Chiral Rings of Two-dimensional Field Theories with (0,2) Supersymmetry

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ABSTRACT

This thesis is devoted to a thorough study of chiral rings in two-dimensional (0,2) theories. We first discuss properties of chiral operators in general two-dimensional (0,2) nonlinear sigma models, both in theories twistable to the A/2 or B/2 model, as well as in non-twistable theories. As a special case, we study the quantum sheaf cohomology of Grassmannians as a deformation of the usual quantum cohomology. The deformation corresponds to a (0,2) deformation of the nonabelian gauged linear sigma model whose geometric phase is associated with the Grassmannian. Combined with the classical result, the quantum ring structure is derived from the one-loop effective potential. Supersymmetric localization is also applicable in this case, which proves to be efficient in computing A/2 correlation functions. We then compute chiral operators in general (0,2) nonlinear sigma models, and apply them to the Gadde-Gukov-Putrov triality proposal, which says that certain triples of (0,2) GLSMs should RG flow to nontrivial IR fixed points. As another application, we extend previous works to construct $(0, 2)$ Toda-like mirrors to the sigma model engineering Grassmannians.

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GENERAL AUDIENCE ABSTRACT

This thesis studies a mathematical concept called the chiral ring, which emerges from string theory. String theory is a conjectured theory that potentially unifies the existing fundamental physical laws. It has connections with many branches of mathematics, especially geometry. Spacetime is ten-dimensional in string theory, of which four dimensions are visible, and the other six are hidden at ordinary energy levels. The chiral ring encodes many geometric properties of the hidden part of spacetime. These properties can in turn affect the visible universe even at low energies. Research on chiral rings has primarily focused on a special class of geometries which have large symmetries and so are easier to handle. In order to tackle more general scenarios, we analyze the chiral rings corresponding to theories with only half the symmetry and give several new results and applications.

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Contents

Chapter 1

Introduction

String theory is a theoretical framework that potentially unifies the standard model and general relativity. The basic idea is to replace traditional point particles with one-dimensional objects called strings. At scales much larger than the string length, strings with various vibration modes behave as all sorts of particles. Upon quantization, string theory gives rise to a supergravity theory at energies much lower than the Planck energy. String theory makes sense only in ten-dimensional spacetime, so in order to recover the four-dimensional spacetime we observe, six of the ten dimensions must be compactified to give a reasonable four-dimensional low energy theory. There are many choices for the internal six-dimensional space on which the theory is compactified.

There are five types of string theories, each of which includes a scalar field Φ, called the dilaton, which defines the string coupling $g_s = \exp(\Phi)$. If the string coupling is small enough, then a perturbative description of string theory is possible. There are also nonperturbative compactifications. For example, in F-theory, the dilaton is not constant and there is no way to make it small everywhere so a perturbative description is not possible. In perturbative string theory, the stringy version of quantum mechanics is described by a two-dimensional field theory with the two-dimensional space being the world sheet of the string. When computing amplitudes, contributions from different world sheets are proportional to different powers of g_s. To make matters more tractable, constraints are usually put on the internal space. For example, in heterotic string theory, if we want to preserve $N = 1$ supersymmetry in the low energy theory, the internal world sheet theory has to possess at least $N = 2$ superconformal symmetry. A large number of such superconformal theories can be described as IR fixed points of two-dimensional field theories with (2,2) or (0,2) supersymmetry such as nonlinear sigma models, Landau-Ginzburg models and gauged linear sigma models (GLSMs).

This thesis concerns chiral states and rings in (0,2) theories. Chiral rings have been extensively studied in two-dimensional (2,2) supersymmetric theories, facilitating developments in algebraic geometry and quantum field theory. They can also be defined in (0,2) supersymmetric theories (see for example [26–29, 50, 52–55, 64, 67]), and are a current research topic.

As one particularly illuminating example, we compute quantum sheaf cohomology of Grassmannians. This is the OPE ring in the A/2 model, which is the (0,2) analogue of quantum cohomology. Quantum cohomology of Grassmannians has been extensively studied and a complete description of its ring structure is known. The ring structure is computed by the A model correlation functions of the nonabelian GLSM with (2,2) supersymmetry. Since the (2,2) locus is embedded in the moduli space of (0,2) theories, it is natural to ask how the quantum cohomology ring is deformed when one goes off the (2,2) locus [50]. In the abelian case, this problem is studied in, for example [67].

The underlying space of the quantum sheaf cohomology is $H^\bullet(Gr(k,n), \wedge^\bullet \mathcal{E}^\vee)$. Here $\mathcal{E}$ is a deformation of the tangent bundle of the Grassmannian, which represents the zero modes of the left moving fermions in the (0,2) theory. This vector bundle fits into a short exact sequence which enables us to study the classical sheaf cohomology in a mathematically rigorous fashion [81]. Combining the result with the one-loop effective potential, we compute the quantum corrections to the generating relations.

Another way to study the quantum sheaf cohomology is supersymmetric localization [77, 79]. After performing the A/2 twist [50], the (0,2) analogue of the A twist, the correlation functions of the generators reduce to Jeffrey-Kirwan residues on the Coulomb branch, which greatly simplify the procedure of computing the A/2 correlation functions. As the analogue of the A-model correlation functions, these A/2 correlation functions completely determine the ring structure of the quantum sheaf cohomology.

We also apply chiral states to discuss triality proposals for two-dimensional (0,2) theories, described in [24, 25]. Specifically, those papers proposed that all points in phase diagrams of certain triples of (0,2) GLSMs should flow to the same conformal theory in the IR limit, resulting in a 'triality' relating three different two-dimensional gauge theories. Many two-dimensional gauge theory dualities can be described as different presentations of the same IR geometry, as discussed in [9], but as also observed there, triality is different – of the six total geometric phases, there are three pairs such that each pair corresponds to the same geometry, but the geometries associated to different pairs are simply different.

Chiral rings are also useful in constructing (0,2) analogue of mirror symmetry. Mirror symmetry has been of great interest to both physicists and mathematicians. In heterotic compactifications, there is a natural generalization, known as $(0,2)$ mirror symmetry. We will see that the chiral ring serves as a guidance when one tries to find the mirror dual.

We begin in chapter 2 with an overview of chiral states and rings in general (0,2) nonlinear sigma models as well as in pseudo-topological models. The correct counting of chiral states in two-dimensional (0,2) nonlinear sigma models utilizes the fact that the Fock vacuum transforms as a section of a line bundle over the target space. We also observe that even in (2,2) theories, the Fock vacuum can be a section of a nontrivial line bundle, encoding choices of spin structure on the target space. When applied to (0,2) theories, the resulting spectra are shown to satisfy basic consistency properties, such as invariance under Serre duality and invariance under dualizing bundle factors.

In chapter 3, we give a detailed derivation of quantum sheaf cohomology of Grassmannians. We first review basic ingredients of gauged linear sigma models with (0,2) supersymmetry in section 3.1. In section 3.2, we discuss general issues regarding (0,2)-deformations on the Grassmannian, its one-loop effective potential on the Coulomb branch and the localization formula. In section 3.3, we derive the classical ring in a mathematically rigorous fashion. In section 3.4, we give a representation for the quantum sheaf cohomology by combining our result in section 3.3 and the one-loop effective potential. In section 3.5, we show how to use the localization technique to compute the generating relations through some examples.

In chapter 4 we apply chiral operators to study examples of triality. We begin with a brief overview of triality, and how in general terms one can keep track of all of the global symmetries in chiral operator computations. Some of the symmetries are realized in an implicit fashion which can only be seen through Bott-Borel-Weil theorem. We also briefly A/2 and B/2 models of some examples of triality. Next, in each of the examples of triality, we compute chiral operators as representations of the global symmetry in two geometric phases in each of two GLSMs related by triality, to better understand triality and (0,2) chiral states.

In the last chapter, we construct a mirror dual of nonlinear sigma models engineering Grassmannians as well as a dual GLSM. In each case, chiral ring plays an important role in finding the dual theory. Consistency checks are made by comparing correlation functions.

Chapter 2

Overview of chiral rings in (0,2) theories

In this chapter we review general properties of chiral rings in (0,2) theories. The chiral ring is a ring of operators annihilated by half of the supersymmetry transformations, which we will explore in more detail in the following.

Let us first address chiral rings in gauged linear sigma models. These are two-dimensional gauge theories, so one might expect that chiral rings should be given as rings of gauge-invariant operators modulo relations determined by the superpotential, just as they are computed in four-dimensional gauge theories. Unfortunately, even in (2,2) theories in two dimensions, this gives an incomplete result.

Consider, for example, the (2,2) GLSM describing the quintic hypersurface in $\mathbb{P}^4$, a $U(1)$ gauge theory with five fields ϕ_i of charge $+1$ and one field p of charge -5. The chiral ring computed as above includes operators of the form

$$p\,(\text{degree 5 polynomial in } \phi_i)$$

modulo relations of the form pdG, where G is the quintic hypersurface. Certainly these form part of the chiral ring – in particular, these encode complex structure deformations in $H^{2,1}$. However, it is well-known that not all complex structure deformations of a hypersurface or complete intersection can be expressed as polynomials of the form above. Such non-algebraic complex structure deformations contribute to cohomology $H^{2,1}$ and to the chiral ring in the nonlinear sigma model. Since the (2,2) chiral ring lies in a topological subsector, in principle those same non-algebraic deformations ought to appear in the complete chiral ring of the GLSM. Unfortunately, it is not known at present how to present those elements of the chiral ring of the GLSM, those non-algebraic complex structure deformations, in terms of gauge-invariant operators.

Although we do not know how to build the complete chiral ring in a (2,2) GLSM, we do know how to build the complete chiral ring in a (2,2) nonlinear sigma model. Thus, in this chapter we shall focus on chiral rings in nonlinear sigma models.

2.1 Review of chiral rings in (2,2) theories and Fock vacuum subtleties

Let's first review the definition of chiral ring in general $N = 2$ superconformal field theories [31]. The $N = 2$ superconformal algebra is defined by the following commutation and anticommutation relations

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + (c/12)m(m^2-1)\delta_{m+n,0}, \\
&[L_n, G_r^{\pm}] = (n/2 - r)G_{n+r}^{\pm},\ [L_n, J_m] = -mJ_{m+n}, \\
&[J_m, J_n] = (c/3)m\delta_{m+n,0},\ [J_n, G_r^{\pm}] = \pm G_{n+r}^{\pm}, \\
&\{G_r^-, G_s^+\} = 2L_{r+s} - (r-s)J_{r+s} + (c/3)(r^2 - 1/4)\delta_{r+s,0}.
\end{aligned} \tag{2.1}$$

Here, L_n is the Virasoro generator, J denotes the $U(1)$ current, G is the supercurrent and c is the central charge. m, n are integers, r, s are integers in the R sector, and half-integers in the NS sector. Chiral states in the NS Hilbert space(which we focus on here for simplicity) are states such that

$$G_{-1/2}^{+}|\phi\rangle = 0, \tag{2.2}$$

and anti-chiral states are defined similarly by replacing G^+ with G^-. Primary chiral states satisfy, in addition to (2.2), the condition

$$G_{n+1/2}^{-}|\phi\rangle = G_{n+1/2}^{+}|\phi\rangle = 0, n \geqslant 0.$$

From the $N = 2$ algebra, one can show that the conformal dimension h satisfies the BPS bound $h \geqslant |q|/2$, where q is the $U(1)$ R-charge. For a primary chiral state the dimension h is one-half its charge q, i.e. $h = q/2$. Similarly, $h = -q/2$ for an anti-chiral primary state. We can define chiral primary operators using the correspondence between operators and states $|\mathcal{O}\rangle = \mathcal{O}|0\rangle$.

Now let's consider the operator algebra of primary chiral operators. For chiral primary operators $\mathcal{O}_1$ and $\mathcal{O}_2$, the product is defined to be

$$\mathcal{O}_1\mathcal{O}_2(z) = \lim_{z' \to z} \mathcal{O}_1(z')\mathcal{O}_2(z). \tag{2.3}$$

The limit is well defined because the $U(1)$ charge of the operators is additive, and the conformal dimensions satisfy

$$h_{\mathcal{O}_1\mathcal{O}_2} \geqslant \frac{1}{2}(q_{\mathcal{O}_1} + q_{\mathcal{O}_2}) = h_{\mathcal{O}_1} + h_{\mathcal{O}_2}. \tag{2.4}$$

Note that the product of two chiral fields is again chiral because

$$G^+_{-1/2}\cdot \mathcal{O}_1\mathcal{O}_2(z) = \frac{1}{2\pi i}\oint_{|w-z|=\varepsilon} dw \lim_{z'\to z} G^+(w)\mathcal{O}_1(z')\mathcal{O}_2(z) = 0.$$

If the product is primary, (2.4) becomes an equality and the singularity in (2.3) is absent because the lowest order term is proportional to $(z-z')^{h_{\mathcal{O}_1\mathcal{O}_2}-h_{\mathcal{O}_1}-h_{\mathcal{O}_2}}$. Otherwise, the operator product defined above is zero. Thus we see (2.3) defines a ring structure on the set of chiral primary operators. This ring is called the chiral ring. The anti-chiral ring is defined similarly using anti-chiral operators.

One can continuously connect the NS sector to the R sector by rotating the boundary conditions, using the spectral flow operator U_θ. The $N=2$ superconformal algebra flows to an isomorphic algebra under the spectral flow:

$$\begin{aligned} U_\theta L_n U_\theta^{-1} &= L_n + \theta J_n + (c/6)\theta^2\delta_{n,0}, \\ U_\theta J_n U_\theta^{-1} &= J_n + (c/3)\theta\delta_{n,0}, \\ U_\theta G^\pm_r U_\theta^{-1} &= G^\pm_{r\pm\theta}. \end{aligned}$$

We see that the spectral flow interpolates between NS and R sectors if θ is a half-integer. If the theory has $N=(2,2)$ superconformal symmetry, i.e. it has $N=2$ superconformal symmetry in both left and right moving sectors, there are two inequivalent rings, the left-chiral right-chiral operators and left-anti-chiral right-chiral operators, which are called (c,c) and (a,c) chiral rings respectively. Under spectral flow, they can be identified with ground state operators in the RR sector. The left and right spectral flow parameters are the same in the case of (c,c) chiral ring, and opposite in sign in the case of (a,c) chiral ring.

Now let us review the pertinent aspects of chiral rings in (2,2) supersymmetric nonlinear sigma models, focusing on some subtleties in Fock vacua that do not seem to be widely appreciated but which will play a crucial role in the (0,2) generalization.

Consider a (2,2) nonlinear sigma model on a complex Kähler manifold X. Let $\phi:\Sigma\to X$ denote the worldsheet scalars with Σ being a Riemann surface. If we pick local coordinates $z,\bar z$ on Σ and ϕ^I on X and let K, $\overline{K}$ be the canonical and anti-canonical line bundles of Σ, then ψ^I_+ is a section of $K^{1/2}\otimes\phi^*(TX)$ and ψ^I_- is a section of $\overline{K}^{1/2}\otimes\phi^*(TX)$. The action of the nonlinear sigma model is

$$L = 2t\int d^2z\left(\frac{1}{2}g_{IJ}(\phi)\partial_z\phi^I\partial_{\bar z}\phi^J + i\psi^{\bar i}_- D_z\psi^i_- g_{\bar i i} + i\psi^{\bar i}_+ D_{\bar z}\psi^i_+ g_{\bar i i} + R_{i\bar i j\bar j}\psi^i_+\psi^{\bar i}_+\psi^j_-\psi^{\bar j}_-\right), \quad (2.5)$$

where t is the coupling constant, and R is the Riemann curvature of X, D is the covariant derivative. The supersymmetry transformation of the model is defined by

$$
\begin{aligned}
\delta\phi^i &= i\alpha_-\psi_+^i + i\alpha_+\psi_-^i, \\
\delta\phi^{\bar{i}} &= i\tilde{\alpha}_-\psi_+^{\bar{i}} + i\tilde{\alpha}_+\psi_-^{\bar{i}}, \\
\delta\psi_+^i &= -\tilde{\alpha}_-\partial_z\phi^i - i\alpha_+\psi_-^j\Gamma^i_{jm}\psi_+^m, \\
\delta\psi_+^{\bar{i}} &= -\alpha_-\partial_z\phi^{\bar{i}} - i\tilde{\alpha}_+\psi_-^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\psi_+^{\bar{m}}, \\
\delta\psi_-^i &= -\tilde{\alpha}_+\partial_{\bar{z}}\phi^i - i\alpha_-\psi_+^j\Gamma^i_{jm}\psi_-^m, \\
\delta\psi_-^{\bar{i}} &= -\alpha_+\partial_{\bar{z}}\phi^{\bar{i}} - i\tilde{\alpha}_-\psi_+^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\psi_-^{\bar{m}},
\end{aligned}
$$

where $\alpha_\pm$ and $\tilde{\alpha}_\pm$ are fermionic parameters of the infinitesimal transformation. The chiral ring we will generalize to (0,2) is the ring of states in (R,R) sectors[1], which following standard methods (see *e.g.* [31]) are of the form

$$
b_{\bar{\imath}_1,\cdots,\bar{\imath}_n,j_1,\cdots,j_m}(\phi)\psi_+^{\bar{\imath}_1}\cdots\psi_+^{\bar{\imath}_n}\psi_-^{j_1}\cdots\psi_-^{j_m}|0\rangle.
$$

The factor

$$
b_{\bar{\imath}_1,\cdots,\bar{\imath}_n,j_1,\cdots,j_m}(\phi)\psi_+^{\bar{\imath}_1}\cdots\psi_+^{\bar{\imath}_n}\psi_-^{j_1}\cdots\psi_-^{j_m}
$$

represents an (m,n)-form, so it has a standard understanding in terms of the Dolbeault cohomology of the space. What we will need in our discussion of (0,2) chiral rings, and may be less widely understood, is that the Fock vacuum $|0\rangle$ might also couple to a nontrivial bundle on the target space[2] and contribute to the state counting. This phenomenon has also been discussed in [7, 33, 34], but as this phenomenon may be obscure, to make this thesis self-contained, we will briefly review it here. We will also take this opportunity to describe how the same phenomenon arises even in (2,2) theories, in describing spin structures on target spaces.

2.1.1 Fock vacua coupling to nontrivial bundles on the target

This subsection was published in [99]. In a chiral R sector, the Fock vacuum couples to $(K_X)^{\pm 1/2}$ in general, where K_X is the canonical line bundle of X. This follows from the

[1] Usually we compute states in R sectors rather than NS because the low-energy states can be obtained via a cohomology computation. For example, from [31][section 2], in an $N = 2$ SCFT, in the R sector

$$
L_0 \;=\; \{G_0^-, G_0^+\} \;+\; c/12,
$$

and so $L_0 - c/12$ lives in cohomology, whereas in the NS sector,

$$
L_0 \;=\; (1/4)\{G^-_{-1/2}, G^+_{+1/2}\} \;+\; (1/4)\{G^-_{+1/2}, G^-_{-1/2}\},
$$

which is not a cohomology computation, but rather an analogue of a harmonic representative computation. The two are related by supersymmetry and should give equivalent results, but the cohomology computation is significantly simpler, especially in the absence of an explicit target space metric.

[2] We should distinguish this from Fock vacua coupling to nontrivial bundles over a moduli space of SCFT's, say, which is well-known, see for example [32, 100].

usual multiplicity of Fock vacua in the presence of periodic fermions. Schematically, if we define two vacua $|0\rangle$, $|0\rangle'$ by

$$\psi_0^i |0\rangle' \;=\; 0, \quad \psi_0^{\bar{\imath}} |0\rangle \;=\; 0,$$

then

$$|0\rangle' \;=\; \left(\prod_i \psi_0^i \right) |0\rangle, \quad |0\rangle \;=\; \left(\prod_{\bar{\imath}} \psi_0^{\bar{\imath}} \right) |0\rangle'.$$

Since

$$\prod_i \psi_0^i \;\sim\; K_X^{-1}, \quad \prod_{\bar{\imath}} \psi_0^{\bar{\imath}} \;\sim\; K_X,$$

then, just as in fractional charges, $|0\rangle'$ transforms as a section of $(K_X)^{-1/2}$ and $|0\rangle$ transforms as a section of $(K_X)^{+1/2}$.

In the case of a (2,2) theory, the Fock vacuum couples to

$$(K_X)^{+1/2} \otimes (K_X)^{-1/2}, \tag{2.6}$$

one factor for left-movers, the other for right-movers.

If X is simply-connected, the square roots $(K_X)^{\pm 1/2}$, if they exist, are uniquely determined, and the Fock vacuum couples to a trivial bundle. (If they do not exist in a given sector, then that sector is physically inconsistent.)

More generally, if X is not simply-connected, then there will be multiple different square roots $(K_X)^{\pm 1}$. These different choices of square roots correspond to different choices of spin structure on the target space X, as spinors on a complex Kähler manifold can be expressed in the form [38][section II.3]

$$\wedge^{\bullet} TX \otimes (K_X)^{+1/2}.$$

If $X = T^2$, for example, the different square roots $(K_X)^{1/2}$ simply correspond to choices of periodic and antiperiodic boundary conditions around the legs of the torus.

To compute the chiral ring, we must specify two square roots, one for left-movers, another for right-movers. Which square roots should appear, associated to left- and right-movers, is part of the specification of the nonlinear sigma model. In other words, just as one must specify a metric and B field on X in order to define a nonlinear sigma model, if X is not simply-connected then in addition one must also specify a spin structure on X, and that choice of spin structure enters worldsheet physics via Fock vacua, as above.

In this thesis, we will work with simply-connected spaces. However, in the (0,2) case, Fock vacua will couple to bundles of the form $(\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2}$, and so even if there is no ambiguity, the bundle can be nontrivial. More generally, suppose $|0\rangle \in \Gamma(L)$ for some line bundle L. If L has no sections at all, this merely implies that the Fock vacuum is not BRST-closed: we still have a Fock vacuum, it merely does not define a state by itself, analogous to tachyonic states projected out of closed bosonic string spectra. At a different extreme, if L

admits multiple sections, this is merely another source of multiplicity beyond that provided by Fermi zero modes.

For completeness, let us explore some of the implications of the statement above on the (2,2) locus.

If X is Calabi-Yau, so that K_X is trivial, then there is a canonical[3] trivial square root, specifically $(K_X)^{+1/2} = \mathcal{O}_X$. (If K_X is nontrivial, then in general there will not be any canonical choice of square root, if square roots in fact exist.) Only in that canonical trivial spin structure on a Calabi-Yau does there exist a nowhere-zero covariantly constant spinor. (For example, on T^2, only in the (R,R) spin structure does $K^{1/2}$ have a section.)

Since the anomaly in the left and right $U(1)_R$ symmetries is determined by K_X and not the spin structure, if X is Calabi-Yau this theory should flow to a nontrivial (2,2) SCFT, even if one chooses nontrivial left- or right-spin structures. However, the target space string theories are not likely well-defined if either spin structure is nontrivial [35, 36]. Even if the target space string theory is well-defined, spacetime supersymmetry must surely be broken if either spin structure is nontrivial. In an SCFT associated to a Calabi-Yau compactification, there is an isomorphism between R and NS sector states: spectral flow rotates one into the other. If $(K_X)^{1/2}$ is nontrivial, however, that isomorphism is broken. After all, the Fock vacuum ambiguity which leads to this interpretation in terms of $(K_X)^{1/2}$ only exists in the R sector, not the NS sector, and so the states that arise in the R sector are necessarily distinct[4]. Put another way, spectral flow relates NS sector states to the R sector states associated with trivial spin structures, instead of the given spin structures. In effect, this is a further condition for spacetime supersymmetry in $N = 2$ SCFT's, beyond the familiar statement that the difference of left and right charges should be integral [31][Eq. (2.3)].

In orbifolds, an example of the effect of having $(K_X)^{1/2}$ nontrivial while K_X is trivial is given by the Scherk-Schwarz mechanism [39], which breaks supersymmetry in orbifolds by assigning different boundary conditions to fermions than to bosons. (In full string theories, such boundary conditions might also contribute to *e.g.* failures of level-matching or modular invariance, but to decide the matter, one would need to specify the rest of the CFT needed for a critical string.)

[3] More generally, the nth roots of the structure sheaf form a finite group which is canonically $\mathrm{Hom}(\pi_1(X), \mu_n)$ where μ_n is the group of nth roots of unity.

[4] The spectral flow operator that takes NS to the relevant R and conversely has $\theta = 1/2$ in the notation of [31], and charge $\pm c/6$, and formally would be associated to a square root of the canonical bundle. That spectral flow operator would be different from the one in [31], which maps NS to the R sector for the canonical spin structure. The one relevant here could not be expressed merely as the exponential of a boson, unlike the one in [31].

2.1.2 A, B model topological field theories and non-simply-connected targets

The discussion of the Fock vacuum and counting of chiral states in this subsection was published in [99].

It is often useful to twist $N=2$ supersymmetric theories to topological field theories [40,41]. Let's denote the generators of the left and right $U(1)$ R-symmetries as J_L and J_R respectively and define $J_V = J_L - J_R$, $J_A = J_L + J_R$. Twisting replaces the group $U(1)_r$ of worldsheet rotation generated by M by the diagonal subgroup of $U(1)_r \times U(1)_V$ or $U(1)_r \times U(1)_A$, considering $M + J_V$ or $M + J_A$ as the new generator of the rotation group. The former is known as the A twist, and the latter is known as the B twist. Due to anomaly considerations, for a non-linear sigma model possibly with a superpotential, A twist is possible when the superpotential is scale invariant, B twist is possible when the first Chern class of the target space vanishes.

Now let's consider A and B twisted non-linear sigma models. In the A model, ψ_+^i and $\psi_-^{\bar{i}}$ are sections of $\phi^*(T^{1,0}X)$ and $\phi^*(T^{0,1}X)$ respectively. It is customary to combine them into a section χ of $\phi^*(TX)$. $\psi_+^{\bar{i}}$ is a (1,0) form on Σ with values in $\phi^*(T^{0,1}X)$, we denote it as $\psi_z^{\bar{i}}$. On the other hand, ψ_-^i is a (0,1) form with values in $\phi^*(T^{1,0}X)$, and will be denoted as $\psi_{\bar{z}}^i$.

The BRST transformation is

$$\begin{aligned}
\delta\phi^i &= i\alpha\chi^i, \\
\delta\phi^{\bar{i}} &= i\alpha\chi^{\bar{i}}, \\
\delta\chi^i &= \delta\chi^{\bar{i}} = 0, \\
\delta\psi_z^{\bar{i}} &= -\alpha\partial_z\phi^{\bar{i}} - i\alpha\chi^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\psi_z^{\bar{m}}, \\
\delta\psi_{\bar{z}}^i &= -\alpha\partial_{\bar{z}}\phi^i - i\alpha\chi^j\Gamma^i_{jm}\psi_{\bar{z}}^m,
\end{aligned} \tag{2.7}$$

where α is the infinitesimal fermionic parameter. Let's denote by Q the generator of this transformation, i.e. $\delta\mathcal{O} = -i\alpha\{Q,\mathcal{O}\}$ for any operator $\mathcal{O}$. Note that $Q^2 = 0$.

Let $W = W_{I_1 I_2 \cdots I_n}(\phi) d\phi^{I_1} d\phi^{I_2} \cdots d\phi^{I_n}$ be an n-form on X, we can define a local operator associated with W

$$\mathcal{O}_{W(P)} = W_{I_1 I_2 \cdots I_n}\chi^{I_1}\chi^{I_2}\cdots\chi^{I_n}(P).$$

From (2.7), it is easy to see

$$\{Q, \mathcal{O}_W\} = -\mathcal{O}_{dW},$$

with d the exterior derivative. Thus the BRST cohomology of the A model is equivalent to the de Rham cohomology of X.

On the other hand, the B twist is possible only when the first Chern class of X vanishes[5], i.e. when X is a Calabi-Yau space. In the B model, $\psi_\pm^{\bar{i}}$ are sections of $\phi^*(T^{0,1}X)$, while ψ_+^i

[5]On a Riemann surface without boundary, the B twist only requires that K_X^2 be trivial, not necessarily K_X itself, see [54] appendix A.

is a section of $K \otimes \phi^*(T^{1,0}X)$, and ψ_-^i is a section of $\overline{K} \otimes \phi^*(T^{1,0}X)$. Let's define

$$\eta^{\bar{i}} = \psi_+^{\bar{i}} + \psi_-^{\bar{i}},$$
$$\theta_i = g_{i\bar{i}}(\psi_+^{\bar{i}} - \psi_-^{\bar{i}}).$$

Also, we combine $\psi_\pm^i$ into a one form ρ with values in $\phi^*(T^{1,0}X)$.

The BRST transformation laws are

$$\delta\phi^i = 0,$$
$$\delta\phi^{\bar{i}} = i\alpha\eta^{\bar{i}},$$
$$\delta\eta^{\bar{i}} = \delta\theta_i = 0,$$
$$\delta\rho^i = -\alpha d\phi^i.$$

For BRST operators, we consider $(0, p)$ forms on X with values in $\wedge^q T^{1,0}X$, such an object can be written

$$V = V_{\bar{i}_1\bar{i}_2\cdots\bar{i}_p}^{j_1 j_2\cdots j_q} d\bar{z}^{i_1} d\bar{z}^{i_2} \cdots d\bar{z}^{i_p} \frac{\partial}{\partial z_{j_1}} \cdots \frac{\partial}{\partial z_{j_q}}.$$

With any such V, we can define the corresponding operator

$$\mathcal{O}_V = V_{\bar{i}_1\bar{i}_2\cdots\bar{i}_p}^{j_1 j_2\cdots j_q} \eta^{\bar{i}_1}\eta^{\bar{i}_2} \cdots \eta^{\bar{i}_p}\theta_{j_1} \cdots \theta_{j_q}.$$

One finds that

$$\{Q, \mathcal{O}_V\} = -\mathcal{O}_{\bar{\partial} V},$$

and thus $V \longrightarrow \mathcal{O}_V$ gives an isomorphism from $\oplus_{p,q} H^p(X, \wedge^q T^{1,0}X)$ to the BRST cohomology of the B model.

The correlation function of any Q-exact operator is zero, thus for any Q-closed operator $\mathcal{O}$, $\langle \mathcal{O}\{Q, \mathcal{O}'\}\rangle = 0$ for any operator $\mathcal{O}'$. It can be shown that the energy-momentum tensor operator is Q-exact in both A model and B model. Thus both of these models are independent of the worldsheet metric. Using the same argument, one can show that A model is independent of the complex structure of X, while B model is independent of the Kähler class of X.

For A model, (2.5) can be written as

$$L = it \int_\Sigma d^2 z\{Q, V\} + t \int_\Sigma \phi^*(K),$$

where

$$V = g_{i\bar{j}} \left(\psi_z^{\bar{i}} \partial_{\bar{z}} \phi^j + \partial_z \phi^{\bar{i}} \psi_{\bar{z}}^j \right),$$

and K is the Kähler form $-i g_{i\bar{j}} dz^i dz^{\bar{j}}$. Thus, for any Q-closed operator $\mathcal{O}$, its correlation function can be written as a sum

$$\langle \mathcal{O} \rangle = \sum_n e^{-2\pi n t} \int_{B_n} D\phi D\chi D\psi e^{-it\{Q, \int V\}} \mathcal{O},$$

where the sum is taken over the degrees of the map, B_n is the component of the field space for maps of degree n. We see the path integral with a fixed degree is independent of t, thus it can be computed in the large t limit. In this limit, the path integral localizes to the holomorphic maps, i.e. maps obeying

$$\partial_{\bar{z}}\phi^i = \partial_z \phi^{\bar{i}} = 0.$$

For B model, (2.5) can be written as

$$L = it \int_\Sigma d^2z \{Q, V\} + tW,$$

where

$$V = g_{i\bar{j}} \left(\rho^i_z \partial_{\bar{z}} \phi^{\bar{j}} + \rho^i_{\bar{z}} \partial_z \phi^{\bar{j}} \right),$$

and

$$W = \int_\Sigma \left(-\theta_i D\rho^i - \frac{i}{2} R_{i\bar{i}j\bar{j}} \rho^j \eta^{\bar{i}} \theta_k g^{k\bar{j}} \right).$$

Here D is the exterior derivative on Σ. Under a change of t, the first term picks up a Q-exact term, while the second term is invariant through a redefinition $\theta \to \theta/t$. Thus, the same arguments apply to B model, showing that the path integral localizes to maps obeying

$$\partial_{\bar{z}}\phi^i = \partial_z \phi^i = 0,$$

i.e. constant maps. In sum, correlation functions in the A model receive nonperturbative corrections labeled by the degree of the map, correlation functions in the B model do not receive any quantum correction.

For completeness, let us also briefly discuss the A and B model topological field theories for non-simply-connected target spaces. In both cases, the Fock vacuum couples to the ratio of square roots in equation (2.6). In conventional treatments of the A and B models such as [41], the two square roots are assumed identical, so that the bundle is trivial. (If X does not admit a spin structure, then $(K_X)^{1/2}$ does not exist as an honest bundle, only a 'twisted' bundle; however, by formally identifying the contributions from left- and right-moving sectors, we can still make sense of the RR sector, and hence the topological field theories.)

More generally, if the square roots are not identical, then the Fock vacuum couples to a nontrivial bundle. The two topological field theories appear to still be well-defined, but their interpretations are slightly different. The operators in the A model continue to be counted by

$$H^q(\Omega^p_X),$$

but the states are now counted by

$$H^q\left(\Omega^p_X \otimes (K_X)^{+1/2} \otimes (K_X)^{-1/2}\right).$$

Similarly, the operators in the B model continue to be counted by

$$H^q(\wedge^p TX),$$

but the states are now counted by

$$H^q\left(\wedge^p TX \otimes (K_X)^{+1/2} \otimes (K_X)^{-1/2}\right).$$

We do not interpret this as a violation of the state-operator correspondence, which refers to the $SL(2,\mathbb{C})$-invariant NS-NS vacuum, but instead in terms of spectral flow. For example, when A model three-point correlation functions are interpreted in a physical theory, the physical correlation function takes the form

$$(\text{spacetime spinor})\,(\text{spacetime boson})\,(\text{spacetime spinor})\,,$$

where the spinor structure is encoded in the Fock vacuum. We interpret the issue above similarly.

That said, correlation functions of local observables are unchanged by the choices of target-space spin structure, as the combination of $|0\rangle$ and $\langle 0|$ result in a factor of

$$\left((K_X)^{+1/2} \otimes (K_X)^{-1/2}\right)^2 \;\cong\; K_X \otimes K_X^{-1} \;\cong\; \mathcal{O}_X.$$

It is possible that nonlocal[6] observables may be able to detect the spin structure.

2.1.3 Quantum cohomology

Given two operators in the A-model, their OPE is again a BRST closed operator, so the A-model OPE defines a ring structure which is known as quantum cohomology of the target space. Since the A-model operators are counted by $H^q(\Omega^p_X)$, we see quantum cohomology is a deformation of the classical cohomology. Given a basis $T_i, i = 1, \cdots, \dim X$ of $H^\bullet(X,\mathbb{Q})$, let T^j be its dual basis with respect to the inner product defined by $g_{ij} = \int_X T_i \wedge T_j$. Let ω denote the Kähler form of the target space X, for any $\beta \in H_2(X,\mathbb{Z})$, define $q^\beta = e^{2\pi i \int_\beta \omega}$, then the operator product of $\mathcal{O}_1$ and $\mathcal{O}_2$ is

$$\mathcal{O}_1\mathcal{O}_2 = \sum_i \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_{T_i}\rangle T^i.$$

As we have seen in the last section, the correlation function can always be written as

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_{T_i}\rangle = \sum_{\beta\in H_2(X,\mathbb{Z})} \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_{T_i}\rangle_\beta q^\beta,$$

which is a sum over all instanton sectors. Since the path integral of A model is localized on holomorphic maps $\phi : \Sigma \longrightarrow X$, the moduli space of such maps is divided into different instanton sectors counted by $H_2(X,\mathbb{Z})$, so we have a contribution from each sector in the sum above. We will see this ring structure can be generalized to quantum sheaf cohomology of (0,2) theories.

[6] We would like to thank H. Jockers for observing this possibility.

2.2 (0,2) chiral states

The discussion of the counting of chiral states in this section was published in [99]. In (0,2) nonlinear sigma models, the left moving fermions couple to a holomorphic vector bundle $\mathcal{E}$ over X. That means other than right moving fermions ψ_+ coupling to $K_\Sigma^{1/2} \otimes \phi^* TX$, we have left moving fermions λ_- coupling to $\overline{K}_\Sigma^{1/2} \otimes \phi^* \bar{\mathcal{E}}$. Also, we need to fix a Hermitian metric on $\mathcal{E}$ and the corresponding curvature F. The action of the theory is

$$L = 2t \int d^2 z \left(\frac{1}{2} g_{IJ}(\phi) \partial_z \phi^I \partial_{\bar{z}} \phi^J + i \lambda_-^{\bar{a}} D_z \lambda_-^b g_{\bar{a}b} + i \psi_+^{\bar{i}} D_{\bar{z}} \psi_+^j g_{\bar{i}j} + R_{i\bar{j}a\bar{b}} \psi_+^i \psi_+^{\bar{i}} \lambda_-^a \lambda_-^{\bar{b}} \right).$$

Let us denote the connection form of $\mathcal{E}$ by A, then the supersymmetry transformations have the following form[7]

$$\begin{aligned}
\delta\phi^i &= i\alpha_- \psi_+^i, \\
\delta\phi^{\bar{i}} &= i\tilde{\alpha}_- \psi_+^{\bar{i}}, \\
\delta\psi_+^i &= -\tilde{\alpha}_- \partial_z \phi^i, \\
\delta\psi_+^{\bar{i}} &= -\alpha_- \partial_z \phi^{\bar{i}}, \\
\delta\lambda_-^a &= -i\alpha_- \psi_+^j A_{jc}^a \lambda_-^c, \\
\delta\lambda_-^{\bar{a}} &= -i\tilde{\alpha}_- \psi_+^{\bar{j}} A_{\bar{j}\bar{c}}^{\bar{a}} \lambda_-^{\bar{c}}.
\end{aligned}$$

Briefly, the (0,2) chiral states in which we are interested are the "massless" or zero-energy elements of the ring of states annihilated by a right supercharge. The set of all states annihilated by a right supercharge, an infinite tower, forms a ring. In this thesis, we will focus on the "massless" elements of that ring, which form a finite-dimensional subset.

The elements of the right-chiral ring with fixed conformal dimension need not form a ring. Surprisingly, however, under certain circumstances [55] it can be shown that the OPE's nevertheless close into themselves. Specifically, if the bundle rank is less than eight, then the massless chiral states in the A/2 model will close into a ring, at least in patches on the moduli space.

Let us make explicit what we mean by the (0,2) chiral states in a (0,2) nonlinear sigma model

[7] In more general (0,2) theories with nonzero E and J functions (E and J are holomorphic sections of $\mathcal{E}$ and $\mathcal{E}^\vee$ respectively such that their pairing vanishes), the action includes terms depending on E and J. The transformation law of λ is modified to

$$\begin{aligned}
\delta\lambda_-^a &= -i\alpha_- \psi_+^j A_{jc}^a \lambda_-^c + i\alpha_- h^{a\bar{b}} \overline{F}_{\bar{b}} + i\tilde{\alpha}_- E^a, \\
\delta\lambda_-^{\bar{a}} &= -i\tilde{\alpha}_- \psi_+^{\bar{j}} A_{\bar{j}\bar{c}}^{\bar{a}} \lambda_-^{\bar{c}} + i\tilde{\alpha}_- h^{\bar{a}b} \overline{F}_b + i\alpha_- \overline{E}^{\bar{a}},
\end{aligned}$$

where h denotes the metric of $\mathcal{E}$.

on a space X with holomorphic vector bundle $\mathcal{E}$, satisfying the conditions[8]

$$\mathrm{ch}_2(\mathcal{E}) \;=\; \mathrm{ch}_2(TX), \quad c_1(\mathcal{E}) \;\equiv\; c_1(TX) \bmod 2,$$

which extend the usual Green-Schwarz anomaly cancellation condition of heterotic string theory.

In the (R,R) sector, slightly generalizing the old analysis of [42], the states in the worldsheet theory are of the form

$$b_{\bar{\imath}_1,\cdots,\bar{\imath}_n,a_1,\cdots,a_m}(\phi)\lambda_-^{a_1}\cdots\lambda_-^{a_m}\psi_+^{\bar{\imath}_1}\cdots\psi_+^{\bar{\imath}_n}|0\rangle$$

for the Fock vacuum defined by

$$\psi_+^i|0\rangle \;=\; 0 \;=\; \lambda_-^{\overline{a}}|0\rangle.$$

The Fock vacuum defined as above transforms as a section of the bundle

$$(\det\mathcal{E})^{-1/2}\otimes K_X^{+1/2},$$

essentially as a consequence of its fractional charges under global symmetries, as discussed in section 2.1. Following standard methods (for example [42]), since the right supercharge can be identified with $\overline{\partial}$, the states above realize a Dolbeault representation of the sheaf cohomology groups

$$H^n\left(X, (\wedge^m\mathcal{E})\otimes(\det\mathcal{E})^{-1/2}\otimes K_X^{+1/2}\right).$$

(This is a special case of the general result for massless spectra of heterotic strings on stacks described in [33][appendix A].)

The ratio of square roots will exist whenever[9]

$$c_1(\mathcal{E}) \;\equiv\; c_1(TX) \text{ mod } 2,$$

[8] The second condition suffices to define the theory in an (R,R) sector. In more general sectors, one would need to separately require that $c_1(\mathcal{E})$ and $c_1(TX)$ vanish mod 2; however, in this thesis we will only be concerned with the RR sector.

[9] In GLSMs, the analogous constraint for a single $U(1)$ would be the statement

$$\sum_\alpha q_{L,\alpha} \;\equiv\; \sum_\beta q_{R,\beta} \text{ mod } 2,$$

relating the sum of charges of left- and right-moving fields. However, note that since

$$\sum_\alpha q_\alpha^2 \;\equiv\; \sum_\alpha q_\alpha \text{ mod } 2,$$

the anomaly cancellation condition

$$\sum_\alpha q_{L,\alpha}^2 \;=\; \sum_\beta q_{R,\beta}^2$$

implies the statement above. See also [33][appendix A.4] for a discussion of this condition as it appears in orbifolds and related theories.

which is typically taken as a consistency condition on heterotic nonlinear sigma models. (In fact, to make sense of the (R,NS) and (NS,R) sectors, we must require that $\det \mathcal{E}$ and K_X separately admit square roots, which requires $c_1(E) \equiv 0$ mod 2 and separately $c_1(TX) \equiv 0$ mod 2. For our purposes in this thesis, we will focus on (R,R) sectors, and so the condition above suffices.) Thus, square roots will exist in cases of interest.

As an aside, in a typical perturbative heterotic compactification, it is taken that both K_X and $\det \mathcal{E}$ are trivial. In this case, each has a canonical trivial square root.

Now, beyond the ambiguities just described, there are different choices one could make for R sector Fock vacua. For example, we could instead consider the Fock vacuum defined by

$$\psi_+^i |0\rangle \;=\; 0 \;=\; \lambda_-^a |0\rangle,$$

which instead couples to

$$(\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2}.$$

In this case, states would be enumerated in the form

$$b_{\bar{\imath}_1,\cdots,\bar{\imath}_n,\bar{a}_1,\cdots,\bar{a}_m} \lambda_-^{\bar{a}_1} \cdots \lambda_-^{\bar{a}_m} \psi_+^{\bar{\imath}_1} \cdots \psi_+^{\bar{\imath}_n} |0\rangle,$$

and counted by

$$H^n \left(X, (\wedge^m \mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2} \right).$$

However, the choice of Fock vacuum should not change the states, and that is reflected in mathematical dualities. For example, using the fact that

$$\wedge^m \mathcal{E}^* \;=\; (\wedge^{r-m} \mathcal{E}) \otimes (\det \mathcal{E}^*)$$

(for r the rank of $\mathcal{E}$), it is easy to check that

$$H^n \left(X, (\wedge^m \mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2} \right) \;=\; H^n \left(X, (\wedge^{r-m} \mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2} \right),$$

and so we see that these are merely two different descriptions of the same set of states:

$$H^\bullet \left(X, (\wedge^\bullet \mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2} \right) \;=\; H^\bullet \left(X, (\wedge^\bullet \mathcal{E}^*) \otimes (\det \mathcal{E})^{+1/2} \otimes K_X^{+1/2} \right). \tag{2.8}$$

Thus, the choice of conventions in picking Fock vacua do not alter the set of states.

Let us list a few consistency checks:

- In the A/2 model, where $\det \mathcal{E}^* \cong K_X$, the states should be counted by $H^\bullet(X, \wedge^\bullet \mathcal{E}^*)$, and in the B/2 model, where $\det \mathcal{E} \cong K_X$, the states should be counted by $H^\bullet(X, \wedge^\bullet \mathcal{E})$.

- On the (2,2) locus, the states should be counted by

$$H^\bullet(X, \wedge^\bullet T^*X) \;=\; H^\bullet(X, \wedge^\bullet TX \otimes K_X).$$

- The states should be counted by sheaf cohomology groups that close into themselves under Serre duality.

