ESTIMATION PROBLEMS CONNECTED WITH STOCHASTIC PROCESSES

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1. INTRODUCTION

1.1 Background Although continuous time processes have been of interest for many years to economists, meteorologists, engineers and others, it is only in the last decade that statisticians have considered, to any extent, the problems associated with the spectral analysis of these processes. References [1] through [8] of the bibliography are some of the more general treatments of the developments in the spectral analysis area. The specific problem of cross-spectral estimation has received but scant attention in the literature, though some consideration of the problem has been given in [4], [5] and [7].

Since the concepts, terminology and methodology of cross-spectral analysis are not widely known, the first chapter of this paper will consist of a brief and non-rigorous exposition of the theory and techniques of this type of analysis, designed both as a general introduction to the subject and as a means of clarifying the basic assumptions upon which the developments of the succeeding chapters depend.

1.2 Basic Concepts Throughout this paper we shall be concerned with two real, continous stochastic (or random) time processes, $\{x(t)\}$ and $\{y(t)\}$, which are assumed to be stationary, jointly Gaussian and ergodic and to have zero expectations.

To clarify these concepts, we note first that a continuous stochastic process $\{z(t)\}$ consists of an ensemble of continuous functions $z(t)$ which, for any sequence of times $t_1, t_2, \ldots, t_n$, possess a joint probability distribution. The process $\{z(t)\}$ is real if z and t are defined on the set of real numbers.

A process is stationary if the joint distribution for any set of times $t_1, t_2, \ldots, t_n$ depends only on the time differences $t_j - t_i$ and is independent of the absolute values of the t_j. Equivalently, a process is stationary if it is invariant under translations of the time axis.

The processes $\{x(t)\}$ and $\{y(t)\}$ are jointly Gaussian if, for any set of times $t_1, t_2, \ldots, t_n$, the 2n random variables,

$$x(t_1), x(t_2), \ldots, x(t_n), y(t_1), y(t_2), \ldots, y(t_n)$$

possess a multivariate normal distribution. Therefore for the given set of times $t_1, t_2, \ldots, t_n$, the processes $\{x(t)\}$

and $\{y(t)\}$ are completely specified by a vector of means, which are zero by assumption, and a variance-covariance matrix Σ. The matrix Σ is 2n by 2n in size and may be written as the partitioned matrix

$$\Sigma = \left[\begin{array}{c|c} \Sigma_{xx} & \Sigma_{xy} \\ \hline \Sigma_{yx} & \Sigma_{yy} \end{array}\right] , \tag{1.2.1}$$

where each of the sub-matrices are n by n and are defined as

$$\left.\begin{aligned} \Sigma_{xx} &= [E\{x(t_i)x(t_j)\}] \\ \Sigma_{xy} &= [E\{x(t_i)y(t_j)\}] \\ \Sigma_{yx} &= [E\{y(t_i)x(t_j)\}] \\ \Sigma_{yy} &= [E\{y(t_i)y(t_j)\}] \end{aligned}\right\} \quad i,j=1,2,\ldots,n. \tag{1.2.2}$$

The expectations in the preceding expression are termed the autocovariances and cross-covariances of $\{x(t)\}$ and $\{y(t)\}$ and are usually written

$$\begin{aligned} E[x(t_i)x(t_j)] &= \rho_x(t_j-t_i) \\ E[y(t_i)y(t_j)] &= \rho_y(t_j-t_i) \\ E[x(t_i)y(t_j)] &= \rho_{xy}(t_j-t_i) \\ E[y(t_i)x(t_j)] &= \rho_{yx}(t_j-t_i) \quad . \end{aligned} \tag{1.2.3}$$

Note that these covariances depend only on the time differences t_j-t_i, which reflects the earlier assumption of stationarity. The matrix Σ is now seen to depend only on the four sets of covariances above for all time differences $t_j - t_i$, $i, j = 1, 2, \ldots, n$.

The autocovariance functions

$$\rho_x(\tau) = E[x(t)x(t+\tau)]$$

and $$\rho_y(\tau) = E[y(t)y(t+\tau)] \tag{1.2.4}$$

and the cross-covariance function

$$\rho_{xy}(\tau) = \rho_{yx}(-\tau) = E[x(t)y(t+\tau)] \tag{1.2.5}$$

are straightforward generalizations to a continuous time difference parameter τ and it is intuitively clear that if these functions were known for all values of τ they would completely determine the Gaussian processes $\{x(t)\}$ and $\{y(t)\}$ since the autocovariances and cross-covariances would be known for all possible sets of times $t_1, t_2, \ldots, t_n$.

Thus far we have dealt only with ensemble properties of the processes $\{x(t)\}$ and $\{y(t)\}$, that is, properties associated with the distributions of the ensembles of functions comprising the processes. In practice we usually have but a single realization (observed time record) of each process and on the basis of this information we wish to estimate population autocovariances and cross-covariances. Since the processes are stationary we may form temporal (time average) estimates, for some time interval

$(-\frac{T}{2},\frac{T}{2})$, of the type

$$R_x(\tau,T) = \frac{1}{T-\tau}\int_{\frac{T}{2}}^{\frac{T}{2}-\tau} x(t)x(t+\tau)dt, \qquad (1.2.6)$$

or suitable numerical approximations, with similar expressions for $R_y(\tau,T)$ and $R_{xy}(\tau,T)$. The ergodic assumption implies that

$$\lim_{T\rightarrow\infty} R_x(\tau,T) = \rho_x(\tau) \qquad (1.2.7)$$

with probability one. The temporal estimates $R_y(\tau,T)$ and $R_{xy}(\tau,T)$ similarly converge in probability to $\rho_y(\tau)$ and $\rho_{xy}(\tau)$. The ergodic assumption, therefore, provides the basis for the estimation of ensemble parameters from the information contained in single realizations.

1.3 <u>The Spectral Density Functions</u> The covariance functions (1.2.4) and (1.2.5) are related, as Fourier transforms, to a second set of functions known as the spectral density functions. The relationships, and their inverses, between the two sets of functions are [5]

$$\rho_x(\tau) = \int_0^\infty \Phi_x(\omega) \cos \omega\tau \, d\omega \; , \; \Phi_x(\omega) = \frac{2}{\pi} \int_0^\infty \rho_x(\tau) \cos \omega\tau \, d\tau \tag{1.3.1}$$

$$\rho_y(\tau) = \int_0^\infty \Phi_y(\omega) \cos \omega\tau \, d\omega \; , \; \Phi_y(\omega) = \frac{2}{\pi} \int_0^\infty \rho_y(\tau) \cos \omega\tau \, d\tau \tag{1.3.2}$$

$$\rho_{xy}(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} \Phi_{xy}(\omega) \, e^{i\omega\tau} d\omega, \; \Phi_{xy}(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \rho_{xy}(\tau) \, e^{-i\omega\tau} d\tau, \tag{1.3.3}$$

where $\Phi_x(\omega)$ and $\Phi_y(\omega)$ are the power spectral density functions for the processes $\{x(t)\}$ and $\{y(t)\}$ and $\Phi_{xy}(\omega)$ is the cross-spectral density function for the two processes. The argument, ω, of the spectral densities is angular frequency, so that if τ is measured in seconds, ω is measured in radians per second. The three spectral density functions completely determine the processes $\{x(t)\}$ and $\{y(t)\}$ as do the three covariance functions.

In this paper, we shall be concerned with certain problems associated with the estimation of the cross-spectral density function $\Phi_{xy}(\omega)$.

1.4 Applications of Cross-Spectral Analysis We digress for a moment to consider briefly two types of problems for which estimates of the cross-spectral density function are required. First, let $\{x(t)\}$ represent a random disturbance and let $\{y(t)\}$ represent the response of some physical system to this disturbance. Furthermore let the response be linear [5]. Information on the characteristics of the physical system can be obtained from the frequency response function $T(\omega)$, where

$$T(\omega) = \frac{\Phi_{xy}(\omega)}{\Phi_x(\omega)} \qquad (1.4.1)$$

This type of problem occurs frequently in aeronautics where the response of an airfoil to random atmospheric turbulence is of concern.

A second application arises in connection with the determination of the power spectrum of the sum of two random processes. If

$$\{z(t)\} = \{x(t)\} + \{y(t)\} \qquad (1.4.2)$$

it can be shown [7] that

$$\Phi_z(\omega) = \Phi_x(\omega) + \Phi_{xy}(\omega) + \Phi_{yx}(\omega) + \Phi_y(\omega). \qquad (1.4.3)$$

This relationship may be generalized to the sum of three or more processes, in which case the expression analagous to (1.4.3) would contain all possible power-spectral and cross-spectral density functions.

1.5 Properties of the Covariance and Spectral Density Functions The autocovariance functions $\rho_x(\tau)$ and $\rho_y(\tau)$ satisfy the following conditions [4] :

$$\rho_x(\tau) = \rho_x(-\tau) \tag{1.5.1}$$

$$\rho_x(0) \geq \rho_x(\tau) \tag{1.5.2}$$

$$\rho_x(0) = \text{Variance of } \{x(t)\} , \tag{1.5.3}$$

where similar expressions hold for $\rho_y(\tau)$.

The power spectral density functions have the following properties [4] :

$$\Phi_x(\omega) \geq 0 \tag{1.5.4}$$

$$\Phi_x(\omega) = \Phi_x(-\omega) \tag{1.5.5}$$

$$\int_0^\infty \Phi_x(\omega)\, d\omega = \rho_x(0), \tag{1.5.6}$$

where similar expressions again hold for $\Phi_y(\omega)$.

The cross-covariance function satisfies

$$\rho_{xy}(\tau) = \rho_{yx}(-\tau). \tag{1.5.7}$$

The cross spectral density function may be expressed as

$$\Phi_{xy}(\omega) = C_{xy}(\omega) + iQ_{xy}(\omega) \tag{1.5.8}$$

where $C_{xy}(\omega)$ and $Q_{xy}(\omega)$ are the co-spectral and quadrature-spectral density functions [4]. These functions are related to the cross-covariance function as follows [5] :

$$\rho_{xy}(\tau) = \int_{0}^{\infty} \left\{ C_{xy}(\omega) \cos \omega\tau - Q_{xy}(\omega) \sin \omega\tau \right\} d\omega \quad (1.5.9)$$

$$C_{xy}(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \rho_{xy}(\tau) \cos \omega\tau \, d\tau \quad (1.5.10)$$

$$Q_{xy}(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \rho_{xy}(\tau) \sin \omega\tau \, d\tau \, . \quad (1.5.11)$$

These functions also have the following properties [4]:

$$C_{xy}(\omega) = C_{yx}(\omega) = C_{xy}(-\omega) \quad (1.5.12)$$

$$Q_{xy}(\omega) = -Q_{yx}(\omega) = -Q_{xy}(-\omega) \quad (1.5.13)$$

$$C^2_{xy}(\omega) + Q^2_{xy}(\omega) \leq \Phi_x(\omega)\, \Phi_y(\omega). \quad (1.5.14)$$

1.6 <u>Discrete Estimation Procedures</u> Let $x(t)$ and $y(t)$ be single realizations of the processes $\{x(t)\}$ and $\{y(t)\}$ which have been evaluated at times t_i, $i = 1,2,\ldots,n$, where

$$t_i = t_1 + (i-1)\,\Delta t. \quad (1.6.1)$$

The constant interval, Δt, between successive determinations is chosen [5] as

$$\Delta t = \frac{\pi}{W} \quad (1.6.2)$$

where W is the folding frequency, i.e., the maximum angular frequency for which the power is not negligible.

The first step is to form estimates of the cross-covariances $\rho_{xy}(\alpha\Delta t)$ for $\alpha = -r, \ldots, r$. The choice of r will be discussed later. The usual cross-covariance estimators are given by

$$\hat{\rho}_{xy}(\alpha\Delta t) = \begin{cases} \dfrac{1}{n-\alpha} \sum\limits_{i=1}^{n-\alpha} x(t_i)y(t_i + \alpha\Delta t), \ \alpha \geq 0 \\[2ex] \dfrac{1}{n-|\alpha|} \sum\limits_{i=|\alpha|+1}^{n} x(t_i)y(t_i + \alpha\Delta t), \ \alpha < 0. \end{cases} \tag{1.6.3}$$

The standard estimator for $C_{xy}(\omega)$ at the point $\omega = \omega_h$ is obtained by numerical integration of (1.5.10), using the trapezoidal rule with $h = \Delta t$, from $-r\Delta t$ to $r\Delta t$. In the resulting expression each $\rho_{xy}(\alpha\Delta t)$ is replaced by the corresponding $\hat{\rho}_{xy}(\alpha\Delta t)$ to give the co-spectral estimator

$$\hat{C}_{xy}(\omega_h) = \sum_{\alpha=-r}^{r} \hat{\rho}_{xy}(\alpha\Delta t)\, H_1(\alpha) \tag{1.6.4}$$

where

$$H_1(\alpha) = \begin{cases} \dfrac{1}{W} \cos \omega_h \alpha\Delta t, \ \alpha = -(r-1), \ldots, (r-1) \\[2ex] \dfrac{1}{2W} \cos \omega_h \alpha\Delta t, \ \alpha = -r, r. \end{cases} \tag{1.6.5}$$

By similar methods we obtain, from (1.5.11), the standard quadrature-spectral estimator

$$\hat{Q}_{xy}(\omega_h) = \sum_{\alpha=-r}^{r} \hat{\rho}_{xy}(\alpha\Delta t)\, H_2(\alpha) \tag{1.6.6}$$

where

$$H_2(\alpha) = \begin{cases} \frac{1}{W} \sin \omega_h \alpha \Delta t, \alpha = -(r-1), \dots, (r-1) \\ \\ \frac{1}{2W} \sin \omega_h \alpha \Delta t, \; \alpha = -r, r. \end{cases} \quad (1.6.7)$$

The cross-spectral estimator is then given by

$$\hat{\Phi}_{xy}(\omega_h) = \hat{C}_{xy}(\omega_h) + i\hat{Q}_{xy}(\omega_h). \quad (1.6.8)$$

1.7 <u>Spectral Windows</u> From (1.2.4) and (1.6.3) it follows that

$$E\left\{\hat{\rho}_{xy}(\alpha\Delta t)\right\} = \rho_{xy}(\alpha\Delta t). \quad (1.7.1)$$

The expectations of the standard estimators are therefore given by

$$E\left\{\hat{C}_{xy}(\omega_h)\right\} = \sum_{\alpha=-r}^{r} \rho_{xy}(\alpha\Delta t) \; H_1(\alpha) \quad (1.7.2)$$

and

$$E\left\{\hat{Q}_{xy}(\omega_h)\right\} = -\sum_{\alpha=-r}^{r} \rho_{xy}(\alpha\Delta t) \; H_2(\alpha). \quad (1.7.3)$$

Making the substitution (1.5.9) with $\tau = \alpha\Delta t$ in (1.7.2) and (1.7.3) and noting, from (1.6.5) and (1.6.7), that $H_1(\alpha)$ and $H_2(\alpha)$ are even and odd functions, respectively, of α we obtain, after interchanging the order of summation and integration,

$$E\left\{\hat{C}_{xy}(\omega_h)\right\} = \int_0^W C_{xy}(\omega) \left\{\sum_{\alpha=-r} \cos \rho\alpha\Delta t \; H_1 \; (\alpha)\right\} d\omega \quad (1.7.4)$$

and

$$E\left\{\hat{Q}_{xy}(\omega_h)\right\} = \int_{o}^{W} Q_{xy}(\omega) \left\{\sum_{\alpha=-r} \sin\omega\alpha\Delta t\ H_2(\alpha)\right\} d\omega, \tag{1.7.5}$$

where the upper limit of integration in (1.5.9) has been replaced by the folding frequency, W.

The expressions in brackets in (1.7.4) and (1.7.5) are continuous weight functions which bias the estimates of the spectral densities at $\omega = \omega_h$. These weight functions are known as "spectral windows" since, in a sense, they are windows through which we view the spectral densities. The spectral windows for the standard co-spectral and quadrature-spectral estimators are both of the type shown in Figure 1. Note that the majority of the weighting occurs in the immediate vicinity of the point $\omega = \omega_h$, but the weight function extends, alternately positive and negative, over the range of ω, ie, $o \le \omega \le W$. The widths of the lobes of the spectral windows are inversely proportional to r. This property, along with the volume of calculations that can be handled, influence the choice of r [4].

