THEOREMS OF WIENER-LEVY TYPE
FOR INTEGRAL OPERATORS IN $C_p$

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DEDICATION

This paper is dedicated to Sandy.
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INTRODUCTION

The relationship between the growth behavior of the Fourier coefficients of a function $f(x)$ and the growth behavior of the Fourier coefficients of some analytic function of $f(x)$ is a question which has been under investigation for quite some time. This study has its roots in a 1932 paper of Wiener [26] in which the following lemma appears:

If $f(x)$ is a function with an absolutely convergent Fourier series, which nowhere vanishes for real arguments, then $1/f(x)$ has an absolutely convergent Fourier series.

Subsequently, in 1934, Lévy [16] established a significant generalization of Wiener's lemma which today bears the name "The Wiener-Lévy Theorem".

If $f(x)$ has an absolutely convergent Fourier series and if $W(z)$ is an analytic function whose domain contains the range of $f(x)$, then $W[f(x)]$ also has an absolutely convergent Fourier series.

Perhaps the most general of the modern versions of the Wiener-Lévy Theorem may be found in the 1970 paper of Alpar [2]. This is essentially the result presented earlier in Zygmund [28] as modified by remarks of Ingham [13].
If \( f(x) \) has an absolutely convergent Fourier series and the range of \( f(x) \) lies on a curve \( C \) in the complex plane, and if \( W(z) \) is analytic (but not necessarily single-valued) at each point of \( C \) such that \( W(z) \) returns to its initial determination after \( z \) travels completely around \( C \), then \( W[f(x)] \) has an absolutely convergent Fourier series.

Recently, it has been observed that many results which have previously been established in the context of Fourier series actually possess analogues in the theory of integral operators. As an example, it is well-known (see Zygmund [28]) that a function has an absolutely convergent Fourier series if and only if it is the convolution of two square-integrable functions. The comparable statement for integral operators is that the \( s \)-numbers of an operator are summable if and only if it is the composition of two Hilbert-Schmidt operators (see Cochran [5] or Schatten [23]).

It is this type of analogy in which the \( s \)-numbers of integral operators take the place of the Fourier coefficients of functions which we intend to exploit even further. In particular we will obtain versions of the classical Wiener-Lévy Theorem which are valid in the general setting of integral operator theory.

The organization of this paper is as follows. In Chapter 1 we introduce the definitions and notation necessary for the sequel. Following these preliminaries, we analyze in Chapter 2 a recent attempt by Cochran [7] to generalize the Wiener-Lévy Theorem and
then we present a new and different theorem which actually contains the classical theorem of Wiener and Lévy as a corollary. Examples are given to demonstrate that the new theorem and Cochran's earlier result are distinct.

In Chapter 3 we consider work of Alpar [2] which extends the Wiener-Lévy Theorem to the case of functions for which the $p^{th}$ powers of their Fourier coefficients are absolutely summable for some $1 < p < 2$. We show how analogues of Alpar's principal result can be developed for integral operators and subsequently obtain his main theorem as a corollary of our generalizations.

We study smoothness conditions of the Lipschitz and bounded variation type in Chapter 4 and demonstrate that these are preserved under the action of analytic transformations. We are then able to offer simple proofs both of Cochran's result and of the generalization of Alpar's theorem. In addition, we extend previous work of Bljumin and Kotljar [3] and Cochran [6] which involves the effect of these types of smoothness conditions on the structure of integral operators.

Finally, in Chapter 5, we investigate several topics which are closely related to the Wiener-Lévy Theorem. Firstly, we consider the converse of the Wiener-Lévy Theorem in the integral operator setting and show by example that it is false in its strongest form. We then prove a weakened version of the converse which applies to real-valued functions. In a second section we examine, and generalize to the case of integral operators, a theorem of Marcinkiewicz [18] in which the assumption of analyticity in the
Wiener-Lévy Theorem is weakened to a certain infinite-differentiability condition, while simultaneously the summability conditions on the Fourier coefficients of $f(x)$ are strengthened. The converse of this result is also investigated. In the last portion of our paper we present an important variation of the Wiener-Lévy Theorem established by Krein [15] and pose an open question as to the validity of an analogous result in the general context of integral operator theory. We also present some related topics which are worthy of further investigation.
In this chapter we present relevant background material which will facilitate the reading of this paper and unify the notation which will be used throughout.

1.1 Functions of a Single Real Variable

If \( f(x) \) is a complex-valued function of the real variable \( x \) with domain the closed interval \( [a,b] \), then we say that \( f \in L^P[a,b] \), where \( 0 < p < \infty \), provided that \( \int_a^b |f(x)|^p dx < \infty \). It is well-known that the collection of all functions belonging to \( L^P[a,b] \) is a Banach space with respect to the norm

\[
||f||_p = \left( \int_a^b |f(x)|^p dx \right)^{1/p}
\]

if equality of functions is interpreted in the almost everywhere sense. We will say that \( f \in L^P[a,b] \) is nontrivial provided that \( ||f||_p \neq 0 \). Frequently we use the notation \( f \in L^P \) when the domain of \( f(x) \) is understood.

In addition to the spaces \( L^P \) it is convenient to define the space \( L^\infty \). We say that \( f \in L^\infty[a,b] \) if there exists a function \( g(x) \) defined on \( [a,b] \) such that \( f = g \) almost everywhere and

\[
\sup_{x \in [a,b]} |g(x)| < \infty.
\]

In this case we define

\[
||f||_\infty = \inf_g \left\{ \sup_{x \in [a,b]} |g(x)| \right\}
\]
where the infimum is taken over all $g(x)$ as described above. The collection of all functions in $L^\infty$ is also a Banach space under the norm $\| \cdot \|_\infty$.

For functions $f(x)$ which are defined on $[-\pi, \pi]$ and are $2\pi$-periodic, that is $f(x+2\pi) = f(x)$ for all $x$, we wish to introduce two additional classes of functions. We say that $f \in \Lambda_\alpha$ if there exists a constant $M$ such that

$$|f(x+h) - f(x)| \leq M|h|^\alpha \text{ for all } x \text{ and } h$$

and $f \in \Lambda_\alpha^P$ if there exists a constant $N$ such that

$$\int_{-\pi}^{\pi} |f(x+h) - f(x)|^p \, dx \leq N|h|^{\alpha p}, \text{ for all } h$$

1.2 Functions of a Complex Variable

The term region shall mean an open connected subset of the complex plane. If $W(z)$ is a complex-valued function defined on a region $R$, then $W(z)$ will be said to be analytic in $R$ if it has a derivative at each point of $R$. Analytic functions will be assumed to be single-valued unless otherwise specified.

1.3 Fourier Series

Whenever the words "Fourier series" are used in the subsequent text we shall always mean the classical complex Fourier series as introduced below. We shall only consider Fourier series on a finite interval $[a,b]$, and for convenience, and with no loss of generality, we take $[a,b] = [-\pi, \pi]$.

If $f \in L^1[-\pi, \pi]$ then the classical Fourier coefficients
of \( f(x) \) are defined by the well-known formula

\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx
\]

for each integer \( n \). The series \( \sum_{n=-\infty}^{\infty} c_n e^{inx} \) is said to be the Fourier series for \( f(x) \).

Although, in general, a function \( f(x) \) need not equal its Fourier series, results of this nature may be obtained if \( f(x) \) is given some additional structure. For example, if \( f \in L^2[-\pi,\pi] \) then the Fourier series converges in mean-square to \( f(x) \) in the sense that

\[
\lim_{m \to \infty} \left( \| f(x) - \sum_{n=-m}^{m} c_n e^{inx} \|_2 \right) = 0.
\]

If, in addition, \( f(x) \) is \( 2\pi \)-periodic and possesses the smoothness condition of continuous differentiability (for instance), then

\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}
\]

uniformly on \([-\pi,\pi]\).

For integrable functions it is well-known that their Fourier coefficients tend to zero as \( n \to \pm \infty \). It proves convenient, therefore, to group functions in \( L^1 \) together on the basis of the rate at which this occurs. The classes \( A_p \), \( 0 < p < \infty \), are defined to contain all those functions \( f(x) \) for which \( \sum_{n=-\infty}^{\infty} |c_n|^p < \infty \). By Parseval's theorem it then follows, for example, that \( f \in L^2[-\pi,\pi] \) implies \( f \in A_p \) for all \( p \geq 2 \). For a comprehensive collection of sufficient conditions for \( f(x) \) to be in \( A_p \) with \( p < 2 \) see Cochran [6].
1.4 Functions of Two Real Variables

As for functions of one real variable there are $L^p$ spaces for functions of two real variables. In particular, if $K(x,y)$ is a complex-valued function of two real variables with domain $[a,b] \times [a,b] = \{(x,y) : a \leq x \leq b, a \leq y \leq b\}$ then we say that $K \in L^p([a,b] \times [a,b])$, where $0 < p < \infty$, if

$$\int_a^b \int_a^b |K(x,y)|^p \, dx \, dy < \infty.$$ 

For all such $K(x,y)$ we define

$$||K||_p \equiv \left( \int_a^b \int_a^b |K(x,y)|^p \, dx \, dy \right)^{1/p}$$

and $L^p$ forms a Banach space with respect to this norm if equality of functions is interpreted in the almost everywhere sense. We will say that $K(x,y)$ is nontrivial provided $||K||_p \neq 0$.

Analogously to the case of one variable functions we define the space $L^\infty([a,b] \times [a,b])$ to consist of all functions $K(x,y)$ for which there exists $M(x,y)$ defined on $[a,b] \times [a,b]$ such that $K(x,y) = M(x,y)$ almost everywhere and $\sup_{a \leq x, y \leq b} |M(x,y)| < \infty$. For all such functions $K(x,y)$ we let

$$||K||_\infty \equiv \inf_M \{ \sup_{a \leq x, y \leq b} |M(x,y)| \}$$

where the infimum is over all $M(x,y)$ as described above. With respect to the norm $||\cdot||_\infty$, the space $L^\infty$ is a Banach space.

We now introduce some special classes of two variable functions for which we will have later use. For the remainder of this section $K(x,y)$ will have the particular domain $[-\pi, \pi] \times [-\pi, \pi]$ and will be $2\pi$-periodic as a function of $x$. We say that $K \in \Lambda_a(x)$ if there exists a nonnegative function
A(y) \in L^2[-\pi, \pi] such that

$$|K(x+h, y) - K(x, y)| \leq A(y)|h|^\alpha$$

for all \(x, y,\) and \(h;\)

\(K \in \Lambda^p(x)\) if there exists a nonnegative \(B(y) \in L^2[-\pi, \pi]\) such that

$$\int_{-\pi}^\pi |K(x+h, y) - K(x, y)|^p dx \leq B^p(y)|h|^\alpha$$

for all \(y\) and \(h;\)

\(K \in BV(x)\) if there exists a nonnegative \(C(y) \in L^2[-\pi, \pi]\) such that

$$\sum_{i=1}^n |K(x_i, y) - K(x_{i-1}, y)| \leq C(y)$$

for all \(y\) and all partitions \(-\pi = x_0 < x_1 < \ldots < x_n = \pi\) of \([-\pi, \pi];\)

and \(K \in C^n(x),\) with \(0 \leq n < \infty,\) if \(\frac{\partial^m}{\partial x^m} K(x, y)\) exists as a continuous function of two variables, \(2\pi\)-periodic in \(x,\) for all \(m \leq n.\)

Of course analogous definitions may be made for the classes \(\Lambda^p_{\alpha}(y),\)

\(\Lambda^p(y),\ BV(y),\) and \(C^n(y).\) We also wish to introduce the definition of the \(L^p\) modulus of continuity of \(K(x, y),\) as a function of \(x,\) namely

$$\omega_p(K, \delta) = \sup_{0 < h \leq \delta} \left\{ \int_{-\pi}^\pi |K(x+h, y) - K(x-h, y)|^p dx \right\}^{1/p}$$

Since our primary interest in two variable functions relates to their role as kernels of Fredholm integral operators we now introduce some appropriate background information from the theory of integral equations.

1.5 Integral Equations

Throughout this paper all two variable functions \(K(x, y),\)

when used as kernels of integral operators, will be assumed to be in \(L^2([a, b] \times [a, b]).\) For such kernels the so-called characteristic
value problem for Fredholm integral equations of the second kind
is to find nontrivial complex-valued functions \( \phi(x) \in L^2[a,b] \) and
complex numbers \( \lambda \) such that

\[
\phi(x) = \lambda \int_a^b K(x,y) \phi(y) dy.
\]

Any such associated pairs of \( \lambda \)'s and \( \phi \)'s are called, respectively,
characteristic values (c.v.'s) and characteristic functions (c.f.'s)
for the integral kernel \( K(x,y) \). As a convention we will also call \(+\infty\) a c.v. for \( K(x,y) \) if there exists a nontrivial \( \phi \in L^2[a,b] \) for
which \( \int_a^b K(x,y) \phi(y) dy = 0 \).

