FERMIONS IN YANG-MILLS GAUGE THEORIES:
INVARIANCE, COVARIANCE AND TOPOLOGY

by

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(ABSTRACT)

I present a study on the invariance and covariance properties of the Dirac operator describing fermions in Yang-Mills fields. This includes the study of anomalies of the gauge currents. We are particularly interested in the geometric and topological features in the problem. The complicated topological structures and properties present in these theories are made clear by elementary calculations in several simple models. We show explicitly how non-trivial phase and sign ambiguities arise to give the so-called anomalies. The Atiyah-Singer index theorem is seen to be a very powerful tool to calculate the topological invariants that characterize the anomalies. The index theorem also gives topological invariants describing the failure of covariance of the fermion propagator.

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1. Introduction

One of the aims of physics is to understand the equations of motion and interactions of matter. It has been helpful in this regard to try to resolve matter into its constituents and understand the interaction among them. The laws of motion of more complex systems are then derived from these. The simplest and most successful theory at present is the standard model. In this model, the basic building blocks that account for the observed phenomena fall into the following categories: quarks, leptons, gauge particles and Higgs fields. The quarks and leptons are sometimes called the matter fields. They seem to come in three repetitive generations, and they are spin-one-half particles. In each generation, a quark has a quantum number called color indexed by 1,2,3; the leptons do not carry this quantum number. Quarks and leptons are further distinguished by an ″iso-spin″ quantum number of an SU(2) group. The interactions in the model are mediated by gauge bosons of the type proposed by Yang and Mills [1]. In this type of theory, interactions are determined to a large extent by the local symmetry principle. In the standard model, the strong interaction between quarks is described by an SU(3) gauge theory with color being the charge. The electroweak force is determined by the gauge group $SU_L(2) \times U(1)$[2,3,4]. The corresponding gauge bosons, the W's and Z, were discovered not long ago.[5,6] The success of the standard model has led to great efforts to unify the forces of nature, notably the Grand Unified Theories (GUTs)[7], Composite Models[8,23] and Superstring Theories[9,10]. All these theories either are or contain local gauge theories. It is therefore important to understand the properties of this type of field theory, especially the non-perturbative aspects.

In general, non-abelian gauge theories are highly non-linear. Perturbative studies cannot answer most of the interesting questions. It is still a formidable task to understand the non-perturbative behavior of these theories in their full generality. In the numerical approach, lattice calculations[11] are promising and have produced many interesting results, but they are limited by the present computational capability. Furthermore, how to do calculations with fermions in the Monte Carlo method is still a problem.

A complementary approach is to study the geometry of the fields. Through this approach, we have gained many insights into the general structures. It has helped, for instance, to establish many important notions about the field configurations, which are indispensable to applications in areas such as GUTs and cosmology. Monopoles[12,13], instantons[14], domain walls, etc., in the GUT models were first identified by the techniques of algebraic topology (homotopy theory, homology and cohomology theories). More intriguing mathematical results such as the Morse theory and index theory concerning the relation between geometry in the large and geometry in the small have proved to be very powerful also. An example is the chiral anomaly, which says the chiral currents in a gauge theory are not conserved. It played the crucial role in the calculation[15,16,20,21] of the decay rate of $\pi^0 \rightarrow \gamma\gamma$ and in resolving the U(1) problem in Quantum Chromodynamics(QCD)[31]. It was later found that the U(1) anomaly equation is actually a version of the index theorem.

Another kind of anomaly is the kind that destroys gauged symmetries. The usual procedure of quantizing the fields are seen to be incompatible with gauge symmetries when these anomalies are present. The current practice is to require this sort of anomaly to be absent in a theoretical model, thereby restricting the possibilities[22]. Another application of these anomalies is the practice of anomaly matching in preon models[23]. The anomalies thus have served as a very important quantitative guide to constructing realistic unified theories.

In this dissertation, I describe a study clarifying the topological picture of the gauge anomalies. On the basis of this, one can look at the covariance of the fermion propagator and obtain generalizations of the anomaly equations.

When I started, there appeared in the literature reports of cohomological studies of the gauge anomalies[24]. They strongly hinted at a topological picture and origin. The central part of these studies is the so-called descent equations[24][35].

These are interlocking equations that relate cocycles of different orders in different dimensions. The lowest order cocycle represents the anomaly. By some cohomological manipulations, one can find cocycles of increasing orders in descending dimensions. These formal manipulations seemed rather mysterious at the time. The natural questions are: a) what exactly is the topological picture of the anomalies and how can topology help? b) what do the other cocycles represent?

We answer the first question completely by combining sample calculations in several models[46] and the results in numerous excellent formal works[38-45]. The most prominent feature that emerges is the resemblance to the structure of magnetic monopoles in the configuration space of gauge fields.

Only a partial answer can be provided to the second question. In this context, the resemblance is to instanton numbers in the configuration space of gauge fields. We can show how zero modes of the Dirac operator affect the invariance and covariance properties of the theory and their topological origin. We can also show how, in the Hamiltonian picture, the spectral flow of the energy levels causes representations of the gauge transformation group on the fermionic Fock space to be ill-defined. We explain how the cocycles describe the zero modes that affect the gauge variations. These I do in the following chapters, preceded by reviews on physical and mathematical background.

2. Anomalies in Gauge Theories

Symmetry considerations have been crucial to the development of quantum field theories and the categorization of elementary particles and their interactions. When there is a symmetry, we can frequently make definitive statements about a system even when a full dynamical understanding is lacking. For example, this method is of utmost importance in the revelation of the quark structure of the hadrons[25,26].

For a classical field theory, to check if a certain transformation is a symmetry, we ordinarily only need to check if the action of the theory is invariant under the transformation. If it is invariant, the equations of motion will retain their form under the transformation and we have a symmetry for the system. However, the naive expectation that the procedure is still valid in the quantized theory can be misleading. When a field theory is quantized, a regularization procedure has to be employed because of the infinite number of degrees of freedom. It may happen that an invariant regularization does not exist and the classical symmetry is not preserved. This is what we call an anomaly. For continuous symmetries of a classical field theory, there are conserved currents which result from the these symmetries (Noether theorem) [22]. Where we have anomalies, these currents are no longer conserved.

Chiral Anomalies

It was found as early as 1949 that chiral symmetry can be broken in this way[15,16,17]. The rediscovery of this fact in the late '60s solved a then great puzzle and gave the correct rate for the neutral pion decay[20,21].

From the point of view of Quantum Chromodynamics (QCD), in the limit of zero quark masses, the action is invariant under independent transformations on the left and right chiral fermion fields, and the associated chiral currents are conserved classically. These currents can be probed by weak interactions. The conservation laws imply the existence of massless bosons, the Nambu-Goldstone modes, which are directly coupled to these currents[32,33]. The partially conserved axial current hypothesis (PCAC) postulates that physical pion amplitudes can be expanded around zero external pion momenta. The coefficients of the expansion are then fixed by the physical parameters of the axial currents. PCAC was well tested by many experimental data[18]. However, it would predict vanishing decay rate for $\pi^0 \rightarrow \gamma\gamma$ in the limit $m_\pi \rightarrow 0$, contradicting the observed life time 0.87×10^{-16} s. Diagrammatic calculations of the triangle graph in the conservation of the currents however show that there is a momentum parametrization ambiguity due to the linear divergence of the diagram[20]. The vector current is conserved for a certain parametrization but the axial current is not for this scheme. The divergence is proportional to the scalar product of the electric and magnetic fields. When the PCAC equations are modified by this term, the neutral pion decay is correctly accounted for. The violation of the would-be chiral symmetry in the massless limit was termed the chiral anomaly. In this case, the chiral anomaly arises from the gauge interaction of the photon with the vector current $\bar{\psi}\gamma_\mu\psi$. In QCD, the interaction of the gluons with the quark color currents gives rise similarly to an anomaly of the chiral U(1) symmetry. This anomaly was used to solve the U(1) problem in QCD[31].

What happens if the gauge symmetries in a gauge theory acquire anomalies? In this case, the consequences can be fatal. The local gauge invariance upon which the gauge theory is constructed will no longer be valid, and the usual quantization and renormalization procedures for the gauge fields cannot be carried through[22]. Let us see how this comes about in detail.

Gauge Theory

In a gauge theory, the gauge potential is a matrix-valued vector A_μ locally. Let dx^μ denote the differential of the local coordinates. They are considered as anti-commuting quantities, *i.e.*, differential forms. $dx^\mu \wedge dx^\nu = - dx^\nu \wedge dx^\mu$. We call $A \equiv A_\mu dx^\mu \equiv A^a_\mu \lambda_a dx^\mu$ the connection one form. The exterior differential of a form is simply $d \equiv dx^\mu \frac{\partial}{\partial x^\mu}$. Under a gauge transformation g, A transforms as

$A \to g(A + d)g^{-1}$. As is well-known, this is an inhomogeneous rule. The field strength (curvature) 2-form $F = F^a_{\mu\nu}\lambda_a dx^\mu{\wedge}dx^\nu$ is defined to be $F \equiv dA + A^2 \equiv dA + A{\wedge}A$. It transforms covariantly under a gauge transformation,

$$F \to gFg^{-1}.$$

From the gauge principle, matter fields can be coupled to the gauge potential using the covariant derivative $D_\mu = \partial_\mu + A_\mu$. For a Dirac fermion in 2n dimensions, the Dirac γ matrices satisfy the Clifford algebra relations,

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\delta^{\alpha\beta} , \tag{2.1}$$

where we have assumed a Euclidean signature metric. When the dimension is even, we can also write in the chiral representation,

$$\gamma^a = \begin{bmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{bmatrix}, a = 1,2,\dots,2n-1, \quad \gamma^{2n} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \gamma^{2n+1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.2}$$

with $\sigma_a\sigma_b + \sigma_b\sigma_a = 2\delta_{ab}$.

The Dirac operator is $iD(A) = ie^\mu_\alpha\gamma^\alpha(\partial_\mu + A_\mu + \Gamma_\mu)$, where e^μ_α are an orthonormal basis for the tangent space with respect to the metric, and Γ_μ is the Riemannian connection for the spinors. In even dimensions, a massless fermion can have a definite chirality, + or −. In this case, the chiral Dirac operator is more useful. $iD_\pm(A) = ie^\mu_\alpha\sigma^\alpha_\pm(\partial_\mu + A_\mu + \Gamma_\mu)$, where $\sigma^a_\pm = \sigma_a\sigma^{2n}_\pm = \pm i$. The action functional for a system of fermions and gauge bosons is of the form

$$S(A,\psi,\overline{\psi}) = \int d^{2n}x\sqrt{\det h}\,[\,-\frac{1}{16\pi\alpha_s}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + i\overline{\psi}D(A)\psi]\,. \tag{2.3}$$

Here, $h_{\mu\nu}$ is the Riemannian metric on the manifold and α_s gives the coupling strength. This expression is invariant for the transformation $\psi \rightarrow g\psi,\ A \rightarrow g(A + d)g^{-1}$. The path-integral measure contains $\exp(-S)$. The Green's functions of the fields are calculated by the expectation values of the product of the fields with this weight. They are generated by the partition function

$$Z[B,\eta,\overline{\eta}] = \int DAD\overline{\psi}D\psi\,\exp(\,-\,S + B\cdot A + \overline{\eta}\cdot\psi + \overline{\psi}\cdot\eta)\,. \tag{2.4}$$

Gauge Anomalies

Formally, the path-integral over the fermi fields in the background of the gauge fields without the sources is equal to the determinant of the Dirac operator,

$$Z[A] = \int D\psi D\overline{\psi}\,\exp[\int\overline{\psi}iD(A)\psi\sqrt{h}\,d^{2n}x] = \mathrm{Det}[D(A)]\ . \tag{2.5}$$

The Dirac operator $D(A)$ is hermitian but the chiral operator $D_{\pm}(A)$ is not. The determinant of $D(A)$ is therefore real. In any dimension, the determinant of the Dirac operator $D(A)$ can always be made invariant and thus free of anomalies. The reason is that gauge invariant regularization schemes exist[51]. Since, in even dimensions, $D(A)$ can be written as

$$D(A) = \begin{bmatrix} 0 & D_-(A) \\ D_+(A) & 0 \end{bmatrix}, \tag{2.6}$$

formally, $\mathrm{Det}[D(A)] = |\mathrm{Det}[D_\pm(A)]|^2$. Therefore, while the chiral determinant can be complex, it has an invariant modulus. The phase however may be gauge variant. Such gauge variations of the phase are symptomatic of the gauge anomalies. For variations under infinitesimal gauge transformations, $\frac{\delta i \ln Z}{\delta \varepsilon} = <DJ>$, where J is the current coupled to the gauge field. This is simply Noether's theorem[22]. Hence any continuous gauge variation can be interpreted as a violation of the naive current conservation law. In a non-chiral theory with $D(A)$, the naive covariant conservation of the chiral current is also violated. Note that this current is not the source of the gauge fields and cannot be written as the variation of the determinant. The U(1) part of the anomaly was used to solve the U(1) problem in QCD. Gauge anomaly can also arise in odd dimensions, where no chirality is defined, as I shall demonstrate on page 62.

