the desired result.

Let $k \in G - H$, and note that $K := \langle k \rangle$ is a finite subgroup of G . Since $f(k^i)$ has finite support for all i , for each i we have that $f(k^i)(k^i g) = 0$ for all but finitely many $g \in G$. Also note that if $f(k^i)(k^i g) = 0$ for all i we get that

$$\alpha(k^i g) = -f(k^i g)(k^i g) = -f(k^i)(k^i g) - k^i f(g)(k^i g) = -f(g)(g) = \alpha(g),$$

and thus α is constant on the coset Kg . This implies that there is a finite subset of G such that α is not constant on Kg . Let H_1 be the finite subgroup of $\langle H, k \rangle$ generated by the $g \in \langle H, k \rangle$ such that α is not constant on Kg . Then each $g \in H_1$ can be written as a word on H and K , and define H_0 to be the subgroup of H generated by the elements of H in each of these words. Since H_1 is finite and each word has finite length, we see that H_0 is a finite subgroup of H . Then, by defining $L = \langle H_0, k \rangle$, we see that for any $g \in \langle H, k \rangle - L$, α is constant on Kg .

We claim that α is constant outside of $H \cap L$, noting that L is finitely generated and therefore finite, and thus $H \cap L$ is a finite subgroup of G . Since L is a finite subgroup of $\langle H, k \rangle$, there exists some $h \in H - L$, and by definition we have $k \notin H$, so L and H are both proper subgroups of $\langle H, k \rangle$. Therefore, if we can show that α is constant on cosets $Hg \neq H$ and $Lg \neq L$, then we can use the lemma to deduce that α is constant on $\langle H, L \rangle - (H \cap L) = \langle H, k \rangle - (H \cap L)$.

To see that α is constant on $Hg \neq H$, we simply use that $f(H) \subseteq \mathbb{C}^H$ to see that $f(h)(hg) = 0$ for all $g \in G - H$. Therefore, given $g \in \langle H, k \rangle - H$ we have

$$\alpha(hg) = -f(hg)(hg) = -f(h)(hg) - h \cdot f(g)(hg) = -f(g)(g) = \alpha(g)$$

and thus α is constant on Hg .

To show that α is constant on $Lg \neq L$, first note that if $lg \in H$ for any $l \in L$, then $Lg = Llg$, and thus we can make the assumption that if $lg \in H$ for any $l \in L$ then $g \in H$. After making this assumption, fix a coset $Lg \neq L$. If $K = L$ then α is constant on $Kg \neq K$ as cosets in $\langle H, k \rangle$ by the definition of L , so we have that α is constant on Lg .

Now assume that there exists $g \in L - K$. Note that α is constant on Lg if and only if $\alpha(lg) = \alpha(g)$ for all $l \in L$, if and only if $\alpha g^{-1}(l) = \alpha g^{-1}(1)$ for all $l \in L$, if and only if αg^{-1} is constant on L . Since we have assumed that there exists $g \in (H \cap L) - K$, and certainly $k \notin H \cap L$, both $H \cap L$ and K are proper subgroups of $L = \langle H \cap L, K \rangle$. Therefore, to show that αg^{-1} is constant on L , we consider the cosets Kl and $(H \cap L)l$ in L , and we aim to show that αg^{-1} is constant on all cosets Kl and on all cosets $(H \cap L)l \neq (H \cap L)$, and then use the second part of the lemma to conclude that αg^{-1} is constant on $\langle K, H \cap L \rangle = L$.

First, we look at the cosets Kl . Since $g \in \langle H, k \rangle - L$, we have that $lg \notin L$ for any $l \in L$. Therefore, by the definition of L we have that α is constant on Kl for all $l \in L$. This gives us that

$$\alpha g^{-1}(kl) = \alpha(klg) = \alpha(lg) = \alpha g^{-1}(l),$$

for all $k \in K$ and $l \in L$, and thus αg^{-1} is constant on Kl for all $l \in L$.

Now we consider the cosets $(H \cap L)l \neq (H \cap L)$. To show that αg^{-1} is constant on these cosets, we use the assumption that if $lg \in H$ for any $l \in L$, then $g \in H$.

If $lg \in H$ for some $l \in L$, then by our assumption we have that $g \in H$, which implies that $l \in H$. Therefore, $(H \cap L)l = (H \cap L)$. If $lg \notin H$ for any $l \in L$, then α is constant on Hlg for all $l \in L$, and hence on $(H \cap L)lg$ for all $l \in L$. Therefore, for any $x \in H \cap L$ and $l \in L$, we have that

$$\alpha g^{-1}(xl) = \alpha(xlg) = \alpha(lg) = \alpha g^{-1}(l),$$

which implies that αg^{-1} is constant on $(H \cap L)l$.

So, αg^{-1} is constant on Kl for all $l \in L$ and on $(H \cap L)l \neq H \cap L$, and we can now apply the second part of the lemma to conclude that αg^{-1} is constant on L , or, equivalently α is constant on Lg . This gives us that α is constant on Lg for all $g \in \langle H, k \rangle - L$ and Hg for all $g \in \langle H, k \rangle - H$, and since H and L are both proper subgroups of $\langle H, L \rangle$, we apply the first part of the lemma to deduce that α is constant on $\langle H, L \rangle - (H \cap L)$.

Finally, let $k' \in G - H$ such that $k \neq k'$. By the same proof we see that α is constant on $\langle H, L' \rangle - (H \cap L')$, so $\alpha(h) = \alpha(k')$ for all but finitely many $h \in H$. We also have that $\alpha(h) = \alpha(k)$ for all but finitely many $h \in H$. Therefore, since

H is countable, there exists $h \in H$ such that $\alpha(k') = \alpha(h) = \alpha(k)$, and thus α is constant on $G - (H \cap L)$, and this concludes the proof. ■

Finally, we arrive at the desired result:

Theorem 3.11. Let G be an uncountable locally-finite group. Then $H^1(G, \mathbb{C}G) = 0$.

Proof. Let $f \in H^1(G, \mathbb{C}G)$ and let H_1 be any countable subgroup of G . Then the support of $f(h)$ is finite for all $h \in H_1$, so we can find a countable subgroup H_2 such that $H_1 \subseteq H_2$ and $f(H_1) \subseteq \mathbb{C}H_2$. By induction, we can create a chain of countable subgroups of G such that $f(H_i) \subseteq \mathbb{C}H_{i+1}$ for all i and $H_1 \subseteq H_2 \subseteq H_3 \dots$. Letting $H = \bigcup_{i=1}^{\infty} H_i$ we see that H is a countable subgroup, and thus a proper subset of G , with $f(H) \subseteq \mathbb{C}H$. Therefore, $f \in B_G$ by the previous theorem which implies that $H^1(G, \mathbb{C}G) = 0$. ■

Corollary 3.12. Uncountable locally finite groups have one end.

Proof. If G is an uncountable locally-finite group, $H^1(G, \mathbb{C}G) = 0$ which implies that $e(G) = 1$. ■

The fact that the cardinality of the group affects the vanishing or non-vanishing of the first cohomology group is what we are most interested in exploring in this paper. When considering G locally finite and coefficients in $\mathbb{C}G$, we see that $H^1(G, \mathbb{C}G) \neq 0$ when G is countable, but $H^1(G, \mathbb{C}G) = 0$ when G is uncountable. Could something similar happen when we consider coefficients in some other G -modules?

Chapter 4

$H^1(G, l^p(G))$

We now turn to the l^p -cohomology of a group G . First note that if G is a finite group, then $l^p(G) = \mathbb{C}G$, and so we again have that $H^1(G, l^p(G)) = 0$. However, if we let G be an infinite group, $l^p(G)$ is the closure of $\mathbb{C}G$ in the l^p -norm, and much changes with regards to the cohomology. We start with trying to determine when the group $H^1(G, l^p(G))$ is even a Hausdorff space (i.e. when is B_G closed in Z_G). The following lemma will give some insight on the issue.

Definition 4.1. For this section, let $l^p(G)^G$ denote the space of functions $\{f: G \rightarrow l^p(G)\}$.

Theorem 4.2. Let G be an infinite group. Then there is a continuous map $f: l^p(G) \rightarrow B_G$ given by $f(e) = g \cdot e - e$, and this map has a continuous inverse if and only if G is non-amenable.

Proof. To say that the map is continuous (where the topology on $l^p(G)^G$ is pointwise convergence) is the same as saying any net e_λ such that $e_\lambda \rightarrow 0$ implies that $g \cdot e_\lambda - e_\lambda \rightarrow 0$. Since $\|e_\lambda\|_p \rightarrow 0$, we see that for any $g \in G$, $\|g \cdot e_\lambda - e_\lambda\|_p \leq \|g \cdot e_\lambda\|_p + \|e_\lambda\|_p \rightarrow 0$, and thus the map is continuous.

Now assume that G is amenable. Let $\{F_\lambda\}_{\lambda \in L}$, be the compact (finite) sets of G , where L is a directed set by inclusion. Let $\epsilon_\lambda = \frac{1}{|F_\lambda|}$ for $\lambda \in L$, and note that this is a net contained in \mathbb{R}^+ with $\lim_{\lambda \in L} \epsilon_\lambda = 0$. Then, by the Følner condition, for each $\lambda \in L$, find U_λ such that $|U_\lambda \Delta g \cdot U_\lambda| / |U_\lambda| < \epsilon_\lambda$ for all $g \in F_\lambda$. Then we can

let

$$e_\lambda = \frac{\sum_{u \in U_\lambda} u}{\|\sum_{u \in U_\lambda} u\|_p},$$

noting that $\|e_\lambda\|_p = 1$. Then, given any $g \in G$, we have that $g \in F_{\lambda'}$ for some λ' , and we get that for $\lambda > \lambda'$ (where $>$ is the operation in L), we have

$$\begin{aligned} \|g \cdot e_\lambda - e_\lambda\|_p &= \frac{\|\sum_{u \in U_\lambda \Delta g \cdot U_\lambda} u\|_p}{\|\sum_{u \in U_\lambda} u\|_p} \\ &= \sqrt[p]{\frac{|U_\lambda \Delta g \cdot U_\lambda|}{|U_\lambda|}} < \sqrt[p]{\epsilon_\lambda} \rightarrow 0. \end{aligned}$$

Therefore, we have a net converging in B_G , but not converging in $l^p(G)$, and thus the inverse map is not continuous.

Now assume that the inverse map is not continuous and, for contradiction, that G is non-amenable. If we fix a finite subset F of G , then we must have a countable subgroup $H = \langle h_1, h_2, h_3, \dots \rangle$ of G containing F with H non-amenable (since if every finitely generated subgroup of G is amenable, then G is amenable). Since the inverse map is not continuous, there exists a net $\{e_\lambda\}_{\lambda \in L} \subset l^p(G)$ with $\lim_{\lambda \in L} g \cdot e_\lambda - e_\lambda = 0$ for all $g \in G$, but $\lim_{\lambda \in L} e_\lambda \neq 0$. Fix $\epsilon > 0$. We aim to show that there exists $b \in l^1(G)$ with $\|b\|_1 = 1$ and $\|g \cdot b - b\|_1 < \epsilon$ for all $g \in F$ (i.e. that H has the Reiter Property).

Since F is finite $F \subset \{h_1, h_2, \dots, h_n\}$ for some n . Since $e_\lambda \not\rightarrow 0$, there exists $\delta > 0$ such that for every $\lambda \in L$, there exists $x > \lambda$ such that $\|e_x\|_p > \delta$. Let $X = \{x \in L : \|e_x\|_p > \delta\} \subset L$, so that $\{e_x\}_{x \in X}$ is a subnet of e_λ . Since

$$\begin{aligned} \|g \cdot e_\lambda - e_\lambda\|_p \rightarrow 0 &\Rightarrow \|g \cdot e_x - e_x\|_p \rightarrow 0 \\ &\Rightarrow \|g \cdot \frac{e_x}{\|e_x\|_p} - \frac{e_x}{\|e_x\|_p}\|_p \rightarrow 0 \end{aligned}$$

for all $g \in G$. So, we can find x_1 such that for $x \geq x_1$, $\|h_1 \cdot \frac{e_x}{\|e_x\|_p} - \frac{e_x}{\|e_x\|_p}\|_p < \frac{\epsilon}{2^{p-1}p}$. Then, we can find $x_2 > x_1$ such that for $x \geq x_2$, we have $\|h_2 \cdot \frac{e_x}{\|e_x\|_p} - \frac{e_x}{\|e_x\|_p}\|_p < \frac{\epsilon}{2^{p-1}p}$. Continuing in this way we arrive at x_n such that $\|h_i \cdot \frac{e_{x_n}}{\|e_{x_n}\|_p} - \frac{e_{x_n}}{\|e_{x_n}\|_p}\|_p < \frac{\epsilon}{2^{p-1}p}$ for all h_i , $1 \leq i \leq n$. Letting $f = \frac{e_{x_n}}{\|e_{x_n}\|_p}$, we see that $\|f\|_p = 1$, $\|g \cdot f - f\|_p < \frac{\epsilon}{2^{p-1}p}$ for all $g \in F$.