In the special case when \( K(x,y) \) is a nontrivial Hermitian
kernel, i.e. \( K(x,y) = \overline{K(y,x)} \), the following are among
the known results (see Cochran [5], Dunford and Schwartz [8],
Gohberg and Krein [9], or Schatten [23], for example):

1) c.v.'s and c.f.'s exist and the c.v.'s are
all real.

2) the c.v.'s form a countable sequence with
no finite limit point, i.e. they may be
subscripted with positive integers and
arranged in the order \( 0 < |\lambda_1| \leq |\lambda_2| \leq \ldots \),
with \( \lim_{n \to \infty} |\lambda_n| = \infty \). Note that if \( K(x,y) \)
has only a finite number of finite c.v.'s
then the rest will be \(+\infty\).

3) c.f.'s corresponding to distinct c.v.'s
are orthogonal with respect to the inner
product \( (f,g) \equiv \int_a^b f(x) \overline{g(x)} dx \).
4) there are only a finite number of linearly independent c.f.'s corresponding to each finite c.v.

Since the set of all c.f.'s corresponding to a fixed c.v. forms a linear space, it is evident from 2), 3), and 4) that an orthonormal set \( \{ \phi_n \}_{n=1}^{\infty} \) may be constructed with the property that each \( \phi_n \) is a c.f. for \( K(x,y) \) and if there exists a function \( \psi \in L^2 \) with \( (\psi, \phi_n) = 0 \) for every \( n \) then \( \psi \) is not a c.f. for \( K(x,y) \).

We shall call such a set \( \{ \phi_n \}_{n=1}^{\infty} \), indexed so that each \( \phi_n \) is associated with its corresponding \( \lambda_n \), a full set of characteristic functions for the kernel \( K(x,y) \). We may then write (see Schatten [23], Smithies [24], or Tricomi [25])

\[
K(x,y) = \sum_{n=1}^{\infty} \lambda_n^{-1} \phi_n(x) \overline{\phi_n(y)}
\]

where the series is mean-square convergent in the sense that

\[
\lim_{m \to \infty} \| K(x,y) - \sum_{n=1}^{m} \lambda_n^{-1} \phi_n(x) \overline{\phi_n(y)} \|_2 = 0.
\]

It should be noted that in the expansion (1) all terms for which \( \lambda_n = \infty \) will of course offer no contribution.

It is now tempting to follow the example of Fourier series and group kernels together according to which values of \( p > 0 \) yield \( \sum_{n=1}^{\infty} |\lambda_n|^{-p} < \infty \). However, if we leave the special class of Hermitian kernels the existence of characteristic values is no longer guaranteed; in fact if \( K(x,y) \) is a Volterra kernel \( (K(x,y) = 0 \) for \( x < y \)) then it is known that \( K(x,y) \) has no characteristic values.
at all (see Hochstadt [12] or Smithies [24]).

In order to develop a special set of numbers which are intrinsic to all kernels, Hermitian or not, the following scheme is used. Construct a new kernel $L(x,y)$ from $K(x,y)$ by the formula

$$L(x,y) = \int_a^b K(x,z)K^*(z,y)dz.$$  

It is easily verified that $L(x,y)$ is Hermitian and therefore $L$ possesses characteristic values $\{\lambda_n\}_{n=1}^\infty$ whenever $K$ is nontrivial. Also readily verifiable is the fact that

$$(L\phi,\phi) = (\int_a^b L(x,y)\phi(y)dy,\phi(x)) \geq 0$$

for all $\phi \in L^2[a,b]$, i.e. $L$ is nonnegative definite. It follows that if $\lambda_n$ and $\phi_n(x)$ are a characteristic pair for $L$, then

$$(\phi_n,\phi_n) = \lambda_n (L\phi_n,\phi_n)$$

and hence $\lambda_n > 0$.

Now define a sequence of positive numbers $\{s_n\}_{n=1}^\infty$ by

$$s_n \equiv \lambda_n^{-1/2} \text{ (by convention if } \lambda_n = \infty \text{ then } s_n = 0).$$

If $\{\phi_n\}_{n=1}^\infty$ is a full set of characteristic functions for $L(x,y)$ then define a new set of functions $\{\psi_n\}_{n=1}^\infty$ by the formula

$$(2) \quad s_n \psi_n(x) \equiv \int_a^b K^*(x,y)\phi_n(y)dy.$$  

Observe that $\{\psi_n\}_{n=1}^\infty$ is also an orthonormal set and that the reciprocal relation

$$(3) \quad s_n \phi_n(x) = \int_a^b K(x,y)\psi_n(y)dy$$

is valid. The $s_n$'s are typically called the $s$-numbers of the original kernel $K(x,y)$ and the two collections $\{\phi_n\}_{n=1}^\infty$ and $\{\psi_n\}_{n=1}^\infty$ a full set.
of singular functions for \( K(x,y) \). For clarification we shall usually write \( s_n = s_n(K) \).

It should be noted at this point that the order in which \( K \) and \( K^* \) appear in the definition of \( L \) is entirely arbitrary; in fact, if we had defined instead \( L'(x,y) \equiv \int_a^b K^*(x,z)K(z,y)dz \), then \( L' \) would have the same set of characteristic values \( \{\lambda_n\}_{n=1}^\infty \) and the set \( \{\psi_n\}_{n=1}^\infty \) would be a full set of characteristic functions for \( L' \). We could then define the set \( \{\phi_n\}_{n=1}^\infty \) via relation (3). In view of the previous remarks we have \( s_n(K^*) = s_n(K) \) for all \( n \).

For the special case in which \( K(x,y) \) is a continuous function of two variables we wish to observe that the full set of singular functions for \( K(x,y) \) may be chosen to consist of continuous functions. This follows from (2) and (3) and the fact that \( \int_a^b K(x,y)g(y)dy \) and \( \int_a^b K^*(x,y)g(y)dy \) are continuous functions for any \( g \in L^2[a,b] \) owing to the uniform continuity of \( K \) and \( K^* \) over \([a,b] \times [a,b]\).

Returning to the general case, once the \( s \)-numbers and the full set of singular functions for \( K(x,y) \) have been defined using either \( L \) or \( L' \) it can be shown that (see Gohberg and Krein [9] or Schatten [23])

\[
(4) \quad K(x,y) = \sum_{n=1}^\infty s_n(K) \phi_n(x) \overline{\psi_n(y)}
\]

with convergence in the mean-square sense. Note that if \( L \) has a finite number \( N \) of finite characteristic values then \( K \) will have only \( N \) nonzero \( s \)-numbers and the expansion (4) may be rewritten

\[
K(x,y) = \sum_{n=1}^N s_n(K) \phi_n(x) \overline{\psi_n(y)}. \text{ Such a kernel is said to be degenerate of rank } N.
\]
Since all nontrivial kernels possess $s$-numbers we are now in a position to complete the operator analogy with the earlier Fourier series classes $A_p$. Following Dunford and Schwartz [8] and Gohberg and Krein [9] the kernel classes $C_p$ are defined to be the collections of all kernels $K$ for which
\[
\sum_{n=1}^{\infty} [s_n(K)]^p < \infty.
\]
For $K \in C_p$ we define
\[
|K|_p = \left\{ \sum_{n=1}^{\infty} [s_n(K)]^p \right\}^{1/p},
\]
If $1 \leq p \leq \infty$ then $|\cdot|_p$ is a norm and $C_p$ is a Banach space (see Dunford and Schwartz [8] and McCarthy [19]). For $0 < p < 1$, $|\cdot|_p$ is no longer a norm but the useful triangle inequality
\[
|K + M|^p_p \leq |K|^p_p + |M|^p_p
\]
implies that $C_p$ is still a linear space with metric $d(K,M) = |K - M|^p_p$.

It is worth noting the important facts that $|K^*_p| = |K|_p$, degenerate kernels are in $C_p$ for all $p > 0$, all kernels in $L^2$ are in $C_p$ for $p \geq 2$, and any kernel in $C_p(p > 0)$ generates a compact integral operator. Indeed, amplifying on these latter observations, if $K(x,y) \in L^2([a,b] \times [a,b])$ serves as the kernel of the integral operator $T$ given by
\[
(T\phi)(x) = \int_a^b K(x,y)\phi(y)dy,
\]
then $T$ is a compact transformation of $L^2[a,b]$ into itself. As is

\[\dagger\text{We will often use the notation } |K|_p \text{ even when } K \text{ is not in } C_p, \text{ in which case } |K|_p = \infty.\]
customary, we will not distinguish between the s-numbers of the operator T and the s-numbers of the inducing kernel K. In particular, we may occasionally say that the operator belongs to a particular class \( C \) when we know the generating kernel \( K \) has this property.

This concludes the introductory material on integral equations but it is appropriate to mention that the above development may also be made using infinite or semi-infinite intervals in place of the finite interval \([a,b]\). In this manuscript, however, we shall only be concerned with the case in which the intervals are finite.
Chapter 2

THE GENERALIZATION OF THE WIENER-LEVY THEOREM

TO INTEGRAL OPERATORS IN $C_p$ -- THE CASE $0 < p \leq 1$

In this chapter we prove a theorem for integral operators in $C_p$ for the case $0 < p \leq 1$ which contains the classical Wiener-Levy Theorem as a corollary. This new theorem is compared and contrasted with the earlier work of Cochran [7].

2.1 The Nature of the Analogy

It is well-known that if $f(x) \in L^2[-\pi,\pi]$ and is $2\pi$-periodic then the difference kernel defined by $K(x,y) \equiv f(x-y)$ has s-numbers \( \{2\pi|c_n|\}_{n=-\infty}^{\infty} \) and a full set of singular functions \( \{(2\pi)^{-1/2}e^{-inx}\}_{n=-\infty}^{\infty} \) and \( \{(2\pi)^{-1/2}e^{-iny}\}_{n=-\infty}^{\infty} \) where \( \{c_n\}_{n=-\infty}^{\infty} \) are the Fourier coefficients for $f(x)$. This fundamental link between Fourier series and integral operators together with the several papers (see Cochran[6]) which contain results for integral operators analogous to known results for Fourier series leads one to suspect that there is a Wiener-Levy type theorem for integral operators, i.e. operators of the form

\[
(T\phi)(x) = \int_{a}^{b} K(x,y)\phi(y)\,dy.
\]

There appear to be at least two natural ways of developing an analogous operator-theoretic result, however, the main distinction being the manner in which the inducing kernels are composed with one another. Dunford and Schwartz [8] consider the general case of compact operators on a Hilbert space with operator composition as
their fundamental operation. In this case, if \( T \) is a compact operator and \( W(z) \) is analytic in some open set \( U \) containing the spectrum of \( T \) then \( W(T) \) is defined by

\[
W(T) = \frac{1}{2\pi i} \int_{\partial U} \frac{W(\lambda)}{\lambda I-T} \, d\lambda
\]

where \( \partial U \) is the boundary of \( U \) and \( I \) is the identity operator. It might be noted that whenever \( W(z) \) has a power series expansion

\[
W(z) = \sum_{i=0}^{\infty} a_i z^i
\]

valid in a neighborhood of the spectrum of \( T \), then the integral representation reduces to simply

\[
W(T) = \sum_{i=0}^{\infty} a_i T^i.
\]

Dunford and Schwartz prove the following analogue of the Wiener-Levy Theorem valid for \( C_1 \) as well as for \( C_p \) for \( p > 1 \).

**Theorem 2.1:** If \( p \geq 1 \), \( T \in C_p \), and \( W(z) \) is analytic on the spectrum of \( T \) with \( W(0) = 0 \), then \( W(T) \in C_p \).

Note that the condition \( W(0) = 0 \) is necessary since compact operators with an infinite spectrum must have zero as an accumulation point of their spectrum and by the spectral mapping theorem (see Dunford and Schwartz [8]) the spectrum of the operator \( W(T) \) is obtained by applying the analytic (and hence continuous) function \( W(z) \) to the spectrum of \( T \).

Further generalizations of Theorem 2.1 have been investigated by Gulick [10] and Wong [27].