Methods of Calculation

There are many different ways to calculate the anomalies using different regularization schemes. A remarkable theorem of Adler and Bardeen[27] says that, in four dimensions, the anomaly described by the current divergence is calculated by the lowest order loop diagram and receives no essential contributions from higher diagrams. Among schemes which have been used are the Pauli-

Villars regularization[22], the point-splitting method[17], the heat kernel method[28] and the related method of complex powers of Seeley[29].

I shall here review only the last method, which has the merit of being straightforward, non-perturbative and simple under scaling.

Seeley[29] constructed complex powers of elliptic operators. For an operator D, the s-th power is represented by a kernel $D^s = \int dy K_s(x,y)$. $K_{-1}(x,y)$ is the usual Green's function of the operator. It is a power of D in the sense that $D^1 = D$, $D^0 =$ *identity* , $D^{s+t} = D^s D^t$ as operators. For fixed $x \neq y$, $K_s(x,y)$ is an entire function of s. For $x = y$, $K_s(x,x)$ is a meromorphic function of s. It is analytic at $s = 0$, which is a little surprising because $K_0(x,y)$ acts like a Dirac delta function. The value of $K_0(x,x)$ is particularly simple. It only depends on the coefficient functions in the operator locally. No long range data are needed. There is a recursion relation to calculate this value. Now, the anomaly can be expressed in terms of this quantity:

$$\delta \ln \mathrm{Det}[D_\pm(A)] = Tr[D_\pm^{-1} \delta D_\pm] = \pm\, Tr[D_\pm^0 \gamma^{2n+1} \delta\varepsilon] . \tag{2.7}$$

Here the chiral Dirac operator is in the modified form, $D_\pm(A) = \gamma^\mu(\partial_\mu + \dfrac{1 \pm \gamma^{2n+1}}{2} A_\mu + c(x)_\mu)$, where $c(x)$ does not depend on A_μ. This modification is needed to define the determinant. Therefore once $K_0(x,x)$ is obtained by Seeley's formulae, the anomaly can be written down explicitly.

The formula for $K_0(x,x)$ calls for a function denoted $b_{-m-n}(x,p,\lambda)$, where m is the order of the elliptic operator, n is the dimension of the manifold. It can be found in the following way. Locally, suppose the elliptic operator D is given by

$$D = \sum_{j=0}^{m} a_{m-j}(x, -i\partial), \tag{2.8}$$

where a's have been arranged such that the x dependent coefficients are moved to the left of the derivatives, and a_j is homogeneous in ∂ of order j. The symbol of D is then $\sigma(D) = \sum_{j=0}^{m} a_{m-j}(x,p)$, which is a function on the phase space. Now consider the operator $(D - \lambda)^{-1}$. Assume the complex parameter λ scales as p^m. The symbol of $(D - \lambda)^{-1}$ can be expanded according to the orders of homogeneity in $(p, \lambda^{1/m})$, $\sum_{l=0}^{\infty} b_{-m-l}(x,p,\lambda)$. $b_j(x, \alpha p, \alpha^{1/m}\lambda) = \alpha^j b_j(x,p,\lambda)$. Seeley gives the following recursion relation[29] to calculate the b's:

$$\begin{aligned} b_{-m}(a_m - \lambda) &= I, \\ b_{-m-l}(a_m - \lambda) &= -\sum \delta^{\mu_1} \ldots \delta^{\mu_h} b_{-m-j} \partial_{\mu_1} \ldots \partial_{\mu_h} a_{m-k}/h!\ , \end{aligned} \tag{2.9}$$

where, in the summation, $j < l,\ j + k + h = l, ;f.0 \le k \le m,$ and $\delta^\mu \equiv \frac{\partial}{\partial p_\mu}$, $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$. Assume the space-time is flat. For the Dirac operator then, $a_1 = -\gamma \cdot p$, $a_0 = i\gamma \cdot A$ or $i\gamma \cdot A(1 \pm \gamma^{d+1})/2$ depending on whether the fermion is chiral or not. Therefore, $m = 1$ and the recursion formula simplifies to

$$\begin{aligned} b_{-1} &= -1/(\gamma \cdot p + \lambda) \\ b_{-1-l} b_{-1}^{-1} &= -\sum_{h=0}^{l-1} \delta^{\mu_1} \ldots \delta^{\mu_h} b_{-l+h} \partial_{\mu_1} \ldots \partial_{\mu_h} a_0/h!. \end{aligned} \tag{2.10}$$

$K_0(x,x)$ is obtained from the following formula[29]:

$$K_0(x,x) = \frac{1}{m(2\pi)^n}\int d\Omega \int_0^\infty dt b_{-m-n}(x, \hat{n}, te^{i\theta}), \tag{2.11}$$

where $\hat{n}$ is the unit vector in the direction of the solid angle $d\Omega$. θ defines a ray on the complex plane with no eigenvalue of the operator on it.

Instanton, Chern Characters

On a 4-dimensional Riemannian manifold, the dual form of F is

$$*F = \frac{1}{2\sqrt{\det h}} F_{\mu\nu}\varepsilon^{\mu\nu\lambda\rho}h_{\lambda\tau}h_{\rho\sigma}dx^\tau \wedge dx^\sigma, \tag{2.12}$$

where $h_{\mu\nu}$ is the Riemannian metric. It also transforms covariantly. The classical action is proportional to $\int \mathrm{tr}(*F \wedge F)$, which is gauge invariant. In Euclidean signature metric, configurations satisfying the equations $F = *F$ are called the instantons[14].

Now, the notion of duality depends on the metric. Without the metric, $*F$ is not defined. For an instanton configuration, however, $\mathrm{tr}(*F \wedge F) = \mathrm{tr}(F \wedge F)$. The right hand side is independent of the metric. More generally, we can write down gauge invariants of the form $\mathrm{tr}F^m$ in arbitrary dimensions, where I have omitted the exterior wedge for brevity. It is easy to show that they are closed forms, *i.e.*, their exterior derivatives vanish. If the connection is defined globally on the space-time manifold, then they are actually exact. $\mathrm{tr}F^m = d\omega$. In general, the "non-exact" part is a topological invariant. It is called the Chern character. We have more on this subject in the next Chapter. In the case that A is defined globally, let $A_t = tA, F_t = tdA + t^2A^2$. Then [37],

$$\mathrm{tr}(F^m) = d\omega, \quad \omega = m\int_0^1 dt \ \ \mathrm{tr}(AF_t^{m-1}) . \tag{2.13}$$

Anomaly and Cohomology

The chiral gauge anomaly represents the non-invariance of the fermion determinant. It should be possible to write down an "effective action" to simulate this behavior[34]. Apparently, we need a functional of the gauge field that changes under infinitesimal gauge transformations. It should not be a local functional of the gauge field because otherwise it would be simply considered as a counter term in the renormalization program and there would not be an anomaly. Also, the anomaly suffers no renormalization and therefore is scale independent. It is thus reasonable to assume that the anomaly does not depend on the metric of space-time. Then the anomaly has to be a differential form. What I shall now describe is a procedure to construct such an effective action satisfying the above conditions which reproduces the anomaly. This is the method of the so-called descent equations[24,35]. In four dimensions, the result was shown to agree with explicit calculations.

To begin, assume that the anomaly for a chiral fermion interacting with the gauge field can be derived using an effective action which is proportional to the integral of ω on a space of one dimension higher. Formally take the Chern character $ch_{n+1} = \frac{1}{(n+1)!}\mathrm{tr}(\frac{i}{2\pi}F)^{n+1}$ with the space-time having dimension $2n$. As was mentioned above, locally, the Chern character is an exact form. Write $ch_{n+1} = d\omega^0_{2n+1}$. Consider ω^0_{2n+1}, which is a $(2n+1)$-form, integrated on a $(2n+1)$ dimensional manifold whose boundary is the space-time. This expression is then taken to be the effective action giving rise to the anomaly. The anomaly is then equal to the variation of this effective action. It can be shown that, the variation of ω^0_{2n+1} is a total divergence, $\delta\omega^0_{2n+1} = d\,\omega^1_{2n}$. The integral then can be evaluated on the boundary, which is the space-time itself. The expression for the anomaly is thus obtained.

$$\delta \ln \mathrm{Det} = \omega^1_{2n}. \tag{2.14}$$

This manipulation can obviously be continued.

Cocycles

Let us think of A as dependent on the transformation g. $A^g = g(A+d)g^{-1}$. Denote an infinitesimal change by the differential $v \equiv \delta g g^{-1}$, $\delta v = v^2$. For convenience, assume δ anti-commutes with d. Then,

$$\delta A = -dv + vA + Av, \tag{2.15}$$

where the omission of the label g on A should not cause confusion. In the above discussion, the relations between the forms are then

$$\begin{aligned} ch_{n+1} &= d\omega^0_{2n+1} \\ \delta\omega^0_{2n+1} &= d\omega^1_{2n} . \end{aligned} \tag{2.16}$$

Applying the operator δ to ω^1_{2n}, we can show that it is of the form $d\omega^2_{2n-1}$. This operation can be continued and we get the descent equations[35].

$$\begin{aligned} ch_{n+1} &= d\omega^0_{2n+1}, \\ \delta\omega^0_{2n+1} &= d\omega^1_{2n}, \\ \delta\omega^1_{2n} &= d\omega^2_{2n-1}, \\ &\dots \\ \delta\omega^i_{2n+1-i} &= d\omega^{i+1}_{2n-i}, \\ &\dots . \end{aligned} \tag{2.17}$$

ω^i_j is an i-form in δ and a j-form in d. $i + j = 2n + 1$. ω^i_j integrated on a manifold of dimension j without boundary is closed for δ. That is, it is a cocycle in δ. As we have seen, $i = 1$ case gives the anomaly. Faddeev has argued the $i = 2$ expression gives the Hamiltonian interpretation of the anomaly. These expressions hint at the possibility of topological interpretation of the origin of the anomalies. This is the main topic of the dissertation. Once this is understood, we study the effects of the other cocycles in the descent equations.

3. Mathematical Background

In this chapter, we start to give the mathematical background that has become an important tool in analyzing the gauge theories, *i.e.* , the subject of algebraic topology. An excellent review intended for physicists can be found in Ref.[36].

In algebraic topology, we are interested in homotopy invariants. These are invariants of spaces under the continuous change of a parameter. Therefore, we consider objects the same if they can be turned into each other by continuous deformations.

Homotopy Groups

The first type of invariants we consider are the homotopy groups $\pi_n(M)$. $\pi_n(M)$ consists of maps from the n-dimensional sphere S^n into M with the north pole mapped to a fixed point p in M. Of course, all maps that are homotopic to each other are considered the same. To multiply two maps, we simply put two images together and this is equivalent to an image of S^n with the equator continuously shrunk to the base point p. It is easy to visualize the inverse of an element in $\pi_n(M)$. For $n \geq 2$, π_n is abelian. π_1 however may be non-abelian, while π_0 denotes the set of disconnected components of M, which does not necessarily have a group structure. If M is a group, then π_1 is also abelian and π_0 does become a group.

The loop space ΩM of M consists of maps $S^1 \to M$ with a fixed point in S^1 mapped into a fixed point in M. $\pi_n(\Omega M) \cong \pi_{n+1}(M)$.

A space which has only one non-vanishing homotopy group is called an Eilenberg-MacLane space. If $\pi_j(K) = \pi$, $\pi_i(K) = 0$, for $i \neq j$, K is an Eilenberg-MacLane space of the type $K(\pi, j)$. It is unique up to homotopy in the category of CW complexes. For example, S^1 is of type $K(Z, 1)$. $\pi_i(S^n) = 0$ for $i < n$. $\pi_n(S^n) = Z$. But generally, S^n is not an Eilenberg-MacLane space. Formally taking $n \to \infty$, we find $\pi_i(S^\infty) = 0$, so S^∞ is contractible! For the classical simple Lie groups, when the dimensions of the groups are large compared with the dimension of the homotopy groups (in the so-called stable range), the homotopy groups follow the Bott periodicity. $\pi_{even}(U) = 0$, $\pi_{odd}(U) = Z$; $\pi_{8n+3,7}(O) = Z$, $\pi_{8n+0,1}(O) = Z_2$, $\pi_{8n=2,4,5,6}(O) = 0$; $\pi_{8n+3,7}(Sp) = Z$, $\pi_{8n+4,5}(Sp) = Z_2$, $\pi_{8n+0,1,2,6}(Sp) = 0$. Outside the stable range, the homotopy groups are rather sporadic.