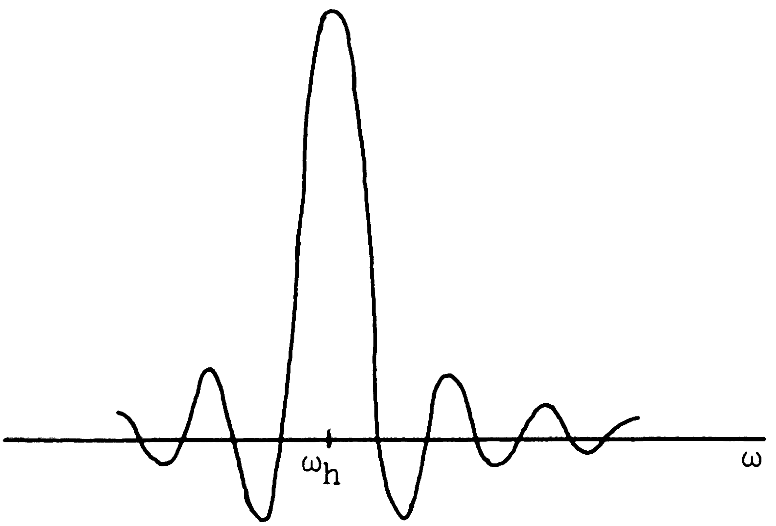

Figure 1: Typical spectral window.

1.8 <u>Other Estimators</u> Tukey [2] has proposed a power spectral estimator which is, essentially, a weighted three-point moving average of the standard power spectral estimators. The spectral window of the proposed estimator has sharply reduced side lobes. This approach is readily extensible to the standard co-spectral and quadrature-spectral estimators.

Bartlett [9] has proposed a power spectral estimator for which the spectral window is non-negative. In chapter 2 of this paper co-spectral and quadrature-spectral estimators with non-negative spectral windows are developed and their properties considered in some detail.

The estimators considered thus far require approximately $(2r + 1)n$ multiplicative operations to obtain a point estimate of co-power or quadrature-power. Even for moderate values of r and n these procedures are practical only on punched-card or other electronic equipment. A simple method for the crude assessment of cross-power has been proposed [4], but this method provides only estimates of average power over the ranges $(\frac{W}{2}, W)$, $(\frac{W}{4}, \frac{W}{2})$, $(\frac{W}{8}, \frac{W}{4})$, etc. In chapter 3 of this paper estimators, requiring only n multiplicative operations, which provide point estimates of the co- and quadrature-spectral densities are presented.

In chapter 4 we consider a different type of estimation problem, that in which certain properties of the given realizations can be measured directly with analog equipment. In the interest of continuity, the introduction to this material, involving generalized harmonic analysis, is deferred to chapter 4.

2. CROSS-SPECTRAL ESTIMATORS WITH NON-NEGATIVE SPECTRAL WINDOWS

2.1 Preliminary Estimators As an extension of the power spectral estimator proposed in [9] we consider the co-spectral and quadrature-spectral estimators

$$C^*_{xy}(\omega_h) = \sum_{\alpha=-r}^{r} \hat{\rho}_{xy}(\alpha\Delta t) \quad (1-\frac{|\alpha|}{r+1})\frac{1}{W} \cos \omega_h\alpha\Delta t \qquad (2.1.1)$$

and

$$Q^*_{xy}(\omega_h) = \sum_{\alpha=-r}^{r} \hat{\rho}_{xy}(\alpha\Delta t)(1-\frac{|\alpha|}{r+1}) \frac{1}{W} \sin \omega_h\alpha\Delta t. \qquad (2.1.2)$$

These estimators are, essentially, the standard estimators given by (1.6.4) and (1.6.6) with a weighting factor $(1-\frac{|\alpha|}{r+1})$ added.

The expectations of these estimators can be derived by the methods of section 1.7, and are given by

$$E\left\{C^*_{xy}(\omega_h)\right\} = \int_0^W C_{xy}(\omega)f^*_C(\omega;\omega_h,r)d\omega \qquad (2.1.3)$$

and

$$E\left\{Q^*_{xy}(\omega_h)\right\} = \int_0^W Q_{xy}(\omega)f^*_Q(\omega;\omega_h,r)d\omega, \qquad (2.1.4)$$

where

$$f_C^*(\omega;\omega_h,r) = \frac{1}{W}\sum_{\alpha=-r}^{r}(1-\frac{|\alpha|}{r+1})\cos\omega\alpha\Delta t\ \cos\omega_h\alpha\Delta t$$

$$= \frac{1}{W}\left\{1+2\sum_{\alpha=1}^{r}(1-\frac{\alpha}{r+1})\cos\omega\alpha\Delta t\ \cos\omega_h\alpha\Delta t\right\} \qquad (2.1.5)$$

and

$$f_Q^*(\omega;\omega_h,r) = \frac{1}{W}\sum_{\alpha=-r}^{r}(1-\frac{|\alpha|}{r+1})\sin\omega\alpha\Delta t\ \sin\omega_h\alpha\Delta t$$

$$= \frac{2}{W}\sum_{\alpha=1}^{r}(1-\frac{\alpha}{r+1})\sin\omega\alpha\Delta t\ \sin\omega_h\alpha\Delta t \qquad (2.1.6)$$

are the co-spectral and quadrature-spectral windows of the estimators.

To simplify the notation we transform to a new variable

$$x = \frac{\pi\omega}{W}, \qquad (2.1.7)$$

and (2.1.5) and (2.1.6) become

$$f_C^*(x;x_h,r) = \frac{1}{\pi}\left\{1+2\sum_{\alpha=1}^{r}(1-\frac{\alpha}{r+1})\cos\alpha x\ \cos\alpha x_h\right\} \qquad (2.1.8)$$

and

$$f_Q^*(x;x_h,r) = \frac{2}{\pi}\sum_{\alpha=1}^{r}(1-\frac{\alpha}{r+1})\sin\alpha x\ \sin\alpha x_h, \qquad (2.1.9)$$

where the range of x is $0 \leq x \leq \pi$.

2.2 Non-negativity of the Co-spectral Window In order to better examine certain properties of the co-spectral window it will be necessary to evaluate the summation appearing in (2.1.8). The co-spectral window may be written

$$\pi f_C^*(x;x_h r) = 1 + 2 \sum_{\alpha=1}^{r} (1-\frac{\alpha}{r+1}) \cos \alpha x \cos \alpha x_h$$

$$= 1 + \sum_{\alpha=1}^{r} (1-\frac{\alpha}{r+1}) \quad \cos \alpha(x+x_h) + \cos \alpha(x-x_h) \tag{2.2.1}$$

$$= 1 + \sum_{\alpha=1}^{r} (1-\frac{\alpha}{r+1}) \cos \alpha\theta + \sum_{\alpha=1}^{r} (1-\frac{\alpha}{r+1}) \cos \alpha\phi$$

$$= 1 + g(\theta) + g(\phi)$$

where

$$\theta = x+x_h, \ \phi = x-x_h \tag{2.2.2}$$

and

$$g(\theta) = \sum_{\alpha=1}^{r} \cos \alpha\theta - \frac{1}{r+1} \sum_{\alpha=1}^{r} \alpha \cos \alpha\theta. \tag{2.2.3}$$

To evaluate (2.2.3) we note that

$$\sum_{\alpha=1}^{r} \cos \alpha\theta = \frac{\cos(\frac{r+1}{2})\theta \sin \frac{r\theta}{2}}{\sin \frac{\theta}{2}}$$

$$= \frac{\cos \theta + \cos r\theta - \cos (r+1)\theta - 1}{2(1- \cos \theta)}. \tag{2.2.4}$$

Furthermore since

$$\sum_{\alpha=1}^{r} \sin \alpha\theta = \frac{\sin (\frac{r+1}{2})\theta \sin \frac{r\theta}{2}}{\sin \frac{\theta}{2}}$$

$$= \frac{\sin \theta + \sin r\theta - \sin(r+1)\theta}{2(1- \cos \theta)}, \tag{2.2.5}$$

we may evaluate the second sum of (2.2.3) as

$$\sum_{\alpha=1}^{r} \alpha \cos \alpha\theta = \frac{d}{d\theta} \sum_{\alpha=1}^{r} \sin \alpha\theta = \frac{(r+1)\cos r\theta - r\cos(r+1)\theta - 1}{2(1-\cos\theta)} \tag{2.2.6}$$

Then from (2.2.4) and (2.2.6) we obtain

$$g(\theta) = \frac{(r+1)\cos\theta - \cos(r+1)\theta - r}{2(r+1)(1-\cos\theta)}, \tag{2.2.7}$$

and, to simplify notation, we define

$$h(\theta) = g(\theta) + \tfrac{1}{2} = \frac{1-\cos(r+1)\theta}{2(r+1)(1-\cos\theta)}. \tag{2.2.8}$$

From (2.2.1) and (2.2.2) we may write

$$f_C^*(x;x_h,r) = \frac{1}{\pi}\left\{h(\theta) + h(\phi)\right\}$$

$$= \frac{1}{2\pi(r+1)}\left\{\frac{1-\cos(r+1)(x+x_h)}{1-\cos(x+x_h)} + \frac{1-\cos(r+1)(x-x_h)}{1-\cos(x-x_h)}\right\}. \tag{2.2.9}$$

In this form $f_C^*(x;x_h,r)$ is not defined for $x = x_h$. However we note from (2.2.1) that for $x = x_h$, $x_h \neq 0$ or π,

$$\pi f_C^*(x;x_h,r) = 1 + g(0) + g(2x_h) \tag{2.2.10}$$

and for $x = x_h$, $x_h = 0$ or π,

$$\pi f_C^*(x;x_h,r) = 1 + 2g(0). \tag{2.2.11}$$

But

$$g(0) = \sum_{\alpha=1}^{r} \left(1 - \frac{\alpha}{r+1}\right) = \frac{r}{2} \qquad (2.2.12)$$

and we may express the co-spectral window as

$$f_C^*(x;x_h,r) = \begin{cases} \frac{1}{\pi}\left\{h(x+x_h) + h(x-x_h)\right\}, & x \neq x_h \\ \frac{1}{\pi}\left\{\frac{r+1}{2} + h(2x_h)\right\}, & x = x_h,\ x_h \neq 0,\ \pi \\ \frac{1}{\pi}\ (r+1), & x = x_h,\ x_h = 0,\ \pi, \end{cases} \qquad (2.2.13)$$

where the h-function is defined by (2.2.8). Also from (2.2.8) we see that

$$h(\theta) \geq 0,\ \theta \neq 0, \qquad (2.2.14)$$

so that

$$f_C^*(x;x_h,r) \geq 0 \qquad (2.2.15)$$

for all admissible values of x, x_h, and r, that is, $0 \leq x, x_h \leq \pi$, $r = 1, 2, 3, \ldots$. Therefore the co-spectral window, as defined in (2.1.5), (2.1.8), and (2.2.13), is non-negative for all relevant values of x, x_h, and r.

2.3 <u>Other Properties of the Co-spectral Window</u> The co-spectral window exhibits symmetry in x and x_h jointly.

For given r, the window for estimating $C_{xy}(\pi - x_h)$ is the reflection, about $\frac{\pi}{2}$, of the window for estimating $C_{xy}(x_h)$, that is

$$f_C^*(x; x_h, r) = f_C^*(\pi - x;\ \pi - x_h,\ r). \tag{2.3.1}$$

The area under the co-spectral window is unity for all x_h and r since, from (2.1.8),

$$\int_0^\pi f_C^*(x; x_x, r)dx = \int_0^\pi \frac{1}{\pi}\left\{1 + 2\sum_{\alpha=1}^{r} \left(1 - \frac{\alpha}{r+1}\right) \cos \alpha x \cos \alpha x_h\right\} dx$$

$$= 1 + 2\sum_{\alpha=1}^{r} \left(1 - \frac{\alpha}{r+1}\right) \cos \alpha x_h \int_0^\pi \cos \alpha x\, dx = 1. \tag{2.3.2}$$

The main lobe of the co-spectral window occurs at $x = x_h$ (or, equivalently, at $\omega = \omega_h$), as in the case of the standard estimator; see Figure 1. The height of the ordinate at $x = x_h$ is, from (2.2.13),

$$f_C^*(x_h; x_h, r) = \begin{cases} \frac{1}{\pi}\left\{\frac{r+1}{2} + h(2x_h)\right\}, & x_h \neq 0, \pi \\ \frac{1}{\pi}(r+1) & x_h = 0, \pi, \end{cases} \tag{2.3.3}$$

where, from (2.2.8)

$$h(2x_h) = \frac{1 - \cos 2(r+1)x_h}{2(r+1)(1 - \cos 2x_h)} \quad \frac{\sin^2(r+1)x_h}{2(r+1)\sin^2 x_h}. \tag{2.3.4}$$

The derivative of $f_C^*(x;x_h,r)$, from (2.2.13), is given by

$$\frac{d}{dx}f_C^*(x;x_h,r) = \begin{cases} \frac{1}{\pi}\{h'(x+x_h) + h'(x-x_h)\}, & x \neq x_h \\ \frac{1}{\pi}h'(2x_h), & x = x_h,\ x_h \neq 0,\ \pi \\ 0, & x = x_h,\ x_h = 0,\ \pi, \end{cases} \tag{2.3.5}$$

where $h'(\theta)$ denotes $\frac{d}{dx}h(\theta)$, so that, from (2.2.8)

$$h'(\theta) = \frac{(r+1)(1-\cos\theta)\sin(r+1)\theta - \sin\theta\{1-\cos(r+1)\theta\}}{2(r+1)(1-\cos\theta)^2}. \tag{2.3.6}$$

In particular $h'(2x_h)$ reduces to

$$h'(2x_h) = \frac{\sin(r+1)x_h}{2(r+1)\sin^3 x_h}\{(r+1)\cos(r+1)x_h \sin x_h - \cos x_h \sin(r+1)x_h\}, \tag{2.3.7}$$

so that, in general, the peak of the main lobe of the co-spectral window does not occur at $x = x_h$, $x_h \neq 0, \pi$.

From (2.1.8) we obtain

$$\frac{d}{dx}f_C^*(x;x_h,r) = \frac{2}{\pi}\sum_{\alpha=1}^{r}\left(\frac{\alpha^2}{r+1} - \alpha\right)\cos\alpha x_h \sin\alpha x, \tag{2.3.8}$$

from which we note that

$$\frac{d}{dx}f_C^*(0;x_h,r) = \frac{d}{dx}f_C^*(\pi;x_h,r) = 0 \tag{2.3.9}$$

for all x_h and r. Hence the spectral window always achieves a relative maximum or minimum value at the end-points of the interval $0 \leq x \leq \pi$.

2.4 <u>The Quadrature-Spectral Window</u> The quadrature-spectral window (2.1.9) can be expressed, for $x \neq x_h$, as

$$\begin{aligned}
f_Q^*(x;x_h,r) &= \frac{2}{\pi} \sum_{\alpha=1}^{r} (1 - \frac{\alpha}{r+1}) \sin \alpha x \sin \alpha x_h \\
&= \frac{1}{\pi} \sum_{\alpha=1}^{r} (1 - \frac{\alpha}{r+1}) \{\cos \alpha(x-x_h) - \cos \alpha(x+x_h)\} \\
&= \frac{1}{\pi} \sum_{\alpha=1}^{r} (1 - \frac{\alpha}{r+1}) (\cos \alpha\phi - \cos \alpha\theta) \\
&= \frac{1}{\pi}\{g(\phi) - g(\theta)\} \\
&= \frac{1}{\pi}\{h(\phi) - h(\theta)\}, \qquad (2.4.1)
\end{aligned}$$

where ϕ, θ, g and h are defined as in section 2.2. The cases where $x = x_h$ may be evaluated as in section 2.2 and $f_Q^*(x;x_h,r)$ may be expressed, analogous to (2.2.13), as

$$f_Q^*(x;x_h,r) = \begin{cases} \frac{1}{\pi}\{h(x - x_h) - h(x + x_h)\}, & x \neq x_h \\ \frac{1}{\pi}\{\frac{r+1}{2} - h(2x_h)\}, & x = x_h,\ x_h \neq 0, \pi \\ 0, & x = x_h,\ x_h = 0, \pi. \end{cases} \qquad (2.4.2)$$

In contrast with the results of section 2.2, the quadrature-spectral window is not non-negative for all admissible values of x, x_h, and r. Note that we may choose, for any value of r,

$$x - x_h = \frac{2\pi}{r+1} \tag{2.4.3}$$

so that, from (2.2.8), $h(x - x_h) = 0$. Then for any choice of x and x_h, satisfying (2.4.3), such that

$$x + x_h \neq \frac{2k\pi}{r+1} \tag{2.4.4}$$

where k is an integer, we obtain $h(x + x_h) > 0$ and hence $f_Q^*(x;x_h,r) < 0$.

For $x \neq x_h$ we have, from (2.2.8) and (2.4.2)

$$2\pi(r+1)f_Q^*(x;x_h,r) = \frac{1-\cos(r+1)(x-x_h)}{1-\cos(x-x_h)} - \frac{1-\cos(r+1)(x+x_h)}{1-\cos(x+x_h)}$$

$$= \{1-\cos(r+1)(x-x_h)\}\left\{\frac{1}{1-\cos(x-x_h)} - \frac{1}{1-\cos(x+x_h)}\right\} + \frac{2\sin(r+1)x\sin(r+1)x_h}{1-\cos(x+x_h)} \tag{2.4.5}$$

$$= \frac{2\sin x\sin x_h\{1-\cos(r+1)(x-x_h)\}}{(\cos x - \cos x_h)^2} + \frac{2\sin(r+1)x\sin(r+1)x_h}{1-\cos(x+x_h)}.$$

We now place a restriction on the values of r which will be considered admissible for given values of x_h.