An approach alternative to the above is to go inside the integral operator and treat the kernel \( K(x,y) \) as a function of two variables and to define \( W[K(x,y)] \) simply as the composition of two
complex-valued functions using kernel multiplication rather than
kernel composition as the fundamental operation. With this
approach Cochran [7] has established the following:

Theorem 2.2: Let \( W(z) \) be analytic in a region \( R \)
and let \( K(x,y) \) be a continuous kernel which maps
the square \([−\pi, \pi] \times [−\pi, \pi]\) into \( R \) and is \( 2\pi \)-periodic
in \( x \). If \( K(x,y) \) is such that for some \( 1 < p \leq 2 \)

\[
\sum_{n=1}^{\infty} \left( \sum_{m=n}^{\infty} m^{-2/q} \left| \omega_p(K, m^{-1}) \right|_2^2 \right)^{1/2} < \infty
\]

where \( p^{-1} + q^{-1} = 1 \) then \( W[K(x,y)] \in C_1 \). (Recall
the definition of the \( L^p \) modulus of continuity \( \omega_p \)).

Unfortunately, although this theorem is interesting in its
own right, it does not contain the classical Wiener-Lévy Theorem
as a special case. For example, the function \( f(x) = \sum_{n=1}^{\infty} 2^{-n} \cos 2^n x \)
satisfies the hypotheses of the Wiener-Lévy Theorem, but the
difference kernel \( K(x,y) \equiv f(x-y) \) does not satisfy the hypotheses
of Theorem 2.2 (see Section 2.3 for details). It is here then that
our work begins. In the next section we show that under kernel
multiplication a direct analogue of the Wiener-Lévy Theorem is
indeed valid for integral operators.

2.2 The Main Result

Following a few preliminary lemmas we shall establish:

Theorem 2.3: Let \( W(z) \) be analytic in a region \( R \)
and let \( K(x,y) \) map the square \( D \equiv [a,b] \times [a,b] \)
continuously into $R$. If $\{\phi_n\}_{n=1}^\infty$ and $\{\psi_n\}_{n=1}^\infty$ are a full set of continuous singular functions for $K(x,y)$ and

\[ \sum_{n=1}^\infty \left[ s_n(K) \right]^p \left\| \phi_n \right\|_\infty^p \left\| \psi_n \right\|_\infty^p = M < \infty \]

for some $0 < p \leq 1$, then $W[K(x,y)] \in C_p$.

It should be remarked that the classical result of Wiener and Lévy is a corollary of this theorem in the case that $p = 1$ and $K$ is a difference kernel. It might also be observed that the hypotheses of our Theorem 2.3 and the earlier Cochran result are of a distinct nature. In Theorem 2.2 various smoothness conditions are placed upon the kernel $K(x,y)$, treated as a function of two real variables, whereas in Theorem 2.3 the only smoothness assumption is continuity and the main hypotheses concern the behavior of $K(x,y)$ as the kernel of an integral operator. We shall make more comparisons between these two theorems in Section 2.3, but first we prove our preliminary lemmas and the main result, Theorem 2.3.

**Lemma 2.4:** Let $0 < p \leq 1$. If $K(x,y)$ is continuous on $D = [a,b] \times [a,b]$ and $\sum_{n=1}^\infty \left[ s_n(K) \right]^p \left\| \phi_n \right\|_\infty^p \left\| \psi_n \right\|_\infty^p < \infty$

where $\{\phi_n\}_{n=1}^\infty$ and $\{\psi_n\}_{n=1}^\infty$ are a full set of continuous singular functions for $K(x,y)$, then

\[ K(x,y) = \sum_{n=1}^\infty s_n(K) \phi_n(x) \psi_n(y) \]

uniformly in $D$. 
Proof: Note that \( \sum_{n=1}^{\infty} [s_n(K)]^p \| \phi_n \|_\infty^p \| \psi_n \|_\infty^p < \infty \)
implies that \( \sum_{n=1}^{\infty} s_n(K) \| \phi_n \|_\infty \| \psi_n \|_\infty < \infty \).

Then since \( \sum_{n=m}^{\infty} s_n(K) \phi_n(x) \bar{\psi}_n(y) \)
and the latter converges to zero as \( m \to \infty \), \( \sum_{n=1}^{\infty} s_n(K) \phi_n(x) \bar{\psi}_n(y) \) is the uniform limit of continuous functions and hence must be continuous. It follows that \( K(x,y) \) and \( \sum_{n=1}^{\infty} s_n(K) \phi_n(x) \bar{\psi}_n(y) \) are both continuous functions which are equal in the \( L^2 \)-norm (see Section 1.5) and therefore are equal pointwise.

Lemma 2.5: If \( K(x,y) = \sum_{n=0}^{\infty} K_n(x,y) \) where the convergence is in \( L^2 \)-norm, then \( |K|_p^p \leq \sum_{n=0}^{\infty} |K_n|_p^p \)
for \( 0 < p \leq 1 \) and \( |K|_p^p \leq \sum_{n=0}^{\infty} |K_n|_p^p \) for \( 1 < p \leq 2 \).

Proof: We only demonstrate the case \( 0 < p \leq 1 \) as the proof for \( 1 < p \leq 2 \) is obtained by omitting all \( p \) exponents in the following argument. Since the \( L^2 \)-norm dominates the operator norm, we have that \( \sum_{n=0}^{\infty} K_n(x,y) \) also converges to \( K(x,y) \) in operator norm. By Lemma 3.1 of McCarthy [19] and the triangle inequality for \( | \cdot |_p^p \),
\[ |K|^p \leq \liminf_{m \to \infty} \left| \sum_{n=0}^{m} K_n \right|^p \]
\[ \leq \liminf_{m \to \infty} \left( \sum_{n=0}^{m} |K_n|^p \right) \]
\[ = \sum_{n=0}^{\infty} |K_n|^p \]

**Lemma 2.6**: Let \( 0 < p \leq 2 \) and \( K(x,y) \in L^2([a,b] \times [a,b]) \).

Then
\[ |K|^p = \inf \sum_{n=1}^{\infty} \left\| \int_a^b K(x,y) \psi_n(y) \, dy \right\|_2^p \]
where the infimum is taken over all orthonormal bases \( \{ \psi_n \}_{n=1}^{\infty} \) of \( L^2[a,b] \).

**Proof**: See Dunford and Schwartz [8], Gohberg and Krein [9], or McCarthy [19].

**Lemma 2.7**: Let \( 0 < p \leq 2 \) and \( K(x,y) \in L^2([a,b] \times [a,b]) \).

a) If \( f, g \in L^\infty[a,b] \) then
\[ \left| f(x)g(y)K(x,y) \right|^p \leq \left\| f \right\|_\infty \left\| g \right\|_\infty \left| K \right|^p \]

b) If \( f_i, g_i \in L^\infty[a,b] \) for \( 1 \leq i \leq n \) and \( K \in C_p \) then
\[ \sum_{i=1}^{n} f_i(x)g_i(y)K(x,y) \in C_p \]

**Proof**: 

a) Let \( \{ \phi_n \}_{n=1}^{\infty} \) and \( \{ \psi_n \}_{n=1}^{\infty} \) be a full set of singular functions for the kernel \( g(y)K(x,y) \). Then by Lemma 2.6 above
\[ \left| f(x)g(y)K(x,y) \right|^p \leq \sum_{n=1}^{\infty} \left\| f(x) \right\|_{a}^{b} g(y)K(x,y) \psi_n(y) \, dy \right\|_2^p \]
\[= \sum_{n=1}^{\infty} \left| \left| f(x) s_n (gK) \phi_n(x) \right| \right|_2^p \]
\[= \sum_{n=1}^{\infty} [s_n(gK)]^p \left| \left| f(x) \phi_n(x) \right| \right|_2^p \]
\[\leq \left| \left| f \right| \right|_\infty^p \left| \left| g(y) K(x,y) \right| \right|_p^p . \]

Since \(|g(y)K(x,y)|_p = |\overline{g(x)}K^*(x,y)|_p\) the argument may be repeated using a full set of singular functions for \(K^*(x,y)\) to yield the desired result.

b) For \(0 < p \leq 1\) it follows from the triangle inequality for \(|\cdot|^p\) and part a) that
\[
\left| \sum_{i=1}^{n} f_i(x) g_i(y) K(x,y) \right|_p^p \leq \sum_{i=1}^{n} \left| f_i(x) g_i(y) K(x,y) \right|_p^p \leq \sum_{i=1}^{n} \left| f_i \right|_\infty^p \left| g_i \right|_\infty^p \left| K \right|_p^p < \infty.
\]

The proof for \(1 < p \leq 2\) again merely involves removing the \(p\) exponents.

For our final preparatory lemma we establish a result which shows that whether or not a kernel is in \(C_p^p\) for \(0 < p \leq 2\) depends on local considerations. The validity of this result was suggested by earlier work of Cochran [7]; the proof shares much in common with an argument of Bochner [4] in Fourier series.

**Lemma 2.8:** Let \(0 < p \leq 2\) and \(D = [a,b] \times [a,b]\). Suppose that \(K(x,y)\) is defined on \(D\) and for each point \((x_0, y_0) \in D\) there is a neighborhood \(N_0\)
containing \((x_0, y_0)\) and a kernel \(K_0(x, y) \in C_p\) such that \(K_0(x, y) = K(x, y)\) for all \((x, y) \in N_0 \cap D\), then \(K(x, y) \in C_p\).

**Proof:** For each \((x_0, y_0) \in D\) we may choose an open rectangle \(N_0\), of the form \(N_0 = A_0 \times B_0\), containing \((x_0, y_0)\) and a kernel \(K_0(x, y) \in C_p\) such that \(K_0(x, y) = K(x, y)\) for all \((x, y) \in N_0 \cap D\). Since the collection of all these open rectangles covers \(D\) and \(D\) is compact we may choose a finite number of these rectangles, say \(\{N_j = A_j \times B_j\}_{j=1}^m\), to cover \(D\).

Let \(\{K_j(x, y)\}_{j=1}^m\) be the corresponding kernels such that \(K_j(x, y) \in C_p\) and \(K_j(x, y) = K(x, y)\) for all \((x, y) \in N_j \cap D\), \(1 \leq j \leq m\).

Now define \(M_j = N_j / \bigcup_{i=1}^{j-1} N_i\) for \(1 \leq j \leq m\). Since each point \((x, y) \in D\) is contained in exactly one \(M_j\), we may write \(K(x, y) = \sum_{j=1}^m \chi_{M_j}(x, y)K_j(x, y)\) for \((x, y) \in D\) where \(\chi_{M_j}\) is the characteristic function of the set \(M_j\).

To prove \(K(x, y) \in C_p\) it suffices to prove that \(\chi_{M_j}(x, y)K_j(x, y) \in C_p\) for \(1 \leq j \leq m\). By Lemma 2.7(b), however, we need only establish that each \(\chi_{M_j}\) is of the form \(\chi_{M_j}(x, y) = \sum_{i=1}^n f_i(x)g_i(y)\) where \(f_i, g_i \in L^\infty[a, b]\), \(1 \leq i \leq n\).

---

† If \(A\) and \(B\) are sets then we use \(A/B\) to designate the set \(\{ a \in A : a \notin B \}\).
Clearly \( x_{M_1} \) is of this form since \( M_1 = N_1 = A_1 \times B_1 \) and \( x_{M_1}(x,y) = x_{A_1}(x) \times x_{B_1}(y) \). If \( j > 1 \) then \( M_j = \bigcap_{i=1}^{j-1} (N_i/N_i) \)

and

\[
\chi_{M_j}(x,y) = \prod_{i=1}^{j-1} x_{N_i/N_i}(x,y) = \prod_{i=1}^{j-1} [x_{N_i}(x,y) - x_{N_i} \cap N_j(x,y)].
\]

Since \( N_i \cap N_j = (A_i \times B_i) \cap (A_j \times B_j) = (A_i \cap A_j) \times (B_i \cap B_j) \),

we have

\[
\chi_{M_j}(x,y) = \prod_{i=1}^{j-1} [x_{A_i}(x) \times x_{B_i}(y) - x_{A_i} \cap A_j(x) \times x_{B_i} \cap B_j(y)]
\]

which, when expanded, is in the desired form.

We now come to the proof of our main result, a generalized Wiener-Lévy Theorem for integral operators. For ease of reference we restate the theorem.

**Theorem 2.3:** Let \( W(z) \) be analytic in a region \( R \) and let \( K(x,y) \) map the square \( D = [a,b] \times [a,b] \) continuously into \( R \). If \( \{\phi_n\}_{n=1}^{\infty} \) and \( \{\psi_n\}_{n=1}^{\infty} \) are a full set of continuous singular functions for \( K(x,y) \) and

\[
\sum_{n=1}^{\infty} [s_n(K)]^p \|\phi_n\|_\infty^p \|\psi_n\|_\infty^p = M < \infty
\]

for some \( 0 < p \leq 1 \) then \( W[K(x,y)] \in C_p \).