Homology and Cohomology Groups

Next, we describe the homology and cohomology groups. To simplify the discussions, we restrict ourselves to CW complexes here. These are spaces made of closed discs in different dimensions. Starting from a set of points, adding arcs to them and attaching the ends to the points, we get the 1-skeleton. Successively add 2-discs (2-cells), 3-cells, and so on. Always let the boundary of a cell be attached to the lower skeleton. The resultant space is then a CW complex. A space may have many different CW decompositions. The homology and cohomology groups we are about to define are unique for a space. Each cell is assigned an orientation. A cellular i-chain is an object that can be written as a linear combination of the i-cells with integer coefficients. In other words, the set of i-chains is the free abelian group generated by the i-cells. Naturally, when we take the boundary (as a linear operation) of an i-chain we get an $(i-1)$-chain,

$$\partial \colon C_i \to C_{i-1} \; .$$

If $\partial a = 0$, a is said to be a closed chain. If $a = \partial b$ for some b, a is said to be a boundary. It is clear that a boundary is closed but not *vice versa* . The homology group consists of closed chains with two identified if they differ by a boundary,

$$H_i(Z) = \frac{\ker(\partial{:}C_i \to C_{i-1})}{\mathrm{im}(\partial{:}C_{i+1} \to C_i)} = \{\text{ closed chains }\}\ \textit{modulo}\ \{\text{ boundaries }\}\,. \qquad (3.1)$$

The cohomology group is defined similarly. A cellular i -cochain is a linear functional of the i-chains. So the group of cochains is the "dual" of the group of chains, $C^i \equiv \mathrm{Hom}(C_i, Z)$. A coboundary operator can be defined going in the opposite direction,

$$d{:}C^i \rightarrow C^{i+1} \ ,$$

by the partial integration formula $dc(a) = c(\partial a)$.

$$H^i(Z) = \frac{\ker(d{:}C^i \rightarrow C^{i+1})}{\mathrm{im}(d{:}C^{i-1} \rightarrow C^i)} = \{\text{ closed cochains }\} \textit{ modulo } \{\text{ coboundaries }\} \quad (3.2)$$

For a compact manifold without boundary, if we replace the integers Z by the real numbers R, the cohomology group is isomorphic to the De Rham cohomology,

$$H^i(R) \cong H^i_{DR} \equiv \frac{\ker(\, d{:}i\text{-forms} \rightarrow (i+1)\text{-forms})}{\mathrm{im}(\, d{:}(i-1)\text{-forms} \rightarrow i\text{-forms})} \ , \quad (3.3)$$

where d is the exterior differential. The linear functionals are simply the integrations of the forms.

For finite complexes, H_i and H^i are finitely generated abelian groups. Therefore each is of the form,

$$Z + Z + \cdots + \quad Z_{m_1} + Z_{m_2} + \cdots .$$

The first part consists of Z's and is called the free part. The number of them is called the dimension or the Betti number. The rest is known as the torsion part. If $H^i(Z)$ has b^i copies of Z, the $H^i(R)$ is equal to b^i copies of R, b^i -dimensional vector space. The same is true for homology groups.

Homology and cohomology groups with coefficients in an arbitrary abelien group are also useful. But they all can be calculated by the groups with integer coefficient. These are given by the Universal Coefficient Theorems. For an abelian group A,

$$H_i(A) = H_i(Z) \otimes A + \mathrm{Tor}(H_{i-1}(Z), A); \tag{3.4}$$

$$H^i(A) = \mathrm{Hom}(H_i(Z),A) + \mathrm{Ext}(H_{i-1}(Z), A)\,. \tag{3.5}$$

$\otimes$, Hom, Tor and Ext are functors linear in both of their arguments.

$\otimes$	Z	Z_n
Z	Z	Z_n
Z_m	Z_m	$Z_{(m,n)}$

Hom	Z	Z_n
Z	Z	Z_n
Z_m	0	$Z_{(m,n)}$

Tor	Z	Z_n
Z	0	0
Z_m	0	$Z_{(m,n)}$

Ext	Z	Z_n
Z	0	0
Z_m	Z_m	$Z_{(m,n)}$

(m,n) denotes the greatest common divisor of m and n. It follows from the second equation that

$$H^i(Z) = \text{free part of } H_i(Z) + \text{ torsion part of } H_{i-1}(Z)\,. \tag{3.6}$$

A Moore space is a space that has only one non-vanishing homology group. $M \cong M(A,i)$, $H_i(M,Z) = A$, $H_j(M,Z) = 0$, $j \neq i$. The Moore space is also unique up to homotopy in the category of CW complexes. $S^n \cong M(Z,n)$.

There is a direct relationship between the homology and the homotopy groups. The Hurewicz theorem states that, for a given space, the lowest non-trivial homotopy group, if not the first, coincides with the lowest non-trivial homology group. $\pi_n \cong H_n, H_i = 0$, if $\pi_i = 0$, for $0 < i < n$ and $n > 1$. If $\pi_1 \neq 0$, $H_1 \cong (\pi_1)_{ab}$, where $(\ )_{ab}$ denotes abelianization, a process forcing the commutativity of the group.

A very powerful theory relating analysis to topology is Morse theory. Suppose we have a smooth compact manifold of dimension n with no boundary. The Betti numbers satisfy the duality relation, $b^i = b^{n-i}$. Now, take a smooth function on this manifold. This function is supposed to have only isolated stationary points where its derivative vanishes. At these points, assume the matrix $\partial\partial f$ of the second partial derivatives is non-degenerate. Morse theory then concludes that the number of stationary points at which there are i decreasing directions (the matrix $\partial\partial f$ has i negative eigenvalues) is at least b^i. The more complicated the manifold is, the more stationary points a function on the manifold must have.

Fiber Bundles

We now turn to the theory of fiber bundles, which is the basis to understand the mathematical properties of gauge theories.

The notion of a fiber bundle generalizes that of the Cartesian product of two spaces. It introduces twists into the product treating the two spaces asymmetrically. Given a base space M and a space as the fiber F, to construct the twisted product, we cover the M with open sets $\{U_\alpha\}$. We think of M as a patch work made of these open sets. On each U_α, we form the usual product $U_\alpha \times F$. To piece together to form the bundle (total space B), we use transition functions on the overlaps. At each point in the overlap, the transition function has value in a transformation group of the fiber. G is called the structure group. The transition functions interconnect the fibers in different patches. The possible twists are introduced by these. On multiple overlaps, there are consistent conditions to insure unambiguous construction. The relationship among the fiber, the base and the total space may be summarized by the following.

$$F \xrightarrow{i} B \xrightarrow{\pi} M\,,$$

where the first map is an inclusion of F above an arbitrary fixed point in M and π is the projection identifying all the points in a fiber.

There is an exact sequence of homotopy groups for the fiber bundle structure.

$$\cdots \to \pi_{n+1}(F) \to \pi_{n+1}(B) \to \pi_{n+1}(M) \to \pi_n(F) \to \pi_n(B) \to \pi_n(M) \to \cdots . \tag{3.6}$$

The arrows represent homomorphisms of the groups. The sequence is exact in the sense that the image of an arrow is equal to the kernel of the next arrow.

A section of a bundle $F \xrightarrow{i} B \xrightarrow{\pi} M$ is formally a map s that is the right inverse of π, $\pi \cdot s = id.$. It is a continuous choice of points in the total space with each fiber contributing one point. Sections can also be considered as fields: functions on patches valued in the fiber with transition functions connecting them.

A fiber bundle with fiber $F = G$ and the action of G on G being the left (or right) translation is called a principal fiber bundle. From a given fiber bundle, we can form an associated principal bundle by replacing its fiber with the structure group G and letting the transition functions act on the fibers as left translations. The original bundle can easily be recovered from this bundle by putting the original fibers back. Two bundles are isomorphic if starting from one we can make transformations of the fibers in open sets and the corresponding transition functions to get to the other. Only the isomorphic classes have geometric significance. To classify bundles according to the isomorphism classes, only the associated principal bundles need to be classified. The most general tool for the classification is the classifying space BG. It is required to have a universal principal G bundle defined on it. The property that characterizes a universal bundle is that it has a contractible total space. The classifying space exists uniquely up to homotopy. The universal bundle has all the basic twists the group G can have.

In this bundle, twists occur wherever possible. To transcribe some of the twists to a given manifold M, we make a map $M \to BG$ and copy the fibers and transition functions through the map to M. The bundle resulted this way is called the pull-back. To be more precise, any principal G bundle is isomorphic to a pull-back of the universal bundle through a map $M \to BG$. Two pull-back bundles are isomorphic if and only if the two maps are homotopic.

$$\{ \text{ isomorphic classes of } G \text{ bundles on } M \} \cong [M , \ BG] \ . \tag{3.7}$$

Homotopically, BG is the inverse to the loop space construction. $\Omega BG \overset{h.}{=} B\Omega G \overset{h.}{=} G$. ΩG is a group with a pointwise rule of multiplication .

Let us look at two simple cases. If the base is a Moore space of the type $M(Z, n) \overset{h.}{=} S^n$, then $[M, BG] = [S^n, BG] = \pi_n(BG) = \pi_{n-1}(\Omega BG) = \pi_{n-1}(G)$. Therefore, G bundles on S^n are classified by the homotopy group $\pi_{n-1}(G)$. Pictorially, S^n can be covered by two discs with overlap homotopy equivalent to the equatorial S^{n-1} and the topological type of the transition functions is given by $\pi_{n-1}(G)$. Now, if G is an Eilenberg-MacLane space of the type $K(A, n)$ with A an abelian group, then BG is an Eilenberg-MacLane space of type $K(A, n+1)$ because of the inverse relation of Ω and B. $[M, \ BG] = [M, \ K(A, n+1)]$. But $[M, \ K(A, n+1)] = H^{n+1}(M, A)$ since the Eilenberg-MacLane space is the classifying space for cohomology. The bundles in this case is classified by cohomology. From the point of view of obstruction theory, this is the only obstruction to the construction of a section of the principal G bundle.

For example, to classify bundles on S^4 with G a simple Lie group, one looks at $\pi_3(G) = Z$, *i.e.*, they are labelled by an integer. In physics, the integer is called the instanton number. Let's look at another example. A real line bundle has group $G = GL(1,R)$ which is the real line with its origin deleted. $Gl(1,R) \overset{h.}{=} Z_2 \overset{h.}{=} K(Z_2, 0)$. Therefore the real line bundles are classified by $H^1(\ ,Z_2)$. The element in $H^1(\ ,Z_2)$ that corresponds to a bundle is called the first Stiefel-Whitney characteristic class of the bundle and it specifies the Möbius twists in the bundle.

In the above examples, the classification problem is reduced to homotopy or cohomology groups. In general, however, $[M,\ BG]$ is not explicitly known. The characteristic classes then are invaluable information to distinguish different bundles. And some characteristic classes are especially nice since they link local geometric data to global topological invariants. Generally speaking, a characteristic class is an element in a cohomology group that corresponds to an isomorphism class of bundles. Characteristic classes in the real cohomology (De Rham cohomology of forms) are usually constructed from curvature invariants. Before going into these, we introduce some geometric notions of fiber bundles.

The gauge transformations are continuous transformations of the fibers by the structure group at all points in the base. In the patch $U_{\alpha,\beta}$, suppose the transformation is given by the function $g_{\alpha,\beta}:U_{\alpha,\beta} \rightarrow G$. In order for the transformations to agree on the overlap $U_\alpha \cap U_\beta$, we must have

$g_\alpha g_{\alpha\beta} = g_{\alpha\beta} g_\beta$, or $g_\alpha = g_{\alpha\beta} g_\beta g_{\alpha\beta}^{-1}$, where $g_{\alpha\beta}$ is the transition function. Therefore the transformations can be identified with the sections $\{ g \}$ of the adjoint bundle.

On a smooth manifold, The local geometry of a fiber bundle is specified by a connection which determines the parallel transport of the fiber. Locally, it is an algebra valued 1-form. Across an overlap, it transforms as,

$$A_\alpha = g_{\alpha\beta}(A_\beta + d)g_{\alpha\beta}^{-1}. \tag{3.8}$$

Under a gauge transformation, the connection is transformed the same way,

$$A^g = g(A + d)g^{-1}. \tag{3.9}$$

Infinitesimal holonomy by A is described by the curvature 2-form $F = dA + A \wedge A$. It transforms covariantly. $F_\alpha = g_{\alpha\beta} F_\beta g_{\alpha\beta}^{-1}$. And $F^g = gFg^{-1}$.

From an invariant polynomial of the curvature, one can construct a characteristic class in the De Rham cohomology. This amounts to showing a) it is a closed form; b) it changes at most by a exact form when the connection is changed for the same bundle. First of all, an invariant polynomial of F is defined globally on the base since the transition functions do not act on it at all. It is a differential form of even rank because the curvature is a 2-form. It is straightforward to show that it is closed by using the Bianchi identity $dF = FA - AF$ and the fact that it is invariant under infinitesimal transformations. On the other hand, a invariant polynomial can be expressed as the linear combination of symmetric homogeneous polynomials. If $P(\ , \ , \dots , \ )$ is a symmetric homogeneous invariant polynomial of degree m, it can easily be checked that [37]

$$P(F_1, F_1, \dots , F_1) - P(F_0, F_0, \dots , F_0) = d\alpha, \tag{3.10}$$

where,

$$\alpha = m\int_0^1 dt P(A_1 - A_0, F_t, \dots , F_t), \ \ F_t = dA_t + A_t^2, \ A_t = (1 - t)A_0 + tA_1. \tag{3.11}$$

Since A_t transforms in the right way, it is also a connection. The difference $A_1 - A_0$ transforms covariantly. It can be concluded that α is a well defined form on M. We see that the cohomology class defined by P is a topological invariant for the bundle. For the group $U(n)$, we may take P to be $c = \det(1 + \frac{i}{2\pi}F)$. The characteristic class is called the Chern class. Expand this into forms of various orders, $c = c_1 + c_2 + \cdots . \ c_i$, a $2i$ -form, is called the i-th Chern class. A nice thing about this invariant is that it is actually an integer invariant: the integral of it over any integral cycle is an integer. Another choice for P is $ch = tr \exp(\frac{i}{2\pi}F) = \sum_i \frac{1}{i!}(\frac{i}{2\pi}F)^i$. This invariant adds no new information. But it is very useful for vector bundles because of its nice algebraic properties. The first Chern character coincides with the first Chern class. The U(1) bundles on a space equivalent to a 2-sphere are completely characterized by $c_1 = ch_1 \in H^2(S^2, Z) \cong Z$. In physics, the integer is the Dirac monopole charge.