Specifically we restrict ourselves to values of $r = r'$ such that, for any given x_h,

$$\sin(r'+1)x_h = 0. \tag{2.4.6}$$

This restricted class of quadrature-spectral windows will be denoted by $f_Q^*(x;x_h,r')$ and may be regarded as a subset, in terms of r, of the set $f_Q^*(x;x_h,r)$. The quadrature-spectral estimators characterized by this subset of spectral windows are obtained from (2.1.2) by replacing r by r'. Such estimators will be indicated by $Q_{xy}^*(\omega_h,r')$.

We see at once that for $x \neq x_h$

$$f_Q^*(x;x_h,r') \geq 0 \tag{2.4.7}$$

since, in (2.4.5), the first term is non-negative and the second term vanishes. Also since $h(2x_h) = 0$ for $r = r'$, see (2.3.4) and (2.4.6), it follows from (2.4.2) that the inequality above holds for all admissible values of x, x_h, and r'. The restricted quadrature-spectral window may be defined, analogous to (2.4.2), as

$$f_Q^*(x;x_h,r') = \begin{cases} \frac{1}{\pi} \; h(x - x_h) - h(x + x_h) \;, & x \neq x_h \\ \frac{1}{\pi}\left(\frac{r'+1}{2}\right), & x = x_h,\ x_h \neq 0,\ \pi \\ 0, & x = x_h,\ x_h = 0,\ \pi. \end{cases} \tag{2.4.8}$$

<u>2.5 The Restricted Co-spectral Estimator</u> Restricted co-spectral estimators $C^*_{xy}(\omega_h,r')$ with spectral windows $f^*_C(x;x_h,r')$ may be defined, analogous to the quadrature-spectral case, in terms of the restriction given in (2.4.6). The pair of restricted estimators, $C^*_{xy}(\omega_h,r')$ and $Q^*_{xy}(\omega_h,r')$, provide the means for obtaining point estimates, with non-negative spectral windows, of both the co-spectral and quadrature-spectral density functions for the same values of $r = r'$.

It will be shown in section 2.6 that the subset of co-spectral windows $f^*_C(x;x_h,r')$ has several desirable properties not shared by the whole set $f^*_C(x;x_h,r)$. In cases where only co-spectral estimates are required, the advantages of the restricted co-spectral window must be weighed against the loss of flexibility associated with the restricted estimator in selecting a method of estimation. As an example of a situation where only co-spectral estimates are required, consider equation (1.4.3). The cross-spectral densities appearing in this expression may be expanded in accordance with (1.5.8) to give

$$\Phi_{xy}(\omega) + \Phi_{yx}(\omega) = C_{xy}(\omega) + iQ_{xy}(\omega) + C_{yx}(\omega) + iQ_{yx}(\omega)$$

$$= 2C_{xy}(\omega) \qquad (2.5.1)$$

from (1.5.12) and (1.5.13).

The loss of flexibility associated with the use of the restricted estimators arises from the dependence of admissible values of r upon x_h. In many cases this dependence need not be a serious handicap. It seems reasonable that many estimation problems might be framed in terms of a requirement for point estimates of the spectral densities at equally spaced points on the x scale, for example at $x_h = 0, \frac{\pi}{10}, \frac{2\pi}{10}, \ldots, \frac{9\pi}{10}, \pi$, or, more generally, at the sequence of points

$$x_h = \frac{p\pi}{n},\ p = 0, 1, \ldots, n. \tag{2.5.2}$$

If such be the case and if we choose r' to satisfy

$$\sin(r'+1)\frac{\pi}{n} = 0 \tag{2.5.3}$$

then r' will also satisfy

$$\sin(r'+1)\frac{p\pi}{n} = 0,\ p = 0, 1, \ldots, n, \tag{2.5.4}$$

so that we may estimate at all the points (2.5.2) for the same values of r'.

The values of r' which satisfy (2.5.3) are given by

$$r' = in - 1,\ i = 1, 2, 3, \ldots . \tag{2.5.5}$$

If $n = 10$, we may estimate both densities at the points $x_h = 0, \frac{\pi}{10}, \frac{2\pi}{10}, \ldots, \frac{9\pi}{10}, \pi$ for $r' = 9, 19, 29, \ldots$.

We shall assume, henceforth, that we are interested in obtaining point estimates of the spectral densities at the points given by (2.5.2).

2.6 Properties of the Restricted Co-spectral Window The restricted co-spectral windows are non-negative and have unit area since they possess the properties of the whole set $f_C^*(x;x_h,r)$.

The height of the main lobe, analogous to (2.3.3) is given by

$$f_C^*(x_h;x_h,r') = \begin{cases} \frac{1}{\pi}(\frac{r'+1}{2}), & x_h \neq 0, \pi \\ \frac{1}{\pi}(r'+1), & x_h = 0, \pi, \end{cases} \tag{2.6.1}$$

since $h(2x_h) = 0$ from (2.4.6) and (2.3.4). Furthermore since

$$\frac{d}{dx}f_C^*(x_h;x_h,r') = 0 \tag{2.6.2}$$

for all x_h, the peak of the main lobe occurs at $x = x_h$. In terms of the set of points (2.5.2), the ordinates at $x_h = \frac{p\pi}{n}$ are given by

$$f_C^*\left(\frac{p\pi}{n};\frac{p\pi}{n},r'\right) = \begin{cases} \frac{1}{\pi}\left(\frac{r'+1}{2}\right), & p = 1, 2, \ldots, n-1 \\ \frac{1}{\pi}(r'+1), & p = 0, n\ . \end{cases} \tag{2.6.3}$$

From (2.2.9) we obtain, for $x \neq x_h$,

$$f_C^*(x;x_h,r) = \frac{1}{2\pi(r+1)}\left\{\frac{1-\cos(r+1)(x+x_h)}{1-\cos(x+x_h)} + \frac{1-\cos(r+1)(x+x_h)}{1-\cos(x-x_h)} - \frac{2\sin(r+1)x\ \sin(r+1)x_h}{1-\cos(x-x_h)}\right\}, \tag{2.6.4}$$

so that

$$f_C^*(x;x_h,r') = \frac{1-\cos(r'+1)(x+x_h)}{2\pi(r'+1)}\left\{\frac{1}{1-\cos(x+x_h)} + \frac{1}{1-\cos(x-x_h)}\right\} \tag{2.6.5}$$

because of (2.4.6). Therefore the values of x which satisfy

$$\cos(r'+1)(x+x_h)-1 = 0 \tag{2.6.6}$$

will be zeros of $f_C^*(x;x_h,r')$. The solutions of (2.6.6) are of the form

$$x = \frac{2j\pi}{r'+1} - x_h \tag{2.6.7}$$

where j takes on all integer values such that $0 \leq x \leq \pi$ and $x \neq x_h$.

The distance between successive zeros, excluding the pair enclosing the point $x = x_h$, is called the lobe width. From (2.6.7) the lobe width is seen to be $\frac{2\pi}{r'+1}$. The width of the main lobe is $\frac{4\pi}{r'+1}$. Note that the lobe width can be made arbitrarily small by choosing r' sufficiently large.

The ordinates for $\pi f_C^*(x;x_h,r')$, with $n = 10$ and $r' = 9$, are listed in Table 1 and plotted in Figure 2 for selected values of x. These ordinates were calculated from the form

$$\pi f_C^*(x;x_h,r') = \begin{cases} \dfrac{1-\cos(r'+1)(x+x_h) \quad 1-\cos x \cos x_h}{(r'+1)(\cos x - \cos x_h)^2}, & x \neq x_h \\ \dfrac{r'+1}{2}, \; x = x_h, \; x_h \neq 0, \pi & \\ r'+1, \; x = x_h, \; x_h = 0, \pi & \end{cases} \qquad (2.6.8)$$

where the expression for $\pi f_C^*(x;x_h,r')$, $x \neq x_h$, was derived from (2.6.5). Ordinates and plots are given only for values of $x_h \leq \frac{\pi}{2}$ because of the symmetry property (2.3.1) of these functions.

	$\pi f_C^*(x;x_n,9)$					
	x_n					
x	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$,
0	10	4.09	0	.475	0	.200
$\frac{\pi}{10}$	4.09	5	2.29	0	.339	0
$\frac{2\pi}{10}$	0	2.29	5	2.01	0	.306
$\frac{3\pi}{10}$	.475	0	2.01	5	2.25	0
$\frac{4\pi}{10}$	0	.339	0	2.25	5	2.09
$\frac{5\pi}{10}$	.200	0	.306	0	2.09	5
$\frac{6\pi}{10}$	0	.174	0	.299	0	2.09
$\frac{7\pi}{10}$	.127	0	.152	0	.299	0
$\frac{8\pi}{10}$	0	.114	0	.152	0	.306
$\frac{9\pi}{10}$	.103	0	.114	0	.174	0
π	0	.103	0	.127	0	.200

Table 1: Ordinates of Co-spectral Windows

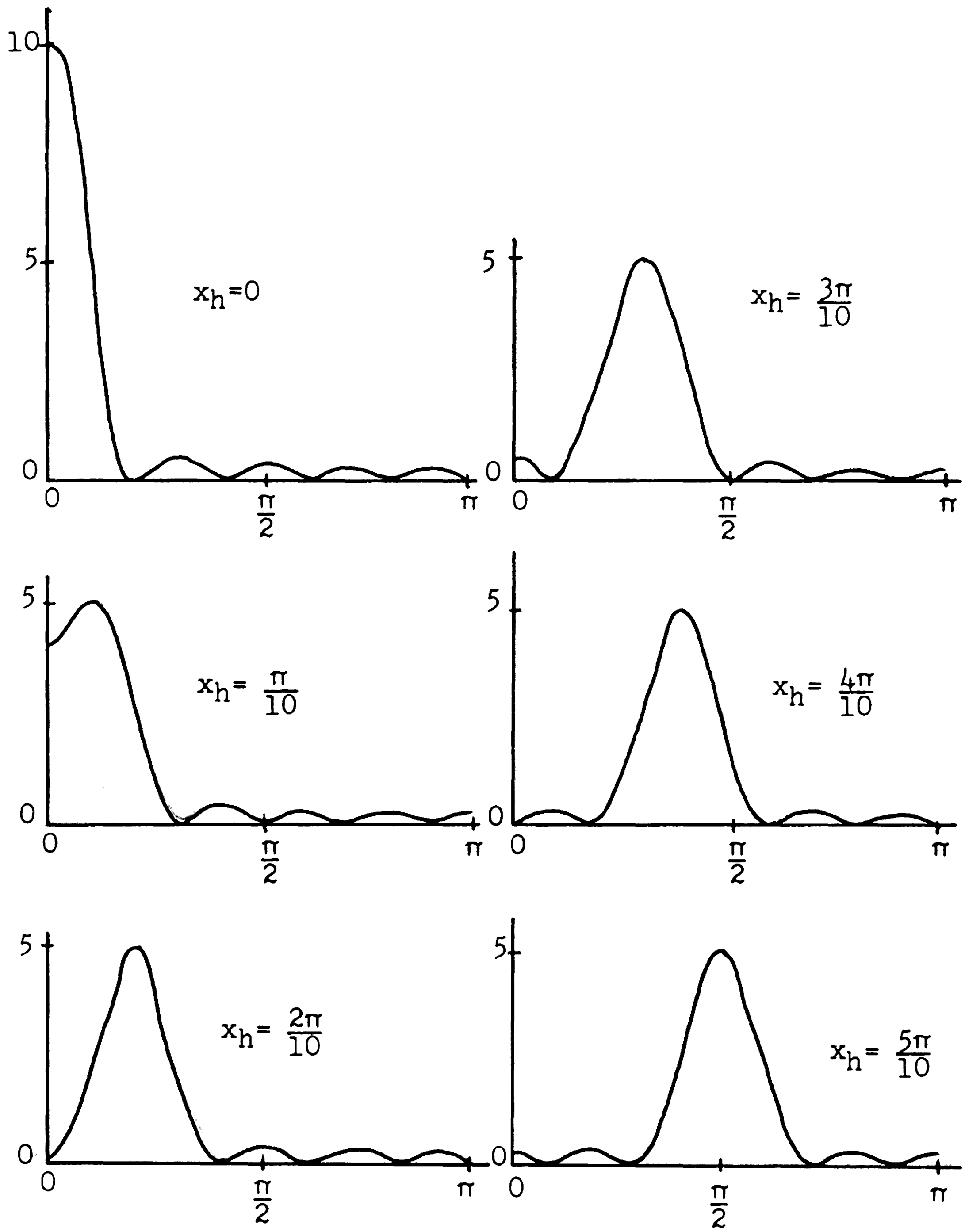

Figure 2: Co-spectral Windows for $n = 10$, $r' = 9$

2.7 <u>Area of Main Lobe</u> The area of the main lobe of the spectral window is a measure of the proportion of the total weighting which occurs in the immediate vicinity of the point under estimation. It is obviously desirable that this area be large, i.e., near unity. An explicit expression for the area of the main lobe of the modified co-spectral window is developed below, where the symbol AML is used to denote the required area. Since the width of the main lobe is $\frac{4\pi}{r'+1}$ and since we require point estimates at points $\frac{\pi}{r'+1}$ units apart, we must consider three cases, see Figure 2,

$$\text{(i)} \quad x_h = 0, \pi$$

$$\text{(ii)} \quad x_h = \frac{\pi}{r'+1}, \frac{r'\pi}{r'+1} \tag{2.7.1}$$

$$\text{(iii)} \quad x_h = \frac{j\pi}{r'+1}, \; j = 2, 3, \ldots, \frac{(r'-1)\pi}{r'+1}.$$

We shall evaluate AML for case (iii) and then indicate the results for cases (i) and (ii). For case (iii)

$$\text{AML} = \int_{x_h - \frac{2\pi}{r'+1}}^{x_h + \frac{2\pi}{r'+1}} f_C^*(x;x_h,r')dx \tag{2.7.2}$$

which, from (2.1.8), is given by

$$AML = \frac{4}{r'+1} + \frac{2}{\pi} \sum_{\alpha=1}^{r'} (1 - \frac{\alpha}{r'+1}) \cos \alpha x_h \int_{x_h - \frac{2\pi}{r'+1}}^{x_h + \frac{2\pi}{r'+1}} \cos \alpha x \, dx$$

$$= \frac{4}{r'+1} + \frac{2}{\pi} \sum_{\alpha=1}^{r'} (\frac{1}{\alpha} - \frac{1}{r'+1}) \cos \alpha x_h \left\{ \sin \alpha (x_h + \frac{2\pi}{r'+1}) - \sin \alpha (x_h - \frac{2\pi}{r'+1}) \right\}$$

$$= \frac{4}{r'+1} + \frac{4}{\pi} \sum_{\alpha=1}^{r'} (\frac{1}{\alpha} - \frac{1}{r'+1}) \cos^2 \alpha x_h \sin \frac{2\alpha\pi}{r'+1}$$

$$= \frac{4}{r'+1} + \frac{4}{\pi} S_1 - \frac{4}{\pi(r'+1)} S_2 \qquad (2.7.3)$$

where

$$S_1 = \sum_{\alpha=1}^{r'} \frac{1}{\alpha} \cos^2 \alpha x_h \sin \frac{2\alpha\pi}{r'+1} \qquad (2.7.4)$$

and

$$S_2 = \sum_{\alpha=1}^{r'} \cos^2 \alpha x_h \sin \frac{2\alpha\pi}{r'+1} . \qquad (2.7.5)$$

The quantity S_2 may be written

$$S_2 = \tfrac{1}{2} \sum_{\alpha=1}^{r'} (1 + \cos 2\alpha x_h) \sin \frac{2\alpha\pi}{r'+1}$$

$$= \tfrac{1}{2} \sum_{\alpha=1}^{r'} \sin \alpha (\frac{2\pi}{r'+1}) + \frac{1}{4} \sum_{\alpha=1}^{r'} \sin \alpha (\frac{2\pi}{r'+1} + 2x_h)$$

$$+ \frac{1}{4} \sum_{\alpha=1}^{r'} \sin \alpha (\frac{2\pi}{r'+1} - 2x_h) .$$

But since

$$\sum_{\alpha=1}^{r'} \sin \alpha\, \theta = \frac{\sin \frac{r'+1}{2}\theta \sin \frac{r'}{2}\theta}{\sin \frac{\theta}{2}} \tag{2.7.6}$$

it follows that $S_2 = 0$ because

$$\sin \frac{r'+1}{2} \cdot \frac{2\pi}{r'+1} = \sin \pi = 0 \tag{2.7.7}$$

and

$$\sin \frac{r'+1}{2}\left(\frac{2\pi}{r'+1} \pm 2x_h\right) = \sin(1 \pm j)\pi = 0 \tag{2.7.8}$$

since j is an integer (2.7.1). Therefore we obtain, for case (iii),

$$\text{AML} = \frac{4}{r'+1} + \frac{4}{\pi}\sum_{\alpha=1}^{r'} \frac{1}{\alpha} \cos^2\alpha x_h \sin \frac{2\alpha\pi}{r'+1}. \tag{2.7.9}$$

Analogous results for cases (i) and (ii) are

$$\text{AML} = \begin{cases} \frac{2}{r'+1}+\frac{2}{\pi} \sum_{\alpha=1}^{r'} \frac{1}{\alpha} \sin \frac{2\alpha\pi}{r'+1}, & x_h = 0,\ \pi \\ \frac{3}{r'+1}+\frac{2}{\pi} \sum_{\alpha=1}^{r'} \frac{1}{\alpha} \cos \frac{\alpha\pi}{r'+1} \sin \frac{3\alpha\pi}{r'+1}, & x_h = \frac{\pi}{r'+1},\ \frac{r'\pi}{r'+1} \end{cases}.$$

Attempts to evaluate the finite series in (2.7.9) and (2.7.10) have not been fruitful. However numerical determinations have been made for selected values of x_h and r'. For $x_h = \frac{\pi}{2}$, the admissible values of r' are $r' =$ 3, 5, 7, ..., i.e., the odd integers. The value $r' = 1$ is not considered admissible since the width of the main lobe is 2π for $r' = 1$. Computed values of AML for $x_h = \frac{\pi}{2}$ are

r'	$AML(x_h = \frac{\pi}{2})$
3	1.0000
5	0.9423
7	0.9244
9	0.9177
19	0.9061
29	0.8996

(2.7.11)

For $x_h = 0$, the admissible values of r' are $r' = 1, 2, \ldots$. Computed values of AML are

r'	$AML(x_h = 0)$
1	1.0000
2	0.9423
3	0.9244

(2.7.12)

This would suggest that the area of the main lobe behaves in the same manner for different values of x_h, with respect to the admissible values of r' for each x_h. This supposition has been verified for other values of x_h and the corresponding first few admissible values of r'. This result, however, has not been proved.