**Proof:** By Lemma 2.8 it suffices to prove that for each point \( (x_0,y_0) \in D \) there exists a neighborhood \( N_0 \) containing \( (x_0,y_0) \) and a kernel \( K_0(x,y) \in C_p \) such that \( K_0(x,y) = W[K(x,y)] \) for all \( (x,y) \in N_0 \cap D \).

Let \( (x_0,y_0) \in D \) and let \( z_0 = K(x_0,y_0) \). Since \( W(z) \) is analytic at \( z_0 \) we may expand \( W(z) \) in a
power series

\[ W(z) = \sum_{m=0}^{\infty} a_m (z-z_0)^m \]

with a radius of convergence \( \rho > 0 \). Choose real \( r \) such that \( 0 < r < \rho \).

For each \( \delta > 0 \) define the truncating functions \( \alpha_\delta(x) \) and \( \beta_\delta(y) \) by:

\[
\begin{align*}
\alpha_\delta(x) &\equiv \begin{cases} 
1, & \text{if } |x-x_0| < \delta \\
0, & \text{otherwise}
\end{cases} \\
\beta_\delta(y) &\equiv \begin{cases} 
1, & \text{if } |y-y_0| < \delta \\
0, & \text{otherwise}
\end{cases}
\end{align*}
\]

Define new kernels \( K_\delta(x,y) \) on \( D \) for each \( \delta > 0 \) by

\[ K_\delta(x,y) = \alpha_\delta(x) \beta_\delta(y) [K(x,y) - K(x_0,y_0)]. \]

Note that since \( K(x,y) \) is continuous at \((x_0,y_0)\), we may choose \( \delta_0 > 0 \) such that \( |K_\delta(x,y)| < r \) for all \((x,y) \in D \) and all \( \delta < \delta_0 \). Thus if

\[ N_\delta \equiv \{ (x,y) : |x-x_0| < \delta \text{ and } |y-y_0| < \delta \} \]

we have

\[ W[K(x,y)] = \sum_{m=0}^{\infty} a_m [K_\delta(x,y)]^m \]

for all \((x,y) \in N_\delta \cap D \) if \( \delta < \delta_0 \).
To establish the theorem we only need to show that

\[ K_0(x, y) = \sum_{m=0}^{\infty} a_m [K_\delta(x, y)]^m \]

is in \( C_p \) for sufficiently small \( \delta \). Towards that end we will first establish

\((*)\) For all \( m \geq 1 \), \( [K_\delta(x, y)]^m \in C_p \) and

\[ |K_\delta^m|_p \leq r^{(m-1)}|K_\delta|_p \]

for all \( \delta < \delta_1 \) where \( \delta_1 \) is independent of \( m \).

**Proof of \( (*) \):** The proof is by induction on \( m \).
To prove that \( K_\delta(x, y) \in C_p \) it suffices to prove that \( K(x, y) \in C_p \) since then \([K(x, y) - K(x_0, y_0)] \in C_p \)
(recall that \( K(x_0, y_0) \) is degenerate and \( C_p \) is a linear space) and consequently \( K_\delta(x, y) \in C_p \) by Lemma 2.7(a). Note that for \( 1 \leq n < \infty \) we may write

\[ 1 = \left\| \phi_n \right\|_2 = \left\{ \int_a^b |\phi_n(x)|^2 \, dx \right\}^{1/2} \leq \left\| \phi_n \right\|_\infty (b-a)^{1/2} \]

and analogously

\[ 1 \leq \left\| \psi_n \right\|_\infty (b-a)^{1/2} \]

Therefore,
$$|K|^p = \sum_{n=1}^{\infty} [s_n(K)]^p$$

$$\leq (b-a)^p \times \sum_{n=1}^{\infty} [s_n(K)]^p \left| \phi_n \right|_\infty^p \left| \psi_n \right|_\infty^p$$

$$< \infty$$

and thus $K(x,y) \in C_p$. The second half of (*) is clearly satisfied for $m = 1$.

Assume now that (*) holds for all $m < j$.

By Lemma 2.4

$$K(x,y) = \sum_{n=1}^{\infty} s_n(K) \phi_n(x) \psi_n(y)$$

and hence using Lemma 2.5 and making other obvious estimates, we find

$$\left| K_0^j \right|^p = \left| \alpha_\delta(x) \beta_\delta(y) \{ K(x,y) - K(x_0, y_0) \} [K_0(x,y)]^{j-1} \right|^p$$

$$= \left| \alpha_\delta(x) \beta_\delta(y) \left\{ \sum_{n=1}^{\infty} s_n(K) \{ \phi_n(x) \psi_n(y) - \phi_n(x_0) \psi_n(y_0) \} \right\} \right|^p$$

$$\times [K_0(x,y)]^{j-1}$$

$$\leq \sum_{n=1}^{\infty} [s_n(K)]^p \left| \alpha_\delta(x) \beta_\delta(y) \{ \phi_n(x) \psi_n(y) - \phi_n(x_0) \psi_n(y_0) \} \right|^p$$

$$\times [K_0(x,y)]^{j-1}$$

$$\leq \sum_{n=1}^{\infty} [s_n(K)]^p \left\{ \{ \alpha_\delta(x) \phi_n(x) \} \beta_\delta(y) \{ \psi_n(y) - \psi_n(y_0) \} \right\}^p$$

$$+ \alpha_\delta(x) \{\phi_n(x) - \phi_n(x_0)\} \beta_\delta(y) \{ \psi_n(y_0) \} [K_0(x,y)]^{j-1}$$
Invoking Lemma 2.7(a) we see that the right-hand side of the above inequality is dominated by:

\[
\sum_{n=1}^{\infty} [s_n(K)]^p c_n |K_\delta^{j-1}|^p
\]

where

\[
c_n \equiv ||\alpha_\delta(x)\phi_n(x)||^p \times ||\beta_\delta(y)(\psi_n(y) - \psi_n(y_0))||^p
\]

\[
+ ||\alpha_\delta(x)(\phi_n(x) - \phi_n(x_0))||^p \times ||\beta_\delta(y)\psi_n(y_0)\psi_n(y)||^p
\]

Note that rather coarsely, \(c_n \leq 4||\phi_n||^p ||\psi_n||^p\).

In view of (1) there thus exists \(N > 1\) such that

\[
\sum_{n=N}^{\infty} [s_n(K)]^p c_n < \epsilon
\]

for arbitrary positive \(\epsilon\). Now choose \(\delta_1 > 0\) so that

\(\phi_n(x) - \phi_n(x_0)\ |^p < \epsilon\) and \(\psi_n(y) - \psi_n(y_0)\ |^p < \epsilon\) for all \(n < N\) whenever \((x,y) \in N_\delta \cap D\) and \(\delta < \delta_1\) which we can do owing to the continuity of the singular functions. Then for \(\delta < \delta_1\), (4) is itself majorized by

\[
\left\{ \sum_{n=1}^{N-1} [s_n(K)]^p c_n + \epsilon \right\} |K_\delta^{j-1}|^p
\]
\[ \leq \epsilon \left\{ \varepsilon \sum_{n=1}^{N-1} [s_n(K)]^P \{ ||\phi_n||_P^p + ||\psi_n||_P^p \} + \varepsilon \right\} |K_0^{j-1}|_P^p \]
\[ \leq \varepsilon \left\{ 2(b-a)^{P/2} \sum_{n=1}^{N-1} [s_n(K)]^P \{ ||\phi_n||_P^p + ||\psi_n||_P^p \} + 1 \right\} |K_0^{j-1}|_P^p \]
\[ \leq \varepsilon \{ 2(b-a)^{P/2} M + 1 \} |K_0^{j-1}|_P^p \]

where we have employed the estimates (2) and (3) and also relation (1).

Since \( \varepsilon \) is arbitrary the above argument holds for \( \varepsilon = r^P/(2(b-a)^{P/2} M + 1) \) in which case we have \( |K_0^{j}|_P \leq r|K_0^{j-1}|_P \) for all \( \delta < \delta_1 \). Note that \( \delta_1 \) is independent of \( j \) so we may apply the inductive hypothesis to yield that \( |K_0^{j}|_P \leq r^{(j-1)} |K_0^{\delta}|_P \) for all \( \delta < \delta_1 \). This completes the proof of (*).

Returning now to \( K_0(x,y) = \sum_{m=0}^{\infty} a_m [K_0(x,y)]^m \) we deduce from Lemmas 2.5, 2.7(a), and (*) that for \( \delta < \delta_1 \),
\[ |K_0|_P = \sum_{m=0}^{\infty} |a_m K_0^m|_P \leq \sum_{m=0}^{\infty} |a_m|_P |K_0^m|_P \]
\[ \leq |a_0|_P |K_0^0|_P + \sum_{m=1}^{\infty} |a_m|_P r^{(m-1)}p |K_0^\delta|_P \]
\[ = |a_0|_P (b-a)^P + r^{-P} |K_0^\delta|_P \sum_{m=1}^{\infty} |a_m|_P (r^P)^m \]

Since \( \rho \) is the radius of convergence of the power series \( \sum_{m=1}^{\infty} a_m z^m \), \( \rho^P \) is the radius of convergence of the series \( \sum_{m=1}^{\infty} |a_m|^P z^m \). Thus since \( r < \rho \), the last expression is
finite (recall that $K_0 \in C_p$), from which it follows that $K_0(x,y) \in C_p$ for $\delta < \delta_1$. Taking $\delta < \min(\delta_0, \delta_1)$, our proof is complete.

An immediate consequence of this main result is:

**Corollary 2.9:** Let $W(z)$ be analytic in a region $R$ and let $K(x,y)$ map the square $[a,b] \times [a,b]$ continuously into $R$. If $K(x,y)$ is in $C_1$ and a full set of continuous singular functions for $K(x,y)$ is uniformly bounded, then $W[K(x,y)]$ is in $C_1$ also.

With this result in hand, it is not difficult to give an alternative proof of the classical Wiener-Lévy Theorem:

**Theorem 2.10 (Wiener-Lévy Theorem):** Let $W(z)$ be analytic in a region $R$ and let $f(x) \in L^2[-\pi, \pi]$ map $[-\pi, \pi]$ into $R$. If $f(x)$ has an absolutely convergent Fourier series then so does $W[f(x)]$.

**Proof:** Since $\sum_{n=-\infty}^{\infty} |a_n| < \infty$, where $\{a_n\}_{n=-\infty}^{\infty}$ are the Fourier coefficients for $f(x)$, the function $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ is necessarily continuous and $2\pi$-periodic. Recalling that $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$ in $L^2$-norm, and hence almost everywhere, we may assume without loss of generality that $f(x)$ is also continuous and $2\pi$-periodic.

To prove the theorem define a difference kernel
\( K(x, y) \equiv f(x-y) \) on \([-\pi, \pi] \times [-\pi, \pi]. \) As previously observed \( K(x, y) \) will have \( s \)-numbers \( \{2\pi|a_n|\}_{n=-\infty}^{\infty} \)

and a full set of singular functions \( \{(2\pi)^{-1/2} e^{inx}\}_{n=-\infty}^{\infty} \) and \( \{(2\pi)^{-1/2} e^{iny}\}_{n=-\infty}^{\infty} \). Clearly, then, \( K(x, y) \) satisfies the hypotheses of Corollary 2.9 and thus \( W[K(x, y)] \) is in \( C_1 \). But, since \( W[K(x, y)] = W[f(x-y)] \) which is a difference kernel, the \( s \)-numbers for \( W[K(x, y)] \) are \( \{2\pi|b_n|\}_{n=-\infty}^{\infty} \) where \( \{b_n\}_{n=-\infty}^{\infty} \) are the Fourier coefficients of \( W[f(x)] \). Therefore \( W[f(x)] \) has an absolutely convergent Fourier series.

2.3 Analysis of the Main Result

As promised earlier we now return to a comparison of Theorems 2.2 and 2.3 (the case \( p = 1 \)). The essential conclusion is that these are distinct results, i.e. neither one is a corollary of the other. To establish this fact it suffices to show that neither one of the following two hypotheses is a special case of the other.