To finish, let $b = f^p$ and note that $b \in l^1(G)$ with $\|b\|_1 = 1$, $b \geq 0$, and given any

$g \in F$,

$$\begin{aligned}
\|g \cdot b - b\|_1 &= \sum_{g \in G} |f(h^{-1}g)^p - f(g)^p| \\
&\leq \sum_{g \in G} p|f(h^{-1}g) - f(g)|^{p-1} |f(h^{-1}g) - f(g)| \quad (\text{Mean Value Theorem}) \\
&\leq p \left(\sum_{g \in G} |f(h^{-1}g) + f(g)|^p \right)^{\frac{p-1}{p}} \|f(h^{-1}g) - f(g)\|_p \quad (\text{H\"older's Inequality}) \\
&= p (\|f(h^{-1}g) + f(g)\|_p)^{p-1} \|f(h^{-1}g) - f(g)\|_p \\
&\leq p (\|f(h^{-1}g)\|_p + \|f(g)\|_p)^{p-1} \|f(h^{-1}g) - f(g)\|_p \\
&= p 2^{p-1} \|f(h^{-1}g) - f(g)\|_p < \epsilon
\end{aligned}$$

since $\|f\|_p = 1$. Therefore, H has the Reiter property and is amenable, which is a contradiction. \blacksquare

This theorem completely determines when a group will be Hausdorff in the case when G is countable. When G is countable, the topology of pointwise convergence on the space $l^p(G)^G$ (the set of all $f: G \rightarrow l^p(G)$) is given by the countable family of seminorms, $\|f(g_i)\|_p$, and thus $l^p(G)^G$ is a Frechet space. So, since a closed subspace of a Frechet space is a Frechet space, and any bijective, continuous map from a Frechet space to a Frechet space has a continuous inverse, we know that if G is amenable then $H^1(G, l^p(G))$ cannot be Hausdorff. When looking at the case when G is non-amenable, we actually don't need G to be countable, as seen in the following Theorem.

Theorem 4.3. Let G be a nonamenable group. Then $H^1(G, l^p(G))$ is Hausdorff.

Proof. The goal is to prove that B_G is a closed subspace of Z_G , and thus the quotient is Hausdorff. If G is nonamenable, then the map $l^p(G) \rightarrow B_G$ has a continuous inverse. When this is the case we note that the topology on $l^p(G)$, which is a Banach space, makes B_G into a complete metric space. So we aim to show that given any $x \in \overline{B_G}$, there exists a Cauchy net in B_G converging to x , and thus $x \in B_G$.

To see this first note that $l^p(G)$ is a locally convex space, since its topology is given by the seminorms $\{p_g\}_{g \in G}$, where for $e \in l^p(G)^G$, $p_g(e) = \|e(g)\|_p$. So if we let $\phi = \{F_\alpha\}_{\alpha \in A}$ be the set of all finite subsets of G directed by inclusion, we can get

a directed family of seminorms, $\{p_{F_\alpha}\}_{\alpha \in A}$ where $p_{F_\alpha} = \sum_{g \in F_\alpha} p_g$, and this directed family defines the same topology (namely, the topology of pointwise convergence) on G .

Since $x \in \overline{B_G}$, any neighborhood of x in $l^p(G)^G$ will intersect with B_G nontrivially. Therefore, for any $\alpha \in A$, we can find x_α such that $p_{F_\alpha}(x_\alpha - x) < \epsilon_\alpha$, where we let $\epsilon_\alpha = \frac{1}{|F_\alpha|}$. Since the entourages on the uniform space $l^2(G)^G$ are given by the directed family of seminorms from the previous paragraph, in order to show that $\{x_\alpha\}$ is Cauchy, given any seminorm p_{F_β} and $\epsilon > 0$, we can find γ such that for $\lambda, \mu > \gamma$, $p_{F_\beta}(f_\lambda - f_\mu) < \epsilon$. Let γ be such that $F_\beta \subset F_\gamma$ and $\epsilon_\gamma < \epsilon/2$. Then we have

$$\begin{aligned} p_{F_\beta}(f_\lambda - f_\mu) &\leq p_{F_\gamma}(f_\lambda - f_\mu) \\ &\leq p_{F_\gamma}(f_\mu - f) + p_{F_\gamma}(f_\lambda - f) \\ &\leq p_{F_\mu}(f_\mu - f) + p_{F_\lambda}(f_\lambda - f) \\ &< \epsilon_\mu + \epsilon_\lambda \leq 2\epsilon_\gamma < \epsilon. \end{aligned}$$

Therefore, x_α is a Cauchy net, and thus, using the fact that B_G is a complete metric space, it converges to a unique element $x \in B_G$. This implies that B_G is closed in $l^p(G)^G$, and thus in Z_G (since Z_G is a closed subspace of $l^p(G)^G$), and we must have that $H^1(G, l^p(G))$ is Hausdorff. ■

In the case when G is countable we can see this using sequences instead of nets, since in this case $l^p(G)^G$ is a metric space, but the result holds even in the more general case. This gives the following corollary.

Corollary 4.4. Let G be a countable group. Then $H^1(G, l^p(G))$ is Hausdorff if and only if G is non-amenable.

Proof. This follows directly from the Theorem and the discussion immediately preceding it. ■

Note that this corollary already gives us an instance of the cohomology of a group with coefficients in $\mathbb{C}G$ differing from the cohomology with coefficients in $l^p(G)$. Namely, if G is a countable, amenable, group which is not a free product with amalgamation, HNN extension, or locally finite, then $H^1(G, \mathbb{C}G) = 0$ but $H^1(G, l^p(G))$ is not even Hausdorff. To see an example of such a group, consider $G = \mathbb{Z} \times \mathbb{Z}$.

This is abelian and therefore amenable, and also is torsion-free. Any non-trivial, torsion-free free product contains $F(x, y)$ (the free group on two generators) as a subgroup, and is therefore non-amenable, so G is certainly not a free product, and the torsion free version of an HNN extension is just the infinite cyclic group, so G is not an HNN extension. Finally, G is certainly not locally finite since it has $\langle(1, 0)\rangle = \mathbb{Z}$ which is infinite, therefore, $H^1(G, \mathbb{C}G) = 0$ but $H^1(G, l^p(G)) \neq 0$.

The following proposition helps us extract further information about $H^1(G, l^p(G))$ from our knowledge of $H^1(G, \mathbb{C}G)$ in the finitely generated case.

Proposition 4.5. Let G be a finitely generated group. Then the G -module embedding of $\mathbb{C}G \hookrightarrow l^p(G)$ induces an embedding of groups $H^1(G, \mathbb{C}G) \hookrightarrow H^1(G, l^p(G))$.

Letting ψ be the identity and ϕ be the natural embedding of $\mathbb{C}G$ into $l^p(G)$, we see that the two homomorphisms are compatible (as in the definition given in the first section of this paper), and therefore we have a group homomorphism $\theta: H^1(G, \mathbb{C}G) \rightarrow H^1(G, l^p(G))$. To see that θ is injective, let $b \in \ker \theta$. Then there is an $\alpha \in l^p(G)$ such that $b(g) = g \cdot \alpha - \alpha$. We claim that $\alpha \in \mathbb{C}G$, i.e. α has finite support.

To see this, let $\alpha = \sum_{g \in G} a_g g$, and note that since $b \in H^1(G, \mathbb{C}G)$, $b(g) \in \mathbb{C}G$ for all $g \in G$. Therefore, if we define $\phi(h) = \{g \in G : a_{h^{-1}g} - a_g \neq 0\}$, we see that $|\phi(h)| < \infty$ for all $h \in G$ (this is precisely the condition that $b(h) \in \mathbb{C}G$ for all $h \in G$). Let S be a generating set for G , and assume it is closed under inverses. Let $F(G) = \bigcup_{s \in S} \phi(s)$, and note that this is still a finite set. Let X be the Cayley graph of G , with vertex set G and edge set $\{(g, sg), s \in S\}$. Since $F(G)$ is finite, we know that $X - F(G)$ has finitely many connected components.

Let q and r be in an infinite component of $X - F(G)$. Then $r = s_1 s_2 \dots s_n q$ for some $s_i \in S$. Since this path is completely outside of $F(G)$, we know that $a_{s_{i-1} s_i \dots s_n q} = a_{s_i \dots s_n q}$ for all i . Therefore, $a_q = a_{s_1 s_2 \dots s_n q} = a_r$, and this holds for any r and q in an infinite component of $X - F(G)$. This implies that α is constant on all infinite components of $X - F(G)$, and since $\alpha \in l^p(G)$, it must be equivalently zero on all infinite components of $X - F(G)$. But since there are only finitely many finite components of $X - F(G)$, we have that $\{g \in G : a_g \neq 0\}$ is a finite set. Therefore, $\alpha \in \mathbb{C}G$, which tells us that $b(g) = g \cdot \alpha - \alpha$ where $\alpha \in \mathbb{C}G$, and thus b is a 1-cocycle and we have that θ is injective. ■

This injection certainly gives us more information on when $H^1(G, l^p(G)) \neq 0$, and we can also extract some information on the number of ends of a group given its l^p -cohomology, as seen in the next two corollaries.

Corollary 4.6. Let G be a finitely generated group and suppose G satisfies one of the following:

1. $G = B *_C D$ where $B \neq C \neq D$ and C is finite;
2. $G = B *_C x$ where C is finite;

then $H^1(G, l^p(G)) \neq 0$.

Corollary 4.7. Let G be a finitely generated group. Then if $H^1(G, l^p(G)) = 0$, G is non-amenable and $e(G) = 1$.

Proof. We know that if G satisfies (1) or (2), $H^1(G, \mathbb{C}G) \neq 0$ (from the last section), so Corollary 1 follows from the embedding of $H^1(G, \mathbb{C}G)$ into $H^1(G, l^p(G))$.

Corollary 4 also follows from the embedding, since $H^1(G, l^p(G)) = 0 \Rightarrow H^1(G, \mathbb{C}G) = 0 \Rightarrow e(G) = 1$. ■

Chapter 5

$l^\infty(G)$ and $c_0(G)$

Many of the results in this section are due to Reznikov [13], and of particular importance in these results is the use of a length function to define elements of the cohomology group.

Definition 5.1. Given a group G , a length function is a function $\ell : G \rightarrow \mathbb{R}^+$ such that the following hold:

1. $\ell(e) = 0$
2. $\ell(g) = \ell(g^{-1})$
3. $\ell(gh) \leq \ell(g) + \ell(h)$.

Using these three properties of a length function, we can see that any length function on the group will yield an element of $H^1(G, l^\infty(G))$.

$$\ell(g^{-1}x) \leq \ell(g^{-1}) + \ell(x) \implies \ell(g^{-1}x) - \ell(x) \leq \ell(g^{-1})$$

Similarly,

$$\begin{aligned} \ell(x) = \ell(gg^{-1}x) &\leq \ell(g) + \ell(g^{-1}x) = \ell(g^{-1}) + \ell(g^{-1}x) \\ &\implies \ell(g^{-1}x) - \ell(x) \geq -\ell(g^{-1}) \end{aligned}$$

So, we have that $|\ell(g^{-1}x) - \ell(x)| \leq \ell(g^{-1})$ which tells us that $\|g \cdot \ell - \ell\|_\infty < \infty$, and thus $g \cdot \ell - \ell$ is an element of $H^1(G, l^\infty(G))$ for any length function ℓ . What is not always so clear is whether or not the length function is a non-zero element of the cohomology group, which is determined by whether or not the length function

is bounded.

The most obvious choice of a length function is the function $\ell : G \rightarrow \mathbb{Z}^+$ which assigns to g the length of the shortest word representing g on some generating set. If G is infinite and finitely generated, this length function yields a non-zero element of the first cohomology group, as shown in the following proposition.

Proposition 5.2. Let G be finitely generated. Then $H^1(G, l^\infty(G)) \neq 0$.

Proof. Given finite generating set S , let $\ell : G \rightarrow \mathbb{Z}^+$ assign to g the length of the shortest word representing g on S . This can easily be seen to be a length function, and by the preceding discussion, we know that $g \cdot \ell - \ell \in l^\infty(G)$ for all $g \in G$. Then note that since G is infinite and S is finite, there must exist words of arbitrary length, therefore $\ell \notin l^\infty(G)$, and $g \cdot \ell - \ell$ is a non-zero element of $H^1(G, l^\infty(G))$. ■

This becomes less clear if G is not finitely generated, but it remains true for G countable by altering the length function in the following way:

Given an infinite, countable group G and a countable generating set S , define a length function $\ell : G \rightarrow \mathbb{Z}^+$ as follows: Let $S = \{x_1, x_2, \dots\}$. Then let $d_n(g) =$ length of the shortest word on x_1, \dots, x_n representing g , and ∞ if no such word exists. Then we let $X_n = \{g \in G \mid d_n(g) \leq n\}$ and note that by throwing out duplicates we can get a strictly ascending chain $X_1 \subset X_2 \subset \dots$ such that $G = \bigcup X_i$. Also note that the chain is necessarily infinite. Then let $\ell(g) = \min\{n \mid g \in X_n\}$. This satisfies all properties of a length function, and since the chain of X_n is infinite, this length function will be unbounded.

Proposition 5.3. Let G be a countable group. Then $H^1(G, l^\infty(G)) \neq 0$.