The asymptotic behavior of a general class of estimators, of which the one under discussion is a member, has been investigated [8] and it has been shown that the spectral windows of this class behave as Dirac functions as r becomes infinite. This behavior is apparent since, from (2.6.8),

$$f_C^*(x;x_h,r') \longrightarrow \begin{cases} 0, & x \neq x_h \\ \infty, & x = x_h \end{cases} \tag{2.7.13}$$

as $r' \longrightarrow \infty$ and since the area under $f_C^*(x;x_h,r')$ is unity for all r'.

2.8 Properties of the Restricted Quadrature-Spectral Window From (2.2.13), (2.6.1), and (2.4.8) we find that

$$f_C^*(x;x_h,r') - f_Q^*(x;x_h,r') = \begin{cases} \frac{1}{\pi}\, 2h(x+x_h)\ , & x \neq x_h \\ 0, & x = x_h,\ x_h \neq 0,\ \pi \\ \frac{1}{\pi}(r'+1), & x = x_h,\ x_h = 0,\ \pi, \end{cases} \tag{2.8.1}$$

and since $h(x+x_h) \geq 0$ it follows that the modified quadrature-spectral window is bounded above by the modified co-spectral window. Also the heights of the main lobes are the same for the two windows, except at $x_h = 0$ or π, in which case the quadrature-spectral window is identically zero. Furthermore for $x_h \neq 0, \pi$, the area under $f_Q^*(x;x_h,r')$ approaches unity as r becomes infinite since

$$\lim_{r' \to \infty} \frac{2}{\pi} h(x+x_h) = \lim_{r' \to \infty} \frac{1-\cos(r'+1)(x+x_h)}{\pi(r+1)\{1-\cos(x+x_h)\}} = 0. \qquad (2.8.2)$$

From (2.8.1) we also infer that the zeros of $f_C^*(x;x_h,r')$ are zeros of $f_Q^*(x;x_h,r)$, though the converse is not necessarily true since $f_Q^*(0;x_h,r) = 0$ for all x_h and r while $f_C^*(0;x_h,r) \geq 0$. Similarly for $x = \pi$.

In Table 2 below ordinates of the modified co- and quadrature-spectral windows are listed for $x_h = \frac{\pi}{2}$ and $r' = 9$.

From (2.8.1) we may also infer that the asymptotic behavior of $f_Q^*(x;x_h,r')$ is the same as that of $f_C^*(x;x_h,r')$.

x	$\pi f_C^*(x;\frac{\pi}{2},9)$	$\pi f_Q^*(x;\frac{\pi}{2},9)$
0	.200	0
$\frac{\pi}{10}$	0	0
$\frac{2\pi}{10}$	.306	.280
$\frac{3\pi}{10}$	0	0
$\frac{4\pi}{10}$	2.09	1.99
$\frac{5\pi}{10}$	5	5
$\frac{6\pi}{10}$	2.09	1.99
$\frac{7\pi}{10}$	0	0
$\frac{8\pi}{10}$	.306	.280
$\frac{9\pi}{10}$	0	0
π	.200	0

Table 2: Ordinates of Modified Co-spectral and Quadrature-spectral Windows

3. A RANDOMIZED CROSS-SPECTRAL ESTIMATOR

3.1 Background The standard methods for estimation of cross-spectra, as outlined in Chapter 1, are feasible, generally, only if high-speed digital computing equipment is available. For example, if $x(t)$ and $y(t)$ are each measured, at intervals Δt, for n points, and if $r\Delta t$ is the maximum time difference for which the cross-covariance function is to be estimated, then the number of multiplicative operations necessary to obtain the required cross-covariance estimators (1.6.3) is of the order $n(2r+1)$. Even for moderate values of n and r, say $n = 500$ and $r = 20$, these methods are beyond the reach of research groups equipped only with desk-type calculators.

A need seems to exist, then, for a method of obtaining point estimates of cross-power which is simple enough to be carried out on desk calculators but which retains the bias properties of the standard estimators. In this chapter such an estimator is proposed and its properties examined.

3.2 Definition of the Estimator The proposed estimator is

$$\Phi'_{xy}(\omega_h) = C'_{xy}(\omega_h) + iQ'_{xy}(\omega_h) \qquad (3.2.1)$$

where

$$C'_{xy}(\omega_h) = \frac{1}{n} \sum_{u=1}^{n} x(t_u)y(t_u+k_u\Delta t)G_1(k_u) \tag{3.2.2}$$

and

$$Q'_{xy}(\omega_h) = -\frac{1}{n} \sum_{u=1}^{n} x(t_u)y(t_u+m_u\Delta t)G_2(m_u), \tag{3.2.3}$$

and where the k_u are discrete random variables, identically and independently distributed according to $p_1(k)$, $k = -r, \ldots, r$; the m_u are also discrete random variables, identically and independently distributed according to $p_2(m)$, $m = -r, \ldots, r$, and are independent of the k_u; and $G_1(k_u)$ and $G_2(m_u)$ are weight functions to be specified later.

In practice, once $p_1(k)$, $p_2(m)$, $G_1(k)$, and $G_2(m)$ were given, the use of this estimator would proceed as follows: random samples of size n on the discrete variables k and m would be obtained, the products

$$x(t_u)y(t_u+k_u\Delta t)G_1(k_u)$$

and (3.2.4)

$$x(t_u)y(t_u+m_u\Delta t)G_2(m_u)$$

formed and summed to provide estimates of $C_{xy}(\omega)$, $Q_{xy}(\omega)$, and $\Phi_{xy}(\omega)$ at the point $\omega = \omega_h$.

This estimator depends, then, on the random selection of two sets of n time differences, $k_u\Delta t$ and $m_u\Delta t$, from the set of (2r+1) admissible time differences, as contrasted to the systematic evaluation of all possible time differences for the standard estimators.

3.3 Expectation of the Estimator The expectation of the randomized cross-spectral estimator may be written, from (3.2.1), as

$$E\left\{\Phi'_{xy}(\omega_h)\right\} = E\left\{C'_{xy}(\omega_h)\right\} + iE\left\{Q'_{xy}(\omega_h)\right\}. \tag{3.3.1}$$

The expectations appearing in the right side of (3.3.1) are readily obtainable. The expectation of the co-spectral estimator, $C'_{xy}(\omega_h)$, is developed in two steps. First, from (3.2.2) and (1.2.3), the conditional expectation of $C'_{xy}(\omega_h)$ for a fixed value of k_u, say $k_u = \alpha$, is given by

$$\begin{aligned} E\left\{C'_{xy}(\omega_h) \middle| k_u = \alpha\right\} &= \frac{1}{n} \sum_{u=1}^{n} \rho_{xy}(\alpha\Delta t)\ G_1(\alpha) \\ &= \rho_{xy}(\alpha\Delta t)\ G_1(\alpha). \end{aligned} \tag{3.3.2}$$

Then the unconditional expectation of $C'_{xy}(\omega_h)$, that is, the expectation on α of (3.3.2), is obtained as

$$E\left\{C'_{xy}(\omega_h)\right\} = \sum_{\alpha=-r}^{r} \rho_{xy}(\alpha\Delta t)\ G_1(\alpha)\ p_1(\alpha). \tag{3.3.3}$$

The expectation of the quadrature-spectral estimator is similarly found to be

$$E\left\{Q'_{xy}(\omega_h)\right\} = -\sum_{\alpha=-r}^{r} \rho_{xy}(\alpha\Delta t)\ G_2(\alpha)\ p_2(\alpha). \tag{3.3.4}$$

Equation (1.5.9) may be put in the form

$$\rho_{xy}(\alpha\Delta t) = \int_0^W \left\{C_{xy}(\omega)\cos \omega\alpha\Delta t - Q_{xy}(\omega)\sin \omega\alpha\Delta t\right\} d\omega.$$

Making this substitution in (3.3.3) and (3.3.4) the expectations of the co-spectral and quadrature-spectral estimators may be written

$$E\left\{C'_{xy}(\omega_h)\right\} = \int_0^W C_{xy}(\omega) \left\{\sum_{\alpha=-r}^{r} \cos \omega\alpha\Delta t \; G_1(\alpha) \; p_1(\alpha)\right\} d\omega \tag{3.3.5}$$

$$- \int_0^W Q_{xy}(\omega) \left\{\sum_{\alpha=-r}^{r} \sin \omega\alpha\Delta t \; G_1(\alpha) \; p_1(\alpha)\right\} d\omega$$

and

$$E\left\{Q'_{xy}(\omega_h)\right\} = - \int_0^W C_{xy}(\omega) \left\{\sum_{\alpha=-r}^{r} \cos \omega\alpha\Delta t \; G_2(\alpha) \; p_2(\alpha)\right\} d\omega \tag{3.3.6}$$

$$+ \int_0^W Q_{xy}(\omega) \left\{\sum_{\alpha=-r}^{r} \sin \omega\alpha\Delta t \; G_2(\alpha) \; p_2(\alpha)\right\} d\omega$$

In order that the expectations of $C'_{xy}(\omega_h)$ and $Q'_{xy}(\omega_h)$ involve only $C_{xy}(\omega)$ and $Q_{xy}(\omega)$ respectively, we impose the restrictions

$$\begin{aligned} G_1(-\alpha) \; p_1(-\alpha) &= G_1(\alpha) \; p_1(\alpha) \\ G_2(-\alpha) \; p_2(-\alpha) &= -G_2(\alpha) \; p_2(\alpha), \end{aligned} \tag{3.3.7}$$

that is, the products G_1p_1 and G_2p_2 are even and odd functions, respectively, of α. With these restrictions we obtain

$$E\left\{C'_{xy}(\omega_h)\right\} = \int_0^W C_{xy}(\omega)\left\{\sum_{\alpha=-r}^{r} \cos\omega\alpha\Delta t\ G_1(\alpha)\ p_1(\alpha)\right\} d\omega \quad (3.3.8)$$

and

$$E\left\{Q'_{xy}(\omega_h)\right\} = \int_0^W Q_{xy}(\omega)\left\{\sum_{\alpha=-r}^{r} \sin\omega\alpha\Delta t\ G_2(\alpha)\ p_2(\alpha)\right\} d\omega. \quad (3.3.9)$$

The quantities in braces above are the continuous weight functions or spectral windows for these two estimators. Note that these expectations depend on the products G_1p_1 and G_2p_2 but not specifically on the forms of p_1 and p_2 alone since G_1 and G_2 may be arbitrarily specified.

3.4 <u>Variance of the Estimator</u> The variance of the cross-spectral estimator may be expressed in terms of the variances and covariance of the co-spectral and quadrature-spectral estimators. From (3.2.1) we have

$$V\left\{\Phi'_{xy}(\omega_h)\right\} = V\left\{C'_{xy}(\omega_h)\right\} - V\left\{Q'_{xy}(\omega_h)\right\}$$

$$+ 2i\ \mathrm{Cov}\left\{C'_{xy}(\omega_h), Q'_{xy}(\omega_h)\right\}. \quad (3.4.1)$$

The first term on the right side of (3.4.1) may be evaluated in terms of the relationship

$$V\left\{C'_{xy}(\omega_h)\right\} = E\left\{C'^{2}_{xy}(\omega_h)\right\} - E^2\left\{C'_{xy}(\omega_h)\right\}. \tag{3.4.2}$$

From (3.2.2) we may write

$$C'^{2}_{xy}(\omega_h) = \tag{3.4.3}$$

$$\frac{1}{n^2}\sum_{u=1}^{n}\sum_{v=1}^{n} x(t_u)y(t_u+k_u\Delta t)x(t_v)y(t_v+k_v\Delta t)G_1(k_u)G_1(k_v)$$

and the conditional expectation of (3.4.3) for fixed values of k_u and k_v, say $k_u = \alpha$ and $k_v = \beta$ is then given by

$$E\left\{C'^{2}_{xy}(\omega_h)\middle| k_u = \alpha,\ k_v = \beta\right\} =$$

$$\frac{1}{n^2}\sum_{u=1}^{n}\sum_{v=1}^{n} E\left\{x(t_u)y(t_u+\alpha\Delta t)x(t_v)y(t_v+\beta\Delta t)\right\}G_1(\alpha)G_1(\beta). \tag{3.4.4}$$

Since the processes $\{x(t)\}$ and $\{y(t)\}$ are assumed to be jointly Gaussian with zero means (see section 1.2), the four variables within the expectation symbol in (3.4.4) possess the four-variate normal distribution with variance-covariance matrix $\Sigma = [\sigma_{ij}]$ where

$$
\begin{aligned}
\sigma_{11} &= \sigma_{33} = \rho_x(0) \\
\sigma_{22} &= \sigma_{44} = \rho_y(0) \\
\sigma_{12} &= \sigma_{21} = \rho_{xy}(\alpha\Delta t) \\
\sigma_{13} &= \sigma_{31} = \rho_x(t_u-t_v) \\
\sigma_{14} &= \sigma_{41} = \rho_{xy}(t_v-t_u+\beta\Delta t) \\
\sigma_{23} &= \sigma_{32} = \rho_{xy}(t_u-t_v+\alpha\Delta t) \\
\sigma_{24} &= \sigma_{42} = \rho_y(t_u-t_v+\alpha\Delta t-\beta\Delta t) \\
\sigma_{34} &= \sigma_{43} = \rho_{xy}(\beta\Delta t).
\end{aligned}
\tag{3.4.5}
$$

The moment generating function for the four-variate normal distribution with zero means is

$$
MGF = e^{\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\theta_i\theta_j\sigma_{ij}} \tag{3.4.6}
$$

from which it follows that the expectation in (3.4.4) is given by $\sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}$, where the σ_{ij} are as in (3.4.5). Noting that

$$
t_u-t_v = (u-v)\Delta t \tag{3.4.7}
$$

from (1.6.1), we may rewrite (3.4.4) as

$$E\left\{C'^2_{xy}(\omega_h) \mid k_u = \alpha,\ k_v = \beta\right\} =$$

$$\frac{1}{n^2} \sum_{u=1}^{n} \sum_{v=1}^{n} \Big\{ \rho_{xy}(\alpha\Delta t)\rho_{xy}(\beta\Delta t) + \rho_x[(u-v)\Delta t]\,\rho_y[(u-v+\alpha-\beta)\Delta t]$$

$$+ \rho_{xy}[(u-v+\alpha)\Delta t]\,\rho_{xy}[(v-u+\beta)\Delta t]\Big\} G_1(\alpha)G_1(\beta). \quad (3.4.8)$$