**Hypothesis A:** \( K(x, y) \) is continuous on \([-\pi, \pi] \times [-\pi, \pi] \) and \( \sum_{n=1}^{\infty} s_n(K) \left\| \phi_n \right\|_\infty \left\| \psi_n \right\|_\infty < \infty \) where \( \{\phi_n\}_{n=1}^{\infty} \) and \( \{\psi_n\}_{n=1}^{\infty} \) are a full set of continuous singular functions for \( K(x, y) \).

**Hypothesis B:**

i) \( K(x, y) \) is continuous on \([-\pi, \pi] \times [-\pi, \pi] \) and \( 2\pi \)-periodic as a function of \( x; \)
Our demonstration is in two parts.

Example of \( K(x,y) \) satisfying A but not B:

Let \( f(x) = \sum_{j=1}^{\infty} 2^{-j} \cos 2^j x \) and define a kernel \( K(x,y) \) by \( K(x,y) \equiv f(x-y) \). Then \( K(x,y) \) is continuous on \([\pi, \pi] \times [\pi, \pi]\) since it is the uniform limit of continuous functions. Also note that \( K(x,y) \), being a difference kernel created from a \( 2\pi \)-periodic function of one variable, has a full set of singular functions \((2\pi)^{-1/2} \sin x\) \( n = -\infty \) and \((2\pi)^{-1/2} \sin y\) \( n = -\infty \) and s-numbers \( (2\pi) c_n \) \( n = -\infty \) where \( c_n \) \( n = -\infty \) are the classical Fourier coefficients of the original function \( f(x) \). Since

\[
c_n = \begin{cases} 
2^{-(j+1)} & \text{if } n = 2^j, j = 1,2, \ldots, \\
0 & \text{otherwise,}
\end{cases}
\]

it is clear that Hypothesis A is satisfied.

To show that the kernel \( K(x,y) \) defined above does not satisfy Hypothesis B we follow an approach suggested by work of Lorentz [17]. Let \( h \equiv \pi 2^{-(k^2+1)} \) for some \( k > 0 \) and \( K_j(x,y) \equiv 2^{-j} \cos 2^j (x-y) \). Then

\[
\Lambda_j \equiv \left\{ \int_{-\pi}^{\pi} |K_j(x+h,y) - K_j(x-h,y)|^p \, dx \right\}^{1/p}
\]
\[
= \left\{ \int_{-\pi}^{\pi} \left| 2^{-j} \cos 2^j (x-y+h) - 2^{-j} \cos 2^j (x-y-h) \right|^p dx \right\}^{1/p}
\]
\[
= 2^{1-j} \left\{ \int_{-\pi}^{\pi} \left| \sin 2^j (x-y) \sin 2^j h \right|^p dx \right\}^{1/p}
\]
\[
= 2^{1-j} \left| \sin 2^j h \right| \left\{ \int_{-\pi}^{\pi} \left| \sin 2^j (x-y) \right|^p dx \right\}^{1/p}
\]
\[
= A 2^{-j} \left| \sin 2^j h \right|
\]
\[
= A 2^{-j} \left| \sin 2^j 2^{-k^2} (\pi/2) \right|
\]

in view of the choice of \( h \), where \( A \equiv 2 \| \sin x \|_p \).

Note that

\[
\Delta_j = \begin{cases} 
0 & \text{if } j > k \\
A 2^{-k} & \text{if } j = k \\
\leq A \pi 2^j 2^{-k^2} 2^{-2} & \text{if } 1 \leq j < k
\end{cases}
\]

where the last inequality is a consequence of the well-known result that \( \sin \theta \leq \theta \) for \( \theta \geq 0 \).

Now, since \( |B| \leq |B+C| + |C| \) for arbitrary constants \( B \) and \( C \), employing Minkowski's inequality in very crude fashion and making the obvious estimates leads to

\[
\Delta \equiv \left\{ \int_{-\pi}^{\pi} \left| K(x+h,y) - K(x-h,y) \right|^p dx \right\}^{1/p}
\]
\[
\geq \Delta_k - \sum_{j \neq k} \Delta_j
\]
\[
= \Delta_k - \sum_{j=1}^{k-1} \Delta_j
\]
\[ \geq A_2^{-k} - A_2 \sum_{j=1}^{k-1} 2^j \]
\[ \geq A_2^{-k} - A_2 \sum_{j=1}^{k-2} \{k_2(k-1)^2\} \]
\[ = A_2^{-k} \left( 1 - k_2(k+1) \right) \]
\[ \geq A_2^{-(k+1)} \text{ if } k \geq 4. \]

Then, \( w_p(K,m^{-1}) = \sup_{0<h<m^{-1}} \Delta \geq A_2^{-(k+1)} \) whenever \( k > \max \left\{ \sqrt{\log_2 \pi m - 1}, 4 \right\} \). Therefore,

if say \( k = \left[ \frac{1}{2} \log_2 m + 4 \right] \), \(|\omega_p(K,m^{-1})| \geq \pi A_2^{-(2k+1)} \geq A^2/256m \) and then

\[ \sum_{m=n}^{\infty} m^{-2/q} \left| \omega_p(K,m^{-1}) \right|^2 \geq (A^2/256) \sum_{m=n}^{\infty} m^{-(2/q + 1)} \]
\[ \geq (A^2/256) \sum_{m=n}^{\infty} m^{-2} \text{ (since } q \geq 2) \]
\[ \geq (A^2/256) \int_n^{\infty} x^{-2} dx \]
\[ = \frac{A^2}{256n}. \]

Consequently, \( \sum_{n=1}^{\infty} \left[ n^{-1} \sum_{m=n}^{\infty} m^{-2/q} \left| \omega_p(K,m^{-1}) \right|^2 \right]^{1/2} \geq (A/16) \sum_{n=1}^{\infty} n^{-1} = \infty \) which proves that Hypothesis B is not satisfied.

Example of \( K(x,y) \) satisfying B but not A:

First we establish the following simple, but helpful result.
Lemma 2.11: Let \( \{ f_n(x) \}_{n=1}^m \) be a set of continuous functions on \([a,b]\), then there is a continuous function \( g(x) \) on \([a,b]\) such that \( (g,f_n) = 0 \) for \( 1 \leq n \leq m \), \( g(a) = g(b) = 0 \), and \( \|g\|_2 = 1 \).

Proof: Since the set \( \{ (x-a)(x-b)f_n(x) \}_{n=1}^m \) is a finite collection of continuous functions, we may choose a continuous function \( p(x) \neq 0 \) such that \( (p(x), (x-a)(x-b)f_n(x)) = 0 \) for \( 1 \leq n \leq m \). Then \( (x-a)(x-b)p(x), f_n(x) = 0 \) for \( 1 \leq n \leq m \) and the desired function \( g(x) \) is given by

\[
g(x) = \frac{(x-a)(x-b)p(x)}{\| (x-a)(x-b)p(x) \|_2}
\]

We are now ready to construct an appropriate \( K(x,y) \) defined on \([-\pi,\pi] \times [-\pi,\pi]\). We begin by defining a sequence of continuous functions \( C_n(x) \) on \([0,1]\) inductively as follows:

Take \( C_1(x) \equiv 1 \) and assume \( C_n(x) \) is defined and continuous on \([0,1]\) for all \( n < m \). Write \( m \) in the form \( m = 1 + \sum_{i=0}^{k-1} 2^i + j \) where \( 1 \leq j \leq 2^k \) and let \([a,b] = [ (j-1)/2^k , j/2^k ] \). By Lemma 2.11 there exists a continuous function \( g(x) \) such that \( \int_a^b g(x)C_n(x)dx = 0 \) for all \( n < m \), \( g(a) = g(b) = 0 \), and \( \int_a^b |g(x)|^2dx = 1 \). Therefore if we let
\[ C_m(x) = \begin{cases} g(x) & \text{if } x \in [a,b] \\ 0 & \text{if } x \notin [a,b] \end{cases} \]

then \( C_m(x) \) is continuous on \([0,1]\), \( \int_0^1 C_m(x)C_n(x)dx = 0 \) for \( n < m \), and \( \int_0^1 |C_m(x)|^2dx = 1 \). Thus, the collection \( \{C_n(x)\}_{n=1}^\infty \) is an orthonormal set of continuous functions on \([0,1]\).

For the kernel \( K(x,y) \) we take
\[
K(x,y) = \sum_{n=2}^{\infty} \frac{\sin nx D_n(y)}{n \log n \|D_n\|_\infty}
\]
on \([-\pi,\pi] \times [-\pi,\pi]\) where \( D_n(y) \equiv C_n[(y+\pi)/2\pi] \).

Note that \( K(x,y) \) does not satisfy Hypothesis A since \( \phi_n(x) = \pi^{-1/2} \sin nx, \psi_n(y) = (2\pi)^{-1/2} D_n(y) \),
\[
s = \frac{\pi^{1/2}}{n \log n \|D_n\|_\infty} \quad \text{and hence} \quad \sum_{n=2}^{\infty} s_n K \|\phi_n\|_\infty \|\psi_n\|_\infty = \sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty.
\]

The kernel \( K(x,y) \) we have constructed above does satisfy Hypothesis B, however. Clearly it is \( 2\pi \)-periodic in \( x \). To prove that it is continuous let \( k \geq 0 \) be fixed.

\[ ^\dagger \text{The method of construction of the set } \{C_n(x)\}_{n=1}^\infty \text{ was prompted by the procedure used to form the Haar orthogonal system (see Alexits [1] or Haar [11]). However, since the Haar functions are discontinuous, considerable modification of the procedure was necessary to obtain functions which have the desired properties.} \]
and consider all integers $n_j$ of the form

$$n_j = 1 + \sum_{i=0}^{k-1} 2^i + j \text{ where } 1 \leq j \leq 2^k.$$ 

If $y \in [-\pi, \pi]$, then by the construction of

$$\{C_n(x)\}_{n=1}^\infty,$$

there is at most one value of $j$, $1 \leq j \leq 2^k$, for which $D_{n_j}(y) \neq 0$. Therefore if $(x,y) \in [-\pi, \pi] \times [-\pi, \pi]$ and $N \geq 4$ with

$$N = 1 + \sum_{i=0}^{k'-1} 2^i + j \text{ and } 1 \leq j \leq 2^{k'},$$

then

$$\sum_{n=N}^{\infty} \frac{\sin nx D_n(y)}{n \log n |D_n|_\infty} \leq \sum_{n=N}^{\infty} \frac{|D_n(y)|}{n |D_n|_\infty} \leq \sum_{k=k'}^{\infty} 2^{-(2k-1)}$$

which may be made arbitrarily small for large enough $N$. Thus $K(x,y)$ is the uniform limit of continuous functions and hence is itself continuous.

To verify that $K(x,y)$ also satisfies part ii) of Hypothesis B we only need show that $\frac{\partial}{\partial x} K(x,y)$ exists and is continuous since then $\omega_p(K,m^{-1}) \leq 2M(2\pi)^{1/p}/m$ where $M \equiv \left| \frac{\partial K}{\partial x} \right|_\infty$ and therefore

$$\sum_{n=1}^{\infty} \left( n^{-1} \sum_{m=n}^{\infty} m^{-2/q} \left| |w_p(K,m^{-1})| \right|^2 \right)^{1/2} \leq (2\pi)^{3/2} (2M) \sum_{n=1}^{\infty} \left( n^{-1} \sum_{m=n}^{\infty} m^{-(2/q + 2)} \right)^{1/2}$$

which is finite. To our final purpose, then, note
that for $N$ and $k'$ chosen as above

$$
\sum_{n=N}^{\infty} \left| \frac{\partial}{\partial x} \left\{ \frac{\sin nx D_n(y)}{n \log n \|D_n\|_\infty} \right\} \right| = \sum_{n=N}^{\infty} \left| \frac{\cos nx D_n(y)}{\log n \|D_n\|_\infty} \right|
$$

\leq \sum_{n=N}^{\infty} \frac{|D_n(y)|}{\log n \|D_n\|_\infty}

\leq \sum_{k=k'}^{\infty} \left( \log 2^{2^{k-1}} \right)^{-1}

which may be made arbitrarily small for large enough $N$.

The series of derivatives for $K(x,y)$, therefore, converges uniformly so

$$\frac{\partial}{\partial x} K(x,y) = \sum_{n=2}^{\infty} \frac{\cos nx D_n(y)}{\log n \|D_n\|_\infty},$$

which is continuous, and hence our demonstration is complete.
In Chapter 2 we extended the classical Wiener-Levy Theorem to integral operators generated by kernels in $C_p$ for $0 < p < 1$. We now wish to study the case $1 < p < 2$ with particular regard to generalizing work of Alpar [2].