There are other types of characteristic classes. There are even characteristic classes in cohomology with other coefficients. For example, the Stiefel-Whitney classes are in Z_2 coefficient. The first class of the tangent bundle decides if a real vector field can be given an orientation while the second class decides if spinor fields can exist.

A useful invariannt from the Riamannian curvature is

$$\hat{A}(M) = \sqrt{\det \frac{\frac{R}{4\pi}}{\sinh \frac{R}{4\pi}}}\,. \tag{3.12}$$

Vector Bundles and K-Theory

Vector bundles are bundles with vector spaces as their fibers. For simplicity, consider complex vector bundles. The fiber is a complex vector space $F \cong C^n$. The structure group can be either $GL(C, n)$ or $U(n)$. The latter corresponds to a choice of a hermitian metric. By operating on the fibers of vector bundles at each point of the base, we can get new vector bundles:

1. direct sum $V_1 \oplus V_2$,

2. tensor product $V_1 \otimes V_2$,

3. complex conjugate V^*,

4. Hom(V_1, V_2),

5. exterior algebra $\Lambda(V)$.

$\mathrm{Hom}(V_1, V_2) \cong V_1^* \otimes V_2$. This isomorphism, though, depends on a choice of a hermitian metric.

Suppose that $V \to M$ is a vector bundle. And a group Γ acts on both V and M. Suppose the action commutes with the bundle projection.

$$\begin{array}{ccc} \Gamma \times V & \to & V \\ \downarrow & & \downarrow \\ \Gamma \times M & \to & M \end{array}$$

And the action is linear in the fibers. Such a bundle is called an equivariant vector bundle under Γ. Because of the special requirements on the equivariant bundles, their classification is different from the classification without the requirements. An non-trivial equivariant bundle may correspond to a trivial bundle in the general category. This construct is very useful for studying gauge covariance in gauge theories.

Let $Vect(M)$ be the set of equivalent classes of vector bundles on M. The direct sum gives $Vect(M)$ an abelian semi-group structure. The tensor product makes it a semi-ring. By formally introduce a subtraction operation, it can be made a group or a ring, at the expense of loosing some information. This group is called the K-group of M, $K(M)$. One way to define K is to take it as $Vect \times Vect$ with elements of the forms (a, b) and $(a + c, b + c)$ identified. The elements of this group are called virtual vector bundles, or vector bundles for short. In this group then, it is sensible to talk about the difference of two vector bundles. Because of the tensor product, $K(\ )$ is actually a ring. This construction can be carried out in the equivariant category also.

The Chern character is very natural for K-theory because it behaves well under the direct sum and the tensor product. If we introduce the so-called cup product into the cohomology group to make it a ring, the Chern character gives a ring homomorphism from $K(\ )$ to $H^*(\ )$. If a virtual bundle V is given by (V_1, V_2), then $ch(V) = ch(V_1) - ch(V_2)$.

The Atiyah-Singer Index Theorems

The Atiyah-Singer index theorems[38] concern the zero modes of elliptic operators. In their simplest form, difference of the numbers of zero modes of the operator and its conjugate is calculated by topological invariants. The difference is a topological invariant although each number is not. For example, on a sphere S^{2n}, for a Dirac operator in the background of a gauge field, the difference of the number of the zero modes of positive and negative chiralities is calculated to be

$$n_+ - n_- = \int_{S^{2n}} \frac{1}{n!} \mathrm{tr}(\frac{i}{2\pi} F)^n \ , \tag{3.13}$$

where F is in the representation of the Dirac operator. This formula is the integrated form of the equation for the U(1) anomaly.

More generally, suppose M is a compact manifold without boundary. An elliptic operator D takes the sections of the vector bundle V_1 to those of V_2, $D{:}\Gamma(V_1) \to \Gamma(V_2)$. The kernel of the operator is the vector space of zero modes, $\ker(D) \equiv \{v_1 \mid v_1 \in \Gamma(V_1), Dv_1 = 0\}$. The cokernel is the complementary part of the image of the operator, $\mathrm{coker}(D) \equiv \{v_2 \mid v_2 \in \Gamma(V_2), v_2 \neq Dv_1\} = \ker(D^*)$. Now, suppose there is a family of elliptic operators parametrized by a space Y. The kernel and cokernel of the operator then depend on a parameter $y \in Y$. Let Ker and Coker denote the family of the kernels and cokernels parametrized by Y. The upshot of the Atiyah-Singer index theorem is that, although the Ker or Coker cannot be made a vector bundle on Y individually, the formal difference Ker (D) $-$ Coker(D), with a slight modification when necessary, is an element of $K(Y)$. This element is called the (family) Index of the operator. Atiyah and Singer showed how this can be calculated by topological methods from the symbol of the elliptic operator. Taking the zero-th Chern character of this element gives the difference of the dimensions dim Ker $-$ dim Coker, which is the difference of the number of zero modes.

Another interpretation of the Index, although less rigorous, is perhaps much more fruitful for our purposes. The sections of V_1 and V_2 give rise to Hilbert spaces H_1 and H_2. When parametrized by Y, they become infinite dimensional vector bundles. The operator D is then a section of the bundle $\mathrm{Hom}(H_1, H_2)$. The Index is then the formal difference $H_1 - H_2$. Since all these constructs are infinite dimensional, regularizations are needed. Conversely, a good regularization procedure that preserves all the continuity requirements has to be compatible with this interpretation.

Spectral Flow and Topological Conservation Laws

Suppose that we have a family of self-adjoint operators D_Y , where Y parametrizes the family. For a particular eigenvalue, the eigenspaces can be thought of forming a vector bundle on Y, a line bundle for a non-degenerate eigenvalue. We can take the direct sum of the eigenspaces for several eigenvalues to get larger vector bundles. The eigenvalue may depend on the parameter $y \in Y$. I call this a spectral band. If two bands become degenerate at some point in Y, we consider the direct sum of the eigenspaces. Let us now imagine introducing another parameter t which runs from 0 to 1, say. As t, changes, some spectral bands simply change gradually. Some, however, can collide with other bands. This seems to be a complicated situation. But if we sum all the participating bands and look at the total eigenbundle, nothing has really happened. More precisely, although the sum at the colliding point may not split whereas it does at other points, an additive invariant in a group does not change. That is, the sum is conserved in the group $K(Y)$. Virtual vector bundles may get exchanged. But the sum is conserved. Thus, we have a topological conservation law. From this, we can conclude that the Chern character, because it is additive, is conserved also.

4. The Topology of the Gauge Configurations

The path integral of the gauge fields is in some sense the summation over all gauge potentials. To define the integral, we work on a compact Riemannian manifold M.

Instanton Sectors and Instantons

Given the gauge group G, or in mathematical terminology the structure group, there may be many instanton sectors. The connections are classified according to the principal bundles they belong to. The functional space of gauge potentials then has disconnected pieces corresponding to the isomorphic classes of principal G bundles on M. Each component is called an instanton sector. The sectors are classified by $[M, BG]$. If $M = S^n$, the classification is given by $\pi_{n-1}(G)$.

If M is four dimensional, the number $\int_M ch_2$ is a constant in each sector, which was indicated in the last Chapter. Suppose it is positive for a certain sector.

Then the self-dual configurations for which $F = *F$ minimize the Euclidean action (or maximize the statistical weight) in that sector. If the number is negative, the same thing is true for the anti-instanton configurations $F = \ - *F$.

The simple version of the Atiyah-Singer index theorem says the difference of the number of solutions for the Dirac equation of positive and negative chiralities is a constant for each sector. The index can be calculated by integrating a form constructed from the curvatures.

In the following, I shall concentrate on one of the instanton sectors and speak of the space of connections which is actually only one sector.

The Space of Connections

Define **A** to be the space of connections. We would like to have linear combinations defined in **A**. If this space is for the trivial bundle, the elements are just global Lie algebra valued differential forms on M. Any linear combinations of two is another connection. Therefore **A** is a vector space in this case. If the bundle is not trivial, then we have to be careful about the properties on the overlaps. Let $A_0, A_1 \in \mathbf{A}$ be two connections. Write $A = sA_0 + tA_1$. In the overlap $U_\alpha \cap U_\beta$, $A_\alpha = g_{\alpha\beta}[A_\beta + (s + t)d]g_{\alpha\beta}^{-1}$. Therefore, if $s = 1 - t$, then A is a connection. Notice **A** is an affine space and hence topologically trivial. It is contractible to a point. The homotopy, homology, cohomology and K groups all vanish. This is not the end of the topological story though. Since we are dealing with gauge theory, it is important to keep gauge invariance in mind.

The Group of Gauge Transformations

Let **G** be the group of gauge transformations on the G bundle on M with a base point. **G** acts on **A** by the formula $g{:}A \to g(A + d)g^{-1}$. This action is free in the sense that no A is left invariant except by the identity[39][40]. No solution exists for $A^g = A, \;\; g \neq id.$. The topology of **G** can be highly non-trivial.

The Gauge Orbit Space and a Universal Bundle

The free action of **G** implies a nice orbit space **A/G**. This space is obtained by identifying the gauge equivalent potentials. Gauge invariant functions of the potentials are actually functions on **A/G**. Local trivializations can be shown to exist for the projection **A** $\to$ **A/G**. It can then be concluded that **G** $\to$ **A** $\to$ **A/G** is a principal **G** bundle[40]. Since **A** is contractible, this is actually the universal bundle. The orbit space is a classifying space B**G**. What was a topologically trivial space becomes, as far as **G** is concerned, the most complicated space possible. The non-trivial topology is forced upon us by the gauge invariance.

The Fermionic Hilbert Spaces

The fermi fields in the background of a gauge potential can be thought of as forming a Hilbert space. The collection of the Hilbert spaces for all $A \in \mathbf{A}$ is then an infinite dimensional vector bundle on **A**[41]. On an even dimensional manifold, this Hilbert bundle splits into two Hilbert bundles, one for each chirality. For each bundle, a single fiber is the Hilbert space of sections of the bundle $S^{\pm} \otimes V$, where $S^{\pm}$ is the chiral spinor bundle and V is the G vector bundle given by the representation of the fermions. As ordinary bundles, they are trivial simply because the base space **A** is trivial. But remember that there is gauge covariance to impose. Now, let P be the projection for either one of the Hilbert bundles. $P:(\psi, A) \to A$. The action of **G** is given by $g:(\psi, A) \to (g\psi, A^{g})$. The two operations obviously commute with each other. The action of **G** is linear on the fibers. Therefore, the bundle is really an equivariant bundle under **G**. Because **A** is a free **G** space, the equivariant **G** bundles on **A** are in one to one correspondence with the ordinary bundles on the orbit space **A**/**G**. The chiral Dirac operator, which is covariant under gauge transformations, then can be regarded as a section of the bundle $\mathrm{Hom}(\mathrm{H}^{+}, \ \mathrm{H}^{-})$ in the equivariant category. The Index of the operator is then the difference $\mathrm{H}^{+} - \mathrm{H}^{-}$ in the equivariant K group. This comes from the second definition of the Index of last Chapter. We can look at the Index from the first definition also. It would only involve finite dimensional vector spaces, but they do not form well behaved spaces like the bundles.

Atiyah and Singer calculated the Chern character of the Index of the chiral Dirac operator[41].

$$ch(Index_+) = \int_M \hat{A}(M)\ ch(E)\,. \tag{4.1}$$

Here, E is a vector bundle on $M \times \mathbf{A}/\mathbf{G}$. In defining the quantum numbers of the fermion fields, a representation of the gauge group was chosen to give the vector bundle V. The vector bundle E is a G vector bundle in the same representation. Its restriction to a point in the orbit space is in fact the vector bundle V. In general, it is obtained from the following principal G bundle. Assume the transition function on the fibers of the principal bundle P defining the gauge theory is the left translation. We have a right action of G on P. G acts on $\mathbf{A}$ trivially. The actions of G and $\mathbf{G}$ commute. Now, G acts on the space $(P \times \mathbf{A})/\mathbf{G}$ freely. Hence, we get a principal G bundle on $M \times \mathbf{A}/\mathbf{G}$.

Transgression and the Cocycles

The Chern characters appearing above are of course even cocycles on the orbit space $\mathbf{A}/\mathbf{G}$. We can think of them as closed forms on the total space $\mathbf{A}$. Or else, we can pull the Chern characters back through the projection of the universal bundle to the total space. On $\mathbf{A}$, they vanish when taken in the fiber direction because they are really forms on the base. Since the total space is contractible, they are exact forms, *i.e.*, they are the exterior derivatives of some odd forms. Now, if we take the fiber direction for these odd forms, they are closed forms on a fiber. This is so because their derivatives have no fiber components. Each fiber is a copy of the gauge transformation group. Thus, we see that these odd forms are cocycles on the group **G** [41]. In fact, they are the odd cocycles that appear in the descent equations. I shall elaborate on these objects in a later Chapter.