The conditional expectation (3.4.8) is a function of α and β, say $g(\alpha,\beta)$, and its expectation with respect to α and β is of the form

$$E\{g(\alpha,\beta)\} = \sum_{\alpha=-r}^{r} g(\alpha,\alpha)p_1(\alpha) + \sum_{\substack{\alpha=-r \\ \alpha\neq\beta}}^{r} \sum_{\beta=-r}^{r} g(\alpha,\beta)p_1(\alpha)p_1(\beta). \quad (3.4.9)$$

The unconditional expectation of $C'^2_{xy}(\omega_h)$ is therefore given by*

$$E\left\{C'^2_{xy}(\omega_h)\right\} = \sum_{\alpha} \frac{1}{n^2} \sum_{u} \sum_{v} \Big\{ \rho^2_{xy}(\alpha\Delta t)$$

$$+\rho_x[(u-v)\Delta t]\,\rho_y[(u-v)\Delta t] + \rho_{xy}[(u-v+\alpha)\Delta t]\,\rho_{xy}[(v-u+\alpha)\Delta t]\Big\} G_1^2(\alpha)p_1(\alpha)$$

$$+\sum_{\substack{\alpha \\ \alpha\neq\beta}} \sum_{\beta} \frac{1}{n^2} \sum_{u} \sum_{v} \Big\{ \rho_{xy}(\alpha\Delta t)\rho_{xy}(\beta\Delta t) + \rho_x[(u-v)\Delta t]\,\rho_y[(u-v+\alpha-\beta)\Delta t]$$

$$+\rho_{xy}[(u-v+\alpha)\Delta t]\,\rho_{xy}[(v-u+\beta)\Delta t]\Big\} G_1(\alpha)G_1(\beta)p_1(\alpha)p_1(\beta). \quad (3.4.10)$$

Adopting the notation

---

* Henceforth the summations on α and β extend from $-r$ to r and summations on u and v from 1 to n whenever the limits of summation are not given.

$$A(\alpha) = \frac{1}{n^2} \sum_u \sum_v \Big\{ \rho_x[(u-v)\Delta t]\, \rho_y[(u-v)\Delta t] + \rho_{xy}[(u-v+\alpha)\Delta t]\, \rho_{xy}[(v-u+\alpha)\Delta t] \Big\} \tag{3.4.11}$$

and

$$B(\alpha,\beta) = \frac{1}{n^2} \sum_u \sum_v \Big\{ \rho_x[(u-v)\Delta t]\, \rho_y[(u-v+\alpha-\beta)\Delta t] + \rho_{xy}[(u-v+\alpha)\Delta t]\; \rho_{xy}[(v-u+\beta)\Delta t] \Big\}, \tag{3.4.12}$$

the expectation of $C'^2_{xy}(\omega_h)$ may be more conveniently expressed as

$$\begin{aligned} E\{C'^2_{xy}(\omega_h)\} &= \sum_\alpha \rho^2_{xy}(\alpha\Delta t) G_1^2(\alpha) p_1(\alpha) + \sum_\alpha A(\alpha) G_1^2(\alpha) p_1(\alpha) \\ &+ \sum_{\substack{\alpha \\ \alpha\neq\beta}} \sum_\beta \rho_{xy}(\alpha\Delta t)\rho_{xy}(\beta\Delta t) G_1(\alpha) G_1(\beta) p_1(\alpha) p_1(\beta) \\ &+ \sum_{\substack{\alpha \\ \alpha\neq\beta}} \sum_\beta B(\alpha,\beta) G_1(\alpha) G_1(\beta) p_1(\alpha) p_1(\beta). \end{aligned} \tag{3.4.13}$$

Since, from (3.3.3),

$$\begin{aligned} E^2\{C_{xy}(\omega_h)\} &= \sum_\alpha \sum_\beta \rho_{xy}(\alpha\Delta t)\rho_{xy}(\beta\Delta t) G_1(\alpha) G_1(\beta) p_1(\alpha) p_1(\beta) \\ &= \sum_\alpha \rho^2_{xy}(\alpha\Delta t) G_1^2(\alpha) p_1^2(\alpha) \\ &+ \sum_{\substack{\alpha \\ \alpha\neq\beta}} \sum_\beta \rho_{xy}(\alpha\Delta t)\rho_{xy}(\beta\Delta t) G_1(\alpha) G_1(\beta) p_1(\alpha) p_1(\beta) \end{aligned} \tag{3.4.14}$$

we obtain, for (3.4.2),

$$V\{C'_{xy}(\omega_h)\} = \sum_{\alpha} \rho^2_{xy}(\alpha\Delta t)G_1^2(\alpha)p_1^2(\alpha)\left\{\frac{1}{p_1(\alpha)} - 1\right\} \tag{3.4.15}$$
$$+ \sum_{\alpha} A(\alpha)G_1^2(\alpha)p_1^2(\alpha)\frac{1}{p_1(\alpha)} + \sum_{\alpha} \sum_{\substack{\beta \\ \alpha\neq\beta}} B(\alpha,\beta)G_1(\alpha)G_1(\beta)p_1(\alpha)p_1(\beta)$$

The variance of $Q'_{xy}(\omega_h)$ is readily obtained. From (3.2.2) and (3.2.3) it is clear that the variance of $Q'_{xy}(\omega_h)$ can be obtained from the variance of $C'_{xy}(\omega_h)$ simply by replacing G_1 and p_1 by G_2 and p_2. Therefore

$$V\left\{Q'_{xy}(\omega_h)\right\} = \sum \rho^2_{xy}(\alpha\Delta t)G_2^2(\alpha)p_2^2(\alpha)\left\{\frac{1}{p_2(\alpha)} - 1\right\} \tag{3.4.16}$$
$$+ \sum_{\alpha} A(\alpha)G_2^2(\alpha)p_2^2(\alpha)\frac{1}{p_2(\alpha)} + \sum_{\alpha} \sum_{\substack{\beta \\ \alpha\neq\beta}} B(\alpha,\beta)G_2(\alpha)G_2(\beta)p_2(\alpha)p_2(\beta).$$

The covariance of $C'_{xy}(\omega_h)$ and $Q'_{xy}(\omega_h)$ may be derived by the same methods used to obtain the variance of $C'_{xy}(\omega_h)$. We note first that

$$C'_{xy}(\omega_h)Q'_{xy}(\omega_h) =$$

$$= \frac{1}{n^2} \sum_{u} \sum_{v} x(t_u)y(t_u+k_u\Delta t)x(t_v)y(t_v+m_v\Delta t)G_1(k_u)G_2(m_v). \tag{3.4.17}$$

Taking first the conditional expectation of (3.4.17) for fixed values of k_u and m_v, and then taking the expectation with respect to these fixed values leads to

$$E\left\{C'_{xy}(\omega_h)Q'_{xy}(\omega_h)\right\} =$$

$$- \sum_{\alpha}\sum_{\beta}\frac{1}{n^2}\sum_{u}\sum_{v}\Big\{\rho_{xy}(\alpha\Delta t)\rho_{xy}(\beta\Delta t)+\rho_x\left[(u-v)\Delta t\right]\rho_y\left[(u-v+\alpha-\beta)\Delta t\right]$$

$$+\rho_{xy}\left[(u-v+\alpha)\Delta t\right]\rho_{xy}\left[(v-u+\beta)\Delta t\right]\Big\}G_1(\alpha)G_2(\beta)p_1(\alpha)p_2(\beta). \tag{3.4.18}$$

The right side of (3.4.18) can be simplified to give

$$E\left\{C'_{xy}(\omega_h)Q'_{xy}(\omega_h)\right\} = -\sum_{\alpha}\left\{\rho^2_{xy}(\alpha)+A(\alpha)\right\}G_1(\alpha)G_2(\alpha)p_1(\alpha)p_2(\alpha)$$

$$-\sum_{\substack{\alpha \\ \alpha\neq\beta}}\sum_{\beta}\left\{\rho_{xy}(\alpha\Delta t)\rho_{xy}(\beta\Delta t)+B(\alpha,\beta)\right\}G_1(\alpha)G_2(\beta)p_1(\alpha)p_2(\beta). \tag{3.4.19}$$

From (3.3.3) and (3.3.4) we obtain

$$E\left\{C'_{xy}(\omega_h)\right\}E\left\{Q'_{xy}(\omega_h)\right\} =$$

$$-\sum_{\alpha}\sum_{\beta}\rho_{xy}(\alpha\Delta t)\rho_{xy}(\beta\Delta t)G_1(\alpha)G_2(\beta)p_1(\alpha)p_2(\beta)$$

$$= -\sum_{\alpha}\rho^2_{xy}(\alpha\Delta t)G_1(\alpha)G_2(\alpha)p_1(\alpha)p_2(\alpha)$$

$$-\sum_{\substack{\alpha \\ \alpha\neq\beta}}\sum_{\beta}\rho_{xy}(\alpha\Delta t)\rho_{xy}(\beta\Delta t)G_1(\alpha)G_2(\beta)p_1(\alpha)p_2(\beta).$$

Hence from (3.4.19) and (3.4.20) we find

$$\mathrm{Cov}\left\{C'_{xy}(\omega_h),Q'_{xy}(\omega_h)\right\} = -\sum_{\alpha}A(\alpha)G_1(\alpha)G_2(\alpha)p_1(\alpha)p_2(\alpha) \tag{3.4.21}$$

$$-\sum_{\substack{\alpha \\ \alpha\neq\beta}}\sum_{\beta}B(\alpha,\beta)G_1(\alpha)G_2(\beta)p_1(\alpha)p_2(\beta).$$

3.5 <u>Flexibility of the Estimator</u> In section 3.3 it was shown that the expectations, or, equivalently, the spectral windows, of the randomized co-spectral and quadrature-spectral estimators depended on the specification of the products G_1p_1 and G_2p_2 but not on the particular forms of p_1 and p_2 since G_1 and G_2 can be arbitrarily chosen.

From section 3.4 we note that the covariance of $C'_{xy}(\omega_h)$ and $Q'_{xy}(\omega_h)$ is similarly dependent only on the products G_1p_1 and G_2p_2. The first two terms of the variances of the estimators, however, depend directly on the forms of the probability distributions p_1 and p_2.

Before considering in general these two levels of specifications, i.e.

(i) the specification of G_1p_1 and G_2p_2 to provide co-spectral and quadrature-spectral windows with desirable properties, and

(ii) the specification of p_1 and p_2 so that the variances of the estimators are small, consistent with the requirement that the forms of p_1 and p_2 be such that random samples from these distributions are readily obtainable,

we shall consider a very important special case, that in which the expectations of the proposed estimators are the same as the expectations for the standard estimators.

3.6 <u>Comparison with the Standard Estimator</u> The standard co-spectral estimator, from (1.6.4), is

$$\hat{C}_{xy}(\omega_h) = \sum_{\alpha=-r}^{r} \hat{\rho}_{xy}(\alpha\Delta t)H_1(\alpha), \tag{3.6.1}$$

where we now choose to define

$$\hat{\rho}_{xy}(\alpha\Delta t) = \frac{1}{n}\sum_{u=1}^{n} x(t_u)y(t_u+\alpha\Delta t),\ \alpha = -r, \ldots, r, \tag{3.6.2}$$

rather than with the abbreviated sums, i.e., $i = 1, \ldots, n-\alpha$, as given in (1.6.3). Note that this modification does not affect the expectation of $\hat{C}_{xy}(\omega_h)$ and it will permit a valid comparison of the variances of the standard and proposed estimators, given equal expectations. This comparison of variances will provide a measure of the loss of precision due to the operational simplification associated with the random sampling feature of the proposed estimator.

Since the expectations of the standard and proposed estimators, from (1.7.2) and (3.3.3), are

$$E\left\{\hat{C}_{xy}(\omega_h)\right\} = \sum_{\alpha=-r}^{r} \rho_{xy}(\alpha\Delta t)H_1(\alpha)$$

and (3.6.3)

$$E\left\{C'_{xy}(\omega_h)\right\} = \sum_{\alpha=-r}^{r} \rho_{xy}(\alpha\Delta t)G_1(\alpha)p_1(\alpha)$$

it is clear that if we specify

$$G_1(\alpha)p_1(\alpha) = H_1(\alpha), \ \alpha = -r, \ldots, r, \qquad (3.6.4)$$

the two estimators will have the same expectation and hence the same spectral window.

The advantage of the proposed estimator lies in its simplicity relative to the standard estimator. The two co-spectral estimators may be expressed as

$$\hat{C}_{xy}(\omega_h) = \sum_{\alpha=-r}^{r} \frac{1}{n} \sum_{u=1}^{n} x(t_u)y(t_u+\alpha\Delta t)H_1(\alpha) \qquad (3.6.5)$$

and

$$C'_{xy}(\omega_h) = \frac{1}{n} \sum_{u=1}^{n} x(t_u)y(t_u+k_u\Delta t)G_1(k_u) \qquad (3.6.6)$$

where, it is recalled, the k_u are randomly selected values of α, where α has probability distribution $p_1(\alpha)$. Hence $G_1(k_u)$ is defined by

$$G_1(k_u) = \frac{H_1(k_u)}{p_1(k_u)} \qquad (3.6.7)$$

for any chosen probability distribution $p_1(\alpha)$. The use of $C'_{xy}(\omega_h)$ instead of $\hat{C}_{xy}(\omega_h)$ reduces the number of multiplicative and summation operations by a factor of $(2r+1)$, which, for r in the usual range of 10-60, represents a substantial reduction in calculations.

The gain in simplicity, however, is associated with a corresponding loss in precision which can be assessed in terms of the variances of the two estimators.

From (3.4.15), with $G_1(\alpha)p_1(\alpha) = H_1(\alpha)$, we have

$$V\left\{C'_{xy}(\omega_h)\right\} = \sum_{\alpha}\rho^2_{xy}(\alpha\Delta t)H_1^2(\alpha)\left\{\frac{1}{p_1(\alpha)} - 1\right\}$$

$$+ \sum_{\alpha}A(\alpha)H_1^2(\alpha)\frac{1}{p_1(\alpha)} + \sum_{\substack{\alpha \\ \alpha\neq\beta}}\sum_{\beta}B(\alpha,\beta)H_1(\alpha)H_1(\beta). \qquad (3.6.8)$$

The variance of $\hat{C}_{xy}(\omega_h)$ may be determined by the methods of section 3.4. We have, from (1.6.4) and (3.6.2),

$$E\left\{\hat{C}^2_{xy}(\omega_h)\right\} =$$

$$\sum_{a}\sum_{\beta}\frac{1}{n^2}\sum_{u}\sum_{v}\left\{E\left[x(t_u)y(t_u+\alpha\Delta t)x(t_v)y(t_v+\beta\Delta t)\right]H_1(\alpha)H_1(\beta)\right\}$$

$$= \sum_{\alpha}\sum_{\beta}\frac{1}{n^2}\sum_{u}\sum_{v}\Big\{\rho_{xy}(\alpha\Delta t)\rho_{xy}(\beta\Delta t)+\rho_x[(u-v)\Delta t]\,\rho_y[(u-v+\alpha-\beta)\Delta t]$$

$$+ \rho_{xy}[(u-v+\alpha)\Delta t]\,\rho_{xy}[(v-u+\beta)\Delta t]\Big\}H_1(\alpha)H_1(\beta)$$

$$= \sum_{\alpha}\sum_{\beta}\rho_{xy}(\alpha\Delta t)\rho_{xy}(\beta\Delta t)H_1(\alpha)H_1(\beta) \qquad (3.6.9)$$

$$+ \sum_{\alpha}A(\alpha)H_1^2(\alpha)+\sum_{\substack{\alpha \\ \alpha\neq\beta}}\sum_{\beta}B(\alpha,\beta)H_1(\alpha)H_1(\beta)$$

where $A(\alpha)$ and $B(\alpha,\beta)$ are given in (3.4.11) and (3.4.12).