3.1 Preliminaries to a Generalization of Alpar's Result

In his 1970 paper, Alpar [2] proved the following result for Fourier series which is an analogue to the classical theorem of Wiener and Lévy for the case $1 < p < 2$.

**Theorem 3.1:** Let $W(z)$ be analytic (but not necessarily single-valued) in a region $R$ and let $f(x)$ be a $2\pi$-periodic function mapping $[-\pi, \pi]$ continuously into $R$. If $f \in \Lambda_{\alpha}^1$ for some $p^{-1} < \alpha \leq 1$, $1 < p < 2$, and the range of $f(x)$ lies on a curve $C$ such that $W(z)$ returns to its initial determination after $z$ travels completely around $C$, then $W(f) \in A_p$.

We might note that the condition "$W(z)$ returns to its initial determination after $z$ travels completely around $C$" was introduced by Ingham [13] in order to correct the version of the Wiener-Lévy Theorem given in Zygmund [28]. Its essential purpose is to
guarantee the $2\pi$-periodicity of $W(f(x))$ when $W(z)$ is multi-valued.

A result analogous to Theorem 3.1 can be established for integral operators, indeed we shall prove the following:

**Theorem 3.2:** Let $W(z)$ be analytic (but not necessarily single-valued) in a region $R$ and let $K(x,y)$ satisfy the following conditions:

1. $K(x,y)$ maps the square $D \equiv [-\pi, \pi] \times [-\pi, \pi]$ continuously into $R$.
2. $K(x,y)$ is $2\pi$-periodic in $x$.
3. $K(x,y) \in \Lambda^1_\alpha(x)$ for some $1 < p < 2$.

Then, if $W(z)$ returns to its initial determination after $z$ travels completely around any curve $C_y$ of the form $z = K(x,y), -\pi \leq x \leq \pi$, then $W[K(x,y)] \in C_p$.

Observe that, as in Theorem 3.1, the condition on the curve $C_y$ assures that $W[K(x,y)]$ is a $2\pi$-periodic function of $x$.

As usual a few preliminary results need to be developed before the proof of Theorem 3.2 is presented.

**Lemma 3.3:** Let $K(x,y)$ and $L(x,y)$ be defined on $[-\pi, \pi] \times [-\pi, \pi]$. If $K(x,y) \in C^2(x)$, (recall the definition from Chapter 1), and $1 < p < 2$, then

$$
|K(x,y)L(x,y)|_p \leq \left\{ ||K||_\infty + 4 \left| \frac{\partial^2 K}{\partial x^2} \right|_\infty \right\} |L|_p
$$

**Proof:** Since $K \in C^2(x)$ it can be expanded, as a function of $x$, in a uniformly convergent Fourier
series, and therefore

\[ K(x,y) = \sum_{m=-\infty}^{\infty} c_m(y) e^{imx} \]

for all \((x,y) \in [-\pi, \pi] \times [-\pi, \pi]\). Then, using Lemmas 2.5 and 2.7(a),

\[ |K(x,y)L(x,y)|_p = \left| \sum_{m=-\infty}^{\infty} c_m(y) e^{imx} L(x,y) \right|_p \]

\[ \leq \sum_{m=-\infty}^{\infty} |c_m(y) e^{imx} L(x,y)|_p \]

\[ \leq \sum_{m=-\infty}^{\infty} |c_m|_\infty |L|_p \]

\[ = |L|_p \sum_{m=-\infty}^{\infty} |c_m|_\infty \]

We now wish to estimate the size of \(||c_m||_\infty\) for each \(m\).

For \(m = 0\) we have \(|c_0(y)| = (2\pi)^{-1} \int_{-\pi}^{\pi} K(x,y) \, dx \leq ||K||_\infty\), and for \(m \neq 0\),

\[ |c_m(y)| = (2\pi)^{-1} \int_{-\pi}^{\pi} K(x,y) e^{-imx} \, dx \]

\[ = (2\pi^{-2})^{-1} \int_{-\pi}^{\pi} \left( \frac{\partial^2 K}{\partial x^2} K(x,y) \right) e^{-imx} \, dx \]

\[ \leq \frac{m^{-2} \left| \frac{\partial^2 K}{\partial x^2} \right|_\infty}{\int_{-\pi}^{\pi} \left( \frac{\partial^2 K}{\partial x^2} \right) e^{-imx} \, dx} \]

where the second equality is obtained by integration by parts using the periodicity of \(K\) and \(\frac{\partial K}{\partial x}\). It follows, therefore, that
Lemma 3.4: If $W(z)$ is twice continuously differentiable and $K(x,y) \in C^2(x)$, then

$$
\left| \frac{\partial^2 W[K(x,y)]}{\partial x^2} \right|_\infty \leq A_1 \left| W'[K(x,y)] \right|_\infty + A_2 \left| W''[K(x,y)] \right|_\infty
$$

where $A_1$ and $A_2$ are constants depending only on $K$.

Proof: By the chain rule,

$$
\frac{\partial^2 W[K(x,y)]}{\partial x^2} = \left( \frac{\partial^2 K}{\partial x^2} \right) W'[K(x,y)] + \left( \frac{\partial K}{\partial x} \right)^2 W''[K(x,y)],
$$

and thus, using obvious estimates,

$$
\left| \frac{\partial^2 W[K(x,y)]}{\partial x^2} \right|_\infty \leq \left| \frac{\partial^2 K}{\partial x^2} \right|_\infty \left| W'[K(x,y)] \right|_\infty + \left| \frac{\partial K}{\partial x} \right|^2 \left| W''[K(x,y)] \right|_\infty.
$$

Choosing $A_1 \equiv \left| \frac{\partial^2 K}{\partial x^2} \right|_\infty$ and $A_2 \equiv \left| \frac{\partial K}{\partial x} \right|^2 \left| W''[K(x,y)] \right|_\infty$ completes the proof.

Lemma 3.5: If $K(x,y)$, $2\pi$-periodic in $x$, is continuous on $[-\pi,\pi] \times [-\pi,\pi]$ and also in $\Lambda_{\alpha}^{-1}(x)$, where $p^{-1} < \alpha \leq 1$, $1 < p < 2$, then

$$
\left| K^n \right|_p \leq B \left| K \right|_\infty^{n-1}
$$
for all $n \geq 1$, where $B$ is some constant independent of $n$.

Proof: Since $K^*(x,y) \equiv K(y,x)$, it is immediate that $K^*(x,y)$ is $2\pi$-periodic in $y$ and a member of $\Lambda_{\alpha}^1(y)$.

For $m \neq 0$, it follows from a change of variable, then, that

$$\int_{-\pi}^{\pi} [K^*(x,y)]^n e^{imy} dy = -\int_{-\pi}^{\pi} [K^*(x,y + \pi/m)]^n e^{imy} dy$$

and hence

$$\int_{-\pi}^{\pi} [K^*(x,y)]^n e^{imy} dy =$$

$$-\frac{1}{2} \int_{-\pi}^{\pi} \left\{ [K^*(x,y + \pi/m)]^n - [K^*(x,y)]^n \right\} e^{imy} dy.$$

Therefore, using some obvious estimates and the fact that $K^* \in \Lambda_{\alpha}^1(y)$, we have

$$\left| \int_{-\pi}^{\pi} [K^*(x,y)]^n e^{imy} dy \right|$$

$$\leq \frac{1}{2} \int_{-\pi}^{\pi} \left| [K^*(x,y + \pi/m)]^n - [K^*(x,y)]^n \right| dy$$

$$\leq \frac{1}{2} \int_{-\pi}^{\pi} \left| K^*(x,y + \pi/m) - K^*(x,y) \right|$$

$$\times \left\{ \sum_{j=1}^{n} \left| [K^*(x,y + \pi/m)]^{n-j} [K^*(x,y)]^{j-1} \right| \right\} dy$$

$$\leq \frac{1}{2} n \left| K \right|_{n-1}^{\infty} \int_{-\pi}^{\pi} \left| K^*(x,y + \pi/m) - K^*(x,y) \right| dy$$

$$\leq \frac{1}{2} n \left| K \right|_{n-1}^{\infty} (\pi/|m|)^{\alpha} A(x)$$

for some nonnegative $A(x) \in L^2[-\pi,\pi]$.

By virtue of Lemma 2.6, the two equalities $|K|_p = |K^*|_p$ and $(K^n)^* = (K^*)^n$, and the inequalities derived above,
\[
|K^n_p|^p \leq \sum_{m=-\infty}^{\infty} \left| \int_{-\pi}^{\pi} [K^*(x,y)]^n (2\pi)^{-1/2} e^{imy} dy \right|^p_2
\]

\[
\leq \left\{ 2\pi |K|_\infty^n \right\}^p + 2\left\{ \frac{\pi^\alpha |A|_2}{2\sqrt{2\pi}} n |K|_\infty^{n-1} \right\}^p \sum_{m=1}^{\infty} m^{-\alpha p}
\]

Since \(\alpha p > 1\), we have finally that \( |K^n_p| \leq B \ n \ |K|_\infty^{n-1} \)
for all \(n \geq 1\), with \(B\) independent of \(n\), which completes the demonstration.

**Lemma 3.6:** Let \(\epsilon > 0\). If \(K(x,y)\) is continuous on
\[D = [-\pi, \pi] \times [-\pi, \pi]\]
and is \(2\pi\)-periodic in \(x\), then there exists a function \(Q(x,y)\) such that

a) \(Q(x,y)\) is continuous on \(D\),

b) \(Q(x,y)\) is \(2\pi\)-periodic in \(x\),

c) \(Q(x,y) \in C^\infty(x)\),

d) \(||K - Q||_\infty < \epsilon\).

**Proof:** Let \(C \equiv \{ (e^{ix}, y) : (x, y) \in D \} \) and define a function \(K_1\) on \(C\) by \(K_1(e^{ix}, y) \equiv K(x,y)\). Note that \(K_1\) is well-defined, and continuous, since \(K(-\pi, y) = K(\pi, y)\).

Let \(A\) be the algebra of continuous functions on \(C\) generated by all functions of the form \(z^m y^n\) where \((z, y) \in C\) and \(m\) and \(n\) are integers with \(n \geq 0\). Since \(A\) satisfies the hypotheses of the Stone-Weierstrass Theorem and \(K_1(z,y)\) is continuous on \(C\), there exists \(Q_1(z,y) \in A\) such that \(||K_1 - Q_1||_\infty < \epsilon\). Defining \(Q(x,y) \equiv Q_1(e^{ix}, y)\) yields a function possessing the required properties.
3.2 A Generalization of Alpar's Result

Having dispensed with the preliminary lemmas we are now ready to prove Theorem 3.2, which we restate here for convenience.

Theorem 3.2: Let $W(z)$ be analytic (but not necessarily single-valued) in a region $R$ and let $K(x,y)$ satisfy the following conditions:

a) $K(x,y)$ maps the square $D \equiv [-\pi, \pi] \times [-\pi, \pi]$ continuously into $R$.

b) $K(x,y)$ is $2\pi$-periodic in $x$.

c) $K(x,y) \in \mathcal{A}^{1/\alpha}(x)$ for some $-1 < \alpha \leq 1$, $1 < p < 2$.

Then, if $W(z)$ returns to its initial determination after $z$ travels completely around any curve $C_y$ of the form $z = K(x,y)$, $-\pi < x < \pi$, then $W[K(x,y)] \in C^p$.

Proof: Since the range of $K$ is a compact subset of $R$, there exists $\rho > 0$ such that the set $A \equiv \{ z : \text{dist}(z,K(D)) \leq 2\rho \}$ is contained in $R$.

By Lemma 3.6 there then exists a function $Q(x,y)$ such that $Q(x,y)$ is continuous on $D$, $Q(x,y)$ is $2\pi$-periodic in $x$, $Q(x,y) \in C^\infty(x)$, and $\|K - Q\|_\infty < \rho/2$.