The Determinant Line Bundle

If $L \in \mathrm{Hom}(V_1, V_2)$ is a linear operator on vector spaces of dimension n, its determinant then is a linear operator on the top exteriors power of the vector spaces $\det L \in \mathrm{Hom}(\Lambda^n V_1, \Lambda^n V_2)$. Similarly, for the chiral Dirac operator $D_+(A)$, we define the determinant line bundle as $\mathrm{Hom}(\Lambda^\infty \Psi^+, \Lambda^\infty \Psi^-)$. The determinant of the Dirac operator is a section of this bundle. The word line means one dimensional vector space. The first Chern character of the determinant line bundle is equal to that of the Index bundle. What happens when the first Chern class is not zero? I shall answer this question in the nest Chapter by an typical example. I show that the non-triviality of the determinant line bundle will give rise to gauge anomalies.

The Hamiltonian Picture

In the canonical quantization scheme of the fermi fields, one works on the Minkowski space. The temporal gauge is usually chosen. The massless single particle Hamiltonian is the Dirac operator in $(n - 1)$ dimensions. There is no chirality in odd dimensions. The fermion Hilbert spaces on each fiber in the universal bundle $\mathbf{A} \rightarrow \mathbf{A}/\mathbf{G}$ give a representation of the group **G**. The eigenvalues of the Dirac operator are the energy levels for the fermion. The operator is self-adjoint and the eigenvectors form a complete set. Roughly speaking, the Hilbert space is generated by the eigenvectors of the Dirac operator. The Hilbert space is considered as varying with the background. On each fiber, since it gives a representation, the Hilbert spaces gives a trivial bundle. This does not mean, however, that any fraction of it is trivial. In particular, we shall be interested in the negative (or positive) energy part of the Hilbert spaces. That is, on a fiber at $[A]$, we look at Φ_- (Φ_+). As a bundle on a gauge orbit, it does not behave well group theoretically as the group element changes within the gauge transformation group. For instance, it does not give a representation. Let us concentrate on the topological questions first. Our interpretation of the even cocycles in the descent equations is that they are the Chern characters of this bundle. This is related to spectral flow of the Hamiltonian. In order to define topological invariants for this infinite dimensional bundle, we need to take a reference trivial bundle on a subspace of the total space. We use a parameter to interpolate the reference bundle and the bundle we wish to study. As the parameter varies, the positive energy states can interact with the negative states by having the energy go above or below zero or collide with other energy levels. The exchange of the topological invariants then determines the bundle of interest. We shall study this problem later.

The Vacuum Line Bundle

If we take the gauge fields as background, then we can second quantize the fermions starting from the so-called vacuum states. The vacuum states are a one dimensional space for a given background. The collection of the vacuum states on a fiber of the universal bundle is what we call the vacuum line bundle. According to Dirac, the vacuum is the state with all the negative energy states filled. Therefore, the vacuum line bundle is equal to $\Lambda^{\infty}\Phi_-$. The Fock space of the fermions is $\Lambda(\Phi_+ \oplus \Phi_-{}^*)$. Under unitary actions, $\Lambda(V^*) \cong \Lambda(V)$. up to a phase. For $V = \Phi_-$, the phase is the phase of the vacuum line bundle. $\Lambda(\Phi_+ \oplus \Phi_-) \cong \Lambda(\Phi_+) \otimes \Lambda(\Phi_-{}^*) \cong \Lambda(\Phi_+) \otimes \Lambda(\Phi_-) \cong \Lambda(\Phi_+ \oplus \Phi_-) = \Lambda(H)$, modulo the vacuum line bundle. The last bundle is trivial as we know. Therefore, whether the Fock bundle is trivial depends on if the vacuum line bundle is trivial[45]. On the other hand, the Gauss law requires that the Fock space be a representation of the time independent transformation group. Topologically, this amounts to checking if the vacuum line bundle is trivial. If the Fock space fails to be a representation, we say this is an anomaly.

When the spaces are required to be real, the Fock space is $\Lambda(\Phi_-)$.

Specialization to Spheres

If we are not interested in exotic space-time topologies, we need only consider the spheres as the one point compactification of the Euclidean space. On S^n, the gauge orbit space **A/G** can be shown [40] to be homotopically equivalent to $\Omega^{n-1}G$. and $\mathbf{G} \overset{h.}{=} \Omega^n G$. In particular, $\pi_i(\mathbf{A/G}) = \pi_{i+n-1}(G)$, and $\pi_i(\mathbf{G}) = \pi_{i+n}(G)$. I shall use these facts to construct the samples to see how the anomalies are related to the non-trivial topological structures. On a sphere, $\hat{A}(S^N) = 1$.

5. SU(2) Theory in Two Dimensions

In this Chapter, I study the chiral Dirac operator on a 2-sphere in the background of SU(2) gauge fields[46,52]. $M = S^2$, $G = SU(2)$. π_0 (A) $= \pi_1(SU(2)) = \{0\}$. There are no non-trivial instanton sectors. The spaces **A** and **A/G** are connected. π_1 (**A/G**) $= \pi_0$(**G**) $= \pi_2(SU(2)) = 0$. The orbit space is simply connected. π_2(**A/G**) $= \pi_1$(**G**). $= \pi_3(SU(2)) = Z$. There is a basic non-contractible image of a 2-sphere in the orbit space and it comes from a non-contractible image of a loop in **G**. From these data, by using the Hurewicz theorem and the universal coefficient theorem, we find H^2(**A/G**,Z) = Z. Now, the U(1) bundles on **A/G** are classified by this cohomology group because U(1) is an Eilenberg-MacLane space of type $K(Z,1)$. Complex line bundles are 1-dimensional vector bundles. The structure group can be taken as the U(1) group. Therefore, they are classified by this cohomology group. On the gauge orbit space, the complex line bundles are classified by an integer. This integer is the first Chern character. The determinant of the chiral Dirac operator is a section of a line bundle on the orbit space. By the result of Atiyah and Singer, for the basic representation, the first Chern character of the Index of the chiral Dirac operator is actually the generator of $H^2(\mathbf{A}/\mathbf{G})$. That is to say, the determinant line bundle corresponds to a basic non-trivial line bundle on the orbit space or it is a basic non-trivial equivariant line bundle on **A**. Let us see what is actually happening to the determinant itself.

The Space-Time 2-Sphere

We use spherical coordinates (θ, φ) on the sphere. We choose a metric with constant curvature,

$$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2 . \tag{5.1}$$

Since the tangent bundle is non-trivial, basis vectors are not globally defined. Let N be the region $0 \leq \theta < \pi$, and S , $0 < \theta \leq \pi$. N (S) is the patch excluding the south (north) pole. We write for an orthonormal basis of the cotangent bundle,

$$\begin{bmatrix} e^1 \\ e^2 \end{bmatrix}^{N,S} = \begin{bmatrix} \cos\varphi & \delta_{N,S}\sin\varphi \\ -\delta_{N,S}\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} d\theta \\ \sin\theta d\varphi \end{bmatrix}, \tag{5.2}$$

where $\delta_N = -1$, $\delta_S = +1$. On the overlap $N \cap S$, the transition function relates the two choices,

$$\begin{bmatrix} e^1 \\ e^2 \end{bmatrix}^N = \begin{bmatrix} \cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & \cos 2\varphi \end{bmatrix} \begin{bmatrix} e^1 \\ e^2 \end{bmatrix}^S. \tag{5.3}$$

Here, the number 2 in the trigonometry functions is actually the Euler characteristic of the 2-sphere. The Riemannian connection is easy to find.

$$d_\wedge \begin{bmatrix} e^1 \\ e^2 \end{bmatrix}^{N,S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \omega_{N,S} \begin{bmatrix} e^1 \\ e^2 \end{bmatrix}^{N,S}, \tag{5.4}$$

where $\omega_{N,S} = (\delta_{N,S} + \cos\theta)\, d\varphi$. The orthonormal basis for the tangent bundle, dual to that for the cotangent bundle, is

$$[e_1, e_2]^{N,S} = [\partial_\theta, \frac{1}{\sin\theta}\partial_\varphi] \begin{bmatrix} \cos\varphi & \delta^{N,S}\sin\varphi \\ -\delta^{N,S}\sin\varphi & \cos\varphi \end{bmatrix}, \tag{5.5}$$

where no differentiation is implied. The spinor fields are the sections of a vector bundle whose associated principal bundle is the square root of that for the tangent bundle. The square root takes two signs giving two chiralities $\pm$. The transition function is then given by

$$\psi^N_\pm = e^{\pm i\varphi}\psi^S_\pm, \tag{5.6}$$

while the Riemannian connection is half of that for the vector fields,

$$\Gamma^{N,S}_\pm = \pm\frac{i}{2}\omega_{N,S}\,. \tag{5.7}$$

The chiral Dirac operator is of the form

$$-iD_{\pm}^{N,S} = -i\sigma_{\pm}^{N,S}\cdot(dP + \Gamma_{\pm}^{N,S} + A), \tag{5.8}$$

where $\sigma_{\pm}^{N,S} = \pm\, ie_1^{N,S} + e_2^{N,S}$, $dP = e^1e_1 + e^2e_2$ and $e_\mu \cdot e^\nu = \delta_{\mu\nu}$. This operator flips the chirality of the spinor field on which it acts. The boundary conditions on the spinors are simply that they are well-defined functions on N and S and related by the transition functions specified. In actual calculations, we can make the following substitution so that we do not deal with N and S separately.

$$\psi_{\pm} = e^{\pm i\delta_{N,S}\varphi/2}\psi_{\pm}^{N,S}. \tag{5.9}$$

With this being done, care should be taken about the boundary conditions. For example, the functions are anti-periodic in φ and there are special conditions on the north and south poles.

Eigenvalues of the Dirac Operator in Absence of the Gauge Field

For $A = 0$, we now solve the eigenvalue problem

$$-i\begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}^{N,S}\psi^{N,S} = \lambda\psi^{N,S}. \tag{5.10}$$

Equivalently, we consider the following problem,

$$-D_-D_+\psi_+ = \lambda^2\psi_+, \tag{5.11}$$

where I have made the substitution to eliminate factors depending on N,S. Explicitly,

$$- D_- D_+ \;=\; - \,(\partial_\theta + \frac{1}{2}\cot\theta)^2 - \frac{1}{\sin^2\theta}\partial_\varphi^2 - i\frac{\cos\theta}{\sin^2\theta}\partial_\theta. \tag{5.12}$$

Let $\psi_+ = e^{i(m+\frac{1}{2})\varphi}\Theta$ to separate the variables. Here, m is an integer and $\frac{1}{2}$ is needed to fit the anti-periodic condition. Let $x = \cos\theta$. The equation now becomes

$$[\frac{d}{dx}(1 - x^2)\frac{d}{dx} - \frac{(2m + 1 + x)^2}{4(1 - x^2)} - \frac{1}{2} + \lambda^2]\Theta = 0. \tag{5.13}$$

This is an equation of the Fuchsian type with three regular singularities, $\pm\, 1$ and ∞. In terms of the Riemann P-function

$$\Theta \sim \left\{ \begin{matrix} 1 & -1 & \infty & \\ \alpha_1 & \beta_1 & \gamma_1\,, & x \\ \alpha_2 & \beta_2 & \gamma_2 & \end{matrix} \right\},$$

where $\alpha_{1,2} = \pm\,\frac{m+1}{2}$, $\beta_{1,2} = \pm\,\frac{m}{2}$, $\gamma_{1,2} = \frac{1}{2} \pm \lambda$. Let $z = \frac{1-x}{2}$,

$$\Theta \sim \left\{ \begin{matrix} 0 & 1 & \infty & \\ \alpha_1 & \beta_1 & \gamma_1\,, & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{matrix} \right\}.$$

One of the solutions is obtained the following way,

$$\Theta \sim \left\{ \begin{array}{cccc} 0 & 1 & \infty & \\ \frac{m+1}{2} & \frac{m}{2} & \frac{1}{2}+\lambda, & z \\ -\frac{m+1}{2} & -\frac{m}{2} & \frac{1}{2}-\lambda & \end{array} \right\}.$$

$$\sim \left\{ \begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & m+1+\lambda, & z \\ -(m+1) & -m & m+1-\lambda & \end{array} \right\} (1-z)^{\frac{m}{2}} z^{\frac{m+1}{2}}.$$

$$\sim z^{\frac{m+1}{2}} (1-z)^{\frac{m}{2}} F(m+1+\lambda, m+1-\lambda, m+2; \ z),$$

where $F(\ , \ , \ ; \ z)$ is the Gauss hypergeometrical series. By interchanging α_1 and α_2 and/or β_1 and β_2 , we get three other solutions in terms of the hypergeometrical series. Altogether, we have four expressions of which any two are linearly independent,

$$\begin{array}{l} z^{\frac{m+1}{2}} (1-z)^{\frac{m}{2}} F(m+1+\lambda, m+1-\lambda, m+2; \ z) \\ z^{\frac{m+1}{2}} (1-z)^{-\frac{m}{2}} F(1+\lambda, 1-\lambda, m+2; \ z) \\ z^{-\frac{m+1}{2}} (1-z)^{-\frac{m}{2}} F(-m+\lambda, -m-\lambda, -m; \ z) \\ z^{-\frac{m+1}{2}} (1-z)^{\frac{m}{2}} F(\lambda, -\lambda, -m; \ z) \end{array} \qquad (5.14)$$

The eigensections have the form $\psi_+ = e^{-i(\frac{\delta_{N,S}-1}{2}-m)\varphi}\Theta_m(z)$. They are non-singular functions on N or S respectively. The boundary conditions then follows.

$$\begin{array}{l} \Theta_{m\neq-1}(z=0)=0, \quad \Theta_{-1}(z=0) = \textit{finite}\,; \\ \Theta_{m\neq 0}(z=1)=0, \quad \Theta_0(z=1) = \textit{finite}\,. \end{array} \qquad (5.15)$$

From these, we determine the eigenvalues and eigenfunctions.

$$\Theta_{lm} = z^{\frac{|m| + sgn(m)}{2}} (1 - z)^{\frac{|m|}{2}} \times$$
$$F(|m| + \frac{1 + sgn(m)}{2} + l,\ |m| + \frac{1 + sgn(m)}{2} - l,\ |m| + 1 + sgn(m);\ z)\,,$$
$$Y^{N,S}_{lm\,\mp} = \exp[\pm i(\frac{\delta^2_{N,S} - 1}{2} - m)\varphi]\,\Theta_{lm}(z), \tag{5.16}$$
$$(-D_\pm D_\mp Y_{lm\,\mp})^{N,S} = l^2 Y^{N,S}_{lm\,\mp},$$

where $sgn(m \geq 0) = 1, sgn(m < 0) = -1;\ l = 1,2,3,\ldots,\ -l \leq m \leq l - 1.$ The Dirac operator then has integers (except zero) as its eigenvalues and the degree of degeneracy for $\lambda = l$ is $2|l|$. There are no zero modes in this case.