From (1.7.2) we obtain

$$E^2\left\{\hat{C}_{xy}(\omega_h)\right\} = \sum_{\alpha}\sum_{\beta}\rho_{xy}(\alpha\Delta t)\rho_{xy}(\beta\Delta t)H_1(\alpha)H_1(\beta), \qquad (3.6.10)$$

so that

$$V\left\{\hat{C}_{xy}(\omega_h)\right\} = \sum_{\alpha}A(\alpha)H_1^2(\alpha)+\sum_{\substack{\alpha \\ \alpha\neq\beta}}\sum_{\beta}B(\alpha,\beta)H_1(\alpha)H_1(\beta). \qquad (3.6.11)$$

Therefore, by (3.6.8) and (3.6.11),

$$V\left\{C'_{xy}(\omega_h)\right\} = V\left\{\hat{C}_{xy}(\omega_h)\right\} + \sum_{\alpha}\left\{\rho^2_{xy}(\alpha\Delta t)+A(\alpha)\right\} H_1^2(\alpha)\left\{\frac{1}{p_1(\alpha)} - 1\right\} \tag{3.6.12}$$

or

$$\frac{V\left\{C'_{xy}(\omega_h)\right\}}{V\left\{\hat{C}_{xy}(\omega_h)\right\}} = 1 + \frac{\sum_{\alpha}\left\{\rho^2_{xy}(\alpha\Delta t)+A(\alpha)\right\} H_1^2(\alpha)\left\{\frac{1}{p_1(\alpha)} - 1\right\}}{V\left\{\hat{C}_{xy}(\omega_h)\right\}} . \tag{3.6.13}$$

A parallel development can be carried out for the proposed and standard quadrature-spectral estimators. By setting

$$G_2(\alpha)p_2(\alpha) = H_2(\alpha), \ \alpha = -r, \ldots, r \tag{3.6.14}$$

in (1.7.3) and (3.3.4), the two estimators have the same expectation. The variances of the estimators are given by

$$\begin{aligned} V\left\{Q'_{xy}(\omega_h)\right\} &= \sum_{\alpha}\rho^2_{xy}(\alpha\Delta t)H_2^2(\alpha)\left\{\frac{1}{p_2(\alpha)} - 1\right\} \\ &\quad + \sum_{\alpha}A(\alpha)H_2^2(\alpha)\,\frac{1}{p_2(\alpha)} + \sum_{\alpha}\sum_{\substack{\beta \\ \alpha\neq\beta}} B(\alpha,\beta)H_2(\alpha)H_2(\beta) \end{aligned} \tag{3.6.15}$$

and

$$V\left\{\hat{Q}_{xy}(\omega_h)\right\} = \sum_{\alpha}A(\alpha)H_2^2(\alpha) + \sum_{\alpha}\sum_{\substack{\beta \\ \alpha\neq\beta}} B(\alpha,\beta)H_2(\alpha)H_2(\beta), \tag{3.6.16}$$

so that

$$V\{Q'_{xy}(\omega_h)\} = V\{\hat{Q}_{xy}(\omega_h)\} \tag{3.6.17}$$
$$+ \sum_{\alpha=-r}^{r} \{\rho^2_{xy}(\alpha\Delta t)+A(\alpha)\} H^2_2(\alpha)\left\{\frac{1}{p_2(\alpha)} - 1\right\}$$

and

$$\frac{V\{Q'_{xy}(\omega_h)\}}{V\{\hat{Q}_{xy}(\omega_h)\}} = 1 + \frac{\sum_{\alpha}\{\rho^2_{xy}(\alpha\Delta t)+A(\alpha)\} H^2_2(\alpha)\left\{\frac{1}{p_2(\alpha)} - 1\right\}}{V\{\hat{Q}_{xy}(\omega_h)\}}. \tag{3.6.18}$$

It may further be shown, in a similar manner, that

$$\mathrm{Cov}\{C'_{xy}(\omega_h), Q'_{xy}(\omega_h)\} = \mathrm{Cov}\{\hat{C}_{xy}(\omega_h), \hat{Q}_{xy}(\omega_h)\}, \tag{3.6.19}$$

from which it follows that

$$V\{\Phi'_{xy}(\omega_h)\} = V\{\hat{\Phi}_{xy}(\omega_h)\} \tag{3.6.20}$$
$$+ \sum_{\alpha}\{\rho^2_{xy}(\alpha\Delta t)+A(\alpha)\}\left\{H^2_1(\alpha)\left[\frac{1}{p_1(\alpha)} - 1\right] - H^2_2(\alpha)\left[\frac{1}{p_2(\alpha)} - 1\right]\right\},$$

where

$$\hat{\Phi}_{xy}(\omega_h) = \hat{C}_{xy}(\omega_h) + i\hat{Q}_{xy}(\omega_h). \tag{3.6.21}$$

3.7 Specification of G_1p_1 and G_2p_2 In the preceding section it was shown that G_1p_1 and G_2p_2 could be chosen so that the randomized co-spectral and quadrature-spectral estimators had the same expectation as the corresponding standard estimators.

If we choose

$$G_1(\alpha)p_1(\alpha) = \frac{1}{W}\left(1 - \frac{|\alpha|}{r+1}\right)\cos \omega_h\alpha\Delta t$$

and (3.7.1)

$$G_2(\alpha)p_2(\alpha) = \frac{1}{W}\left(1 - \frac{|\alpha|}{r+1}\right)\sin \omega_h\alpha\Delta t$$

it follows, from (3.3.3), (3.3.4), (2.1.1), and (2.1.2), that the resulting randomized estimators will have the same expectations as the estimators developed in Chapter 2, i.e., they will be characterized by non-negative spectral windows.

In general if $\tilde{C}_{xy}(\omega_h)$ is a co-spectral estimator of the type

$$\tilde{C}_{xy}(\omega_h) = \sum_{\alpha=-r}^{r} \hat{\rho}_{xy}(\alpha\Delta t)K_1(\alpha), \tag{3.7.2}$$

that is, if it is expressible as a linear function of the sample cross-covariances, then the specification

$$G_1(\alpha)p_1(\alpha) = K_1(\alpha) \tag{3.7.3}$$

defines a randomized estimator, say $C''_{xy}(\omega_h)$, such that

$$E\left\{C''_{xy}(\omega_h)\right\} = E\left\{\tilde{C}_{xy}(\omega_h)\right\}. \tag{3.7.4}$$

The same argument holds for the quadrature-spectral estimators and therefore for the cross-spectral estimators.

3.8 Specification of p_1 and p_2 From section 3.5 we recall that if the products G_1p_1 and G_2p_2 are specified, p_1 and p_2 may be chosen arbitrarily and the expectations of the estimators will be unaffected. The variances of the estimators depend on the choices of p_1 and p_2 however. We consider first the specification of p_1 and p_2 which will yield minimum variance within this class of estimators.

The variance of the randomized co-spectral estimator may be written, from (3.4.15), as

$$V\left\{C'_{xy}(\omega_h)\right\} = \sum_{\alpha} \frac{R_1(\alpha)}{p_1(\alpha)} + K_1(\alpha,\beta) \tag{3.8.1}$$

where

$$R_1(\alpha) = \left\{\rho^2_{xy}(\alpha\Delta t)+A(\alpha)\right\} G_1^2(\alpha)p_1^2(\alpha) \tag{3.8.2}$$

and

$$\begin{aligned} K_1(\alpha,\beta) = &- \sum_{\alpha}\rho^2_{xy}(\alpha\Delta t)G_1^2(\alpha)p_1^2(\alpha) \\ &+ \sum_{\substack{\alpha \\ \alpha\neq\beta}}\sum_{\beta} B(\alpha,\beta)G_1(\alpha)G_1(\beta)p_1(\alpha)p_1(\beta). \end{aligned} \tag{3.8.3}$$

Note that $R_1(\alpha)$ and $K_1(\alpha,\beta)$ are independent of the particular form assigned to p_1, given that the product G_1p_1 is specified.

From (3.8.1) we may infer that

$$R_1(\alpha) \geq 0, \ \alpha = -r, \ldots, r. \tag{3.8.4}$$

To show this assume that $R_1(\alpha) < 0$ for one or more values of α, $\alpha = -r, \ldots, r$. In particular let $R_1(\alpha_0)$ be negative. If we then define $p_1(\alpha)$ as

$$p_1(\alpha) = \begin{cases} \delta, & \alpha = \alpha_0 \\ \frac{1-\delta}{2r}, & \alpha = -r, \ldots, r, \ \alpha \neq \alpha_0, \end{cases} \tag{3.8.5}$$

we can always choose δ small enough to make (3.8.1) negative. But the variance of $C'_{xy}(\omega_h)$ cannot be negative and hence we conclude that (3.8.4) must be true.

The values of $p_1(\alpha)$, $\alpha = -r, \ldots, r$, which minimize (3.8.1) are those values which minimize

$$S = \sum_\alpha \frac{R_1(\alpha)}{p_1(\alpha)} \tag{3.8.6}$$

subject to the conditions

$$\sum_\alpha p_1(\alpha) = 1 \tag{3.8.7}$$

and

$$p_1(\alpha) > 0, \ \alpha = -r, \ldots, r. \tag{3.8.8}$$

The minimization of (3.8.6) subject to the restraint (3.8.7) can be accomplished by equating to zero the derivative, with respect to $p_1(\alpha)$, of

$$S' = \sum_\alpha \frac{R_1(\alpha)}{p_1(\alpha)} + \lambda\left\{\sum_\alpha p_1(\alpha) - 1\right\}. \tag{3.8.9}$$

This procedure yields the set of equations

$$\frac{R_1(\alpha)}{p_1^2(\alpha)} = \lambda, \ \alpha = -r, \ldots, r. \tag{3.8.10}$$

From (3.8.4) it follows that $\lambda \geq 0$ and we may take the positive square roots of (3.8.10) to obtain the set of equations

$$\frac{R_1^{\frac{1}{2}}(\alpha)}{p_1(\alpha)} = \lambda^{\frac{1}{2}}, \ \alpha = -r, \ldots, r, \tag{3.8.11}$$

where $p_1(\alpha)$ now satisfies (3.8.8). We may evaluate $\lambda^{\frac{1}{2}}$ as

$$\lambda^{\frac{1}{2}} = \frac{\sum_\alpha R_1^{\frac{1}{2}}(\alpha)}{\Sigma p_1(\alpha)} = \sum_\alpha R_1^{\frac{1}{2}}(\alpha) \tag{3.8.12}$$

and hence

$$p_1(\alpha) = \frac{R_1^{\frac{1}{2}}(\alpha)}{\sum_\alpha R_1^{\frac{1}{2}}(\alpha)}, \ \alpha = -r, \ldots, r \tag{3.8.13}$$

is the required probability distribution.

By a similar argument it can be shown that the values

$$p_2(\alpha) = \frac{R_2^{\frac{1}{2}}(\alpha)}{\sum_\alpha R_2^{\frac{1}{2}}(\alpha)}, \ \alpha = -r, \ldots, r, \tag{3.8.14}$$

where

$$R_2(\alpha) = \left\{\rho_{xy}^2(\alpha\Delta t) + A(\alpha)\right\} G_2^2(\alpha) p_2^2(\alpha), \tag{3.8.15}$$

will minimize the variance of the randomized quadrature-spectral estimator $Q'_{xy}(\omega_h)$.

The results (3.8.13) and (3.8.14) are not useful results from the practical standpoint since the minimizing probability distributions depend not only on process parameters but also on the point ω_h at which the spectral densities are being estimated.

The practical approach requires that $p_1(\alpha)$ and $p_2(\alpha)$ be such that random samples from these distributions are readily obtainable. The obvious choice, for simplicity in sampling, is

$$p_1(\alpha) = p_2(\alpha) = \frac{1}{2r+1}, \ \alpha = -r, \ldots, r, \qquad (3.8.16)$$

in which case the required samples can be obtained from a table of random numbers.

The choice of the uniform distribution (3.8.16) is also defensible in terms of the variances of the estimators. The variance of the co-spectral estimator depends on the $R_1(\alpha)$, $\alpha = -r, \ldots, r$, given in (3.8.2). The $R_1(\alpha)$ are functions of the process parameters and are hence unknown. Furthermore, since the $R_1(\alpha)$ depend on the $A(\alpha)$, see (3.4.11), the estimation of the $R_1(\alpha)$ in terms of sample autocovariances and cross-covariances is prohibitive. In the absence of information as to the magnitudes of the $R_1(\alpha)$ the safest choice of $p_1(\alpha)$, to limit the first term of (3.8.1), is certainly the uniform distribution (3.8.16). If the $R_1(\alpha)$ are all equal, (3.8.13) reduces to the uniform distribution, and, in general, if the $R_1(\alpha)$ are relatively uniform in magnitude then the variance of the estimator, where $p_1(\alpha)$ is the uniform distribution, will not greatly exceed the minimum variance.

4. CROSS-SPECTRAL ESTIMATION FOR THE ANALOG CASE

4.1 Background Let $x(t)$ and $y(t)$ be realizations, on the time interval $(-\frac{T}{2},\frac{T}{2})$, of the processes $\{x(t)\}$ and $\{y(t)\}$ which are stationary, ergodic, jointly Gaussian and have zero means. Furthermore let these realizations possess the following Fourier series expansions [5] on the interval $(-\frac{T}{2},\frac{T}{2})$:

$$x(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} \{a_j \cos \omega_j t + b_j \sin \omega_j t\} \tag{4.1.1}$$

and

$$y(t) = \frac{a_0'}{2} + \sum_{j=1}^{\infty} \{a_j' \cos \omega_j t + b_j' \sin \omega_j t\} \tag{4.1.2}$$

where

$$a_j = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos \omega_j t \, dt, \; j = 0, 1, 2, \ldots, \tag{4.1.3}$$

$$b_j = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin \omega_j t \, dt, \; j = 1, 2, \ldots, \tag{4.1.4}$$

$$a_j' = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t) \cos \omega_j t \, dt, \; j = 0, 1, 2, \ldots, \tag{4.1.5}$$

$$b'_j = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t) \sin \omega_j t \, dt, \; j = 1, 2, \ldots, \tag{4.1.6}$$

and where

$$\omega_j = \frac{2\pi j}{T} \tag{4.1.7}$$

in angular frequency.

4.2 <u>Estimators</u> It is shown [5] that if the Fourier coefficients a_j, b_j, a'_j, and b'_j are given then the following quantities provide estimates of the power spectral, co-spectral, and quadrature-spectral density functions at $\omega = \omega_j$:

$$\hat{\Phi}_x(\omega_j;T) = \frac{T}{4\pi}(a_j^2+b_j^2) \tag{4.2.1}$$

$$\hat{\Phi}_y(\omega_j;T) = \frac{T}{4\pi}(a_j'^2+b_j'^2) \tag{4.2.2}$$

$$\hat{C}_{xy}(\omega_j;T) = \frac{T}{4\pi}(a_j a'_j+b_j b'_j) \tag{4.2.3}$$

$$\hat{Q}_{xy}(\omega_j;T) = \frac{T}{4\pi}(a'_j b_j-a_j b'_j). \tag{4.2.4}$$

The Fourier coefficients a_j, b_j, a'_j, and b'_j can be determined by means of suitable analog equipment in many cases. In such cases the estimators above provide extremely simple point estimates of the spectral densities. We propose to derive the expectations, variances, and co-variances of these estimators.

4.3 <u>Notation for Definite Integrals</u> In the development of the variances and covariances of the estimators of section 4.2, certain definite integrals occur with sufficient frequency to warrant the adoption of a special notation. One such integral is the following:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos \omega t \cos \omega_j t \, dt = \left. \frac{\sin(\omega-\omega_j)t}{2(\omega-\omega_j)} + \frac{\sin(\omega+\omega_j)t}{2(\omega+\omega_j)} \right|_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= \frac{\sin \frac{T}{2}(\omega-\omega_j)}{\omega-\omega_j} \quad \frac{\sin \frac{T}{2}(\omega+\omega_j)}{\omega+\omega_j} = H(\omega)+K(\omega) \tag{4.3.1}$$

where we define

$$H(\omega) = \frac{\sin \frac{T}{2}(\omega-\omega_j)}{\omega-\omega_j} \tag{4.3.2}$$

and

$$K(\omega) = \frac{\sin \frac{T}{2}(\omega+\omega_j)}{\omega+\omega_j} \, . \tag{4.3.3}$$

In terms of this notation we similarly obtain

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin \omega t \sin \omega_j t \, dt = H(\omega) - K(\omega), \qquad (4.3.4)$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin \omega t \cos \omega_j t \, dt = 0, \qquad (4.3.5)$$

$$\int\int_{-\frac{T}{2}}^{\frac{T}{2}}{}^{*} \cos \omega(t-s) \cos \omega_j (t-s) ds \, dt = 2\left\{H^2(\omega) + K^2(\omega)\right\}, \qquad (4.3.6)$$

$$\int\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin \omega(t-s) \sin \omega_j (t-s) ds \, dt = 2\left\{H^2(\omega) - K^2(\omega)\right\}, \qquad (4.3.7)$$

and

$$\int\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin \omega(t-s) \cos \omega_j (t-s) ds \, dt = 0. \qquad (4.3.8)$$

* The use of multiple integral signs with a single set of integration limits indicates that each integral has the given limits.