As a consequence, for each $(x,y) \in D$ we have

$W[K(x,y)] = W[Q(x,y) + \{K(x,y) - Q(x,y)\}]$

$= \sum_{m=0}^{\infty} \frac{1}{m!} W^{(m)} [Q(x,y)] [K(x,y) - Q(x,y)]^m$

and thus, in view of Lemma 2.5,
Note that for each fixed $y$, $-\pi \leq y \leq \pi$, the curve $z = Q(x,y)$, $-\pi \leq x \leq \pi$, lies within the set $A \subset \mathbb{R}$ and is sufficiently close to the curve $C_y$ so that it has the same winding number as $C_y$ with respect to all branch points of $W(z)$, i.e., it shares the property of preserving the initial determination of $W(z)$ after $z$ travels completely around the curve. $W^{(m)}[Q(x,y)]$, therefore, is in $C^2(x)$ from which it follows, using Lemma 3.3, that

\begin{equation}
\sum_{m=0}^{\infty} \frac{1}{m!} \left| W^{(m)}[Q(x,y)] [K(x,y) - Q(x,y)]^m \right|_p
\end{equation}

Since by Lemma 3.4,

\begin{equation}
\left| \frac{\partial^2}{\partial x^2} W^{(m)}[Q(x,y)] \right|_\infty \leq A_1 \left| W^{(m+1)}[Q(x,y)] \right|_\infty
\end{equation}

\begin{equation}
+ A_2 \left| W^{(m+2)}[Q(x,y)] \right|_\infty ,
\end{equation}

we see that we need to estimate $\left| W^{(m+\ell)}[Q(x,y)] \right|_\infty$ for $\ell = 0, 1, \text{ and } 2$.

By Cauchy's integral formula

$$W^{(m+\ell)}[Q(x,y)] = \frac{(m+\ell)!}{2\pi i} \oint \frac{W(z)}{|z-Q(x,y)|=\rho} \frac{dz}{[z-Q(x,y)]^{m+\ell+1}}$$
and hence

\[ |W^{(m)}(Q(x,y))| \leq (m+\ell)! \, \rho^{-(m+\ell)} \beta(x,y) \]

where \( \beta(x,y) \equiv \max_{z \sim Q(x,y)} |W(z)| \). Combining these estimates with (1), (2), and (3), and then applying Lemma 3.5 yields

\[ |W(K(x,y))|_p \leq \sum_{m=0}^{\infty} \frac{1}{m!} \left( m! \, \rho^{-m} \|\beta\|_\infty + 4A_1 (m+1)! \, \rho^{-(m+1)} \|\beta\|_\infty \right. \]

\[ + 4A_2 (m+2)! \, \rho^{-(m+2)} \|\beta\|_\infty \left. \right) \|K(x,y) - Q(x,y)\|_p^m \]

\[ \leq \sum_{m=0}^{\infty} \frac{1}{m!} \left( N(m+2)! \, \rho^{-m} \|\beta\|_\infty \right) \|K(x,y) - Q(x,y)\|_p^m \]

\[ \leq 2N \|\beta\|_\infty \|K(x,y) - Q(x,y)\|^0_\infty \]

\[ + \sum_{m=1}^{\infty} \frac{1}{m!} \left( N(m+2)! \, \rho^{-m} \|\beta\|_\infty \right) B \, m \, (\rho/2)^{m-1} \]

\[ \leq 4\pi N \|\beta\|_\infty + BN\rho^{-1} \|\beta\|_\infty \sum_{m=1}^{\infty} \frac{(m+2)(m+1)(m)}{2^{m-1}} \]

\[ < \infty \]

where \( N \) is some appropriate constant. \( W[K(x,y)] \) is therefore in \( C_p \) and the proof is complete.

Before concluding this chapter we wish to make a few observations. First note that there was no loss of generality in stating our hypotheses for Theorem 3.2 in terms of the
variable $x$; indeed, the theorem is equally valid if the smoothness conditions are placed on $y$. Secondly, recalling our remarks in Section 2.2, observe that the hypotheses of Theorem 3.2 are of the smoothness variety and do not explicitly involve the structure of the integral operator generated by the kernel $K(x, y)$ as in Theorem 2.3. Finally, note that Alpar's result (Theorem 3.1) is a direct corollary of our Theorem 3.2 in the case when $K(x, y) = f(x - y)$. 
Chapter 4

WIENER-LÉVY TYPE RESULTS FOR KERNELS

SATISFYING VARIOUS SMOOTHNESS REQUIREMENTS

4.1 Smoothness Conditions Which are Preserved
Under Analytic Transformations

Several of the Wiener-Lévy type results we have discussed earlier, such as Theorems 2.2 and 3.2, involve the assumption of certain smoothness conditions upon the kernel of the integral operator of interest. For single-valued analytic transformations the proofs of these theorems may be greatly simplified by observing that such transformations preserve the smoothness conditions in question. The following lemmas state these results more precisely. They also provide the foundation for further Wiener-Lévy type results concerning kernels satisfying various combinations of Lipschitz or bounded variation smoothness conditions.

Lemma 4.1: Let \( W(z) \) be analytic in a region \( R \) and let \( D \) be a compact subset of \( R \). Then there exists a constant \( M \) such that

\[
|W(z_2) - W(z_1)| \leq M |z_2 - z_1|
\]

for all \( z_1, z_2 \in D \).
Proof: By the compactness of $D$ there exists $\epsilon > 0$ such that the set $E \equiv \{ z : \text{dist}(z,D) \leq \epsilon \}$ is a subset of $R$. Define $M_1 = \max_{z \in E} |W(z)|$, $M_2 = \max_{z \in E} |W'(z)|$, and $M = \max \{ M_2, 2M_1/\epsilon \}$.

Now let $z_1, z_2 \in D$ and define $B$ to be the open disc about $z_2$ of radius $\epsilon$. If $z_1 \in B$ then the line segment joining $z_1$ to $z_2$ is contained in $B$ and hence $|W(z_2) - W(z_1)| = \int_{z_1}^{z_2} W'(z)dz \leq M_2 |z_2 - z_1|$. If $z_1 \notin B$ then $|z_1 - z_2| \geq \epsilon$ and therefore $|W(z_2) - W(z_1)| \leq 2M_1 \leq (2M_1/\epsilon) |z_2 - z_1|$. Clearly, then, $|W(z_2) - W(z_1)| \leq M |z_2 - z_1|$ for all $z_1, z_2 \in D$.

Lemma 4.2: Let $W(z)$ be analytic in a region $R$ and let $K(x,y)$ map the square $[-\pi, \pi] \times [-\pi, \pi]$ continuously into $R$. If $K(x,y) \in \Lambda_{\alpha}^P(x)$, then $W[K(x,y)] \in \Lambda_{\alpha}^P(x)$ also. (The result is equally true for $\Lambda_{\alpha}^P(y)$.)

Proof: From the compactness of the range of $K$, and Lemma 4.1,

$$\int_{-\pi}^{\pi} |W[K(x+h,y)] - W[K(x,y)]|^p dx \leq M^p \int_{-\pi}^{\pi} |K(x+h,y) - K(x,y)|^p dx \leq M^p B^p(y) |h|^\alpha p$$

for some nonnegative $B \in L^2$. 
Lemma 4.3: Let $W(z)$ be analytic in a region $R$ and let $K(x,y)$ map the square $[-\pi, \pi] \times [-\pi, \pi]$ continuously into $R$. If $K(x,y) \in BV(x)$, then $W[K(x,y)] \in BV(x)$ also. (The result is equally true for BV(y).)

Proof: Let $-\pi = x_0 < x_1 < \ldots < x_n = \pi$ be a partition of $[-\pi, \pi]$. By virtue of the compactness of the range of $K$, and Lemma 4.1, we have

$$\sum_{i=1}^{n} |W[K(x_i,y)] - W[K(x_{i-1},y)]| \leq M \sum_{i=1}^{n} |K(x_i,y) - K(x_{i-1},y)| \leq M C(y)$$

for some nonnegative $C \in L^2$.

Lemma 4.4: Let $W(z)$ be analytic in a region $R$ and let $K(x,y)$ map the square $[-\pi, \pi] \times [-\pi, \pi]$ continuously into $R$. Then the respective $L^p$ moduli of continuity of $K$ and $W[K]$ satisfy

$$\omega_p(W[K],\delta) \leq M \omega_p(K,\delta)$$

for some constant $M$.

Proof: From the compactness of the range of $K$, and Lemma 4.1,
\[ \omega_p(W[K], \delta) \equiv \sup_{0<h \leq \delta} \left\{ \int_{-\pi}^{\pi} |W[K(x+h,y)] - W[K(x-h,y)]|^p \, dx \right\}^{1/p} \]
\[ \leq M \sup_{0<h \leq \delta} \left\{ \int_{-\pi}^{\pi} |K(x+h,y) - K(x-h,y)|^p \, dx \right\}^{1/p} \]
\[ = M \omega_p(K, \delta). \]

4.2 The Consequences of Our Lemmas

We are now prepared to reexamine certain of the results in Chapters 2 and 3. We begin with a restated version of Theorem 3.2 for the case of a single-valued analytic transformation.

**Theorem 4.5:** Let \( W(z) \) be analytic in a region \( R \) and let \( K(x,y) \) satisfy the following conditions:

a) \( K(x,y) \) maps the square \([-\pi, \pi] \times [-\pi, \pi]\) continuously into \( R \).

b) \( K(x,y) \) is \( 2\pi \)-periodic in \( x \).

c) \( K(x,y) \in \Lambda^1_{\alpha}(x) \) for some \( p^{-1} < \alpha \leq 1, 1 < p < 2. \)

Then \( W[K(x,y)] \in C_p. \) (The result is also valid if the smoothness conditions are placed on the variable \( y \).)

**Proof:** In view of Lemma 4.2 it suffices to prove that any kernel satisfying conditions a), b), and c) is in \( C_p. \) This, however, is an immediate consequence of Lemma 3.5 for the special case \( n = 1. \)

In similar fashion we can greatly simplify the proof of Theorem 2.2 given previously by Cochran [7]. Before doing so we first restate the theorem for easy referral.
Theorem 2.2: Let $W(z)$ be analytic in a region $R$ and let $K(x,y)$ be a continuous kernel which maps the square $[-\pi,\pi] \times [-\pi,\pi]$ into $R$ and is $2\pi$-periodic in $x$. If $K(x,y)$ is such that for some $1 < p < 2$

$$\sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} m^{-2/q} \left| \omega_p(K,m^{-1}) \right|^2 \right)^{1/2} < \infty$$

where $p^{-1} + q^{-1} = 1$, then $W[K(x,y)] \in C_1$.

Proof: In light of Lemma 4.4 it is clear that $W[K(x,y)]$ satisfies the same hypotheses as $K(x,y)$. Therefore it suffices to prove that these hypotheses are sufficient for $K(x,y)$ to be in $C_1$. The sufficiency follows from Lemma 3 and inequality (16) of [7].

In addition to the above examples, there are other known sufficient conditions for kernels $K(x,y)$ to be in $C_\alpha$ (see Bljumin and Kotljari [3] and Cochran [6]). Since most of these sufficient conditions are smoothness requirements on $K(x,y)$ to which Lemmas 4.2 and 4.3 apply, these results may also be carried over immediately to single-valued analytic functions of $K(x,y)$. In the remainder of this chapter we state these results. The proofs of the theorems are direct consequences of Lemmas 4.2 and 4.3 and known results for $K(x,y)$ found in [3] and [6]. For convenience in the sequel, we assume that $W(z)$ is analytic in a region $R$, $K(x,y)$ maps the square $[-\pi,\pi] \times [-\pi,\pi]$ continuously into $R$, and the smoothness conditions hold in either variable.
Theorem 4.6: If $K \in \Lambda_\alpha$, then $W[K] \in C_\rho$ for all $\rho > 2/(2\alpha + 1)$. (Note that $\Lambda_\alpha \subseteq \Lambda^p_\alpha$ for all $p$.)

Theorem 4.7: If $K \in \Lambda^p_\alpha$, then $W[K] \in C_\rho$ for all $\rho > \left\{ \begin{array}{ll} \frac{p}{p(\alpha + 1) - 1}, & \text{if } 1 < p \leq 2 \\ \frac{2}{2\alpha + 1}, & \text{if } p > 2 \end{array} \right.$

Theorem 4.8: If $K \in \Lambda^p_\alpha$ and $K \in \Lambda^q_\beta$ with $p < q$, then $W[K] \in C_\rho$ for all $\rho > \left\{ \begin{array}{ll} \frac{p}{p(\alpha + 1) - 1}, & \text{if } pq(\alpha - \beta) > q - p \\ \frac{q}{q(\beta + 1) - 1}, & \text{if } pq(\alpha - \beta) \leq q - p \end{array} \right.$

Case 1: $(q \leq 2)$

Case 2: $(p \leq 2 < q)$

Case 3: $(p > 2)$

Theorem 4.9: If $K \in BV$, then $W[K] \in C_\rho$ for all $\rho > 1$.

Theorem 4.10: If $K \in BV$ and $K \in \Lambda_\beta$, $0 < \beta < 1$, then $W[K] \in C_\rho$ for all $\rho > 2/(\beta + 2)$.