- The structure above is compatible with the left Ramond elliptic genus of (0,2) nonlinear sigma models. As discussed in *e.g.* [9, 43, 44], the leading term in the elliptic genus of a sigma model on X with bundle $\mathcal{E}$ is proportional to

$$\int_X \hat{A}(TX) \wedge \mathrm{ch}\left((\det \mathcal{E})^{-1/2} \wedge_{-1} \mathcal{E} \right) \; = \; \int_X \mathrm{td}(TX) \wedge \mathrm{ch}\left(K_X^{+1/2} \otimes (\det \mathcal{E})^{-1/2} \wedge_{-1} \mathcal{E} \right),$$

  which is the Hirzebruch-Riemann-Roch index appropriate for the sheaf cohomology groups above.

- The states should be invariant under $\mathcal{E} \mapsto \mathcal{E}^*$ (which should swap the A/2, B/2 models) (see *e.g.* [54]).

- If the bundle $\mathcal{E}$ is reducible, the states should be invariant under separately dualizing factors. We can check this explicitly as follows. Suppose $\mathcal{E}$ splits holomorphically, $\mathcal{E} = \mathcal{A} \oplus \mathcal{B}$, of ranks r_1, r_2, respectively. Using

$$\wedge^{\bullet} \mathcal{E} \; = \; \sum_{i+j=\bullet} \wedge^i \mathcal{A} \otimes \wedge^j \mathcal{B},$$

  we have

$$\begin{aligned} & H^{\bullet}\left(X, (\wedge^i \mathcal{A}) \otimes (\wedge^j \mathcal{B}) \otimes (\det \mathcal{A})^{-1/2} \otimes (\det \mathcal{B})^{-1/2} \otimes K_X^{+1/2} \right) \\ & \quad = \; H^{\bullet}\left(X, (\wedge^i \mathcal{A}) \otimes (\wedge^{r_2 - j} \mathcal{B}^*) \otimes (\det \mathcal{B}) \otimes (\det \mathcal{A})^{-1/2} \otimes (\det \mathcal{B})^{-1/2} \otimes K_X^{+1/2} \right) \\ & \quad = \; H^{\bullet}\left(X, (\wedge^i \mathcal{A}) \otimes (\wedge^{r_2 - j} \mathcal{B}^*) \otimes (\det \mathcal{A})^{-1/2} \otimes (\det \mathcal{B}^*)^{-1/2} \otimes K_X^{+1/2} \right). \end{aligned}$$

  Thus, the spectrum remains invariant if we replace $\mathcal{A} \oplus \mathcal{B}$ by $\mathcal{A} \oplus \mathcal{B}^*$. At some level, this reflects the fact that shuffling between Fock vacua does not change the set of states, and so is a self-consistency test.

The sheaf cohomology groups above in (2.8) can straightforwardly be shown to satisfy all of the conditions above.

So far we have only discussed the additive structure of the chiral states; however, in special cases, there are also results on product structures. For example, in the special cases $\det \mathcal{E} \cong K_X^*$, there exists a pseudo-topological twists known as the A/2 model, for which nonperturbative corrections to product structures have been computed for X a toric variety and $\mathcal{E}$ a deformation of the tangent bundle, see for example [26–28, 50, 54, 55, 67].

Finally, let us discuss the behavior of these states under deformations and RG flow. In a (2,2) theory, the chiral states live in a topologically protected subsector, and so one expects to have the same additive structure in the chiral rings everywhere along RG flow and under deformations. For example, this is the physics reason why the Hodge numbers of Calabi-Yau's are the same in different geometric phases of the same GLSM. (In mathematics, this result is a consequence of motivic integration.)

By contrast, in (0,2) theories, the chiral ring Q-cohomology computation is protected only against perturbative corrections, for the same reasons that the (0,2) superpotential is not perturbatively renormalized. In applications such as [42], where RG flow stays in weakly-coupled regimes, Q-cohomology can be reliably used to count states. By contrast, in this thesis we compute Q-cohomology in weakly-coupled UV nonlinear sigma models which RG flow to strong coupling. As a result, one should expect that our Q-cohomology computations above will not necessarily give the correct IR spectrum, but rather additional states could enter or leave along the RG flow, and in fact that is precisely what we find.

2.2.1 Pseudo-topological twist

A (0,2) theory cannot be twisted to give a completely topological theory, but when the theory has a left-moving $U(1)$ R-symmetry, we can still do a pseudo-topological twist such that the twisted theory has most of the corresponding results as topological theories discussed in section 2.1.2. Under some conditions, the right-chiral operators saturating the left BPS bound form a topological subsector such that they reproduce (a,c) or (c,c) chiral ring on the (2,2) locus.

In order to get a ring structure from these operators, their OPE should be non-singular. The appearance of poles in the OPE requires the existence of right-chiral operators violating the left BPS bound. It is shown in [55] that this can not happen at least in a small neighborhood of the (2,2) locus. It is also shown that, when the number of left-moving fermions is smaller than eight, this can not happen either at any point of the (0,2) moduli space with or without a (2,2) locus. The twists are again implemented by regarding $M + J_V$ or $M + J_A$ as the new generator of the rotation group, and are called A/2 and B/2 twists respectively. The two distinct chiral rings correspond to the BRST cohomology of A/2 and B/2 twisted theories.

These twists of a (0,2) nonlinear sigma model on a space X with bundle $\mathcal{E}$ exist when [50,53]

$$\det \mathcal{E} \;\cong\; K_X^{\pm 1}, \;\; \mathrm{ch}_2(\mathcal{E}) \;=\; \mathrm{ch}_2(TX).$$

The first condition guarantees that the path integral measure is a scalar with the plus sign for B/2 twist and the minus sign for A/2 twist. The second is the Green-Schwarz anomaly cancellation condition. The A/2 twisted sigma model is defined by coupling fermions to bundles as follows

$$\begin{aligned}
\psi_+^i &\in \Gamma(\phi^* T^{1,0}X), \\
\psi_+^{\bar{i}} &\in \Gamma(K_\Sigma \otimes (\phi^* T^{1,0}X)^\vee), \\
\lambda_-^a &\in \Gamma(\overline{K}_\Sigma \otimes (\phi^* \overline{\mathcal{E}})^\vee), \\
\lambda_-^{\bar{a}} &\in \Gamma(\phi^* \overline{\mathcal{E}}).
\end{aligned}$$

Likewise, the B/2 twist is defined by

$$\begin{aligned}
\psi_+^i &\in \Gamma(K_\Sigma \otimes \phi^* T^{1,0}X), \\
\psi_+^{\bar{i}} &\in \Gamma((\phi^* T^{1,0}X)^\vee), \\
\lambda_-^a &\in \Gamma(\overline{K}_\Sigma \otimes (\phi^* \overline{\mathcal{E}})^\vee), \\
\lambda_-^{\bar{a}} &\in \Gamma(\phi^* \overline{\mathcal{E}}).
\end{aligned}$$

Note that on the (2,2) locus, where $\mathcal{E} = TX$, the twists above are equivalent to the A and B twist. OPE's in these pseudo-topological theories define 'quantum sheaf cohomology' rings, generalizing ordinary quantum cohomology rings. For example, in A/2 theories, the chiral operators are counted by

$$\oplus H^\bullet\left(X, \wedge^\bullet \mathcal{E}^*\right),$$

and quantum sheaf cohomology encodes nonperturbative quantum corrections to the product structure on the sheaf cohomology above. The ring relations are defined in the same way as section 2.1.3 except that the correlation functions are computed in the A/2 model.

Quantum sheaf cohomology of toric varieties was computed in [26–28]. Let $W = H^2(X, \mathbb{C})$, then $\mathcal{E}$ fits into a short exact sequence

$$0 \to \mathcal{O}_X \otimes_{\mathbb{C}} W^\vee \to \bigoplus_\rho \mathcal{O}_X(D_\rho) \to \mathcal{E} \to 0,$$

where the direct sum is over all one-dimensional cones of the fan defining the toric variety and D_ρ is the divisor corresponding to ρ. Then it is shown in [26] that the quantum sheaf cohomology is a quotient

$$QH^*_{\mathcal{E}}(X) = (Sym^* W \otimes \mathbb{C}[q^\beta])/QSR(X, \mathcal{E}),$$

where $\mathbb{C}[q^\beta]$ is the Novikov ring and $QSR(X, \mathcal{E})$ is the quantum Stanley-Reisner ideal, both are defined in [26]. In the next chapter, we will generalize the idea to non-abelian theories by studying the quantum sheaf cohomology of Grassmannians.

Chapter 3

Quantum sheaf cohomology of Grassmannians

In this chapter, we study (0,2) chiral rings in detail in the example of nonabelian GLSMs. The material in sections 3.2-3.5 was published in [81, 96], and represents new work. On the (2,2) locus, when the gauge connection is determined by the spin connection, charged matter couplings such as the $\overline{\mathbf{27}}^3$ and $\mathbf{27}^3$ in compactifications to four dimensions are computed by the A and B model topological field theories, and their values are by now well-understood via mirror symmetry. Off the (2,2) locus, much less is known.

In principle, charged matter couplings off the (2,2) locus can be computed by the A/2 and B/2 pseudo-topological field theories, and work has been done in that direction, starting with [50] (motivated by the mirror symmetry analysis of [51]). This work continued in *e.g.* [52–67], and culminated in a description of quantum sheaf cohomology rings on toric varieties with gauge bundles given by deformations of the tangent bundle, as described physically in GLSMs in [67, 68] and mathematically in [26, 27]. (See also [69–75] for more recent discussions, and [76] for a recent discussion of perturbative contributions to Yukawa couplings.)

Although those results are an important step, computing nonperturbative corrections and quantum sheaf cohomology for compact Calabi-Yau's with bundles that are not deformations of tangent bundles remains an open question.

As a stepping-stone towards that goal, we have been considering quantum sheaf cohomology on Grassmannians. These have technical complications beyond those of toric varieties, yet also have enough symmetries to make one hope that a tractable solution exists. In terms of GLSMs, this involves understanding nonabelian cases, whereas all previous work in quantum sheaf cohomology in (0,2) models has been in abelian GLSMs. In terms of the underlying mathematics, this becomes a story about nontrivial sheaves on Quot schemes, a technical step beyond toric cases, for which the pertinent moduli spaces are again toric varieties and induced sheaves are locally-free. In this thesis, we are only considering the physics approach (nonabelian GLSMs), not the math approach (Quot schemes).

3.1 Two-dimensional (0,2) gauge theories

To set up the background for our discussion in this chapter, we review the basics of 2d gauged linear sigma models (GLSM) with (0,2) supersymmetry in this section.

A (0,2) superfield is a function of the coordinate of the 2d world sheet and two Grassmannian variables $\theta^+, \overline{\theta}^+$. The superspace covariant derivatives are defined by

$$\begin{aligned} D_+ &= \frac{\partial}{\partial\theta^+} - i\overline{\theta}^+ \left(\partial_0 + \partial_1\right), \\ \overline{D}_+ &= -\frac{\partial}{\partial\overline{\theta}^+} + i\theta^+ \left(\partial_0 + \partial_1\right). \end{aligned}$$

The (0,2) supersymmetry in two dimensions admits three types of supermultiplets. The first is the chiral multiplet Φ. It satisfies $\overline{D}_+\Phi = 0$, and has expansion

$$\Phi = \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\overline{\theta}^+\partial_+\phi,$$

where ϕ is a complex scalar, ψ_+ is a right-moving Weyl fermion. The second multiplet is the Fermi multiplet Ψ, it satisfies $\overline{D}_+\Psi = \sqrt{2}E(\Phi)$, where E is a holomorphic function of Φ. The components of the Fermi multiplet are

$$\Psi = \psi_- - \sqrt{2}\theta^+ G - i\theta^+\overline{\theta}^+\partial_+\psi_- - \sqrt{2}\overline{\theta}^+ E.$$

The only on-shell degree of freedom is the left-moving fermion ψ_-. When we want to construct a gauge theory, we need the third type of supermultiplet, the vector multiplet. It is a real superfield with the expansion

$$V = v - 2i\theta^+\lambda_- - 2i\overline{\theta}^+\overline{\lambda}_- + 2\theta^+\overline{\theta}^+ D,$$

where v is the gauge field, λ_- is the gaugino, D is the auxiliary field. In addition to the E functions, one can also add a superpotential defined by J functions, which are also holomorphic functions of the chiral superfields:

$$\int d\theta^+ \Psi_a J^a(\Phi)|_{\overline{\theta}^+=0}.$$

Supersymmetry requires the holomorphic E and J functions obey

$$\sum_a E_a(\Phi) J^a(\Phi) = 0.$$

In a gauge theory, one needs to modify our definition for chiral and Fermi multiplets by replacing $\overline{D}_+$ with

$$\overline{\mathcal{D}}_+ = \exp(\overline{\Lambda})\overline{D}_+ \exp(-\overline{\Lambda}),$$

where $\Lambda = \theta^+\overline{\theta}^+(v_0 + v_1)$. Likewise,

$$\mathcal{D}_+ = \exp(-\Lambda)D_+ \exp(\Lambda).$$

The field strength is $\Upsilon = [\overline{\mathcal{D}}_+, \mathcal{D}_0 - \mathcal{D}_1]$, where $\mathcal{D}_i$ is the gauge covariant derivative along the bosonic directions.

The Fayet-Illiopoulos term is added to the gauge theory as

$$\frac{t}{4}\int d\theta^+ \Upsilon|_{\overline{\theta}^+=0},$$

where $t = ir + \theta/2\pi$ combines the FI parameter and the θ-angle.

As an example, we write down the Lagrangian of an abelian (0,2) gauge theory with chiral multiplets Φ_i of charge q_i and Fermi multiplets Ψ_a of charge q_a:

$$\mathcal{L} = \mathcal{L}_V + \mathcal{L}_\Phi + \mathcal{L}_\Psi + \mathcal{L}_{FI} + \mathcal{L}_J,$$

where

$$\begin{aligned}
\mathcal{L}_V &= \frac{1}{8e^2}\int d^2\theta \overline{\Upsilon}\Upsilon = \frac{1}{e^2}\left(\frac{1}{2}F_{01}^2 + i\overline{\lambda}_-\partial_+\lambda_- + \frac{1}{2}D^2\right), \\
\mathcal{L}_\Phi &= -\frac{i}{2}\int d^2\theta \overline{\Phi}_i D_- \Phi^i, \\
&= -|D_\mu\phi_i|^2 + i\overline{\psi}_{+i}D_-\psi_+^i - \sqrt{2}iq_i\overline{\phi}_i\lambda_-\psi_+^i + \sqrt{2}iq_i\phi^i\overline{\psi}_{+i}\overline{\lambda}_- + q_i|\phi_i|^2 D, \\
\mathcal{L}_\Psi &= -\frac{1}{2}\int d^2\theta \overline{\Psi}_a \Psi^a, \\
&= i\overline{\psi}_{-a}D_+\psi_-^a + |G_a|^2 - |E_a|^2 - \overline{\psi}_{-a}\frac{\partial E^a}{\partial \phi_i}\psi_{+i} - \frac{\partial \overline{E}_a}{\partial \overline{\phi}_i}\overline{\psi}_{+i}\psi_-^a, \\
\mathcal{L}_{FI} &= \frac{t}{4}\int d\theta^+ \Upsilon|_{\overline{\theta}^+=0} + c.c. = -rD + \frac{\theta}{2\pi}F_{01}, \\
\mathcal{L}_J &= \int d\theta^+ \Psi_a J^a(\Phi)|_{\overline{\theta}^+=0} + c.c. = \sqrt{2}G_a J^a(\phi) + \psi_{-a}\psi_{+i}\frac{\partial J^a}{\partial \phi^i} + c.c..
\end{aligned}$$

After integrating out the auxiliary fields the potential for the scalars ϕ_i is

$$V = \frac{e^2}{2}\left(\sum_i q_i|\phi_i|^2 - r\right)^2 + \sum_a |E_a(\phi)|^2 + \sum_a |J_a(\phi)|^2.$$

3.2 Nonabelian A/2 models

3.2.1 (0,2)-deformation

The gauged linear sigma model can be used to implement various geometric settings. On the (2,2) locus, the Grassmannian $G(k,n)$ is described by a two-dimensional supersymmetric $U(k)$ gauge theory with n chirals in the fundamental representation. Let's denote these chiral fields by $\Phi^i_\alpha, \alpha = 1, \cdots, k, i = 1, \cdots, n$. The (2,2) vector supermultiplet decomposes into a (0,2) vector multiplet V and a chiral multiplet Σ. The bosonic component of Σ is an adjoint valued scalar σ. The chiral supermultiplet, Φ^i_a, decomposes into a (0,2) chiral multiplet $\Phi^i_\alpha = (\phi^i_\alpha, \psi^i_{+\alpha})$ and a (0,2) Fermi multiplet $\Lambda^i_\alpha = (\psi^i_{-\alpha}, F^i_\alpha)$, obeying

$$\overline{D}_+ \Lambda^i_\alpha \;=\; \sigma^\beta_\alpha \Phi^i_\beta. \tag{3.1}$$

In a (0,2) theory, the covariant derivative of the Fermi superfield can be any function annihilated by the covariant derivative, i.e., (3.1) is generalized to

$$\overline{D}_+ \Lambda^i_\alpha \;=\; E^i_\alpha, \tag{3.2}$$

where E is a holomorphic function of the chiral superfields satisfying

$$\overline{D}_+ E = 0.$$

In particular, we can deform off the (2,2) locus by taking

$$\overline{D}_+ \Lambda^i_\alpha \;=\; A^i_j \sigma^\beta_\alpha \Phi^j_\beta \;+\; B^i_j (\mathrm{Tr}\,\sigma) \Phi^j_\alpha,$$

where A and B are n by n matrices. For simplicity, in this thesis we will assume A is invertible, which will guarantee our models can be deformed to the (2,2) locus. In principle, one could also imagine nonlinear deformations, functions of say

$$\epsilon^{\alpha_1 \cdots \alpha_k} \Phi^{i_1}_{\alpha_1} \cdots \Phi^{i_k}_{\alpha_k},$$

but as conjectured in [67] and later demonstrated in [26, 27, 71, 75], the A/2 model correlation functions and quantum sheaf cohomology ring relations are independent of nonlinear deformations, so we only consider linear deformations.

The left moving fermion is now a section of the vector bundle $\phi^* \mathcal{E}$, where $\mathcal{E}$ is a vector bundle on $G(k,n)$ defined by the short exact sequence

$$0 \to \mathcal{S} \otimes \mathcal{S}^* \xrightarrow{g} \mathcal{V} \otimes \mathcal{S}^* \to \mathcal{E} \to 0, \tag{3.3}$$

where g can be represented as

$$\omega^\beta_\alpha \mapsto A^i_j \omega^\beta_\alpha x^j_\beta + \omega^\beta_\beta B^i_j x^j_\alpha.$$

The dual of (3.3) is

$$0 \to \mathcal{E}^* \xrightarrow{i} \mathcal{V}^* \otimes \mathcal{S} \xrightarrow{f} \mathcal{S}^* \otimes \mathcal{S} \to 0, \tag{3.4}$$

where f can be represented as

$$t_i^\alpha \mapsto t_i^\alpha f_\beta^i = t_i^\alpha A_j^i x_\beta^j + \delta_\beta^\alpha t_i^\gamma B_j^i x_\gamma^j.$$

Our goal is to study the quantum sheaf cohomology ring

$$\bigoplus_{r \geqslant 0} H^r(G(k,n), \wedge^r \mathcal{E}^*).$$

The number of bundle moduli is equal to $h^1(X, \mathrm{End}\, TX)$. In the case at hand, $X = G(k,n)$, and $TX = \mathcal{S}^* \otimes \mathcal{Q}$, where $\mathcal{S}$ is the universal vector bundle and $\mathcal{Q}$ is the universal quotient bundle. Applying the Borel-Weil-Bott theorem, one can compute

$$h^1(G(k,n), \mathrm{End}\, TG(k,n)) \;=\; \begin{cases} n^2-1 & 1 < k < n-1, \\ 0 & \text{else.} \end{cases}$$

In other words, projective spaces have no tangent bundle moduli, but other Grassmannians do. Let us see how this number emerges from our description of the deformation.

Our description above encodes moduli in the two $n \times n$ matrices A, B. The invertible matrix A can be transformed into the identity matrix using a $GL(n)$ field redefinition, so that in effect only one matrix (B, or rather BA^{-1}) encodes the moduli. However, the overall trace in B is trivial, and does not define any bundle deformations, which we can see as follows. Without loss of generality, take A to be the identity. Denote by i the imbedding of $\mathcal{S}$ in $\mathcal{V}$. Given a local section of $\mathcal{V}^* \otimes \mathcal{S} = \mathcal{H}om(\mathcal{V}, \mathcal{S})$, denoted by t, $f(t)$ can be written as $ti + \mathrm{Tr}(tBi) I_{k\times k}$, where $I_{k\times k}$ is the $k \times k$ identity matrix. If t is in the kernel of f and $B = \varepsilon I_{n \times n}$, then

$$ti + \varepsilon \;\mathrm{Tr}(ti)\; I_{k\times k} = 0. \tag{3.5}$$

Taking the trace, we get

$$(1 + \varepsilon k)\; \mathrm{Tr}(ti) = 0.$$

For generic ε, this implies $\mathrm{Tr}(ti) = 0$, but then $ti = 0$ by (3.5). This means t is in the kernel of f_0 (f with $B = 0$). The converse is also true. We conclude that $\mathcal{E}^* \cong \Omega$, the holomorphic cotangent bundle, when $B = \varepsilon I_{n\times n}$. Thus, we see the number of nontrivial deformations is $n^2 - 1$, encoded in B (or BA^{-1} if A is nontrivial), modulo an overall trace.

Not all $n \times n$ matrices define a vector bundle through equation (3.3). In fact we will show in section 3.3.2 that a B-deformation fails to give rise to a vector bundle on $G(k,n)$ if and only if there exist k eigenvalues of B (or BA^{-1}, if A is nontrivial) that sum to -1. Physically, if this condition is satisfied, then the GLSM develops a noncompact branch, independent of the value of the Fayet-Iliopoulos parameter. In any event, this criterion gives us the discriminant locus along which the $A/2$ correlation functions diverge.

3.2.2 One-loop effective potential

We will derive the quantum sheaf cohomology ring relations from the one-loop effective potential on the Coulomb branch, which we review in this section.

For the GLSM corresponding to $G(k, n)$, the gauge group is $U(k)$, which is generically[1] broken to $U(1)^k$ along the Coulomb branch. For σ the adjoint-valued field in the (2,2) vector multiplet, Take $\sigma_a, a = 1, \cdots, k$, to be the components of σ in the Cartan subalgebra. These will act as coordinates along the Coulomb branch. On this branch, the charge for Φ_a^i is δ_a^b under the b-th $U(1)$. Notice that all the Φ_a^i's with the same a have the same charges under all the $U(1)$'s. For fixed a, we can rewrite (3.2) as

$$\overline{\mathcal{D}}_+ \Lambda_a^i = E_j^i(\sigma_a) \Phi_a^j,$$

where the $n \times n$ matrix E_j^i is given by

$$E_j^i(\sigma_a) = \sigma_a A_j^i + \mathrm{Tr}(\sigma) B_j^i$$

for general A, or for A taken to be the identity,

$$E_j^i(\sigma_a) = \sigma_a \delta_j^i + \mathrm{Tr}(\sigma) B_j^i.$$

According to [50], the one-loop effective J function is

$$\tilde{J}_a = -\ln\left[-q^{-1} \det(E_a)\right]. \tag{3.6}$$

(Here a minus sign is inserted to comply with the convention in mathematical literature, this corresponds to an overall shift in the theta angle.) The equations of motion are $\tilde{J}_a = 0$ for each a, or more simply, for each a,

$$\det(E(\sigma_a)) = \det(\sigma_a A + \mathrm{Tr}(\sigma) B) = -q \tag{3.7}$$

for general matrices A.

3.2.3 Supersymmetric localization

We shall check the predictions for the quantum sheaf cohomology ring by computing correlation functions in examples, using supersymmetric localization. Now, it is not known how

[1] For our computations, we will be able to essentially ignore loci with enhanced gauge symmetry. For example, in supersymmetric localization computations, residues vanish along such loci, because of *e.g.* factors of the form

$$\prod_{a \neq b} (\sigma_a - \sigma_b)$$

in the numerator of the integrand. As a result, such loci do not contribute to our computations, and will be ignored in this thesis.

to apply supersymmetric localization to an untwisted (0,2) theory, but in this thesis we are concerned with a twist of the (0,2) theory, known as the A/2 model. In a (0,2) nonlinear sigma model on a space X with bundle $\mathcal{E}$, we can understand the A/2 twist as follows. Before the twist, the right moving fermion ψ_+ is a section of $K^{1/2} \otimes \phi^* TX$, and the left moving fermion λ_- is a section of $\overline{K}^{1/2} \otimes \phi^* \mathcal{E}^*$, where K is the canonical line bundle of the worldsheet. In an A/2 twisted nonlinear sigma model [50], for example, we have

$$\begin{aligned}
\psi_+^i &\in \Gamma(\phi^* T^{1,0} X), \\
\psi_+^{\bar{\imath}} &\in \Gamma(K \otimes \phi^* T^{0,1} X), \\
\lambda_-^a &\in \Gamma(\overline{K} \otimes \phi^* \mathcal{E}^*), \\
\lambda_-^{\bar{a}} &\in \Gamma(\phi^* \overline{\mathcal{E}}^*),
\end{aligned}$$

with the chiral ring being isomorphic to

$$\bigoplus_{r \geqslant 0} H^r(X, \wedge^r \mathcal{E}^*).$$

In the UV GLSM for the Grassmannian $G(k,n)$, the gauge-invariant chiral ring operators are of the form $\operatorname{Tr} \sigma^k$ for integers k and σ the bosonic field of the chiral multiplet in the adjoint representation. We will express these in terms of symmetric polynomials in commuting elements forming a basis along the Coulomb branch, denoted $\sigma_a = \sigma_1, \sigma_2, \cdots, \sigma_k$. A/2 correlation functions of symmetric polynomials in the σ_a then suffice to determine the quantum sheaf cohomology associated with the chiral ring. In terms of these commuting elements, the bosonic potential becomes of the form

$$\begin{aligned}
&\sum_{i,a} |A^i_j \sigma_a \phi^j_a \,+\, B^i_j (\operatorname{Tr}\sigma) \phi^j_a|^2 \\
&\quad = \sum_{i,a} \overline{\phi}^j_a \phi^k_a \left(A^i_j \sigma_a \,+\, B^i_j (\operatorname{Tr}\sigma) \right)^* \left(A^i_k \sigma_a \,+\, B^i_k (\operatorname{Tr}\sigma) \right) \\
&\quad = \sum_{i,a} \overline{\phi}^j_a \phi^k_a (E^i_{j,a})^* E^i_{k,a},
\end{aligned}$$

where

$$E^i_{j,a} \;=\; A^i_j \sigma_a \,+\, B^i_j (\operatorname{Tr}\sigma).$$

The Yukawa couplings have the form

$$-\overline{\psi}^j_{-a} \psi^i_{+a} E^i_{j,a} \,+\, \text{c.c.}.$$

These couplings – the bosonic potential and Yukawa couplings – define what amount to σ-dependent masses that play a crucial role in the one-loop partition function in supersymmetric localization.

Supersymmetric localization in the A/2 model for (0,2) theories given by deformations of (2,2) theories was discussed in [75]. It is shown that, on the Coulomb branch, the correlation functions of the A/2 theory can be computed explicitly as a sum over flux sectors, with each summand given by a generalized Jeffrey-Kirwan-Grothendieck residue (JKG-Res) defined in [75]. The result is

$$\langle \mathcal{O} \rangle = \frac{(-1)^N}{|W|} \sum_{\mathfrak{m} \in \Gamma^\vee} q^{\mathfrak{m}} \mathrm{JKG} - \mathrm{Res} \left\{ Z_{\mathfrak{m}}^{\text{one-loop}}(\sigma) \mathcal{O}(\sigma) \; d\sigma_1 \wedge \cdots \wedge d\sigma_{\mathrm{rk(G)}} \right\},$$

where W is the Weyl group, N is the number of chiral multiplets of R-charge 2, $\Gamma^\vee$ is the dual weight lattice of the gauge group G, $Z_{\mathfrak{m}}^{\text{one-loop}}$ is the one-loop determinant obtained by performing the path integral over non-zero modes. For the non-abelian theory we are considering,

$$Z_{\mathfrak{m}}^{\text{one-loop}} \;=\; \prod_{a=1}^{k} \left(\frac{1}{\det \tilde{E}(\sigma_a)} \right)^{\mathfrak{m}_a + 1},$$

where

$$\tilde{E}^i_j(\sigma_a) \;=\; A^i_j \sigma_a \;+\; B^i_j \left(\sum_b \sigma_b \right).$$

This implies that for any polynomial f in $\sigma_a, a = 1, \cdots, k$, the correlation functions should have the form

$$\langle f(\sigma) \rangle =$$
$$\frac{1}{k!} \sum_{\mathfrak{m}_1, \cdots, \mathfrak{m}_k \in \mathbb{Z}} \mathrm{JKG} - \mathrm{Res} \left\{ (-1)^{(n-1)\sum \mathfrak{m}_i} q^{\sum \mathfrak{m}_i} \left(\prod_{a \neq b} (\sigma_a - \sigma_b) \right) \prod_{a=1}^{k} \left(\frac{1}{\det \tilde{E}(\sigma_a)} \right)^{\mathfrak{m}_a + 1} f(\sigma) \right\}. \tag{3.8}$$

In principle, given the $A/2$ correlation functions, the quantum sheaf cohomology ring is defined in the same way as the ordinary quantum cohomology. If we take a basis e_i for $\bigoplus_{r \geqslant 0} H^r(G(k,n), \wedge^r \mathcal{E}^*)$ as a vector space, and a dual basis $\hat{e}_i$ in the sense that

$$\langle e_i \hat{e}_j \rangle = \delta_{ij},$$

the generating relations read

$$\sigma = \sum_i \langle \sigma e_i \rangle \hat{e}_i$$

for any σ. More to the point, the quantum (sheaf) cohomology ring relations define identities in the correlation functions: if in the ring, some quantity R is set to zero, then any correlation function containing R should vanish. We will use localization to check the ring structure in examples in section 3.5.

3.3 Ring structure of the classical sheaf cohomology

3.3.1 Representation of the ring as a quotient

Let's first summarize the idea we use and how it leads to the classical ring relations of $\bigoplus_{r\geqslant 0} H^r(G(k,n),\wedge^r\mathcal{E}^\vee)$. First, in order to determine the ring relations at order r, we take the rth exterior power of (3.4),

$$\begin{aligned}0 &\to \wedge^r\mathcal{E}^\vee \to \wedge^r(\mathcal{V}^*\otimes\mathcal{S}) \to \wedge^{r-1}(\mathcal{V}^*\otimes\mathcal{S})\otimes(\mathcal{S}^*\otimes\mathcal{S}) \to \wedge^{r-2}(\mathcal{V}^*\otimes\mathcal{S})\otimes\mathrm{Sym}^2(\mathcal{S}^*\otimes\mathcal{S})\\ &\to \cdots \to \mathcal{V}^*\otimes\mathcal{S}\otimes\mathrm{Sym}^{r-1}(\mathcal{S}^*\otimes\mathcal{S}) \to \mathrm{Sym}^r(\mathcal{S}^*\otimes\mathcal{S}) \to 0.\end{aligned} \tag{3.9}$$

Breaking up the exact sequence above into r short exact sequences

$$\begin{aligned}&0 \to S_r \to Z_r \to S_{r-1} \to 0,\\ &0 \to S_{r-1} \to Z_{r-1} \to S_{r-2} \to 0,\\ &...,\\ &0 \to S_1 \to Z_1 \to S_0 \to 0,\end{aligned} \tag{3.10}$$

where $Z_j = \wedge^j(\mathcal{V}^*\otimes\mathcal{S})\otimes\mathrm{Sym}^{r-j}(\mathcal{S}^*\otimes\mathcal{S})$, $S_j = \mathrm{Ker}\ (Z_j \to Z_{j-1})$, and $S_0 = \mathrm{Sym}^r(\mathcal{S}^*\otimes\mathcal{S})$. They induce connecting maps on cohomology $\delta : H^j(S_j) \to H^{j+1}(S_{j+1})$, $j = 0, ..., r-1$. Thus we get a connecting map δ_r from $H^0(\mathrm{Sym}^r(\mathcal{S}^*\otimes\mathcal{S}))$ to $H^r(\wedge^r\mathcal{E}^\vee)$ by composing the connecting maps associated with all the short exact sequences. Thus the ring relations are encoded in the kernel of δ_r.

$\mathrm{Sym}^r(\mathcal{S}^*\otimes\mathcal{S})$ can be written as a direct sum in the form $\bigoplus_{\mu}(K_\mu\mathcal{S}^*\otimes K_\mu\mathcal{S})$, where each μ is some Young diagram standing for an irreducible representation of $U(k)$ and K_μ is the corresponding Schur functor. The direct sum ranges over all the Young diagrams with r boxes and at most k rows. Since $K_\mu\mathcal{S}^*\otimes K_\mu\mathcal{S} \cong Hom(K_\mu\mathcal{S}, K_\mu\mathcal{S})$, and each $H^0(K_\mu\mathcal{S}^*\otimes K_\mu\mathcal{S})$ is one-dimensional, a basis for $H^0(\mathrm{Sym}^r(\mathcal{S}^*\otimes\mathcal{S}))$ can be taken to be $\{\sigma_\mu \mid \mu$ is a Young diagram with r boxes and at most k rows $\}$, where σ_μ is the identity bundle map on $K_\mu\mathcal{S}$.

The product on $\bigoplus_{r\geqslant 0} H^0(\mathrm{Sym}^r(\mathcal{S}^*\otimes\mathcal{S}))$ is defined to be the tensor product of bundle maps. Because

$$K_\lambda\mathcal{S}\otimes K_\mu\mathcal{S} = \bigotimes_{\nu} N_{\lambda\mu\nu}K_\nu\mathcal{S},$$

we see

$$\sigma_\lambda\otimes\sigma_\mu = \bigotimes_{\nu} N_{\lambda\mu\nu}\sigma_\nu. \tag{3.11}$$

The numbers $N_{\lambda\mu\nu}$ are determined by the Littlewood-Richardson rule. $N_{\lambda\mu\nu}$ is the number of ways the Young diagram λ can be expanded to the Young diagram ν by a strict μ-expansion. Note that (3.11) would remain unchanged if one replaced each bundle map σ_μ with the Schur polynomial corresponding to μ, and the tensor product with the usual product of polynomials. This implies that $\bigoplus_{r\geqslant 0} H^0(\mathrm{Sym}^r(\mathcal{S}^*\otimes\mathcal{S}))$ is isomorphic to the ring of symmetric polynomials with k variables. And these variables are just the diagonal elements of the σ field. Let's denote this ring by $A(k)$.

Because the connecting map from $A(k)$ to $\bigoplus_{r\geqslant 0} H^r(\wedge^r\mathcal{E}^\vee)$ is a surjective homomorphism of $\mathbb{C}$-algebras, to determine the ring structure of the latter, we only need to find $\ker(\delta)$, the kernel of the connecting map. Obviously, the polynomials with degree greater than $k(n-k)$ are in the kernel because the Grassmannian has dimension $k(n-k)$. Let $I_m(k)$ be the ideal of polynomials with degree at least m, from the argument above, we can focus on the quotient ring $A_{k(n-k)}(k) = A(k)/I_{k(n-k)+1}(k)$, which is the ring of symmetric polynomials with degree at most $k(n-k)$. Let's still use $\ker(\delta)$ to denote the kernel of the connecting map descending to $A_{k(n-k)}(k)$. Thus we see, the sheaf cohomology has the following representation

$$\bigoplus_{r\geqslant 0} H^r(G(k,n),\wedge^r\mathcal{E}^\vee) \cong A_{k(n-k)}(k)/\ker(\delta). \tag{3.12}$$

Let's denote by $\sigma_{(r)}$ the Schur polynomial corresponding to the Young diagram with r boxes in a row. We will show that, for a generic B deformation, $\ker(\delta)$ is generated by $R_r, r = n-k+1,\cdots,k(n-k)$, where

$$R_r = \sum_{i=0}^{\min\{r,n\}} I_i\sigma_{(r-i)}\sigma_{(1)}^i, \tag{3.13}$$

and I_i is the ith characteristic polynomial of B, which is defined through

$$\det(tI+B) = \sum_{i=0}^{n} I_{n-i}t^i.$$

This gives us the classical sheaf cohomology. In section 3.4.2, we will derive the quantum sheaf cohomology. The conclusion is, when quantum corrections are taken into account, we should replace $\ker(\delta)$ with the ideal generated by

$$R_r + q\sigma_{(r-n)}, \tag{3.14}$$

$r = n-k+1,\cdots,k(n-k)$, and $\sigma_{(m)}$ is defined to be zero when $m<0$.

3.3.2 The degenerate locus

By (3.3), the deformed tangent bundle $\mathcal{E}$ is determined by the map $f \in Hom(\mathcal{S}\otimes\mathcal{S}^*, \mathcal{V}\otimes\mathcal{S}^*)$. One can show

$$Hom(\mathcal{S}\otimes\mathcal{S}^*,\mathcal{V}\otimes\mathcal{S}^*) \cong H^0(\mathcal{S}^*\otimes\mathcal{S}\otimes\mathcal{V}\otimes\mathcal{S}^*) \cong \mathcal{V}^*\otimes\mathcal{V}\oplus\mathcal{V}^*\otimes\mathcal{V}. \tag{3.15}$$

Note that here we used our assumption that $k > 1$.

The map f can be written down explicitly. Locally we have

$$f : \lambda \mapsto \lambda_a^b A_j^i \phi_b^j + (tr\lambda) B_j^i \phi_a^j, \tag{3.16}$$

where a, b are S indices, and i, j are V indices. When (A_j^i), $i, j = 1, ..., n$ form an invertible matrix, we can always set $A_j^i = \delta_j^i$ (the Kronecker delta), using the $GL(V)$ action on $\mathcal{V}$. So it remains to consider the B-deformations. We will write f as f_B to indicate the B-dependence and view B as a $n \times n$ matrix.

The *degenerate locus* of B-deformations is the set of B such that the cokernel of f_B fails to be a deformed tangent bundle.

In this section we work out the degenerate locus.

Lemma 3.3.1. *Let $\mathbb{B}$ be a linear operator acting on an n-dimensional vector space V. Then for any k eigenvalues (counting multiplicity) $\lambda_1, ..., \lambda_k$ of $\mathbb{B}$, one can always find a k dimensional invariant subspace $V_k \subset V$ such that $\lambda_1, ..., \lambda_k$ are the eigenvalues of $\mathbb{B}|_{V_k}$.*

Proof: Since there exists a basis under which $\mathbb{B}$ is represented by its Jordan canonical form, it follows that V has an invariant subspace V_λ for each eigenvalue λ, and $\mathbb{B}|_{V_\lambda}$ has a matrix representation $B_\lambda = \text{diag}\ \{\lambda, ..., \lambda, \mathrm{J}_{\lambda,\mathrm{l}_1}, ..., \mathrm{J}_{\lambda,\mathrm{l}_\mathrm{m}}\}$, where $J_{\lambda,l}$ is the $l \times l$ Jordan block

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \cdots & 1 \\ & & & \lambda \end{pmatrix}.$$

Let $\nu(\lambda) = \dim V_\lambda$. Since B_λ is upper-triangular, $\mathbb{B}|_{V_\lambda}$ has an invariant subspace $W_j \subset V_\lambda$ of dimension j, for any $0 \le j \le \nu(\lambda)$. Hence $\mathbb{B}$ has an invariant subspace $W_{\lambda,j} \subset V_\lambda$ of dimension j, for any $0 \le j \le \nu(\lambda)$.

Now, group the eigenvalues $\lambda_1, ..., \lambda_k$ by multiplicity and write them as

$$(\lambda_{i_1}, \mu(\lambda_{i_1})), ..., (\lambda_{i_s}, \mu(\lambda_{i_s})),$$

then $\oplus_{j=1}^s W_{\lambda_{i_j}, \mu(\lambda_{i_j})}$ is the desired invariant subspace of V under the action of $\mathbb{B}$. $\square$

Dual to (3.16), $f_B : \mathcal{V}^* \otimes \mathcal{S} \to \mathcal{S}^* \otimes \mathcal{S}$ can be written as

$$f_B : c_i^a \mapsto c_i^a v_b^i + c_i^d B_j^i v_d^j \delta_b^a. \tag{3.17}$$

For the kernel to be a deformed cotangent bundle, we need to ensure the map is of rank k^2 at every point of the Grassmannian $G(k, n)$.

Denote the image of f_B as a tuple $(\sigma_b^a)_{a,b=1,...,k}$, then

$$\sigma_b^a = c_{i'}^{a'} (\delta_{a'}^a v_b^{i'} + \delta_b^a B_j^{i'} v_{a'}^j). \tag{3.18}$$

So f_B is represented by a big $k^2 \times kn$ matrix M. We use (a,b) as the row index of M, and (a',i') as the column index.

Write M as $M_1 + M_2$, where $M_1 = \mathrm{diag}\{\mathbf{V},...,\mathbf{V}\}$ with $\mathbf{V}_{b,i'} = v_b^{i'}$, corresponding to $\delta^a_{a'} v_b^{i'}$, and M_2 has non-vanishing rows only when $a = b$, and each such row has entry $B^{i'}_j v^j_{a'}$ at place (a',i').

Now we want to know the equivalent condition for $\mathrm{rank}(M) < k^2$.

For the case $k = 1$, this is equivalent to $B^i_j v^j_1 + v^i_1 = 0, \forall i$. In matrix language, this says there are solutions for $\mathbf{V(B{+}I)} = 0$. So the condition is

$$\det \mathbf{(I{+}B)} = 0. \tag{3.19}$$

When $k \geq 2$, we first perform a partial Gauss elimination on M: for each $b = 2, 3, ..., k$, subtract the first row from row (b,b). The result matrix M' is identical to M_1, except the first row and the first n columns.

Note that rank(M) < k^2 iff the rows of M' are linearly dependent.

Write down the linear-dependence condition $\sum c_{ab} M'_{(a,b)} = 0$, where $M'_{(a,b)}$ is the $(a,b) - th$ row. Observe that the undeformed $B = 0$ case implies that we can assume

$$c_{11} = 1.$$

Then, because of the 'almost-diagonal' nature of M', we can spell the conditions out for each column of M', and repackage them into

$$\mathbf{CV} = \mathbf{VB}, \tag{3.20}$$

where we have

$$\mathbf{C}_{ab} = \begin{cases} -\sum_{j=1}^k c_{jj}, & a = b = 1, \\ c_{ab}, & \textit{otherwise}. \end{cases}$$

and $\mathbf{B}_{ji'} = B^{i'}_j$.

Hence we conclude

Theorem 3.3.2. *The B-deformation fails to define a vector bundle iff there exists at least one point in $G(k,n)$ such that (3.20) has non-zero solutions.*

Note that the constraint on $\mathbf{C}$ is equivalent to tr $\mathbf{C} = -1$. It is independent of the choice of the Stiefel coordinates $\mathbf{V}$. Moreover, it suffices to consider the Jordan canonical form of B since $\mathbf{CV} = \mathbf{VB}$ is equivalent to $\mathbf{CVN} = \mathbf{VNN^{-1}BN}$, $\mathbf{N} \in GL(n)$.

Theorem 3.3.3. *An $n \times n$ matrix B is in the degenerate locus for $G(k,n)$ iff*

$(*)$ *there exists k eigenvalues $\lambda_1, ..., \lambda_k$ of B such that $\sum \lambda_i = -1$.*

Proof. The $k = 1$ case is done before, since this is equivalent to (3.19). For $k \geq 2$ Theorem 3.3.2 shows that we need to consider the solutions of **CV=VB** for each $\mathbf{V}$, which is a Stiefel coordinate of the point $[\mathbf{V}] \in G(k,n)$.

For each $\mathbf{V}$, we can always find a $g \in GL(V)$ such that $\mathbf{V} = \mathbf{V}_0 g$, where $\mathbf{V}_0 = (\mathbf{I_k}\ \mathbf{0})$ when written as a block matrix. Let $\tilde{\mathbf{B}} = \mathbf{B}_g = g\mathbf{B}g^{-1}$ So it suffices to consider $\mathbf{CV_0} = \mathbf{V_0}\tilde{\mathbf{B}}$, for all $g \in GL(V)/\mathfrak{B}$, where $\mathfrak{B}$ is the Borel subgroup that leaves $[\mathbf{V}_0]$ fixed.

Observe that $\mathbf{CV_0} = \mathbf{V_0}\tilde{\mathbf{B}}$ has a solution with tr $\mathbf{C} = -1$ is equivalent to $\tilde{\mathbf{B}} = \begin{pmatrix} \mathbf{J}_{11} & 0 \\ \mathbf{J}_{21} & \mathbf{J}_{22} \end{pmatrix}$ in block matrix notion with tr $\mathbf{J}_{11} = -1$.

Recall that we have the (strange) notation conversion $B = \mathbf{B}^T$. So we can reformulate the equivalent condition for the B-deformation failing to give rise to a vector bundle on $G(k,n)$ as

(∗∗) *there exists* $g \in GL(V)$ *such that* $\tilde{B} = B_g = g^{-1}Bg = \begin{pmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{pmatrix}$ *with* $\mathrm{tr}\ \mathrm{J}_{11} = -1$.