4.4 Expectations and Spectral Windows From (4.2.1) we have

$$E\left\{\hat{\Phi}_x(\omega_j;T)\right\} = \frac{T}{4\pi} E(a_j^2 + b_j^2), \tag{4.4.1}$$

where

$$a_j^2 = \frac{4}{T^2} \iint_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(s)\cos \omega_j t \cos \omega_j s \, ds \, dt \tag{4.4.2}$$

and

$$b_j^2 = \frac{4}{T^2} \iint_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(s)\sin \omega_j t \sin \omega_j s \, ds \, dt \tag{4.4.3}$$

from (4.1.3) and (4.1.4). Therefore

$$a_j^2+b_j^2 = \frac{4}{T^2} \iint_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(s)\cos \omega_j(t-s)ds \, dt \tag{4.4.4}$$

and

$$E\left\{\hat{\Phi}_x(\omega_j;T)\right\} = \frac{1}{\pi T} \iint_{-\frac{T}{2}}^{\frac{T}{2}} \rho_x(s-t)\cos \omega_j(s-t)ds \, dt. \tag{4.4.5}$$

But

$$\rho_x(s-t) = \int_0^{\infty} \Phi_x(\omega)\cos \omega(s-t)d\omega \tag{4.4.6}$$

so that

$$E\left\{\hat{\Phi}_x(\omega_j;T)\right\} = \int_0^\infty \Phi_x(\omega) f_p(\omega;\omega_j,T)\,d\omega, \tag{4.4.7}$$

where

$$\begin{aligned} f_p(\omega;\omega_j,T) &= \frac{1}{\pi T} \int\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos \omega(s-t) \cos \omega_j(s-t)\,ds\,dt \\ &= \frac{2}{\pi T}\left\{H^2(\omega)+K^2(\omega)\right\} \end{aligned} \tag{4.4.8}$$

from (4.3.6), is the spectral window for the power spectral estimator (4.2.1).

From the definitions of $H(\omega)$ and $K(\omega)$ we may put (4.4.8) in the form

$$f_p(\omega;\omega_j,T) = \begin{cases} \dfrac{2(1-\cos T\omega)(\omega^2+\omega_j^2)}{\pi T(\omega^2-\omega_j^2)^2}, & \omega \neq \omega_j \\ \dfrac{T}{2\pi}, & \omega = \omega_j \neq 0 \\ \dfrac{T}{\pi}, & \omega = \omega_j = 0. \end{cases} \tag{4.4.9}$$

We note at once that

$$f_p(\omega;\omega_j,T) \geq 0 \tag{4.4.10}$$

for all ω and ω_j and for $T > 0$.

The expectation of $\hat{\Phi}_y(\omega_j;T)$ is given by the right-hand side of (4.4.7) with $\Phi_x(\omega)$ replaced by $\Phi_y(\omega)$. The two power spectral estimators have the same spectral window as given by (4.4.9).

The area under the power spectral window may be shown to be unity. From (4.4.9).

$$\int_0^\infty f\;(\omega;\omega\;,T)d\omega = \frac{2}{\pi T}\int_0^\infty \frac{(1-\cos T\omega)(\omega^2+\omega_j^2)}{(\omega^2-\omega_j^2)^2}\,d\omega$$

$$= \frac{2}{\pi T}\int_0^\infty \frac{\omega^2+\omega_j^2}{(\omega^2-\omega_j^2)^2}\,d\omega - \int_0^\infty \frac{\omega^2\cos T\omega}{(\omega^2-\omega_j^2)^2}\,d\omega - \omega_j^2\int_0^\infty \frac{\cos T\omega}{(\omega^2-\omega_j^2)^2}\,d\omega$$

$$= \frac{2}{\pi T}\left\{0 + \frac{\pi T}{4} + \frac{\pi T}{4}\right\} = 1, \tag{4.4.11}$$

where the values for the last two integrals were obtained from [10], page 248.

The expectation of the co-spectral estimator (4.2.3) may be derived in a manner analogous to the development of the expectation of the power spectral estimator. From (4.2.3) we have

$$E\left\{\hat{C}_{xy}(\omega_j;T)\right\} = \frac{T}{4\pi}\,E(a_j a_j' + b_j b_j'). \tag{4.4.12}$$

Since

$$a_j a_j' + b_j b_j' = \frac{4}{T^2} \iint\limits_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(s)\cos\omega_j(s-t)ds\,dt \qquad (4.4.13)$$

we obtain

$$E\left\{\hat{C}_{xy}(\omega_j;T)\right\} = \frac{1}{\pi T} \iint\limits_{-\frac{T}{2}}^{\frac{T}{2}} \rho_{xy}(s-t)\cos\omega_j(s-t)ds\,dt. \qquad (4.4.14)$$

But

$$\rho_{xy}(s-t) = \int_0^\infty \left\{C_{xy}(\omega)\cos\omega(s-t) - Q_{xy}(\omega)\sin\omega(s-t)\right\}d\omega \qquad (4.4.15)$$

so that

$$E\left\{\hat{C}_{xy}(\omega_j;T)\right\} = \frac{1}{\pi T}\int_0^\infty C_{xy}(\omega)\left\{\iint\limits_{-\frac{T}{2}}^{\frac{T}{2}} \cos\omega(s-t)\cos\omega_j(s-t)ds\,dt\right\}d\omega \qquad (4.4.16)$$

$$- \frac{1}{\pi T}\int_0^\infty Q_{xy}(\omega)\left\{\iint\limits_{-\frac{T}{2}}^{\frac{T}{2}} \sin\omega(s-t)\cos\omega_j(s-t)ds\,dt\right\}d\omega.$$

The second term on the right-hand side of (4.4.16) vanishes by (4.3.8). Hence the co-spectral window is identical to the power spectral window and is given by

$$f_C(\omega;\omega_j,T) = \begin{cases} \dfrac{2(1-\cos T\omega)(\omega^2+\omega_j^2)}{\pi T(\omega^2-\omega_j^2)^2}, & \omega \neq \omega_j \\ \dfrac{T}{2\pi}, & \omega = \omega_j \neq 0 \\ \dfrac{T}{\pi}, & \omega = \omega_j = 0. \end{cases} \tag{4.4.17}$$

From the equivalence of the co-spectral and power spectral windows we note that

$$f_C(\omega;\omega_j,T) \geq 0 \tag{4.4.18}$$

for all ω and ω_j and for $T > 0$, and also that

$$\int_0^\infty f_C(\omega;\omega_j,T)d\omega = 1. \tag{4.4.19}$$

The expectation of the quadrature-spectral estimator (4.2.4) may be similarly obtained and is given by

$$E\left\{\hat{Q}_{xy}(\omega_j;T)\right\} = \int_0^\infty Q_{xy}(\omega)f_Q(\omega;\omega_j,T)d\omega \tag{4.4.20}$$

where

$$f_Q(\omega;\omega_j,T) = \frac{1}{\pi T}\iint_{-\frac{T}{2}}^{\frac{T}{2}} \sin\omega(t-s)\sin\omega_j(t-s)ds\,dt$$

$$= \frac{2}{\pi T}\left\{H^2(\omega)-K^2(\omega)\right\} \tag{4.4.21}$$

or, analogous to (4.4.17),

$$f_Q(\omega;\omega_j,T) = \begin{cases} \dfrac{4\omega\omega_j(1-\cos T\omega)}{\pi T(\omega^2-\omega_j^2)^2}, & \omega \neq \omega_j \\ \dfrac{T}{2\pi}, & \omega = \omega_j \neq 0 \\ 0, & \omega = \omega_j = 0. \end{cases} \qquad (4.4.22)$$

From (4.3.22) it is clear that

$$f_Q(\omega;\omega_j,T) \geq 0 \qquad (4.4.23)$$

for ω, ω_j, and $T > 0$. Furthermore since

$$f_C(\omega;\omega_j,T)-f_Q(\omega;\omega_j,T) = \begin{cases} \dfrac{2(1-\cos\omega T)}{\pi T(\omega^2-\omega_j^2)^2}(\omega+\omega_j)^2, & \omega \neq \omega_j \\ 0, & \omega = \omega_j \neq 0 \\ \dfrac{T}{\pi}, & \omega = \omega_j = 0 \end{cases} \qquad (4.4.24)$$

it follows that $f_Q(\omega;\omega_j,T)$ is bounded above by $f_C(\omega;\omega_j,T)$. Excluding the case $\omega = \omega_j = 0$, we note that $(f_C-f_Q) \longrightarrow 0$ as $T \longrightarrow \infty$, from which we infer that the area under $f_Q \longrightarrow 1$ as $T \longrightarrow \infty$.

From (4.4.17) and (4.4.22) we see that

$$\left.\begin{matrix} f_C(\omega;\omega_j,T) \\ f_Q(\omega;\omega_j,T) \end{matrix}\right\} \longrightarrow \begin{cases} 0, & \omega \neq \omega_j \\ \infty, & \omega = \omega_j \neq 0 \end{cases} \qquad (4.4.25)$$

as $T \longrightarrow \infty$, so that the spectral windows behave like Dirac functions in the limit and the estimators of section 4.2 are asymptotically unbiased.

From (4.4.17) we note that for $\omega \neq \omega_j$, $f_C(\omega;\omega_j,T) = 0$ for those values of ω for which

$$1-\cos \omega T = 0. \tag{4.4.26}$$

The solutions of (4.4.26) are given by

$$\omega = \frac{2k\pi}{T} \tag{4.4.27}$$

where k takes on all non-negative integer values except $k = j$.

The lobe width of $f_C(\omega;\omega_j,T)$ is therefore $\frac{2\pi}{T}$ and the width of the main lobe is $\frac{4\pi}{T}$. The lobe width can be made arbitrarily small by choosing T sufficiently large. From (4.4.22) and (4.4.26) it follows that the lobe width of the quadrature spectral window is the same as for the co-spectral window.

4.5 Variances and Covariances In order to simplify the evaluations of the variances and covariances of the estimators of section 4.2, we shall extend the notation of section 4.3 to the evaluation of more complex definite integrals. To do this let the P_i define the following trigonometric products:

$$P_1 = \cos\omega(s-u)\cos\lambda(t-v)$$
$$P_2 = \cos\omega(s-u)\sin\lambda(t-v)$$
$$P_3 = \sin\omega(s-u)\cos\lambda(t-v)$$
$$P_4 = \sin\omega(s-u)\sin\lambda(t-v)$$
$$P_5 = \cos\omega(t-u)\cos\lambda(s-v)$$
$$P_6 = \cos\omega(t-u)\sin\lambda(s-v)$$
$$P_7 = \sin\omega(t-u)\cos\lambda(s-v)$$
$$P_8 = \sin\omega(t-u)\sin\lambda(s-v)$$
$$P_9 = \cos\omega_j(t-s)\cos\omega_j(u-v)$$
$$P_{10} = \cos\omega_j(t-s)\sin\omega_j(u-v)$$
$$P_{11} = \sin\omega_j(t-s)\cos\omega_j(u-v)$$
$$P_{12} = \sin\omega_j(t-s)\sin\omega_j(u-v)$$
$$P_{13} = \cos\omega(t-s)\cos\lambda(u-v)$$
$$P_{14} = \cos\omega(t-s)\sin\lambda(u-v)$$
$$P_{15} = \sin\omega(t-s)\cos\lambda(u-v)$$
$$P_{16} = \sin\omega(t-s)\sin\lambda(u-v) \qquad (4.5.1)$$

In this section we shall be faced with the problem of evaluating integrals of the type

$$I(P_iP_j) = \iiiint_{-\frac{T}{2}}^{\frac{T}{2}} P_iP_j\,dsdtdudv. \qquad (4.5.2)$$

The product $P_i P_j$ is the product of four sines and/or cosines of differences of two variables. If the trigonometric functions of differences are expanded completely the integration of (4.5.2) may be performed term by term. Within each term the variables are separable and each term may be written as the product of integrals of the types given by (4.3.6), (4.3.7), and (4.3.8) and hence evaluated in terms of the quantities $H(\omega)$ and $K(\omega)$ given by (4.3.2) and (4.3.3). The following values of $I(P_i P_j)$ were evaluated in this manner:

$$I(P_1P_9) = 2[H^2(\omega)+K^2(\omega)]\,[H^2(\lambda)+K^2(\lambda)] + 8H(\omega)K(\omega)H(\lambda)K(\lambda)$$
$$I(P_1P_{10}) = 0$$
$$I(P_1P_{11}) = 0$$
$$I(P_1P_{12}) = -2[H^2(\omega)+K^2(\omega)]\,[H^2(\lambda)+K^2(\lambda)] + 8H(\omega)K(\omega)H(\lambda)K(\lambda)$$
$$I(P_2P_9) = 0$$
$$I(P_2P_{10}) = 2[H^2(\omega)+K^2(\omega)]\,[H^2(\lambda)-K^2(\lambda)]$$
$$I(P_2P_{11}) = 2[H^2(\omega)+K^2(\omega)]\,[H^2(\lambda)-K^2(\lambda)]$$
$$I(P_2P_{12}) = 0$$
$$I(P_3P_9) = 0$$
$$I(P_3P_{10}) = -2[H^2(\omega)-K^2(\omega)]\,[H^2(\lambda)+K^2(\lambda)]$$
$$I(P_3P_{11}) = -2[H^2(\omega)-K^2(\omega)]\,[H^2(\lambda)+K^2(\lambda)]$$
$$I(P_3P_{12}) = 0$$
$$I(P_4P_9) = 2[H^2(\omega)-K^2(\omega)]\,[H^2(\lambda)-K^2(\lambda)]$$
$$I(P_4P_{10}) = 0$$

$$I(P_4P_{11}) = 0$$
$$I(P_4P_{12}) = -2[H^2(\omega)-K^2(\omega)][H^2(\lambda)-K^2(\lambda)]$$
$$I(P_5P_9) = 2[H^2(\omega)+K^2(\omega)][H^2(\lambda)+K^2(\lambda)] +8H(\omega)K(\omega)H(\lambda)K(\lambda)$$
$$I(P_5P_{10}) = 0$$
$$I(P_5P_{11}) = 0$$
$$I(P_5P_{12}) = 2[H^2(\omega)+K^2(\omega)][H^2(\lambda)+K^2(\lambda)] -8H(\omega)K(\omega)H(\lambda)K(\lambda)$$
$$I(P_{13}P_9) = 4[H^2(\omega)+K^2(\omega)][H^2(\lambda)+K^2(\lambda)]$$
$$I(P_{13}P_{10}) = 0$$
$$I(P_{13}P_{11}) = 0$$
$$I(P_{13}P_{12}) = 0$$
$$I(P_{14}P_9) = 0$$
$$I(P_{14}P_{10}) = 4[H^2(\omega)+K^2(\omega)][H^2(\lambda)-K^2(\lambda)]$$
$$I(P_{14}P_{11}) = 0$$
$$I(P_{14}P_{12}) = 0$$
$$I(P_{15}P_9) = 0$$
$$I(P_{15}P_{10}) = 0$$
$$I(P_{15}P_{11}) = 4[H^2(\omega)-K^2(\omega)][H^2(\lambda)+K^2(\lambda)]$$
$$I(P_{15}P_{12}) = 0$$
$$I(P_{16}P_9) = 0$$
$$I(P_{16}P_{10}) = 0$$
$$I(P_{16}P_{11}) = 0$$
$$I(P_{16}P_{12}) = 4[H^2(\omega)-K^2(\omega)][H^2(\lambda)-K^2(\lambda)] \qquad (4.5.3)$$

We are now prepared to evaluate $V\left\{\hat{\Phi}_x(\omega_j;T)\right\}$. The expectation of the square of the estimator is given by

$$E\left\{\hat{\Phi}_x^2(\omega_j;T)\right\} = \frac{T^2}{16\pi^2}\, E\left\{(a_j^2+b_j^2)^2\right\}$$

$$= \frac{T^2}{16\pi^2}\, E\left\{\frac{16}{T^4}\iiiint_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(s)x(u)x(v)\cos\omega_j(t-s)\cos\omega_j(u-v) \cdot dsdtdudv\right\}$$

$$= \frac{1}{(\pi T)^2}\iiiint_{-\frac{T}{2}}^{\frac{T}{2}} E\left\{x(t)x(s)x(u)x(v)\right\} \cdot P_9 dsdtdudv. \qquad (4.5.4)$$

Since the process $\{x(t)\}$ and $\{y(t)\}$ are stationary and jointly Gaussian we obtain

$$E\left\{x(t)x(s)x(u)x(v)\right\}$$

$$= \rho_x(s-t)\rho_x(v-u)+\rho_x(u-s)\rho_x(v-t)+\rho_x(u-t)\rho_x(v-s)$$

$$= \rho_x(t-s)\rho_x(u-v)+\rho_x(s-u)\rho_x(t-v)+\rho_x(t-u)\rho_x(s-v)$$

$$= \int_0^\infty\int_0^\infty \Phi_x(\omega)\Phi_x(\lambda)\left\{\cos\omega(t-s)\cos\lambda(u-v)+\cos\omega(s-u)\cos\lambda(t-v) + \cos\omega(t-u)\cos\lambda(s-v)\right\}d\omega d\lambda$$

$$= \int_0^\infty\int_0^\infty \Phi_x(\omega)\Phi_x(\lambda)\left\{P_{13}+P_1+P_5\right\}d\omega d\lambda\ . \qquad (4.5.5)$$