A Two Parameter Family of Gauge Potentials

Let us recall that π_1 (A/G) $= \pi_0$(G) $= 0$. π_2 (A/G) $= \pi_1$(G) $= \pi_3(SU(2)) = Z = H^2$ (A/G). The non-triviality of the determinant line bundle is from a basic 2-sphere in the orbit space. I now construct such a sphere in the following order : a representative of the generator of $\pi_3(SU(2))$ $\rightarrow$ a representative of the generator of π_1(G) $\rightarrow$ a representative of the generator of π_2(A/G). The final object is the 2-sphere we are seeking and is a two parameter family of the gauge potentials.

The simplest choice for the representative of the generator of $\pi_3(SU(2))$ is the usual identification of S^3 to SU(2).

$$S^3 = \{(x_1, x_2, x_3, x_4)\,|\,x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

$$U = \begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix} : S^3 \rightarrow SU(2)\,. \tag{5.17}$$

It is convenient to use spherical coordinates,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \cos\frac{s}{2} \\ \sin\frac{s}{2}\cos\theta \\ \sin\frac{s}{2}\sin\theta\cos\varphi \\ \sin\frac{s}{2}\sin\theta\sin\varphi \end{bmatrix}, \quad \begin{array}{c} 0 \le s \le 2\pi \\ 0 \le \theta \le \pi, \;\; \varphi \sim \varphi + 2\pi. \end{array} \tag{5.18}$$

$(\theta,\ \varphi)$ will be identified with the spherical coordinates on the space-time 2-sphere. To make the SU(2) matrix the identity at $\theta = 0$, the north pole , we multiply this by its inverse at the pole. The basic loop in **G** is thus found.

$$g_s = \begin{bmatrix} e^{-is/2} & 0 \\ 0 & e^{is/2} \end{bmatrix} \begin{bmatrix} \cos\frac{s}{2} + i\sin\frac{s}{2}\cos\theta & \sin\frac{s}{2}\sin\theta e^{i\varphi} \\ -\sin\frac{s}{2}\sin\theta e^{-i\varphi} & \cos\frac{s}{2} - i\sin\frac{s}{2}\cos\theta \end{bmatrix}. \tag{5.19}$$

Now pick an arbitrary gauge potential, in this case $A = 0$. Use g_s to transform 0. $\{A_s = g_s^{-1}dg_s\}$ is a non-contractible loop in the fiber at $[A = 0]$ of the universal bundle **A**$\rightarrow$**A/G**. But anything is contractible in the total space **A**. Arbitrarily choose another connection A_0. Then $\{A(s,t) = (1 - t)A_0 + tA_s \mid 0 \le t \le 1\}$ is a disc in **A**. Its projection in **A/G** is the representative of the generator of π_2(**A/G**). For simplicity, A_0 is chosen to be

$$A_0 = i\vec{\sigma} \cdot d\vec{x} =$$

$$\begin{bmatrix} -i\sin\theta & e^{i\varphi}\cos\theta \\ -e^{-i\varphi}\cos\theta & i\sin\theta \end{bmatrix} d\theta + \begin{bmatrix} 0 & ie^{i\varphi} \\ -ie^{-i\varphi} & 0 \end{bmatrix} \sin\theta d\varphi. \tag{5.20}$$

$$iA(s,t) =$$

$$d\theta \begin{bmatrix} p_1 \sin\theta & e^{i\varphi}(p_2 + ip_1\cos\theta) \\ e^{-i\varphi}(p_2 - ip_1\cos\theta) & -p_1\sin\theta \end{bmatrix} + \tag{5.21}$$

$$\sin\theta d\varphi \begin{bmatrix} p_2\sin\theta & -e^{i\varphi}(p_1 - ip_2\cos\theta) \\ -e^{-i\varphi}(p_1 + ip_2\cos\theta) & -p_2\sin\theta \end{bmatrix},$$

where $p_1 = \frac{t}{2}\sin s + 1 - t$, $p_2 = \frac{t}{2}(1 - \cos s)$. The chiral Dirac operator, with the strange boundary condition, in background of these potentials is

$$iD_\pm(s,t) = \pm[-\partial_\theta \pm \frac{i}{\sin\theta}\partial_\varphi - \frac{1}{2}\cot\theta \pm p_\pm \begin{bmatrix} \sin\theta & ie^{i\varphi}(\pm 1 + \cos\theta) \\ ie^{-i\varphi}(\pm 1 - \cos\theta) & -\sin\theta \end{bmatrix}], \tag{5.22}$$

where $p_\pm = \pm ip_1 + p_2$.

The Eigenvalues and the Determinants

Formally, the determinant of the chiral Dirac operator is the infinite product of the eigenvalues of the chiral operator. But the operator flips chirality. An equation of the form $-iD_\pm\psi_\pm = \lambda\psi_\pm$ would not make sense, especially when the tangent bundle is non-trivial. The following recipe is then used to obtain the determinant, keeping in mind that adding a constant factor to the determinant is immaterial. Consider the eigenvalues defined by

$$- D_{\pm}(B)D_{\mp}(A)\psi_{\mp} = \lambda_{\mp}(A,B)\psi_{\mp}, \tag{5.23}$$

with B a fixed connection. The determinant is given by

$$\mathrm{Det}(D_{\pm}(A)) = \prod_i \frac{\lambda_{i\pm}(A,B)}{\lambda_{i\pm}(B,B)}. \tag{5.24}$$

This is equivalent to the modified version in Chapter 3 . I choose both A and B from the family from above, labelled by p and p' respectively. The equation then is

$$- D_{\pm}(p')D_{\mp}(p)\psi_{\mp} = \lambda\psi_{\mp}. \tag{5.25}$$

Guided by the results of the case without the gauge potential, there are six cases with the corresponding factorizations to be solved.

Case $\begin{matrix}1\\2\end{matrix}$ $:m > 0, \quad D_{\pm}D_{\mp}$

$$\psi_{\mp} = \begin{bmatrix} u z^{\frac{m}{2} \mp \frac{1}{4} + \frac{1}{4}} (1 - z)^{\frac{m}{2} \mp \frac{1}{4} - \frac{1}{4}} e^{i(\mp m + \frac{1}{2})\varphi} \\ v z^{\frac{m}{2} \pm \frac{1}{4} + \frac{1}{4}} (1 - z)^{\frac{m}{2} \pm \frac{1}{2} - \frac{1}{2}} e^{i(\mp m - \frac{1}{2})\varphi} \end{bmatrix}; \tag{5.26}$$

Case $\begin{matrix}3\\4\end{matrix}$ $:m < 0, \quad D_{\pm}D_{\mp}$

$$\psi_{\mp} = \begin{bmatrix} uz^{-\frac{m}{2} \pm \frac{1}{4} - \frac{1}{4}}(1-z)^{-\frac{m}{2} \pm \frac{1}{4} + \frac{1}{4}} e^{i(\mp + \frac{1}{2})\varphi} \\ vz^{-\frac{m}{2} \mp \frac{1}{4} - \frac{1}{4}}(1-z)^{-\frac{m}{2} \mp \frac{1}{4} + \frac{1}{4}} e^{i(\mp m - \frac{1}{2})\varphi} \end{bmatrix}; \tag{5.27}$$

Case 5 :$m = 0, \quad D_+D_-$

$$\psi_- = \begin{bmatrix} u(1-z)^{\frac{1}{2}} e^{\frac{i}{2}\varphi} \\ vz^{\frac{1}{2}} e^{-\frac{i}{2}\varphi} \end{bmatrix}; \tag{5.28}$$

Case 6 :$m = 0, \quad D_-D_+$

$$\psi_+ = \begin{bmatrix} uz^{\frac{1}{2}} e^{\frac{i}{2}\varphi} \\ v(1-z)^{\frac{1}{2}} e^{-\frac{i}{2}\varphi} \end{bmatrix}. \tag{5.29}$$

If $\begin{bmatrix} u \\ v \end{bmatrix}$ is finite from $z = 0$ to $z = 1$, then the solutions satisfy the boundary conditions. We will look for polynomial solutions. Since we shall find that the solutions include all the solutions for $p = p' = 0$ case, we shall have found all the solutions. We look at one case at a time.

Case 1 $m > 0, \quad -D_+(p')D_-(p)$. We assume the solution to be of the form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \sum_{i=0}^{n} \begin{bmatrix} f_i z^i \\ g_i z^{i-1} \end{bmatrix}. \tag{5.30}$$

Of course, $g_0 = 0$. The recursion equation for the coefficients is then

$$\{2p_-[-(l+m)+2p'_+]+2p'_+(l+m-1)\}\begin{bmatrix} 1 & -i \\ -i-1 \end{bmatrix}\begin{bmatrix} f_{l-1} \\ g_{l-1} \end{bmatrix}+$$

$$\{\lambda+[-(l+m)+2p'_+\begin{bmatrix} -1 & 2i \\ 0 & 1 \end{bmatrix}](l+m)+2p_-\begin{bmatrix} l+1 & -2ip'_+ \\ 0 & l \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i-1 \end{bmatrix}\}\begin{bmatrix} f_l \\ g_l \end{bmatrix}+ \tag{5.31}$$

$$\begin{bmatrix} l+1 & -2ip'_+ \\ 0 & l \end{bmatrix}(l+m+1)\begin{bmatrix} f_{l+1} \\ g_{l+1} \end{bmatrix}=0.$$

Now, $f_l = g_l = 0$, for $l > n$. The first equation having the $n+2$, $n+1$, n-th terms gives $g_n = -if_n$. Substitute this into the next equation. It becomes

$$[2p_-(-n-m+2p'_+)+2p'_+(n+m-1)]\begin{bmatrix} 1 & -i \\ -i-1 \end{bmatrix}\begin{bmatrix} f_{n-1} \\ g_{n-1} \end{bmatrix}+[\lambda+(2p'_+-n-m)(n+m)]f_n\begin{bmatrix} 1 \\ -i \end{bmatrix}=0. \tag{5.32}$$

The third equation, multiplied by $\begin{bmatrix} 1 & -i \\ -i-1 \end{bmatrix}$, gives

$$[\lambda-(n+m-1+2p'_+)(n+m-1)+2p_-(1-2p'_+)]\begin{bmatrix} 1 & -i \\ -1-1 \end{bmatrix}\begin{bmatrix} f_{n-1} \\ g_{n-1} \end{bmatrix}+(1-2p'_+)(n+m)f_n\begin{bmatrix} 1 \\ -i \end{bmatrix}=0. \tag{5.33}$$

We thus find the eigenvalues to the solution of the following quadratic equation by requiring the compatibility of the above two equations.

$$0=\begin{vmatrix} \lambda+(2p'_+-n-m)(n+m) & 2p_-(2p'_+-n-m)+2p'_+(n+m-1) \\ (1-2p'_+)(n+m) & \lambda-(n+m-1+2p'_+)(n+m-1)+2p_-(1-2p'_+) \end{vmatrix}. \tag{5.34}$$

The product of the two roots is given by the above determinant with λ set to 0. The value of the product is then $(n+m-1)^2(n+m)^2$, which is independent of $p_\pm$ and $p'_\pm$. The integer is non-negative. The $n = 0$ case is not allowed because $f_0 = ig_0$, and $g_0 = 0$ has only zero solution. $n = 1,2,3,....$

Case 2 $m > 0, \; - D_-(p')D_+(p)$. The solutions in this case can be gotten from Case 1 by the following interchanges.

$$f \to g, \; g \to f, \; p_- \to p_+, \; p'_+ \to p'_- .$$

The eigenvalues are the complex conjugate of those in Case 1.

Case 3 $m < 0, \quad - D_+(p')D_-(p)$. Let $z \to 1 - z$. Then the equation is compared with that of Case 2. The eigenvalues are then found to those from Case 1 with $m \to -m$. In other words, the eigenvalue for m and $-m$ are the same.