View B as the matrix representation of a linear operator $\mathbb{B}$ on V under the standard basis $\{e_1, ..., e_n\}$.Then $\tilde{B}$ is the matrix representation of the same linear operator in the new basis $\{\tilde{e}_1, ..., \tilde{e}_n\} = \{ge_1, ..., ge_n\}$. Also note that $\tilde{B}$ is of the block upper triangular form $\begin{pmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{pmatrix}$ iff $\mathbb{B}V_k \subset V_k$, where $V_i = \mathrm{span}\{\tilde{\mathrm{e}}_1, ..., \tilde{\mathrm{e}}_\mathrm{i}\}$.

So the problem reduces to the determination of k dimensional invariant subspaces of V under the operator $\mathbb{B}$.

Note that $\mathbb{B}|_{V_k}$ is a linear operator whose eigenvalues are also eigenvalues of $\mathbb{B}$. On the other hand, Lemma 3.3.1 says that for any k eigenvalues (counting multiplicity) $\lambda_1, ..., \lambda_k$ of $\mathbb{B}$, one can always find a k dimensional invariant subspace $V_k \subset V$ such that $\lambda_1, ..., \lambda_k$ are the eigenvalues of $\mathbb{B}|_{V_k}$. This implies that tr J_{11} will always be a sum of k eigenvalues of B, and any k eigenvalues of $\mathbb{B}$ can be the eigenvalues of J_{11}. Hence (∗∗) is equivalent to (∗). □

Remark 3.3.1. Unlike results in later sections, this result is true for all B-deformations, not just generic deformations.

Theorem 3.3.4. *For* $G(k,n)$*, the degenerate locus can be described as*

$$\det(\wedge^k I + \sum_{j=0}^{k-1} (\wedge^j I) \wedge B \wedge (\wedge^{k-1-j} I)) = 0. \tag{3.21}$$

In particular, when $k = 1, 2, 3$, the expression is

$$\begin{aligned} &\det(I + B) = 0, \\ &\det(I \wedge I + B \wedge I + I \wedge B) = 0, \\ &\det(I \wedge I \wedge I + B \wedge I \wedge I + I \wedge B \wedge I + I \wedge I \wedge B) = 0, \end{aligned} \tag{3.22}$$

respectively.

Proof of Theorem 3.3.4. View B as the matrix representation of a linear operator $\mathbb{B}$ on V under the standard basis $\{e_1, ..., e_n\}$. It suffices to prove the case when B is of the Jordan canonical form. Suppose the diagonal elements of B are $\lambda_1, ..., \lambda_n$. They are also the eigenvalues of B. $\{e_{i_1 ... i_k} := e_{i_1} \wedge ... \wedge e_{i_k}, i_1 < ... < i_k\}$ is a basis of $\wedge^k V$ and we order the base vectors lexicographically. Note that $Be_i = \lambda_i e_i + \epsilon_i e_{i+1}$, where ϵ_i is either 0 or 1 and $(B \wedge I)(e_i \wedge e_j) = \lambda_i e_i \wedge e_j + \delta_i e_{i+1} \wedge e_j$, etc. It is then easy to see that

$$\wedge^k I + \sum_{j=0}^{k-1} (\wedge^j I) \wedge B \wedge (\wedge^{k-1-j} I)$$

is an upper-triangular matrix and the diagonal element in the row corresponding to $e_{i_1 ... i_k}$ is $1 + \lambda_{i_1} + ... + \lambda_{i_k}$. So the determinant is exactly $\prod(1 + \lambda_{i_1} + ... + \lambda_{i_k})$. $\square$

3.3.3 Mathematical proof of the classical ring structure

We then want to describe $H^r(\wedge^r \mathcal{E}^*)$ via $\delta_r : H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})) \to H^r(\wedge^r \mathcal{E}^*)$. Denote the kernel of $\delta_r : H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})) \to H^r(\wedge^r \mathcal{E}^*)$ as $\mathbb{K}_r$.

Theorem 3.3.5. *The kernel $\mathbb{K}_r$ of $\delta_r : H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})) \to H^r(\wedge^r \mathcal{E}^*)$ only depends on the equivalence class of B modulo similarity transformations $B \mapsto gBg^{-1}, g \in GL(V)$.*

Proof. Let $g \in GL(V)$, then g induces an isomorphism for each naturally defined vector bundle built from $\mathcal{S}, \mathcal{Q}, \mathcal{V}$. Denote the map $\mathcal{V}^* \otimes \mathcal{S} \to \mathcal{S}^* \otimes \mathcal{S}$, $c_i^a \mapsto c_i^a v_b^i + c_i^d B_j^i v_d^j \delta_b^a$ by f_B. Then we have a commutative diagram

$$\begin{tikzcd} \mathcal{V}^* \otimes \mathcal{S} \arrow[r, "f_B"] \arrow[d, "g"', "\cong"] & \mathcal{S}^* \otimes \mathcal{S} \arrow[d, "\cong"', "g"] \\ \mathcal{V}^* \otimes \mathcal{S} \arrow[r, "f_{gBg^{-1}}"] & \mathcal{S}^* \otimes \mathcal{S}. \end{tikzcd} \tag{3.23}$$

So when f_B is surjective, g induces an isomorphism of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}_1^* \arrow[r] \arrow[d, "g"', "\cong"] & \mathcal{V}^* \otimes \mathcal{S} \arrow[r, "f_B"] \arrow[d, "g"', "\cong"] & \mathcal{S}^* \otimes \mathcal{S} \arrow[r] \arrow[d, "g"', "\cong"] & 0 \\ 0 \arrow[r] & \mathcal{E}_2^* \arrow[r] & \mathcal{V}^* \otimes \mathcal{S} \arrow[r, "f_{gBg^{-1}}"] & \mathcal{S}^* \otimes \mathcal{S} \arrow[r] & 0. \end{tikzcd} \tag{3.24}$$

Notice that the g map on any naturally defined vector bundle $\mathcal{W}$ is exactly the familiar $g \in GL(V)$ action, and it induces actions on cohomologies, *i.e.*, for any s in the $GL(V)$-

module $H^*(\mathcal{W})$, $g(s) = g \cdot s$. In particular, Since $H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S}))$ is a trivial $GL(V)$-module [2] , the map $g : H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})) \to H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S}))$ is the identity map. So the commutativity of the diagram

$$\begin{tikzcd} H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})) \arrow[r, "\delta_B"] \arrow[d, equal, "g"] & H^r(\wedge^r \mathcal{E}_1^*) \arrow[d, "g"] \\ H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})) \arrow[r, "\delta_{gBg^{-1}}"] & H^r(\wedge^r \mathcal{E}_2^*). \end{tikzcd} \tag{3.25}$$

implies Ker $\delta_B =$ Ker $\delta_{gBg^{-1}}$. □

Now we consider the image of $\sigma \in H^0(\mathrm{Sym}^2(\mathcal{S}^* \otimes \mathcal{S}))$ under δ_B. To track the B_{ij}-dependence, we make the following definition.

Definition 3.3.1. Let each B_{ij} be a degree one variable and denote the *total B degree* of each cocycle ω as $\deg \omega$.

For any $n \times n$ matrix B, consider the characteristic polynomial (with the sign changed) $\det(\lambda I + B)$. Denote the coefficient of λ^{n-i} as $I_i(B) = I_i$, so that $I_0 = 1, I_1 = tr(B), I_2 = \frac{1}{2}(tr(B)^2 - tr(B^2)))$, and so forth. We have $\deg I_i = i$.

Theorem 3.3.6. *Every $\sigma \in \mathbb{K}_r$ is determined by some $\gamma \in \mathrm{Ker}(H^{j-1}(Z_j^{(r)}) \to H^{j-1}(Z_{j-1}^{(r)}))$, where $Z_j^{(r)} = \wedge^j(\mathcal{V}^* \otimes \mathcal{S}) \otimes \mathrm{Sym}^{r-j}(\mathcal{S}^* \otimes \mathcal{S})$.[3] If γ is B-independent, then σ can be represented by a cocycle γ_0 such that $\deg \gamma_0 \leq r$.*

Proof. Under the assumption we have $\deg \lambda = 0$ as it is B-independent, and $\deg q_i = 1$ by linearity. The key observation here is $\deg d^{-1} = 0$, i.e. $\deg \gamma^0 = \deg \gamma^1$. Assume this is not true, then $\deg \gamma^0 > \deg \gamma^1$. Take the sum of the terms of γ^0 whose degrees are larger than $\deg \gamma^1$, call it γ^0_+, then $d(\gamma^0_+)$ has to be $0 \in C^1((\mathcal{V}^* \otimes \mathcal{S}) \otimes (\mathcal{S}^* \otimes \mathcal{S}))$ since there is no term of the corresponding degree there. Hence one can simply drop γ^0_+ when choosing γ^0. □

Corollary 3.3.7. *When γ is B-independent, the kernel $\mathbb{K}_r$ is generated by elements with coefficients that are polynomials of $I_1, ..., I_r$ whose total B degrees are less than or equal to r.*

The first case we expect a nontrivial kernel of $\delta : H^0(S_0) \to H^r(S_r)$ is the case when $r = n - k + 1$. Here we used the notation S_i as in the short exact sequences

$$0 \to S_j \to Z_j \to S_{j-1} \to 0, \tag{3.26}$$

[2]Easy to prove from Borel-Weil-Bott: the components $K_\lambda \mathcal{S}^*$ of $\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})$ satisfies $|\lambda| = 0$. If $\lambda \neq 0$, then $\lambda_k < 0$. Hence $(\lambda_1, ..., \lambda_k, 0, ..., 0)$ is not non-increasing and it won't contribute to H^0.

[3]For any σ, there always exists such γ.

$j = 1, ..., r$, which are generated from the long exact sequence (3.9) as in (3.10). In particular, $S_0 = \mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})$, $S_r = \wedge^r \mathcal{E}^*$, $Z_j = \wedge^j(\mathcal{V}^* \otimes \mathcal{S}) \otimes \mathrm{Sym}^{r-j}(\mathcal{S}^* \otimes \mathcal{S})$.

We find that

Theorem 3.3.8. *When $r = n - k + 1$, $\mathbb{K}_r$, the kernel of $\delta_r : H^0(S_0) \to H^r(S_r)$ is generated by the image of a $GL(V)$-invariant element in $H^{r-1}(\wedge^r(\mathcal{V}^* \otimes \mathcal{S}))$, for any $B-$deformed $\mathcal{E}^*$.*

Proof. Consider the morphism of complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}^* \arrow[r] \arrow[d, "g"] & \mathcal{V}^* \otimes \mathcal{S} \arrow[r, "f_B"] \arrow[d, equal, "id"] & \mathcal{S}^* \otimes \mathcal{S} \arrow[r] \arrow[d, "g_0"] & 0 \\ 0 \arrow[r] & \mathcal{E}_0^* \arrow[r] & \mathcal{V}^* \otimes \mathcal{S} \arrow[r, "f_0"] & End_0\ \mathcal{S} \arrow[r] & 0. \end{tikzcd} \tag{3.27}$$

Take the induced long exact sequences, we have

$$\begin{tikzcd} 0 \arrow[r] & \wedge^r \mathcal{E}^* \arrow[r] \arrow[d, "g"] & \cdots \arrow[r] & Z_j \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \wedge^r \mathcal{E}_0^* \arrow[r] & \cdots \arrow[r] & Z_{0,j} \arrow[r] & \cdots \arrow[r] & \mathrm{Sym}^r(End_0\ \mathcal{S}) \arrow[r] & 0, \end{tikzcd} \tag{3.28}$$

where $Z_{0,j} = \wedge^j(\mathcal{V}^* \otimes \mathcal{S}) \otimes \mathrm{Sym}^{r-j}(End_0\ \mathcal{S})$.

We claim that the vertical arrows induces isomorphisms on cohomologies, for $j = 1, ..., r$. First, for $j = r$ this is identity. Then, for $j = 1, ..., r-1$, note that $\mathcal{S}^* \otimes \mathcal{S} \cong End_0\mathcal{S} \oplus \mathcal{O}$. Hence

$$\begin{aligned} \mathrm{Sym}^{r-j}(\mathcal{S}^* \otimes \mathcal{S}) &\cong \mathrm{Sym}^{r-j}(End_0\mathcal{S}) \oplus ... \oplus \mathrm{Sym}^2(End_0\mathcal{S}) \oplus End_0\mathcal{S} \oplus \mathcal{O}, \\ &\cong \mathrm{Sym}^{r-j}(End_0\mathcal{S}) \oplus \mathrm{Sym}^{r-j-1}(\mathcal{S}^* \otimes \mathcal{S}). \end{aligned} \tag{3.29}$$

Since $H^\bullet(\wedge^j(\mathcal{V}^* \otimes \mathcal{S}) \otimes \mathrm{Sym}^{r-j-1}(\mathcal{S}^* \otimes \mathcal{S})) = 0$, the claim is proved by tensoring (3.29) with $\wedge^j(\mathcal{V}^* \otimes \mathcal{S})$.

This implies that the kernels of $H^{j-1}(Z_j) \to H^{j-1}(Z_{j-1})$ for $\wedge^r\mathcal{E}^*$ are all isomorphic to the corresponding ones for Ω^r, via the squares

$$\begin{tikzcd} H^{j-1}(Z_j) \arrow[r] \arrow[d, "\cong"] & H^{j-1}(Z_{j-1}) \arrow[d, "\cong"] \\ H^{j-1}(Z_{0,j}) \arrow[r] & H^{j-1}(Z_{0,j-1}). \end{tikzcd} \tag{3.30}$$

We then proceed to check the Ω^r case.

Only the λ's with $\lambda_1 > n - k$ will contribute to $\mathrm{Ker}\,\delta$. When $r = n - k + 1$, this means we only need to consider the case $\lambda = (r)$, i.e. the long exact sequence reduces to

$$\begin{aligned} 0 &\to 0 \to \wedge^r\mathcal{V}^* \otimes \mathrm{Sym}^r\mathcal{S} \to ... \to \wedge^j\mathcal{V}^* \otimes \mathrm{Sym}^{r-j}\mathcal{S}^* \otimes \mathrm{Sym}^r\mathcal{S} \\ &\to ... \to \mathrm{Sym}^r\mathcal{S}^* \otimes \mathrm{Sym}^r\mathcal{S} \to 0 \end{aligned} \tag{3.31}$$

for the purpose of computing $\operatorname{Ker}\delta$.

By Borel-Weil-Bott, $H^i(Z_j) = H^i(\mathcal{V}^*\otimes \mathrm{Sym}^{r-j}\mathcal{S}^*\otimes \mathrm{Sym}^r\mathcal{S}) = 0$ for $i < n-k$, $j = 1, ..., n-k$[4].

So the only contribution to $\operatorname{Ker}\delta$ comes from the kernel of

$$H^{r-1}(Z_r) \xrightarrow{\bar{f}_B} H^{r-1}(Z_{r-1}),$$

which is the $GL(V)$ invariant part of $H^{r-1}(Z_r) = \wedge^r V^* \otimes \wedge^r V$ (which is $K_0V^* = \mathbb{C}$). Note that the identity map in the middle column of (3.27) induces an identity map

$$H^{r-1}(Z_r) \to H^{r-1}(Z_{0,r}). \tag{3.32}$$

Hence we have

$$\begin{tikzcd} \mathrm{Ker} f_B \arrow[r] \arrow[d] & H^{r-1}(Z_r) \arrow[r, "\bar{f}_B"] \arrow[d, equal] & H^{r-1}(Z_{r-1}) \arrow[d, "\cong"] \\ \mathrm{Ker} f_0 \arrow[r] & H^{r-1}(Z_{0,r}) \arrow[r, "\bar{f}_0"] & H^{r-1}(Z_{0,r-1}). \end{tikzcd} \tag{3.33}$$

So we have $\operatorname{Ker} \bar{f}_B = \operatorname{Ker} \bar{f}_0$, for any B.

$\square$

Let $V = V_1 \oplus L$ be an n dimensional vector space. Consider the inclusion of Grassmannians $X = G(k-1, V_1) \hookrightarrow Y = G(k, V)$, with $[S_1] \mapsto [S_1 \oplus L]$. Note that in this case we have $\mathcal{V}|_X = \mathcal{V}_1 \oplus \mathcal{L}$, $\mathcal{S}|_X = \mathcal{S}_1 \oplus \mathcal{L}$, and similarly for their duals. We extend the $GL(V_1)$ action to V by making L a trivial $GL(V_1)$ module. This will be implicitly used when considering the $GL(V_1)$ invariant parts of cohomologies.

Lemma 3.3.9. *Let* $B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}$. *Then there is a commutative diagram:*

$$\begin{tikzcd} \mathcal{V}^* \otimes \mathcal{S}|_X \arrow[r, "f_B"] \arrow[d, "\pi"] & \mathcal{S}^* \otimes \mathcal{S}|_X \arrow[d, "\pi"] \\ \mathcal{V}_1^* \otimes \mathcal{S}_1 \arrow[r, "f_{B_1}"] & \mathcal{S}_1^* \otimes \mathcal{S}_1, \end{tikzcd} \tag{3.34}$$

given by the natural projections as vertical maps.

[4] $\mathrm{Sym}^{r-j}\mathcal{S}^* \otimes \mathrm{Sym}^r\mathcal{S} = \mathrm{Sym}^{r-j}\mathcal{S}^* \otimes \otimes K_{(r^{k-1})}\mathcal{S}^* \otimes (\wedge^k\mathcal{S})^r$ can be completely determined by Pieri's formula. We just need the fact that, when $j = 1, ..., n-k$, for any component $K_\lambda\mathcal{S}^*$ of $\mathrm{Sym}^{r-j}\mathcal{S}^* \otimes \mathrm{Sym}^r\mathcal{S}$, we have $|\lambda| = \sum_{i=1}^k \lambda_i = -j$. So $\lambda_k < 0$. So it takes at least $n-k$ steps to mutate $(\lambda_1, ..., \lambda_k, 0^{n-k})$ to a decreasing sequence.

Proof. We take the standard basis for $V = V_1 \oplus L$, with $V_1 = \langle e_1, ..., e_{n-1}\rangle$ and $L = \langle e_n\rangle$. On X, each S is generated by $v_1, ..., v_k$ with $v_k = (0, ..., 0, 1)^T$, and $v_b^n = 0$ for $b \le k-1$.

As before, we know the map f_B and f_{B_1} explicitly.

$$f_B : c_i^a \mapsto c_i^a v_b^i + c_i^d B_j^i v_d^j \delta_b^a, \tag{3.35}$$

and similarly for f_{B_1} with a, i indices runs to $k-1, n-1$ instead of k, n. Hence

$$\begin{aligned} \pi \circ f_B - f_{B_1} \circ \pi \quad &= c_n^a v_b^n + c_i^k B_j^i v_k^j \delta_b^a + c_n^d B_j^n v_d^j \delta_b^a + c_i^d B_n^i v_d^n \delta_b^a \\ &= c_i^k B_n^i v_k^n \delta_b^a \quad (\text{since } v_k^j = \delta_n^j) \\ &= 0. \end{aligned} \tag{3.36}$$

□

Together with the commutative diagram

$$\begin{tikzcd} \mathcal{V}^* \otimes \mathcal{S} \arrow[r] \arrow[d] & \mathcal{S}^* \otimes \mathcal{S} \arrow[d] \\ \mathcal{V}^* \otimes \mathcal{S}|_X \arrow[r] & \mathcal{S}^* \otimes \mathcal{S}|_X \end{tikzcd}$$

from natural restrictions, we get

$$\begin{tikzcd} \mathcal{V}^* \otimes \mathcal{S} \arrow[r, "f_B"] \arrow[d] & \mathcal{S}^* \otimes \mathcal{S} \arrow[d, "q"] \\ \mathcal{V}^* \otimes \mathcal{S}|_X \arrow[r, "f_B"] \arrow[d] & \mathcal{S}^* \otimes \mathcal{S}|_X \arrow[d] \\ \mathcal{V}_1^* \otimes \mathcal{S}_1 \arrow[r, "f_{B_1}"] & \mathcal{S}_1^* \otimes \mathcal{S}_1. \end{tikzcd} \tag{3.37}$$

Note that each horizontal line is surjective for suitable B or B_1, with a vector bundle as its kernel. In particular, the second line is so because both the first line and q are surjective. This can also be seen from restricting the first line to X as vector bundles directly. So this induces maps of Koszul complexes similar to (3.28), and further the following commutative diagram:

$$\begin{tikzcd} H^{r-1}(\wedge^r(\mathcal{V}^* \otimes \mathcal{S}))_0 \arrow[r] \arrow[d, "q_{r-1}"] & H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})) \arrow[d, "q_0"] \\ H^{r-1}(\wedge^r(\mathcal{V}^* \otimes \mathcal{S})|_X)_0 \arrow[d, "\pi_{r-1}"] & H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})|_X) \arrow[d, "\pi_0"] \\ H^{r-1}(\wedge^r(\mathcal{V}_1^* \otimes \mathcal{S}_1))_0 \arrow[r] & H^0(\mathrm{Sym}^r(\mathcal{S}_1^* \otimes \mathcal{S}_1)), \end{tikzcd} \tag{3.38}$$

where the 0 in the first line indicates $GL(V)$ invariance and the 0's in the second and third line indicate $GL(V_1)$ invariance.

The first line and the third line are clear from Theorem 3.3.8, with the induced map $H^{r-1}(\wedge^r(\mathcal{V}^* \otimes \mathcal{S}))_0 \to H^{r-1}(\wedge^r(\mathcal{V}_1^* \otimes \mathcal{S}_1))_0$. Observe that the map factors through a subspace of $H^{r-1}(\wedge^r(\mathcal{V}^* \otimes \mathcal{S})|_X)$, which is the preimage of $H^{r-1}(\wedge^r(\mathcal{V}_1^* \otimes \mathcal{S}_1))_0$. So it has to be $H^{r-1}(\wedge^r(\mathcal{V}^* \otimes \mathcal{S})|_X)_0$.

Recall that κ_λ is the canonical generator of $H^0(K_\lambda \mathcal{S}^* \otimes K_\lambda \mathcal{S})$. We will use $\kappa_{\lambda,Y}$ to indicate the base manifold Y.

Lemma 3.3.10. *The map $\pi_0 \circ q_0$ maps $\kappa_{\lambda,Y}$ to $\kappa_{\lambda,X}$.*

Proof. The natural decomposition $\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S}) \cong \sum_\lambda K_\lambda \mathcal{S}^* \otimes K_\lambda \mathcal{S}$ implies that it suffices to consider

$$K_\lambda \mathcal{S}^* \otimes K_\lambda \mathcal{S} \xrightarrow{q_0} K_\lambda \mathcal{S}^* \otimes K_\lambda \mathcal{S}|_X \xrightarrow{\pi_0} K_\lambda \mathcal{S}_1^* \otimes K_\lambda \mathcal{S}_1.$$

We give the explicit expression of κ_λ using the normalized Young symmetrizer

$$c_\lambda = n_\lambda \sum_{g \in R(T), h \in C(T)} \mathrm{sgn}(h) e_{gh}$$

for a Young tableau T of shape λ (we actually do not impose any increasing row / column condition on T, so T is just a filling of λ with $1, ..., r$). Recall that one way to define $K_\lambda S$ over complex numbers is $K_\lambda S = \mathrm{Im}\ c_\lambda(S^{\otimes r})$ (see Section 6.1 of Fulton-Harris [90]), where n_λ is a number.

It is straightforward to verify that

$$\kappa_\lambda = \frac{n_\lambda}{r!} \sum_{g,h,\tau} \mathrm{sgn}(h) \delta^{a_{\tau(1)}}_{b_{\tau\rho(1)}} \cdots \delta^{a_{\tau(r)}}_{b_{\tau\rho(r)}} v_{a_1} \otimes \cdots \otimes v^{b_r}, \tag{3.39}$$

where the summation is for $g \in R(T), h \in C(T), \tau \in S_r$ with $\rho = gh$.

Then we observe that $q_0(\kappa_\lambda) = \kappa_\lambda$, and the effect of the projection π_0 is just changing the summation ranges of a_i, b_i from $\{1, ..., k\}$ to $\{1, ..., k-1\}$. This proves $\pi_0 \circ q_0(\kappa_{\lambda,Y}) = \kappa_{\lambda,X}$.

To understand $\pi_{r-1} \circ q_{r-1}$, we consider the following commutative diagram:

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\ \mathcal{S}|_X \arrow[r] \arrow[d] & \mathcal{V}|_X \arrow[d] \\ \mathcal{S}_1 \arrow[r] & \mathcal{V}_1. \end{tikzcd} \tag{3.40}$$

The first line induces

$$0 \to \mathrm{Sym}^r\mathcal{S} \to \mathrm{Sym}^{r-1}\mathcal{S} \otimes \mathcal{V} \to ... \to \wedge^r\mathcal{V} \to \wedge^r\mathcal{Q} = 0 \tag{3.41}$$

and hence an isomorphism $H^{r-1}(\mathrm{Sym}^r\mathcal{S}) \cong H^0(\wedge^r\mathcal{V})$ (from the vanishing of the cohomologies of the terms in between), which in turn indicates

$$H^{r-1}(\mathrm{Sym}^r\mathcal{S} \otimes \wedge^r\mathcal{V}^*) \cong H^0(\wedge^r\mathcal{V} \otimes \wedge^r\mathcal{V}^*).$$

We then have

$$\begin{tikzcd}
H^0(\wedge^r\mathcal{V} \otimes \wedge^r\mathcal{V}^*) \arrow[r, "\cong"] \arrow[d] & H^{r-1}(\mathrm{Sym}^r\mathcal{S} \otimes \wedge^r\mathcal{V}^*) \arrow[d] \\
H^0(\wedge^r\mathcal{V} \otimes \wedge^r\mathcal{V}^*|_X) \arrow[d] & H^{r-1}(\mathrm{Sym}^r\mathcal{S} \otimes \wedge^r\mathcal{V}^*|_X) \arrow[d] \\
H^0(\wedge^r\mathcal{V}_1 \otimes \wedge^r\mathcal{V}_1^*) \arrow[r, "\cong"] & H^{r-1}(\mathrm{Sym}^r\mathcal{S}_1 \otimes \wedge^r\mathcal{V}_1^*).
\end{tikzcd} \tag{3.42}$$

Since $H^{r-1}(\mathrm{Sym}^r\mathcal{S} \otimes \wedge^r\mathcal{V}^*)$ is the only non-vanishing part of $H^{r-1}(\wedge^r(\mathcal{V}^* \otimes \mathcal{S}))$, we actually have

$$\begin{tikzcd}
H^0(\wedge^r\mathcal{V} \otimes \wedge^r\mathcal{V}^*)_0 \arrow[r, "\cong"] \arrow[d, "q'_{r-1}"] & H^{r-1}(\wedge^r(\mathcal{V}^* \otimes \mathcal{S}))_0 \arrow[d, "q_{r-1}"] \\
H^0(\wedge^r\mathcal{V} \otimes \wedge^r\mathcal{V}^*|_X)_0 \arrow[d, "\pi'_{r-1}"] & H^{r-1}(\wedge^r(\mathcal{V}^* \otimes \mathcal{S})|_X)_0 \arrow[d, "\pi_{r-1}"] \\
H^0(\wedge^r\mathcal{V}_1 \otimes \wedge^r\mathcal{V}_1^*)_0 \arrow[r, "\cong"] & H^{r-1}(\wedge^r(\mathcal{V}_1^* \otimes \mathcal{S}_1))_0.
\end{tikzcd} \tag{3.43}$$

Note that we take the $GL(V)$ and $GL(V_1)$ invariant parts as before.

Lemma 3.3.11. *$\pi_{r-1} \circ q_{r-1}$ is an isomorphism.*

Proof. It suffices to prove that $\pi'_{r-1} \circ q'_{r-1}$ is an isomorphism. This can be done by direct computation. Note that q'_{r-1} is induced from the identity map

$$H^0(\wedge^r\mathcal{V} \otimes \wedge^r\mathcal{V}^*) \to H^0(\wedge^r\mathcal{V} \otimes \wedge^r\mathcal{V}^*|_X),$$

which is

$$\wedge^r V \otimes \wedge^r V^* \to \wedge^r V \otimes \wedge^r V^*,$$

and π'_{r-1} is induced from the projection to $(\wedge^r V_1 \otimes \wedge^r V_1^*)$.

It is then straight forward to observe that $\pi'_{r-1} \circ q'_{r-1}$ maps

$$\sum_{a_i,b_i=1}^{n} \sum_{\rho \in S_r} (-1)^\rho \delta^{a_1}_{b_{\rho(1)}} \cdots \delta^{a_r}_{b_{\rho(r)}} e_{a_1} \otimes \cdots \otimes e_{a_r} \otimes e^{b_1} \otimes \cdots \otimes e^{b_r},$$

the generator of the one dimensional space $H^0(\wedge^r \mathcal{V} \otimes \wedge^r \mathcal{V}^*)_0$, to

$$\sum_{a_i,b_i=1}^{n-1} \sum_{\rho \in S_r} (-1)^\rho \delta^{a_1}_{b_{\rho(1)}} \cdots \delta^{a_r}_{b_{\rho(r)}} e_{a_1} \otimes \cdots \otimes e_{a_r} \otimes e^{b_1} \otimes \cdots \otimes e^{b_r},$$

the generator of $H^0(\wedge^r \mathcal{V}_1 \otimes \wedge^r \mathcal{V}_1^*)_0$. □

Theorem 3.3.12. *When $r = n - k + 1$, the kernel κ of*

$$\Delta : H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})) \to H^r(\wedge^r \mathcal{E}^*)$$

takes the same form for all $G(k + c, n + c)$, in terms of $I_1, ..., I_r$.

Proof. Applying Lemma 3.3.11 to (3.38), we find that $\pi_0 \circ q_0(\kappa_B) = \kappa_{B_1}$, when $B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $\kappa_B = \sum s^{\lambda,B} \kappa_\lambda$, $\kappa_{B_1} = \sum s^{\lambda,B_1} \kappa_\lambda$, then $s^{\lambda,B} = s^{\lambda,B_1}$. Now let

$$A_r = \{\alpha = (\alpha_1, ..., \alpha_r) \in \mathbb{Z}^r_{\geq 0} | \sum_{i=1}^{r} \alpha_i \leq r\}$$

and $I^\alpha = \prod_{i=1}^r I_i^{\alpha_i}$. Observe that $I_j(B) = I_j(B_1)$, so we can write

$$\begin{aligned} s^{\lambda,B} &= \sum_{\alpha \in A_r} s^{\lambda,n}_\alpha I^\alpha, \\ s^{\lambda,B_1} &= \sum_{\alpha \in A_r} s^{\lambda,n-1}_\alpha I^\alpha. \end{aligned} \tag{3.44}$$

Note that this holds for arbitrary B_1 with $B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}$. For $I_1, I_2, ..., I_r$, we can always solve the equation $t^r + \sum_{i=1}^r t^{r-i}(-1)^i I_i = 0$ and get r roots $t_1, ..., t_r$. Then the matrix $\mathrm{diag}\{t_1, ..., t_r\}$ has invariants $I_1, ..., I_r$. We can take $I_1, I_2, ..., I_r$ sufficiently small such that our matrix $\mathrm{diag}\{t_1, ..., t_r\}$ is not in the degenerate locus. This implies that

$$\sum_{\alpha \in A_r} s^{\lambda,n}_\alpha I^\alpha = \sum_{\alpha \in A_r} s^{\lambda,n-1}_\alpha I^\alpha \tag{3.45}$$

as an equality of two holomorphic functions of variables $I_1, .., I_r$ holds on an open set. So it holds in general by the identity theorem of holomorphic functions of several variables.

This shows that $s^{\lambda,n}_\alpha = s^{\lambda,n-1}_\alpha$ for arbitrary n and finishes the proof of the theorem. □

For later use, we require the following

Definition 3.3.2.

$$\tilde{\kappa}_{(r)} \equiv \sum_{i=0}^{\min\{r,n\}} I_i \ \kappa_{(r-i)} \cdot \kappa^i_{(1)}.$$

For the special case $B = \varepsilon I$, we can derive the expression of κ_B directly.

In this case, we have a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \Omega \arrow[r] \arrow[d] & \mathcal{V}^* \otimes \mathcal{S} \arrow[r, "f"] \arrow[d, equal] & \mathcal{S}^* \otimes \mathcal{S} \arrow[r] \arrow[d, "h"', "\cong"] & 0 \\ 0 \arrow[r] & \mathcal{E}^* \arrow[r] & \mathcal{V}^* \otimes \mathcal{S} \arrow[r, "f_B"] & \mathcal{S}^* \otimes \mathcal{S} \arrow[r] & 0, \end{tikzcd} \tag{3.46}$$

where h is given by $h : \sigma^a_b \mapsto \sigma^a_b + \varepsilon(\text{tr } \sigma)\delta^a_b$, for any local section σ^a_b of $\mathcal{S}^* \otimes \mathcal{S}$.

Theorem 3.3.13. *When $B = \varepsilon I$,*

$$h(\kappa_{(r)}) = \sum_{j=0}^{k+r-n-1} \varepsilon^j \binom{k+r-n-1}{j} \kappa^j_{(1)} \tilde{\kappa}_{(r-j)}. \tag{3.47}$$

Proof. $\kappa_{(r)}$ is the identity bundle map on $\text{Sym}^r\mathcal{S}$. For any section σ of $\mathcal{S}^* \otimes \mathcal{S}$, h is defined by

$$h(\sigma^a_b) = \sigma^a_b + \varepsilon(\text{Tr}\sigma)\delta^a_b,$$

where $h(\sigma^a_b)$ represents the component of the image of σ under h. More generally, given a section T of $(\mathcal{S}^* \otimes \mathcal{S})^{\otimes r}$, the tensor product of r copies of h is given by $h^r = h_1 \otimes h_2 \cdots \otimes h_r$, where h_i acts only on the i'th factor of $(\mathcal{S}^* \otimes \mathcal{S})^{\otimes r}$, specifically,

$$h_i(T^{a_1 \cdots a_r}_{b_1 \cdots b_r}) = T^{a_1 \cdots a_r}_{b_1 \cdots b_r} + \varepsilon T^{a_1 \cdots a_{i-1} c a_{i+1} \cdots a_r}_{b_1 \cdots b_{i-1} c b_{i+1} \cdots b_r} \delta^{a_i}_{b_i}.$$

$\kappa_{(r)}$ has components $(r!)^{-1}\delta^{(a_1 \cdots a_r)}_{b_1 \cdots b_r}$, $a_i = 1, \cdots, k, b_i = 1, \cdots, k$, where $\delta^{(a_1 \cdots a_r)}_{b_1 \cdots b_r}$ denotes $\delta^{(a_1}_{b_1}\delta^{a_2}_{b_2} \cdots \delta^{a_r)}_{b_r}$, and $(\cdots)$ denotes the symmetrization of indices. We denote $h_1 \otimes h_2 \cdots \otimes h_i$ by h^i for $i = 1, \cdots r$. Thus one can compute, $h^r(\kappa_{(r)})$ has components

$$\begin{aligned} h^r((r!)^{-1}\delta^{(a_1 \cdots a_r)}_{b_1 \cdots b_r}) = (r!)^{-1}h^{r-1}\left(\delta^{(a_1 \cdots a_r)}_{b_1 \cdots b_r} + \varepsilon\delta^{(a_1 \cdots a_{r-1} c)}_{b_1 \cdots b_{r-1} c}\delta^{a_r}_{b_r}\right) \\ =(r!)^{-1}h^{r-1}\left(\delta^{(a_1 \cdots a_r)}_{b_1 \cdots b_r} + \varepsilon\delta^{(a_1 \cdots a_{r-1})}_{b_1 \cdots b_{r-1}}\delta^c_c\delta^{a_r}_{b_r} + \varepsilon\sum_{i=1}^{r-1}\delta^{(a_1 \cdots a_{r-1})}_{b_1 \cdots \hat{b}_i \cdots b_{r-1} c}\delta^c_{b_i}\delta^{a_r}_{b_r}\right) \\ =(r!)^{-1}h^{r-1}\left(\delta^{(a_1 \cdots a_r)}_{b_1 \cdots b_r} + \varepsilon(k+r-1)\delta^{(a_1 \cdots a_{r-1})}_{b_1 \cdots b_{r-1}}\delta^{a_r}_{b_r}\right). \end{aligned} \tag{3.48}$$

Define Y_s to have components

$$\delta^{(a_1 \cdots a_r)}_{b_1 \cdots b_r} + \sum_{t=1}^{s} \sum_{r-s+1 \leqslant i_1 < \cdots < i_t \leqslant r} \varepsilon^t \frac{(k+r-1)!}{(k+r-t-1)!}\delta^{(a_1 \cdots \hat{a}_{i_1}\hat{a}_{i_2} \cdots \hat{a}_{i_t} \cdots a_r)}_{b_1 \cdots \hat{b}_{i_1}\hat{b}_{i_2} \cdots \hat{b}_{i_t} \cdots b_r}\delta^{a_{i_1}}_{b_{i_1}} \cdots \delta^{a_{i_t}}_{b_{i_t}}.$$

We claim that

$$h^r(\kappa_{(r)}) = (r!)^{-1}h^{r-s}(Y_s), \tag{3.49}$$

for $s = 1, 2, \cdots, r$. This is true for $s = 1$ due to (3.48). Let's assume the claim is true for some s, and prove that it is also true for $s+1$. This can be shown through a direct computation as follows:

$$\begin{aligned}
&h^{r-s}((Y_s)^{a_1\cdots a_r}_{b_1\cdots b_r})\\
&\qquad =h^{r-s-1}\Bigg((Y_s)^{a_1\cdots a_r}_{b_1\cdots b_r}+\\
&\qquad \sum_{t=0}^{s}\sum_{r-s+1\leqslant i_1<\cdots<i_t\leqslant r}\varepsilon^{t+1}\frac{(k+r-1)!}{(k+r-t-2)!}\delta^{(a_1\cdots\hat{a}_{r-s}\hat{a}_{i_1}\hat{a}_{i_2}\cdots\hat{a}_{i_t}\cdots a_r)}_{b_1\cdots\hat{b}_{r-s}\hat{b}_{i_1}\hat{b}_{i_2}\cdots\hat{b}_{i_t}\cdots b_r}\delta^{a_{r-s}}_{b_{r-s}}\delta^{a_{i_1}}_{b_{i_1}}\cdots\delta^{a_{i_t}}_{b_{i_t}}\Bigg),\\
&\qquad =h^{r-s-1}((Y_{s+1})^{a_1\cdots a_r}_{b_1\cdots b_r}).
\end{aligned}$$

Thus we can take $s = r$ to have

$$h^r(\kappa_{(r)}) = (r!)^{-1}Y_r. \tag{3.50}$$

Because $(r-t)^{-1}\delta^{(a_1\cdots\hat{a}_{i_1}\hat{a}_{i_2}\cdots\hat{a}_{i_t}\cdots a_r)}_{b_1\cdots\hat{b}_{i_1}\hat{b}_{i_2}\cdots\hat{b}_{i_t}\cdots b_r}$ are the components of $\kappa_{(r-t)}$, (3.50) can be written as

$$\begin{aligned}
h^r(\kappa_{(r)}) =&\kappa_{(r)} + \sum_{t=1}^{r}\sum_{1\leqslant i_1<\cdots<i_t\leqslant r}\varepsilon^t\frac{(k+r-1)!(r-t)!}{(k+r-t-1)!r!}\kappa_{(r-t)}\kappa^t_{(1)},\\
=&\kappa_{(r)} + \sum_{t=1}^{r}\varepsilon^t\binom{r}{t}\frac{(k+r-1)!(r-t)!}{(k+r-t-1)!r!}\kappa_{(r-t)}\kappa^t_{(1)},\\
=&\sum_{t=0}^{r}\binom{k+r-1}{t}\kappa_{(r-t)}\kappa^t_{(1)}\varepsilon^t.
\end{aligned} \tag{3.51}$$

With the aid of the combinatorial formula

$$\binom{m+n}{l} = \sum_{i=0}^{m}\binom{m}{i}\binom{n}{l-i},$$

where $\binom{n}{i} = 0$ when $i < 0$ or $i > n$, one can compute, for $r > n - k$,

$$
\begin{aligned}
h(\kappa_{(r)}) &= \sum_{i=0}^{r} \varepsilon^i \binom{k+r-1}{i} \kappa_{(r-i)} \kappa_{(1)}^i \\
&= \sum_{i=0}^{r} \sum_{j=0}^{k+r-n-1} \varepsilon^j \varepsilon^{i-j} \binom{k+r-n-1}{j} \binom{n}{i-j} \kappa_{(r-i)} \kappa_{(1)}^i \\
&= \sum_{j=0}^{k+r-n-1} \sum_{i=j}^{\min\{n+j,r\}} \varepsilon^j \binom{k+r-n-1}{j} I_{i-j} \kappa_{(r-i)} \kappa_{(1)}^i \\
&= \sum_{j=0}^{k+r-n-1} \varepsilon^j \binom{k+r-n-1}{j} \left(\sum_{i=0}^{\min\{n,r-j\}} I_i \kappa_{(r-j-i)} \kappa_{(1)}^i \right) \kappa_{(1)}^j \\
&= \sum_{j=0}^{k+r-n-1} \varepsilon^j \binom{k+r-n-1}{j} \kappa_{(1)}^j \tilde{\kappa}_{(r-j)}.
\end{aligned}
\tag{3.52}
$$

□

We then have an algorithm to compute the kernel for $r = n - k + 1$.

From Theorem 3.3.6, 3.3.5 and 3.3.8 we know that the kernel κ is of the form

$$\kappa = \sum s^\lambda \kappa_\lambda,$$

where s^λ is a polynomial in I_i, $i = 1, ..., r$, the similarity invariants of the characteristic polynomial of B. Moreover Theorem 3.3.6 and 3.3.8 guarantee that the degree of the polynomial is no more than r.

From Theorem 3.3.12 we know that s^λ has the form

$$s^\lambda = \sum_{\alpha \in A_r} s^\lambda_\alpha I^\alpha, \tag{3.53}$$

where $s^\lambda_\alpha, \alpha \in A_r$, are independent of n.

To determine s^λ_α, we consider specific choices of B on $G(k, n)$ with $k \geq r$. We take $k \geq r$ because this is the 'stable-range'. Namely, when $k < r$, some κ_λ might be 0 for dimension reason, hence cannot be seen in the kernel relation even if they are there for $k \geq r$.

We first work out the general h maps making the following diagram commute:

$$
\begin{tikzcd}
\mathcal{V}^* \otimes \mathcal{S} \arrow[r, "f_B"] \arrow[d, equal] & \mathcal{S}^* \otimes \mathcal{S} \arrow[d, "h"] \\
\mathcal{V}^* \otimes \mathcal{S} \arrow[r, "f_{\tilde{B}}"] & \mathcal{S}^* \otimes \mathcal{S}\ .
\end{tikzcd}
\tag{3.54}
$$

Lemma 3.3.14. *If $\tilde{B} = (1+k\varepsilon)B + \varepsilon I, 1 + k\varepsilon \neq 0$, then diagram (3.54) commutes.*

Proof. Since $\mathrm{Hom}(\mathcal{S}^* \otimes \mathcal{S}, \mathcal{S}^* \otimes \mathcal{S}) \cong H^0(\mathrm{Sym}^2(\mathcal{S}^* \otimes \mathcal{S}) \oplus \wedge^2(\mathcal{S}^* \otimes \mathcal{S})) \cong H^0(\mathrm{Sym}^2(\mathcal{S}^* \otimes \mathcal{S})) \cong \mathbb{C}^2$, we have two parameters and a general map h can be written as $\sigma^a_b \mapsto k_1 \sigma^a_b + k_2(\mathrm{tr}\ \sigma)\delta^a_b$, where δ^a_b is the Kronecker delta function. It is an isomorphism when $k_1(k_1 + k_2 k) \neq 0$, and the inverse is

$$(k_1', k_2') = (\frac{1}{k_1}, -\frac{k_2}{k_1(k_1 + k_2 k)}).$$

Writing the condition $h \circ f_B = f_{\tilde{B}}$ in coordinates, we have

$$k_1(c^a_i v^i_b + c^d_i B^i_j v^j_d \delta^a_b) + k_2 \delta^a_b (c^d_i v^i_d + k c^d_i B^i_j v^j_d \delta^a_b) = c^a_i v^i_b + c^d_i \tilde{B}^i_j v^j_d \delta^a_b. \tag{3.55}$$

Take $a \neq b$, we find $k_1 = 1$. Take $a = b$, we have

$$(1 + k k_2) c^d_i B^i_j v^j_d + k_2 c^d_i v^i_d = c^d_i \tilde{B}^i_j v^j_d,$$

i.e. $\tilde{B}^i_j = (1 + k k_2) B^i_j + k_2 \delta^i_j$. □

This is the second transformation on B that produces isomorphic vector bundles, in addition to the similarity transformation. We will refer this as ε-Transformation, and write

$$\tilde{B}^i_j = ET(B)^i_j = (1 + k\varepsilon) B^i_j + \varepsilon \delta^i_j, 1 + k\varepsilon \neq 0. \tag{3.56}$$

Now let's see how to use this lemma and results for diagonal B to determine the general form of the kernel.