Therefore

$$E\left\{\hat{\Phi}_x^2(\omega_j;T)\right\}$$

$$= \frac{1}{(\pi T)^2}\int_0^\infty\int_0^\infty \Phi_x(\omega)\Phi_x(\lambda)\left\{\iiiint_{-\frac{T}{2}}^{\frac{T}{2}} (P_{13}P_9+P_1P_9+P_5P_9)\,ds\,dt\,du\,dv\right\}d\omega d\lambda$$

$$= \frac{1}{(\pi T)^2}\int_0^\infty\int_0^\infty \Phi_x(\omega)\Phi_x(\lambda)\Big\{8[H^2(\omega)+K^2(\omega)][H^2(\lambda)+K^2(\lambda)]$$
$$+ 16H(\omega)K(\omega)H(\lambda)K(\lambda)\Big\}\, d\omega d\lambda$$

$$= 2\int_0^\infty\int_0^\infty \Phi_x(\omega)\Phi_x(\lambda)\left\{(\tfrac{2}{\pi T})^2[H^2(\omega)+K^2(\omega)][H^2(\lambda)+K^2(\lambda)]\right\}d\omega d\lambda$$

$$+ \int_0^\infty\int_0^\infty \Phi_x(\omega)\Phi_x(\lambda)\left\{(\tfrac{4}{\pi T})^2 H(\omega)K(\omega)H(\lambda)K(\lambda)\right\}d\omega d\lambda$$

$$= 2E^2\left\{\hat{\Phi}_x(\omega_j;T)\right\} + \left\{\int_0^\infty \Phi_x(\omega)[\tfrac{4}{\pi T}H(\omega)K(\omega)]\,d\omega\right\}^2. \qquad (4.5.6)$$

The variance of the estimator is then given by

$$V\left\{\hat{\Phi}_x(\omega_j;T)\right\} = E^2\left\{\hat{\Phi}_x(\omega_j;T)\right\} + \left\{\int_0^\infty \Phi_x(\omega)[\tfrac{4}{\pi T}H(\omega)K(\omega)]\,d\omega\right\}^2. \qquad (4.5.7)$$

From (4.3.2) and (4.3.3) it can be shown that

$$\frac{4}{\pi T}H(\omega)K(\omega) = \frac{2(1-\cos T\omega)}{\pi T(\omega^2-\omega_j^2)}. \qquad (4.5.8)$$

The derivation of the variance of the co-spectral estimator (4.2.3) is similar but somewhat more laborious. The expectation of the square of the estimator is

$$E\left\{\hat{C}^2_{xy}(\omega_j;T)\right\} = \frac{T^2}{16\pi^2} E\left\{(a_j a'_j + b_j b'_j)^2\right\}$$

$$= \frac{T^2}{16\pi^2} E\left\{\frac{4}{T^2}\iiiint_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(s)\cos\omega_j(t-s)dsdt\right\}^2$$

$$= \frac{1}{(\pi T)^2}\iiiint_{-\frac{T}{2}}^{\frac{T}{2}} E\left\{x(t)y(s)x(u)y(v)\right\}\cos\omega_j(t-s)\cos\omega_j(u-v) \cdot dsdtdudv$$

$$= \frac{1}{(\pi T)^2}\iiiint_{-\frac{T}{2}}^{\frac{T}{2}} E\left\{x(t)y(s)x(u)y(v)\right\}\cdot P_9 dsdtdudv. \qquad (4.5.9)$$

The last expectation is given by

$$E\left\{x(t)y(s)x(u)y(v)\right\}$$

$$= \rho_{xy}(s-t)\rho_{xy}(v-u)+\rho_{xy}(s-u)\rho_{xy}(v-t)+\rho_x(t-u)\rho_y(s-v), \quad (4.5.10)$$

where

$$\rho_{xy}(s-t)\rho_{xy}(v-u)$$

$$= \int_0^\infty \int_0^\infty \left\{C_{xy}(\omega)\cos\,\omega(s-t)-Q_{xy}(\omega)\sin\,\omega(s-t)\right\}$$

$$\cdot\ \left\{C_{xy}(\lambda)\cos\,\lambda(u-v)-Q_{xy}(\lambda)\sin\,\lambda(v-u)\right\}d\omega d\lambda$$

$$= \int_0^\infty \int_0^\infty \left\{C_{xy}(\omega)\cos\,\omega(t-s)+Q_{xy}(\omega)\sin\,\omega(t-s)\right\}$$

$$\cdot\ \left\{C_{xy}(\lambda)\cos\,\lambda(u-v)+Q_{xy}(\lambda)\sin\,\lambda(u-v)\right\}d\omega d\lambda$$

$$= \int_0^\infty \int_0^\infty \left\{C_{xy}(\omega)C_{xy}(\lambda)\cdot P_{13}+C_{xy}(\omega)Q_{xy}(\lambda)\cdot P_{14}\right.$$

$$\left. +\ Q_{xy}(\omega)C_{xy}(\lambda)\cdot P_{15}+Q_{xy}(\omega)Q_{xy}(\lambda)\cdot P_{16}\right\}d\omega d\lambda, \qquad (4.5.11)$$

$$\rho_{xy}(s-u)\rho_{xy}(v-t)$$

$$= \int_0^\infty \int_0^\infty \left\{C_{xy}(\omega)\cos\,\omega(s-u)-Q_{xy}(\omega)\sin\,\omega(s-u)\right\} \cdot\left\{C_{xy}(\lambda)\cos\,\lambda(v-t)-Q_{xy}(\lambda)\sin\,\lambda(v-t)\right\}d\omega d\lambda$$

$$= \int_0^\infty \int_0^\infty \left\{C_{xy}(\omega)\cos\,\omega(s-u)-Q_{xy}(\omega)\sin\,\omega(s-u)\right\} \cdot\left\{C_{xy}(\lambda)\cos\,\lambda(t-v)+Q_{xy}(\lambda)\sin\,\lambda(t-v)\right\}d\omega d\lambda$$

$$= \int_0^\infty \int_0^\infty \left\{C_{xy}(\omega)C_{xy}(\lambda)\cdot P_{1}+C_{xy}(\omega)Q_{xy}(\lambda)\cdot P_{2} - Q_{xy}(\omega)C_{xy}(\lambda)\cdot P_{3}-Q_{xy}(\omega)Q_{xy}(\lambda)\cdot P_{4}\right\}d\omega d\lambda, \qquad (4.5.12)$$

and

$$\rho_x(t-u)\rho_y(s-v) = \int_0^\infty \int_0^\infty \Phi_x(\omega)\Phi_y(\lambda)\cos\omega(t-u)\cos\lambda(s-v)d\omega d\lambda$$

$$= \int_0^\infty \int_0^\infty \Phi_x(\omega)\Phi_y(\lambda)\cdot P_5 d\omega d\lambda. \qquad (4.5.13)$$

Therefore we obtain

$$E\left\{x(t)y(s)x(u)y(v)\right\}$$

$$= \int_0^\infty \int_0^\infty \Big\{ C_{xy}(\omega)C_{xy}(\lambda)(P_{13}+P_1)+C_{xy}(\omega)Q_{xy}(\lambda)(P_{14}+P_2)$$

$$+ Q_{xy}(\omega)C_{xy}(\lambda)(P_{15}-P_3)+Q_{xy}(\omega)Q_{xy}(\lambda)(P_{16}-P_4)$$

$$+ \Phi_x(\omega)\Phi_y(\lambda)\cdot P_5\Big\} d\omega d\lambda. \qquad (4.5.14)$$

The expectation of the square of the co-spectral estimator may now be expressed as

$$E\left\{\hat{C}^2_{xy}(\omega_j;T)\right\}$$

$$= \int_0^\infty \int_0^\infty C_{xy}(\omega)C_{xy}(\lambda)\left\{\iiiint_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{(\pi T)^2}(P_{13}P_9+P_1P_9)\,dsdtdudv\right\}d\omega d\lambda$$

$$+ \int_0^\infty \int_0^\infty C_{xy}(\omega)Q_{xy}(\lambda)\left\{\iiiint_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{(\pi T)^2}(P_{14}P_9+P_2P_9)\,dsdtdudv\right\}d\omega d\lambda$$

$$+ \int_0^\infty \int_0^\infty Q_{xy}(\omega)C_{xy}(\lambda)\left\{\iiiint_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{(\pi T)^2}(P_{15}P_9-P_3P_9)\,dsdtdudv\right\}d\omega d\lambda$$

$$+ \int_0^\infty \int_0^\infty Q_{xy}(\omega)Q_{xy}(\lambda)\left\{\iiiint_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{(\pi T)^2}(P_{16}P_9-P_4P_9)\,dsdtdudv\right\}d\omega d\lambda$$

$$+ \int_0^\infty \int_0^\infty \Phi_x(\omega)\Phi_y(\lambda)\left\{\iiiint_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{(\pi T)^2}(P_5P_9)\,dsdtdudv\right\}d\omega d\lambda. \tag{4.5.15}$$

From (4.5.3) the preceding expression may be rewritten as

$$E\left\{\hat{C}^2_{xy}(\omega_j;T)\right\}$$

$$= \int_0^\infty \int_0^\infty C_{xy}(\omega)C_{xy}(\lambda)\left\{\frac{6}{(\pi T)^2}[H^2(\omega)+K^2(\omega)]\,[H^2(\lambda)+K^2(\lambda)] + \frac{8}{(\pi T)^2}\,H(\omega)K(\omega)H(\lambda)K(\lambda)\right\}d\omega d\lambda$$

$$+ \int_0^\infty \int_0^\infty Q_{xy}(\omega)Q_{xy}(\lambda)\left\{-\frac{2}{(\pi T)^2}[H^2(\omega)-K^2(\omega)]\,[H^2(\lambda)-K^2(\lambda)]\right\}d\omega d\lambda$$

$$+ \int_0^\infty \int_0^\infty \Phi_x(\omega)\Phi_y(\lambda)\left\{\frac{2}{(\pi T)^2}[H^2(\omega)+K^2(\omega)]\,[H^2(\lambda)+K^2(\lambda)] + \frac{8}{(\pi T)^2}\,H(\omega)K(\omega)H(\lambda)K(\lambda)\right\}d\omega d\lambda, \quad (4.5.16)$$

which, in turn, may be expressed as

$$E\left\{\hat{C}^2_{xy}(\omega_j;T)\right\} = \frac{3}{2}\,E^2\left\{\hat{C}_{xy}(\omega_j;T)\right\} + \frac{1}{2}\left\{\int_0^\infty C_{xy}(\omega)[\frac{4}{\pi T}\,H(\omega)K(\omega)]\,d\omega\right\}^2$$
$$- \frac{1}{2}\,E^2\left\{\hat{Q}_{xy}(\omega_j;T)\right\} + \frac{1}{2}\,E\left\{\hat{\Phi}_x(\omega_j;T)\right\}E\left\{\hat{\Phi}_y(\omega_j;T)\right\}$$
$$+ \frac{1}{2}\left\{\int_0^\infty \Phi_x(\omega)[\frac{4}{\pi T}\,H(\omega)K(\omega)]\,d\omega\right\}\left\{\int_0^\infty \Phi_y(\omega)[\frac{4}{\pi T}\,H(\omega)K(\omega)]\,d\omega\right\}. \quad (4.5.17)$$

Finally the variance of the co-spectral estimator is given by

$$V\left\{\hat{C}_{xy}(\omega_j;T)\right\}$$

$$= \tfrac{1}{2}\left\{E^2\left[\hat{C}_{xy}(\omega_j;T)\right] + E\left[\hat{\Phi}_x(\omega_j;T)\right]\cdot E\left[\hat{\Phi}_y(\omega_j;T)\right] - E^2\left[\hat{Q}_{xy}(\omega_j;T)\right]\right\}$$

$$+ \tfrac{1}{2}\left\{\int_0^\infty \Phi_x(\omega)\left[\frac{4}{\pi T} H(\omega)K(\omega)\right] d\omega\right\}\left\{\int_0^\infty \Phi_y(\omega)\left[\frac{4}{\pi T} H(\omega)K(\omega)\right] d\omega\right\}$$

$$+ \tfrac{1}{2}\left\{\int_0^\infty C_{xy}(\omega)\left[\frac{4}{\pi T} H(\omega)K(\omega)\right] d\omega\right\}^2. \qquad (4.5.18)$$

By similar methods the following results were obtained:

$$V\left\{\hat{Q}_{xy}(\omega_j;T)\right\}$$

$$= \tfrac{1}{2}\left\{E^2\left[\hat{Q}_{xy}(\omega_j;T)\right] + E\left[\hat{\Phi}_x(\omega_j;T)\right]\cdot E\left[\hat{\Phi}_y(\omega_j;T)\right] - E^2\left[\hat{C}_{xy}(\omega_j;T)\right]\right\}$$

$$- \tfrac{1}{2}\left\{\int_0^\infty \Phi_x(\omega)\left[\frac{4}{\pi T} H(\omega)K(\omega)\right] d\omega\right\}\left\{\int_0^\infty \Phi_y(\omega)\left[\frac{4}{\pi T} H(\omega)K(\omega)\right] d\omega\right\}$$

$$+ \tfrac{1}{2}\left\{\int_0^\infty C_{xy}(\omega)\left[\frac{4}{\pi T} H(\omega)K(\omega)\right] d\omega\right\}^2 \qquad (4.5.19)$$

and

$$\operatorname{Cov}\left[\hat{C}_{xy}(\omega_j;T), \hat{Q}_{xy}(\omega_j;T)\right] = E\left[\hat{C}_{xy}(\omega_j;T)\right]\cdot E\left[\hat{Q}_{xy}(\omega_j;T)\right]. \qquad (4.5.20)$$

The variances (4.5.18) and (4.5.19) reduce to simpler forms in certain special cases, but the given forms are the simplest general results obtained.

SUMMARY

Although continuous time processes have been of interest for many years to economists, meteorologists, engineers and others, it is only in the last decade that statisticians have considered, to any extent, the problems associated with the spectral analysis of these processes. Since the concepts and terminology of spectral analysis are not widely known, Chapter 1 is devoted to a brief introduction to the theory of this type of analysis followed by a review of the standard methods of cross-spectral estimation for discrete time-history data. The remaining three chapters deal with specific problems associated with cross-spectral estimation.

Point estimates of the cross-spectral density function are obtained in terms of its two components, the co-spectral and quadrature-spectral density functions. Estimators for these quantities are biased, the bias taking the form of a continuous weight function or spectral window. The form of such a spectral window is a very important property of the spectral estimator. In Chapter 2 co-spectral and quadrature-spectral estimators are presented which are characterized by non-negative spectral windows. Other properties of these estimators are also examined.

The standard methods of cross-spectral estimation for discrete data from continuous processes are feasible,

generally, only if high-speed digital computing equipment is available because of the large number of multiplicative operations required. Even then it is very time consuming and expensive. In Chapter 3 randomized co-spectral and quadrature-spectral estimators are developed which substantially reduce the number of required arithmetic operations. These estimators may be specified so that they have the same spectral windows as the standard estimators, the estimators of Chapter 2, or any of a general class of systematic estimators. The variances of the estimators are derived and the solutions for minimum variance randomized estimators with given expectations are developed.

Chapter 4 treats the continuous case where it is possible to determine, by means of suitable analog equipment, the coefficients of the Fourier series expansions of realizations of the basic processes over a finite time interval. For such a case estimators, which provide extremely simple point estimates of the spectral densities, are presented. The expectations, variances and covariances of these estimators are derived and the properties of their spectral windows discussed.

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ABSTRACT

ESTIMATION PROBLEMS CONNECTED WITH

STOCHASTIC PROCESSES

by

Alfred Edward Garratt, A.B., M.S.

Thesis submitted to the Graduate Faculty of the

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A brief introduction to the concepts and terminology of spectral analysis and a review of the standard methods for cross-spectral estimation, based on discrete time-history data, are incorporated in Chapter 1.

Co-spectral and quadrature-spectral estimators which are characterized by non-negative spectral windows are developed in Chapter 2. While the spectral windows for the co-spectral estimators are non-negative for all relevant values of the assignable constants, certain restrictions on these constants are necessary to assure the non-negativity of the quadrature-spectral window. The properties of these estimators are considered in detail.

In Chapter 3 randomized co-spectral and quadrature-spectral estimators are presented. These estimators depend on the random selection of sets of time differences, as opposed to the systematic evaluation of all possible time differences for the standard estimators. By suitable choices of probability distributions for the time differences and of weight functions, the expectations of the randomized estimators can be made equivalent to the expectations of the standard estimators or the estimators of Chapter 2. Since the randomized estimator is much simpler to use than the standard estimator, these estimators are compared in terms of their variances, given that they have equal expectations. The choice of probability distributions to yield minimum variance, given that the expectation is specified, is considered.

Extremely simple co-spectral and quadrature-spectral estimators, for the case where the coefficients of the Fourier series expansions of realizations of the processes over a finite time interval can be obtained by means of suitable analog equipment, are developed in Chapter 4. The expectations, variances and covariances of these estimators are derived.