Case 4 As expected, eigenvalues are the complex conjugate of those in Case 3. Or they are obtained from those by making the substitution $m \to |m|$.

Case 5 $m = 0, \quad - D_+(p')D_-(p)$.

$$\begin{bmatrix} u \\ v \end{bmatrix} = \sum_{l=0}^{n} \begin{bmatrix} f_l \\ g_l \end{bmatrix} z^l. \tag{5.35}$$

The recursion relations produce the following quadratic equation for the eigenvalues.

$$0 = \begin{vmatrix} \lambda + (n+1)(-n-1+2p_-) - 2p'_+(2p_- - 1) & 2p_-(n+1) - 2p'_+(n+2p_-) \\ -n(-1+2p_-) & \lambda - (n+2p_-)n \end{vmatrix}. \tag{5.36}$$

The product for the two roots is $n^2(n+1)^2$. Again, they are independent of the parameters. These eigenvalues are for $n = 1,2,3,\ldots$. For $n = 1$, the recursion relations give the equations

$$g_0 = if_0$$
$$\{\lambda + [-1 + 2p_-\begin{bmatrix} 1 & 0 \\ 2i & -1 \end{bmatrix} - 2p'_+\begin{bmatrix} -1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2ip_- & -1 \end{bmatrix}]\}\begin{bmatrix} f_0 \\ g_0 \end{bmatrix} = 0. \tag{5.37}$$

Only one solution exists, $\lambda = (1 - 2p_-)(1 - 2p'_+)$.

Case 6 $m = 0, \quad -D_-(p')D_+(p)$. The eigenvalues are the complex conjugate of those above in Case 5.

In summary, all eigenvalues for chiral Dirac operator pair up, except one, to give constant products. The unpaired one is equal to $(1 - 2p'_\mp)(1 - 2p_\pm)$ for the operator $-D_\mp(p')D_\pm(p)$.

The Determinants

The determinant of the chiral Dirac operator is obtained by taking the product of all eigenvalues. Since the product is infinite, regularization is needed. In the particular case at hand, the simplest procedure is to take the product pairwise and factor the constants out. This way, we are only left with the unpaired eigenvalue. For the chiral case, p' is considered a constant. The factor $(1 - p'_\pm)$ can be factored out also. The final result is then

$$\mathrm{Det}[-iD_\pm(p)] = 1 - 2p_\pm = (1-t)(1 \pm 2i) + te^{\pm is}. \tag{5.38}$$

The determinant for the non-chiral case is found by letting $p = p'$,

$$\mathrm{Det}[-iD(p)] = \lambda_0(p,p) = 5(1-t)^2 + t^2 + 2t(1-t)(\cos s + 2\sin s). \tag{5.39}$$

The Anomaly and Topological Phase Winding

In the polar coordinates of (t,s), the two-parameter family of gauge potentials is the unit disc with t being the radial coordinate and s the angular coordinate. On the boundary of the disc, $t = 1$, the points represent a circle of gauge equivalent potentials. The determinant is $\exp(\pm is)$ for $t = 1$. The fact that this is not a constant means the determinant is anomalous. Further, the modulus is a constant and the phase winds around the boundary with winding number ± 1. This is what a section of non-trivial equivariant bundle looks like. When projected down to the orbit space, the disc becomes a 2-sphere. The phase of the determinant becomes multivalued at the point corresponding to the boundary under the projection. The phase winds around that point and this exactly is what happens to the wave function of a particle moving around a Dirac monopole. The determinant vanishes at a point inside the unit disc

$$(t\cos s,\ t\sin s) = (1,\ 2)\frac{-1}{1+\sqrt{5}}. \tag{5.40}$$

The phase winding starts here but because of continuity, it persists to the boundary where the gauge potentials are gauge equivalent. This zero mode is actually dictated by the first definition of the Index of the Dirac operator. Thus we have shown the relation between the anomaly, the determinant line bundle of the Hilbert bundles and the zero modes of the Dirac operator. Since we can always find the zero mode by constructing a two parameter family which is a non-contractible 2-sphere in the orbit space, these zero modes form a subspace in the space of connections of codimension 2. It can also be easily verified that the 1-cocycle in **G** integrated around the boundary circle gives the winding number of the phase.

The determinant of the non-chiral operator is equal to 1 at the boundary and is therefore gauge invariant.

6. Anomalies in the Hamiltonian Formalism

In the last Chapter, we saw that the anomaly in the path-integral formalism is due to the topological winding of the eigenvalues of the chiral Dirac operator. In this Chapter, I am going to show that topological winding of the eigenvectors of the Dirac Hamiltonian makes the Fock space ill behaved as a representation space of the time independent gauge transformation group. The first model I consider is the same as the one in last Chapter[46]. The formalism is different though. The second model is in $2+1$ dimensions where a real structure (a Majorana condition) exists. The anomaly in this case is an example of a non-perturbative anomaly[46,47,48].

SU(2) Theory in 1 + 1 Dimensions

The compactified spatial manifold is the circle S^1. The fiber of the universal bundle $\mathbf{A} \rightarrow \mathbf{A}/\mathbf{G}$ is a copy of the group $\mathbf{G}$. We would like to see if the vacuum line bundle is trivial and if not what it looks like. Now, $\pi_1(\mathbf{G})$

$= \pi_{1+1}(SU(2)) = 0$, $\pi_2(\mathbf{G}) = \pi_{2+1}(SU(2)) = Z$. Again, the Hurewicz theorem and the universal coefficient theorem gives $H^2(\mathbf{G}) = Z$. The bundles are therefore characterized by the first Chern character. The bundles live on a basic 2-sphere in the group. A 2-sphere is given by

$$g(\theta, \varphi; x) = \begin{bmatrix} e^{-ix/2} & 0 \\ 0 & e^{ix/2} \end{bmatrix} \begin{bmatrix} \cos\frac{x}{2} + i\sin\frac{x}{2}\cos\theta & \sin\frac{x}{2}\sin\theta e^{i\varphi} \\ -\sin\frac{x}{2}\sin\theta e^{-i\varphi} & \cos\frac{x}{2} - i\sin\frac{x}{2}\cos\theta \end{bmatrix}, \tag{6.1}$$

where θ, φ parameterize the 2-sphere and x is the spatial coordinate, $x \sim x + 2\pi$. We would like to study the eigenbundles on $\{A \mid A = gdg^{-1}\}$. Since we shall use the technique of spectral flow, a parameter t is used. $\{A \mid A = tgdg^{-1},\ 0 \le t \le 1\}$ represents the generator of $\pi_3(\mathbf{A}/\mathbf{G})$. The chiral Hamiltonian is then, with the Riemannian connection chosen to be $\Gamma = i/2dx$,

$$H = -i\frac{d}{dx} + \frac{1}{2} + t\sin\frac{\theta}{2}\begin{bmatrix} \sin\theta & i\cos\frac{\theta}{2}e^{-i(x-\theta)} \\ -i\cos\frac{\theta}{2}e^{i(x-\theta)} & -\sin\frac{\theta}{2} \end{bmatrix}. \tag{6.2}$$

The eigenvalues and the eigenvectors can easily be solved. The eigenvalues are

$$E_n^{\pm} = n \pm \sqrt{\frac{1}{4} - t(1-t)\sin^2\frac{\theta}{2}}. \tag{6.3}$$

The energy bands have two kinds of degeneracies. The first kind is the degeneracies occuring between E_n^- and E_{n-1}^+ at $\theta = 0$. This has nothing to do with the topological properties and can be lifted by a simple perturbation. The second kind is the degeneracies that occur at $t = 1/2$, $\theta = \pi$. By calculating the first Chern character for each band, we find that, for $0 \le t < 1/2$, it is 0; for

$1/2 < t \leq 1$, it is ± 1. From the discussion in Chapter 3 on spectral flows, this can easily be interpreted as exchange of the first Chern characters at $t = 1/2$ between the band E_n^+ and E_n^-. Since we are interested in the vacuum line bundle, we look at the space spanned by the negative energy states. For $n < 0$, all the exchanges are within Φ_-. They do not affect the sum. The only one that has an exchange across zero is E_0^-. The vacuum line bundle is isomorphic to the line bundle for this band. What is wrong with this line bundle? Or what is wrong with the eigenvector for this band? At $t = 1$, the background is a 2-sphere in a fiber where the connections are gauge transformation of each other. The eigenvector is

$$\begin{bmatrix} \cos\dfrac{\theta}{2} e^{-ix} \\ i\sin\dfrac{\theta}{2} e^{-i\varphi} \end{bmatrix}$$

up to a phase. The point is that this is not well defined at the "south pole" $\theta = \pi$. Even if we choose a phase function on the 2-sphere, we can only move the singularity around but never get rid of it. The eigenvector cannot be a function on the group of gauge transformations, and therefore the Fock space fails to be a representation of the group. The culprit is a phase. It can be shown, however, that it is a representation space for the centrally extended group by the U(1) phase. This is reflected in the algebra of charge densities as a central term. The central term is given by the second cocycle[49].

SU(2) Theory in 2 + 1 Dimensions

There is a Majorana condition that can be imposed in this case. Everything can be made real. The vacuum line bundle is a real line bundle now. In the temporal gauge, we consider gauge potentials in the two spatial dimensions and time independent gauge transformations. $\pi_0\,(\mathbf{G}) = 0$. $\pi_1(\mathbf{G}) = \pi_3(SU(2)) = Z$. By the Hurewicz theorem and the universal coefficient theorem, $H^1(\mathbf{G}, Z_2) = Z_2$. So there are two kinds of line bundles on the group: the trivial one and non-trivial one. The non-trivial one essentially lives on the basic non-trivial loop in the group. I choose the same loop as in last Chapter and extend it the same way to the two-parameter family of gauge potentials. The interpretation is of course different. The Hamiltonian is just the two-dimensional non-chiral Dirac operator,

$$H = \begin{bmatrix} 0 & -iD_-(p) \\ -iD_+(p) & 0 \end{bmatrix}. \tag{6.4}$$

The eigenvalue problem

$$H\psi = E\psi, \qquad \psi = \frac{1}{\sqrt{2}}\begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}. \tag{6.5}$$

is equivalent to

$$-D_+(p)D_-(p)\psi_- = E^2\psi_-, \;\; \psi_+ = -\frac{i}{E}D_-\psi_-. \tag{6.6}$$

We have solved this equation in the last Chapter. We set $p' = p$, $E = \pm \sqrt{\lambda}$. To study the vacuum line bundle, we only need the eigenvalues that can become zero. In this case, they are $E = \pm \sqrt{\lambda_0(p,p)}$. The eigenvectors are

$$\psi^{N,S} = \frac{1}{\sqrt{8\pi}} \begin{bmatrix} \pm\, e^{i\alpha - i\delta^{N,S}\varphi/2} \begin{bmatrix} \sin\frac{\theta}{2} e^{i\varphi/2} \\ -i\cos\frac{\theta}{2} e^{-i\varphi/2} \end{bmatrix} \\ e^{i\delta^{N,S}\varphi/2} \begin{bmatrix} \cos\frac{\theta}{2} e^{i\varphi/2} \\ i\sin\frac{\theta}{2} e^{-i\varphi/2} \end{bmatrix} \end{bmatrix}, \tag{6.7}$$

where α is the phase of $1 - 2p_-$. This can be put into the real form by the following matrix and an overall phase.

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ -i & 0 & 0 & -i \end{bmatrix}. \tag{6.8}$$

The result is then

$$\psi_r^{N,S} = \frac{1}{2\sqrt{\pi}} \begin{bmatrix} \sin(\frac{\delta^{N,S}-1}{2}\varphi - \frac{\alpha}{2} \pm \frac{\pi}{4}) \sin\frac{\theta}{2} \\ \cos(\frac{\delta^{N,S}-1}{2}\varphi - \frac{\alpha}{2} \pm \frac{\pi}{4}) \cos\frac{\theta}{2} \\ \sin(\frac{\delta^{N,S}+1}{2}\varphi - \frac{\alpha}{2} \pm \frac{\pi}{4}) \cos\frac{\theta}{2} \\ \cos(\frac{\delta^{N,S}-1}{2}\varphi - \frac{\alpha}{2} \pm \frac{\pi}{4}) \sin\frac{\theta}{2} \end{bmatrix}. \tag{6.9}$$

The important feature in this expression is that it depends on the phase α as a half angle. For $t < \sqrt{5}/(1 + \sqrt{5})$, as s changes 2π, the eigenvectors do not change; for $t > \sqrt{5}/(1 + \sqrt{5})$, the eigenvectors change the sign. The two bands touch each other exactly at $t = \sqrt{5}/(1 + \sqrt{5})$. Here, this can be interpreted as the exchange of the first Stiefel-Whitney class. Since only one of these go into the vacuum bundle, the vacuum (real) line bundle must be non-trivial at $t = 1$. The eigenvector is multivalued on the group of gauge transformation and is analogous to a square root branch cut in complex analysis. We conclude therefore that the vacuum line bundle is non-trivial and the Fock space is not a representation of the gauge transformation group[46]. It is rather a representation of the extended group by Z_2. This cannot show up in the commutation relations of the charge densities by the discrete nature of the group Z_2. It can only show up in the cohomology with Z_2 coefficient, which cannnot be expressed in terms of differential forms. The term non-perturbative anomaly arises because of this.