Theorem 3.3.15. *For a generic deformed tangent bundle $\mathcal{E}$ over $X = G(k, n)$, when $r = n - k + 1$, the kernel of $H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})) \to H^r(\wedge^r \mathcal{E}^*)$ is generated by*

$$\tilde{\kappa}_{(r)} = \sum_{i=0}^{r} I_i\ \kappa_{(r-i)} \cdot \kappa^i_{(1)}. \tag{3.57}$$

Proof. Theorem 3.3.13 tells us that, on $G(k, n)$, when $B = \varepsilon_1 I, \varepsilon_1 \neq -1/k$, the kernel is spanned by

$$\tilde{\kappa}_{(r),B} = \sum_{i=0}^{r} I_i(B)\ \kappa_{(r-i)} \cdot \kappa^i_{(1)}.$$

Thus, by Lemma 3.3.11 and (3.38), we see, when $B' = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$, the kernel is spanned by

$$\tilde{\kappa}_{(r),B} = \sum_{i=0}^{r} I_i(B)\ \kappa_{(r-i)} \cdot \kappa^i_{(1)} \ = \sum_{i=0}^{r} I_i(B')\ \kappa_{(r-i)} \cdot \kappa^i_{(1)}$$

on $G(k+1,n+1)$. Lemma 3.3.14 shows that $h(\tilde{\kappa}_{(r),B})$ is in the kernel for

$$B'' = \begin{pmatrix} (1+(k+1)\varepsilon_2)B + \varepsilon_2 I & 0 \\ 0 & \varepsilon_2 \end{pmatrix}$$

on $G(k+1,n+1)$. Note that

$$\begin{aligned} \det(tI + B'') &= (t+\varepsilon_2)\det((t+\varepsilon_2)I + (1+(k+1)\varepsilon_2)B) \\ &= \sum_{i=0}^{n} (t+\varepsilon_2)^{n+1-i} I_i(B)(1+(k+1)\varepsilon_2)^i \\ &= \sum_{m=0}^{n+1} t^{n+1-m} \sum_{i=0}^{n} \varepsilon_2^{m-i} \binom{n+1-i}{n+1-m} I_i(B)(1+(k+1)\varepsilon_2)^i, \end{aligned} \tag{3.58}$$

hence

$$I_m(B'') = \sum_{i=0}^{n} \varepsilon_2^{m-i} \binom{n+1-i}{n+1-m} I_i(B)(1+(k+1)\varepsilon_2)^i.$$

From Theorem 3.3.13, one can compute

$$\begin{aligned} h(\tilde{\kappa}_{(r),B}) &= \sum_{i=0}^{r} I_i(B) h(\kappa_{(1)})^i h(\kappa_{(r-i)}) \\ &= \sum_{i=0}^{r} I_i(B)(1+(k+1)\varepsilon_2)^i \kappa_{(1)}^i \sum_{j=0}^{r-i} \varepsilon_2^j \binom{n+1-i}{j} \kappa_{(r-i-j)} \kappa_{(1)}^j \\ &= \sum_{m=0}^{r} \kappa_{(1)}^m \kappa_{(r-m)} \sum_{i+j=m} \varepsilon_2^j I_i(B)(1+(k+1)\varepsilon_2)^i \binom{n+1-i}{j} \\ &= \sum_{m=0}^{r} \kappa_{(1)}^m \kappa_{(r-m)} I_m(B''), \end{aligned}$$

which shows the kernel for B'' has the same form as B, namely (3.57). The same method can be applied to B'' in place of B, and induction shows the kernel contains $\tilde{\kappa}_{(r),\tilde{B}} = \sum_{i=0}^{r} I_i(\tilde{B})\ \kappa_{(r-i)} \cdot \kappa_{(1)}^i$ for $\tilde{B} = ET^l(B), l = 0,1,2...$, where $ET^0(B) = B$ and $ET^l(B) = ET(ET^{l-1}(B))$. In particular, if we take $\tilde{\varepsilon}_i = 1 + (k+i-1)\varepsilon_i$, then

$$\begin{aligned} ET^0(B) &= \varepsilon_1 I, \\ ET^1(B) &= \operatorname{diag}((\tilde{\varepsilon}_2\varepsilon_1 + \varepsilon_2)I, \varepsilon_2), \\ ET^2(B) &= \operatorname{diag}((\tilde{\varepsilon}_3\tilde{\varepsilon}_2\varepsilon_1 + \tilde{\varepsilon}_3\varepsilon_2 + \varepsilon_3)I, \tilde{\varepsilon}_3\varepsilon_2 + \varepsilon_3, \varepsilon_3), \\ &\vdots \\ ET^r(B) &= \operatorname{diag}((\varepsilon_{r+1} + \tilde{\varepsilon}_{r+1}\varepsilon_r + \cdots + \tilde{\varepsilon}_{r+1}\cdots\tilde{\varepsilon}_2\varepsilon_1)I, \\ &\qquad \varepsilon_{r+1} + \tilde{\varepsilon}_{r+1}\varepsilon_r + \cdots + \tilde{\varepsilon}_{r+1}\cdots\tilde{\varepsilon}_3\varepsilon_2, \cdots, \varepsilon_{r+1}), \end{aligned}$$

where the parameters $\varepsilon_1, \cdots, \varepsilon_{r+1}$ are such that all the matrices above are not on the degenerate locus. For any $\xi_1, \cdots, \xi_r \in \mathbb{C}$ such that $0 < |\xi_i| \ll 1$ and $\xi_i \neq \xi_j$ for $i \neq j$, there is a unique set of solutions to

$$\begin{cases} \varepsilon_{r+1} = \xi_1, \\ \varepsilon_{r+1} + \tilde{\varepsilon}_{r+1}\varepsilon_r = \xi_2, \\ \quad \cdots \\ \varepsilon_{r+1} + \tilde{\varepsilon}_{r+1}\varepsilon_r + \cdots + \tilde{\varepsilon}_{r+1} \cdots \tilde{\varepsilon}_3 \varepsilon_2 = \xi_r, \\ \varepsilon_{r+1} + \tilde{\varepsilon}_{r+1}\varepsilon_r + \cdots + \tilde{\varepsilon}_{r+1} \cdots \tilde{\varepsilon}_2 \varepsilon_1 = 0. \end{cases} \tag{3.59}$$

This implies the expression (3.57) is in the kernel for any deformation given by

$$B = \mathrm{diag}(0, 0, \cdots, 0, \xi_r, \cdots, \xi_1)$$

with $0 < |\xi_i| \ll 1$ and $\xi_i \neq \xi_j$ for $i \neq j$. Since this means the expression of the kernel is given by (3.57) for all $I_1, I_2, \cdots, I_r$ in a small open set in $\mathbb{C}^r$, we see the kernel at order $n-k+1$ is generated by

$$\tilde{\kappa}_{(r)} = \sum_{\lambda,\alpha} s^{\lambda}_{\alpha} I^{\alpha} \kappa_{\lambda} = \sum_{i=0}^{r} I_i \ \kappa_{(r-i)} \cdot \kappa^{i}_{(1)}$$

for a generic deformation. □

Here are some examples. For $G(n-1, n), r = 2$, the result is

$$\kappa = (1 + I_1 + I_2)\kappa_{(2)} + (I_1 + I_2)\kappa_{(1,1)}. \tag{3.60}$$

For $G(n-2, n), r = 3$, the result is

$$\kappa = (1 + I_1 + I_2 + I_3)\kappa_{(3)} + (I_1 + 2I_2 + 2I_3)\kappa_{(2,1)} + (I_2 + I_3)\kappa_{(1,1,1)}. \tag{3.61}$$

Now let's determine elements in the kernel of the connecting map for higher orders. Let $V = V_1 \oplus L$ be an n dimensional vector space. Consider the inclusion of Grassmannians $X = G(k, V_1) \hookrightarrow Y = G(k, V)$ induced by $V_1 \hookrightarrow V$, with $[S] \mapsto [S]$, where $S \subset V_1$ is a subspace. Note that in this case we have $\mathcal{V}|_X = \mathcal{V}_1 \oplus \mathcal{L}$, $\mathcal{S}|_X = \mathcal{S}$, and similarly for their duals.

Lemma 3.3.16. *Let* $B = \begin{pmatrix} B_1 & * \\ 0 & \varepsilon \end{pmatrix}$. *Then there is a commutative diagram:*

$$\begin{tikzcd} \mathcal{V}^* \otimes \mathcal{S} \arrow[r, "f_B"] \arrow[d] & \mathcal{S}^* \otimes \mathcal{S} \arrow[d] \\ \mathcal{V}^* \otimes \mathcal{S}|_X \arrow[r, "f_B"] \arrow[d, "\pi"] & \mathcal{S}^* \otimes \mathcal{S}|_X \arrow[d, "\pi"] \\ \mathcal{V}_1^* \otimes \mathcal{S} \arrow[r, "f_{B_1}"] & \mathcal{S}^* \otimes \mathcal{S}, \end{tikzcd} \tag{3.62}$$

given by the natural projections as vertical maps.

Proof. This is entirely analogous to Lemma 3.3.9. The first square is obviously commutative. For the second one, we take the standard basis for $V = V_1 \oplus L$, with $V_1 = \langle e_1, ..., e_{n-1}\rangle$ and $L = \langle e_n \rangle$. On X, each S is generated by $v_1, ..., v_k$ with $v_b^n = 0$ for $b \leq k$.

As before, we know the map f_B and f_{B_1} explicitly.

$$f_B : c_i^a \mapsto c_i^a v_b^i + c_i^d B_j^i v_d^j \delta_b^a, \tag{3.63}$$

and similarly for f_{B_1} with a, i indices runs to $k, n-1$ instead of k, n. Hence

$$\begin{aligned} \pi \circ f_B - f_{B_1} \circ \pi \quad &= c_n^a v_b^n + \textstyle\sum_{j=1}^{n-1} c_n^d B_j^n v_d^j \delta_b^a + \sum_{i=1}^{n} c_i^d B_n^i v_d^n \delta_b^a \\ &= 0. \end{aligned} \tag{3.64}$$

$\square$

Theorem 3.3.17. *On $G(k, n)$, with B outside the degenerate locus, we have $\tilde{\kappa}_{(r)} \in \mathbb{K}_r$ for any $r \geq n - k + 1$.*

Proof. To specify the dependence of the underlying variety $G(k, n)$ and that of the map f_B, we write $\mathbb{K}_r$ as $\mathbb{K}_r(k, n, B)$. Namely, $\mathbb{K}_r(k, n, B)$ is the kernel of

$$H^0(G(k, n), \mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})) \to H^r(G(k, n), \wedge^r \mathcal{E}^*)$$

for $\mathcal{E}^*$ defined as the kernel of f_B in (3.23).

We compare $\mathbb{K}_r(k, n, B)$ with $\mathbb{K}_r(k, n-1, B_1)$. The idea here is similar to that of Theorem 3.3.12. As a corollary of Lemma 3.3.16, we have

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{K}_r(k, n, B) \arrow[r] \arrow[d, hook] & H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})) \arrow[r] \arrow[d, equal] & H^r(\wedge^r \mathcal{E}^*) \arrow[d] \\ 0 \arrow[r] & \mathbb{K}_r(k, n-1, B_1) \arrow[r] & H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})) \arrow[r] & H^r(\wedge^r \mathcal{E}_1^*). \end{tikzcd} \tag{3.65}$$

Taking $r = n - k + 1$, we have that the image of $\tilde{\kappa}_{(r)} \in \mathbb{K}_r(k, n, B) \subset \mathbb{K}_r(k, n-1, B_1)$ is of the form

$$\imath\left(\tilde{\kappa}_{(n-k+1)}\right) = \tilde{\kappa}_{(n-k+1)} + \varepsilon \tilde{\kappa}_{(n-k)} \kappa_{(1)}.$$

We know $\imath\left(\tilde{\kappa}_{(n-k+1)}\right) \in \mathbb{K}_r(k, n-1, B_1)$ by (3.65), and from Theorem 3.3.15 we know the second term $\varepsilon \tilde{\kappa}_{(n-k)} \kappa_{(1)}$ is also in the same kernel, hence $\tilde{\kappa}_{(n-k+1)}$ must be in the kernel. Up to a change of notation, this shows that on $G(k, n)$, we have $\tilde{\kappa}_{(r)} \in \mathbb{K}_r$ for $r = n - k + 2$. Then the theorem holds in general by induction. $\square$

We know that $\tilde{\kappa}_{(r)} \cdot \kappa_\mu \in \mathbb{K}_s$, for any $r \geq n - k + 1$ and $|\mu| = s - r$. For generic deformations of the tangent bundles, they generate the kernel $\mathbb{K}_s$.

Theorem 3.3.18. *For generic deformed tangent bundles, the kernel $\mathbb{K}_s$ is generated by $\tilde{\kappa}_{(r)} \cdot \kappa_\mu$, $r \geq n-k+1$ and $|\mu| = s-r$.*

Proof. Since linear independence is an open condition, it suffices to show that for the tangent bundle ($B = 0$), $\mathbb{K}_s$ is generated by $\kappa_r \cdot \kappa_\mu$, $r \geq n-k+1$ and $|\mu| = s-r$.

Let $\lambda = (\lambda_1, \cdots, \lambda_k)$. Notice that the Giambelli formula for Schur functions applies to $\kappa_\lambda \in H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S}))$. In general it says

$$\kappa_\lambda = \det \begin{pmatrix} \kappa_{(\lambda_1)} & \kappa_{(\lambda_1+1)} & \cdots & \kappa_{(\lambda_1+k-1)} \\ \kappa_{(\lambda_2-1)} & \kappa_{(\lambda_2)} & \cdots & \kappa_{(\lambda_2+k-2)} \\ & \cdots\cdots\cdots\cdots & & \\ \kappa_{(\lambda_k-k+1)} & \kappa_{(\lambda_k-k+2)} & \cdots & \kappa_{(\lambda_k)} \end{pmatrix}. \tag{3.66}$$

Recall that for the tangent bundle, $\mathbb{K}_s$ is generated by κ_λ, with $\lambda = (\lambda_1, ..., \lambda_k)$, $|\lambda| = s$ and $\lambda_1 \geq n-k+1$. Each such κ_λ is of the form $\sum \kappa_{(r)} \cdot \kappa_\mu$ by the Giambelli formula, with $r \geq n-k+1$.

It remains to show that $h^s(\Omega^s) = h^s(\wedge^s \mathcal{E}^*)$ for a generic deformation. Elements in the kernel correspond to $H^{j-1}(\wedge^j(\mathcal{V}^* \otimes \mathcal{S}) \otimes \mathrm{Sym}^{r-j}(\mathcal{S}^* \otimes \mathcal{S}))$ with $1 \leqslant j \leqslant s$, thus we have

$$h^s(\Omega^s) \leq h^s(\wedge^s \mathcal{E}^*).$$

From semicontinuity, for a generic deformation,

$$h^s(\Omega^s) \geq h^s(\wedge^s \mathcal{E}^*),$$

and hence $h^s(\Omega^s) = h^s(\wedge^s \mathcal{E}^*)$. An alternative derivation of this result is as follows. Note that when $q \neq p$, by semicontinuity,

$$h^q(\wedge^p \mathcal{E}^*) \leq h^q(\Omega^p) = 0,$$

hence

$$h^q(\wedge^p \mathcal{E}^*) = h^q(\Omega^p) = 0.$$

Since the holomorphic Euler characteristic of $\wedge^p \mathcal{E}^*$ does not change across the flat family, we have that $h^s(\Omega^s) = h^s(\wedge^s \mathcal{E}^*)$ for generic deformations. This completes the proof. $\square$

This is a full description of the graded module structure of the classical sheaf cohomology. Moreover, it implies the following description of the ring structure. Combining theorems 3.3.17 and 3.3.18, we get

Theorem 3.3.19. *For generic deformed tangent bundles, the classical sheaf cohomology ring is the ring of symmetric polynomials in k indeterminates modulo the ideal generated by the $\tilde{\kappa}$'s, which can be given explicitly as*

$$\mathbb{C}[\kappa_{(1)}, \kappa_{(2)}, \cdots]/\langle D_{k+1}, D_{k+2}, \cdots, \tilde{\kappa}_{(n-k+1)}, \tilde{\kappa}_{(n-k+2)}, \cdots \rangle, \tag{3.67}$$

where

$$D_m = \det\left(\kappa_{(1+j-i)}\right)_{1\leq i,j\leq m}, \tag{3.68}$$

and $\tilde{\kappa}_r$ is defined in Definition 3.3.2.

3.4 Quantum sheaf cohomology

Having determined the classical sheaf cohomology, let us move on to consider the quantum sheaf cohomology.

The quantum sheaf cohomology ring is the OPE ring of an A/2-twisted theory, just as the ordinary quantum cohomology ring is the OPE ring of an A-twisted theory – quantum sheaf cohomology is the (0,2) generalization of ordinary quantum cohomology. In this section we will describe it for Grassmannians with deformations of the tangent bundle, and give a physics-based derivation.

Also, so far we have given results for general deformation matrices A and B, but as previously observed, the matrix A is redundant. In the rest of this chapter, we will assume without loss of generality that A is the identity. The general case can be reconstructed by replacing B (in results derived for $A = I$) with BA^{-1}.

3.4.1 Gauge invariant operators

The Coulomb branch arguments given in the section 3.2, both one-loop effective actions and supersymmetric localization, involve for a $U(k)$ gauge theory a set of k mutually commuting fields $\sigma_1, \cdots, \sigma_k$ which act as local coordinates on the Coulomb branch. However, these individually are not quite invariant under $U(k)$, as there is still a residual Weyl group action.

The complete group invariants are symmetric polynomials in $\sigma_1, \cdots, \sigma_k$, and these can be naturally associated to Young diagrams, via what are known as Schur polynomials (see [82][chapter 6] or [9][appendix B] for an introduction). For example, if $k = 2$, then

$$\begin{aligned}
\sigma_{\yng(1)} &= \sigma_1 + \sigma_2, \\
\sigma_{\yng(2)} &= \sigma_1^2 + \sigma_2^2 + \sigma_1\sigma_2, \\
\sigma_{\yng(3)} &= \sigma_1^3 + \sigma_1^2\sigma_2 + \sigma_1\sigma_2^2 + \sigma_2^3, \\
\sigma_{\yng(1,1)} &= \sigma_1\sigma_2, \\
\sigma_{\yng(2,1)} &= \sigma_1^2\sigma_2 + \sigma_1\sigma_2^2, \\
\sigma_{\yng(2,2)} &= \sigma_1^2\sigma_2^2,
\end{aligned}$$

and so forth. Each polynomial is homogeneous, of degree equal to the number of boxes in the Young diagram.

As we have seen in the last section, Young diagrams as above correspond mathematically to elements of sheaf cohomology groups

$$H^\bullet\left(G(k,n), \wedge^\bullet \mathcal{E}^*\right),$$

for $\mathcal{E}$ the pertinent tangent bundle deformation, which arise in nonlinear-sigma-model-based analyses. For example, there is a well-known correspondence between generators of cohomology of the Grassmannian $G(k,n)$ of fixed degree, and Young diagrams that fit inside a $k \times (n-k)$ box. (Young diagrams that extend outside of that box would correspond mathematically to cohomology classes of too-high degree, which classically vanish.)

Now, for the purposes of describing the ring, including all the Young diagrams is redundant, as there are relations between their products. For example, in the $k = 2$ case above,

$$\sigma_{\square}^2 \;=\; \sigma_1^2 + 2\sigma_1\sigma_2 + \sigma_2^2 \;=\; \sigma_{\square\square} + \sigma_{\substack{\square\\\square}},$$

and so $\sigma_{\substack{\square\\\square}}$ is determined algebraically by $\sigma_{\square}^2$ and $\sigma_{\square\square}$. More generally, the symmetric polynomials corresponding to any Young diagram that extends past the first row can be expressed algebraically in terms of Young diagrams that run along the first row only. This is known as the Giambelli formula (see *e.g.* [82][section 9.4]), which for a Young diagram λ, reads

$$\sigma_\lambda \;=\; \det\left(\sigma_{(\lambda_i + j - i)}\right)_{1 \leq i,j \leq r}$$

for r the number of boxes in λ, λ_i the number of boxes in the ith row, and $\sigma_{(n)}$ corresponding to a Young diagram with one horizontal row of n boxes, *e.g.*

$$\sigma_{(1)} \;=\; \sigma_{\square}, \quad \sigma_{(2)} \;=\; \sigma_{\square\square}, \quad \sigma_{(3)} \;=\; \sigma_{\square\square\square},$$

and so forth, in conventions in which $\sigma_{(m)} = 0$ for $m < 0$, and is 1 if $m = 0$. For example, the Giambelli formula says

$$\sigma_{\substack{\square\\\square}} \;=\; \det\left[\begin{array}{cc} \sigma_{\square} & \sigma_{\square\square} \\ 1 & \sigma_{\square} \end{array}\right] \;=\; \sigma_{\square}^2 \;-\; \sigma_{\square\square},$$

which we verified explicitly above. For another example,

$$\sigma_{\substack{\square\square\\\square}} \;=\; \det\left[\begin{array}{ccc} \sigma_{\square\square} & \sigma_{\square\square\square} & \sigma_{\square\square\square\square} \\ 1 & \sigma_{\square} & \sigma_{\square\square} \\ 0 & 0 & 1 \end{array}\right] \;=\; \sigma_{\square}\sigma_{\square\square} \;-\; \sigma_{\square\square\square},$$

which is easily checked.

Altogether, the classical cohomology ring of the Grassmannian $G(k,n)$ can be expressed in terms of generators corresponding to Young diagrams with only a single horizontal row, as [83–88]

$$\mathbb{C}\left[\sigma_{(1)}, \cdots, \sigma_{(n-k)}\right] / \left\langle D_{k+1}, \cdots, D_n \right\rangle, \tag{3.69}$$

where

$$D_m \;=\; \det\left(\sigma_{(1+j-i)}\right)_{1\leq i,j\leq m},$$

in conventions in which $\sigma_{(m)} = 0$ if $m < 0$ or $m > n-k$, as each D_m should only be constructed from the available generators.

It should be mentioned that the classical cohomology ring can also be written in the presentation

$$\mathbb{C}\left[\sigma_{(1)},\cdots,\sigma_{(k)}\right] / \langle D_{n-k+1},\cdots,D_n\rangle \,.$$

These two presentations define equivalent rings. When describing the ordinary cohomology ring of the Grassmannian, it is often convenient to think of the generators as Chern classes: Chern classes of the universal quotient bundle in (3.69), and Chern classes of the universal subbundle above (see *e.g.* [87]). For the ordinary cohomology ring, they can be related by transposing Young diagrams, though that description is not symmetric in the quantum case. In any event, in this chapter we will primarily refer back to presentation (3.69).

Now, we are interested in computing both quantum and (0,2) modifications to the classical Grassmannian cohomology ring structure, so *a priori*, it might happen that Young diagrams extending past the first row are needed. Nevertheless, it turns out that they are not, the quantum sheaf cohomology ring can be determined solely by Young diagrams along the first row only.

As a special case, the standard result for the ordinary quantum cohomology ring of $G(k,n)$ is (*e.g.* [83–88])

$$\mathbb{C}\left[\sigma_{(1)},\cdots,\sigma_{(n-k)}\right] / \left\langle D_{k+1},\cdots,D_{n-1},D_n-(-)^{n-k-1}q\right\rangle,$$

or, in terms of the other presentation of the classical cohomology ring,

$$\mathbb{C}\left[\sigma_{(1)},\cdots,\sigma_{(k)}\right] / \left\langle D_{n-k+1},\cdots,D_n-(-)^{k-1}q\right\rangle.$$

3.4.2 Derivation of the quantum correction

In this section we will describe how the quantum sheaf cohomology ring relations can be computed from the one-loop effective action, and in the next section we will check our results against A/2 correlation functions computed via supersymmetric localization. (A purely mathematical derivation of the quantum sheaf cohomology ring relations here is left for future work.)

Before computing quantum sheaf cohomology for general bundle deformations, we shall begin by deriving the ordinary quantum cohomology, along the (2,2) locus, from the one-loop effective action, as a warm-up exercise.

As before, let us take the diagonal elements of the σ field to be $\sigma_i, i = 1,\cdots,k$. On the (2,2) locus, where $B = 0$, we see from the one-loop effective potential (3.6) that the σ_i obey equation (3.7), or more simply

$$\sigma_i^n = -q, \quad i = 1, \cdots, k.$$

Because these equations are of order n, all relations with order lower than n are not affected, *i.e.*

$$\sigma_{(i)} = 0, \quad i = n-k+1, \cdots, n-1.$$

Then, for example, from

$$\sigma_{(n-1)}\sigma_1 = \sum_{\substack{\alpha_1+\cdots+\alpha_k=n \\ \alpha_1 \neq 0}} \sigma_1^{\alpha_1} \cdots \sigma_k^{\alpha_k} = 0,$$

we get

$$\sigma_{(n)} = \sum_{\alpha_1+\cdots+\alpha_k=n} \sigma_1^{\alpha_1} \cdots \sigma_k^{\alpha_k} = \sum_{\alpha_2+\cdots+\alpha_k=n} \sigma_2^{\alpha_2} \cdots \sigma_k^{\alpha_k}.$$

Similarly, from $\sigma_{(n-2)}\sigma_1 = 0$, we have

$$\sigma_{(n-1)} = \sum_{\alpha_2+\cdots+\alpha_k=n-1} \sigma_2^{\alpha_2} \cdots \sigma_k^{\alpha_k}.$$

Then $\sigma_{(n-1)}\sigma_2 = 0$ shows that

$$\sigma_{(n)} = \sum_{\alpha_3+\cdots+\alpha_k=n} \sigma_3^{\alpha_3} \cdots \sigma_k^{\alpha_k}.$$

This procedure stops in $k-1$ steps, and we get our first quantum corrected relation

$$\sigma_{(n)} = \sigma_k^n = -q. \tag{3.70}$$

To derive the equation above, we arbitrarily made use of $\sigma_1, \cdots, \sigma_{k-1}$; by picking a different set of $k-1$ σ_i's, we arrive at equation (3.70) for each value of k.

We can use the same method to deduce the higher order relations, for example

$$\sigma_{(n)}\sigma_1 = -q\sigma_1 = \sum_{\substack{\alpha_1+\cdots+\alpha_k=n+1 \\ \alpha_1 \neq 0}} \sigma_1^{\alpha_1} \cdots \sigma_k^{\alpha_k},$$

which implies

$$\sigma_{(n+1)} = \left(\sum_{\alpha_2+\cdots+\alpha_k=n+1} \sigma_2^{\alpha_2} \cdots \sigma_k^{\alpha_k} \right) - q\sigma_1.$$

Repeatedly using lower order relations leads to

$$\sigma_{(n+1)} \;=\; -q\sigma_1 - q\sigma_2 - \cdots - q\sigma_k \;=\; -q\sigma_{(1)}.$$

Similarly, one can compute

$$\sigma_{(n+l)} = -q\sigma_{(l)}, \ l \geqslant 1.$$

In this fashion, we find that the one-loop effective action implies a quantum cohomology ring of the form (3.81), which we show in section 3.4.3 matches standard presentations of the ordinary quantum cohomology ring.

So far we have shown how one-loop effective action arguments can be used to derive the ordinary quantum cohomology ring along the (2,2) locus. Next we shall leave the (2,2) locus and consider more general (0,2) cases by turning on a nonzero B deformation.

First, we note that the relations

$$D_{k+1} \ = \ D_{k+2} \ = \ \cdots \ = \ 0$$

are trivially satisfied for all $\sigma_{(k)}$ constructed as Schur polynomials in k variables $\sigma_1, \cdots, \sigma_k$. It remains to derive the relations

$$R_{(n-k+1)} \ = \ \cdots \ = \ R_{(n-1)} \ = \ 0, \quad R_{(n)} \ = \ -q, \quad R_{(n+1)} \ = \ -q\sigma_{(1)}, \quad \cdots$$

For any $n \times n$ matrix B, the quantum corrected relations are encoded in

$$\det(E(\sigma_\alpha)) = \det(\sigma_\alpha I + B\sigma_{(1)}) = -q \tag{3.71}$$

due to the one-loop effective potential (3.6). Note that, by definition of I_i we have

$$\det(E(\sigma_a)) = \sum_{i=0}^{n} I_i \sigma_{(1)}^i \sigma_a^{n-i}. \tag{3.72}$$

Again, the relations with dimension smaller than n do not receive quantum corrections, *i.e.* the relations

$$R_{(n-k+1)} = R_{(n-k+2)} = \cdots = R_{(n-1)} = 0 \tag{3.73}$$

still hold in the quantum case. Now let's compute the relation at order n. First, note

$$R_{(n-1)}\sigma_1 = \sum_{i=0}^{n-1} I_i \sigma_{(n-i-1)} \sigma_{(1)}^i \sigma_1 = \sum_{i=0}^{n-1} I_i \left(\sigma_{(n-i)} - \sum_{\substack{|\alpha|=n-i \\ \alpha_1=0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i = 0,$$

where $\sigma^{[\alpha]}$ denotes $\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\cdots\sigma_k^{\alpha_k}$, α is the corresponding multi-index, and we have used the relation $R_{(n-1)} = 0$. This implies that

$$R_{(n)} = \sum_{i=0}^{n} I_i \sigma_{(n-i)} \sigma_{(1)}^i = \sum_{i=0}^{n} I_i \left(\sum_{\substack{|\alpha|=n-i \\ \alpha_1=0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i.$$

Similarly, $R_{(n-2)}\sigma_1 = 0$ implies

$$R_{(n-1)} = \sum_{i=0}^{n-1} I_i \left(\sum_{\substack{|\alpha|=n-i-1 \\ \alpha_1=0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i,$$

and $R_{(n-1)}\sigma_2 = 0$ leads to

$$R_{(n)} = \sum_{i=0}^{n} I_i \left(\sum_{\substack{|\alpha|=n-i \\ \alpha_1=\alpha_2=0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i.$$

Because we have $k-1$ relations in (3.73), induction shows that this procedure allows us to eliminate σ_1 through σ_{k-1} in the expression of R_n, *i.e.*

$$R_{(n)} = \sum_{i=0}^{n} I_i \sigma_k^{n-i} \sigma_{(1)}^i,$$

which is equal to $-q$ due to (3.71) and (3.72). Thus the relation at order n is

$$R_{(n)} + q = 0. \tag{3.74}$$

We can follow the same procedure to compute the quantum correction to $R_{(n+1)}$. Since now there are k relations at hand including (3.74), all the σ_i dependence can be eliminated except those proportional to q. One can compute

$$R_{(n+1)} + q\sigma_{(1)} = 0. \tag{3.75}$$

In general, we can show

$$R_{(n+\ell)} = -q\sigma_{(\ell-1)}P_1 + q\sigma_{(\ell-2)}P_2 - q\sigma_{(\ell-3)}P_3 + \cdots + (-)^k q\sigma_{(\ell-k)}P_k, \tag{3.76}$$

where $\ell > 0$ and P_i is the elementary symmetric polynomial of order i in $\sigma_1, \cdots, \sigma_k$. Again, we define $\sigma_{(s)} = 0$ when $s < 0$ in the above formula. From the fact that

$$\prod_{i=1}^{k} (1+\sigma_i t)^{-1} = \sum_{j=0}^{\infty} (-)^j \sigma_{(j)} t^j,$$
$$\prod_{i=1}^{k} (1+\sigma_i t) = \sum_{j=0}^{k} P_j t^j,$$

we have

$$\sigma_{(\ell)} - \sigma_{(\ell-1)}P_1 + \sigma_{(\ell-2)}P_2 + \cdots + (-)^k \sigma_{(\ell-k)}P_k = 0,$$

which implies

$$R_{(n+\ell)} + q\sigma_{(\ell)} = 0, \quad \ell \geqslant 0. \tag{3.77}$$

Actually, (3.77), or equivalently (3.76), can be proved by induction with the same method as for $R_{(n)}$ and $R_{(n+1)}$. Indeed, if we assume (3.77) is true for l replaced with any positive integer smaller than l, we get

$$R_{(n+\ell-1)}\sigma_1 \;=\; \sum_{i=0}^{n} I_i \left(\sum_{\substack{|\alpha|=n+\ell-i \\ \alpha_1 \neq = 0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i,$$

must match

$$-q\sigma_{(\ell-1)}\sigma_1,$$

(by the inductive assumption) and hence, if we define $E_{s,t}$ to be the elementary polynomial of order t in $\sigma_1, \cdots, \sigma_s$, for $0 \leqslant s \leqslant k$ and $t \leqslant s$, then

$$R_{(n+\ell)} = \sum_{i=0}^{n} I_i \left(\sum_{\substack{|\alpha|=n+\ell-i \\ \alpha_1 = 0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i - q\sigma_{(\ell-1)}E_{1,1}.$$

Let's suppose that

$$\begin{aligned} R_{(n+\ell-s)} = &\sum_{i=0}^{n} I_i \left(\sum_{\substack{|\alpha|=n+\ell-s-i \\ \alpha_1 = \cdots = \alpha_{u-s} = 0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i - q\sigma_{(\ell-s-1)}E_{u-s,1} + q\sigma_{(\ell-s-2)}E_{u-s,2} + \\ &\cdots + (-)^{u-s} q\sigma_{(\ell-t)}E_{u-s,u-s} \end{aligned} \tag{3.78}$$

for any $u \leqslant t < k$ and $0 \leqslant s \leqslant u$ (we have seen this is true for $t = 1$). Starting with

$$R_{(n+\ell-t)} = \sum_{i=0}^{n} I_i \left(\sum_{\substack{|\alpha|=n+\ell-t-i \\ \alpha_1 = 0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i - q\sigma_{(\ell-t-1)}\sigma_1,$$

which is obtained from $R_{(n+\ell-t-1)}\sigma_1 = -q\sigma_{(\ell-t-1)}\sigma_1$, induction on s shows (3.78) is valid for $u \leqslant t+1$ and $s \leqslant u$. Thus we can take $u = k$ and $s = 0$ in (3.78) to get

$$R_{(n+\ell)} = -q\sigma_{(\ell-1)}E_{k,1} + q\sigma_{(\ell-2)}E_{k,2} + \cdots + (-)^k q\sigma_{(\ell-k)}E_{k,k},$$

which is exactly (3.76), hence proving (3.77).

3.4.3 Specialization to ordinary quantum cohomology

We have seen that the quantum sheaf cohomology ring (the OPE ring of the A/2 twist) of a (0,2) deformation of the Grassmannian $G(k,n)$ is given *generically* by

$$\begin{aligned}\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \langle D_{k+1}, D_{k+2}, \cdots, R_{(n-k+1)}, \cdots, R_{(n-1)}, \\ R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, R_{(n+2)} + q\sigma_{(2)}, \cdots \rangle,\end{aligned} \tag{3.79}$$

specializing for $k = 1$ to

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \langle D_{k+1}, D_{k+2}, \cdots, R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, \cdots \rangle,$$

and for $k = n-1$ to

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \langle D_n, D_{n+1}, \cdots, R_{(2)}, \cdots, R_{(n-1)}, R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, \cdots \rangle,$$

where

$$\begin{aligned} D_m &= \det\left(\sigma_{(1+j-i)}\right)_{1 \leq i,j \leq m}, \\ R_{(r)} &= \sum_{i=0}^{\min(r,n)} I_i \sigma_{(r-i)} \sigma_{(1)}^i, \end{aligned}$$

for I_i the coefficients of the characteristic polynomial of B, given by

$$\det(tI + B) = \sum_{i=0}^{n} I_{n-i} t^i.$$

For example, $I_0 = 1$, independent of B, but the other I_i depend upon B. In particular,

$$I_1 = \operatorname{tr} B, \quad I_n = \det B.$$

In passing, it will sometimes be helpful to define a generalization of $R_{(r)}$. For a Young diagram μ, we define R_μ to be

$$R_\mu = \det \begin{bmatrix} R_{(\mu_1)} & R_{(\mu_1+1)} & R_{(\mu_1+2)} & \cdots & R_{(\mu_1+k-1)} \\ \sigma_{(\mu_2-1)} & \sigma_{(\mu_2)} & \sigma_{(\mu_2+1)} & \cdots & \sigma_{(\mu_2+k-2)} \\ \sigma_{(\mu_3-2)} & \sigma_{(\mu_3-1)} & \sigma_{(\mu_3)} & \cdots & \sigma_{(\mu_3+k-3)} \\ \vdots & & & & \ddots \\ \sigma_{(\mu_k-k+1)} & \sigma_{(\mu_k-k+2)} & \sigma_{(\mu_k-k+3)} & \cdots & \sigma_{(\mu_k)} \end{bmatrix}.$$

In the special case that the Young diagram μ consists of a single horizontal row of r boxes, which we would label (r), note that

$$R_\mu = R_{(r)},$$

and it is in this sense that R_μ generalizes $R_{(r)}$.

The description of the ring above holds generically in the space of tangent bundle deformations, but does break down along certain loci. Specifically, the description of the classical sheaf cohomology ring, described by the limit $q \to 0$, breaks down along

$$X \cup V_{n-k+1} \cup V_{n-k+2} \cup \cdots ,$$

where X is the discriminant locus of the tangent bundle deformation (meaning, the locus where the bundle degenerates), and V_m is a locus defined by R_m, as follows. First, for every Young diagram μ of size $|\mu| = m$, such that μ_1, the number of boxes in the first row, is greater than $n-k$, and no column has more than k boxes, expand the determinant below in a sum of Schur polynomials for Young diagrams of the same size:

$$R_\mu \ = \ \sum_\nu C^m_{\mu\nu} \sigma_\nu,$$

where $|\nu| = m = |\mu|$. In this fashion, we define a matrix $(C^m_{\mu\nu})$. Then, we define V_m to be the locus where the rank of the (not necessarily square) matrix $(C^m_{\mu\nu})$ drops.

Along the V_m for

$$n-k+1 \ \leq \ m \ \leq \ k(n-k),$$

the V_m define loci where the dimensions of the sheaf cohomology groups may jump. For $m > k(n-k)$, the V_m merely define loci where the presentation breaks down, where the given relations may not suffice, but the dimensions of the sheaf cohomology groups do not jump.

It is useful to note that the locus above is codimension at least one, and so the presentation of the classical sheaf cohomology ring is pertinent for generic tangent bundle deformations.

It is also important to notice that the locus above does not intersect the (2,2) locus. On the (2,2) locus, where $B = 0$ and $R_{(n)} = \sigma_{(n)}$, from the Giambelli formula $R_\mu = \sigma_\mu$, and so we see that along the (2,2) locus, $C^m_{\mu\nu} = \delta_{\mu\nu}$, whose rank does not drop, and so V_m is the empty set. Thus, V_m never intersects the (2,2) locus, and neither does the discriminant.

In passing, note that the result above is consistent with claims of [55] that in a sufficiently small neighborhood of the (2,2) locus, the OPE's can be consistently defined within the topological subsector.

We conjecture that for $m > k(n-k)$, the loci V_m are all identical to one another and to the discriminant locus, so that the total number of components of the locus where the quantum sheaf cohomology ring relations break down in some fashion is finite. We will see this in examples later, though we do not yet have a general proof for all cases.

So far we have described the loci along which the presentation of the classical sheaf cohomology ring degenerates. The degeneration loci of the presentation of the quantum sheaf cohomology ring are not completely understood by us at present, though we conjecture that the same loci V_m are involved, as we shall see in examples later.

In the remainder of this section, we will check that the ansatz above correctly specializes to the ordinary classical and quantum cohomology rings. We will derive the quantum sheaf cohomology ring above from the one-loop effective action later in section 3.4.2.

First, as one extreme, let us reduce to the classical cohomology ring of $G(k,n)$. Here, $B = 0$ and $q = 0$. In this case,

$$R_{(r)} \;=\; \sigma_{(r)},$$

and the quantum sheaf cohomology ring above becomes

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_{k+1}, D_{k+2}, \cdots, \sigma_{(n-k+1)}, \sigma_{(n-k+2)}, \cdots \right\rangle,$$

or more simply,

$$\mathbb{C}\left[\sigma_{(1)}, \cdots, \sigma_{(n-k)}\right] / \left\langle D_{k+1}, D_{k+2}, \cdots \right\rangle.$$

This is almost identical to the presentation of the ordinary cohomology ring of the Grassmannian given in equation (3.69), except that here the relations involve all D's of degree greater than n, rather than going only to D_n. However, we can establish that the two sets of relations are equivalent, as follows.

We will show that D_{n+1} and all higher relations are linear combinations of the relations $\{D_{k+1}, \cdots, D_n\}$, so that the ring presented above is equivalent to (3.69). To do this, we expand down the first column of the determinant in the Giambelli formula to derive the recursion relation

$$\sigma_{(\ell,1,\cdots,1)} \;=\; \sigma_{(\ell)} D_m \;-\; \sigma_{(\ell+1,1,\cdots,1)},$$

where $\sigma_{(\ell,1,\cdots,1)}$ denotes the Schur polynomial associated to a Young tableau with ℓ boxes in the first row and 1 box in the next m rows, and $\sigma_{(\ell+1,1,\cdots,1)}$ denotes a similar Schur polynomial, albeit associated to a Young diagram with $m-1$ rows with one box. Applying this recursively, one can quickly show

$$D_{n+1} \;=\; \sigma_{(1)} D_n \;-\; \sigma_{(2)} D_{n-1} \;+\; \cdots \;+\; (-)^{n-k+1} \sigma_{(n-k)} D_{k+1} \;+\; (-)^{n-k} \sigma_{(n-k+1,1,\cdots,1)}. \quad (3.80)$$

However, from the Giambelli formula, $\sigma_{(n-k+1,1,\cdots,1)}$ is given by a determinant whose first row vanishes (since it involves σ's all of which are outside the range of the generators), hence $\sigma_{(n-k+1,1,\cdots,1)} = 0$. Thus, we see that D_{n+1} is a linear combination of the relations $\{D_{k+1}, \cdots, D_n\}$, and one can similarly demonstrate the same result for all D_m for $m > n$. In this fashion, we see that the ring above is isomorphic to the ordinary cohomology ring of the Grassmannian given in equation (3.69).

Next, let us verify that the quantum sheaf cohomology ring (3.79) reduces to the ordinary quantum cohomology ring of the Grassmannian $G(k,n)$ along the (2,2) locus. This is the case $B = 0$, but $q \neq 0$. As before,

$$R_{(r)} \;=\; \sigma_{(r)},$$

and so the quantum sheaf cohomology ring above becomes

$$\begin{aligned}\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \big\langle D_{k+1}, D_{k+2}, \cdots, \sigma_{(n-k+1)}, \cdots, \sigma_{(n-1)},& \\ \sigma_{(n)} + q, \sigma_{(n+1)} + q\sigma_{(1)}, \cdots \big\rangle,&\end{aligned} \tag{3.81}$$

specializing for $k = 1$ to

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \big\langle D_2, D_3, \cdots, \sigma_{(n)} + q, \sigma_{(n+1)} + q\sigma_{(1)}, \cdots \big\rangle,$$

and for $k = n - 1$ to

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \big\langle D_n, D_{n+1}, \cdots, \sigma_{(2)}, \cdots, \sigma_{(n-1)}, \sigma_{(n)} + q, \sigma_{(n+1)} + q\sigma_{(1)}, \cdots \big\rangle.$$

The expression above for the quantum cohomology ring is not yet in a standard form, and can be simplified to such a form. First, we show that, for $\ell \geqslant 1$,

$$D_{n+\ell} \;=\; 0, \tag{3.82}$$

in the sense that it is redundant, defining no new relations, and hence can be removed from the presentation above. This can be proved by induction on ℓ. First, we will need a small identity. By expanding the determinant in the definition of D_m across the first row, (and then expanding the determinant of each submatrix along the first column), we find

$$D_m \;=\; \sigma_{(1)} D_{m-1} - \sigma_{(2)} D_{m-2} \;+\; \cdots \;+\; (-)^m \sigma_{(m-1)} D_1 \;+\; (-)^{m+1} \sigma_{(m)}.$$

Now, we proceed with the induction. For $\ell = 1$, we have

$$\begin{aligned} D_{n+1} \;&=\; \sigma_{(1)} D_n + \cdots + (-)^{n-k+1} \sigma_{(n-k)} D_{k+1} + (-)^{n-k+2} \sigma_{(n-k+1)} D_k \\ &\quad + \cdots + (-)^n \sigma_{(n-1)} D_2 + (-)^{n+1} \sigma_{(n)} D_1 + (-)^{n+2} \sigma_{(n+1)}, \\ &=\; (-)^{n+1} (\sigma_{(n)} D_1 - \sigma_{(n+1)}), \\ &=\; (-)^n q (D_1 - \sigma_{(1)}) \;=\; 0, \end{aligned}$$

where we have used the ring relations

$$D_{k+1} \;=\; D_{k+2} \;=\; \cdots \;=\; 0, \quad \sigma_{(n-k+1)} \;=\; \cdots \;=\; \sigma_{(n-1)} \;=\; 0.$$

Next, assume that (3.82) is true for all $\ell \leqslant m$. When $m < k$, we have

$$\begin{aligned} D_{n+m+1} \;&=\; \sigma_{(1)} D_{n+m} - \sigma_{(2)} D_{n+m-1} + \cdots + (-)^{n+m-k+1} \sigma_{(n+m-k)} D_{k+1} + \\ &\quad (-)^{n+m-k+2} \sigma_{(n+m-k+1)} D_k + \cdots + (-)^{n+1} \sigma_{(n)} D_{m+1} + \cdots + (-)^{n+m+2} \sigma_{(n+m+1)}, \\ &=\; (-)^{n+1} \sigma_{(n)} D_{m+1} + \cdots + (-)^{n+m+2} \sigma_{(n+m+1)}, \\ &=\; (-)^n q \left(D_{m+1} - \sigma_{(1)} D_m + \cdots + (-)^{m+1} \sigma_{(m+1)} \right) \;=\; 0. \end{aligned}$$

When $m \geqslant k$, we have

$$\begin{aligned}
D_{n+m+1} &= \sigma_{(1)} D_{n+m} - \sigma_{(2)} D_{n+m-1} + \cdots + (-)^n \sigma_{(n-1)} D_{m+2} \\
&\quad +(-)^{n+1} \sigma_{(n)} D_{m+1} + \cdots + (-)^{n+m+2} \sigma_{(n+m+1)}, \\
&= (-)^{n+1} \sigma_{(n)} D_{m+1} + \cdots + (-)^{n+m+2} \sigma_{(n+m+1)}, \\
&= (-)^n q (D_{m+1} - \sigma_{(1)} D_m + \cdots + (-)^{m+1} \sigma_{(m+1)}) \;=\; 0.
\end{aligned}$$

Thus, we have shown (3.82).