Proof. If ℓ is the length function defined above, then $g \cdot \ell - \ell$ is a non-zero element of $H^1(G, l^\infty(G))$. ■

In Chapter 7 we prove this same result using another technique, as well as exploring how far we can extend this proposition. We also use the length function defined above to show that the map $\phi : H^1(G, c_0(G)) \rightarrow H^1(G, l^\infty(G))$ is not injective for

countable groups G . And finally, the following proposition is proved in Chapter 7, but is easily seen using length functions as well.

Proposition 5.4. Let G be a countable group. Then $H^1(G, c_0(G)) \neq 0$.

Proof. Define $f : G \rightarrow \mathbb{C}$ by $f(x) = \ell(x)^{1/2}$, where ℓ is the special unbounded length function defined above. We see that

$$\begin{aligned} |(g \cdot f - f)(x)| &= |\ell(g^{-1}x)^{1/2} - \ell(x)^{1/2}| \\ &\leq \frac{1}{2(\min\{\ell(g^{-1}x), \ell(x)\})^{1/2}} |\ell(g^{-1}x) - \ell(x)| \\ &\leq \frac{1}{2(\min\{\ell(x) - \ell(g^{-1}), \ell(x)\})^{1/2}} \ell(g^{-1}) \end{aligned}$$

and since there are only finitely many words $x \in G$ of any given length, $g \cdot f - f \in c_0(G)$ for all $g \in G$ and is a non-zero element of $H^1(G, c_0(G))$. ■

Chapter 6

$\overline{H^1}(G, A)$

Before moving on to the results section of the paper, we want to discuss a few interesting results in reduced cohomology. Recall that the reduced cohomology is defined to be $Z_G/\overline{B_G}$, and thus the reduced cohomology is no different than regular cohomology when the space $H^1(G, A)$ is Hausdorff. Since this is not the case in general, we now want to look at whether or not any non-zero elements of $H^1(G, A)$ exist that are not pointwise limits of elements of B_G .

Again, much work has been done in the area of reduced cohomology for groups that are finitely generated, and a useful result is presented in the next chapter of this paper, namely Lemma 7.30, that enables us to extend many of the well-known vanishing results in reduced cohomology to groups that are not necessarily finitely generated. Gournay discusses some of these results in [21], but for this chapter of the paper we just want to present a couple of results that will be used in Chapter 7. We start with a result due to Holopainen and Soardi [20] which gives us the ability to only consider bounded functions in reduced cohomology when we deal with coefficients in $l^p(G)$ or $c_0(G)$. The statement of the theorem and the proof only consider coefficients in $l^p(G)$, but the proof works for $c_0(G)$ without much adjustment.

Lemma 6.1. $\overline{H^1}(G, l^p(G)) = 0$ if and only if for every bounded function $f : G \rightarrow \mathbb{C}$ such that $g \cdot f - f \in l^p(G)$ for all $g \in G$, $g \cdot f - f$ is 0 in $\overline{H^1}(G, l^p(G))$.

Proof. One implication is trivial, for the other, assume for contradiction that all bounded function $f : G \rightarrow \mathbb{C}$ such that $g \cdot f - f \in l^p(G)$ for all $g \in G$ are trivial in $\overline{H^1}(G, l^p(G))$, but that $\overline{H^1}(G, l^p(G)) \neq 0$. Let $g \cdot f - f \neq 0$ in $\overline{H^1}(G, l^p(G))$, and

define

$$f_t(x) = \begin{cases} f(x) & |f(x)| < t \\ t \frac{f(x)}{|f(x)|} & |f(x)| \geq t. \end{cases}$$

Fix $g \in G$ and $0 < \epsilon < 1$, and let $S \subset G$ be such that $\|(g \cdot f - f)|_{G \setminus S}\|_p < \epsilon$ (which can be done since $g \cdot f - f \in l^p(G)$). Define $S_t \subset G$ to be the set where $|f(x)| < t$, noting that $S_t \rightarrow G$ as $t \rightarrow \infty$. Then choose $t' > 1$ such that $S \subset S_{t'}$. Then we see that

$$\begin{aligned} & \|(g \cdot f - f)(x) - (g \cdot f_{t'} - f_{t'})(x)\|_p \\ &= \|((g \cdot f - f)(x) - (g \cdot f_{t'} - f_{t'})(x))|_{G \setminus S_{t'}} + ((g \cdot f - f)(x) - (g \cdot f_{t'} - f_{t'})(x))|_{S_{t'}}\|_p \\ &\leq \|(g \cdot f - f)(x)|_{G \setminus S}\| < \epsilon \end{aligned}$$

by noting that $f(x) = f_{t'}(x)$ for $x \in S_{t'}$ and $(g \cdot f_{t'} - f_{t'})(x) = 0$ for $x \in G \setminus S_{t'}$. The latter is true since if there exists an x such that this does not hold, then $|(g \cdot f - f)(x)| \geq |(g \cdot f_{t'} - f_{t'})(x)| = 2t' > \epsilon$, a contradiction since $S \subset S_{t'}$. Therefore $g \cdot f_{t'} - f_{t'}$ converges to $g \cdot f - f$ for all $g \in G$, and $g \cdot f - f$ is trivial in $\overline{H^1}(G, l^p(G))$, which is the contradiction. \blacksquare

Note again that the proof works when we consider coefficients in $c_0(G)$, and this is actually the case that we use the lemma for in the next chapter. The last result we wish to present shows that not only do the cohomology groups not vanish for any finitely generated group with coefficients in $l^\infty(G)$, but the reduced cohomology groups vanish as well. This is a fact that we will extend slightly in the next chapter to include some uncountable groups, but the framework of the proof will remain virtually unchanged.

Theorem. Let G be a finitely generated group. Then $\overline{H^1}(G, l^\infty(G)) \neq 0$.

Proof. Let S be the generating set for G . Consider the non-zero element of $H^1(G, l^\infty(G))$ given by $g \cdot f - f$ where $f(g) = \ell(g)$ where $\ell : G \rightarrow \mathbb{N}$ is the traditional length function. Assume for contradiction that there exists a sequence f_n such that $f_n \in l^\infty(G)$ and $\|g \cdot f_n - f_n - (g \cdot f - f)\|_\infty \rightarrow 0$ for all $g \in G$. Then we can find N large enough so that $\|g_i \cdot f_N - f_N - (g_i \cdot f - f)\|_\infty < 1/2$ for all $g_i \in S$. Let $\|f_N\|_\infty = M$ and $f_N(1) = M'$.

Since G is infinite, finitely generated, there exists a word on S of length $L > 2(M - M')$, call it $x = g_{i_1} \dots g_{i_L}$. Since $\|g_{i_j}^{-1} \cdot f - f\|_\infty \geq |\ell(g_{i_j} \dots g_{i_L}) - \ell(g_{i_{j+1}} \dots g_{i_L})| = 1$

for all $1 \leq j \leq L-1$, we must have that $f_N(g_{i_j} \dots g_{i_L}) - f_N(g_{i_{j+1}} \dots g_{i_L}) > 1/2$ for all $1 \leq j \leq L-1$. Therefore, $f_N(g_{i_1} \dots g_{i_L}) > L/2 + M' = M$, which is a contradiction. Therefore, $\overline{H^1}(G, l^\infty(G)) \neq 0$. ■

Chapter 7

Results

7.1 Vanishing of $H^1(G, A)$ for Uncountable G

During the first chapters we have seen several interesting results for finitely generated groups, and we would like to explore some of these without the condition that G is finitely generated. The first proof that we will look at extends the result by Bourdon, Martin, and Valette in [2], and their proof will not need to be altered much. However, the result gives us a first glance into the l^p -cohomology of an uncountable group.

Proposition 6. Let G be a group with an infinite, countable, normal subgroup N . If $H^1(N, l^p(N)) = 0$, then $H^1(G, l^p(G)) = 0$.

Proof. To see the proof of the proposition, consider the following exact sequence from group cohomology

$$0 \rightarrow H^1(G/N, A^N) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(N, A|_N)^{G/N} \quad (7.1)$$

where A is any G -module, and $H^1(N, A|_N)^{G/N}$ denotes the elements of $H^1(N, A|_N)$ fixed by G/N action. Applying this exact sequence to $A = l^p(G)$, and noting that since N is infinite, $l^p(G)^N = 0$, we see that the restriction map is injective. So we aim to show that $H^1(N, l^p(G)|_N) = 0$, which will then give us that $H^1(G, l^p(G)) = 0$.

Claim: Let G be a group with an infinite, countable, normal subgroup N . Then if $H^1(N, l^p(N)) = 0$, we must have $H^1(N, l^p(G)|_N) = 0$.

Proof of Claim. Let $l^p(G) = \bigoplus_{t \in T} l^p(tN)$ where T is a subset of G ranging over all cosets of N in G . Then let $b \in Z^1(N, l^p(G)|_N)$. Denumerate N as $N = \{n_i\}$, and we see that $b(n_i) \in l^p(G)|_N$, and thus has countable support. Therefore, for each i , there exists a countable subset of T where $b(n_i)$ is non-zero, and since N is countable, we have $\{t_j\}_1^\infty \subset T$ such that $(b(n_i))(x) = 0$ for all $x \notin t_j N$ for some j and all i .

Therefore, we can write $b(n) = (t_1 b_1(n), t_2 b_2(n), \dots)$ where $b_i(n) \in l^p(N)$, and thus there exists $f_i \in l^p(N)$ such that $b_i(n) = n f_i - f_i$. Then let $F_M = (t_1 f_1, t_2 f_2, \dots, t_M f_M, 0, 0, \dots) \in l^p(G)|_N$, and define $B_M(n) = n F_M - F_M$. We see that this converges pointwise to b and $B_M \in B^1(N, l^p(G)|_N)$ for all M . This gives us that $H^1(N, l^p(G)|_N) = \overline{B^1(N, l^p(G)|_N)}$. But since we already have N non-amenable, which implies that $H^1(N, l^p(G)|_N)$ is Hausdorff, $B^1(N, l^p(G)|_N)$ is closed and thus $H^1(N, l^p(G)|_N) = 0$. \blacksquare

While this result does give us the vanishing of the l^p -cohomology for some uncountable groups, since we need a countable normal subgroup with vanishing cohomology, that normal subgroup must be non-amenable, and thus the group G must be non-amenable. So, this result does not directly produce an obvious example of an uncountable group with cohomology differing from its finitely generated counterpart.

To produce such an example, we wish to highlight Holt's result in [3] regarding the vanishing of $H^1(G, \mathbb{C}G)$ for G uncountable locally finite. The result is interesting because $H^1(G, \mathbb{C}G) \neq 0$ for G countable locally finite. When we instead consider the G -module $l^p(G)$, we have seen that $H^1(G, l^p(G)) \neq 0$ for G any countably infinite, amenable group, in fact, the space $H^1(G, l^p(G))$ is not even Hausdorff. So, does this mean that $H^1(G, l^p(G)) \neq 0$ for uncountable amenable groups as well, or could letting G be uncountable change the first cohomology as it did in the case where the G -module was $\mathbb{C}G$? Crucial to Holt's argument was the ability to find a countable subgroup H of G such that $h \cdot f - f \subseteq \mathbb{C}^H$ for all $h \in H$ where $g \cdot f - f \in H^1(G, \mathbb{C}G)$. We can still follow Holt's process when we consider $l^p(G)$, and the following slight generalization of the lemma will be useful in a later theorem when we use a cardinality argument. For Holt's argument he uses \mathfrak{c} is any uncountable ordinal, $\mathfrak{d} = \aleph_0$ and $A = \mathbb{C}G$.

Lemma 7.1. Let G be a group of cardinality \mathfrak{c} . Then let \mathfrak{d} be a cardinality such that $\aleph_0 \leq \mathfrak{d} < \mathfrak{c}$. Let A be a G -submodule of \mathbb{C}^G such that f has countable support for any $f \in A$. Then given $g \cdot f - f \in H^1(G, A)$ we can find a subgroup of cardinality \mathfrak{d} such that $h \cdot f - f \subseteq \mathbb{C}^H$ for all $h \in H$.

Proof. Let H_1 be a subgroup of cardinality \mathfrak{d} . Then $h \cdot f - f$ has countable support for all $h \in H_1$, so we can generate a subgroup H_2 of cardinality \mathfrak{d} from H_1 and the supports of $h \cdot f - f$ for all $h \in H_1$. Then H_2 has the property that $h \cdot f - f \subseteq \mathbb{C}^{H_2}$ for all $h \in H_1$. Continuing by induction, we arrive at a chain of subgroups $H_1 \subseteq H_2 \subseteq H_3 \subseteq \dots$ such that $h \cdot f - f \subseteq \mathbb{C}^{H_i}$ for all $h \in H_{i-1}$. Then, define $H = \bigcup H_i$, and we see that $|H| = \mathfrak{d}$ and $h \cdot f - f \subseteq \mathbb{C}^H$ for all $h \in H$. ■

Given $f : G \rightarrow \mathbb{C}$ such that $g \cdot f - f \in H^1(G, A)$, the important property of this subgroup H is that f is constant on right cosets of H in G , a fact that will be shown in more detail in the following theorem. When the group is abelian, or at least has an infinite center, these cosets overlap enough to show that f must have countable support. So, before we follow Holt's argument for locally finite groups, we will take a quick look at groups with infinite center.

Recall that the center of G , denoted $Z(G)$, is the subgroup $Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\}$.