We conclude, for this Chapter, that anomalies can arise when the vacuum line bundle on a copy of the group of gauge transformations is non-trivial, and the Fock space fails to be a representation. The question of triviality of the vacuum line bundle can be determined by the technique of spectral flows. The exchange of topological indices at zero energy (zero modes !) is crucial in this sort of determination.

7. Covariance and Higher Codimensional Zero Modes

Just like the chiral currents in a non-chiral theory, there are other Green's functions that are not contained in the determinant of the Dirac operator. Treating the gauge field as the background, the basic function is the two point-function for the fermion, or the propagator in the gauge field background.

$$G_A(x,y) = \frac{\int D\overline{\psi}D\psi \quad \psi(x)\overline{\psi}(y)\exp[\int \overline{\psi} iD(A)\psi\sqrt{g}\ dx]}{\int D\overline{\psi}D\psi \quad \exp[\int \overline{\psi} iD(A)\psi\sqrt{g}\ dx]} = <\overline{\psi}(x)\psi(y)>. \qquad (7.1)$$

This propagator is expected to be covariant,

$$G_{A^g}(x,y) = g(x)G_A(x,y)g^{-1}(y). \qquad (7.2)$$

However, for a chiral fermion, this can fail[41]. There are topological reasons for this to fail. We have seen that a non-zero first Chern character of the Index implies a non-zero first Chern character of the determinant line bundle and this in turn implies that there is an anomaly. While it is true that there is no further obstruction to a trivial line bundle, there are further obstructions for the covariance of the propagator. The Chern characters of the Index bundle are non-zero generally. The first Chern character gives a U(1) phase ambiguity for the covariance. The phase winding starts from a codimension 2 zero mode and its projection to the orbit space is analogous to the wave function in the neighborhood of a Dirac magnetic monopole. The higher Chern characters describe the windings of SU(2), SU(3), ... on higher dimensional spheres. These windings start from higher codimensional zero modes and persist to an gauge orbit causing the propagator to be non-covariant. $ch_l(\text{Index}) \neq 0$ implies the existence of the zero mode of codimension $2l$. This can always be found by constructing a non-trivial $2l$-sphere in the orbit space.

The Chiral Propagator and the Cocycles

Since the cocycles come from the Chern characters by transgression to a fiber and the covariance properties are defined on the fiber, we should be able to find a relation between these objects. Recall that the Hilbert bundles of the fermions are equivariant bundles over the space of the connections under the group of gauge transformations. Essentially this means that the relation between fibers of the Hilbert bundle on a gauge orbit is fixed. To get from the fiber at A^g from that at A, we simply transform it by g. The bundle restricted to a gauge orbit is then trivial. Consistent with this, I introduce a connection ω^+ for H^+ such that when restricted to a gauge orbit, the parallel transport by this connection on a gauge orbit does not depend on the path and coincides with the action of **G**. The connection is then Maurer-Cartan in terms of the representation of **G** and the curvature vanishes when restricted to a gauge orbit. Then we have a connection for H^- related to ω^+ by

$$D_-\omega^- = (\omega^+ + \Delta)D_+, \tag{7.3}$$

where Δ is the exterior differential on **A**. Let $\Omega^\pm = \Delta\omega^\pm + \omega^\pm\omega^\pm$ be the curvatures. The Chern character of the Index then is

$$ch_r(\mathrm{Index}(D_+)) = \frac{i^r}{r!(2\pi)^r}\mathrm{TR}(\Omega^r_+ - \Omega^r_-). \tag{7.4}$$

By transgression, we get a $2r - 1$ form on a gauge orbit. Keeping in mind that the curvature is zero when restricted to the orbit, we find

$$t_{2r-1} = -\frac{(-i)^r}{(2\pi)^r}\frac{(r-1)!}{(2r-1)!}\mathrm{TR}(D_+^{-1}\delta D_+)^{2r-1}. \tag{7.5}$$

Here, δ is the differential in the orbit direction. This expression is determined up to an exact form. By the result of Atiyah and Singer, this is equal to, in cohomology, $\int_{S^{2n}}\omega_{2n}^{2r-1}$. We therefore find a relation between the chiral propagator and the cocycles. For $r = 1$, we recover the familiar anomaly equation. More explicitly, $\delta D_+ = \sigma^\mu_+ v^a D^b_{\mu a}$, $D^b_{\mu a} \equiv \delta_{ab}\partial_\mu + f_{acb}A^c_\mu$ and $v = \delta g g^{-1}$. Define $D_+^{-1}(x,y) = -iG_+(x,y)$. Then we have

$$-i\frac{(-i)^r}{(2\pi)^r}\frac{(r-1)!}{(2r-1)!}\int_{x_1,\dots,x_{2r-1}\in S^{2n}} v^{a_1}(x_1)\wedge\dots\wedge v^{a_{2r-1}}(x_{2r-1}) \times$$

$$D^{b_1}_{\mu_1 a_1} \ldots D^{b_{2r-1}}_{\mu_{2r-1} a_{2r-1}} \mathrm{tr}[G_+(x_{2r-1}, x_1)\sigma_+^{\mu_1}\lambda^{b_1} \ldots G_+(x_{2r-2}, x_{2r-1})\sigma_+^{\mu_{2r-1}}\lambda^{b_{2r-1}}] \tag{7.6}$$

$$= \int_{S^{2n}} \omega_{2n}^{2r-1}.$$

Taking the interior product with tangent vectors gives this quantity in components. We can simply take the integral sign and the v's away, antisymmetrize the whole expression. The result is the antisymmtrized diagrams of the covariant divergence of the current correlations. In contrast, the current correlation functions are the symmetrized sum of the diagrams. They coincide only when there is only one current, where the equation is the anomaly equation. With labels $[1,2,\ldots,2r-1]$ totally antisymmetrized, from the expressions for the cocycles [50] we get the following identities,

$$D^{b_1}_{\mu_1 a_1} \ldots D^{b_{2r-1}}_{\mu_{2r-1} a_{2r-1}} \mathrm{tr}[G_+(x_{2r-1}, x_1)\sigma_+^{\mu_1}\lambda^{b_1} \ldots G_+(x_{2r-2}, x_{2r-1})\sigma_+^{\mu_{2r-1}}\lambda^{b_{2r-1}}] = \tag{7.7}$$

for $1 \leq r \leq n$,

$$\frac{i^{n+r}}{(2\pi)^{n+r}} \frac{1}{(2r-1)!}(n-r)! \int_0^1 dt (1-t)^{2r-1}$$

$$\times \quad \int_{x \in S^{2n}} \mathrm{Str}[d\delta(x-x_1)\lambda^{a_1}, \ldots, d\delta(x-x_{2r-1})\lambda^{a_{2r-1}}, A, (tdA + t^2A^2)^{n-r}];$$

for $r < n$,

$$\frac{(-1)^{r-n-1}(r+n-1)!\, i^{r+n}}{(2n)!(2r-1)!(2\pi)^{r+n}} \int_{x \in S^{2n}} d\delta(x-x_1) \ldots d\delta(x-x_{2n}) \;\; \delta(x-x_{2n+1}) \ldots \delta(x-x_{2r-1})$$

$$\mathrm{Str}[\lambda_{a_1}, \ldots, \lambda^{a_{2n+1}};\ \lambda^{b_1}, \ldots, \lambda^{b_{r-n-1}}] f_{b_1 a_{2n+2}} \cdots f_{b_{r-n-1} a_{2r-2} a_{2r-1}},$$

where Str stands for the symmetrized trace. This is our interpretation of the odd cocycles. The even cocycles have a different interpretation as we shall see next.

Positive(or Negative) Energy States and the Even Cocycles

Here, we consider a chiral Hamiltonian in the background of time independent gauge transformations. The spatial manifold is assumed to be an odd dimensional sphere S^{2n-1}. The Hilbert bundle of the fermion on a gauge orbit is, as before, trivial. This does not mean though that if we take the positive energy part or the negative part we get a trivial bundle also. But the sum of all states is trivial. Actually, if we look at a single gauge orbit alone, there is no way of knowing if the bundles are trivial or not. This is because we are dealing with infinite dimensional Hilbert spaces. We have to rely on the continuity requirements in the total space of the universal bundle **A** and use the spectral flow to determine the topological invariants of these bundles. Since the eigenvalues move on the space **A**, the bundle of positive energy cannot be defined on the whole space. It can only be defined on subspaces with no zero energy state. In particular, if we pick a generic gauge orbit, there will be no zero modes and the eigenvalues do not change. We shall define $\Phi_\pm$ to be the bundle of positive(negative) energy on the orbit at $[A]$. Let $\{\psi_{l\pm}\}$ be the eigenvectors of positive(negative) energy.

$$\sum_l \psi_{l+}(x)\overline{\psi}_{l+}(y) + \sum_m \psi_{m-}(x)\overline{\psi}_{m+}(y) = 1\delta(x,y). \tag{7.8}$$

We write a connection in terms of these wave functions,

$$\omega^{\pm}_{mn} = \int_x \overline{\psi}_{m\pm}(x)\delta\psi_{n\pm}(x). \tag{7.9}$$

The curvature is, by using the formula above,

$$\Omega^{+}_{mn} = \int_{xy} \delta\overline{\psi}_{m+}(x)\psi_{l-}(x)\wedge\overline{\psi}_{l-}(y)\delta\psi_{n+}(y). \tag{7.10}$$

The Chern characters are given by

$$ch_r(\Phi_+) = -\, ch_r(\Psi_-) = \frac{1}{r!}\frac{i^r}{(2\pi)^r}\mathrm{TR}(\Omega^+)^r. \tag{7.11}$$

We now arrive at the sought after identity[44],

$$ch_r(\Phi_\pm) = \pm \int_{S^{2n-1}} \omega^{2r}_{2n-1}. \tag{7.12}$$

The charge density in second quantization is the normal ordered operator $\rho^a(x) \equiv :\overline{\psi}(x)\lambda^a\psi(x):$. The normal ordering is taken with respect to the vacuum at A. Formally carrying out the commutation, we find

$$[\rho_a(x),\ \rho^b(y)] = f_{abc}\rho^c(x)\delta(x,y) + c_{ab}(x,y), \tag{7.13}$$

where the anomalous term is

$$c_{ab}(x,y) = \sum_{mn}\overline{\psi}_{m-}(x)\lambda^a\psi_{n+}(x)\overline{\psi}_{n+}(y)\lambda^b\psi_{m-}(y) - \quad (m-,n+)\leftrightarrow(n+,m-).$$

By using the above identity for the first Chern character, we find that this term is given by the second cocycle. This is one of the verifications of the central extension conjectured by Faddeev[49].

Finally, I list below formulae given by Zumino for explicit expressions for the even cocycles[50].

For $r \leq n - 1$,

$$\omega^{2r}_{2n-1} = \frac{1}{(2r)!}(n - r - 1)!\frac{i^{n+r}}{(2\pi)^{n+r}}\int_0^1 dt(1 - t)^{2r}\mathrm{Str}[(dv)^{2r}, A,(tdA + t^2A^2)^{n-r-1}];$$

for $r \geq n$,

$$\omega^{2r}_{2n-1} = (-1)^{r-n}\frac{(r + n - 1)!}{(2n - 1)!}(2r)!\frac{i^{n+r}}{(2\pi)^{n+r}}\mathrm{Str}[(dv)^{2n-1}, v,(v^2)^{r-n}]. \tag{7.14}$$

8. Conclusions and Speculations

In this dissertation, I have described a study on the invariance and covariance problems for fermions in gauge fields. I first reviewed the topological tools for the study and generalized the notion of spectral flow. Then, in particular, it is shown that the non-trivial topology of the group of gauge transformations or the gauge orbit space provides the basis for phase windings in the fermionic path-integral. The phase winding starts from a zero mode of the Dirac operator and persists to a gauge orbit causing the fermion determinant to vary when gauge transformations are applied. In the Hamiltonian picture, the second quantization introduces a twist into the fermionic Fock space. The phase resides in the vacuum line bundle. The non-triviality of the line bundle is determined by the continuity in the space of connections and the spectral flow across the orbit space. When the anomaly is present, the Fock space fails to be a representation for the gauge transformation group but rather becomes one for its U(1) central extension in the complex case, and its Z_2 extension in the real case. For the complex case, this pathological behavior is reflected on the level of Lie algebra as a central term in the commutation relations of the charge densities.

As for the higher cocycles, they were related to the fermion propagator and fermion wave functions. Explicit formulae were given. No relation is to be expected for the Jacobi identities of the charge densities and the third cocycle. Since the determinant line bundle can never detect the higher cocycles, they do not show up in any Green's function without external fermions. They should have significance in a deconfined gauge theory. The gauge fields are treated as background fields so far. The problems of higher cocycles cannot be fully understood until the quantization of the gauge fields is understood. For example, how to treat covariant quantities in the first place? In a hypothetical theory built on quaternions, the third cocycle would show up in the determinant since it describes the winding of an SU(2) phase.

The main results of Chapter 5 and Chapter 6 are published in Communications of Mathematical Physics[46]. I also presented these results at the XXIII International Conference on High Energy Physics, Berkeley, 1986[53]. The results in Chapter 7 will be the subject of a forthcoming publication.

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