Next, using the relations

$$\sigma_{(n+\ell)} \;=\; -q \sigma_{(\ell)},$$

we can express the $\sigma_{(i)}$'s with $i > n-k$ as polynomials of q and $\sigma_{(i)}$'s with $i \leqslant n-k$, and so we can rewrite the ring in terms of generators

$$\sigma_{(1)}, \cdots, \sigma_{(n-k)}.$$

Finally, we derive an expression for D_n. Starting with

$$\begin{aligned}
D_n &= \sigma_{(1)} D_{n-1} - \sigma_{(2)} D_{n-2} + \cdots + (-)^{n-k} \sigma_{(n-k-1)} D_{k+1} + (-)^{n-k-1} \sigma_{(n-k)} D_k \\
&\quad (-)^{n-k+2} \sigma_{(n-k+1)} D_{k-1} + \cdots + (-)^n \sigma_{(n-1)} D_1 + (-)^{n+1} \sigma_{(n)},
\end{aligned}$$

we use the ring relations

$$D_{k+1} \;=\; \cdots \;=\; D_{k(n-k)} \;=\; 0, \quad \sigma_{(n-k+1)} \;=\; \cdots \;=\; \sigma_{(n-1)} \;=\; 0$$

and the fact that

$$n-1 \;\leq\; k(n-k)$$

to simplify D_n to

$$D_n \;=\; (-)^{n-k-1} \sigma_{(n-k)} D_k + (-)^{n+1} \sigma_{(n)},$$

which using further ring relations can be written as

$$D_n \;=\; (-)^{n-k-1} \sigma_{(n-k)} D_k + (-)^n q.$$

Finally, let us simplify the ring presentation (3.81). We have shown that inside that quotient ring, D_i is redundant for $i > n$, and given our expression for D_n above, it is straightforward to see that the ring (3.81) can be reduced to

$$\mathbb{C}\left[\sigma_{(1)}, \cdots, \sigma_{(n-k)}\right] / \left\langle D_{k+1}, \cdots, D_{n-1}, D_n + (-)^n q \right\rangle, \tag{3.83}$$

where the "D_n" above is the 'classical' D_n, namely

$$(-)^{n-k-1} \sigma_{(n-k)} D_k$$

in the notation of (3.81). This new presentation is a standard representation of the quantum cohomology ring of $G(k,n)$ (see *e.g.* [83–88]).

3.5 Examples

In this section, we will perform consistency tests on the quantum sheaf cohomolgy ring (3.79) by using supersymmetric localization to compute A/2 correlation functions in examples, and check that the predictions of the quantum sheaf cohomology ring are consistent with those correlation functions.

In each example, we will begin by describing correlation functions and quantum cohomology along the (2,2) locus, and will generalize to (0,2). Furthermore, in all our (0,2) examples, we will take B to be diagonal:

$$B \; = \; \mathrm{diag}(b_1, \cdots, b_n)$$

on $G(k,n)$. The methods of this chapter apply to general B; however, the resulting formulas for general B are rather complex, and it suffices to consider the special case of B diagonal for the purposes of illustrative examples.

We will begin by looking at examples of projective spaces as special cases of the construction described here, and then will turn to Grassmannians which are not projective spaces.

3.5.1 $G(2,4)$

Let's first compute the correlation functions for the (2,2) theory engineering $G(2,4)$. By computing the correlation functions, we want to explicitly show that

$$R_{(3)} = \sigma_{\square\square\square} = 0, \quad R_{(4)} + q = \sigma_{\square\square\square\square} + q = 0,$$

as implied by our general result. According to (3.8), the four-point correlation functions in the theory are given by

$$\langle f(\sigma) \rangle \; = \; \frac{1}{2!} \mathrm{JKG} - \mathrm{Res} \left\{ -(\sigma_1 - \sigma_2)^2 \frac{1}{\sigma_1^4} \frac{1}{\sigma_2^4} f(\sigma) \right\}.$$

The Jeffrey-Kirwan-Grothendieck residues in this case are merely iterated ordinary contour integrals about $\sigma_1 = 0$ and $\sigma_2 = 0$, *i.e.*

$$\langle f(\sigma) \rangle \; = \; \frac{1}{2!} \oint d\sigma_2 \oint d\sigma_1 \left\{ -(\sigma_1 - \sigma_2)^2 \frac{1}{\sigma_1^4} \frac{1}{\sigma_2^4} f(\sigma) \right\}.$$

It is straightforward to show that

$$\begin{gathered} \langle \sigma_1^4 \rangle = 0, \; \langle \sigma_1^3 \sigma_2 \rangle = -\frac{1}{2!}, \\ \langle \sigma_1^2 \sigma_2^2 \rangle = +\frac{2}{2!}, \; \langle \sigma_1 \sigma_2^3 \rangle = -\frac{1}{2!}, \\ \langle \sigma_2^4 \rangle = 0. \end{gathered}$$

Now, let us interpret this in terms of the cohomology of $G(2,4)$. In principle, the cohomology classes correspond to Young tableaux sitting inside the $2 \times (4-2)$ box

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$$

and so in particular are given by

$$
\begin{aligned}
\sigma_{\square} &= \sigma_1 + \sigma_2, \\
\sigma_{\square\square} &= \sigma_1^2 + \sigma_2^2 + \sigma_1\sigma_2, \\
\sigma_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} &= \sigma_1\sigma_2, \\
\sigma_{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}} &= \sigma_1^2\sigma_2 + \sigma_1\sigma_2^2, \\
\sigma_{\begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}} &= \sigma_1^2\sigma_2^2
\end{aligned}
$$

with relations, for example

$$\sigma_{\square\square\square} \;=\; \sigma_1^3 + \sigma_2^3 + \sigma_1^2\sigma_2 + \sigma_1\sigma_2^2 \;=\; 0.$$

Let us check that the relation $R_3 = \sigma_{\square\square\square} = 0$ is encoded in the correlation functions above. Since it involves third-order powers, and the correlation functions involve fourth-order powers, we need to multiply by single copies of σ_1, σ_2. In other words, we claim the following statements should be true:

$$\langle \sigma_1 \sigma_{\square\square\square} \rangle \;=\; 0 \;=\; \langle \sigma_2 \sigma_{\square\square\square} \rangle.$$

Explicitly,

$$
\begin{aligned}
\langle \sigma_1 \sigma_{\square\square\square} \rangle &= \langle \sigma_1 (\sigma_1^3 + \sigma_2^3 + \sigma_1^2\sigma_2 + \sigma_1\sigma_2^2) \rangle \\
&= \langle \sigma_1^4 \rangle + \langle \sigma_1\sigma_2^3 \rangle + \langle \sigma_1^3\sigma_2 \rangle + \langle \sigma_1^2\sigma_2^2 \rangle
\end{aligned}
$$

and it is easy to check that this does indeed vanish. One can similarly verify $\langle \sigma_2 \sigma_{\square\square\square} \rangle = 0$.

Correlation functions in the one-instanton sector are of the form

$$
\begin{aligned}
\langle f(\sigma) \rangle &= \frac{1}{2!}\mathrm{JKG}-\mathrm{Res}\left\{ q(\sigma_1-\sigma_2)^2 \frac{1}{\sigma_1^8}\frac{1}{\sigma_2^4} f(\sigma) \right\} \\
&\quad + \frac{1}{2!}\mathrm{JKG}-\mathrm{Res}\left\{ q(\sigma_1-\sigma_2)^2 \frac{1}{\sigma_1^4}\frac{1}{\sigma_2^8} f(\sigma) \right\}
\end{aligned}
$$

(where the JKG residues again reduce to iterated ordinary contour integrals about $\sigma_1 = 0$ and $\sigma_2 = 0$), from which we compute that the nonvanishing correlation functions are

$$\langle \sigma_1^5 \sigma_2^3 \rangle \;=\; q/2 \;=\; \langle \sigma_1^7 \sigma_2 \rangle, \quad \langle \sigma_1^6\sigma_2^2 \rangle \;=\; -q,$$

$$\langle \sigma_1^3 \sigma_2^5 \rangle \;=\; q/2 \;=\; \langle \sigma_1 \sigma_2^7 \rangle, \;\; \langle \sigma_1^2 \sigma_2^6 \rangle \;=\; -q.$$

Using the fact that

$$\sigma_{\yng(4)} = \sigma_1^4 \;+\; \sigma_1^3\sigma_2 \;+\; \sigma_1^2\sigma_2^2 \;+\; \sigma_1\sigma_2^3 \;+\; \sigma_2^4,$$

we compute

$$\begin{aligned}
\langle \sigma_{\yng(4)} \rangle \;&=\; 0, \\
\langle \sigma_1^4 \sigma_{\yng(4)} \rangle \;&=\; 0 \;=\; \langle \sigma_2^4 \sigma_{\yng(4)} \rangle, \\
\langle \sigma_1^3 \sigma_2 \sigma_{\yng(4)} \rangle \;&=\; q/2 \;=\; \langle \sigma_1 \sigma_2^3 \sigma_{\yng(4)} \rangle, \\
\langle \sigma_1^2 \sigma_2^2 \sigma_{\yng(4)} \rangle \;&=\; -q.
\end{aligned}$$

From the expressions

$$\sigma_{\yng(3,1)} = \sigma_1^3\sigma_2 \;+\; \sigma_1^2\sigma_2^2 \;+\; \sigma_1\sigma_2^3,$$

$$\sigma_{\yng(2,2)} = \sigma_1^2\sigma_2^2,$$

we get

$$\begin{aligned}
\langle \sigma_{\yng(4)} \, \sigma_{\yng(4)} \rangle &= 0, \\
\langle \sigma_{\yng(4)} \, \sigma_{\yng(3,1)} \rangle &= 0, \\
\langle \sigma_{\yng(4)} \, \sigma_{\yng(2,2)} \rangle &= -q.
\end{aligned}$$

Thus we see the relation $R_{(4)}$ is valid because $\sigma_{\yng(4)} = \langle \sigma_{\yng(4)} \, \sigma_{\yng(2,2)} \rangle \cdot 1 = -q$. The other relation at fourth order can be read off immediately,

$$\sigma_{\yng(3,1)} = \sigma_{\yng(3)} \, \sigma_{\yng(1)} - \sigma_{\yng(4)} = q.$$

We can study the (0,2) theories following the same procedure. Again, after absorbing a sign in q, the localization formula (3.8) reads

$$\langle f(\sigma) \rangle \;=\; \frac{1}{2!} \sum_{\mathfrak{m}_1, \cdots, \mathfrak{m}_k \in \mathbb{Z}} \mathrm{JKG} - \mathrm{Res} \left\{ (-q)^{\sum \mathfrak{m}_i} \left((-)(\sigma_1 - \sigma_2) \right) \prod_{a=1}^{2} \left(\frac{1}{\det \tilde{E}(\sigma_a)} \right)^{\mathfrak{m}_i + 1} f(\sigma) \right\},$$

where

$$\tilde{E}^i_j(\sigma_a) \;=\; \delta^i_j \sigma_a \;+\; B^i_j \left(\sum_b \sigma_b \right).$$

In this case, the JKG residue gives us the following iterated residue prescription for generic b_j:

1. First, we perform a contour integral over σ_1, summing over the residues at the four loci
$$\sigma_1 \;=\; -\sigma_2 \frac{b_j}{1+b_j}$$
for $j \in \{1,2,3,4\}$, corresponding to the roots of $\det \tilde{E}(\sigma_1)$,
2. then, we perform a contour integral over σ_2, taking the residue at $\sigma_2 = 0$.

In this case, the results for the classical correlation functions ($\mathfrak{m}_1 = 0 = \mathfrak{m}_2$) are as follows:

$$\begin{aligned}
\langle \sigma_1^4 \rangle &= \Delta^{-1}\left[I_1 + 2I_1^2 + 4I_1I_2 - 2I_3 + 2I_2^2 + 2I_1I_3 - 4I_4 + 2I_2I_3 - 2I_1I_4\right], \\
\langle \sigma_1^3\sigma_2 \rangle &= \Delta^{-1}\left[-1 - 3I_1 - 2I_1^2 - 3I_2 - 4I_1I_2 - 2I_2^2 - I_3 - 2I_1I_3 + 4I_4 - 2I_2I_3 + 2I_1I_4\right], \\
\langle \sigma_1^2\sigma_2^2 \rangle &= \Delta^{-1}\left[2 + 4I_1 + 2I_1^2 + 4I_2 + 4I_1I_2 + 2I_3 - 4I_4 + 2I_2^2 + 2I_1I_3 + 2I_2I_3 - 2I_1I_4\right], \\
\langle \sigma_1\sigma_2^3 \rangle &= \langle \sigma_1^3\sigma_2 \rangle, \\
\langle \sigma_2^4 \rangle &= \langle \sigma_1^4 \rangle,
\end{aligned}$$

or

$$\begin{aligned}
\langle \sigma_{\square\square\square\square} \rangle &= \langle \sigma_1^4 \rangle + \langle \sigma_1^3\sigma_2 \rangle + \langle \sigma_1^2\sigma_2^2 \rangle + \langle \sigma_1\sigma_2^3 \rangle + \langle \sigma_2^4 \rangle, \\
&= 2\Delta^{-1}\left[-I_2 + I_1^2 + 2I_2I_1 - 2I_3 + I_2^2 - 2I_4 + I_1I_3 - I_1I_4 + I_2I_3\right],
\end{aligned}$$

$$\begin{aligned}
\langle \sigma^2_{\square\square} \rangle &= \langle \sigma_1^4 \rangle + 2\langle \sigma_1^3\sigma_2 \rangle + 3\langle \sigma_1^2\sigma_2^2 \rangle + 2\langle \sigma_1\sigma_2^3 \rangle + \langle \sigma_2^4 \rangle, \\
&= \Delta^{-1}\left[2 + 2I_1 + 2I_1^2 + 4I_1I_2 - 2I_3 - 4I_4 + 2I_2^2 + 2I_1I_3 + 2I_2I_3 - 2I_1I_4\right],
\end{aligned}$$

$$\begin{aligned}
\langle \sigma_{\square}\sigma_{\square\square\square} \rangle &= \langle \sigma_1^4 \rangle + 2\langle \sigma_1^3\sigma_2 \rangle + 2\langle \sigma_1^2\sigma_2^2 \rangle + 2\langle \sigma_1\sigma_2^3 \rangle + \langle \sigma_2^4 \rangle, \\
&= \Delta^{-1}\left[-2I_1 - 4I_2 - 4I_3\right],
\end{aligned}$$

$$\begin{aligned}
\langle \sigma^2_{\square}\sigma_{\square\square} \rangle &= \langle \sigma_1^4 \rangle + 3\langle \sigma_1^3\sigma_2 \rangle + 4\langle \sigma_1^2\sigma_2^2 \rangle + 3\langle \sigma_1\sigma_2^3 \rangle + \langle \sigma_2^4 \rangle, \\
&= \Delta^{-1}\left[2 - 2I_2 - 2I_3\right],
\end{aligned}$$

$$\begin{aligned}
\langle \sigma^4_{\square} \rangle &= \langle \sigma_1^4 \rangle + 4\langle \sigma_1^3\sigma_2 \rangle + 6\langle \sigma_1^2\sigma_2^2 \rangle + 4\langle \sigma_1\sigma_2^3 \rangle + \langle \sigma_2^4 \rangle, \\
&= \Delta^{-1}\left[4 + 2I_1\right],
\end{aligned}$$

$$\begin{aligned}
\langle \sigma_{\substack{\square\square\square \\ \square}} \rangle &= \langle \sigma_1^3\sigma_2 \rangle + \langle \sigma_1^2\sigma_2^2 \rangle + \langle \sigma_1\sigma_2^3 \rangle, \\
&= \Delta^{-1}\left[-2I_1 - 2I_2 - 2I_1^2 + 4I_4 - 2I_2^2 - 2I_3I_1 - 2I_3I_2 + 2I_1I_4 - 4I_2I_1\right],
\end{aligned}$$

$$\begin{aligned}
\langle \sigma_{\substack{\square\square \\ \square\square}} \rangle &= \langle \sigma_1^2\sigma_2^2 \rangle, \\
&= \Delta^{-1}\left[2 + 4I_1 + 2I_1^2 + 4I_2 + 4I_1I_2 + 2I_3 - 4I_4 + 2I_2^2 + 2I_1I_3 + 2I_2I_3 - 2I_1I_4\right],
\end{aligned}$$

where the characteristic polynomials of B are given explicitly as

$$\begin{aligned} I_1 &= \sum_i b_i \;=\; \mathrm{tr}\, B, \\ I_2 &= \sum_{i<j} b_i b_j, \\ I_3 &= \sum_{i<j<k} b_i b_j b_k, \\ I_4 &= b_1 b_2 b_3 b_4 \;=\; \det B, \end{aligned}$$

and

$$\begin{aligned} \Delta &= 2\prod_{i<j}(1+b_i+b_j)\,, \\ &= 2\left(1+3I_1+3I_1^2+2I_2+I_1^3+4I_1I_2+2I_1^2I_2+I_2^2+I_1I_3-4I_4+I_1I_2^2 \right. \\ &\qquad \left. +I_1^2I_3+I_1I_2I_3-4I_1I_4-I_1^2I_4-I_3^2\right). \end{aligned}$$

We see the discriminant locus is given by $\Delta = 0$, this is consistent with our general result in [81], which says that the B-deformation fails to define a vector bundle on $G(k,n)$ if and only if there exits k eigenvalues of B whose sum is -1.

Now, the quantum sheaf cohomology ring for this model is predicted by (3.79) to be

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_3, D_4, \cdots, R_{(3)}, R_{(4)}+q, R_{(5)}+q\sigma_{(1)}, \cdots \right\rangle,$$

where

$$\begin{aligned} R_{(3)} &= \sum_{i=0}^{3} I_i \sigma_{(3-i)} \sigma_{(1)}^i, \\ &= \sigma_{(3)} + I_1\sigma_{(2)}\sigma_{(1)} + (I_2+I_3)\sigma_{(1)}^3, \\ R_{(4)} &= \sum_{i=0}^{4} I_i \sigma_{(r-i)} \sigma_{(1)}^i, \\ &= \sigma_{(4)} + I_1\sigma_{(3)}\sigma_{(1)} + I_2\sigma_{(2)}\sigma_{(1)}^2 + (I_3+I_4)\sigma_{(1)}^4. \end{aligned}$$

As a consistency test, it is straightforward to check that the relations above are reflected in the correlation functions. For example, the classical correlation functions are easily demonstrated to obey

$$\langle \sigma_{\square} D_3 \rangle \;=\; \langle D_4 \rangle \;=\; \langle \sigma_{\square} R_{(3)} \rangle \;=\; 0.$$

Now, we also have the relation $R_{(4)} = -q$, which for the purely classical correlation functions implies

$$\langle R_{(4)} \rangle \;=\; 0,$$

which is also easily checked to be true. By including instanton sectors, one can see the full quantum-corrected relation, as we shall discuss next.

The relation $R_{(4)} = -q$ can be derived from the quantum cohomology ring relation derived from the Jeffrey-Kirwan-Grothendieck residue expression, namely,

$$\det \tilde{E}(\sigma_1) \;=\; -q \;=\; \det \tilde{E}(\sigma_2),$$

where

$$\tilde{E}(x) \;=\; Ix \;+\; B(\sigma_1 + \sigma_2).$$

Now, it is straightforward to expand

$$\det \tilde{E}(x) \;=\; x^4 \;+\; I_1(\sigma_1+\sigma_2)x^3 \;+\; I_2(\sigma_1+\sigma_2)^2x^2 \;+\; I_3(\sigma_1+\sigma_2)^3x \;+\; I_4(\sigma_1+\sigma_2)^4$$

so

$$\begin{aligned}
\langle \det \tilde{E}(\sigma_1)\rangle \;&=\; \langle \sigma_1^4\rangle(1+I_1+I_2+I_3+2I_4) \;+\; \langle \sigma_1^3\sigma_2\rangle(I_1+2I_2+4I_3+8I_4) \\
&\quad + \langle \sigma_1^2\sigma_2^2\rangle(I_2+3I_3+6I_4), \\
&=\; \langle \det \tilde{E}(\sigma_2)\rangle,
\end{aligned}$$

which implies

$$\begin{aligned}
2R_{(4)} - R_3\sigma_{\ydiagram{1}} \;&= \sigma_{\ydiagram{4}}(1+I_1+I_2+I_3+2I_4) \;+\; \langle \sigma_{\ydiagram{3,1}}\rangle(-1+I_2+3I_3+6I_4) \\
&\quad + \; \langle \sigma_{\ydiagram{2,2}}\rangle(-I_1+2I_3+4I_4) \;=\; -2q,
\end{aligned}$$

or simply $R_{(4)} + q = 0$ as expected.

Now, let us turn to the interpretation of the loci V_m in this example. First, consider the V_3 locus. It is straightforward to compute

$$\left[R_{(3)}\right] \;=\; [1+I_1+I_2+I_3, I_1+2I_2+2I_3] \begin{bmatrix} \sigma_{(3)} \\ \sigma_{(2,1)} \end{bmatrix},$$

hence

$$\begin{aligned}
V_3 \;&=\; \{1+I_1+I_2+I_3 = 0 \text{ and } I_1+2I_2+2I_3 = 0\}, \\
&=\; \{I_2+I_3 = +1,\; I_1 = -2\}.
\end{aligned}$$

Note that this locus does not intersect the (2,2) locus, as expected on general grounds, as the I_i never all become zero.

Next, we compute the V_4 locus. It is straightforward to compute

$$\begin{bmatrix} R_{(4)} \\ R_{(3,1)} \end{bmatrix} \;=\; (C^4_{\mu\nu}) \begin{bmatrix} \sigma_{(4)} \\ \sigma_{(3,1)} \\ \sigma_{(2,2)} \end{bmatrix},$$

where

$$(C^4_{\mu\nu}) \;=\; \left[\begin{array}{ccc} 1+I_1+I_2+I_3+I_4 & I_1+2I_2+3I_3+3I_4 & I_2+2I_3+2I_4 \\ -I_4 & 1+I_1+I_2-3I_4 & I_1+I_2-2I_4 \end{array}\right].$$

Let M_{12}, M_{13}, M_{23} denote the three 2×2 submatrices of $(C^4_{\mu\nu})$ formed by omitting a column, then the locus V_4 is defined as

$$V_4 \;=\; \{M_{12}=0=M_{13}=M_{23}\}.$$

For completeness, we also list here results for V_5 and V_6. First, V_5 is computed from the relation

$$\left[\begin{array}{c} R_{(5)} \\ R_{(4,1)} \\ R_{(3,2)} \end{array}\right] = \left(C^5_{\mu\nu}\right) \left[\begin{array}{c} \sigma_{(5)} \\ \sigma_{(4,1)} \\ \sigma_{(3,2)} \end{array}\right]$$

for

$$(C^5_{\mu\nu}) \;=\; \left[\begin{array}{ccc} 1+I_1+I_2+I_3+I_4 & I_1+2I_2+3I_3+4I_4 & I_2+3I_3+5I_4 \\ 0 & 1+I_1+I_2+I_3 & I_1+2I_2+2I_3 \\ -I_4 & -I_3-4I_4 & 1+I_1-2I_3-5I_4 \end{array}\right],$$

and V_6 is computed from the relation

$$\left[\begin{array}{c} R_{(6)} \\ R_{(5,1)} \\ R_{(4,2)} \\ R_{(3,3)} \end{array}\right] = \left(C^6_{\mu\nu}\right) \left[\begin{array}{c} \sigma_{(6)} \\ \sigma_{(5,1)} \\ \sigma_{(4,2)} \\ \sigma_{(3,3)} \end{array}\right]$$

for $(C^6_{\mu\nu})$ given by

$$\left[\begin{array}{cccc} 1+I_1+I_2+I_3+I_4 & I_1+2I_2+3I_3+4I_4 & I_2+3I_3+6I_4 & I_3+3I_4 \\ 0 & 1+I_1+I_2+I_3+I_4 & I_1+2I_2+3I_3+3I_4 & I_2+2I_3+2I_4 \\ 0 & -I_4 & 1+I_1+I_2-3I_4 & I_1+I_2-2I_4 \\ -I_4 & -I_3-4I_4 & -I_2-3I_3-6I_4 & 1-I_2-2I_3-3I_4 \end{array}\right].$$

At least when B is diagonal, it is straightforward to check that

$$\det(C_5) \;=\; \det(C_6) \;=\; \Delta,$$

or equivalently,

$$V_5 \;=\; V_6 \;=\; X,$$

consistent with our expectation that for m larger than the dimension of the Grassmannian, V_m matches the discriminant locus.

3.5.2 G(2,5)

In the previous sections we described the results for the Grassmannian $G(2,4)$. Although this is not a projective space, it can be described as a hypersurface in a projective space, so in this section we give one additional nonabelian example, one which is not related or dual to an abelian GLSM, to demonstrate the results. Specifically, in this section we will consider the theory for $G(2,5)$.

The (classical) six-point correlation functions are given by

$$\langle f(\sigma) \rangle \;=\; \frac{1}{2!}\mathrm{JKG}-\mathrm{Res}\left\{ (-)(\sigma_1-\sigma_2)^2 \frac{1}{\sigma_1^5}\frac{1}{\sigma_2^5} f(\sigma) \right\}.$$

We can compute these as iterated ordinary contour integrals about $\sigma_1 = 0$ and $\sigma_2 = 0$. It is straightforward to show that

$$\begin{aligned} \langle \sigma_1^6 \rangle = 0,\ \langle \sigma_1^5 \sigma_2 \rangle &= 0, \\ \langle \sigma_1^4 \sigma_2^2 \rangle = -\frac{1}{2!},\ \langle \sigma_1^3 \sigma_2^3 \rangle &= +\frac{2}{2!}, \\ \langle \sigma_1^2 \sigma_2^4 \rangle = -\frac{1}{2!},\ \langle \sigma_1 \sigma_2^5 \rangle &= 0, \\ \langle \sigma_2^6 \rangle &= 0. \end{aligned}$$

All the nonzero cohomology classes should be defined by Young diagrams fitting inside the 2×3 box

$$\yng(3,3)$$

Using the correlation functions above, it is straightforward to compute the ring relations

$$\begin{aligned} \sigma_{\yng(4)} &= \sigma_1^4 + \sigma_1^3\sigma_2 + \sigma_1^2\sigma_2^2 + \sigma_1\sigma_2^3 + \sigma_2^4 \;=\; 0, \\ \sigma_{\yng(4,1)} &= \sigma_1^4\sigma_2 + \sigma_1^3\sigma_2^2 + \sigma_1^2\sigma_2^3 + \sigma_1\sigma_2^4 \;=\; 0, \\ \sigma_{\yng(4,2)} &= \sigma_1^4\sigma_2^2 + \sigma_1^3\sigma_2^3 + \sigma_1^2\sigma_2^4 \;=\; 0, \end{aligned}$$

which matches the ring relations one expects from the cohomology theory. In each case, one multiplies in arbitrary powers of σ_1, σ_2 to get a six-point function, and in each case, the sum amounts to a scan through values that sum to zero. The top-form, described by $\sigma_{\yng(3,3)} = \sigma_1^3\sigma_2^3$, has nonzero vev, as expected.

Correlation functions in the one-instanton sector are of the form

$$\begin{aligned} &\langle f(\sigma) \rangle \;= \\ &\quad \frac{1}{2!}\mathrm{JKG}-\mathrm{Res}\left\{ q(\sigma_1-\sigma_2)^2 \frac{1}{\sigma_1^{10}}\frac{1}{\sigma_2^5} f(\sigma) \right\} + \frac{1}{2!}\mathrm{JKG}-\mathrm{Res}\left\{ q(\sigma_1-\sigma_2)^2 \frac{1}{\sigma_1^5}\frac{1}{\sigma_2^{10}} f(\sigma) \right\}, \end{aligned}$$

from which one can compute the nonzero correlation functions at order 11 to be

$$\langle \sigma_1^9 \sigma_2^2 \rangle = \frac{q}{2}, \quad \langle \sigma_1^8 \sigma_2^3 \rangle = -q, \quad \langle \sigma_1^7 \sigma_2^4 \rangle = \frac{q}{2},$$
$$\langle \sigma_1^4 \sigma_2^7 \rangle = \frac{q}{2}, \quad \langle \sigma_1^3 \sigma_2^8 \rangle = -q, \quad \langle \sigma_1^2 \sigma_2^9 \rangle = \frac{q}{2}.$$

Thus we see

$$\langle \sigma_{\yng(5)} \, \sigma_{\yng(3,3)} \rangle = -q, \quad \langle \sigma_{\yng(6)} \, \sigma_{\yng(3,2)} \rangle = -q,$$

which impies

$$R_{(5)} + q = 0, \quad R_{(6)} + q\sigma_{\yng(1)} = 0$$

because

$$\langle \sigma_{\yng(3,3)} \rangle = 1, \quad \langle \sigma_{\yng(3,2)} \, \sigma_{\yng(1)} \rangle = 1.$$

Next, let us consider the (0,2) theory defined by deformations of the tangent bundle. In this case, classical correlation functions are given by

$$\begin{aligned} \langle f(\sigma) \rangle \; &= \\ &\frac{1}{2!} \mathrm{JKG} - \mathrm{Res} \left\{ (-)(\sigma_1 - \sigma_2)^2 \left(\frac{1}{\det \tilde{E}(\sigma_1)} \right) \left(\frac{1}{\det \tilde{E}(\sigma_2)} \right) f(\sigma) \right\}, \end{aligned}$$

and for B diagonal, computing much as in previous examples, we find:

$$\begin{aligned} \langle \sigma_1^6 \rangle \;\; = \;\; & \Delta^{-1} \left[-I_1^2 + I_2 - 2I_1^3 + I_1 I_2 + I_3 + 3I_2^2 + 4I_1 I_3 - I_4 - 6I_1^2 I_2 - 6I_1 I_2^2 - 4I_1^2 I_3 \right. \\ & +10 I_2 I_3 - 8 I_1 I_2 I_3 - 2I_2^3 + 7I_3^2 - 4I_2^2 I_3 + 5I_4 I_1 - 5I_5 + 4I_1 I_5 \\ & +8I_1^2 I_5 + 9I_4 I_2 - 2I_1 I_3^2 - 2I_2 I_3^2 - 2I_1 I_2 I_4 - 2I_2^2 I_4 + 11 I_3 I_4 - 2I_2 I_3 I_4 \\ & \left. +4I_4^2 + 2I_1 I_4^2 + I_2 I_5 + 8I_1 I_2 I_5 + 2I_2^2 I_5 - 2I_3 I_5 - 2I_4 I_5 \right], \end{aligned}$$

$$\begin{aligned} \langle \sigma_1^5 \sigma_2 \rangle \;\; = \;\; & \Delta^{-1} \left[I_1 + 3I_1^2 + 2I_1^3 + 5I_1 I_2 - 2I_3 + 6I_1^2 I_2 + 2I_2^2 + I_1 I_3 - 4I_4 + 6I_1 I_2^2 \right. \\ & +4I_1^2 I_3 - 5I_1 I_4 - 5I_5 + 2I_2^3 - 2I_3^2 + 8I_1 I_2 I_3 - 4I_2 I_4 - 14 I_1 I_5 \\ & +4I_2^2 I_3 + 2I_1 I_3^2 + 2I_1 I_2 I_4 - 6I_3 I_4 - 8I_1^2 I_5 - 6I_2 I_5 + 2I_2 I_3^2 + 2I_2^2 I_4 \\ & \left. -4I_4^2 - 8I_1 I_2 I_5 + 2I_3 I_5 + 2I_2 I_3 I_4 - 2I_1 I_4^2 - 2I_2^2 I_5 + 2I_4 I_5 \right], \end{aligned}$$

$$\begin{aligned} \langle \sigma_1^4 \sigma_2^2 \rangle \;\; = \;\; & \Delta^{-1} \left[-1 - 4I_1 - 5I_1^2 - 4I_2 - 2I_1^3 - 10 I_1 I_2 - 2I_3 - 6I_1^2 I_2 - 5I_2^2 - 6I_1 I_3 + 3I_4 \right. \\ & -6I_1 I_2^2 - 4I_1^2 I_3 - 6I_2 I_3 + 3I_1 I_4 + 13 I_5 - 2I_2^3 - 8I_1 I_2 I_3 - I_3^2 + I_2 I_4 \\ & +20 I_1 I_5 - 4I_2^2 I_3 - 2I_1 I_3^2 - 2I_1 I_2 I_4 + 3I_3 I_4 + 8I_1^2 I_5 + 9I_2 I_5 - 2I_2 I_3^2 \\ & \left. -2I_2^2 I_4 + 4I_4^2 + 8I_1 I_2 I_5 - 2I_3 I_5 - 2I_2 I_3 I_4 + 2I_1 I_4^2 + 2I_2^2 I_5 - 2I_4 I_5 \right], \end{aligned}$$

$$\begin{aligned}
\langle \sigma_1^3 \sigma_2^3 \rangle \;&=\; \Delta^{-1} \big[2 + 6I_1 + 6I_1^2 + 6I_2 + 2I_1^3 + 12 I_1 I_2 + 4I_3 + 6I_1^2 I_2 + 6I_2^2 - 2I_4 + 8I_1 I_3 \\
&\quad +6I_1 I_2^2 - 16 I_5 + 4I_1^2 I_3 + 8I_2 I_3 - 2I_1 I_4 + 2I_2^3 - 22 I_1 I_5 + 8 I_1 I_2 I_3 \\
&\quad +2I_3^2 + 4I_2^2 I_3 + 2I_1 I_3^2 + 2I_1 I_2 I_3 - 2I_3 I_4 - 2I_1^2 I_5 - 10 I_2 I_5 + 2I_2 I_3^2 \\
&\quad +2I_2^2 I_4 - 4I_4^2 - 8I_1 I_2 I_5 + 2I_3 I_5 + 2I_2 I_3 I_4 - 2I_1 I_4^2 - 2I_2^2 I_5 + 2I_4 I_5 \big] ,
\end{aligned}$$

$$\langle \sigma_1^2 \sigma_2^4 \rangle \;=\; \langle \sigma_1^4 \sigma_2^2 \rangle, \quad \langle \sigma_1 \sigma_2^5 \rangle \;=\; \langle \sigma_1^5 \sigma_2 \rangle, \quad \langle \sigma_2^6 \rangle \;=\; \langle \sigma_1^6 \rangle,$$

where

$$\Delta \;=\; 2 \prod_{i<j} (1 + b_i + b_j),$$

and

$$\begin{aligned}
I_5 \;&=\; b_1 b_2 b_3 b_4 b_5 \;=\; \det B, \\
I_4 \;&=\; \sum_{i<j<k<\ell} b_i b_j b_k b_\ell, \\
I_3 \;&=\; \sum_{i<j<k} b_i b_j b_k, \\
I_2 \;&=\; \sum_{i<j} b_i b_j, \\
I_1 \;&=\; \sum_i b_i \;=\; \operatorname{tr} B.
\end{aligned}$$

The quantum sheaf cohomology ring (3.79) in this case is given by

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_3, D_4, \cdots, R_{(4)}, R_{(5)} + q, R_{(6)} + q\sigma_{(1)}, \cdots \right\rangle ,$$

where

$$\begin{aligned}
R_{(4)} \;&=\; \sum_{i=0}^{4} I_i \sigma_{(4-i)} \sigma_{(1)}^i, \\
&=\; \sigma_{(4)} + I_1 \sigma_{(3)} \sigma_{(1)} + I_2 \sigma_{(2)} \sigma_{(1)}^2 \;+\; (I_3 + I_4) \sigma_{(1)}^4, \\
R_{(5)} \;&=\; \sum_{i=0}^{5} I_i \sigma_{(5-i)} \sigma_{(1)}^i, \\
&=\; \sigma_{(5)} + I_1 \sigma_{(4)} \sigma_{(1)} + I_2 \sigma_{(3)} \sigma_{(1)}^2 + I_3 \sigma_{(2)} \sigma_{(1)}^3 + (I_4 + I_5) \sigma_{(1)}^5, \\
R_{(6)} \;&=\; \sum_{i=0}^{5} I_i \sigma_{(6-i)} \sigma_{(1)}^i, \\
&=\; \sigma_{(6)} + I_1 \sigma_{(5)} \sigma_{(1)} + I_2 \sigma_{(4)} \sigma_{(1)}^2 + I_3 \sigma_{(3)} \sigma_{(1)}^3 + I_4 \sigma_{(2)} \sigma_{(1)}^4 + I_5 \sigma_{(1)}^6.
\end{aligned}$$

As a consistency test, it is straightforward to check that the relations above are reflected in the correlation functions. For example, the classical correlation functions are easily demonstrated to obey

$$\langle \sigma_{\Box}^3 D_3 \rangle \;=\; \langle \sigma_{\Box}^2 D_4 \rangle \;=\; \langle \sigma_{\Box} D_5 \rangle \;=\; \langle D_6 \rangle \;=\; \langle \sigma_{\Box}^2 R_{(4)} \rangle \;=\; 0.$$

(Other vanishings of classical correlation functions are also implied by the ring relations; for example, in the line above, one could replace any instance of $\sigma_{\Box}^2$ with $\sigma_{\Box\Box}$ to get another vanishing correlation function. Our intent above is merely to list some examples, not to list every possible example.)

In addition, we can also use the classical correlation functions to check the classical limits of the relations

$$R_{(5)} \;=\; -q, \quad R_{(6)} \;=\; -q\sigma_{(1)}.$$

In particular, it is straightforward to show that the classical correlation functions obey

$$\langle \sigma_{\Box} R_{(5)} \rangle \;=\; 0 \;=\; \langle R_{(6)} \rangle,$$

verifying the classical limit of the relations above.

One of the relations should be the classical limit of the quantum cohomology ring relation derived from the Jeffrey-Kirwan-Grothendieck residue expression, namely,

$$\det \tilde{E}(\sigma_1) \;=\; q \;=\; \det \tilde{E}(\sigma_2)$$

where

$$\tilde{E}(x) \;=\; Ix \;+\; B(\sigma_1 + \sigma_2)$$

As before, it is straightforward to expand

$$\det \tilde{E}(x) \;=\; x^5 + I_1(\sigma_1+\sigma_2)x^4 + I_2(\sigma_1+\sigma_2)^2x^3 + I_3(\sigma_1+\sigma_2)^3x^2 + I_4(\sigma_1+\sigma_2)^4x + I_5(\sigma_1+\sigma_2)^5$$

so, for example,

$$\begin{aligned}\det \tilde{E}(\sigma_1) \;&=\; \sigma_1^5(1 + I_1 + I_2 + I_3 + I_4 + I_5) \;+\; \sigma_1^4\sigma_2(I_1 + 2I_2 + 3I_3 + 4I_4 + 5I_5) \\ &\quad + \sigma_1^3\sigma_2^2(I_2 + 3I_3 + 6I_4 + 10I_5) \;+\; \sigma_1^2\sigma_2^3(I_3 + 4I_4 + 10I_5) \\ &\quad + \sigma_1\sigma_2^4(I_4 + 5I_5) \;+\; \sigma_2^5(I_5).\end{aligned}$$

From this we derive

$$\begin{aligned}&\sigma_{\Box\Box\Box\Box\Box}(1 + I_1 + I_2 + I_3 + I_4 + 2I_5) \;+\; \sigma_{\substack{\Box\Box\Box\Box\\ \Box}}(-1 + I_2 + 2I_3 + 4I_4 + 8I_5) \\ &\qquad + \sigma_{\substack{\Box\Box\Box\\ \Box\Box}}(-I_1 - I_2 + I_3 + 5I_4 + 10I_5) \;=\; 2q.\end{aligned}$$

It is straightforward to check, via multiplication by $\sigma_{1,2}$, that the classical limit of the relation above is indeed a property of the correlation functions given in this section.

Chapter 4

A test of triality

In this chapter, we discuss how to use chiral rings to test triality of 2d (0,2) theories. This chapter was published in [99], and represents new work.

4.1 Overview of triality

It was proposed in [24] that triples of (0,2) GLSMs might flow to the same IR fixed point. In [24], the authors claim to get a nontrivial IR fixed point, from a (0,2) theory on a geometry that's not Calabi-Yau, which makes it an unexpected result. The theory they considered was a (0,2) $U(k)$ GLSM:

	type	multiplicity	$su(k)$	$u(1)$
Φ	chiral	n	$\mathbf{k}$	1
P	chiral	B	$\overline{\mathbf{k}}$	-1
Γ	Fermi	nB	$\mathbf{1}$	0
Ψ	Fermi	A	$\overline{\mathbf{k}}$	-1
λ	fermion	1	ad	0
Ω	Fermi	2	$\mathbf{1}$	k

with a (0,2) superpotential $W = \Gamma P \Phi$, where $B = 2k + A - n$. This GLSM was argued to be dual to a (0,2) $U(n-k)$ GLSM:

	type	multiplicity	$su(k)$	$u(1)$
$\tilde{\Phi}$	chiral	n	$\mathbf{k}$	1
$\tilde{P}$	chiral	A	$\overline{\mathbf{k}}$	-1
$\tilde{\Gamma}$	Fermi	nA	$\mathbf{1}$	0
$\tilde{\Psi}$	Fermi	B	$\overline{\mathbf{k}}$	-1
$\tilde{\lambda}$	fermion	1	ad	0
$\tilde{\Omega}$	Fermi	2	$\mathbf{1}$	k

with a (0,2) superpotential $\tilde{W} = \tilde{\Gamma}\tilde{P}\tilde{\Phi}$. A further step of duality move leads to yet a third (0,2) GLSM with a $U(A-n+k)$ gauge group:

	type	multiplicity	$su(k)$	$u(1)$
Φ'	chiral	B	$\mathbf{k}$	1
P'	chiral	n	$\overline{\mathbf{k}}$	-1
Γ'	Fermi	nB	$\mathbf{1}$	0
Ψ'	Fermi	A	$\overline{\mathbf{k}}$	-1
λ'	fermion	1	ad	0
Ω'	Fermi	2	$\mathbf{1}$	k

with a (0,2) superpotential $W' = \Gamma' P' \Phi'$.

This phenomenon, labelled "triality," can be understood as follows. If we integrate out the gauge field, then the large-radius limit of the first (0,2) theory can be understood as a nonlinear sigma model on

$$X_1 \;=\; G(k,n)$$

with bundle

$$\mathcal{E}_1 \;=\; S^A \oplus (Q^*)^{2k+A-n} \oplus (\det S^*)^2,$$

where S denotes the universal subbundle on $G(k,n)$, and Q the universal quotient bundle on $G(k,n)$. This theory has four flavor symmetries, three of which rotate bundle factors

$$SU(A) \times SU(2k+A-n) \times SU(2),$$

and the fourth of which, $SU(n)$, acts on the base. The other theories related by triality can be obtained by cyclically permuting

$$A, \quad 2k+A-n, \quad n,$$

and simultaneously replacing k by $n-k$. For example, the large-radius limit of the second theory is given by a nonlinear sigma model on

$$X_2 \;=\; G(n-k,A)$$

with bundle

$$\mathcal{E}_2 \;=\; S^{2k+A-n} \oplus (Q^*)^n \oplus (\det S^*)^2,$$

and the large-radius limit of the third theory is given by a nonlinear sigma model on

$$X_3 \;=\; G(A-n+k, 2k+A-n)$$

with bundle

$$\mathcal{E}_3 \;=\; S^n \oplus (Q^*)^A \oplus (\det S^*)^2.$$

Now, in order for the geometric description above to make sense, the values of n, A, and k are constrained. For example, to make sense of $G(A-n+k, 2k+A-n)$, we require

$$0 \;<\; A-n+k \;<\; 2k+A-n.$$

In addition, in order for the triality to be interesting, we would also like supersymmetry to remain unbroken, which can be checked by *e.g.* computing elliptic genera as refined Witten indices. Happily, these two requirements – that the geometric description be sensible, and that supersymmetry be unbroken – coincide in these theories.