Theorem 7.2. Let G be a group such that $|Z(G)| = \infty$ and let A be a G -submodule of $c_0(G)$. Then given $g \cdot f - f \in H^1(G, A)$, $g \cdot f - f = g \cdot f' - f'$ where f' has countable support.

Proof. First, we need to follow the argument in Lemma 7.1 to find a countable subgroup H such that $h \cdot f - f \subseteq \mathbb{C}^H$ for all $h \in H$ and $|H \cap Z(G)| = \infty$. So, let H_1 be a countable subgroup of G such that $|H_1 \cap Z(G)| = \infty$. Since $f(g)$ has countable support for all g , we can find a countable subgroup $H_2 \supset H_1$ such that $f(H_1) \subset l^2(H_2)$. Continuing inductively, we arrive at a chain of countable subgroups $H_1 \subset H_2 \subset H_3 \subset \dots$. Let $H = \bigcup_1^\infty H_i$. Then we have $h \cdot f - f \subseteq \mathbb{C}^H$ for all $h \in H$ where H is countable and $|H \cap Z(G)| = \infty$.

Now, since $(h \cdot f - f)(g) = 0$ for any $h \in H$ and $g \in G \setminus H$, we see that $f(h^{-1}g) = f(g)$ for all $h \in H$ and $g \in G \setminus H$. Therefore, f is constant on cosets $Hg \neq H$.

Now, let $f(Hg_1) = c_1$ and $f(Hg_2) = c_2$ where $Hg_1 \neq Hg_2$ and $g_1, g_2 \in G \setminus H$.

Then note that for any $h \in Z(G)$,

$$\begin{aligned} (g_1 g_2^{-1} f - f)(h g_1) &= f(g_2 g_1^{-1} h g_1) - f(h g_1) \\ &= f(h g_2) - f(h g_1) = c_2 - c_1. \end{aligned}$$

Since $|H \cap Z(G)| = \infty$ and $A \subseteq c_0(G)$, we must have $c_1 = c_2$. By setting $f'(g) = f(g) - c_1$, we get $g \cdot f - f = g \cdot f' - f'$ where f' has countable support. ■

We now use Holt's argument to arrive at a similar result for groups that are periodic with an extra cardinality requirement. At this point we are uncertain whether this cardinality property is necessary, but the following theorem certainly shows that it is sufficient.

Theorem 7.3. Let G be an uncountable periodic group such that $|G| > \aleph_1$ and let A be a G -submodule of $c_0(G)$. Then given $g \cdot f - f \in H^1(G, A)$, $g \cdot f - f = g \cdot f' - f'$ where f' has countable support.

Proof. By Lemma 7.1 we can find a proper subgroup H of G with $|H| = \aleph_1$ such that $h \cdot f - f \subseteq \mathbb{C}^H$ for all $h \in H$. Let $k \in G - H$, and note that $K := \langle k \rangle$ is a finite subgroup of G . Since $k^i \cdot f - f$ has countable support for all i , we have that $(k^i \cdot f - f)(g) = 0$ for all but countably many $g \in G$ for each i . When $(k^i \cdot f - f)(g) = 0$ we see that $f(k^{-i}g) = f(g)$, so there is a countable subset of G such that f is not constant on Kg . Let H_1 be the countable subgroup of $\langle H, k \rangle$ generated by the $g \in \langle H, k \rangle$ such that f is not constant on Kg . Then each $g \in H_1$ can be written as a word on H and K , and define H_0 to be the subgroup of H generated by the elements of H in each of these words. Since H_1 is countable and each word has finite length, we see that H_0 is a countable subgroup of H . Then, by defining $L = \langle H_0, k \rangle$, we see that for any $g \in \langle H, k \rangle - L$, f is constant on Kg .

We claim that f is constant outside of $H \cap L$, noting that L is countably generated and therefore countable, and thus $H \cap L$ is a countable subgroup of G . Since L is a countable subgroup of $\langle H, k \rangle$, there exists some $h \in H - L$, and by definition we have $k \notin H$, so L and H are both proper subgroups of $\langle H, k \rangle$. Therefore, if we can show that f is constant on cosets $Hg \neq H$ and $Lg \neq L$, then we can use Lemma 3.1 to deduce that f is constant on $\langle H, L \rangle - (H \cap L) = \langle H, k \rangle - (H \cap L)$.

To see that f is constant on $Hg \neq H$, we note that $(h \cdot f - f)(g) = 0$ for all $g \in G - H$, so $f(hg) = f(g)$ for all $h \in H$ and $g \in G - H$. Therefore, for

$g \in \langle H, k \rangle - H$, f is constant on Hg .

To show that f is constant on $Lg \neq L$, first note that if $lg \in H$ for any $l \in L$, then $Lg = Llg$, and thus we can make the assumption that if $lg \in H$ for any $l \in L$ then $g \in H$. After making this assumption, fix a coset $Lg \neq L$. If $K = L$ then f is constant on $Kg \neq K$ as cosets in $\langle H, k \rangle$ by the definition of L , so the result is clear. So we can assume that there exists $g \in L - K$. Then note that f is constant on Lg if and only if $f(lg) = f(g)$ for all $l \in L$, if and only if $f \cdot g^{-1}(l) = f \cdot g^{-1}(1)$ for all $l \in L$, if and only if $f \cdot g^{-1}$ is constant on L . Since we have assumed that there exists $g \in (H \cap L) - K$, and certainly $k \notin H \cap L$, both $H \cap L$ and K are proper subgroups of L . Therefore, to show that $f \cdot g^{-1}$ is constant on L , we consider the cosets Kl and $(H \cap L)l$ in L , and we aim to show that $f \cdot g^{-1}$ is constant on all cosets Kl and on all cosets $(H \cap L)l \neq (H \cap L)$, and then use the second part of the lemma to conclude that $f \cdot g^{-1}$ is constant on $\langle K, H \cap L \rangle = L$.

First, we look at the cosets Kl . Since $g \in \langle H, k \rangle - L$, we have that $lg \notin L$ for any $l \in L$. Therefore, by the definition of L we have that f is constant on Klg for all $l \in L$. This gives us that

$$f \cdot g^{-1}(kl) = f(klg) = f(lg) = f \cdot g^{-1}(l),$$

for all $k \in K$ and $l \in L$, and thus $f \cdot g^{-1}$ is constant on Kl for all $l \in L$.

Now we consider the cosets $(H \cap L)l \neq (H \cap L)$. To show that $f \cdot g^{-1}$ is constant on these cosets, we use the assumption that if $lg \in H$ for any $l \in L$, then $g \in H$.

If $lg \in H$ for some $l \in L$, then by our assumption we have that $g \in H$, which implies that $l \in H$. Therefore, $(H \cap L)l = (H \cap L)$. If $lg \notin H$ for any $l \in L$, then f is constant on Hlg for all $l \in L$, and hence on $(H \cap L)lg$ for all $l \in L$. Therefore, for any $x \in H \cap L$ and $l \in L$, we have that

$$f \cdot g^{-1}(xl) = f(xlg) = f(lg) = f \cdot g^{-1}(l),$$

which implies that $f \cdot g^{-1}$ is constant on $(H \cap L)l$.

So, $f \cdot g^{-1}$ is constant on Kl for all $l \in L$ and on $(H \cap L)l \neq H \cap L$, and we can now apply the second part of the lemma to conclude that $f \cdot g^{-1}$ is constant on L , or, equivalently f is constant on Lg . This gives us that f is constant on Lg for all $g \in \langle H, k \rangle - L$ and Hg for all $g \in \langle H, k \rangle - H$, and since H and L are both

proper subgroups of $\langle H, L \rangle$, we apply the first part of the lemma to deduce that f is constant on $\langle H, L \rangle - (H \cap L)$.

Finally, let $k' \in G - H$ such that $k \neq k'$. By the same proof we see that f is constant on $\langle H, L' \rangle - (H \cap L')$, so $f(h) = f(k')$ for all but countably many $h \in H$. We also have that $f(h) = f(k)$ for all but countably many $h \in H$. Therefore, since $|H| > \aleph_0$, there exists $h \in H$ such that $f(k') = f(h) = f(k)$, and thus f is constant on $G - (H \cap L)$. By subtracting a constant, we can write $g \cdot f - f = g \cdot f' - f'$ where f' has countable support. ■

So, if a group G has an infinite center or is periodic with cardinality larger than \aleph_1 and A is a G -submodule of $c_0(G)$, we have shown that if $g \cdot f - f \in H^1(G, A)$, then f has countable support. This turns out to imply the vanishing of the cohomology group for several common G -submodules of $c_0(G)$, in particular $l^p(G)$, the main G -module that we are considering. The following propositions discuss the vanishing of the first cohomology groups for the G -modules $l^p(G)$, $\mathcal{N}(G)$, $C_{red}^*(G)$, $l^2(G)l^2(G)$, and $c_0(G)$. For definitions of the G -modules, see the Preliminaries section.

Proposition 7.4. Let G be an uncountable group. If f has countable support and $g \cdot f - f \in l^p(G)$ for all $g \in G$, then $f \in l^p(G)$.

Proof. Assume for contradiction that $\|f\|_p = \infty$. Then let H be the subgroup generated by the support of f . H is countable, so there exists $g \in G - H$. Then $\|g \cdot f - f\|_p^p = \|g \cdot f\|_p^p + \|f\|_p^p = \infty$, contradicting $g \cdot f - f \in l^p(G)$. Therefore, $f \in l^p(G)$. ■

Proposition 7.5. Let G be an uncountable group. If f has countable support and $g \cdot f - f \in c_0(G)$ for all $g \in G$, then $f \in c_0(G)$.

Proof. Let H be the subgroup generated by the support of f , and let $g \in G - H$. Then $g \cdot f - f \in c_0(G)$ implies that $(g \cdot f - f)|_H \in c_0(G)$, by the definition of $c_0(G)$. Since $(g \cdot \alpha - \alpha)|_H = \alpha$, we have that $\alpha \in c_0(G)$. ■

Proposition 7.6. Let G be an uncountable group. If f has countable support and $g \cdot f - f \in \mathcal{N}(G)$ for all $g \in G$, then $f \in \mathcal{N}(G)$.

Proof. $\mathcal{N}(G)$ can be characterized as all $f \in l^2(G)$ such that $ff' \in l^2(G)$ for all $f' \in l^2(G)$. So, since we know that $g \cdot f - f \in \mathcal{N}(G)$ for all $g \in G$, we have that $g \cdot ff' - ff' \in l^2(G)$ for all $f' \in l^2(G)$ and $g \in G$.

Given $f' \in l^2(G)$, let H be the subgroup generated by the support of f and f' . Then H is countable, so we can find $g \in G - H$. Since $\|g \cdot ff' - ff'\|_2^2 = \|g \cdot ff'\|_2^2 + \|ff'\|_2^2$ we must have that $ff' \in l^2(G)$. This gives us that $f \in \mathcal{N}(G)$ ■

Proposition 7.7. Let G be an uncountable group. If f has countable support and $g \cdot f - f \in C_{red}^*(G)$ for all $g \in G$, then $f \in C_{red}^*(G)$.

Proof. C_{red}^* is the closure of $\mathbb{C}G$ under the operator norm (by considering $\mathbb{C}G$ as a subset of $\mathcal{B}(l^2(G))$). Therefore, C_{red}^* is all $f \in l^2(G)$ such that $\sup_{\|f'\|_2=1} \{\|ff'\|_2\} < \infty$.

So assume for contradiction that $f \notin C_{red}^*$. Then for every $N \in \mathbb{N}$, we can find f'_N with $\|f'_N\|_2 = 1$ such that $\|ff'_N\|_2^2 > N$. Let H be the subgroup generated by the support of f and f'_N for all N . Then H is countable, so we can find $g \in G - H$. For any N , we have that $\|f'_N\|_2 = 1$ and $\|(g \cdot f - f)f'_N\|_2^2 = \|g \cdot ff'\|_2^2 + \|ff'\|_2^2 > 2N$. But this contradicts that $g \cdot f - f \in C_{red}^*$ for all $g \in G$, so we must have $f \in C_{red}^*$. ■

Proposition 7.8. Let G be an uncountable group. If f has countable support and $g \cdot f - f \in l^2(G)l^2(G)$ for all $g \in G$, then $f \in l^2(G)l^2(G)$.

Proof. Let H be the subgroup generated by the support of f . Since H is countable, there exists $g \in G - H$. Since $g \cdot f - f \in l^2(G)l^2(G)$, we have that $g \cdot f - f \in c_0(G)$, and thus $(g \cdot f - f)|_H \in c_0(H)$. So, $(g \cdot f - f)|_H \in l^2(H)l^2(H) \subset l^2(G)l^2(G)$, and since $(g \cdot f - f)|_H = f$, we have $f \in l^2(G)l^2(G)$. ■

Theorem 7.9. Let G be an uncountable group with infinite center or a peri-

odic group with $|G| > \aleph_1$. Then if $A = l^p(G), \mathcal{N}(G), C_{red}^*(G), c_0(G), l^2(G), l^2(G), H^1(G, A) = 0$.

Proof. This follows immediately from Theorems 7.2 and 7.3 along with Propositions 7.4-7.8. ■

So, we see that the first cohomology group vanishes for all of these G -modules with countable support if the group is uncountable with infinite center or periodic with $|G| > \aleph_1$. Note that it is still unclear whether or not the cardinality condition can be removed for uncountable periodic groups. We use it for a subset argument in Theorem 7.3, but we have no reason to believe that the condition is necessary.