Theorem 4.11: If $K \in BV$ and $K \in \Lambda_\beta^q$, $\beta > 0$, $q \geq 1$, then $W[K] \in C_\rho$ for all

$$\rho > \begin{cases} 
1, & \text{if } \beta q < 1 \\
\frac{q}{q(\beta + 1) - 1}, & \text{if } \beta q \geq 1, q \leq 2 \\
\frac{2(q - 1)}{q(\beta + 2) - 3}, & \text{if } \beta q \geq 1, q > 2
\end{cases}$$
Chapter 5

TOPICS RELATED TO THE WIENER-LEVY THEOREM

5.1 The Converse of the Wiener-Lévy Theorem

In the preceding chapters we have investigated various properties which are preserved under the action of an analytic transformation. The question naturally arises as to whether or not the full strength of analyticity is necessary for the validity of these results. The answer is in the negative for the case of the classical Wiener-Lévy Theorem. To see this, let \( f \in A_1 \) and \( W(z) \equiv \text{Re}(z) \), then we may write \( W(f) = \frac{1}{2}(f + \overline{f}) \) which is clearly in \( A_1 \), but \( W(z) \) is not an analytic function. The answer becomes positive, however, if we restrict our attention to real-valued functions \( f(x) \) and \( W(t) \). Indeed, Katznelson [14] has shown the following:

**Theorem 5.1:** Let \( I \) be an open interval on the real axis and let \( W(t) \) be defined on \( I \). If for every \( f \in A_1 \) whose range is contained in \( I \) it follows that \( W(f) \in A_1 \), then \( W(t) \) is real-analytic on \( I \). (Real-analytic means that \( W(t) \) has a Taylor series expansion valid in a neighborhood of each point \( t \in I \)).

A related converse result in which the hypotheses on the
function $W(t)$ are considerably weaker has been established by Rudin [22].

**Theorem 5.2:** Let $I$ be an open interval on the real axis and let $W(t)$ be defined on $I$. If for each $f \in A_1$ whose range is contained in $I$ there is a $p < 2$ (depending on $f$) such that $W(f) \in A_p$, then $W(t)$ is real-analytic on $I$.

In view of the relationship between the $s$-numbers of difference kernels and the Fourier coefficients of their generating functions, it is a simple matter to prove an analogue of Theorem 5.2 in the integral operator context.

**Theorem 5.3:** Let $I$ be an open interval on the real axis and let $W(t)$ be defined on $I$. If for each kernel $K(x,y)$ which maps the square $[-\pi, \pi] \times [-\pi, \pi]$ into $I$, and is a member of $C_1$, there is a $p < 2$ (depending on $K$) such that $W[K(x,y)] \in C_p$, then $W(t)$ is real-analytic on $I$.

**Proof:** Let $f(x)$ map $[-\pi, \pi]$ into $I$ and belong to $A_1$. Since we may assume that $f(x)$ is $2\pi$-periodic (see the proof of Theorem 2.10), the difference kernel $K(x,y) \equiv f(x-y)$ possesses $s$-numbers $\{2\pi |c_n| \}_{n=-\infty}^{\infty}$ where $\{c_n\}_{n=-\infty}^{\infty}$ are the Fourier coefficients of $f(x)$. Then since $f(x) \in A_1$, we have that $K(x,y) \in C_1$ and therefore
there exists $p < 2$ such that $W[K(x,y)] \in C_p$.

Recognizing that $W[K(x,y)] = W[f(x-y)]$ is also a difference kernel and hence has $s$-numbers

$$\{2\pi|b_n|\}_{n=-\infty}^\infty$$

where $\{b_n\}_{n=-\infty}^\infty$ are the Fourier coefficients of $W[f(x)]$, it follows that

$W[f(x)] \in A_p$. By Theorem 5.2 our proof is complete.

It should be remarked that the hypotheses of Theorem 5.3 can be weakened since we only need the appropriate behavior for difference kernels. In any event, like the earlier theorems, this result is no longer true in the general case of a complex-valued function $W(z)$ defined on a region $R$, and complex-valued kernels $K(x,y)$ whose range is contained in $R$, if the conclusion is changed to read "then $W(z)$ is analytic in $R$". Modeling the earlier analysis, for example, if $K(x,y) \in C_1$ and $W(z) = \text{Re}(z)$ then we may write $W[K(x,y)] = \frac{1}{2} \{K(x,y) + \overline{K(x,y)}\}$, which is in $C_1$ since $|\overline{K}|_1 = |K|_1$ and $C_1$ is a linear space. The function $W(z)$, however, is not analytic.

5.2 A Theorem of Marcinkiewicz and Its Converse

We begin this section by introducing some new notation.

Let $I$ be an open interval on the real axis and let $W(t)$ be a real-valued function defined for $t \in I$. We say that $W \in G_p$.

To see this, take the complex conjugate of equations (2) and (3) in Section 1.5.
provided that for each compact subset \( J \) of \( I \) there exists a constant \( B \) (depending upon \( J \)) such that 
\[
|W^{(n)}(t)| \leq B^n (n!)^{1/p}
\]
for all \( t \in J \) and all \( n \geq 1 \). (Note that \( W \in C_1 \) if and only if \( W \) is real-analytic).

Using the concept of functions in \( C_p \), Marcinkiewicz [18] proved the following extension of the classical Wiener-Levy Theorem in which the conditions on \( f(x) \) are strengthened while the conditions on \( W(t) \) are weakened.

**Theorem 5.4:** Let \( I \) be an open interval on the real axis and let \( W(t) \) be defined on \( I \) with 
\( W \in C_p \) for some \( 0 < p \leq 1 \). If \( f(x) \) maps \([-\pi, \pi]\) into \( I \) and \( f \in A_p \) then \( W[f] \in A_1 \). (Zygmund has noted that actually \( W[f] \in A_p \)).

Subsequently, Riviere and Sagher [21] established the following converse to Marcinkiewicz's theorem.

**Theorem 5.5:** Let \( I \) be an open interval on the real axis and let \( W(t) \) be defined on \( I \).

Furthermore, let \( 0 < p \leq 1 \) and suppose that for each \( f \in A_p \) whose range is contained in \( I \), \( W[f] \in A_p \) for some \( p < 2 \) (depending on \( f \)). Then \( W \in C_p \).

Note that Theorem 5.2 is the special case of Theorem 5.5 when \( p = 1 \).
As was the case for Theorem 5.2, Theorem 5.5 is easily generalized to the situation involving integral operators with real-valued kernels.

**Theorem 5.6:** Let $I$ be an open interval on the real axis and let $W(t)$ be defined on $I$. Furthermore, let $0 < p \leq 1$ and suppose that for each kernel $K(x,y)$ which maps the square $[-\pi, \pi] \times [-\pi, \pi]$ into $I$, and is a member of $C_p$, there is a $\rho < 2$ (depending on $K$) such that $W[K(x,y)] \in C_\rho$. Then $W \in G_p$.

**Proof:** The argument is sufficiently similar to the proof of Theorem 5.3 that it will be omitted.

Again, the hypotheses of Theorem 5.6 may be weakened to apply to only difference kernels and, as expected, Theorem 5.6 is not generally true for the complex case, as the example subsequent to Theorem 5.3 demonstrates.

5.3 **The Krein Result and Some Open Questions**

In a 1973 paper, McLaughlin [20] presented numerous sufficiency conditions for the Fourier coefficients $\{c_n\}_{n=-\infty}^{\infty}$ of a function $f(x)$ to satisfy summability conditions of the form

$$\sum_{n=-\infty}^{\infty} |n|^{\alpha} |c_n|^{\beta} < \infty.$$
The case $\alpha = 0, \beta = p$ has been of principle interest in this paper, since then $f(x) \in A_p$.

The case $\alpha = 1$ and $\beta = 2$ is also worthy of consideration however. Indeed, if $f(x)$ is a continuous, $2\pi$-periodic function, Krein [15] has shown that the set of all such functions $f(x)$ for which $\sum_{n=-\infty}^{\infty} |n| |c_n|^2 < \infty$ forms a Banach algebra with respect to the norm

$$\{\{f\}\} \equiv ||f||_{\infty} + \left( \sum_{n=-\infty}^{\infty} |n| |c_n|^2 \right)^{1/2}.$$ 

In fact, Krein has established the following interesting variation of the classical Wiener-Lévy Theorem.

**Theorem 5.7:** Let $W(z)$ be analytic in a region $R$ and let $f(x)$ be a $2\pi$-periodic function mapping $[-\pi, \pi]$ continuously into $R$. If $\{\{f\}\} < \infty$, then $\{\{W[f]\}\} < \infty$ also.

In view of our good success in generalizing Wiener-Lévy type results to the integral operator setting, it is reasonable to conjecture the existence also of analogues of this Krein result in the more general context. This remains as an open question which certainly warrants further investigation. On the other hand, the more general case of integral operators whose kernels have s-numbers satisfying $\sum_{n=1}^{\infty} n^\alpha s_n^\beta < \infty$, with $\alpha \neq 0$ or 1, undoubtedly constitutes a far deeper problem since the theorem of Krein still stands as a singular result in Fourier series.

Another class of kernels upon which the action of
analytic transformations could be investigated are those kernels for which \[ \sum_{n=1}^{\infty} s_n^\beta \|\phi_n\|_\gamma \|\psi_n\|_\gamma < \infty. \] In fact, our main result (Theorem 2.3) considers the special case \( \beta = \gamma = p \). More generally, it may be of interest to incorporate these hypotheses with the McLaughlin type conditions and consider those kernels satisfying \[ \sum_{n=1}^{\infty} n^\alpha s_n^\beta \|\phi_n\|_\gamma \|\psi_n\|_\gamma < \infty. \] We leave these questions as topics for future research.


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THEOREMS OF WIENER-LEVY TYPE
FOR INTEGRAL OPERATORS IN $C_p$

by

Christopher Alan Steel

(ABSTRACT)

The classical theorem of N. Wiener and P. Lévy states that if $f(x)$ has an absolutely convergent Fourier series and $W(z)$ is an analytic function whose domain contains the range of $f(x)$, then $W[f(x)]$ also has an absolutely convergent Fourier series. The main result of this paper is an analogue of the Wiener-Lévy Theorem in which we consider analytic transformations acting upon kernels of integral operators of the form $(T\phi)(x) = \int_{-\pi}^{\pi} K(x,y)\phi(y)dy$ and the so-called $s$-numbers of $K$ and $W[K]$ take the place of the classical Fourier coefficients of $f$ and $W[f]$.

Theorem: Let $W(z)$ be analytic in a region $R$ and let $K(x,y)$ map the square $[-\pi,\pi] \times [-\pi,\pi]$ continuously into $R$. If $\{\phi_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$ are a full set of continuous singular functions for $K(x,y)$ and

$$\sum_{n=1}^{\infty} [s_n(K)]^p \|\phi_n\|^p \|\psi_n\|^p < \infty$$

for some $0 < p < 1$, then $\sum_{n=1}^{\infty} [s_n(W[K])]^p < \infty$.
or, expressed alternatively, \( W[K] \) belongs to the Schatten class \( C_p \).

The classical Wiener–Lévy Theorem is obtained as a corollary in the special situation when \( p = 1 \) and \( K(x,y) \equiv f(x-y) \) is a difference kernel.

For the case \( 1 < p < 2 \) we generalize a Fourier series result of L. Alpar to the following theorem in integral operator theory.

**Theorem:** Let \( W(z) \) be analytic (but not necessarily single-valued) in a region \( R \) and let \( K(x,y) \) satisfy the following conditions:

a) \( K(x,y) \) maps the square \([-\pi, \pi] \times [-\pi, \pi]\) continuously into \( R \).

b) \( K(x,y) \) is \( 2\pi \)-periodic in \( x \) (or \( y \)).

c) \( K(x,y) \) satisfies an integrated Lipschitz condition of order \( \alpha \) relatively uniformly in \( x \) (or \( y \)) where \( p^{-1} < \alpha \leq 1, 1 < p < 2 \).

Then, if \( W(z) \) returns to its initial determination after \( z \) travels completely around any curve \( C_y \) (or \( C_x \)) of the form \( z = K(x,y), -\pi \leq x \) (or \( y \)) \( \leq \pi \), then

\[ W[K(x,y)] \in C_p. \]

To round out the paper, we show that analytic transformations preserve smoothness conditions of Lipschitz and bounded variation type, and consequently we are able to give a number of sufficiency
conditions for analytic functions of kernels to be in various kernel classes.

Finally, we investigate the converse of the Wiener-Lévy Theorem and how analogues of it relate to integral operators. The paper concludes with suggestions of several interesting questions warranting further study.