In the UV, in addition to some nonanomalous $U(1)$'s, the theories above have nonanomalous

$$SU(n) \times SU(A) \times SU(2k+A-n) \times SU(2)$$

symmetries, as discussed above. It was proposed in [25] that in the IR, the theories above flow to a common nontrivial SCFT, in which the global flavor symmetries above are enhanced to affine symmetries [25][equ'n (3.1)]

$$SU(n)_{k+A-n} \times SU(A)_k \times SU(2k+A-n)_{n-k} \times SU(2)_1.$$

By examining chiral states among the UV theories above, we will give nontrivial evidence that the different UV theories flow to the same IR fixed point and have the IR affine symmetries indicated above.

4.2 General remarks on chiral states

4.2.1 UV physics

Let us focus on the first model in the last section, a heterotic nonlinear sigma model described by the space and bundle

$$X \;=\; G(k,n), \quad \mathcal{E} \;=\; S^A \oplus (Q^*)^{2k+A-n} \oplus (\det S^*)^2.$$

To compute the chiral states, we first need to compute the bundle to which the Fock vacuum couples. To that end, in the model above,

$$\det \mathcal{E} \;=\; \mathcal{O}(-(A-2) \;-\; (2k+A-n)) \;=\; \mathcal{O}(-2k-2A+n+2),$$

and

$$K_X \;=\; \mathcal{O}(-n),$$

hence

$$(\det \mathcal{E}) \otimes K_X \;=\; \mathcal{O}(-2k - 2A + 2), \quad (\det \mathcal{E})^{-1} \otimes K_X \;=\; \mathcal{O}(2k + 2A - 2n - 2),$$

and so a square root always exists. Furthermore, since the Grassmannian is simply-connected, that square root is unique, and defines the bundle to which the Fock vacuum couples.

Now, in this chapter we are interested in more than merely counting states – we also want to keep track of global symmetry representations. To that end, it will be useful to describe X and $\mathcal{E}$ in terms of vector spaces defining fundamental representations of global symmetry groups. For example, we will describe the model above as

$$X \;=\; G(k, \tilde{V}^*), \quad \mathcal{E} \;=\; U \otimes S \oplus V \otimes Q^* \oplus W \otimes \det S^*,$$

where U, V, W, and $\tilde{V}$ are vector spaces of dimensions A, $2k+A-n$, 2, and n, respectively.

The original UV GLSM has a

$$SU(A) \times SU(2k + A - n) \times SU(2) \times SU(n) \;=\; SU(U) \times SU(V) \times SU(W) \times SU(\tilde{V})$$

symmetry, but in the sheaf cohomology groups, naively only the

$$SU(A) \times SU(2k + A - n) \times SU(2) \;=\; SU(U) \times SU(V) \times SU(W)$$

subgroup is explicit, in its action on the left-moving fermions. The remaining $SU(n) = SU(\tilde{V})$, which acts on the base, is made manifest via the Bott-Borel-Weil theorem, which expresses sheaf cohomology of homogeneous vector bundles (including the $\mathcal{E}$ above) on Grassmannians $G(k,n)$ in terms of representations of $U(n)$, making the relationship explicit.

To that end, when computing chiral rings, it will be important to keep track of the difference between factors of *e.g.* S and Q^*. As holomorphic bundles, for example $\det S \cong \det Q^*$, but they define different representations of the parabolic subgroup $GL(k) \times GL(n-k)$ of $GL(n)$, and the difference will manifest via Bott-Borel-Weil in terms of the precise representation of $U(n)$ appearing. For example, if $n = 2$, then on $G(1, \tilde{V}^*) = \mathbb{P}^1$, $S^* \otimes Q^* \cong \mathcal{O}$. However, applying Bott-Borel-Weil, we find

$$\begin{aligned} H^\bullet(\mathcal{O}) \;&=\; \mathbb{C}\delta^{\bullet,0}, \\ H^\bullet(S^* \otimes Q^*) \;&=\; \wedge^2 \tilde{V} \delta^{\bullet,0}. \end{aligned}$$

The two sheaf cohomology groups have the same dimension – as they should, since the bundles are isomorphic as holomorphic bundles – but encode different representations of $GL(\tilde{V})$. Thus, for example, when specifying Fock vacuum bundles, we must specify not only a holomorphic line bundle, but in addition a precise representation of the parabolic subgroup, *i.e.* a precise description as powers of S^* and Q^*.

4.3 Pseudo-topological twists

As we discussed in section 2.2.1, theories with (0,2) supersymmetry admit pseudo-topological twists, resulting in theories known as the A/2 and B/2 models [28, 50]. The A/2 model twist of a (0,2) nonlinear sigma model on a space X with bundle $\mathcal{E}$ is well-defined when, in addition to the Green-Schwarz anomaly cancellation condition,

$$\det \mathcal{E}^* \;\cong\; K_X,$$

and the B/2 model twist is well-defined when, in addition to Green-Schwarz,

$$\det \mathcal{E} \;\cong\; K_X.$$

Dualizing the bundle $\mathcal{E}$ yields an isomorphic quantum field theory, in which the A/2 and B/2 twists are exchanged.

In principle, if the bundle $\mathcal{E}$ is reducible, then there are further variants, further topological twists, obtained by dualizing the various individual factors. Dualizing those bundle factors produces an isomorphic quantum field theory, but modifies the twists. If we think of the twists as twisting along a $U(1)$ symmetry of the theory, then the point here is that if the bundle is reducible, then there are additional $U(1)$ symmetries (corresponding to different phase factors on different factors) which yield different pseudo-topological twists, or equivalently, the A/2 and B/2 twist but for a bundle obtained by dualizing some of the factors. We will speak of a theory 'admitting an A/2 or B/2 twist' when the particular choice of $\mathcal{E}$ satisfies one of the conditions above.

Now, let us turn to the examples appearing in triality. As a warm-up, consider a (0,2) nonlinear sigma model on the Grassmannian $G(k,n)$ with bundle

$$\mathcal{E} \;=\; S^{\oplus A} \oplus (Q^*)^{\oplus(2k+A-n)} \oplus (\det S^*)^{\oplus 2},$$

from which we derive

$$\begin{aligned} \det \mathcal{E} &\cong (\det S)^{A-2} \otimes (\det Q^*)^{2k+A-n}, \\ &\cong (\det S)^{2A+2k-n-2}, \end{aligned}$$

using $\det Q^* \cong \det S$. For the tangent bundle,

$$0 \longrightarrow S^* \otimes S \longrightarrow S^* \otimes \mathcal{O}^n \longrightarrow T \longrightarrow 0,$$

hence $K_X \cong (\det S)^n$. Putting this together, we find that this model will admit an A/2 twist when

$$0 \;=\; A + k - 1,$$

and the same model will admit a B/2 twist when

$$n \;=\; A + k - 1.$$

For bundles of this particular form, examples admitting an A/2 twist will be rather rare, as it requires $A + k = 1$, but models admitting a B/2 twist are less uncommon.

Next, let us take advantage of the fact that $\mathcal{E}$ is reducible. If we dualize the middle and last factors, we get the bundle

$$\mathcal{E}' \;=\; S^{\oplus A} \oplus (Q)^{\oplus(2k+A-n)} \oplus (\det S)^{\oplus 2},$$

for which we compute

$$\det \mathcal{E}' \;\cong\; (\det S)^{A+2} \otimes (\det Q)^{2k+A-n} \;\cong\; (\det Q)^{2k-n-2},$$

and so

$$(\det \mathcal{E}')^{-1} \otimes K_X \;\cong\; (\det S)^{2k-n-2} \otimes (\det S)^{n} \;\cong\; (\det S)^{2k-2}.$$

Clearly, if $k = 1$, then this presentation admits a B/2 twist. However, if we are willing to make the global symmetry rotating Q^*'s more obscure and dualize pairs of them individually, then we can build an alternative bundle $\mathcal{E}''$ which admits a B/2 twist.

Thus, by suitably dualizing gauge bundle factors, we can find a B/2 twist of any (0,2) theory related by triality. That said, a given B/2 twist is not invariant under triality, as even dualizing a bundle will replace the original B/2 twist with something different.

4.4 First example

In this section we will compare chiral states in examples of different UV NLSM's that are related by triality – some as different phases of the same GLSM, others from different GLSM's. We will find in the examples we compute that all the different presentations have some states in common, and a few states that differ between presentations. However, the states that are in common, all are defined by integrable representations of the global symmetry groups, integrable with respect to the proposed IR affine algebras. Furthermore, the mismatched states will not contribute to elliptic genera and are defined by nonintegrable representations, strongly suggesting that they become massive along the RG flow, indirectly verifying the triality proposal, and also giving a very clean example of how non-protected operators can change along RG flow.

4.4.1 First GLSM

We shall begin with a computation of chiral states in the two phases of a (0,2) GLSM pertinent to triality. Let's take $k = 1$, $A = 3$, and $n = 3$, so $2k + A - n = 2$. The two phases are defined by

$$\mathcal{E} \;\equiv\; U \otimes S \;+\; V \otimes Q^* \;+\; W \otimes \det S^* \;\longrightarrow\; \mathbb{P}^2 \;=\; \mathbb{P}\tilde{V}^*$$

for $r \gg 0$, and

$$\mathcal{F} \; \equiv \; U \otimes S^* \; + \; \tilde{V} \otimes Q^* \; + \; W \otimes \det S \; \longrightarrow \; \mathbb{P}^1 \; = \; \mathbb{P}V^*$$

for $r \ll 0$, where in both phases,

$$U \; = \; \mathbb{C}^3, \;\; V \; = \; \mathbb{C}^2, \;\; W \; = \; \mathbb{C}^2, \;\; \tilde{V} \; = \; \mathbb{C}^3.$$

(For brevity, we only list the underlying geometries, rather than the full matter content of each GLSM. It may be worth observing that although two of the triality phases are given by abelian GLSM's, the third is given by a GLSM with gauge group $U(2)$.)

According to triality, these two nonlinear sigma models should flow in the IR to the same point, hence, on the face of it, one would expect them to have isomorphic chiral states.

It is straightforward to check that, in both cases, anomaly cancellation holds, and furthermore each phase admits a B/2 twist, so that the Fock vacuum line bundle is trivial. As remarked earlier, for our purposes it is important to give a precise presentation of the Fock vacuum line bundle in terms of powers of S^* and Q^*, and in both phases we will present it as the canonical trivial bundle, *i.e.* $K_{(0)}S^* \otimes K_{(0,\cdots,0)}Q^*$. Hence we can compute the chiral states in the form

$$H^\bullet(X, \wedge^\bullet \mathcal{E}).$$

Each phase has a set of global nonanomalous (chiral) $U(1)^3$ symmetries, which are given by

	$\tilde{V}$	U	V	W
$U(1)_{(1)}$	0	0	-1	-1
$U(1)_{(2)}$	1	0	0	$-3/2$
$U(1)_{(3)}$	0	1	0	$+3/2$

In a slight variation from [24, 25], we have assigned the same charge to all elements of W, for simplicity in comparing states.

For the purposes of correctly comparing symmetries, it is useful to distinguish $\mathbb{P}V$ from $\mathbb{P}V^*$, for example. Briefly, on the space $\mathbb{P}V$, the homogeneous coordinates naturally transform under V^*. On that space,

$$0 \longrightarrow S \longrightarrow V \otimes \mathcal{O} \longrightarrow Q \longrightarrow 0,$$

so dualizing, taking the long exact sequence, and using $H^\bullet(Q^*) = 0$, we find that the homogeneous coordinates are given by

$$H^0(S^*) \; = \; H^0(V^* \otimes \mathcal{O}) \; = \; V^*.$$

Here, for $r \gg 0$, the homogeneous coordinates naturally transform under $\tilde{V}$, and for $r \ll 0$, V, hence the two geometries are naturally $\mathbb{P}\tilde{V}^*$ and $\mathbb{P}V^*$, respectively.

In the remaining tables in this section we list all of the states in the two phases, beginning with all of the matching states. Now, to match states in principle we need only match representations of nonanomalous symmetries. However, in this example, the bulk of the matching states match full (anomalous) $GL(U) \times GL(V) \otimes GL(W) \otimes GL(\tilde{V})$ representations. In table 4.1 we list all such states which match exactly, as representations of the anomalous symmetry above, between the two phases. The state column lists the representation of

$$GL(U) \times GL(V) \times GL(W) \times GL(\tilde{V})$$

obtained from the Bott-Borel-Weil computation. As overall $\mathbb{C}^\times$ factors are individually anomalous, we separately list the nonanomalous

$$SL(U) \times SL(V) \times SL(W) \times SL(\tilde{V}) \; = \; SU(3) \times SU(2) \times SU(2) \times SU(3)$$

representations and nonanomalous global

$$U(1)_{(1)} \times U(1)_{(2)} \times U(1)_{(3)}$$

charges in the last two columns. Immediately after the state listing, the next two columns list in which wedge power of $\mathcal{E}$ the state was obtained, and the cohomological degree, respectively, in the $r \gg 0$ phase, and the next two after that give the same information for the $r \ll 0$ phase.

The $U(1)^3$ charges listed include the fractional charges from the Fock vacua. For the $r \gg 0$ phase, for example, the first and third $U(1)$'s act linearly on the left-moving fermions, and so the fractional charges can be computed directly using standard[1] fractional fermion computations. The second $U(1)$ does not act linearly on the right-moving fermions, and so the corresponding vacuum charge is a bit more subtle to compute. We computed it as $-1/2$ of the charge of the top-degree state, the Serre dual to the state in $\wedge^0 \mathcal{E}$. This guaranteed that the first and last entries in table 4.1 have opposite charges. In any event, the result is that for both the $r \gg 0$ and $r \ll 0$ phases, the fractional $U(1)^3$ charge of the vacuum was taken to be $(+3, 0, -3)$. It is a highly nontrivial consistency check that, with that choice, all other states related by Serre duality also have opposite charges.

[1] Alternatively, they can also be computed from our expression for the Fock vacuum line bundle

$$(\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2}.$$

For the $r \gg 0$ phase, where

$$\mathcal{E} \; = \; U \otimes S \; + \; V \otimes Q^* \; + \; W \otimes S^* \; \longrightarrow \; G(1, \tilde{V}^*),$$

it is straightforward to compute that

$$(\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2} \; = \; \left(\wedge^3 U\right)^{-1/2} \otimes \left(\wedge^2 V\right)^{-1} \otimes \left(\wedge^2 W\right)^{-1/2} \otimes \left(\wedge^3 \tilde{V}^*\right)^{-1/2},$$

for which the first and third $U(1)$ charges are computed to match those one would obtain by standard fractional fermion techniques. We caution against applying the same method for the fractional charge under the second $U(1)$ in the $r \gg 0$ phase, as its action is not linear on the right-moving fermions. The method above will sometimes give the correct result in that case, but will also often not.

	$r \gg 0$		$r \ll 0$			
State	$\wedge^\bullet \mathcal{E}$	$H^\bullet(\mathbb{P}^2)$	$\wedge^\bullet \mathcal{F}$	$H^\bullet(\mathbb{P}^1)$	Rep'	$U(1)^3$
1	0	0	0	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(+3,0,-3)$
$W\otimes\tilde{V}$	1	0	2	1	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{3})$	$(+2,-1/2,-3/2)$
$\wedge^2V\otimes\wedge^2\tilde{V}$	2	1	2	1	$(\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{3}})$	$(+1,+2,-3)$
$U\otimes V$	2	1	1	0	$(\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1})$	$(+2,0,-2)$
$U\otimes W$	2	0	2	0	$(\mathbf{3},\mathbf{1},\mathbf{2},\mathbf{1})$	$(+2,-3/2,-1/2)$
$V\otimes W\otimes\wedge^2\tilde{V}$	2	0	3	1	$(\mathbf{1},\mathbf{2},\mathbf{2},\overline{\mathbf{3}})$	$(+1,+1/2,-3/2)$
$U\otimes\wedge^2V\otimes\tilde{V}$	3	1	2	0	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{3})$	$(+1,+1,-2)$
$U\otimes\wedge^2W\otimes\tilde{V}$	3	0	4	1	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{3})$	$(+1,-2,+1)$
$V\otimes\wedge^2V\otimes\wedge^3\tilde{V}$	3	1	3	1	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	$(0,+3,-3)$
$\mathrm{Sym}^2V\otimes W\otimes\wedge^3\tilde{V}$	3	0	4	1	$(\mathbf{1},\mathbf{3},\mathbf{2},\mathbf{1})$	$(0,+3/2,-3/2)$
$\wedge^2U\otimes\mathrm{Sym}^2V$	4	2	2	0	$(\overline{\mathbf{3}},\mathbf{3},\mathbf{1},\mathbf{1})$	$(+1,0,-1)$
$\wedge^2U\otimes V\otimes W$	4	1	3	0	$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2},\mathbf{1})$	$(+1,-3/2,+1/2)$
$\wedge^2U\otimes\wedge^2W$	4	0	4	0	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})$	$(+1,-3,+2)$
$U\otimes\wedge^2V\otimes W\otimes\wedge^2\tilde{V}$	4	1	4	1	$(\mathbf{3},\mathbf{1},\mathbf{2},\overline{\mathbf{3}})$	$(0,+1/2,-1/2)$
$U\otimes V\otimes\wedge^2W\otimes\wedge^2\tilde{V}$	4	0	5	1	$(\mathbf{3},\mathbf{2},\mathbf{1},\overline{\mathbf{3}})$	$(0,-1,+1)$
$\wedge^2U\otimes\wedge^2V\otimes V\otimes\tilde{V}$	5	2	3	0	$(\overline{\mathbf{3}},\mathbf{2},\mathbf{1},\mathbf{3})$	$(0,+1,-1)$
$\wedge^2U\otimes W\otimes\wedge^2V\otimes\tilde{V}$	5	1	4	0	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{2},\mathbf{3})$	$(0,-1/2,+1/2)$
$U\otimes(\wedge^2V)^2\otimes\wedge^3\tilde{V}$	5	2	4	1	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$	$(-1,+3,-2)$
$U\otimes W\otimes\wedge^2V\otimes V\otimes\wedge^3\tilde{V}$	5	1	5	1	$(\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{1})$	$(-1,+3/2,-1/2)$
$U\otimes\mathrm{Sym}^2V\otimes\wedge^2W\otimes\wedge^3\tilde{V}$	5	0	6	1	$(\mathbf{3},\mathbf{3},\mathbf{1},\mathbf{1})$	$(-1,0,+1)$
$\wedge^3U\otimes\mathrm{Sym}^2V\otimes W$	6	2	4	0	$(\mathbf{1},\mathbf{3},\mathbf{2},\mathbf{1})$	$(0,-3/2,+3/2)$
$\wedge^3U\otimes V\otimes\wedge^2W$	6	1	5	0	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	$(0,-3,+3)$
$\wedge^2U\otimes(\wedge^2V)^2\otimes\wedge^2\tilde{V}$	6	2	4	0	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\overline{\mathbf{3}})$	$(-1,+2,-1)$
$\wedge^2U\otimes\wedge^2V\otimes\wedge^2W\otimes\wedge^2\tilde{V}$	6	1	6	1	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\overline{\mathbf{3}})$	$(-1,-1,+2)$
$\wedge^3U\otimes\wedge^2V\otimes V\otimes W\otimes\tilde{V}$	7	2	5	0	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{3})$	$(-1,-1/2,+3/2)$
$\wedge^2U\otimes(\wedge^2V)^2\otimes W\otimes\wedge^3\tilde{V}$	7	2	6	1	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{2},\mathbf{1})$	$(-2,+3/2,+1/2)$
$\wedge^2U\otimes\wedge^2V\otimes V\otimes\wedge^2W\otimes\wedge^3\tilde{V}$	7	1	7	1	$(\overline{\mathbf{3}},\mathbf{2},\mathbf{1},\mathbf{1})$	$(-2,0,+2)$
$\wedge^3U\otimes\wedge^2V\otimes\wedge^2W\otimes\tilde{V}$	7	1	6	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})$	$(-1,-2,+3)$
$\wedge^3U\otimes(\wedge^2V)^2\otimes W\otimes\wedge^2\tilde{V}$	8	2	6	0	$(\mathbf{1},\mathbf{1},\mathbf{2},\overline{\mathbf{3}})$	$(-2,+1/2,+3/2)$
$\wedge^3U\otimes(\wedge^2V)^2\otimes\wedge^2W\otimes\wedge^3\tilde{V}$	9	2	8	1	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(-3,0,+3)$

Table 4.1: List of states shared between the two phases.

$r \gg 0$ State, $\wedge^\bullet \mathcal{E}, H^\bullet(\mathbb{P}^2)$	$r \ll 0$ State, $\wedge^\bullet \mathcal{F}, H^\bullet(\mathbb{P}^1)$	Rep'	$U(1)^3$
$\wedge^3 U \otimes \wedge^3 \tilde{V}^*$, $3, 2$	$\wedge^2 W \otimes \wedge^2 V^*$, $2, 1$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(+3,-3,0)$
$\wedge^3 U \otimes V \otimes \wedge^2 \tilde{V}^*$, $4, 2$	$\tilde{V} \otimes \wedge^2 W \otimes V^*$, $3, 1$	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{3})$	$(+2,-2,0)$
$\mathrm{Sym}^2 V \otimes \wedge^2 W \otimes \tilde{V} \otimes \wedge^3 \tilde{V}$, $4, 0$	$\wedge^3 U \otimes \tilde{V} \otimes \wedge^2 V \otimes \mathrm{Sym}^2 V$, $4, 0$	$(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{3})$	$(-1,+1,0)$
$\wedge^3 U \otimes \mathrm{Sym}^2 V \otimes \wedge^3 \tilde{V}^* \otimes \wedge^2 \tilde{V}$, $5, 2$	$\wedge^2 \tilde{V} \otimes \wedge^2 W \otimes \wedge^2 V^* \otimes \mathrm{Sym}^2 V$, $4, 1$	$(\mathbf{1},\mathbf{3},\mathbf{1},\overline{\mathbf{3}})$	$(+1,-1,0)$
$\wedge^2 V \otimes V \otimes \wedge^2 W \otimes \wedge^3 \tilde{V} \otimes \wedge^2 \tilde{V}$, $5, 0$	$\wedge^3 U \otimes \wedge^2 \tilde{V} \otimes (\wedge^2 V)^2 \otimes V$, $5, 0$	$(\mathbf{1},\mathbf{2},\mathbf{1},\overline{\mathbf{3}})$	$(-2,+2,0)$
$(\wedge^2 V)^2 \otimes \wedge^2 W \otimes (\wedge^3 \tilde{V})^2$, $6, 0$	$\wedge^3 U \otimes \wedge^3 \tilde{V} \otimes (\wedge^2 V)^3$, $6, 0$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(-3,+3,0)$

Table 4.2: Additional shared states defined by matching representations of anomaly-free global symmetries.

State	$r \gg 0$ $\wedge^\bullet \mathcal{E}$	$H^\bullet(\mathbb{P}^2)$	$r \ll 0$ $\wedge^\bullet \mathcal{F}$	$H^\bullet(\mathbb{P}^1)$	Rep'	$U(1)^3$
$\wedge^2 W \otimes \mathrm{Sym}^2 \tilde{V}$	2	0	–	–	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{6})$	$(+1,-1,0)$
$V \otimes \wedge^2 W \otimes K_{(2,1,0)} \tilde{V}$	3	0	–	–	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{8})$	$(0,0,0)$
$\wedge^2 V \otimes \wedge^2 W \otimes K_{(2,2,0)} \tilde{V}$	4	0	–	–	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{6})$	$(-1,+1,0)$
$\wedge^3 U \otimes \wedge^2 V \otimes \wedge^3 \tilde{V}^* \otimes \mathrm{Sym}^2 \tilde{V}$	5	2	–	–	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{6})$	$(+1,-1,0)$
$\wedge^3 U \otimes K_{(2,1)} V \otimes K_{(1,0,-1)} \tilde{V}$	6	2	–	–	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{8})$	$(0,0,0)$
$\wedge^3 U \otimes (\wedge^2 V)^2 \otimes \wedge^3 \tilde{V} \otimes \mathrm{Sym}^2 \tilde{V}^*$	7	2	–	–	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{6})$	$(-1,+1,0)$
$\wedge^3 U \otimes \mathrm{Sym}^3 V$	–	–	3	0	$(\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{1})$	$(0,0,0)$
$\wedge^3 \tilde{V} \otimes \wedge^2 W \otimes \wedge^2 V^* \otimes \mathrm{Sym}^3 V$	–	–	5	1	$(\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{1})$	$(0,0,0)$

Table 4.3: List of all states which are not shared between the two phases.

As a consistency check, note that all of the states in all of the tables in this section come in Serre-dual pairs, exchanging not only cohomology degrees but also dualizing representations and $U(1)^3$ charges. (Recall that in $SU(2)$, $\mathbf{2} = \overline{\mathbf{2}}$, so the dualization only acts nontrivially on $SU(3)$ representations.)

A few additional states have matching representations of anomaly-free global symmetries, *i.e.* the same $SU(3) \times \cdots$ representations and $U(1)^3$ charges, but are expressed differently in terms of anomalous representations. These are listed in table 4.2.

Finally, there are a few remaining states in each phase that do not match any states in the other phase at all, listed in table 4.3. Note in particular that the states in neither phase are a proper subset of the states in the other: both phases have states not in the other.

To aid the reader, as the methods are not commonly used in the physics community, let us take a moment to illustrate how, for example, the next-to-last entry in table 4.1 was computed, in the $r \gg 0$ phase. This entry arose as the only nonzero contribution to $H^\bullet(\mathbb{P}^2, \wedge^8 \mathcal{E})$.

Now,

$$\begin{aligned}\wedge^8\mathcal{E} &= \wedge^8\left(U\otimes S \,+\, V\otimes Q^* \,+\, W\otimes S^*\right),\\ &= \wedge^2(U\otimes S)\otimes\wedge^4(V\otimes Q^*)\otimes\wedge^2(W\otimes S^*)\\ &\qquad\qquad +\ \wedge^3\,(U\otimes S)\otimes\wedge^3(V\otimes Q^*)\otimes\wedge^2(W\otimes S^*)\\ &\qquad\qquad +\ \wedge^3\,(U\otimes S)\otimes\wedge^4(V\otimes Q^*)\otimes\wedge^1(W\otimes S^*),\end{aligned}$$

since $U\otimes S$ has rank 3, $V\otimes Q^*$ has rank 4, and $W\otimes S^*$ has rank 2. (More generally, we sum over all combinations of wedge powers adding up to the given power, which in this case is eight.) To compute each of the wedge powers appearing, we use the identity

$$\wedge^r(A\otimes B) \;=\; \sum_{|\lambda|=r} K_\lambda A\otimes K_{\lambda^T}B,$$

where the sum is over Young diagrams λ with r boxes, and $K_\lambda A$ denotes a tensor product of A's determined by the Young diagram, for example

$$K_{\ydiagram{2}}A \;=\; \mathrm{Sym}^2 A,\quad K_{\ydiagram{1,1}}A \;=\; \wedge^2 A.$$

Thus, for example,

$$\begin{aligned}\wedge^2(W\otimes S^*) &= K_{\ydiagram{2}}W\otimes K_{\ydiagram{1,1}}S^* \,+\, K_{\ydiagram{1,1}}W\otimes K_{\ydiagram{2}}S^*,\\ &= \mathrm{Sym}^2W\otimes\wedge^2S^* \,+\, \wedge^2W\otimes\mathrm{Sym}^2S^*.\end{aligned}$$

However, since S^* has rank one, $\wedge^2S^*=0$, and so

$$\wedge^2(W\otimes S^*) \;=\; \wedge^2W\otimes\mathrm{Sym}^2S^*.$$

Similarly,

$$\wedge^3(V\otimes Q^*) \;=\; K_{\ydiagram{2,1}}V\otimes K_{\ydiagram{2,1}}Q^*,\quad \wedge^4(V\otimes Q^*) \;=\; K_{\ydiagram{2,2}}V\otimes K_{\ydiagram{2,2}}Q^*,$$

since both V and Q^* have rank 2, eliminating most possible contributions, and

$$\wedge^2(U\otimes S) \;=\; \wedge^2U\otimes\mathrm{Sym}^2S,\quad \wedge^3(U\otimes S) \;=\; \wedge^3U\otimes\mathrm{Sym}^3S,$$

since S has rank 1, eliminating most possible contributions. Putting this together, we find

$$\begin{aligned}\wedge^8\mathcal{E} &= \wedge^2U\otimes K_{\ydiagram{2,2}}V\otimes\wedge^2W\otimes\mathrm{Sym}^2S\otimes K_{\ydiagram{2,2}}Q^*\otimes\mathrm{Sym}^2S^*\\ &\qquad +\ \wedge^3\,U\otimes K_{\ydiagram{2,1}}V\otimes\wedge^2W\otimes\mathrm{Sym}^3S\otimes K_{\ydiagram{2,1}}Q^*\otimes\mathrm{Sym}^2S^*\\ &\qquad +\ \wedge^3\,U\otimes K_{\ydiagram{2,2}}V\otimes W\otimes\mathrm{Sym}^3S\otimes K_{\ydiagram{2,2}}Q^*\otimes S^*.\end{aligned}$$

Thus,

$$
\begin{aligned}
H^\bullet(\mathbb{P}^2, \wedge^8 \mathcal{E}) \;\; = \;\; & \wedge^2 U \otimes K_{\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \, \\ \hline \end{array}} V \otimes \wedge^2 W \otimes H^\bullet\left(\mathbb{P}^2, K_{(2,2)} Q^*\right) \\
& + \wedge^3 U \otimes K_{\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}} V \otimes \wedge^2 W \otimes H^\bullet\left(\mathbb{P}^2, K_{(-1)} S^* \otimes K_{(2,1)} Q^*\right) \\
& + \wedge^3 U \otimes K_{\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \, \\ \hline \end{array}} V \otimes W \otimes H^\bullet\left(\mathbb{P}^2, K_{(-2)} S^* \otimes K_{(2,2)} Q^*\right),
\end{aligned}
$$

and from Bott-Borel-Weil,

$$
H^\bullet\left(\mathbb{P}^2, K_{(2,2)} Q^*\right) \; = \; 0 \; = \; H^\bullet\left(\mathbb{P}^2, K_{(-1)} S^* \otimes K_{(2,1)} Q^*\right),
$$

$$
H^\bullet\left(\mathbb{P}^2, K_{(-2)} S^* \otimes K_{(2,2)} Q^*\right) \; = \; K_{(1,1,0)} \tilde{V} \, \delta^{\bullet,2}.
$$

The next-to-last entry in table 4.1 follows.

Note that the states in table 4.3 might not make a net contribution to elliptic genera. From [9][section 2.1], since this theory is B/2-twistable, the leading term in any refined NLSM elliptic genus will be of the form

$$
(-)^{r/2} q^{+(r-n)/12} \sum_{s=0}^{r} (-)^s \chi_y(\wedge^s \mathcal{E}), \tag{4.1}
$$

where y represents the refinement by any nonanomalous symmetry, r is the rank of $\mathcal{E}$, and n is the dimension of the base space. Since in our example the rank is greater than the dimension of the base, the unrefined elliptic genus will vanish [9]. However, since there are a number of nonanomalous symmetries, the elliptic genus can be refined, and it is straightforward to check that by adding suitable refinements, the elliptic genus can certainly be made nonzero.

Now, the contribution of any state to the weighted sum (4.1) is weighted in part by a sign determined by the wedge power of the bundle and the degree of the cohomology group in which the state appears. However, all of the mismatched states in table 4.3 come in pairs with matching representations of nonanomalous symmetries but different signs, and so cancel out. Consider as a prototypical example the two states in the $r \ll 0$ phase. They both live in the same representation $(\mathbf{1}, \mathbf{4}, \mathbf{1}, \mathbf{1})$, and have the same $U(1)^3$ charges, but one enters with a $+$ and the other with a $-$, and so their net contribution to the leading term of any elliptic genus refined by any of the symmetries listed is necessarily zero. Thus, the mismatched states listed in table 4.3 make no net contribution to the leading term of any refined elliptic genus we shall consider. Of course, that guarantees neither that higher order contributions will also vanish, nor that there are no other nonanomalous symmetries whose refinements might receive a contribution. However, we do find it to be a suggestive observation, supporting the idea that these (mismatched) states might all pair up and become massive along the RG flow.

Now, let us examine these states from the perspective of the exact IR limit proposed in [25].

In the notation of that reference, in the IR there is an affine[2]

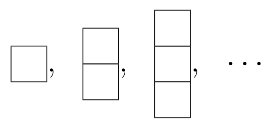
$$SU(U)_1 \times SU(V)_2 \times SU(W)_1 \times SU(\tilde{V})_1 \;=\; SU(3)_1 \times SU(2)_2 \times SU(2)_1 \times SU(3)_1$$

symmetry. The corresponding left-chiral states in the IR should be determined in part by integrable representations of the affine symmetry above.

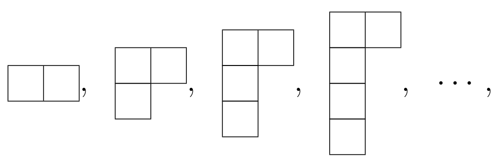
In terms of Young diagrams, a given representation of $SU(N)_k$ is integrable if the Young diagram has no row with more than k boxes [47][section 16.6]. For example, for $k = 1$, there are N integrable representations, given by

$$\mathbf{1}, \;\; \mathbf{N}, \;\; \wedge^2\mathbf{N}, \;\; \cdots, \;\; \wedge^{N-1}\mathbf{N}.$$

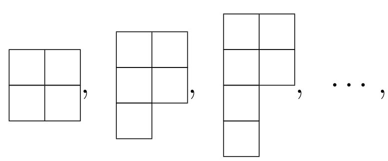
corresponding to Young diagrams

1, 

In the case $k = 2$, in addition to the N diagrams above, there are $N - 1$ diagrams of the form



plus $N - 2$ diagrams of the form



and so forth, for a total of

$$1 \,+\, 2 \,+\, \cdots \,+\, N-1 \,+\, N \;=\; \frac{1}{2}N(N+1)$$

integrable representations of $su(N)$ at level $k = 2$.

[2] The $SU(W)$ affine contribution is not mentioned explicitly in [25]. The level can be computed from the trace anomaly

$$k_\Omega \;=\; \mathrm{tr}\, \gamma^3 J J \;=\; -1/2$$

(in their conventions). The corresponding level of the IR current algebra is

$$2|k_\Omega| \;=\; 1$$

and the sign indicates that it is left-moving.

In the present case, $SU(3)_1$ has integrable representations

$$\mathbf{1}, \quad \mathbf{3}, \quad \overline{\mathbf{3}},$$

$SU(2)_1$ has integrable representations

$$\mathbf{1}, \mathbf{2},$$

and $SU(2)_2$ has integrable representations

$$\mathbf{1}, \quad \mathbf{2}, \quad \mathbf{3}.$$

Note that all of the states in tables 4.1 and 4.2 have integrable representations of the non-abelian UV global symmetry groups. On the other hand, all of the mismatched states in table 4.3 have at least one non-integrable representation. If any of the mismatched states survived to the IR, they would then appear to contradict the assertion of [25] that these theories have a nontrivial IR fixed point of the form described there. However, given that their contributions to refined elliptic genera vanish, we find it much more likely that they become massive, leaving only states which are both common across presentations and defined by suitable representations. In this fashion, we have an indirect test of triality.

4.4.2 Other GLSMs

In the previous subsection we analyzed the chiral states in the two geometric phases of one of three GLSMs that is believed to flow to a single fixed point. Next we shall repeat the same analysis for another GLSM related to the first by triality. We shall find an analogous structure – states in integrable representations match between the phases, and moreover, we shall see that the states that match between phases also match between GLSMs.

The other two GLSMs can be obtained by cyclically permuting

$$U \longrightarrow V \longrightarrow \tilde{V}^* \longrightarrow U \longrightarrow \cdots .$$

The three large-radius phases correspond to the bundles and spaces given by

$$\begin{array}{llcl}
(1): & U \otimes S \,+\, V \otimes Q^* \,+\, W \otimes \det S^* & \longrightarrow & G(1, \tilde{V}^*), \\
(2): & V \otimes S \,+\, \tilde{V}^* \otimes Q^* \,+\, W \otimes \det S^* & \longrightarrow & G(2, U), \\
(3): & \tilde{V}^* \otimes S \,+\, U \otimes Q^* \,+\, W \otimes \det S^* & \longrightarrow & G(1, V),
\end{array}$$

and the three $r \ll 0$ phases are described by

$$\begin{array}{llcl}
(1): & U \otimes S^* \,+\, \tilde{V} \otimes Q^* \,+\, W \otimes \det S & \longrightarrow & G(1, V^*), \\
(2): & V \otimes S^* \,+\, U^* \otimes Q^* \,+\, W \otimes \det S & \longrightarrow & G(2, \tilde{V}), \\
(3): & \tilde{V}^* \otimes S^* \,+\, V^* \otimes Q^* \,+\, W \otimes \det S & \longrightarrow & G(1, U^*).
\end{array}$$

(As a consistency check, the $U(1)^3$ global symmetry is nonanomalous in each of the six phases above.) In the previous section, we computed the chiral rings in both phases of GLSM (1), and note that the $r \ll 0$ phase of (1) is closely related by mathematical duality to the $r \gg 0$ phase of (3). The remaining geometry can be described as either the $r \gg 0$ phase of (2) or the $r \ll 0$ phase of (3). Let us therefore focus on GLSM (3), and compute the chiral rings in the two NLSM phases.

The reader should note that the phases of GLSM (3) are closely related to those of (1). Specifically, we can get the $r \ll 0$ phase of (3) from the $r \gg 0$ phase of (1), and the $r \gg 0$ phase of (3) from the $r \ll 0$ phase of (1), by making the substitutions

$$U \leftrightarrow \tilde{V}, \quad V \leftrightarrow V^*, \quad W \leftrightarrow W^*,$$

so rather than re-compute spectra from scratch, we can simply re-use the existing tables of states by making the replacements above.

The $U(1)^3$ charges are computed from a slightly different action than in the first GLSM, given by

	$\tilde{V}$	U	V	W
$U(1)_{(1)}$	0	0	+1	+1
$U(1)_{(2)}$	1	0	0	$-3/2$
$U(1)_{(3)}$	0	1	0	$+3/2$

(These are almost the same as in the first GLSM, except that there is a sign flip on the charges of the first $U(1)$.) With these (nonanomalous) charge assignments, we shall see that the states we compute in this GLSM which are shared between the two geometric phases, are also shared with the first GLSM. In any event, following the same procedure as in the last section, we compute $(+3, -3, 0)$ for both the $r \gg 0$ and $r \ll 0$ phases.

By making the substitutions described above, we can immediately write down the chiral states in the two phases of this GLSM. These states are encoded in tables 4.4, 4.5, and 4.6, which are precise analogues of the corresponding tables 4.1, 4.2, 4.3 for the previous GLSM related by triality. As a consistency check, it is straightforward to check that all states come in Serre dual pairs in which representations are dualized. Furthermore, all of the states in table 4.6 cancel out of the leading term in any elliptic genus refined by any of the displayed nonanomalous global symmetries, as before.

Finally, we compare the chiral states from the first GLSM, in tables 4.1 and 4.2, with those from the second GLSM, in tables 4.4 and 4.5. It is straightforward to check that they match – the states which are believed to flow to the IR in the first GLSM, are isomorphic to states in the second GLSM which are believed to flow to the IR. (States that mismatch between two phases of the GLSM, we do not consider, as we do not believe they flow to the IR.) This supports the triality proposal, in that it is a check that not only phases of a single GLSM, but phases of multiple GLSMs, all have the same IR limit.

State	$r \ll 0$ $\wedge^\bullet \mathcal{E}$	$H^\bullet(\mathbb{P}^2)$	$r \gg 0$ $\wedge^\bullet \mathcal{F}$	$H^\bullet(\mathbb{P}^1)$	Rep'	$U(1)^3$
1	0	0	0	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(+3,-3,0)$
$W^* \otimes U$	1	0	2	1	$(\mathbf{3},\mathbf{1},\mathbf{2},\mathbf{1})$	$(+2,-3/2,-1/2)$
$\wedge^2 V^* \otimes \wedge^2 U$	2	1	2	1	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\mathbf{1})$	$(+1,-3,+2)$
$\tilde{V} \otimes V^*$	2	1	1	0	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{3})$	$(+2,-2,0)$
$\tilde{V} \otimes W^*$	2	0	2	0	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{3})$	$(+2,-1/2,-3/2)$
$V^* \otimes W^* \otimes \wedge^2 U$	2	0	3	1	$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2},\mathbf{1})$	$(+1,-3/2,+1/2)$
$\tilde{V} \otimes \wedge^2 V^* \otimes U$	3	1	2	0	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{3})$	$(+1,-2,+1)$
$\tilde{V} \otimes \wedge^2 W^* \otimes U$	3	0	4	1	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{3})$	$(+1,+1,-2)$
$V^* \otimes \wedge^2 V^* \otimes \wedge^3 U$	3	1	3	1	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	$(0,-3,+3)$
$\mathrm{Sym}^2 V^* \otimes W^* \otimes \wedge^3 U$	3	0	4	1	$(\mathbf{1},\mathbf{3},\mathbf{2},\mathbf{1})$	$(0,-3/2,+3/2)$
$\wedge^2 \tilde{V} \otimes \mathrm{Sym}^2 V^*$	4	2	2	0	$(\mathbf{1},\mathbf{3},\mathbf{1},\overline{\mathbf{3}})$	$(+1,-1,0)$
$\wedge^2 \tilde{V} \otimes V^* \otimes W^*$	4	1	3	0	$(\mathbf{1},\mathbf{2},\mathbf{2},\overline{\mathbf{3}})$	$(+1,+1/2,-3/2)$
$\wedge^2 \tilde{V} \otimes \wedge^2 W^*$	4	0	4	0	$(\mathbf{1},\mathbf{1},\mathbf{1},\overline{\mathbf{3}})$	$(+1,+2,-3)$
$\tilde{V} \otimes \wedge^2 V^* \otimes W^* \otimes \wedge^2 U$	4	1	4	1	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{2},\mathbf{3})$	$(0,-1/2,+1/2)$
$\tilde{V} \otimes V^* \otimes \wedge^2 W^* \otimes \wedge^2 U$	4	0	5	1	$(\overline{\mathbf{3}},\mathbf{2},\mathbf{1},\mathbf{3})$	$(0,+1,-1)$
$\wedge^2 \tilde{V} \otimes \wedge^2 V^* \otimes V^* \otimes U$	5	2	3	0	$(\mathbf{3},\mathbf{2},\mathbf{1},\overline{\mathbf{3}})$	$(0,-1,+1)$
$\wedge^2 \tilde{V} \otimes W^* \otimes \wedge^2 V^* \otimes U$	5	1	4	0	$(\mathbf{3},\mathbf{1},\mathbf{2},\overline{\mathbf{3}})$	$(0,+1/2,-1/2)$
$\tilde{V} \otimes (\wedge^2 V^*)^2 \otimes \wedge^3 U$	5	2	4	1	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})$	$(-1,-2,+3)$
$\tilde{V} \otimes W^* \otimes \wedge^2 V^* \otimes V^* \otimes \wedge^3 U$	5	1	5	1	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{3})$	$(-1,-1/2,+3/2)$
$\tilde{V} \otimes \mathrm{Sym}^2 V^* \otimes \wedge^2 W^* \otimes \wedge^3 U$	5	0	6	1	$(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{3})$	$(-1,+1,0)$
$\wedge^3 \tilde{V} \otimes \mathrm{Sym}^2 V^* \otimes W^*$	6	2	4	0	$(\mathbf{1},\mathbf{3},\mathbf{2},\mathbf{1})$	$(0,+3/2,-3/2)$
$\wedge^3 \tilde{V} \otimes V^* \otimes \wedge^2 W^*$	6	1	5	0	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})$	$(0,+3,-3)$
$\wedge^2 \tilde{V} \otimes (\wedge^2 V^*)^2 \otimes \wedge^2 U$	6	2	4	0	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\overline{\mathbf{3}})$	$(-1,-1,+2)$
$\wedge^2 \tilde{V} \otimes \wedge^2 V^* \otimes \wedge^2 W^* \otimes \wedge^2 U$	6	1	6	1	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},\overline{\mathbf{3}})$	$(-1,+2,-1)$
$\wedge^3 \tilde{V} \otimes \wedge^2 V^* \otimes V^* \otimes W^* \otimes U$	7	2	5	0	$(\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{1})$	$(-1,+3/2,-1/2)$
$\wedge^2 \tilde{V} \otimes (\wedge^2 V)^2 \otimes W^* \otimes \wedge^3 U$	7	2	6	1	$(\mathbf{1},\mathbf{1},\mathbf{2},\overline{\mathbf{3}})$	$(-2,+1/2,+3/2)$
$\wedge^2 \tilde{V} \otimes \wedge^2 V^* \otimes V^* \otimes \wedge^2 W^* \otimes \wedge^3 U$	7	1	7	1	$(\mathbf{1},\mathbf{2},\mathbf{1},\overline{\mathbf{3}})$	$(-2,+2,0)$
$\wedge^3 \tilde{V} \otimes \wedge^2 V^* \otimes \wedge^2 W^* \otimes U$	7	1	6	0	$(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})$	$(-1,+3,-2)$
$\wedge^3 \tilde{V} \otimes (\wedge^2 V^*)^2 \otimes W^* \otimes \wedge^2 U$	8	2	6	0	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{2},\mathbf{1})$	$(-2,+3/2,+1/2)$
$\wedge^3 \tilde{V} \otimes (\wedge^2 V^*)^2 \otimes \wedge^2 W^* \otimes \wedge^3 U$	9	2	8	1	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(-3,+3,0)$

Table 4.4: List of states shared between the two phases.