7.2 Coefficients in $l^\infty(G)$

We now want to consider $l^\infty(G)$ and decide whether or not the above result could hold for this G -module. Already, however, we see that $l^\infty(G)$ contains functions without countable support, so our previously used techniques will not be of much use. In fact, these results do not hold for the G -module $l^\infty(G)$, and to this point we do not know of a group with vanishing first cohomology when considering coefficients in $l^\infty(G)$. Length functions are the non-zero elements of $H^1(G, l^\infty(G))$ that we will construct, and these are introduced more generally in Chapter 5.

We first look at the case when G is countable, because in this case we can consider whether or not the map $l^\infty(G) \rightarrow B_G$ has a continuous inverse to determine whether or not the cohomology group vanishes (since $l^\infty(G)^G$ is a Frechet Space).

Theorem 7.10. Let G be a group. Then the map $f: l^\infty(G) \rightarrow B_G$ does not have a continuous inverse.

Note. We make a note here that when we talk about the map from $A = l^\infty(G)$, we really are talking about A/N where N is the normal subgroup of A made up of constant functions. Since A is a Banach space and N is a closed subset, A/N is also a Banach space and nothing changes. If we do not make this distinction, then the inverse map is not well-defined.

Proof. To show that the inverse map is not continuous, we aim to show that if \mathcal{O} is a system of basic neighborhoods of $0 \in B_G$, then for every $O \in \mathcal{O}$ we can find a map $e \in l^\infty(G)$ such that $\|e\| = 1$ and $f(e) \in O$. Note that neighborhoods of the form $O_{g_1, g_2, \dots, g_n, \epsilon} := \{z \in B_G : \|z(g_i)\| < \epsilon, 1 \leq i \leq n\}$ give a system of basic neighborhoods of 0 in B_G .

So, let $O := O_{g_1, \dots, g_n, \epsilon}$ be given, and we intend to construct $e \in l^\infty(G)$ such that $\|e\| = 1$ and $f(e) \in O$. Let N be such that $\frac{1}{N} < \epsilon$ and let $H := \langle g_1, \dots, g_n \rangle$. We now use the function $\ell(x) : H \rightarrow \mathbb{N}$ which assigns to x the length of the shortest word on $\{g_1, \dots, g_n\}$ that can be used to represent x . Then define $e : G \rightarrow \mathbb{C}$ by

$$e(x) = \begin{cases} 1 - \frac{\ell(x)}{N} & : x \in H \text{ such that } \ell(x) < N, \\ 0 & : \text{otherwise.} \end{cases}$$

Note here that $\|e\|_\infty = 1$ and since H is finitely generated there are only finitely many $x \in H$ such that $\ell(x) = m$ for any m . So, if $x \notin H$, then $g_i^{-1}x \notin H$ for $1 \leq i \leq n$, and we get that

$$|(g_i \cdot e - e)(x)| = |e(g_i^{-1}x) - e(x)| = |0 - 0| = 0,$$

and if $x \in H$, then $g_i^{-1}x \in H$, and since $\ell(g_i^{-1}x) = 1$ for all $1 \leq i \leq n$, we see that

$$|(g_i \cdot e - e)(x)| = |e(g_i^{-1}x) - e(x)| \leq \frac{\ell(g_i^{-1}x)}{N} = \frac{1}{N} < \epsilon.$$

Therefore, $\|g_i \cdot e - e\| < \epsilon$ for $1 \leq i \leq n$, which implies that $f(e) \in O$, and thus the inverse map is not continuous. ■

Since the function $e(x)$ constructed in the proof is also an element of $c_0(G)$ with norm 1, the same proof can be used to show the following theorem:

Theorem 7.11. Let G be a group. Then the map $f : c_0(G) \rightarrow B_G$ does not have a continuous inverse.

And now we can conclude the following for countable groups G .

Corollary 7.12. Let G be a countable group and $A = l^\infty(G)$ or $c_0(G)$. Then $H^1(G, A) \neq 0$.

Proof. Since G is countable, A^G is a Frechet Space under the topology of pointwise convergence (since its topology is generated by the countable family of seminorms $p_g(f) = \|g \cdot f - f\|_\infty$ for each $g \in G$). So, B_G is a closed subset of A^G if and only if the continuous map from $A \rightarrow B_G$ has a continuous inverse. By Theorems 7.10 and 7.11, it never has a continuous inverse, and therefore B_G is never closed in A^G . Z_G is a closed subset of A^G , therefore $B_G \neq Z_G$, and thus $H^1(G, A) = Z_G/B_G \neq 0$. ■

If we drop the countability of G , however, then A^G is no longer a Frechet space, and we can no longer say that $H^1(G, A) \neq 0$ directly from Theorems 7.10 and 7.11 (as seen by the result in Theorem 7.9). But when $A = l^\infty(G)$, we still get that $H^1(G, A) \neq 0$ when G is solvable or when G contains a normal subgroup N such that G/N is infinite, finitely generated. We start with the proof when G is abelian, and for the proof we will use the following lemmas.

Lemma 7.13. Let G be a group such that $G = \bigcup_{i=1}^\infty H_i$ with $H_i \leq G$ and $H_i \subsetneq H_{i+1}$ for all i . Then $H^1(G, l^\infty(G)) \neq 0$.

Proof. Define $f: G \rightarrow \mathbb{C}$ by $f(g) = n$ for $g \in H_n \setminus H_{n-1}$. Then certainly $f \notin l^\infty(G)$ (since the chain of subgroups is infinite), however, we aim to show that $g \cdot f - f \in l^\infty(G)$ for all $g \in G$.

To see this, let $g \in G$ be given, and note that $|(g \cdot f - f)(x)| = |f(g^{-1}x) - f(x)|$. Let n be such that $g \in H_n \setminus H_{n-1}$. Then if $x \in H_n$, we have that

$$|f(g^{-1}x) - f(x)| \leq n$$

since $g^{-1}x \in H_n$. If $x \notin H_n$, then $x \in H_j \setminus H_{j-1}$ for some $j > n$, and since $g^{-1} \in H_{j-1}$, $g^{-1}x \in H_j \setminus H_{j-1}$. Therefore, we have that $|f(g^{-1}x) - f(x)| = 0$.

So, $\|g \cdot f - f\|_\infty \leq n$, and since $g \in G$ was arbitrary, we have that $g \cdot f - f \in l^\infty(G)$ for all $g \in G$, and thus $g \cdot f - f$ represents a non-zero element of $H^1(G, l^\infty(G))$. ■

Lemma 7.14. Let M be an R -module and $I \triangleleft R$ such that $I^e = 0$ for some exponent $e > 0$. If $S = \{m_\alpha\}_{\alpha \in A}$ generate M/MI , then S generates M as an R -module.

Proof. This is a well-known fact, but the proof is included for completeness. We know that any element of $m \in M$ can be written as $m = m_1r_1 + \dots + m_nr_n + m_{n+1}i$ where $m_j \in S$, $r_j \in R$ and $i \in I$. Let N be the R -submodule generated by S , and we want to show that $mi \in N$. First, we aim to show that $mi^{e-1} \in N$. $m = m_1r_1 + \dots + m_nr_n + m_{n+1}i$ so

$$\begin{aligned} mi^{e-1} &= m_1r_1i^{e-1} + \dots + m_nr_ni^{e-1} + m_{n+1}i^e \\ &= m_1r_1i^{e-1} + \dots + m_nr_ni^{e-1} \end{aligned}$$

and so $mi^{e-1} \in N$. Now consider mi^{e-2} . Again $m = m_1r_1 + \dots + m_nr_n + m_{n+1}i$, so

$$mi^{e-2} = m_1r_1i^{e-2} + \dots + m_nr_ni^{e-2} + m_{n+1}i^{e-1}$$

and we have already shown that $mi^{e-1} \in N$, therefore $mi^{e-2} \in N$. Continuing in this way we get that $mi \in N$, and therefore $M = N$. \blacksquare

Theorem 7.15. Let G be an infinite abelian group. Then $H^1(G, l^\infty(G)) \neq 0$.

Proof. Let T be the torsion subgroup of G . We will denote all of the elements of G with order p^n by $G[p^n]$, and all of the elements with order a power of p by G_p . First we consider the case where $G = T$. If $G = T$, then we can write $G = \bigoplus_{p \text{ prime}} G_p$. If $|G_p| < \infty$ for all p , then we can write $G = \bigcup_{p \text{ prime}} \langle G_2, G_3, G_5, \dots, G_p \rangle$. We know that this chain of subgroups cannot be written as a finite chain since G is infinite, so by throwing out the unnecessary subgroups, we can use the above lemma to see that $H^1(G, l^\infty(G)) \neq 0$.

Now consider the case when $G_{p'}$ is infinite for some p' . If $G_{p'}$ contains elements of arbitrarily large order, then let $H = \bigoplus_{\{p \text{ prime}\} \setminus \{p'\}} G_p$, and we see that $G/H \cong G_{p'}$. We can write $G_{p'} = \bigcup_{n=1}^{\infty} (\bigcup_{i=1}^n G[p'^n])$, and this corresponds to a chain of subgroups of G containing H that cannot be written as a finite chain (since $G_{p'}$ has elements of arbitrarily large order). Therefore, we can again use the above lemma to show that $H^1(G, l^\infty(G)) \neq 0$.

If, however, $G_{p'}$ has bounded exponent, then $p'^n G_{p'} = 0$ for some smallest n . Since $p'^n G_{p'} = 0$, we can view $G_{p'}$ as a $\mathbb{Z}/p'^n\mathbb{Z}$ -module. A generating set for $G/p'G$ also generates G as a $\mathbb{Z}/p'^n\mathbb{Z}$ -module by lemma 7.14, so since G is infinite, we must have that $G/p'G$ is infinite. But we can view $G/p'G$ as a $\mathbb{Z}/p'\mathbb{Z}$ -vector space, and so we

have an infinite number of generators, say $S := \{x_a\}_{a \in A}$. By choosing a countable number of generators S' , and letting $H = \langle S \setminus S' \rangle$, we have that G/H is infinite and countably generated, so we can write G/H as a strictly increasing chain of subgroups, which corresponds to an infinite increasing chain of subgroups whose union is G . Therefore, we use the above lemma to show that $H^1(G, l^\infty(G)) \neq 0$.

Finally, assume that $|G/T| = \infty$ (it cannot be finite since G/T is torsion free). Let $A := G/T$ for simplicity. We first define $\mathbb{Q}A$ to be the module of fractions $S^{-1}A$ where $S = \mathbb{Z} \setminus \{0\}$. Since A is torsion free, we can embed \mathbb{Q} in $\mathbb{Q}A$, and \mathbb{Q} is divisible which gives us that $\mathbb{Q}A = Q \oplus X$ where $Q \cong \mathbb{Q}$. Note that elements of Q are of the form $x = a/s$ for $a \in A$ and $s \in S$, so if $x \in Q$ then $sx \in A$. Further, since A is torsion free, we can see that $|Q \cap A| = \infty$. Therefore, since $(A \cap Q) \oplus (A \cap X) \leq A$, we have that

$$[A : A \cap X] \geq [(A \cap X) \oplus (A \cap Q) : A \cap X] = \infty.$$

Then we also note that

$$\frac{A}{X \cap A} \cong \frac{A + X}{X} \leq \frac{Q \oplus X}{X} \cong \mathbb{Q}$$

and therefore we have that $[A : A \cap X]$ is infinite and countable.

Now, if $A/(A \cap X)$ is not finitely generated, then $A/(A \cap X) = \langle x_1(A \cap X), x_2(A \cap X), \dots \rangle$, and so we can write $A = \bigcup_{i=1}^{\infty} A_i$ where $A_i = \langle A \cap X, x_1, \dots, x_i \rangle$. This chain of subgroups then corresponds to a chain of subgroups of G containing T , and so we can write G as an infinite, strictly ascending chain of subgroups and use the above lemma to show that $H^1(G, l^\infty(G)) \neq 0$.

So all that remains is to show that $H^1(G, l^\infty(G)) \neq 0$ if $A/(A \cap X)$ is finitely generated. In this case, let $A/(A \cap X) = \langle x_1(A \cap X), \dots, x_n(A \cap X) \rangle$. Then $A \cap X$ corresponds to a subgroup of G containing T , call it H , and we can write $G = \langle x_1H, \dots, x_nH \rangle$. In this case, we define $f: G \rightarrow \mathbb{C}$ by $\alpha(g) = \ell(gH)$ where $\ell(gH)$ denotes the length of the shortest word on $\{x_1H, \dots, x_nH\}$ representing the coset gH . Since $[A : (A \cap X)] = \infty$, we see that $[G : H] = \infty$, and so $f \notin l^\infty(G)$. However, we do have that for any $g \in G$,

$$|(g \cdot f - f)(x)| = |\ell(g^{-1}xH) - \ell(xH)| \leq \ell(g^{-1}H),$$

and so $g \cdot f - f \in l^\infty(G)$ for all $g \in G$. Therefore, f represents a non-zero element of $H^1(G, l^\infty(G))$, which concludes the proof. ■

We now look to extend to the case when G is solvable, but we will prove a lemma which will be useful beyond the solvable case.