$r \ll 0$ State, $\wedge^\bullet \mathcal{E}, H^\bullet(\mathbb{P}^2)$	$r \gg 0$ State, $\wedge^\bullet \mathcal{F}, H^\bullet(\mathbb{P}^1)$	Rep'	$U(1)^3$
$\wedge^3 \tilde{V} \otimes \wedge^3 U^*$, $3, 2$	$\wedge^2 W^* \otimes \wedge^2 V$, $2, 1$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(+3, 0, -3)$
$\wedge^3 \tilde{V} \otimes V^* \otimes \wedge^2 U^*$, $4, 2$	$U \otimes \wedge^2 W^* \otimes V$, $3, 1$	$(\mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	$(+2, 0, -2)$
$\mathrm{Sym}^2 V^* \otimes \wedge^2 W^* \otimes K_{(2,1,1)} U$, $4, 0$	$\wedge^3 \tilde{V} \otimes U \otimes K_{(3,1)} V^*$, $4, 0$	$(\mathbf{3}, \mathbf{3}, \mathbf{1}, \mathbf{1})$	$(-1, 0, +1)$
$\wedge^3 \tilde{V} \otimes \mathrm{Sym}^2 V^* \otimes K_{(0,0,-1)} U$, $5, 2$	$\wedge^2 U \otimes \wedge^2 W^* \otimes K_{(1,-1)} V$, $4, 1$	$(\overline{\mathbf{3}}, \mathbf{3}, \mathbf{1}, \mathbf{1})$	$(+1, 0, -1)$
$K_{(2,1)} V^* \otimes \wedge^2 W^* \otimes K_{(2,2,1)} U$, $5, 0$	$\wedge^3 \tilde{V} \otimes \wedge^2 U \otimes K_{(3,2)} V^*$, $5, 0$	$(\overline{\mathbf{3}}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	$(-2, 0, +2)$
$(\wedge^2 V^*)^2 \otimes \wedge^2 W^* \otimes (\wedge^3 U)^2$, $6, 0$	$\wedge^3 \tilde{V} \otimes \wedge^3 U \otimes (\wedge^2 V^*)^3$, $6, 0$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(-3, 0, +3)$

Table 4.5: Additional shared states defined by matching representations of anomaly-free global symmetries.

	$r \ll 0$		$r \gg 0$			
State	$\wedge^\bullet \mathcal{E}$	$H^\bullet(\mathbb{P}^2)$	$\wedge^\bullet \mathcal{F}$	$H^\bullet(\mathbb{P}^1)$	Rep'	$U(1)^3$
$\wedge^2 W^* \otimes \mathrm{Sym}^2 U$	2	0	–	–	$(\mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(+1, 0, -1)$
$V^* \otimes \wedge^2 W^* \otimes K_{(2,1,0)} U$	3	0	–	–	$(\mathbf{8}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	$(0, 0, 0)$
$\wedge^2 V^* \otimes \wedge^2 W^* \otimes K_{(2,2,0)} U$	4	0	–	–	$(\mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(-1, 0, +1)$
$\wedge^3 \tilde{V} \otimes \wedge^2 V^* \otimes \wedge^3 U^* \otimes \mathrm{Sym}^2 U$	5	2	–	–	$(\mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(+1, 0, -1)$
$\wedge^3 \tilde{V} \otimes K_{(2,1)} V^* \otimes K_{(1,0,-1)} U$	6	2	–	–	$(\mathbf{8}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	$(0, 0, 0)$
$\wedge^3 \tilde{V} \otimes (\wedge^2 V^*)^2 \otimes \wedge^3 U \otimes \mathrm{Sym}^2 U^*$	7	2	–	–	$(\mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(-1, 0, +1)$
$\wedge^3 \tilde{V} \otimes \mathrm{Sym}^3 V^*$	–	–	3	0	$(\mathbf{1}, \mathbf{4}, \mathbf{1}, \mathbf{1})$	$(0, 0, 0)$
$\wedge^3 U \otimes \wedge^2 W^* \otimes \wedge^2 V \otimes \mathrm{Sym}^3 V^*$	–	–	5	1	$(\mathbf{1}, \mathbf{4}, \mathbf{1}, \mathbf{1})$	$(0, 0, 0)$

Table 4.6: List of all states which are not shared between the two phases.

In passing, we have seen how non-protected operators can pair up and become massive along RG flow, but in principle the opposite can also happen – pairs of massive operators can become massless and enter the RG flow. We do not seem to observe this in any of the examples discussed in this chapter, and we leave open the question of whether more general triality examples exhibit that phenomenon, or whether for some reason it does not happen in UV presentations of triality.

4.5 Second example

4.5.1 First GLSM

Next, we shall consider a (0,2) GLSM with $k = 1$, $A = 4$, and $n = 2$, so $2k + A - n = 4$. Its two phases are defined by

$$\mathcal{E} \;\equiv\; U \otimes S \,+\, V \otimes Q^* \,+\, W \otimes \det S^* \;\longrightarrow\; \mathbb{P}^1 \;=\; \mathbb{P}\tilde{V}^*$$

for $r \gg 0$, and

$$\mathcal{F} \;\equiv\; U \otimes S^* \,+\, \tilde{V} \otimes Q^* \,+\, W \otimes \det S \;\longrightarrow\; \mathbb{P}^3 \;=\; \mathbb{P}V^*$$

for $r \ll 0$, where in both phases,

$$U \;=\; \mathbb{C}^4, \;\; V \;=\; \mathbb{C}^4, \;\; W \;=\; \mathbb{C}^2, \;\; \tilde{V} \;=\; \mathbb{C}^2.$$

Each phase has a set of global nonanomalous $U(1)^3$ symmetries, which are given by

	$\tilde{V}$	U	V	W
$U(1)_{(1)}$	0	0	-1	-2
$U(1)_{(2)}$	1	0	0	-1
$U(1)_{(3)}$	0	1	0	$+2$

In a slight variation from [24, 25], we have assigned the same charge to all elements of W, for simplicity in comparing states.

As in the previous example, the $\mathbb{P}^1$ of the $r \gg 0$ phase is identified with $\mathbb{P}\tilde{V}^*$ rather than $\mathbb{P}\tilde{V}$.

These nonlinear sigma models are neither A/2 nor B/2-twistable; the Fock vacuum line bundle is nontrivial. In the $r \gg 0$ phase, we take the Fock vacuum line bundle to be

$$\mathcal{L} \;=\; (Q^*)^{-2}.$$

In the $r \ll 0$ phase, we take the Fock vacuum line bundle to be

$$\mathcal{L} \;=\; K_{(-3)}S^* \otimes K_{(-1,-1,-1)}Q^*.$$

This vacuum is somewhat more interesting, as it has no cohomology, hence there are no chiral states in $H^\bullet(\wedge^0 \mathcal{F} \otimes \mathcal{L})$. The theory still has a Fock vacuum, but it does not define a nontrivial element of BRST cohomology.

In our previous examples, many of the states matched 'on the nose,' in representations of not only anomaly-free symmetries but also anomalous symmetries. In this example, there are no states that match in representations of anomalous symmetries, only in anomaly-free symmetries.

Matching states are listed in tables 4.7 and 4.8. (Because of the sheer number of states, they could not all be listed in a single table, so they were broken into two sets related by Serre duality.) Displayed are states, the wedge power of the gauge bundle involved, the cohomology degree in which they appear, the representation of

$$SL(U) \times SL(V) \times SL(W) \times SL(\tilde{V}) \;=\; SU(4) \times SU(4) \times SU(2) \times SU(2)$$

and charges under the nonanomalous global

$$U(1)_{(1)} \times U(1)_{(2)} \times U(1)_{(3)}$$

symmetries, in the same format as for the previous example. The $U(1)^3$ charges listed include the fractional contributions from the Fock vacua. For $r \gg 0$, we compute that the Fock vacuum has charge $(+4, +2, -4)$, whereas for $r \ll 0$, we compute that the Fock vacuum has charge $(0, -2, -4)$.

Furthermore, as before, each geometric phase has a few states not possessed by the other phase. These are listed in table 4.9.

As a consistency check, note that all of the states come in Serre dual pairs, just as in the previous example. In addition, the mismatched states come in pairs that cancel out of refined elliptic genus computations, as before, which means they can plausibly become massive along the RG flow.

In the IR, these theories are believed to flow to an SCFT in which the nonanomalous global symmetry

$$SU(U) \times SU(V) \times SU(W) \times SU(\tilde{V})$$

is enhanced to an affine symmetry

$$SU(U)_1 \times SU(V)_1 \times SU(W)_1 \times SU(\tilde{V})_3 \;=\; SU(4)_1 \times SU(4)_1 \times SU(2)_1 \times SU(2)_3,$$

and the states in the IR should all be associated with integrable representations of the affine algebra above. Using the fact that the integrable representations of $SU(2)_3$ are given by

$$\mathbf{1},\;\; \mathbf{2},\;\; \mathbf{3},\;\; \mathbf{4},$$

and the integrable representations of $SU(4)_1$ are given by

$$\mathbf{1},\;\; \mathbf{4},\;\; \wedge^2\mathbf{4} \;=\; \mathbf{6},\;\; \wedge^3\mathbf{4} \;=\; \overline{\mathbf{4}},$$

$r \gg 0$ State, $\wedge^\bullet \mathcal{E}, H^\bullet(\mathbb{P}^1)$	$r \ll 0$ State, $\wedge^\bullet \mathcal{F}, H^\bullet(\mathbb{P}^3)$	Rep'	$U(1)^3$
$\mathrm{Sym}^2\tilde{V}^*$, $0,0$	$\mathrm{Sym}^2\tilde{V} \otimes (\wedge^4 V)^{-1}$, $2,2$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})$	$(+4,0,-4)$
$U \otimes (\wedge^2\tilde{V})^{-1} \otimes \tilde{V}^*$, $1,0$	$U \otimes \tilde{V} \otimes (\wedge^4 V)^{-1}$, $2,1$	$(\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{2})$	$(+4,-1,-3)$
$V \otimes \tilde{V}^*$, $1,0$	$(\wedge^2\tilde{V}) \otimes \tilde{V} \otimes (\wedge^4 V)^{-1} \otimes V$, $3,2$	$(\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{2})$	$(+3,+1,-4)$
$W \otimes K_{(1,-2)}\tilde{V}$, $1,0$	$W \otimes \mathrm{Sym}^3\tilde{V} \otimes (\wedge^4 V)^{-1}$, $4,3$	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{4})$	$(+2,0,-2)$
$\wedge^2 U \otimes (\wedge^2\tilde{V})^{-2}$, $2,0$	$\wedge^2 U \otimes (\wedge^4 V)^{-1}$, $2,0$	$(\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1})$	$(+4,-2,-2)$
$U \otimes V \otimes (\wedge^2\tilde{V})^{-1}$, $2,0$	$U \otimes \wedge^2\tilde{V} \otimes (\wedge^4 V)^{-1} \otimes V$, $3,1$	$(\mathbf{4},\mathbf{4},\mathbf{1},\mathbf{1})$	$(+3,0,-3)$
$U \otimes W \otimes \mathrm{Sym}^2\tilde{V}^*$, $2,0$	$U \otimes W \otimes \mathrm{Sym}^2\tilde{V} \otimes (\wedge^4 V)^{-1}$, $4,2$	$(\mathbf{4},\mathbf{1},\mathbf{2},\mathbf{3})$	$(+2,-1,-1)$
$\wedge^2 V$, $2,0$	$(\wedge^2\tilde{V})^2 \otimes (\wedge^4 V)^{-1} \otimes \wedge^2 V$, $4,2$	$(\mathbf{1},\mathbf{6},\mathbf{1},\mathbf{1})$	$(+2,+2,-4)$
$V \otimes W \otimes K_{(1,-1)}\tilde{V}$, $2,0$	$W \otimes K_{(3,1)}\tilde{V} \otimes K_{(0,-1,-1,-1)}V$, $5,3$	$(\mathbf{1},\mathbf{4},\mathbf{2},\mathbf{3})$	$(+1,+1,-2)$
$\wedge^2 U \otimes W \otimes K_{(-1,-2)}\tilde{V}$, $3,0$	$\wedge^2 U \otimes \tilde{V} \otimes W \otimes (\wedge^4 V)^{-1}$, $4,1$	$(\mathbf{6},\mathbf{1},\mathbf{2},\mathbf{2})$	$(+2,-2,0)$
$U \otimes V \otimes W \otimes \tilde{V}^*$, $3,0$	$U \otimes W \otimes K_{(2,1)}\tilde{V} \otimes K_{(0,-1,-1,-1)}V$, $5,2$	$(\mathbf{4},\mathbf{4},\mathbf{2},\mathbf{2})$	$(+1,0,-1)$
$U \otimes \wedge^2 W \otimes K_{(+1,-2)}\tilde{V}$, $3,0$	$U \otimes \wedge^2 W \otimes \mathrm{Sym}^3\tilde{V} \otimes (\wedge^4 V)^{-1}$, $6,3$	$(\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{4})$	$(0,-1,+1)$
$\wedge^2 V \otimes W \otimes \tilde{V}$, $3,0$	$W \otimes K_{(3,2)}\tilde{V} \otimes K_{(0,0,-1,-1)}V$, $6,3$	$(\mathbf{1},\mathbf{6},\mathbf{2},\mathbf{2})$	$(0,+2,-2)$
$V \otimes \wedge^2 W \otimes K_{(2,-1)}\tilde{V}$, $3,0$	$\wedge^4 U \otimes \mathrm{Sym}^3\tilde{V} \otimes V$, $7,0$	$(\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{4})$	$(-1,+1,0)$
$\wedge^4 U \otimes K_{(-3,-3)}\tilde{V}$, $4,1$	$\wedge^2 W \otimes K_{(-2,-2,-2,-2)}V$, $2,3$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(+4,-4,0)$
$\wedge^3 U \otimes V \otimes K_{(-2,-2)}\tilde{V}$, $4,1$	$\wedge^3 U \otimes K_{(0,-1,-1,-1)}V$, $3,0$	$(\overline{\mathbf{4}},\mathbf{4},\mathbf{1},\mathbf{1})$	$(+3,-2,-1)$
$\wedge^3 U \otimes W \otimes K_{(-2,-2)}\tilde{V}$, $4,0$	$\wedge^3 U \otimes W \otimes (\wedge^4 V)^{-1}$, $4,0$	$(\overline{\mathbf{4}},\mathbf{1},\mathbf{2},\mathbf{1})$	$(+2,-3,+1)$
$\wedge^2 U \otimes \wedge^2 V \otimes \wedge^2\tilde{V}^*$, $4,1$	$\wedge^2 U \otimes \wedge^2\tilde{V} \otimes \wedge^2 V^*$, $4,1$	$(\mathbf{6},\mathbf{6},\mathbf{1},\mathbf{1})$	$(+2,0,-2)$
$\wedge^2 U \otimes V \otimes W \otimes \wedge^2\tilde{V}^*$, $4,0$	$\wedge^2 U \otimes W \otimes \wedge^2\tilde{V} \otimes \wedge^3 V^*$, $5,1$	$(\mathbf{6},\mathbf{4},\mathbf{2},\mathbf{1})$	$(+1,-1,0)$
$\wedge^2 U \otimes \wedge^2 W \otimes \mathrm{Sym}^2\tilde{V}^*$, $4,0$	$\wedge^2 U \otimes \wedge^2 W \otimes K_{(2,0)}\tilde{V} \otimes (\wedge^4 V)^{-1}$, $6,2$	$(\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{3})$	$(0,-2,+2)$
$U \otimes \wedge^3 V$, $4,1$	$U \otimes K_{(2,2)}\tilde{V} \otimes V^*$, $5,2$	$(\mathbf{4},\overline{\mathbf{4}},\mathbf{1},\mathbf{1})$	$(+1,+2,-3)$
$U \otimes \wedge^2 V \otimes W$, $4,0$	$U \otimes W \otimes K_{(2,2)}\tilde{V} \otimes \wedge^2 V^*$, $6,2$	$(\mathbf{4},\mathbf{6},\mathbf{2},\mathbf{1})$	$(0,+1,-1)$
$U \otimes V \otimes \wedge^2 W \otimes K_{(1,-1)}\tilde{V}$, $4,0$	$U \otimes \wedge^2 W \otimes K_{(3,1)}\tilde{V} \otimes \wedge^3 V^*$, $7,3$	$(\mathbf{4},\mathbf{4},\mathbf{1},\mathbf{3})$	$(-1,0,+1)$
$\wedge^4 V \otimes \wedge^2\tilde{V}$, $4,1$	$(\wedge^2\tilde{V})^3$, $6,3$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,+4,-4)$
$\wedge^3 V \otimes W \otimes \wedge^2\tilde{V}$, $4,0$	$W \otimes K_{(3,3)}\tilde{V} \otimes V^*$, $7,3$	$(\mathbf{1},\overline{\mathbf{4}},\mathbf{2},\mathbf{1})$	$(-1,+3,-2)$
$\wedge^2 V \otimes \wedge^2 W \otimes \mathrm{Sym}^2\tilde{V}$, $4,0$	$\wedge^4 U \otimes K_{(3,1)}\tilde{V} \otimes \wedge^2 V$, $8,0$	$(\mathbf{1},\mathbf{6},\mathbf{1},\mathbf{3})$	$(-2,+2,0)$
$\wedge^4 U \otimes V \otimes K_{(-2,-3)}\tilde{V}$, $5,1$	$\wedge^2 W \otimes \tilde{V} \otimes K_{(-1,-2,-2,-2)}V$, $3,3$	$(\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{2})$	$(+3,-3,0)$
$\wedge^3 U \otimes \wedge^2 V \otimes K_{(-1,-2)}\tilde{V}$, $5,1$	$\wedge^3 U \otimes \tilde{V} \otimes \wedge^2 V^*$, $4,0$	$(\overline{\mathbf{4}},\mathbf{6},\mathbf{1},\mathbf{2})$	$(+2,-1,-1)$
$\wedge^3 U \otimes \wedge^2 W \otimes K_{(-1,-2)}\tilde{V}$, $5,0$	$\wedge^3 U \otimes \wedge^2 W \otimes \tilde{V} \otimes \wedge^4 V^*$, $6,1$	$(\overline{\mathbf{4}},\mathbf{1},\mathbf{1},\mathbf{2})$	$(0,-3,+3)$
$\wedge^2 U \otimes \wedge^3 V \otimes \tilde{V}^*$, $5,1$	$\wedge^2 U \otimes K_{(2,1)}\tilde{V} \otimes V^*$, $5,1$	$(\mathbf{6},\overline{\mathbf{4}},\mathbf{1},\mathbf{2})$	$(+1,+1,-2)$
$\wedge^2 U \otimes V \otimes \wedge^2 W \otimes \tilde{V}^*$, $5,0$	$\wedge^2 U \otimes \wedge^2 W \otimes K_{(2,1)}\tilde{V} \otimes \wedge^3 V^*$, $7,2$	$(\mathbf{6},\mathbf{4},\mathbf{1},\mathbf{2})$	$(-1,-1,+2)$
$U \otimes \wedge^4 V \otimes \tilde{V}$, $5,1$	$U \otimes K_{(3,2)}\tilde{V}$, $6,2$	$(\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{2})$	$(0,+3,-3)$
$U \otimes \wedge^2 V \otimes \wedge^2 W \otimes \tilde{V}$, $5,0$	$U \otimes \wedge^2 W \otimes K_{(3,2)}\tilde{V} \otimes \wedge^2 V^*$, $8,3$	$(\mathbf{4},\mathbf{6},\mathbf{1},\mathbf{2})$	$(-2,+1,+1)$
$\wedge^3 V \otimes \wedge^2 W \otimes K_{(2,1)}\tilde{V}$, $5,0$	$\wedge^4 U \otimes K_{(3,2)}\tilde{V} \otimes \wedge^3 V$, $9,0$	$(\mathbf{1},\overline{\mathbf{4}},\mathbf{1},\mathbf{2})$	$(-3,+3,0)$

Table 4.7: First half of shared states defined by matching representations of anomaly-free global symmetries.

$r \gg 0$ State, $\wedge^\bullet \mathcal{E}, H^\bullet(\mathbb{P}^1)$	$r \ll 0$ State, $\wedge^\bullet \mathcal{F}, H^\bullet(\mathbb{P}^3)$	Rep'	$U(1)^3$
$\wedge^4 U \otimes \wedge^2 V \otimes K_{(-1,-3)}\tilde{V}$, 6, 1	$\wedge^2 W \otimes \mathrm{S}^2\tilde{V} \otimes K_{(2,2,1,1)}V^*$, 4, 3	$(\mathbf{1}, \mathbf{6}, \mathbf{1}, \mathbf{3})$	$(+2, -2, 0)$
$\wedge^4 U \otimes V \otimes W \otimes K_{(-2,-2)}\tilde{V}$, 6, 1	$\wedge^4 U \otimes W \otimes \wedge^3 V^*$, 5, 0	$(\mathbf{1}, \mathbf{4}, \mathbf{2}, \mathbf{1})$	$(+1, -3, +2)$
$\wedge^4 U \otimes \wedge^2 W \otimes K_{(-2,-2)}\tilde{V}$, 6, 0	$\wedge^4 U \otimes \wedge^2 W \otimes \wedge^4 V^*$, 6, 0	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(0, -4, +4)$
$\wedge^3 U \otimes \wedge^3 V \otimes \mathrm{Sym}^2\tilde{V}^*$, 6, 1	$\wedge^3 U \otimes \mathrm{Sym}^2\tilde{V} \otimes V^*$, 5, 0	$(\overline{\mathbf{4}}, \overline{\mathbf{4}}, \mathbf{1}, \mathbf{3})$	$(+1, 0, -1)$
$\wedge^3 U \otimes \wedge^2 V \otimes W \otimes \wedge^2\tilde{V}^*$, 6, 1	$\wedge^3 U \otimes W \otimes \wedge^2\tilde{V} \otimes \wedge^2 V^*$, 6, 1	$(\overline{\mathbf{4}}, \mathbf{6}, \mathbf{2}, \mathbf{1})$	$(0, -1, +1)$
$\wedge^3 U \otimes V \otimes \wedge^2 W \otimes \wedge^2\tilde{V}^*$, 6, 0	$\wedge^3 U \otimes \wedge^2 W \otimes \wedge^2\tilde{V} \otimes \wedge^3 V^*$, 7, 1	$(\overline{\mathbf{4}}, \mathbf{4}, \mathbf{1}, \mathbf{1})$	$(-1, -2, +3)$
$\wedge^2 U \otimes \wedge^4 V \otimes K_{(1,-1)}\tilde{V}$, 6, 1	$\wedge^2 U \otimes K_{(3,1)}\tilde{V}$, 6, 1	$(\mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{3})$	$(0, +2, -2)$
$\wedge^2 U \otimes \wedge^3 V \otimes W$, 6, 1	$\wedge^2 U \otimes W \otimes K_{(2,2)}\tilde{V} \otimes V^*$, 7, 2	$(\mathbf{6}, \overline{\mathbf{4}}, \mathbf{2}, \mathbf{1})$	$(-1, +1, 0)$
$\wedge^2 U \otimes \wedge^2 V \otimes \wedge^2 W$, 6, 0	$\wedge^2 U \otimes \wedge^2 W \otimes (\wedge^2\tilde{V})^2 \otimes \wedge^2 V^*$, 8, 2	$(\mathbf{6}, \mathbf{6}, \mathbf{1}, \mathbf{1})$	$(-2, 0, +2)$
$U \otimes \wedge^4 V \otimes W \otimes \wedge^2\tilde{V}$, 6, 1	$U \otimes W \otimes K_{(3,3)}\tilde{V}$, 8, 3	$(\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{1})$	$(-2, +3, -1)$
$U \otimes \wedge^3 V \otimes \wedge^2 W \otimes \wedge^2\tilde{V}$, 6, 0	$U \otimes K_{(3,3)}\tilde{V} \otimes \wedge^2 W \otimes V^*$, 9, 3	$(\mathbf{4}, \overline{\mathbf{4}}, \mathbf{1}, \mathbf{1})$	$(-3, +2, +1)$
$\wedge^4 V \otimes \wedge^2 W \otimes K_{(2,2)}\tilde{V}$, 6, 0	$\wedge^4 U \otimes K_{(3,3)}\tilde{V} \otimes \wedge^4 V$, 10, 0	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(-4, +4, 0)$
$\wedge^4 U \otimes \wedge^3 V \otimes \mathrm{Sym}^3\tilde{V}^*$, 7, 1	$\wedge^2 W \otimes \mathrm{S}^3\tilde{V} \otimes K_{(2,1,1,1)}V^*$, 5, 3	$(\mathbf{1}, \overline{\mathbf{4}}, \mathbf{1}, \mathbf{4})$	$(+1, -1, 0)$
$\wedge^4 U \otimes \wedge^2 V \otimes W \otimes K_{(2,1)}\tilde{V}^*$, 7, 1	$\wedge^4 U \otimes \tilde{V} \otimes W \otimes \wedge^2 V^*$, 6, 0	$(\mathbf{1}, \mathbf{6}, \mathbf{2}, \mathbf{2})$	$(0, -2, +2)$
$\wedge^3 U \otimes \wedge^4 V \otimes K_{(1,-2)}\tilde{V}$, 7, 1	$\wedge^3 U \otimes \mathrm{Sym}^3\tilde{V}$, 6, 0	$(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{4})$	$(0, +1, -1)$
$\wedge^3 U \otimes \wedge^3 V \otimes W \otimes \tilde{V}^*$, 7, 1	$\wedge^3 U \otimes W \otimes K_{(2,1)}\tilde{V} \otimes V^*$, 7, 1	$(\overline{\mathbf{4}}, \overline{\mathbf{4}}, \mathbf{2}, \mathbf{2})$	$(-1, 0, +1)$
$\wedge^2 U \otimes \wedge^4 V \otimes W \otimes \tilde{V}$, 7, 1	$\wedge^2 U \otimes W \otimes K_{(3,2)}\tilde{V}$, 8, 2	$(\mathbf{6}, \mathbf{1}, \mathbf{2}, \mathbf{2})$	$(-2, +2, 0)$
$\wedge^4 U \otimes \wedge^3 V \otimes W \otimes \mathrm{Sym}^2\tilde{V}^*$, 8, 1	$\wedge^4 U \otimes W \otimes \mathrm{Sym}^2\tilde{V} \otimes V^*$, 7, 0	$(\mathbf{1}, \overline{\mathbf{4}}, \mathbf{2}, \mathbf{3})$	$(-1, -1, +2)$
$\wedge^4 U \otimes \wedge^2 V \otimes \wedge^2 W \otimes \wedge^2\tilde{V}^*$, 8, 1	$\wedge^4 U \otimes \wedge^2 W \otimes \wedge^2\tilde{V} \otimes \wedge^2 V^*$, 8, 1	$(\mathbf{1}, \mathbf{6}, \mathbf{1}, \mathbf{1})$	$(-2, -2, +4)$
$\wedge^3 U \otimes \wedge^4 V \otimes W \otimes K_{(1,-1)}\tilde{V}$, 8, 1	$\wedge^3 U \otimes W \otimes K_{(3,1)}\tilde{V}$, 8, 1	$(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{3})$	$(-2, +1, +1)$
$\wedge^3 U \otimes \wedge^3 V \otimes \wedge^2 W$, 8, 1	$\wedge^3 U \otimes \wedge^2 W \otimes K_{(2,2)}\tilde{V} \otimes V^*$, 9, 2	$(\overline{\mathbf{4}}, \overline{\mathbf{4}}, \mathbf{1}, \mathbf{1})$	$(-3, 0, +3)$
$\wedge^2 U \otimes \wedge^4 V \otimes \wedge^2 W \otimes \wedge^2\tilde{V}$, 8, 1	$\wedge^2 U \otimes K_{(3,3)}\tilde{V} \otimes \wedge^2 W$, 10, 3	$(\mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(-4, +2, +2)$
$\wedge^4 U \otimes \wedge^4 V \otimes W \otimes K_{(1,-2)}\tilde{V}$, 9, 1	$\wedge^4 U \otimes \mathrm{Sym}^3\tilde{V} \otimes W$, 8, 0	$(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{4})$	$(-2, 0, +2)$
$\wedge^4 U \otimes \wedge^3 V \otimes \wedge^2 W \otimes \tilde{V}^*$, 9, 1	$\wedge^4 U \otimes \wedge^2 W \otimes K_{(2,1)}\tilde{V} \otimes V^*$, 9, 1	$(\mathbf{1}, \overline{\mathbf{4}}, \mathbf{1}, \mathbf{2})$	$(-3, -1, +4)$
$\wedge^3 U \otimes \wedge^4 V \otimes \wedge^2 W \otimes \tilde{V}$, 9, 1	$\wedge^3 U \otimes \wedge^2 W \otimes K_{(3,2)}\tilde{V}$, 10, 2	$(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{1}, \mathbf{2})$	$(-4, +1, +3)$
$\wedge^4 U \otimes \wedge^4 V \otimes \wedge^2 W \otimes K_{(1,-1)}\tilde{V}$, 10, 1	$\wedge^4 U \otimes \wedge^2 W \otimes K_{(3,1)}\tilde{V}$, 10, 1	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3})$	$(-4, 0, +4)$

Table 4.8: Second half of shared states defined by matching representations of anomaly-free global symmetries. We have occasionally used the symbol S as an abbreviation for Sym.

State	$r \gg 0$ $\wedge^\bullet \mathcal{E}$	$H^\bullet(\mathbb{P}^2)$	$r \ll 0$ $\wedge^\bullet \mathcal{F}$	$H^\bullet(\mathbb{P}^1)$	Rep'	$U(1)^3$
$\wedge^2 W \otimes K_{(2,-2)}\tilde{V}$	2	0	–	–	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{5})$	$(0,0,0)$
$\wedge^4 U \otimes \wedge^4 V \otimes K_{(1,-3)}\tilde{V}$	8	1	–	–	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{5})$	$(0,0,0)$
$\wedge^4 U \otimes K_{(+1,-1,-1,-1)}V$	–	–	4	0	$(\mathbf{1},\mathbf{10},\mathbf{1},\mathbf{1})$	$(+2,-2,0)$
$\wedge^2 W \otimes \wedge^2\tilde{V} \otimes K_{(0,-2,-2,-2)}V$	–	–	4	3	$(\mathbf{1},\overline{\mathbf{10}},\mathbf{1},\mathbf{1})$	$(+2,-2,0)$
$\wedge^4 U \otimes \tilde{V} \otimes K_{(+1,0,-1,-1)}V$	–	–	5	0	$(\mathbf{1},\mathbf{20},\mathbf{1},\mathbf{2})$	$(+1,-1,0)$
$\wedge^2 W \otimes K_{(2,1)}\tilde{V} \otimes K_{(0,-1,-2,-2)}V$	–	–	5	3	$(\mathbf{1},\mathbf{20},\mathbf{1},\mathbf{2})$	$(+1,-1,0)$
$\wedge^4 U \otimes \wedge^2\tilde{V} \otimes K_{(+1,+1,-1,-1)}V$	–	–	6	0	$(\mathbf{1},\mathbf{20},\mathbf{1},\mathbf{1})$	$(0,0,0)$
$\wedge^4 U \otimes \mathrm{Sym}^2\tilde{V} \otimes K_{(+1,0,0,-1)}V$	–	–	6	0	$(\mathbf{1},\mathbf{15},\mathbf{1},\mathbf{3})$	$(0,0,0)$
$\wedge^2 W \otimes K_{(3,1)}\tilde{V} \otimes K_{(0,-1,-1,-2)}V$	–	–	6	3	$(\mathbf{1},\mathbf{15},\mathbf{1},\mathbf{3})$	$(0,0,0)$
$\wedge^2 W \otimes K_{(2,2)}\tilde{V} \otimes K_{(0,0,-2,-2)}V$	–	–	6	3	$(\mathbf{1},\mathbf{20},\mathbf{1},\mathbf{1})$	$(0,0,0)$
$\wedge^4 U \otimes K_{(2,1)}\tilde{V} \otimes K_{(1,1,0,-1)}V$	–	–	7	0	$(\mathbf{1},\mathbf{20},\mathbf{1},\mathbf{2})$	$(-1,+1,0)$
$\wedge^2 W \otimes K_{(3,2)}\tilde{V} \otimes K_{(0,0,-1,-2)}V$	–	–	7	3	$\mathbf{1},\mathbf{20},\mathbf{1},\mathbf{2})$	$(-1,+1,0)$
$\wedge^4 U \otimes K_{(2,2)}\tilde{V} \otimes K_{(1,1,1,-1)}V$	–	–	8	0	$(\mathbf{1},\mathbf{10},\mathbf{1},\mathbf{1})$	$(-2,+2,0)$
$K_{(3,3)}\tilde{V} \otimes \wedge^2 W \otimes \mathrm{Sym}^2 V^*$	–	–	8	3	$(\mathbf{1},\overline{\mathbf{10}},\mathbf{1},\mathbf{1})$	$(-2,+2,0)$

Table 4.9: List of all states which are not shared between the two phases.

it is straightforward to check explicitly that all of the matching states listed in tables 4.7 and 4.8 are indeed associated with integrable representations, whereas by contrast all of the non-matching states in table 4.9 are associated with non-integrable representations. As before, it is our expectation that the non-matching states listed in table 4.9 do not survive to the IR.

4.5.2 Other GLSMs

Now, let us compute the chiral states in another GLSM related to the previous one by triality. We shall see, as before, that states in integrable representations match between phases, and also that those same states match between GLSMs.

Just as in section 4.4.2, we can obtain the other GLSMs related by triality by cyclically permuting

$$U \longrightarrow V \longrightarrow \tilde{V}^* \longrightarrow U \longrightarrow \cdots$$

(and changing the gauge group). The three large-radius phases correspond to the bundles and spaces given by

$$\begin{array}{rlcl}
(1): & U \otimes S \,+\, V \otimes Q^* \,+\, W \otimes \det S^* & \longrightarrow & G(1,\tilde{V}^*), \\
(2): & V \otimes S \,+\, \tilde{V}^* \otimes Q^* \,+\, W \otimes \det S^* & \longrightarrow & G(1,U), \\
(3): & \tilde{V}^* \otimes S \,+\, U \otimes Q^* \,+\, W \otimes \det S^* & \longrightarrow & G(3,V),
\end{array}$$

and the three $r \ll 0$ phases are described by

$$\begin{array}{lllll} (1): & U \otimes S^* \,+\, \tilde{V} \otimes Q^* \,+\, W \otimes \det S & \longrightarrow & G(1, V^*), \\ (2): & V \otimes S^* \,+\, U^* \otimes Q^* \,+\, W \otimes \det S & \longrightarrow & G(1, \tilde{V}), \\ (3): & \tilde{V}^* \otimes S^* \,+\, V^* \otimes Q^* \,+\, W \otimes \det S & \longrightarrow & G(3, U^*). \end{array}$$

(As a consistency check, the $U(1)^3$ global symmetry is nonanomalous in each of the six phases above.)

Note that the $r \gg 0$ phase of the first GLSM and the $r \ll 0$ phase of the second GLSM are closely related: if we exchange

$$U \,\leftrightarrow\, V^*, \;\; W \,\leftrightarrow\, W^*, \;\; \tilde{V} \,\leftrightarrow\, \tilde{V}^*,$$

then we can map one theory into the other. Similarly, the $r \ll 0$ phase of the first GLSM and the $r \gg 0$ phase of the second. Therefore, rather than compute a new set of chiral states from scratch, we can re-use our existing computations to derive the states for the second GLSM.

The $U(1)^3$ action that matches that of the previous GLSM is given by

	$\tilde{V}$	U	V	W
$U(1)_{(1)}$	0	0	-1	-2
$U(1)_{(2)}$	-1	0	0	$+1$
$U(1)_{(3)}$	0	1	0	$+2$

The vacuum charges in the $r \ll 0$ and $r \gg 0$ phases are given by, respectively, $(-4, +2, +4)$ and $(-4, -2, 0)$.

Our results for the second GLSM are listed in tables 4.10, 4.11, 4.12, and 4.13. Tables 4.10, 4.11, and 4.12 list states that match between the $r \gg 0$ and $r \ll 0$ phases; table 4.13 lists the remainder. It is straightforward to check that all states come in Serre dual pairs, that all of the matching states lie in integrable representations, and that the mismatched states do not lie in integrable representations. In addition, as before, the mismatched states naturally come in pairs such that they can make no net contribution to the (leading term of the) elliptic genus, refined by any of the listed nonanomalous symmetries.

In addition, it is straightforward to check that all of the matching states in this GLSM, are in one-to-one correspondence with matching states in the previous GLSM related by triality – for any matching state in this GLSM, one can find a matching state in the previous GLSM in the same representation of nonanomalous symmetries.

$r \ll 0$ State, $\wedge^\bullet \mathcal{E}, H^\bullet(\mathbb{P}^1)$	$r \gg 0$ State, $\wedge^\bullet \mathcal{F}, H^\bullet(\mathbb{P}^3)$	Rep'	$U(1)^3$
$\mathrm{Sym}^2\tilde{V},\ 0,0$	$\mathrm{Sym}^2\tilde{V}^* \otimes \wedge^4 U,\ 2,2$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})$	$(-4,0,+4)$
$V^* \otimes \wedge^2\tilde{V} \otimes \tilde{V},\ 1,0$	$V^* \otimes \tilde{V}^* \otimes \wedge^4 U,\ 2,1$	$(\mathbf{1},\overline{\mathbf{4}},\mathbf{1},\mathbf{2})$	$(-3,-1,+4)$
$U^* \otimes \tilde{V},\ 1,0$	$\wedge^2\tilde{V}^* \otimes \tilde{V}^* \otimes \wedge^4 U \otimes U^*,\ 3,2$	$(\overline{\mathbf{4}},\mathbf{1},\mathbf{1},\mathbf{2})$	$(-4,+1,+3)$
$W^* \otimes K_{(1,-2)}\tilde{V}^*,\ 1,0$	$W^* \otimes \mathrm{Sym}^3\tilde{V}^* \otimes \wedge^4 U,\ 4,3$	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{4})$	$(-2,0,+2)$
$\wedge^2 V^* \otimes (\wedge^2\tilde{V})^2,\ 2,0$	$\wedge^2 V^* \otimes \wedge^4 U,\ 2,0$	$(\mathbf{1},\mathbf{6},\mathbf{1},\mathbf{1})$	$(-2,-2,+4)$
$V^* \otimes U^* \otimes \wedge^2\tilde{V},\ 2,0$	$V^* \otimes \wedge^2\tilde{V}^* \otimes \wedge^4 U \otimes U^*,\ 3,1$	$(\overline{\mathbf{4}},\overline{\mathbf{4}},\mathbf{1},\mathbf{1})$	$(-3,0,+3)$
$V^* \otimes W^* \otimes \mathrm{Sym}^2\tilde{V},\ 2,0$	$V^* \otimes W^* \otimes \mathrm{Sym}^2\tilde{V}^* \otimes \wedge^4 U,\ 4,2$	$(\mathbf{1},\overline{\mathbf{4}},\mathbf{2},\mathbf{3})$	$(-1,-1,+2)$
$\wedge^2 U^*,\ 2,0$	$(\wedge^2\tilde{V}^*)^2 \otimes \wedge^4 U \otimes \wedge^2 U^*,\ 4,2$	$(\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1})$	$(-4,+2,+2)$
$U^* \otimes W^* \otimes K_{(1,-1)}\tilde{V}^*,\ 2,0$	$W^* \otimes K_{(3,1)}\tilde{V}^* \otimes \wedge^3 U,\ 5,3$	$(\overline{\mathbf{4}},\mathbf{1},\mathbf{2},\mathbf{3})$	$(-2,+1,+1)$
$\wedge^2 V^* \otimes W^* \otimes K_{(2,1)}\tilde{V},\ 3,0$	$\wedge^2 V^* \otimes \tilde{V}^* \otimes W^* \otimes \wedge^4 U,\ 4,1$	$(\mathbf{1},\mathbf{6},\mathbf{2},\mathbf{2})$	$(0,-2,+2)$
$V^* \otimes U^* \otimes W^* \otimes \tilde{V},\ 3,0$	$V^* \otimes W^* \otimes K_{(2,1)}\tilde{V}^* \otimes \wedge^3 U,\ 5,2$	$(\overline{\mathbf{4}},\overline{\mathbf{4}},\mathbf{2},\mathbf{2})$	$(-1,0,+1)$
$V^* \otimes \wedge^2 W^* \otimes K_{(1,-2)}\tilde{V}^*,\ 3,0$	$V^* \otimes \wedge^2 W^* \otimes \mathrm{Sym}^3\tilde{V}^* \otimes \wedge^4 U,\ 6,3$	$(\mathbf{1},\overline{\mathbf{4}},\mathbf{1},\mathbf{4})$	$(+1,-1,0)$
$\wedge^2 U^* \otimes W^* \otimes \tilde{V}^*,\ 3,0$	$W^* \otimes K_{(3,2)}\tilde{V}^* \otimes \wedge^2 U,\ 6,3$	$(\mathbf{6},\mathbf{1},\mathbf{2},\mathbf{2})$	$(-2,+2,0)$
$U^* \otimes \wedge^2 W^* \otimes K_{(2,-1)}\tilde{V}^*,\ 3,0$	$\wedge^4 V^* \otimes \mathrm{Sym}^3\tilde{V}^* \otimes U^*,\ 7,0$	$(\overline{\mathbf{4}},\mathbf{1},\mathbf{1},\mathbf{4})$	$(0,+1,-1)$
$\wedge^4 V^* \otimes K_{(3,3)}\tilde{V},\ 4,1$	$\wedge^2 W^* \otimes K_{(2,2,2,2)}U,\ 2,3$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,-4,+4)$
$\wedge^3 V^* \otimes U^* \otimes K_{(2,2)}\tilde{V},\ 4,1$	$\wedge^3 V^* \otimes \wedge^3 U,\ 3,0$	$(\overline{\mathbf{4}},\mathbf{4},\mathbf{1},\mathbf{1})$	$(-1,-2,+3)$
$\wedge^3 V^* \otimes W^* \otimes K_{(2,2)}\tilde{V},\ 4,0$	$\wedge^3 V^* \otimes W^* \otimes \wedge^4 U,\ 4,0$	$(\mathbf{1},\mathbf{4},\mathbf{2},\mathbf{1})$	$(+1,-3,+2)$
$\wedge^2 V^* \otimes \wedge^2 U^* \otimes \wedge^2\tilde{V},\ 4,1$	$\wedge^2 V^* \otimes \wedge^2\tilde{V}^* \otimes \wedge^2 U,\ 4,1$	$(\mathbf{6},\mathbf{6},\mathbf{1},\mathbf{1})$	$(-2,0,+2)$
$\wedge^2 V^* \otimes U^* \otimes W^* \otimes \wedge^2\tilde{V},\ 4,0$	$\wedge^2 V^* \otimes W^* \otimes \wedge^2\tilde{V}^* \otimes \wedge^3 U,\ 5,1$	$(\overline{\mathbf{4}},\mathbf{6},\mathbf{2},\mathbf{1})$	$(0,-1,+1)$
$\wedge^2 V^* \otimes \wedge^2 W^* \otimes \mathrm{Sym}^2\tilde{V},\ 4,0$	$\wedge^2 V^* \otimes \wedge^2 W^* \otimes K_{(2,0)}\tilde{V}^* \otimes \wedge^4 U,\ 6,2$	$(\mathbf{1},\mathbf{6},\mathbf{1},\mathbf{3})$	$(+2,-2,0)$
$V^* \otimes \wedge^3 U^*,\ 4,1$	$V^* \otimes K_{(2,2)}\tilde{V}^* \otimes U,\ 5,2$	$(\mathbf{4},\overline{\mathbf{4}},\mathbf{1},\mathbf{1})$	$(-3,+2,+1)$
$V^* \otimes \wedge^2 U^* \otimes W^*,\ 4,0$	$V^* \otimes W^* \otimes K_{(2,2)}\tilde{V}^* \otimes \wedge^2 U,\ 6,2$	$(\mathbf{6},\overline{\mathbf{4}},\mathbf{2},\mathbf{1})$	$(-1,+1,0)$
$V^* \otimes U^* \otimes \wedge^2 W^* \otimes K_{(1,-1)}\tilde{V}^*,\ 4,0$	$V^* \otimes \wedge^2 W^* \otimes K_{(3,1)}\tilde{V}^* \otimes \wedge^3 U,\ 7,3$	$(\overline{\mathbf{4}},\overline{\mathbf{4}},\mathbf{1},\mathbf{3})$	$(+1,0,-1)$
$\wedge^4 U^* \otimes \wedge^2\tilde{V}^*,\ 4,1$	$(\wedge^2\tilde{V}^*)^3,\ 6,3$	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(-4,+4,0)$
$\wedge^3 U^* \otimes W^* \otimes \wedge^2\tilde{V}^*,\ 4,0$	$W^* \otimes K_{(3,3)}\tilde{V}^* \otimes U,\ 7,3$	$(\mathbf{4},\mathbf{1},\mathbf{2},\mathbf{1})$	$(-2,+3,-1)$
$\wedge^2 U^* \otimes \wedge^2 W^* \otimes \mathrm{Sym}^2\tilde{V}^*,\ 4,0$	$\wedge^4 V^* \otimes K_{(3,1)}\tilde{V}^* \otimes \wedge^2 U^*,\ 8,0$	$(\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{3})$	$(0,+2,-2)$
$\wedge^4 V^* \otimes U^* \otimes K_{(3,2)}\tilde{V},\ 5,1$	$\wedge^2 W^* \otimes \tilde{V}^* \otimes K_{(2,2,2,1)}U,\ 3,3$	$(\overline{\mathbf{4}},\mathbf{1},\mathbf{1},\mathbf{2})$	$(0,-3,+3)$
$\wedge^3 V^* \otimes \wedge^2 U^* \otimes K_{(2,1)}\tilde{V},\ 5,1$	$\wedge^3 V^* \otimes \tilde{V}^* \otimes \wedge^2 U,\ 4,0$	$(\mathbf{6},\mathbf{4},\mathbf{1},\mathbf{2})$	$(-1,-1,+2)$
$\wedge^3 V^* \otimes \wedge^2 W^* \otimes K_{(2,1)}\tilde{V},\ 5,0$	$\wedge^3 V^* \otimes \wedge^2 W^* \otimes \tilde{V}^* \otimes \wedge^4 U,\ 6,1$	$(\mathbf{1},\mathbf{4},\mathbf{1},\mathbf{2})$	$(+3,-3,0)$
$\wedge^2 V^* \otimes \wedge^3 U^* \otimes \tilde{V},\ 5,1$	$\wedge^2 V^* \otimes K_{(2,1)}\tilde{V}^* \otimes U,\ 5,1$	$(\mathbf{4},\mathbf{6},\mathbf{1},\mathbf{2})$	$(-2,+1,+1)$
$\wedge^2 V^* \otimes U^* \otimes \wedge^2 W^* \otimes \tilde{V},\ 5,0$	$\wedge^2 V^* \otimes \wedge^2 W^* \otimes K_{(2,1)}\tilde{V}^* \otimes \wedge^3 U,\ 7,2$	$(\overline{\mathbf{4}},\mathbf{6},\mathbf{1},\mathbf{2})$	$(+2,-1,-1)$
$V^* \otimes \wedge^4 U^* \otimes \tilde{V}^*,\ 5,1$	$V^* \otimes K_{(3,2)}\tilde{V}^*,\ 6,2$	$(\mathbf{1},\overline{\mathbf{4}},\mathbf{1},\mathbf{2})$	$(-3,+3,0)$
$V^* \otimes \wedge^2 U^* \otimes \wedge^2 W^* \otimes \tilde{V}^*,\ 5,0$	$V^* \otimes \wedge^2 W^* \otimes K_{(3,2)}\tilde{V}^* \otimes \wedge^2 U,\ 8,3$	$(\mathbf{6},\overline{\mathbf{4}},\mathbf{1},\mathbf{2})$	$(+1,+1,-2)$
$\wedge^3 U^* \otimes \wedge^2 W^* \otimes K_{(2,1)}\tilde{V}^*,\ 5,0$	$\wedge^4 V^* \otimes K_{(3,2)}\tilde{V}^* \otimes \wedge^3 U^*,\ 9,0$	$(\mathbf{4},\mathbf{1},\mathbf{1},\mathbf{2})$	$(0,+3,-3)$

Table 4.10: First half of shared states defined by matching representations of anomaly-free global symmetries.