Lemma 7.16. Let N be a normal subgroup of G . Then, if $H^1(G/N, l^\infty(G/N)) \neq 0$, then $H^1(G, l^\infty(G)) \neq 0$.

Proof. Let $f : G/H \rightarrow \mathbb{C}$ be such that $g \cdot f - f$ is a non-zero element of $H^1(G/H, l^\infty(G/H))$. Then define $f^* : G \rightarrow \mathbb{C}$ by $f^*(x) = f(xH)$. Then $f^* \notin l^\infty(G)$ since $f^* \notin l^\infty(G/H)$. However, we note that for any $g \in G$,

$$\begin{aligned} |(g \cdot f^* - f^*)(x)| &= |f^*(g^{-1}x) - f^*(x)| = |f(g^{-1}xH) - f(xH)| \\ &= |(gH \cdot f - f)(xH)| \leq N \end{aligned}$$

for all $x \in G$ since $gH \cdot f - f \in l^\infty(G/H)$ for all $gH \in G/H$. Therefore, $g \cdot f^* - f^*$ is a non-zero element of $H^1(G, l^\infty(G))$. ■

Lemma 7.17. Let H be a subgroup of G . If $[G : H] < \infty$, then $H^1(G, l^p(G)) \cong H^1(H, l^p(H))$ for $1 \leq p \leq \infty$.

Proof. Use Shapiro's Lemma [7] noting that the induced G -module $\text{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, l^p(H)) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} l^p(H) \cong l^p(G)$ when $[G : H] < \infty$.

Theorem 7.18. Let G be an infinite solvable group. Then $H^1(G, l^\infty(G)) \neq 0$.

Proof. Since G is solvable we have a chain of subgroups $1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G$ where G_i/G_{i-1} is abelian for all i . Since G is an infinite group, we know that there must exist a largest j such that $[G_j : G_{j-1}] = \infty$. Since G_j/G_{j-1} is an infinite abelian group, we have that $H^1(G_j/G_{j-1}, l^\infty(G_j, G_{j-1})) \neq 0$ by Theorem 7.15. Therefore, by Lemma 7.16, we see that $H^1(G_j, l^\infty(G_j)) \neq 0$. Since $[G_i : G_{i-1}] < \infty$ for all $i > j$, Lemma 7.17 gives us that $H^1(G, l^\infty(G)) \cong H^1(G_j, l^\infty(G_j)) \neq 0$. ■

7.3 Induced Maps

We now shift to the study of maps between cohomology groups. In the first chapter we discussed when maps between G -modules induce maps on cohomology groups, and one case of this is when there is an embedding of G -modules. Then, in chapter 3 we looked at how the embedding of $\mathbb{C}G$ into $l^p(G)$ induces an injective map $H^1(G, \mathbb{C}G) \hookrightarrow H^1(G, l^p(G))$ when G is finitely generated. We also now know that this result does not hold for G not finitely generated, since for G uncountably infinite locally finite, $H^1(G, \mathbb{C}G) \neq 0$ but $H^1(G, l^p(G)) = 0$. In fact, the map is not injective even if we let G be countably generated locally finite, as seen in the following proposition.

Proposition 7.19. Let G be a countably infinite locally finite group. Then the map $\phi : H^1(G, \mathbb{C}G) \rightarrow H^1(G, l^p(G))$ induced from the embedding $\mathbb{C}G \hookrightarrow l^p(G)$ is not injective.

Proof. G is countable, so let $G = \{g_i\}_{i \in \mathbb{N}}$. Since G is locally finite, we can write $G = \bigcup G_i$ where G_i is finite for all i and $G_{i-1} \subset G_i$ by defining $G_i = \langle g_1, \dots, g_i \rangle$. We can also assume we have strict inclusion at each step by throwing out duplicates. Then, given $g \in G$, $g \in G_i - G_{i-1}$ for some i , so define $f : G \rightarrow \mathbb{C}$ by

$$f(g) = \frac{2^{-i}}{|G_i - G_{i-1}|}.$$

We aim to show that $g \cdot f - f$ is a nonzero element of $H^1(G, \mathbb{C}G)$ that maps to 0 under ϕ . The fact that the image of $g \cdot f - f$ is 0 in $H^1(G, l^p(G))$ is clear since $\sum_{g \in G} f(g) = \sum_{i=1}^{\infty} 2^{-i} < \infty$, so since f is summable it is certainly p -summable. Also, f does not have finite support, so if we can show that $g \cdot f - f \in \mathbb{C}G$ for any $g \in G$, then $g \cdot f - f$ is a nonzero element of $H^1(G, \mathbb{C}G)$ whose image under ϕ is 0.

Given $g \in G$, $g \in G_i - G_{i-1}$ for some i . Then note that if $x \notin G_i$, then $x \in G_j - G_{j-1}$ for some $j > i$. Since $g \in G_{j-1}$, then $g^{-1}x \notin G_{j-1}$, thus $g^{-1}x \in G_j - G_{j-1}$. Therefore, $f(x) = f(g^{-1}x)$, and we have that

$$(g \cdot f - f)(x) = f(g^{-1}x) - f(x) = 0$$

when $x \notin G_i$. Therefore, the support of f is contained in G_i , thus $g \cdot f - f \in \mathbb{C}G$ for any $g \in G$, and we have that $g \cdot f - f \neq 0$ in $H^1(G, \mathbb{C}G)$ and thus ϕ is not injective. ■

We then want to consider the induced maps from the embeddings $l^p(G) \hookrightarrow l^q(G)$ where $p < q$ and $c_0(G) \hookrightarrow l^\infty(G)$. We will first consider the countable case before commenting on whether or not the results hold for uncountable groups G . So for the following results we assume G to be countable, and these results will not extend to uncountable G in general.

Proposition 7.20. Let G be a countable locally finite group. Then the map $\phi : H^1(G, l^p(G)) \rightarrow H^1(G, l^q(G))$ induced from the embedding $l^p(G) \hookrightarrow l^q(G)$ when $1 \leq p < q < \infty$ is not injective.

Proof. We can write $G = \bigcup_{i=1}^{\infty} G_i$ where $G_1 \subset G_2 \subset \dots$ and G_i is finite for all i . Then define $f : G \rightarrow \mathbb{C}$ as follows. Given any $g \in G$, $g \in G_i \setminus G_{i-1}$ for some i . Then

$$f(g) = \frac{1}{((|G_i| - |G_{i-1}|)i)^{1/p}}.$$

Then f is certainly not in $l^p(G)$, however, we claim that $g \cdot f - f \in l^p(G)$ for all $g \in G$, and $f \in l^q(G)$. Therefore, if the claim holds, we have a non-zero element of $H^1(G, l^p(G))$ whose image is 0 under ϕ .

To see that $g \cdot f - f \in l^p(G)$ for all $g \in G$, let $g \in G_i \setminus G_{i-1}$. Then if $x \notin G_i$, then $x \in G_j \setminus G_{j-1}$ for some $j > i$, so $g^{-1} \in G_{j-1}$ implies that $g^{-1}x \in G_j \setminus G_{j-1}$. Therefore, $f(g^{-1}x) = \alpha(x)$ which gives us that

$$|(g \cdot f - f)(x)| = |f(g^{-1}x) - f(x)| = 0$$

for all $x \notin G_i$. Therefore, $g \cdot f - f$ has finite support and is an element of $l^p(G)$.

All that remains is to show that $f \in l^q(G)$, but this is easily seen by noting that if $g \in G_i \setminus G_{i-1}$, then

$$|f(g)|^q = \frac{1}{(|G_i| - |G_{i-1}|)^{q/p} i^{q/p}} \leq \frac{1}{(|G_i| - |G_{i-1}|) i^{q/p}}$$

and so we see that when we sum over all $g \in G$, we get

$$\sum_{g \in G} |f(g)|^q \leq \sum_{i=1}^{\infty} i^{q/p} < \infty.$$

Therefore $f \in l^q(G)$, concluding the proof. ■

Note that in the statement of the proposition $l^q(G)$ can be replaced with $l^\infty(G)$ or $c_0(G)$ since $l^q(G) \subset c_0(G) \subset l^\infty(G)$. This can also be said for the following result, which extends Proposition 7.20 to all amenable groups.

Theorem 7.21. Let G be a countably infinite amenable group. Then the map $\phi : H^1(G, l^p(G)) \rightarrow H^1(G, l^q(G))$ induced from the embedding $l^p(G) \hookrightarrow l^q(G)$ where $1 \leq p < q < \infty$ is not an embedding.

Proof. Let $S = \{g_1, g_2, \dots\}$ (finite or infinite) be a generating set for G . Since G is countable amenable, we can get a Følner exhaustion $F_1 \subset F_2 \subset \dots$ such that $G = \bigcup F_i$, $|F_i| < \infty$ for all i and $\frac{|F_i \cdot g \Delta F_i|}{|F_i|} \rightarrow 0$ for all $g \in G$ (where Δ denotes the symmetric difference between sets) [8]. Note here that given any finite set $S' \subset G$, $\{F_i \cup S'\}$ will also be a Følner exhaustion of G .

We now create a special Følner exhaustion in the following way. Let $X_1 = F_1$. Given X_n , note that $S' = X_n \cdot g_i^{\pm 1}$ $1 \leq i \leq n$ is a finite set and $\{F'_i\} = \{F_i \cup g_1 \cup g_1^{-1} \cup \dots \cup g_n \cup g_n^{-1}\}$ is a Følner exhaustion of G . So we can choose i such that $S' \subset F'_i$, $|F'_i| > 2|X_n|$ and $|F'_i \cdot g_j^{\pm 1} \Delta F'_i| < 1/2^n |F'_i|$ for $1 \leq j \leq n$. Then let $X_{n+1} = F'_i$. This will give us a Følner exhaustion such that

- (i) $X_n g_i^{\pm 1} \subset X_{n+1}$ for $1 \leq i \leq n$
- (ii) $|X_{n+1}| > 2|X_n|$
- (iii) $|X_n \cdot g_i^{\pm 1} \Delta X_n| < 1/2^n |X_n|$ for $1 \leq i \leq n$.

Now define the function $\ell : G \rightarrow \mathbb{Z}^+$ by $\ell(x) = n$ if $x \in X_n \setminus X_{n-1}$. The key property of this function is that if $\ell(x) = n$, then for $1 \leq i \leq n$, $|\ell(xg_i) - \ell(x)| \leq 1$. We can see this since $g_i \cdot X_n \subset X_{n+1}$ for $1 \leq i \leq n$, and since $xg_i g_i^{-1} \notin X_{n-1}$ we can't have $xg_i \in X_{n-2}$.

Now let $e_n = |X_n \setminus X_{n-1}|$ and define $f : G \rightarrow \mathbb{C}$ by

$$f(x) = \frac{1}{\ell(x)^{(1/p)} e_{\ell(x)}}$$

Clearly $f \in l^q(G) \setminus l^p(G)$, and we aim to show that $g \cdot f - f \in l^p(G)$ for all $g \in G$, giving us that $g \cdot f - f$ is a non-zero element in $H^1(G, l^p(G))$ that maps to 0 under ϕ . To show this, it is enough to show that it holds for the generating set. So

consider $h(x) = (g_i \cdot f - f)(x) = f(xg_i^{-1}) - f(x)$. Since the set of $x \in G$ where $\ell(x) \leq i$ is finite, we can assume that $\ell(x) > i$. Let $\ell(x) = n$. If $\ell(xg_i^{-1}) = n$, then $h(x) = 0$. So we only need to consider where $\ell(xg_i^{-1}) = n + 1$ and where $\ell(xg_i^{-1}) = n - 1$. First note that the number of x such that $\ell(xg_i^{-1}) = n + 1$ is $|X_n \cdot g_i^{-1} \Delta X_n| < |X_n|/2^n < e_n/2^{n-1}$ since $e_n = |X_n| - |X_{n-1}| > |X_n| - |X_n|/2 = |X_n|/2$. Therefore, we have that

$$\sum_{x \in X_n | xg_i^{-1} \in X_{n+1}} h(x) \leq \sum_{x \in X_n | xg_i^{-1} \in X_{n+1}} f(x) \leq \frac{e_n}{2^{n-1} n^{(1/p)} e_n} \leq \frac{1}{2^{n-1}}$$

Similarly, the number of $x \in X_n$ such that $\ell(xg_i^{-1}) = n - 1$ is $|X_{n-1} \cdot g_i \Delta X_{n-1}| < e_{n-1}/2^{n-2}$ and so we get that

$$\sum_{x \in X_n | xg_i^{-1} \in X_{n-1}} h(x) \leq \sum_{x \in X_n | xg_i^{-1} \in X_{n-1}} f(xg_i^{-1}) \leq \frac{e_{n-1}}{2^{n-2} (n-1)^{(1/p)} e_{n-1}} \leq \frac{1}{2^{n-2}}$$

and so we arrive at

$$\begin{aligned} \sum_{x \in G} h(x) &= \sum_{x \in X_{n-2}} h(x) + \sum_{x \in G \setminus X_{n-2}} h(x) \\ &< \sum_{x \in X_{n-2}} h(x) + \sum_{i=1}^{\infty} (1/2^{i-1}) + (1/2^{i-2}) < \infty \end{aligned}$$

since $|X_{n-2}| < \infty$. Thus, $g \cdot f - f \in l^1(G) \subset l^p(G)$ for all $g \in G$, which concludes the proof. \blacksquare

This turns out to be the only time when the induced map is not injective for G countably infinite, as shown by the following theorem. Note, however, that we are only showing this for $p > 1$, although it is possible the result holds for $p = 1$.

Theorem 7.22. Let G be a non-amenable group. Then the induced map $\phi : H^1(G, l^p(G)) \rightarrow H^1(G, l^q(G))$ from the embedding $l^p(G) \hookrightarrow l^q(G)$ is injective for $1 < p < q < \infty$.

Proof. Let $f \in l^q(G)$ such that $g \cdot f - f \in l^p(G)$ for all $g \in G$. We know that there exists a finitely-generated non-amenable subgroup H in G . Note that for all $h \in H$, $\|h \cdot f|_H - f|_H\|_p^p = \sum_{x \in H} |f(h^{-1}x) - f(x)|^p \leq \sum_{x \in G} |f(h^{-1}x) - f(x)|^p = \|h \cdot f - f\|_p^p < \infty$. Thus, $h \cdot f|_H - f|_H \in H^1(H, l^p(H))$ and we can write $f|_H = u + v$

where $u \in l^p(H)$ (since H is non-amenable so $\overline{B_G} = B_G$) and v is p -harmonic. We know that $v \in c_0(H)$ since $f \in l^q(G)$, so by Lemma 6.1 in [22], $v = 0$. Therefore, $f|_H \in l^p(H)$.

Now, given a neighborhood of 0 in $l^p(G)^G$, call it $\mathcal{O}(g \cdot f - f)_{g_1, \dots, g_n, \epsilon}$, we must find f' such that $g \cdot f' - f' \in \mathcal{O}(g \cdot f - f)_{g_1, \dots, g_n, \epsilon}$. Since $g_i \cdot f - f \in l^p(G)$ for all i , $g_i \cdot f - f$ has countable support, and we can find a finite subset $S \subset G$ such that $(g_i \cdot f - f)|_{G \setminus S} < \epsilon$ for all i . So let H' be the subgroup generated by S, g_1, \dots, g_n . By noting that $(g_i \cdot f' - f')(x) = (g_i \cdot f - f)(x)$ for all $x \in H'$ and all i , we get that

$$\begin{aligned} \|(g_i \cdot f' - f') - (g_i \cdot f - f)\|_p &= \|(g_i \cdot f - f)|_{G \setminus H'}\|_p \\ &\leq \|(g_i \cdot f - f)|_{G \setminus S}\|_p < \epsilon. \end{aligned}$$

So, $g \cdot f' - f' \in \mathcal{O}(g \cdot f - f)_{g_1, \dots, g_n, \epsilon}$, and we have that $g \cdot f - f \in \overline{B_G} = B_G$, concluding the proof. \blacksquare

We can again replace $l^q(G)$ with $c_0(G)$ and the result still holds, and so the corollary that follows will also remain true when we replace $l^q(G)$ with $c_0(G)$. However, Theorem 7.22 will not hold in general if we replace $l^q(G)$ with $l^\infty(G)$, and the Fox derivative [12] gives an example of a non-zero element of $H^1(G, l^p(G))$ that maps to 0 in $H^1(G, l^\infty(G))$ when G is a free group on at least two generators (a non-amenable group). But when we consider the embedding $l^p(G) \hookrightarrow l^q(G)$ for $1 < p < q < \infty$ or the embedding $l^p(G) \hookrightarrow c_0(G)$, the following corollary completely characterizes when the induced map is injective.

Corollary 7.23. Let G be a countably infinite group. Then the induced map $\phi : H^1(G, l^p(G)) \rightarrow H^1(G, l^q(G))$ from the embedding $l^p(G) \hookrightarrow l^q(G)$ is injective if and only if G is non-amenable for $1 < p < q < \infty$.

Proof. This follows immediately from Theorems 7.21 and 7.22. \blacksquare

The last induced map we look at in this paper is the map $\phi : H^1(G, c_0(G)) \rightarrow H^1(G, l^\infty(G))$ (induced from the natural embedding of $c_0(G) \hookrightarrow l^\infty(G)$). When G is countably infinite, a length function will help define a non-zero element of $H^1(G, c_0(G))$ that maps to 0 under ϕ . Most often, a length function will be defined by assigning to an element $g \in G$ the length of the shortest word representing g on some generating set. We take a slightly different but similar approach here,

the important property being that the resulting length function will be unbounded regardless of the generating set. This length function was introduced in Chapter 5.

Definition. Given an infinite, countable group G and a countable (possibly finite) generating set S , define a length function $\ell : G \rightarrow \mathbb{N} \cup \{0\}$ as follows: Let $S = \{x_1, x_2, \dots\}$. Then let $d_n(g)$ = length of the shortest word on x_1, \dots, x_n representing g , and ∞ if no such word exists. Then we let $X_n = \{g \in G \mid d_n(g) \leq n\}$ and note that by throwing out duplicates we can get a strictly ascending chain $X_1 \subset X_2 \subset \dots$ such that $G = \bigcup X_i$. Also note that the chain is necessarily infinite. Then let $\ell(g) = \min\{n \mid g \in X_n\}$. This can easily be checked to be a length function on G (i.e. $\ell(e) = 0$, $\ell(g) = \ell(g^{-1})$ and $\ell(g_1g_2) \leq \ell(g_1) + \ell(g_2)$).

Theorem 7.24. Let G be a countably infinite group. Then the map $\phi : H^1(G, c_0(G)) \rightarrow H^1(G, l^\infty(G))$ induced from the embedding $c_0(G) \hookrightarrow l^\infty(G)$ is not injective.

Proof. First let S be a countable (possibly finite) generating set for G , and construct the length function as above. Define $f : \mathbb{R}^+ \rightarrow \mathbb{C}$ as follows: Let $f(x) = 0$ for $x \in [0, 1)$. Otherwise, if $2^n \leq x < 2^{n+1}$, then

$$f(x) = \begin{cases} \frac{2^{n+1}-x}{2^n} & \text{when } n \text{ is even} \\ \frac{x-2^n}{2^n} & \text{when } n \text{ is odd.} \end{cases}$$

Finally, we define $\alpha : G \rightarrow \mathbb{C}$ by $\alpha(g) = f(\ell(g))$. Since by the definition of the length function we have that for any n there exists a $g \in G$ such that $\ell(g) = n$, we can see that $\alpha(g) = 1$ for infinitely many $g \in G$ ($\ell(g) = 2^n$, n even) and $f(g) = 0$ for infinitely many $g \in G$ ($\ell(g) = 2^n$, n odd). However, $0 \leq \alpha(g) \leq 1$ for all $g \in G$, therefore, $f \in l^\infty(G) \setminus c_0(G)$.

So, if we can show that $x \cdot f - f \in c_0(G)$ for all $x \in S$, then $g \cdot f - f$ is a non-zero element of $H^1(G, c_0(G))$ that maps to 0 in $H^1(G, l^\infty(G))$. To see this, note that

$$\begin{aligned} |(x \cdot \alpha - \alpha)(g)| &= |\alpha(gx^{-1}) - \alpha(g)| \\ &= |f(\ell(gx^{-1})) - f(\ell(g))| \\ &\leq \max\{|f'(c)|\} |\ell(gx^{-1}) - \ell(g)| \\ &\leq \max\{|f'(c)|\} \ell(x^{-1}) \end{aligned}$$

where $c \in [\min\{\ell(g), \ell(gx^{-1})\}, \max\{\ell(g), \ell(gx^{-1})\}]$. We see that $|f'(x)| = 1/2^n \rightarrow 0$, and since there are only finitely many $g \in G$ such that $\ell(g) \leq N$ for any N , we

see that $x \cdot \alpha - \alpha \in c_0(G)$ for all $x \in S$, and thus all $g \in G$. Therefore, ϕ is not injective. ■

Reznikov does this for finitely generated groups in [reference], and it is worth noting that the result does not hold for general groups G , since Theorem 7.9 shows that there are uncountable groups such that $H^1(G, c_0(G)) = 0$, and therefore ϕ is trivially injective.

7.4 Reduced Cohomology

Finally, we turn to results regarding reduced cohomology. As discussed in Chapter 6, if the space $H^1(G, A)$ is Hausdorff (with the topology of pointwise convergence), then the reduced cohomology group, $\overline{H^1}(G, A)$, is no different than the group $H^1(G, A)$. Also, note that a non-zero element of $\overline{H^1}(G, A)$ is certainly non-zero in $H^1(G, A)$, and so $\overline{H^1}(G, A) \neq 0$ implies that $H^1(G, A) \neq 0$. We begin by proving a result about the non-vanishing of $\overline{H^1}(G, l^\infty(G))$ for certain groups G , noting that this also implies that $H^1(G, l^\infty(G))$ is non-zero as a corollary. For the proof, we follow the outline of Gournay's argument in [21], but using cosets to show that the result can hold for groups not necessarily finitely generated.

Proposition 7.25. Let G be a group with a normal subgroup such that G/H is infinite and finitely generated. Then $\overline{H^1}(G, l^\infty(G)) \neq 0$.

Proof. Let S be the generating set for G/H . Consider the non-zero element of $H^1(G/H, l^\infty(G/H))$ given by $g \cdot f - f$ where $f(gH) = \ell(gH)$ where $\ell : G/H \rightarrow \mathbb{N}$ is the traditional length function using the generating set S . Assume for contradiction that there exists a sequence f_n such that $f_n \in l^\infty(G)$ and $\|g \cdot f_n - f_n - (g \cdot f - f)\|_\infty \rightarrow 0$ for all $g \in G$. Then we can find N large enough so that $\|g_i \cdot f_N - f_N - (g_i \cdot f - f)\|_\infty < 1/2$ for all $g_i \in S$. Let $\|f_N\|_\infty = M$ and $f_N(1) = M'$.

Since G/H is infinite, finitely generated, there exists a word on S of length $L > 2(M - M')$, call it $x = g_{i_1}H \dots g_{i_L}H$. Since $\|g_{i_j}^{-1} \cdot f - f\|_\infty \geq |\ell(g_{i_j}H \dots g_{i_L}H) - \ell(g_{i_{j+1}}H \dots g_{i_L}H)| = 1$ for all $1 \leq j \leq L - 1$, we must have that $f_N(g_{i_j}H \dots g_{i_L}H) - f_N(g_{i_{j+1}}H \dots g_{i_L}H) > 1/2$ for all $1 \leq j \leq L - 1$. Therefore, $f_N(g_{i_1}H \dots g_{i_L}H) > L/2 + M' = M$, which is a contradiction. Therefore, $\overline{H^1}(G/H, l^\infty(G/H)) \neq 0$. ■

Corollary 7.26. Let G be a group with a normal subgroup such that G/H is infinite and finitely generated. Then $H^1(G, l^\infty(G)) \neq 0$.

Proof. This follows immediately from Proposition 7.25 and the comment in the preceding paragraph.

Despite our inability to construct a group with vanishing first l^∞ -cohomology in section 7.2, we can give an example of a class of groups with vanishing reduced l^∞ -cohomology, namely, locally finite groups. In fact, all infinite locally finite groups prove to have vanishing reduced cohomology when we consider coefficients in $l^p(G)$ for any $1 \leq p \leq \infty$, as seen in the following theorem. To show this, given an arbitrary element of $H^1(G, l^p(G))$, we simply construct an element of B_G that lies in an arbitrary open neighborhood.

Theorem 7.27. Let G be an infinite, locally finite group. Then $\overline{H^1}(G, l^p(G)) = 0$ for $1 \leq p \leq \infty$.

Proof. Let $\theta(g) = g \cdot f - f$, with $f \in \mathbb{C}^G$, be an element of $H^1(G, l^p(G))$ (so $\theta(g) \in l^p(G)$ for all $g \in G$). To show that $\theta = 0$ in $\overline{H^1}(G, l^p(G))$, given any $\mathcal{O}_{g_1, \dots, g_n, \epsilon}(\theta) := \{\alpha \in l^p(G)^G : \|\alpha(g_i) - \theta(g_i)\|_p < \epsilon \text{ for } 1 \leq i \leq n\}$, we need to find $\phi \in B_G$ such that $\phi \in \mathcal{O}_{g_1, \dots, g_n, \epsilon}(\theta)$.

Let $H = \langle g_1, \dots, g_n \rangle$, which is a finite subgroup since G is locally finite. Then let K be a right transversal of H in G so that $G = HK$. We define $\beta_H(hk) = f(hk) - f(k) = (h^{-1} \cdot f - f)(k)$. We claim that $\beta_H \in l^p(G)$. If $p = \infty$, then note that since H is finite and $h^{-1} \cdot f - f \in l^\infty(G)$ for all $h \in H$, we can find M such that $(h^{-1} \cdot f - f)(g) \leq M$ for all $h \in H$ and $g \in G$. Therefore $\beta_H(hk) = (h^{-1} \cdot f - f)(k) \leq M$ for all $h \in H$ and $k \in K$, which gives us that $\beta_H \in l^\infty(G)$. If $1 \leq p < \infty$, then we can find M such that $\sum_{g \in G} (h^{-1} \cdot f - f)(g)^p < M$ for all $h \in H$ (again since H is finite), so we see that

$$\begin{aligned} \|\beta_H\|_p^p &= \sum_{h \in H} \sum_{k \in K} \beta_H(hk)^p \\ &= \sum_{h \in H} \sum_{k \in K} (h^{-1} \cdot f - f)(k)^p \\ &\leq \sum_{h \in H} \sum_{g \in G} (h^{-1} \cdot f - f)(g)^p < M|H| < \infty. \end{aligned}$$

Therefore, the element $\phi(g) := g \cdot \beta_H - \beta_H \in B_G$.