$r \ll 0$ State, $\wedge^\bullet \mathcal{E}, H^\bullet(\mathbb{P}^1)$	$r \gg 0$ State, $\wedge^\bullet \mathcal{F}, H^\bullet(\mathbb{P}^3)$	Rep'	$U(1)^3$
$\wedge^4 V^* \otimes \wedge^2 U^* \otimes K_{(3,1)}\tilde{V}$, 6, 1	$\wedge^2 W^* \otimes S^2 \tilde{V}^* \otimes K_{(2,2,1,1)}U$, 4, 3	$(\mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{3})$	$(0, -2, +2)$
$\wedge^4 V^* \otimes U^* \otimes W^* \otimes K_{(2,2)}\tilde{V}$, 6, 1	$\wedge^4 V^* \otimes W^* \otimes \wedge^3 U$, 5, 0	$(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2}, \mathbf{1})$	$(+2, -3, +1)$
$\wedge^4 V^* \otimes \wedge^2 W^* \otimes K_{(2,2)}\tilde{V}$, 6, 0	$\wedge^4 V^* \otimes \wedge^2 W^* \otimes \wedge^4 U$, 6, 0	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(+4, -4, 0)$
$\wedge^3 V^* \otimes \wedge^3 U^* \otimes \mathrm{Sym}^2 \tilde{V}$, 6, 1	$\wedge^3 V^* \otimes \mathrm{Sym}^2 \tilde{V}^* \otimes U$, 5, 0	$(\mathbf{4}, \mathbf{4}, \mathbf{1}, \mathbf{3})$	$(-1, 0, +1)$
$\wedge^3 V^* \otimes \wedge^2 U^* \otimes W^* \otimes \wedge^2 \tilde{V}$, 6, 1	$\wedge^3 V^* \otimes W^* \otimes \wedge^2 \tilde{V}^* \otimes \wedge^2 U$, 6, 1	$(\mathbf{6}, \mathbf{4}, \mathbf{2}, \mathbf{1})$	$(+1, -1, 0)$
$\wedge^3 V^* \otimes U^* \otimes \wedge^2 W^* \otimes \wedge^2 \tilde{V}$, 6, 0	$\wedge^3 V^* \otimes \wedge^2 W^* \otimes \wedge^2 \tilde{V}^* \otimes \wedge^3 U$, 7, 1	$(\overline{\mathbf{4}}, \mathbf{4}, \mathbf{1}, \mathbf{1})$	$(+3, -2, -1)$
$\wedge^2 V^* \otimes \wedge^4 U^* \otimes K_{(1,-1)}\tilde{V}^*$, 6, 1	$\wedge^2 V^* \otimes K_{(3,1)}\tilde{V}^*$, 6, 1	$(\mathbf{1}, \mathbf{6}, \mathbf{1}, \mathbf{3})$	$(-2, +2, 0)$
$\wedge^2 V^* \otimes \wedge^3 U^* \otimes W^*$, 6, 1	$\wedge^2 V^* \otimes W^* \otimes K_{(2,2)}\tilde{V}^* \otimes U$, 7, 2	$(\mathbf{4}, \mathbf{6}, \mathbf{2}, \mathbf{1})$	$(0, +1, -1)$
$\wedge^2 V^* \otimes \wedge^2 U^* \otimes \wedge^2 W^*$, 6, 0	$(\wedge^2 V \otimes \wedge^2 W \otimes (\wedge^2 \tilde{V})^2)^* \otimes \wedge^2 U$, 8, 2	$(\mathbf{6}, \mathbf{6}, \mathbf{1}, \mathbf{1})$	$(+2, 0, -2)$
$V^* \otimes \wedge^4 U^* \otimes W^* \otimes \wedge^2 \tilde{V}^*$, 6, 1	$V^* \otimes W^* \otimes K_{(3,3)}\tilde{V}^*$, 8, 3	$(\mathbf{1}, \overline{\mathbf{4}}, \mathbf{2}, \mathbf{1})$	$(-1, +3, -2)$
$V^* \otimes \wedge^3 U^* \otimes \wedge^2 W^* \otimes \wedge^2 \tilde{V}^*$, 6, 0	$V^* \otimes K_{(3,3)}\tilde{V}^* \otimes \wedge^2 W^* \otimes U$, 9, 3	$(\mathbf{4}, \overline{\mathbf{4}}, \mathbf{1}, \mathbf{1})$	$(+1, +2, -3)$
$\wedge^4 U^* \otimes \wedge^2 W^* \otimes K_{(2,2)}\tilde{V}^*$, 6, 0	$\wedge^4 V^* \otimes K_{(3,3)}\tilde{V}^* \otimes \wedge^4 U^*$, 10, 0	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(0, +4, -4)$

Table 4.11: Second portion of shared states defined by matching representations of anomaly-free global symmetries. We have occasionally used the symbol S as an abbreviation for Sym.

$r \ll 0$ State, $\wedge^\bullet \mathcal{E}, H^\bullet(\mathbb{P}^1)$	$r \gg 0$ State, $\wedge^\bullet \mathcal{F}, H^\bullet(\mathbb{P}^3)$	Rep'	$U(1)^3$
$\wedge^4 V^* \otimes \wedge^3 U^* \otimes \mathrm{S}^3 \tilde{V}$, 7, 1	$\wedge^2 W^* \otimes \mathrm{S}^3 \tilde{V}^* \otimes K_{(2,1,1,1)}U$, 5, 3	$(\mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{4})$	$(0, -1, +1)$
$\wedge^4 V^* \otimes \wedge^2 U^* \otimes W^* \otimes K_{(2,1)}\tilde{V}$, 7, 1	$\wedge^4 V^* \otimes \tilde{V}^* \otimes W^* \otimes \wedge^2 U$, 6, 0	$(\mathbf{6}, \mathbf{1}, \mathbf{2}, \mathbf{2})$	$(+2, -2, 0)$
$\wedge^3 V^* \otimes \wedge^4 U^* \otimes K_{(2,-1)}\tilde{V}$, 7, 1	$\wedge^3 V^* \otimes \mathrm{Sym}^3 \tilde{V}^*$, 6, 0	$(\mathbf{1}, \mathbf{4}, \mathbf{1}, \mathbf{4})$	$(-1, +1, 0)$
$\wedge^3 V^* \otimes \wedge^3 U^* \otimes W^* \otimes \tilde{V}$, 7, 1	$\wedge^3 V^* \otimes W^* \otimes K_{(2,1)}\tilde{V}^* \otimes U$, 7, 1	$(\mathbf{4}, \mathbf{4}, \mathbf{2}, \mathbf{2})$	$(+1, 0, -1)$
$\wedge^2 V^* \otimes \wedge^4 U^* \otimes W^* \otimes \tilde{V}^*$, 7, 1	$\wedge^2 V^* \otimes W^* \otimes K_{(3,2)}\tilde{V}^*$, 8, 2	$(\mathbf{1}, \mathbf{6}, \mathbf{2}, \mathbf{2})$	$(0, +2, -2)$
$\wedge^4 V^* \otimes \wedge^3 U^* \otimes W^* \otimes \mathrm{Sym}^2 \tilde{V}$, 8, 1	$\wedge^4 V^* \otimes W^* \otimes \mathrm{Sym}^2 \tilde{V}^* \otimes U$, 7, 0	$(\mathbf{4}, \mathbf{1}, \mathbf{2}, \mathbf{3})$	$(+2, -1, -1)$
$\wedge^4 V^* \otimes \wedge^2 U^* \otimes \wedge^2 W^* \otimes \wedge^2 \tilde{V}$, 8, 1	$\wedge^4 V^* \otimes \wedge^2 W^* \otimes \wedge^2 \tilde{V}^* \otimes \wedge^2 U$, 8, 1	$(\mathbf{6}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(+4, -2, -2)$
$(\wedge^3 V \otimes \wedge^4 U \otimes W \otimes K_{(1,-1)}\tilde{V})^*$, 8, 1	$\wedge^3 V^* \otimes W^* \otimes K_{(3,1)}\tilde{V}^*$, 8, 1	$(\mathbf{1}, \mathbf{4}, \mathbf{2}, \mathbf{3})$	$(+1, +1, -2)$
$\wedge^3 V^* \otimes \wedge^3 U^* \otimes \wedge^2 W^*$, 8, 1	$(\wedge^3 V \otimes \wedge^2 W \otimes K_{(2,2)}\tilde{V})^* \otimes U$, 9, 2	$(\mathbf{4}, \mathbf{4}, \mathbf{1}, \mathbf{1})$	$(+3, 0, -3)$
$\wedge^2 V^* \otimes \wedge^4 U^* \otimes \wedge^2 W^* \otimes \wedge^2 \tilde{V}^*$, 8, 1	$\wedge^2 V^* \otimes K_{(3,3)}\tilde{V}^* \otimes \wedge^2 W^*$, 10, 3	$(\mathbf{1}, \mathbf{6}, \mathbf{1}, \mathbf{1})$	$(+2, +2, -4)$
$\wedge^4 V^* \otimes \wedge^4 U^* \otimes W^* \otimes K_{(2,-1)}\tilde{V}$, 9, 1	$\wedge^4 V^* \otimes \mathrm{Sym}^3 \tilde{V}^* \otimes W^*$, 8, 0	$(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{4})$	$(+2, 0, -2)$
$\wedge^4 V^* \otimes \wedge^3 U^* \otimes \wedge^2 W^* \otimes \tilde{V}$, 9, 1	$(\wedge^4 V \otimes \wedge^2 W \otimes K_{(2,1)}\tilde{V})^* \otimes U$, 9, 1	$(\mathbf{4}, \mathbf{1}, \mathbf{1}, \mathbf{2})$	$(+4, -1, -3)$
$\wedge^3 V^* \otimes \wedge^4 U^* \otimes \wedge^2 W^* \otimes \tilde{V}^*$, 9, 1	$\wedge^3 V^* \otimes \wedge^2 W^* \otimes K_{(3,2)}\tilde{V}^*$, 10, 2	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})$	$(+3, +1, -4)$
$(\wedge^4 V \otimes \wedge^4 U \otimes \wedge^2 W \otimes K_{(1,-1)}\tilde{V})^*$, 10, 1	$\wedge^4 V^* \otimes \wedge^2 W^* \otimes K_{(3,1)}\tilde{V}^*$, 10, 1	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3})$	$(+4, 0, -4)$

Table 4.12: Third portion of shared states defined by matching representations of anomaly-free global symmetries. We have occasionally used the symbol S as an abbreviation for Sym.

State	$r \ll 0$ $\wedge^\bullet \mathcal{E}$	$H^\bullet(\mathbb{P}^2)$	$r \gg 0$ $\wedge^\bullet \mathcal{F}$	$H^\bullet(\mathbb{P}^1)$	Rep'	$U(1)^3$
$\wedge^2 W^* \otimes K_{(2,-2)}\tilde{V}^*$	2	0	–	–	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{5})$	$(0,0,0)$
$\wedge^4 V^* \otimes \wedge^4 U^* \otimes K_{(3,-1)}\tilde{V}$	8	1	–	–	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{5})$	$(0,0,0)$
$\wedge^4 V^* \otimes K_{(1,1,1,-1)}U$	–	–	4	0	$(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,-2,+2)$
$\wedge^2 W^* \otimes \wedge^2 \tilde{V}^* \otimes K_{(2,2,2,0)}U$	–	–	4	3	$(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,-2,+2)$
$\wedge^4 V^* \otimes \tilde{V}^* \otimes K_{(1,1,0,-1)}U$	–	–	5	0	$(\mathbf{20},\mathbf{1},\mathbf{1},\mathbf{2})$	$(0,-1,+1)$
$\wedge^2 W^* \otimes K_{(2,1)}\tilde{V}^* \otimes K_{(2,2,1,0)}U$	–	–	5	3	$(\mathbf{20},\mathbf{1},\mathbf{1},\mathbf{2})$	$(0,-1,+1)$
$\wedge^4 V^* \otimes \wedge^2 \tilde{V}^* \otimes K_{(1,1,-1,-1)}U$	–	–	6	0	$(\mathbf{20},\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,0,0)$
$\wedge^4 V^* \otimes \mathrm{Sym}^2 \tilde{V}^* \otimes K_{(1,0,0,-1)}U$	–	–	6	0	$(\mathbf{15},\mathbf{1},\mathbf{1},\mathbf{3})$	$(0,0,0)$
$\wedge^2 W^* \otimes K_{(3,1)}\tilde{V}^* \otimes K_{(2,1,1,0)}U$	–	–	6	3	$(\mathbf{15},\mathbf{1},\mathbf{1},\mathbf{3})$	$(0,0,0)$
$\wedge^2 W^* \otimes K_{(2,2)}\tilde{V}^* \otimes K_{(2,2,0,0)}U$	–	–	6	3	$(\mathbf{20},\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,0,0)$
$\wedge^4 V^* \otimes K_{(2,1)}\tilde{V}^* \otimes K_{(1,1,0,-1)}U^*$	–	–	7	0	$(\mathbf{20},\mathbf{1},\mathbf{1},\mathbf{2})$	$(0,+1,-1)$
$\wedge^2 W^* \otimes K_{(3,2)}\tilde{V}^* \otimes K_{(2,1,0,0)}U$	–	–	7	3	$(\mathbf{20},\mathbf{1},\mathbf{1},\mathbf{2})$	$(0,+1,-1)$
$\wedge^4 V^* \otimes K_{(2,2)}\tilde{V}^* \otimes K_{(1,-1,-1,-1)}U$	–	–	8	0	$(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,+2,-2)$
$K_{(3,3)}\tilde{V}^* \otimes \wedge^2 W^* \otimes \mathrm{Sym}^2 U$	–	–	8	3	$(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,+2,-2)$

Table 4.13: List of all states which are not shared between the two phases.

4.6 Third example: T_{222}

4.6.1 First GLSM

This example is the case in which $k = 1$, $A = 2$, and $n = 2$, so $2k + A - n = 2$. There is only one geometry appearing in this example, namely the space $G(1,2) = \mathbb{P}^1$, with bundle

$$\mathcal{E} \;=\; S^2 \oplus (Q^*)^2 \oplus (\det S^*)^2.$$

In the notations above, we will consider the GLSM which for $r \gg 0$ describes

$$\mathcal{E} \;=\; U \otimes S \;+\; V \otimes Q^* \;+\; W \otimes \det S^* \;\longrightarrow\; G(1, \tilde{V}^*),$$

for

$$U \;=\; \mathbb{C}^2 \;=\; V \;=\; W \;=\; \tilde{V},$$

and which for $r \ll 0$ is described by

$$\mathcal{F} \;=\; U \otimes S^* \;+\; \tilde{V} \otimes Q^* \;+\; W \otimes \det S \;\longrightarrow\; G(1, V^*).$$

Both of the phases above are B/2-twistable without further dualization; the Fock vacuum line bundle is trivial. As noted earlier, we need to make a choice of presentation as powers of S^* and Q^* – although the choice does not enter the final nonanomalous representations, we must make a choice in order to initially compute the chiral states. We choose the canonical trivial presentation, as $K_{(0)}S^* \otimes K_{(0)}Q^*$ in both phases.

We will take the nonanomalous (chiral) global $U(1)^3$ to be defined by

	$\tilde{V}$	U	V	W
$U(1)_{(1)}$	0	0	-1	-1
$U(1)_{(2)}$	1	0	0	-1
$U(1)_{(3)}$	0	1	0	1

For both the $r \gg 0$ and $r \ll 0$ phases, we compute that the Fock vacuum has charge $(+2, 0, -2)$.

Table 4.14 lists all the states which match between the two phases of this GLSM. A few states do not match; these are listed in table 4.15. As a consistency check, note that both tables are invariant under Serre duality as expected.

It was predicted in [25][Eq. (3.1)] that the $SU(2)^4$ global flavor symmetries of this model should be promoted to an

$$SU(2)_1 \times SU(2)_1 \times SU(2)_1 \times SU(2)_1$$

affine symmetry in the IR SCFT, and indeed, note that all of the matching representations in table 4.14 are integrable, whereas all of the mismatched representations in table 4.15 are non-integrable, suggesting that the mismatched representations do not survive to the IR. In addition, it is also easy to check that the mismatched states in table 4.15 cancel out of the leading term in elliptic genera, refined by any listed nonanomalous symmetry, which is consistent with the prediction that they do not survive to the IR.

Another prediction of [25] is that in this particular model, the $SU(2)_1^4$ affine symmetry of the IR SCFT should be enhanced to an E_6 symmetry at level 1. Briefly, the $SU(W)$ of the two Fermi fields can combine with the $SU(U)$ to form $SU(4)$, and by triality, any one of the other $SU(2)$'s at a time. The resulting structure can be naturally interpreted as a subgroup of E_6.

We can see some evidence for this in the states above. In particular, the total number of matching states is 54, twice the dimension of the $\mathbf{27}$ representation, suggesting that the states above flow to copies of either the $\mathbf{27}$ or $\overline{\mathbf{27}}$ representations. Indeed, the only integrable representations of E_6 at level 1 are $\mathbf{1}$, $\mathbf{27}$, and $\overline{\mathbf{27}}$.

Now that we have verified the dimensions are consistent, let us see if the precise representations appearing above match those in the decomposition of a $\mathbf{27}$ or $\overline{\mathbf{27}}$ under an $su(2)^4$

$r \gg 0$ State, $\wedge^\bullet \mathcal{E}, H^\bullet(\mathbb{P}^1)$	$r \ll 0$ State, $\wedge^\bullet \mathcal{F}, H^\bullet(\mathbb{P}^1)$	Rep'	$U(1)^3$
1, 0, 0	1, 0, 0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(+2,0,-2)$
$W \otimes \tilde{V}$, 1, 0	$W \otimes \tilde{V}$, 2, 1	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})$	$(+1,0,-1)$
$\wedge^2 U \otimes K_{(-1,-1)}\tilde{V}$, 2, 1	$\wedge^2 W \otimes K_{(-1,-1)}V$, 2, 1	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(+2,-2,0)$
$U \otimes V$, 2, 1	$U \otimes V$, 1, 0	$(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$	$(+1,0,-1)$
$U \otimes W$, 2, 0	$U \otimes W$, 2, 0	$(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})$	$(+1,-1,0)$
$\wedge^2 V \otimes \wedge^2 \tilde{V}$, 2, 1	$\wedge^2 V \otimes \wedge^2 \tilde{V}$, 2, 1	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,+2,-2)$
$V \otimes W \otimes \wedge^2 \tilde{V}$, 2, 0	$V \otimes W \otimes \wedge^2 \tilde{V}$, 3, 1	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})$	$(0,+1,-1)$
$\wedge^2 U \otimes V \otimes \tilde{V}^*$, 3, 1	$\tilde{V} \otimes \wedge^2 W \otimes V^*$, 3, 1	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})$	$(+1,-1,0)$
$U \otimes \wedge^2 V \otimes \tilde{V}$, 3, 1	$U \otimes \tilde{V} \otimes \wedge^2 V$, 2, 0	$(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})$	$(0,+1,-1)$
$U \otimes \wedge^2 W \otimes \tilde{V}$, 3, 0	$U \otimes \wedge^2 W \otimes \tilde{V}$, 4, 1	$(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})$	$(0,-1,+1)$
$V \otimes \wedge^2 W \otimes K_{(2,1)}\tilde{V}$, 3, 0	$\wedge^2 U \otimes \tilde{V} \otimes K_{(2,1)}V$, 3, 0	$(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})$	$(-1,+1,0)$
$\wedge^2 U \otimes V \otimes W$, 4, 1	$\wedge^2 U \otimes V \otimes W$, 3, 0	$(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})$	$(0,-1,+1)$
$\wedge^2 U \otimes \wedge^2 W$, 4, 0	$\wedge^2 U \otimes \wedge^2 W$, 4, 0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(0,-2,+2)$
$U \otimes \wedge^2 V \otimes W \otimes \wedge^2 \tilde{V}$, 4, 1	$U \otimes \wedge^2 V \otimes W \otimes \wedge^2 \tilde{V}$, 4, 1	$(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})$	$(-1,+1,0)$
$U \otimes V \otimes \wedge^2 W \otimes \wedge^2 \tilde{V}$, 4, 0	$U \otimes V \otimes \wedge^2 W \otimes \wedge^2 \tilde{V}$, 5, 1	$(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$	$(-1,0,+1)$
$\wedge^2 V \otimes \wedge^2 W \otimes K_{(2,2)}\tilde{V}$, 4, 0	$\wedge^2 U \otimes \wedge^2 \tilde{V} \otimes K_{(2,2)}V$, 4, 0	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(-2,+2,0)$
$\wedge^2 U \otimes \wedge^2 V \otimes W \otimes \tilde{V}$, 5, 1	$\wedge^2 U \otimes \tilde{V} \otimes W \otimes \wedge^2 V$, 4, 0	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})$	$(-1,0,+1)$
$\wedge^2 U \otimes \wedge^2 V \otimes \wedge^2 W \otimes \wedge^2 \tilde{V}$, 6, 1	$\wedge^2 U \otimes \wedge^2 V \otimes \wedge^2 W \otimes \wedge^2 \tilde{V}$, 6, 1	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(-2,0,+2)$

Table 4.14: Shared states defined by matching representations of anomaly-free global symmetries.

	$r \gg 0$		$r \ll 0$			
State	$\wedge^\bullet \mathcal{E}$	$H^\bullet(\mathbb{P}^1)$	$\wedge^\bullet \mathcal{F}$	$H^\bullet(\mathbb{P}^1)$	Rep'	$U(1)^3$
$\wedge^2 W \otimes K_{(2,0)}\tilde{V}$	2	0	–	–	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})$	$(0,0,0)$
$\wedge^2 U \otimes \wedge^2 V \otimes K_{(1,-1)}\tilde{V}$	4	1	–	–	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{3})$	$(0,0,0)$
$\wedge^2 U \otimes K_{(2,0)}V$	–	–	2	0	$(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})$	$(0,0,0)$
$\wedge^2 \tilde{V} \otimes \wedge^2 W \otimes K_{(1,-1)}V$	–	–	4	1	$(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})$	$(0,0,0)$

Table 4.15: List of all states which are not shared between the two phases.

subalgebra[3]. We can think of the $su(2)^4$ subalgebra of E_6 as obtained from a maximal $(SU(2) \times SU(6))/\mathbb{Z}_2$ subgroup, and the $SU(6)$ naturally contains an $SU(2)^3$ from omitting two of the nodes in its Dynkin diagram. The $SU(W)$, which can recombine with any one other $SU(2)$, can be understood as the middle node in the E_6 Dynkin diagram, or equivalently the middle node in the $SU(6)$ Dynkin diagram. Under the $(SU(2) \times SU(6))/\mathbb{Z}_2$ subgroup, the $\mathbf{27}$ of E_6 decomposes as [48][table 15]

$$\mathbf{27} \; = \; (\mathbf{2}, \overline{\mathbf{6}}) \; + \; (\mathbf{1}, \mathbf{15}).$$

Furthermore, under the $SU(2)^3$ subgroup of $SU(6)$,

$$\overline{\mathbf{6}} \; = \; (\mathbf{2}, \mathbf{1}, \mathbf{1}) \; + \; (\mathbf{1}, \mathbf{2}, \mathbf{1}) \; + \; (\mathbf{1}, \mathbf{1}, \mathbf{2}).$$

(The $\mathbf{6}$ has the same decomposition – the $su(2)^4$ subalgebra cannot distinguish the $\mathbf{6}$ from $\overline{\mathbf{6}}$.) Similarly, the $\mathbf{15} = \wedge^2 \mathbf{6}$ should decompose as

$$\begin{aligned} \mathbf{15} \; &= \; \wedge^2(\mathbf{2}, \mathbf{1}, \mathbf{1}) \; + \; \wedge^2(\mathbf{1}, \mathbf{2}, \mathbf{1}) \; + \; \wedge^2(\mathbf{1}, \mathbf{1}, \mathbf{2}) \; + \; (\mathbf{2}, \mathbf{2}, \mathbf{1}) \; + \; (\mathbf{2}, \mathbf{1}, \mathbf{2}) \; + \; (\mathbf{1}, \mathbf{2}, \mathbf{2}), \\ &= \; 3(\mathbf{1}, \mathbf{1}, \mathbf{1}) \; + \; (\mathbf{2}, \mathbf{2}, \mathbf{1}) \; + \; (\mathbf{2}, \mathbf{1}, \mathbf{2}) \; + \; (\mathbf{1}, \mathbf{2}, \mathbf{2}). \end{aligned}$$

Thus, under the $(SU(2)^4)/\mathbf{Z}_2$ subgroup of E_6,

$$\begin{aligned} \mathbf{27} \; = \; & (\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}) \; + \; (\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}) \; + \; (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2}) \; + \; (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \; + \; (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}) \; + \; (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}) \\ & + 3(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}). \end{aligned}$$

(The $\overline{\mathbf{27}}$ has the same decomposition – the $su(2)^4$ subalgebra cannot distinguish $\mathbf{27}$ from $\overline{\mathbf{27}}$.) It is straightforward to check that the states in table 4.14 do indeed form two copies of the decomposition above, partially verifying that in the IR, the states transform (two) $\mathbf{27}$'s or $\overline{\mathbf{27}}$'s.

It may well be possible to say more. For example, [25][Eq. (3.28)] contains a prediction for the left NS partition function, from which one could extract a counting of left NS states. However, our chiral ring computations above are in the left R sector, and for reasons discussed earlier, it is not entirely clear that those two sectors should match in these examples, so we will not pursue that direction further.

4.6.2 Other GLSMs

Other GLSMs related by triality can be constructed by cyclically exchanging the vector spaces

$$U \; \mapsto \; V \; \mapsto \; \tilde{V}^* \; \mapsto \; U \; \mapsto \; \cdots .$$

It is straightforward to check that the same $U(1)^3$ is nonanomalous for each GLSM so obtained; however, to match charges, we need to pick a slightly different $U(1)^3$ action for the

[3] We would like to thank A. Knutson for a discussion of the corresponding group theory.

different GLSMs. Moreover, since the dimensions of these vector spaces all match, we can immediately derive the chiral states from tables 4.14 and 4.15 for the first GLSM for $T_{2,2,2}$.

For completeness, we list in tables 4.16 and 4.17 the corresponding chiral states for one cyclic rotation, *i.e.* U replaced with V and so forth. The global $U(1)^3$ charges listed are those for the action defined by

	$\tilde{V}$	U	V	W
$U(1)_{(1)}$	0	0	-1	-1
$U(1)_{(2)}$	-1	0	0	$+1$
$U(1)_{(3)}$	0	1	0	1

For both geometric phases, we compute that the Fock vacuum has charge $(+2, -2, 0)$. As before, all states come in Serre-dual pairs which dualize representations. Also as before, all the representations appearing amongst matched states are integrable, whereas the unmatched states in table 4.17 have nonintegrable representations. Furthermore, the unmatched states cancel out of leading terms in refined elliptic genera, because they come in pairs with opposite chirality and matching representations of global symmetries.

With the $U(1)^3$ charges above, it is straightforward to check that all of the states below in table 4.16, which match between the two phases of the second GLSM and are expected to survive to the IR, also appear in table 4.14, which listed the states of the first GLSM that are expected to survive to the IR. Thus, these rings provide evidence not only that different phases of each GLSM flow to the same IR fixed point, but that in addition, phases of related distinct GLSMs also flow to the same IR fixed point.

In these examples, we see that the states do not all match. However, the states that match are all consistent with the proposed affine symmetry algebras of the IR fixed point. Furthermore, the mismatched states are all in non-integrable representations, and appear in pairs which cancel out of refined elliptic genera, suggesting that they are lifted in RG flow. In this fashion, we are able to confirm triality. (In principle, it could also happen that pairs of massive states become massless, but we did not observe this in any of the computed examples.) We are also able to support predictions for enhanced IR symmetries in certain theories, such as the enhanced E_6 in the $T_{2,2,2}$ theory, by confirming that the matching UV states fall into the decomposition of integrable representations of the E_6 algebra.

$r \gg 0$ State, $\wedge^\bullet \mathcal{E}, H^\bullet(\mathbb{P}^1)$	$r \ll 0$ State, $\wedge^\bullet \mathcal{F}, H^\bullet(\mathbb{P}^1)$	Rep'	$U(1)^3$
$1,\, 0, 0$	$1,\, 0, 0$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(+2, -2, 0)$
$W \otimes U^*,\, 1, 0$	$W \otimes U^*,\, 2, 1$	$(\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1})$	$(+1, -1, 0)$
$\wedge^2 V \otimes K_{(-1,-1)} U^*,\, 2, 1$	$\wedge^2 W \otimes K_{(-1,-1)} \tilde{V}^*,\, 2, 1$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(0, -2, +2)$
$V \otimes \tilde{V}^*,\, 2, 1$	$V \otimes \tilde{V}^*,\, 1, 0$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})$	$(+1, -1, 0)$
$V \otimes W,\, 2, 0$	$V \otimes W,\, 2, 0$	$(\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1})$	$(0, -1, +1)$
$\wedge^2 \tilde{V}^* \otimes \wedge^2 U^*,\, 2, 1$	$\wedge^2 \tilde{V}^* \otimes \wedge^2 U^*,\, 2, 1$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(+2, 0, -2)$
$\tilde{V}^* \otimes W \otimes \wedge^2 U^*,\, 2, 0$	$\tilde{V}^* \otimes W \otimes \wedge^2 U^*,\, 3, 1$	$(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})$	$(+1, 0, -1)$
$\wedge^2 V \otimes \tilde{V}^* \otimes U,\, 3, 1$	$U^* \otimes \wedge^2 W \otimes \tilde{V},\, 3, 1$	$(\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2})$	$(0, -1, +1)$
$V \otimes \wedge^2 \tilde{V}^* \otimes U^*,\, 3, 1$	$V \otimes U^* \otimes \wedge^2 \tilde{V}^*,\, 2, 0$	$(\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	$(+1, 0, -1)$
$V \otimes \wedge^2 W \otimes U^*,\, 3, 0$	$V \otimes \wedge^2 W \otimes U^*,\, 4, 1$	$(\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})$	$(-1, 0, +1)$
$\tilde{V}^* \otimes \wedge^2 W \otimes K_{(2,1)} U^*,\, 3, 0$	$\wedge^2 V \otimes U^* \otimes K_{(2,1)} \tilde{V}^*,\, 3, 0$	$(\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2})$	$(0, +1, -1)$
$\wedge^2 V \otimes \tilde{V}^* \otimes W,\, 4, 1$	$\wedge^2 V \otimes \tilde{V}^* \otimes W,\, 3, 0$	$(\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})$	$(-1, 0, +1)$
$\wedge^2 V \otimes \wedge^2 W,\, 4, 0$	$\wedge^2 V \otimes \wedge^2 W,\, 4, 0$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(-2, 0, +2)$
$V \otimes \wedge^2 \tilde{V}^* \otimes W \otimes \wedge^2 U^*,\, 4, 1$	$V \otimes \wedge^2 \tilde{V}^* \otimes W \otimes \wedge^2 U^*,\, 4, 1$	$(\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1})$	$(0, +1, -1)$
$V \otimes \tilde{V}^* \otimes \wedge^2 W \otimes \wedge^2 U^*,\, 4, 0$	$V \otimes \tilde{V}^* \otimes \wedge^2 W \otimes \wedge^2 U^*,\, 5, 1$	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})$	$(-1, +1, 0)$
$\wedge^2 \tilde{V}^* \otimes \wedge^2 W \otimes K_{(2,2)} U^*,\, 4, 0$	$\wedge^2 V \otimes \wedge^2 U^* \otimes K_{(2,2)} \tilde{V}^*,\, 4, 0$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(0, +2, -2)$
$\wedge^2 V \otimes \wedge^2 \tilde{V}^* \otimes W \otimes U^*,\, 5, 1$	$\wedge^2 V \otimes U^* \otimes W \otimes \wedge^2 \tilde{V}^*,\, 4, 0$	$(\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1})$	$(-1, +1, 0)$
$\wedge^2 V \otimes \wedge^2 \tilde{V}^* \otimes \wedge^2 W \otimes \wedge^2 U^*,\, 6, 1$	$\wedge^2 V \otimes \wedge^2 \tilde{V}^* \otimes \wedge^2 W \otimes \wedge^2 U^*,\, 6, 1$	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(-2, +2, 0)$

Table 4.16: Shared states defined by matching representations of anomaly-free global symmetries.

	$r \gg 0$		$r \ll 0$			
State	$\wedge^\bullet \mathcal{E}$	$H^\bullet(\mathbb{P}^1)$	$\wedge^\bullet \mathcal{F}$	$H^\bullet(\mathbb{P}^1)$	Rep'	$U(1)^3$
$\wedge^2 W \otimes K_{(2,0)} U^*$	2	0	–	–	$(\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(0, 0, 0)$
$\wedge^2 V \otimes \wedge^2 \tilde{V}^* \otimes K_{(1,-1)} U^*$	4	1	–	–	$(\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	$(0, 0, 0)$
$\wedge^2 V \otimes K_{(2,0)} \tilde{V}^*$	–	–	2	0	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3})$	$(0, 0, 0)$
$\wedge^2 U^* \otimes \wedge^2 W \otimes K_{(1,-1)} \tilde{V}^*$	–	–	4	1	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3})$	$(0, 0, 0)$

Table 4.17: List of all states which are not shared between the two phases.

Chapter 5

Application to Toda-like duality

Chiral rings can also be used to check certain dualities. In this chapter we consider Toda-like (0,2) mirror symmetry. The contents of this chapter will appear as one part of the paper [101], and represents new work.

Ordinary mirror symmetry exchanges A twists with B twists, which means an A twisted nonlinear sigma model is equivalent to a B twisted nonlinear sigma model on the mirror Calabi-Yau manifold [41]. Similarly, $(0,2)$ mirror symmetry exchanges $A/2$ twists with $B/2$ twists, meaning an $A/2$ twisted nonlinear sigma model on $(X, \mathcal{E})$ is equivalent to the $B/2$ twisted nonlinear sigma model on the mirror $(X', \mathcal{E}')$.

A version of $(0,2)$ mirrors also exists for Fano spaces. An A twisted (respectively $A/2$ twisted) nonlinear sigma model on a Fano manifold X is equivalent to a B twist (respectively $B/2$ twist) of a Landau-Ginzburg model. The $(2,2)$ version of this duality is well studied and is given by Toda dual to Fano manifolds [94]. Comparatively, little is known about $(0,2)$ analogues. $(0,2)$ Toda-like mirrors to products of projective spaces are constructed in [93], generalizing the only example previously in the literature [92]. The goal of this chapter is to extend the construction of (0,2) Landau-Ginzburg mirrors to Grassmanians, as deformations of (2,2) Landau-Ginzburg mirrors, to help pave the way for a more systematic understanding of (0,2) mirror symmetry.

5.1 Dual GLSM

As in chapter 3, we consider deformed tangent bundle defined over Grassmannian $G(k,n)$ defined by matrices A and B. In this section, we assume A to be the identity matrix (one can always find such a basis because A is invertible). Let $\lambda_i, i = 1, \cdots, n$, denote the diagonal entries in the Jordan canonical form of B. In this section, we construct another GLSM which is dual to the original one as A/2 models.

The dual theory is defined as follows. It has $U(1)^k$ gauge symmetry and consists of chiral fields Σ_a and Y_i^a for $a = 1, \cdots, k, i = 1, \cdots, n$ and corresponding Fermi fields with the same indices and charges. The Σ's are neutral under the gauge symmetry, while the charge of Y_i^a under the bth $U(1)$ is δ_a^b. They are subject to vanishing E functions and the following J functions:

$$\begin{aligned} J_a^i &= \Sigma_a + \left(\sum_b \Sigma_b\right) \lambda_i - \exp(-Y_i^a), \\ J^a &= \sum_{i=1}^n Y_i^a - t, \end{aligned} \tag{5.1}$$

where J_a^i, J^a couple to the companion Fermi fields of Y_i^a and Σ_a respectively and t is the FI parameter ($q = \exp(-t)$).

We can reproduce the one-loop effective potential (3.6) by integrating out all the Y fields. Indeed, the equation of motion for the Y fields derived from J_a^i reads

$$Y_i^a = -\log\left(\Sigma_a + \left(\sum_b \Sigma_b\right)\lambda_i\right).$$

Plugging this into J^a, we recover (3.6). However, this doesn't match the correlation functions (5.3) because the theory at hand is Abelian, there are no contributions from off-diagonal fields in the vector multiplet, namely the factor $\prod_{a\neq b}(\sigma_a - \sigma_b)$ in the localization formula. To resolve this discrepancy, one can simply modify the measure of the path integral in the following way

$$\langle \mathcal{O} \rangle = \int [D\Sigma][DY] \prod_{a\neq b}(\Sigma_a - \Sigma_b)\mathcal{O}e^{-S} \tag{5.2}$$

with S the action (the integral over the Fermi fields is implicit). See the appendix of [94] for more discussion on this measure.

5.2 (0,2) mirror dual

Now we construct a (0,2) mirror dual for Grassmannians. In the appendix of [94], Hori and Vafa gave a proposal of the Toda dual theories to A twisted NLSM on Grassmannian manifolds defined by the following superpotential:

$$W = \sum_i \Sigma_i(Q_i^\alpha Y_\alpha - t) + \sum_\alpha e^{-Y_\alpha},$$

where t is the FI parameter and Y_α are the fundamental fields. Take $G(2,4)$ as an example:

$$W = \Sigma_1(Y_1 + Y_2 + Y_3 + Y_4 - t) + \Sigma_2(Y_5 + Y_6 + Y_7 + Y_8 - t) + \sum_{i=1}^8 e^{-Y_i} + \ln(\Sigma_1 - \Sigma_2).$$

After integrating out the Σ's, the effective superpotential becomes:

$$W = -1 - \ln(t - (Y_1 + Y_2 + Y_3 + Y_4)) + \sum_{i=1}^{7} e^{-Y_i} + e^{-(2t - Y_1 - \cdots - Y_7)}.$$

This implies that the vacua are given by

$$\exp(-\tilde{Y}) - \exp(-Y) + \frac{1}{2\tilde{Y} - 2Y} = 0,$$

where

$$\begin{aligned} Y &\equiv Y_1 = Y_2 = Y_3 = Y_4, \\ \tilde{Y} &\equiv Y_5 = Y_6 = Y_7. \end{aligned}$$

However, this equation has infinite numbers of solutions. So this proposal is not suitable for our purpose to construct $(0,2)$ Toda Theory.

Therefore, in this section, instead we will propose a (0,2) mirror dual of the model introduced in section 3.2. For any operator $\mathcal{O}$ of the nonabelian GLSM, the correlation function of the A/2 twisted theory is computed by the localization formula [95]

$$\langle \mathcal{O} \rangle = \sum_{J=0} (\mathcal{O} \prod_{a \neq b} (\sigma_a - \sigma_b) H^{-1}), \tag{5.3}$$

where

$$H = \det_{a,b} \left(\frac{\partial J_a}{\partial \sigma_b} \right) \prod_a \det(M_a),$$

$$M_a = \sigma_a A + (\sum_b \sigma_b) B.$$

Any dual theory should reproduce the chiral ring structure given by this formula.This means the B/2 model of the proposed theory should reproduce the A/2 chiral ring and the correlation function (5.3) of the original theory. For this purpose, note that we can rewrite H in (5.3) as

$$H = \det_{a,b} \left(- \frac{\partial \det(M_a)}{\partial \sigma_b} \right).$$

So we define the dual theory to be a (0,2) Landau-Ginzburg model with chiral fields $\Sigma_a, a = 1, \cdots, k$ and corresponding Fermi fields Λ_a. The J function coupling to Λ_a is

$$\tilde{J}^a = - \det(M_a) + q. \tag{5.4}$$

The constant q is inserted to ensure that the J functions of the two theories have the same zero set. Given an operator defined by a symmetric polynomial in the σ's, the B/2 correlation function of this dual theory is

$$\langle \mathcal{O} \rangle = \sum_{J=0} (\mathcal{O} H^{-1}).$$

Again, to produce (5.3) we need to define the measure of this dual theory to be given by

$$\langle \mathcal{O} \rangle = \int [D\Sigma] \prod_{a \neq b} (\Sigma_a - \Sigma_b) \mathcal{O} e^{-S}. \tag{5.5}$$

Chapter 6

Conclusions

This thesis is composed of several topics concerning chiral rings of two-dimensional field theories with (0,2) supersymmetry. The starting point is the definition and properties we reviewed and explored in Chapter 2. By identifying the Fock vacuum with a section of the line bundle $(\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2}$, the chiral ring, defined to be the RR ground states, is counted by the sheaf cohomology groups $H^n\left(X, (\wedge^m \mathcal{E}) \otimes (\det \mathcal{E})^{-1/2} \otimes K_X^{+1/2}\right)$. When the A/2 pseudo-topological twist is possible, its correlation functions define the quantum sheaf cohomology $H^\bullet(X, \wedge^\bullet \mathcal{E}^*)$. We have shown how to use techniques of homological algebra to derive the classical ring structure for Grassmannians. The quantum sheaf cohomology of Grassmannians was also computed via one-loop effective J-functions on the Coulomb branch. We used the supersymmetric localization formula to test our result.

Chiral rings have various applications in physics and mathematics. In this thesis, we provided two examples. First, we used Bott-Borel-Weil theorem to give an indirect test of the triality proposal. Second, we gave two Toda-like dualities for the nonabelian theory engineering the Grassmannians by matching the chiral rings.

In [101], we also found (0,2) mirror duals for the deformed tangent bundles of Hirzebruch and del Pezzo surfaces. One could in principle find more (0,2) dualities based on the same method. It is unknown whether our method for computing the quantum sheaf cohomology of Grassmannians can be generalized to more complicated nonabelian theories. We should also point out that, for theories without (2,2) locus, concrete results about quantum sheaf cohomology and (0,2) mirror symmetry are still lacking. We leave these problems to future works.

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