Now, given any $h \in H$ and $g \in G$, noting that $g = jk$ for some $j \in H$ and $k \in K$, we see that

$$\begin{aligned}\phi(h)(g) &= \beta_H(h^{-1}jk) - \beta_H(jk) \\ &= f(h^{-1}jk) - f(k) - (f(jk) - f(k)) \\ &= (h \cdot f - f)(jk) = \theta(h)(g).\end{aligned}$$

Therefore, since $g_i \in H$ for $1 \leq i \leq n$, we have that $\|\phi(g_i) - \theta(g_i)\|_p = 0 < \epsilon$. Thus, $\phi \in \mathcal{O}_{g_1, \dots, g_n, \epsilon}(\theta)$ and $\overline{H^1}(G, l^p(G)) = 0$. \blacksquare

We also note here that the proof of this works when we consider coefficients in $c_0(G)$, however, the following theorem and lemma show that the reduced cohomology with coefficients in $c_0(G)$ vanishes for all groups G . We start by proving this for finitely generated groups (a fact conjectured by Gournay in [21]), then extend to all groups using the lemma.

Theorem 7.28. Let G be a finitely generated group. Then $\overline{H^1}(G, c_0(G)) = 0$.

Proof. Let $g \cdot f - f$ where $f : G \rightarrow \mathbb{R}$ be an element of $H^1(G, c_0(G))$. We can assume $f(x) \geq 0$ for all $x \in G$ by assuming f is bounded using Lemma 2.1 [Gournay] (reference back to earlier in paper) and then adding an appropriate constant.

So, assume $f \geq 0$ and let $\ell : G \rightarrow \mathbb{Z}^+$ be the traditional length function on the finite generating set S . We then define $B_n = \{x \in G : \ell(x) \leq n\}$ and $M_n = \max_{s \in S} \{\max_{x \in G \setminus B_{n-1}} \{|f(s^{-1}x) - f(x)|\}\}$. We claim that M_n is finite for all n , and that $M_n \rightarrow 0$ as $n \rightarrow \infty$.

To see this, note that $s \cdot f - f \in c_0(G)$ for all $s \in S$, so given any $\epsilon > 0$, we can find a finite subset $A \subset G$ such that $|f(s^{-1}x) - f(x)| < \epsilon$ for all $x \in G \setminus A$. But because A is finite, we can find n such that $A \subset B_n$, therefore $|f(s^{-1}x) - f(x)| < \epsilon$ for all $x \in G \setminus B_n$. Since there are only finitely many $s \in S$, we can find N such that $|f(s^{-1}x) - f(x)| < \epsilon$ for all $s \in S$ and $s \in G \setminus B_N$. Therefore, $M_N < \epsilon$, showing that $M_n \rightarrow 0$ as $n \rightarrow \infty$.

Now, given $x \in G$ and some n , let $b \in B_n \setminus B_{n-1}$ be such that $\max_{b \in B_n \setminus B_{n-1}} \{f(b) -$

$M_n \ell(xb^{-1})\} = f(b) - M_n \ell(xb^{-1})$. We define $f_n(x)$ as follows:

$$f_n(x) = \begin{cases} f(x) & : x \in B_n \\ f(b) - M_n \ell(xb^{-1}) & : f(b) - M_n \ell(xb^{-1}) > 0 \\ 0 & : \text{otherwise.} \end{cases}$$

To see that $f_n(x) \in c_0(G)$, let $b' \in B_n$ be such that $\max_{b \in B_n} \{|f(b)|\} = |f(b')|$, and let N such that $\min_{b \in B_n} \{\min_{x \in G \setminus B_N} \{\ell(xb^{-1})\}\} > |f(b')|/M_n$. Then we see that for $x \in G \setminus B_N$, $|f(b)| - M_n \ell(xb^{-1}) \leq 0$ for all $b \in B_n$, and so we have that $f_n(x) = 0$ on $G \setminus B_N$, and since B_N is finite, $f_n \in c_0(G)$.

So all that remains to be shown is that $\|(g \cdot f_n - f_n) - (g \cdot f - f)\|_\infty \rightarrow 0$ as $n \rightarrow \infty$ for all $g \in G$. It is enough to show this for all $s \in S$. So let $s \in S$ be given. First note that $f(x) = f_n(x)$ for all $x \in B_n$, so $|(f_n(s^{-1}x) - f_n(x)) - (f(s^{-1}x) - f(x))| = 0$ if $x, s^{-1}x \in B_n$.

Now assume both $x, s^{-1}x \notin B_n$. There exists $b_1 \in B_n \setminus B_{n-1}$ such that $f_n(x) = f(b_1) - M_n \ell(xb_1^{-1})$. So, we must have $f_n(s^{-1}x) \geq f(b_1) - M_n \ell(s^{-1}xb_1^{-1}) \geq f(b_1) - M_n(\ell(s^{-1}) + \ell(xb_1^{-1})) = f_n(x) - M_n$. There also exists $b_2 \in B_n \setminus B_{n-1}$ such that $f_n(s^{-1}x) = f(b_2) - M_n \ell(s^{-1}xb_2^{-1})$. So, we have $f_n(x) \geq f(b_2) - M_n \ell(xb_2^{-1}) \geq f(b_2) - M_n(\ell(s) + \ell(s^{-1}xb_2^{-1})) = f_n(s^{-1}x) - M_n$. Thus, $|f_n(s^{-1}x) - f_n(x)| \leq M_n$.

Finally, without loss of generality, assume $x \in B_n$ and $s^{-1}x \notin B_n$. Since $\ell(s^{-1}x) \leq 1 + \ell(x)$, we have that $x \in B_n \setminus B_{n-1}$, so certainly $f_n(s^{-1}x) \geq f(x) - M_n \ell(s^{-1}xx^{-1}) = f_n(x) - M_n$.

Now, there exists $b \in B_n \setminus B_{n-1}$ such that $f_n(s^{-1}x) = f(b) - M_n \ell(s^{-1}xb^{-1})$. We know $\ell(xb^{-1}) \leq 1 + \ell(s^{-1}xb^{-1})$, and since both $b, x \in G \setminus B_{n-1}$, we have that $|f(b) - f(x)| \leq \ell(xb^{-1})M_n$. Therefore, we have

$$\begin{aligned} f_n(s^{-1}x) &= f(b) - M_n \ell(s^{-1}xb^{-1}) \\ &\leq f(x) - \ell(xb^{-1})M_n - M_n \ell(s^{-1}xb^{-1}) \\ &\leq f_n(x) - M_n \end{aligned}$$

which gives us that $|f_n(s^{-1}x) - f_n(x)| = M_n$.

So, since $|f_n(s^{-1}x) - f_n(x)| \leq M_n$ for all $x \in G$, and certainly $|f(xs^{-1}) - f(x)| \leq M_n$

by the definition of M_n , we have

$$\begin{aligned} |(f_n(s^{-1}x) - f_n(x)) - (f(s^{-1}x) - f(x))| &\leq |f_n(s^{-1}x) - f_n(x)| + |f(s^{-1}x) - f(x)| \\ &\leq M_n + M_n \end{aligned}$$

Therefore, $\|(s \cdot f_n - f_n) - (s \cdot f - f)\|_\infty \leq 2M_n \rightarrow 0$ as $n \rightarrow \infty$. Thus, $g \cdot f_n - f_n$ is a sequence in B_G converging to $g \cdot f - f$ for every $g \in G$.

To finish, let $g \cdot f - f$ be an element of $H^1(G, c_0(G))$ where $f : G \rightarrow \mathbb{C}$. We can write $f(x) = u(x) + iv(x)$ where $u : G \rightarrow \mathbb{R}$ and $v : G \rightarrow \mathbb{R}$. Since $(g \cdot f - f)(x) = (g \cdot u - u)(x) + i(g \cdot v - v)(x)$, we see that $(g \cdot u - u), (g \cdot v - v) \in H^1(G, c_0(G))$, so we can find sequences $g \cdot u_n - u_n$ and $g \cdot v_n - v_n$ in B_G converging to $g \cdot u - u$ and $g \cdot v - v$. Then $g \cdot (u_n + iv_n) - (u_n + iv_n)$ will converge to $g \cdot f - f$, thus $g \cdot f - f = 0$ in $\overline{H^1}(G, c_0(G))$, concluding the proof. \blacksquare

To extend this result to all groups, we use the following lemma, which is useful in extending many results in reduced l^p -cohomology to the not necessarily finitely generated case. We prove the lemma for coefficients in $l^p(G)$ where $1 \leq p < \infty$, but the proof is easily adjusted to accommodate coefficients in $c_0(G)$. Also, to present the lemma in the greatest generality, we make the following definition:

Definition 7.29. We say that G has locally vanishing reduced l^p -cohomology (or c_0 -cohomology) if for any finite subset S there exists a finitely generated subgroup H containing S such that $\overline{H^1}(H, l^p(H)) = 0$ (or $\overline{H^1}(H, c_0(H)) = 0$). In particular, if every finitely generated subgroup of G has vanishing reduced cohomology, then G has locally vanishing reduced cohomology.

Lemma 7.30. If G has locally vanishing reduced l^p -cohomology, then $\overline{H^1}(G, l^p(G)) = 0$.

Proof. Let $f \in \mathbb{C}^G$ be such that $g \cdot f - f \in l^p(G)$ for all $g \in G$. Given an open neighborhood $\mathcal{O}(g \cdot f - f)_{g_1, \dots, g_n, \epsilon}$ of $g \cdot f - f$ in Z_G , we aim to show that we can find $g \cdot f' - f' \in B_G$ in the given neighborhood. Since any element of Z_G can be written as $g \cdot f - f$ for some $f \in \mathbb{C}^G$, this will show that $\overline{H^1}(G, l^p(G)) = 0$.

Since $g_i \cdot f - f \in l^p(G)$ for all i , we can find a finite subset S_i for each i such that $\|(g_i \cdot f - f)|_{G \setminus S_i}\|_p < \epsilon/2$. Then, letting $S = \bigcup S_i$, we have that $\|(g_i \cdot f - f)|_{G \setminus S}\|_p < \epsilon/2$ for all i . Then, let H be the finitely generated subgroup

of G containing S and the g_i such that $\overline{H^1}(H, l^p(H)) = 0$.

Note that $g \cdot f|_H - f|_H$ is an element of $H^1(H, l^p(H))$, so by the hypothesis we can find $f' \in l^p(H)$ such that $\|(g_i \cdot f' - f') - (g_i \cdot f|_H - f|_H)\|_p < \epsilon/2$ for all i .

Now, by defining $f' = 0$ on $G \setminus H$, $f' \in l^p(G)$ and we have

$$\begin{aligned} & \|(g_i \cdot f' - f') - (g_i \cdot f - f)\|_p \\ & \leq \|(g_i \cdot f' - f') - (g_i \cdot f|_H - f|_H)\|_p + \|(g_i \cdot f|_H - f|_H) - (g_i \cdot f - f)\|_p \\ & < \|(g_i \cdot f' - f') - (g_i \cdot f|_H - f|_H)\|_p + \|(g_i \cdot f - f)|_{G \setminus S}\|_p < \epsilon \end{aligned}$$

for all i since $(g_i \cdot f|_H - f|_H)(x) = (g_i \cdot f - f)(x)$ for all $x \in H$. This concludes the proof. \blacksquare

While this lemma can be used to extend several results from other papers regarding reduced cohomology, including Gournay's [21], we conclude by characterizing the c_0 -cohomology of any group G .

Corollary 7.31. Let G be a group. Then $\overline{H^1}(G, c_0(G)) = 0$.

Proof. By Theorem 7.28, given any finitely generated subgroup H , $\overline{H^1}(H, c_0(H)) = 0$. Therefore G has locally vanishing reduced c_0 -cohomology. Since Lemma 7.30 remains true for coefficients in $c_0(G)$, we have that $\overline{H^1}(G, c_0(G)) = 0$. \blacksquare

Chapter 8

Symbols and Notation

		page
RG	group ring	(3)
A^G	G -fixed elements of A	(3)
$C^0(G, A)$	n -cochains	(4)
d_n	n th coboundary homomorphism	(5)
$Z^n(G, A)$	n -cocycles	(5)
$B^n(G, A)$	n -coboundaries	(5)
$H^n(G, A)$	n th cohomology group with coefficients in A	(5)
Z_G	1-cocycles	(5)
B_G	1-coboundaries	(5)
\mathbb{C}^G	Set of functions $f : G \rightarrow \mathbb{C}$	(5)
$l^p(G)$	p -summable functions on G	(6)
$l^\infty(G)$	bounded functions on G	(6)
$c_0(G)$	functions on G vanishing at infinity	(6)
$\mathcal{N}(G)$	Von-Neumann algebra	(6)
C_{red}^*	C-star reduced algebra	(6)
$l^2l^2(G)$	Fourier-Stieltjes algebra	(6)
a	almost equivalence	(12)
Δ	symmetric difference	(12)
$e(G)$	ends of a group	(12)

Chapter 9

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